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Abstract

I started my PhD studies in August 2014 with a strong desire to push my own limits without knowing precisely the areas I wanted to cover in detail. To me, it was clear that I was interested by many different fields, however, I was particularly concerned with behavioural finance and with the fact that simple actions could be followed by strong market reactions.

It is in this context that my supervisor, Prof. Semyon Malamud, advised me to derive/measure the consequences of the large acceptance of the RiskMetrics variance model on the price of financial assets. Indeed, this method has the advantage of providing a simple formula to estimate the volatility of any financial asset, but above all, has been used significantly by practitioners in the financial industry. The question then arises, “Is there a link between this method and the price of financial assets?” In order to answer this question, I have designed a simple portfolio optimization model in which agents update volatility estimates with the RiskMetrics formula. Thanks to this simple idea my first project was born and I quickly realized that I could design an elegant model. With this framework I have been able to establish the existence of a risk factor of which the economic literature was unaware. Moreover, the empirical strategy allows me to estimate the relative risk aversion coefficient independently from established procedures. Importantly, my estimates are in line with the ones obtained with these (standard) approaches.

Meanwhile, I was also interested in a topic that covers a large part of all trades and is known as “over-the-counter markets”. These markets are characterized by their high level of decentralization. Indeed, every transaction is settled directly between a buyer and a seller. In these markets, the only way to secure a trade is to find another agent that is willing to take the counter-party. I became very interested in a series of books and articles that were modeling financial assets traded under these conditions. Hence, I have started to work in this field by solving different models. I was particularly interested in understanding how the price of assets traded with this constraint would react under stressful situations, that is, when agents had to liquidate their investments. After a trial and error process, I found that my model generated puzzling results. Indeed, this model made predictions that were against traditional wisdom. Above all, that model predicted that a large level of capital mobility could impair welfare.

Hence, I had eventually found the link between all the fields I wanted to cover in my thesis, where I debate the optimal allocation of capital under different types of frictions. While my first article treats the case of a centralized market with an agent who forecasts volatility using a particular method, the second article concerns how capital flows across markets when agents are subject to searching frictions. My third article is based on the second and discusses the interaction between innovation and the competition between firms supplying the same products. This article focuses on how capital is used by firms to innovate and how firms grow.

Key words: Asset Pricing, General Equilibrium, Volatility, Risk Premium, Search and Matching Frictions, OTC Markets, Oligopolistic Competition, Firm Size Dynamics, Game Theory.

Résumé

J'ai commencé mes études doctorales en automne 2014 avec l'ambition de repousser mes limites, sans avoir pour autant une idée précise des sujets dont je voulais traiter. Il était évident que je m'intéressais à beaucoup de domaines, à commencer par la finance comportementale et l'impact que de simples décisions prises par les intervenants financiers pouvaient engendrer sur les prix des actifs financiers.

C'est dans cette dynamique que mon superviseur, Prof. Semyon Malamud, m'a proposé de tenter d'estimer l'impact que l'adoption à grande échelle de la base de données Riskmetrics pouvait avoir sur le prix des actifs. En effet, la procédure utilisée pour établir cette base de données a le grand avantage de conduire à une formule de mise à jour des estimations de volatilité très simple et d'être largement utilisée dans l'industrie financière. Existe-t-il donc un lien entre cette méthode et le prix des actifs financier ? Pour répondre à cette question, j'ai développé un modèle d'optimisation de portefeuille ou les agents utilisent cette technique pour estimer le risque des actifs. Grâce à cette simple idée, mon premier projet était né. Je me suis rapidement rendu compte qu'il était possible de dériver un modèle très élégant mathématiquement. Ce modèle m'a permis d'établir l'existence d'un lien entre les primes de risques et un facteur qui n'était pas connu dans la littérature économique. De plus, l'évaluation empirique de mon modèle permet d'estimer l'aversion moyenne au risque des agents financier indépendamment des méthodes utilisées jusqu'à ce jour.

En parallèle, je me suis intéressé à un sujet couvrant une large partie des échanges, tant financiers que commerciaux : les marchés de gré à gré. Ces marchés sont caractérisés par une importante décentralisation. En effet, une transaction y est conclue directement entre un acheteur et un vendeur. Les individus intervenant sur

de tels marchés doivent donc trouver d'autres acteurs avant de pouvoir effectuer une transaction. J'ai rapidement été passionné par toute une série d'articles et de livres modélisant les actifs financiers lorsqu'ils sont échangés dans ces circonstances. En accord avec mon superviseur, j'ai donc commencé à travailler dans ce domaine, et à résoudre différents modèles. En particulier, je m'intéressais à comprendre comment réagissent le prix des actifs financiers échangés dans ce type de marchés en situation de stress, c'est-à-dire lorsqu'un individu, pour des raisons exogènes, doit liquider ses positions. Après de multiples tentatives, mon modèle a généré des résultats particulièrement troublants, car allant à l'encontre de certaines croyances fondamentales en économie. En effet, mon modèle prévoyait que dans certains cas, une grande mobilité du capital pouvait avoir des effets dommageables.
Le lien entre ces sujets, pourtant si différents au premier abord, était tout trouver : ma thèse allait traiter de l'allocation optimale du capital lorsque les agents économiques sont soumis à diverses contraintes. En effet, alors que mon premier sujet traite d'allocation du capital dans un marché centralisé, les intervenants estiment le risque de leurs positions à partir d'une unique approche. Le second sujet s'intéresse à la façon dont le capital est alloué lorsque les agents sont soumis à des frictions de recherche. Mon troisième article étend mon second afin de déterminer les effets de l'interaction entre l'activité de recherche et développement avec la concurrence entre produits finaux. Cet article se focalise sur la façon dont le capital (physique dans ce cas) est utilisé par les entreprises dans le but d'accroître leur taille et leur importance.

Mots-clés : Evaluation d'actifs financiers, modèle d'équilibre général, volatilité, prime de risque, frictions de recherche, marchés OTC, concurrence oligopolistique, taille des entreprises, théorie des jeux.

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1 The RiskMetrics Anomaly

This paper develops an overlapping-generations economy where the agents forecast future variance with the RiskMetrics variance model. I show that this feature induces a two-factor asset pricing model in which risk premiums are determined by both the market beta and the "RiskMetrics beta" (this latter results from an interaction between the past returns of the asset and the market). I confirm this prediction by estimating the model with panel regressions; importantly, the RiskMetrics beta provides a direct estimate of the relative risk aversion. The predicted range is consistent with previous studies. Furthermore, this effect is found in each of the major financial markets, except in Europe. The RiskMetrics anomaly is stronger when the forecasting horizon is short. The effect is robust when considering the three Fama-French factors and the Momentum factor.

1.1 Introduction

An elementary belief at the foundation of mean-variance analysis is that investors expect a higher return for increased risk. However, assessing the riskiness of a financial security requires adopting a methodology that can capture this feature efficiently. The RiskMetrics variance model, which was first established as a J.P. Morgan's internal risk management resource, was quickly endorsed by a large percentage of the market participants. In fact, the simplicity of this technology makes it highly transparent and easily reproducible. The success of this apparatus opens a channel for distortions in the pricing of financial assets. The goal of the current paper is to develop a theoretical model that accounts for this channel and then empirically test this model.

To this end, I develop an overlapping-generations (OLG) model in which mean-variance agents live two periods. In the first period, an agent is born with some initial capital and solves a portfolio allocation problem. In the second period the agent consumes the proceeds of the investment and dies, while a new agent is born into the economy. Importantly, I take into consideration the case when the conditional volatility, which is an important state variable into the agent's optimization problem, is computed with the RiskMetrics Variance model. Thus, the price of the financial security today feeds back into future prices via the volatility channel. I show that this behaviour induces a two-factor asset pricing model in which risk premiums are determined by both the market beta and the "RiskMetric beta" (this latter results from an interaction between the past returns of the asset and the market). Lastly, I validate this prediction by estimating the model with panel regressions. Importantly, my empirical strategy allows me to estimate the relative risk aversion coefficient independently from established procedures. My estimates are in line with the ones obtained with those approaches.

My model predicts that the expected risk premiums are linearly related to the conditional volatility. However, this relation, known as the risk-return tradeoff, is weaker than mainstream theories usually predict. With a simple adjustment to my model, I extend the analysis to the case where the conditional volatility is instead computed with a generalized autoregressive conditional heteroskedasticity (GARCH) model. This approach allows me to compute the risk-return tradeoff in closed-form. My findings suggest that the slope of the relation decreases monotonically with the parameter that controls for the persistence of the variance. Most strikingly, the relation can turn negative, which is consistent with the intense debate on this subject. Finally, my model uses an alternative methodology to test for market anomalies; the main innovation resides in the fact that I use panel regressions instead of the more standard Fama-MacBeth procedure. Since some variables are indexed both by asset and time (i.e. heterogenous), while the control variables are only indexed by time (i.e. homogenous), I orthogonalize sequentially for all homogenous variables and use the residual of these regressions in a second-step panel regression.

This paper is organized as follows: Section 1.2 discusses the related literature, Section 1.3 describes the model and derives the main predictions, Section 1.4 introduces the data and provides empirical support for the model's predictions and Section 1.5 presents the conclusions.

1.2 Related Literature

Because my paper provides an alternative to the capital asset pricing model (CAPM), it undeniably belongs to the literature that test this benchmark. Tests of the CAPM can be traced back to Black et al. (1972) and Fama and MacBeth (1973) who find, as predicted by the theory, a positive relation between average stock returns and asset betas. Following these findings, researchers discovered patterns in contradiction with this paradigm (which are nowadays known as "anomalies") and are surveyed in depth in Schwert (2003). The author defines an anomaly as a pattern that seems to be inconsistent with asset pricing theories. This includes for instance the value and the size effects. The value effect refers to the relation between asset returns and value-related variables. Empirical evaluation of this anomaly started with Basu (1977), who finds that low price-earnings portfolios have earned higher returns than the high price-earnings securities. This finding is in contradiction with the efficient market hypothesis. Basu's paper has been followed by many others, which include for instance Rosenberg et al. (1985) and Bondt and Thaler (1987). Another well-known anomaly is the size effect, first discussed by Banz (1981). The author shows that smaller firms provide higher risk adjusted returns on average than larger firms and concludes that the capital asset pricing model is misspecified. Tests of the ICAPM are also subject to empirical peculiarities. In particular, Boguth et al. (2010) demonstrate that unconditional alphas are biased when conditional beta covaries with the volatility. Fortunately, this finding does not affect my results since the equation estimated in my paper is different than the one pointed out here. (In fact, although conditional beta covaries with volatility, all the necessary information required to compute beta is available to investors ex ante. Furthermore, the relative risk aversion coefficient, which is the parameter estimated in my model, is unconditional and does not covary with the RiskMetrics factor).

A cornerstone of the asset pricing literature is provided by the arbitrage pricing theory from Ross (1976) which offers an alternative to the standard capital asset pricing model and importantly builds the theoretical foundations of factor models, which include seminal contributions such as Fama and French (1992), Fama and French (1993) and Carhart (1997). More recently, Frazzini and Pedersen (2013) argue that because many investors are constrained in the leverage that they can take, they overweight risky securities. Consistent with that, they show that a betting against beta factor, which is long leveraged low-beta assets and short high-beta assets, produces significant positive risk-adjusted returns. Malkhozov et al. (2014) study how funding constraints affect asset prices. Consistent with their framework they find that by holding betas constant, stocks with higher illiquidity earn higher alphas and Sharpe ratios. Finally, Malamud and Vilkov (2018) develop an OLG model in which investors differ in their investment horizons. They predict that in equilibrium the hedging demand of non-myopic investors leads to a two factor intertemporal capital asset pricing model and find evidence for their conjectures.

My model is based on the idea that agents take for granted the fact that volatility features clustering and strong persistence in time. Hence, my article belongs to the literature that analyses the dynamics of the volatility of financial returns. This body of literature is based on Engle (1982) and comprises notable contributions such as Bollerslev (1986), Engle and Bollerslev (1986), Glosten et al. (1993) and Andersen et al. (2001). Interestingly, Christoffersen et al. (1998) established that volatility forecastability declines quickly with the horizon; while the volatility fluctuations are highly predictable for daily horizons, the forecastability vanishes beyond horizons of two weeks. This is of primary importance in my paper since the volatility is one of the main variables used to design the agent's portfolio. Hence, investors with long holding periods cannot take advantage of the predictability of volatility. Ma et al. (2007) show that estimated standard errors of the GARCH(1,1) model are in general biased downward, implying that the persistence of volatility is less strong than it usually appears. This is also of great importance here since my model predicts that these parameters have a direct effect on the slope of the risk-return tradeoff.

My model is also closely related to Bacchetta et al. (2010). The authors design a model with which they analyze the large spikes in asset price risk during the recent financial crisis. My framework shares some important features with their model. In particular, all their findings are based on the link between the current asset price and the risk about the future asset price, which is also key in my model. Nevertheless, while their focus is on financial crisis and self-fulfilling shifts in risk, mine is on equilibrium risk premiums, which is complementary to their analysis.

My paper belongs to the risk-return tradeoff literature. Virtually all models in asset pricing exhibit a tradeoff between the risk premium and the conditional volatility. However, the empirical evidence in favour of this relationship is weak. While many authors find a positive relationship between the expected excess market return and conditional variance (for example, French et al. (1987), Chou (1988), Campbell and Hentschel (1992) and Bansal and Lundblad (2002)), others find the opposite (for instance, Baillie and DeGennaro (1990), Nelson (1991) and Glosten et al. (1993)). Interestingly, Lundblad (2007) shows with simulations that even 100 years of data constitute a small sample that may easily lead to a negative association between risk premiums and conditional volatility, even though the true relation is positive. Hence, the author uses nearly two centuries of history of returns and concludes in favour of a positive and significant risk-return tradeoff. Hedegaard and Hodrick (2016) proposes an alternative to increase the sample size. On the theory side, some models have emerged such as Bandi and Perron (2008) and Bonomo et al. (2015). The first paper found that the tradeoff is mild for short horizons, but increases with the time horizon, while the second paper proposed a model to reconcile stylized facts. My article contributes to this field of literature by providing an analytical characterization of this tradeoff when agents forecast future volatility with a GARCH(1,1) model. Importantly, the relation is positive in general, however, it is weaker than expected and can even turn negative for some parameterizations.

1.3 Model

The model presented here is an overlapping-generations economy in which a new representative agent is born at each time t with wealth W_t and lives for two periods. There are N securities, each one paying a dividend D_t^n and having w_n shares outstanding. While the old agent consumes the proceeds of their investment and dies, the young representative agent chooses a portfolio and invests the remainder of their wealth in the risk-free asset (which pays the risk-free return r_f) to maximize their utility:

$$\max_{\{x_n\}_1^N} \sum_{n=1}^{N} x_n \left(\mathbb{E}_t[D_{t+1}^n + P_{t+1}^n] - (1+r_f)P_t^n\right) - \frac{\gamma^A}{2} \sum_{n=1}^{N} \sum_{j=1}^{N} x_n x_j P_t^n P_t^j \sigma_{t+1}^{n,j}, \quad (1.1)$$

where γ^A is the representative's absolute risk aversion, P_t^n is the time t price of security n, P_{t+1}^n is the time $t+1$ price of security n, D_{t+1}^n is the time $t+1$ dividend paid by security n and $\sigma_{t+1}^{i,j}$ is the covariance of the returns between securities i and j at time $t+1$. Each agent estimates the time $t+1$ covariance between the returns of securities i and j with an exponential weighted moving average (EWMA),

$$\sigma_{t+1}^{i,j} = \phi \sigma_t^{i,j} + (1-\phi) r_t^i r_t^j \quad \forall i, j, t. \quad (1.2)$$

where ϕ is a constant to be defined. For instance, the RiskMetrics database (produced by JP Morgan, now known as ISS and maintained by Wharton Research

Data Services) uses an EWMA with $\phi = 0.94$ for updating daily volatility.[1] Finally, I am interested in the properties of the competitive equilibrium in which the demand for securities equals the supply, $x_n = w_n \quad \forall n$.

1.3.1 The Benchmark Case

I consider first the case when $\phi = 1$ and $\sigma^{i,j}_{t+1} = \sigma^{i,j}\ \forall t$. This benchmark is a simple textbook portfolio optimization problem, which will lead to the capital asset pricing model (CAPM). To derive equilibrium, let us consider the agent's first order condition:

$$\mathbb{E}_t[D^n_{t+1} + P^n_{t+1}] - (1 + r_f)P^n_t - \gamma^A P^n_t \sum_{j=1}^{N} w_j P^j_t \sigma^{n,j} = 0 \quad \forall n, t.$$

In this economy, the total market capitalization is given by $\Omega := \sum_{i=1}^{N} P_i \times w_i$. It follows that the market portfolio weight for security n is given by $w_n P^n_t/\Omega$. Thus, dividing the first-order condition (FOC) by P^n_t implies that it can be rearranged as follows,

$$\mathbb{E}_t[r^n_{t+1} - r_f] = \gamma \sum_{j=1}^{N} \pi_j \sigma^{n,j} \quad \forall n, t, \tag{1.3}$$

where $\sum_{j=1}^{N} \pi_j \sigma^{n,j}$ is the covariance between the market portfolio and asset n. Thanks to equation (1.3), which must hold for any asset and at any time t, one

[1] One important remark is that equation (1.2) is closely related to,

$$\sigma^{i,j}_{t+1} = \omega_{i,j} + \alpha_{i,j}\sigma^{i,j}_t + \beta_{i,j} r^i_t r^j_t \quad \forall i, j, t.$$

which is a *diagonal vech GARCH(1,1)* equation. Hence, equation (1.2) is simply a particular case with $\omega_{i,j} = 0$, $\alpha_{i,j} = \phi$, and $\beta_{i,j} = (1 - \phi)$, $\forall i, j$. All the following computations could have been made with GARCH, nevertheless, the main advantage of EWMA resides in the fact that practitioners use some well-defined constants (i.e. RiskMetrics variance model), which reduces the dimensionality of the estimation. Moreover, I find an equilibrium relation which is significantly more comprehensible.

can construct the expected market risk premium; multiplying (1.3) by the market portfolio weights for each security n and summing over all assets gives:

$$\mathbb{E}_t[r^M_{t+1} - r_f] = \gamma \sum_{i=1}^{N} \sum_{j=1}^{N} \pi_j \pi_i \sigma^{i,j}, \tag{1.4}$$

where $\sum_{i=1}^{N} \sum_{j=1}^{N} \pi_j \pi_i \sigma^{i,j}$ is the variance of the market portfolio. Thus, using the fact that $\beta^n := \frac{\sigma^{n,M}}{\sigma^2_M}$, the CAPM relation follows from dividing (1.3) by (1.4),

$$\mathbb{E}_t[r^n_{t+1} - r_f] = \beta^n \times \mathbb{E}_t[r^M_{t+1} - r_f] \quad \forall n, t. \tag{1.5}$$

Finally, analogous to the Intertemporal CAPM (ICAPM), the expected risk premium depends linearly on the (conditional) variance,

$$\frac{\partial \mathbb{E}_t[r^n_{t+1} - r_f]}{\partial \sigma^2} = \gamma.$$

1.3.2 The Full Case

Here, I consider the case when $\phi < 1$. It follows that the representative agent who is born at time t believes that the covariance between assets i and j at time $t+1$ is provided by equation (1.2). To derive equilibrium, let us consider the representative agent's (born at time t) first order condition for security n after having replaced σ^2_{t+1} by equation (1.2):

$$\mathbb{E}_t[D^n_{t+1} + P^n_{t+1}] - (1 + r_f)P^n_t =$$
$$\gamma^A P^n_t \left[\phi \sum_{j=1}^{N} w_j P^j_t \sigma^{n,j}_t + (1-\phi) r^n_t \sum_{j=1}^{N} w_j P^j_t r^j_t \right] = 0 \quad \forall n, t.$$

In this economy, the total market capitalization is still given by $\Omega := \sum_{i=1}^{N} P_i \times w_i$. Thus, the market portfolio weight for security n is given by $w_n P_t^n / \Omega$. Since the market return is given by $r_t^M := \sum_{j=1}^{N} \pi_j r_t^j$, dividing the FOC by P_t^n implies that it can be rearranged as follows,

$$\mathbb{E}_t[r_{t+1}^n - r_f] = \phi\gamma \sum_{j=1}^{N} \pi_j \sigma_t^{n,j} + (1-\phi)\gamma r_t^n r_t^M \quad \forall n, t, \tag{1.6}$$

where $\sum_{j=1}^{N} \pi_j \sigma_t^{n,j}$ is the covariance between the returns of the market portfolio and the returns of the asset n at time t. Multiplying this equation by the market portfolio's weight for security n and summing over all assets gives the expected market risk premium,

$$\mathbb{E}_t[r_{t+1}^M - r_f] = \phi\gamma \sum_{i=1}^{N} \sum_{j=1}^{N} \pi_i \pi_j \sigma_t^{i,j} + (1-\phi)\gamma r_t^M \times r_t^M, \tag{1.7}$$

where $\sum_{i=1}^{N} \sum_{j=1}^{N} \pi_i \pi_j \sigma_t^{i,j}$ is the variance of the market portfolio at time t. Substituting equation (1.7) into equation (1.6) and using the fact that $\beta_t^n := \frac{\sigma_t^{n,M}}{\left(\sigma_t^M\right)^2}$ leads to Proposition 1.1.

Proposition 1.1 *The equilibrium expected risk premium of security n at time t is given by,*

$$\mathbb{E}_t\left[r_{t+1}^n - r_f\right] = \beta_t^n \times \mathbb{E}_t\left[r_{t+1}^M - r_f\right] + (1-\phi)\gamma r_t^M \left[r_t^n - \beta_t^n r_t^M\right] \quad \forall n, t. \tag{1.8}$$

In this framework, not only the systemic risk is compensated for, but there is an additional factor (which comprises two parts) that derives from the above equations. The first part is the square of the last holding period return of the market r_t^M. The higher this quantity, the lower the expected return of any financial asset during the next period. The second part is the product between the last holding period returns of the market and asset n $r_t^M \times r_t^n$. If both quantities have the same sign, the expected return of security n will increase with the product of the returns. On the other hand, the expected return of security n will decrease with that product if returns have opposite signs. Lastly, the betas are time-varying; because both the variance of the market and the covariance between financial assets and the market are not constant the sensitivity to the market factor changes over time.

1.3.3 Risk-Return Tradeoff

Here, I consider the case where there is only one risky asset (equivalently, where each of the assets are independent from each other). To derive equilibrium I substitute equation (1.2) into the first order condition of the agent's problem. Then, because demand equals supply $x = w$. Hence, the following identity must hold at any time,

$$\mathbb{E}_t[r_{t+1}] = r_f + \gamma \left[\phi\sigma_t^2 + (1-\phi)r_t^2\right]. \tag{1.9}$$

The risk-return tradeoff characterizes the relation between expected returns and financial securities' risk. Thus, this association can be illustrated by deriving the right-hand side of equation (1.9) by σ_t^2 which gives,

$$\frac{\partial \mathbb{E}_t[r_{t+1}]}{\partial \sigma_t^2} := \gamma\phi + \gamma(1-\phi)\frac{\partial r_t^2}{\partial \sigma_t^2}.$$

Then, I use the implicit function theorem to pin down $\frac{\partial r_t^2}{\partial \sigma_t^2}$. Let us define $F(\sigma_t^2, r_t^2) := \mathbb{E}_t[r_{t+1}] - r_f - \gamma\phi\sigma_t^2 - \gamma(1-\phi)r_t^2$. Equation (1.9) implies that $F(\sigma_t^2, r_t^2) = 0$. Furthermore, because the partial derivative of F with respect to $\mathbb{E}_t[r_{t+1}]$ never vanishes (as long as $\mathbb{E}_t[r_{t+1}] \neq 0$), the IFT holds, which implies that,

$$\frac{\partial r_t^2}{\partial \sigma_t^2} = -\frac{1-\phi}{\phi}.$$

Proposition 1.2 follows from substituting this quantity into $\frac{\partial \mathbb{E}_t[r_{t+1}]}{\partial \sigma_t^2}$.

Proposition 1.2 *The sensitivity of expected returns to conditional variance is given by,*

$$\frac{\partial \mathbb{E}_t[r_{t+1}]}{\partial \sigma_t^2} = \gamma \left(\phi - \frac{(1-\phi)^2}{\phi} \right). \tag{1.10}$$

Importantly, this proposition predicts that the risk-return tradeoff is constant; if σ_t^2 increases by Δ the expected return will increase by $\Delta\gamma \left(\phi - \frac{(1-\phi)^2}{\phi} \right)$. This is strictly smaller than the prediction of the benchmark case (i.e. $\Delta\gamma$). Hence, when agents use the exponentially weighted moving average to assess the risk in the future, the risk-return tradeoff is weaker than the one predicted by Merton's ICAPM. This relation can even turn negative if $\phi < 1/2$, where the larger the risk the less the expected return.

1.3.4 Risk-Return Tradeoff, a Generalization

In the previous section I showed that when agents forecast future volatility with the RiskMetrics variance model, the relation between expected returns and conditional variance is weaker than the relation predicted by mainstream theories. Here, I generalize this result by extending the analysis to the GARCH(1,1) model instead of the EWMA. The aim of this extension is to provide a potential justification for why researchers tend to find a weaker relation between excess returns and conditional volatility when they use that methodology with other methods (see for instance Lundblad (2007)). Accordingly, the GARCH(1,1) one-step ahead variance forecast is given by,

$$\sigma_{t+1}^2 = \omega + \alpha\sigma_t^2 + \beta r_t^2.$$

Hence, the counterpart of equation (1.9) is given by,

$$\mathbb{E}_t[r_{t+1}] = r_f + \gamma\left[\omega + \alpha\sigma_t^2 + \beta r_t^2\right].$$

Thus, deriving the right-hand side of this relation gives, $\frac{\partial \mathbb{E}_t[r_{t+1}]}{\partial \sigma_t^2} := \gamma\alpha + \gamma\beta\frac{\partial r_t^2}{\partial \sigma_t^2}$. Then, I use the implicit function theorem to determine $\frac{\partial r_t^2}{\partial \sigma_t^2}$. Let us define $F(\sigma_t^2, r_t^2) := \mathbb{E}_t[r_{t+1}] - r_f - \gamma\alpha\sigma_t^2 - \gamma\beta r_t^2$. The IFT implies that $\frac{\partial r_t^2}{\partial \sigma_t^2} = -\frac{\beta}{\alpha}$. Proposition 1.3 follows from substituting this quantity into $\frac{\partial \mathbb{E}_t[r_{t+1}]}{\partial \sigma_t^2}$.

Proposition 1.3 *The sensitivity of expected returns to conditional variance is given by,*

$$\frac{\partial \mathbb{E}_t[r_{t+1}]}{\partial \sigma_t^2} = \gamma \times \frac{\alpha^2 - \beta^2}{\alpha}. \tag{1.11}$$

Here, α is the parameter that controls for the persistence of the variance (i.e. the GARCH effect), while β controls the sensitivity to the surprise (i.e. the ARCH effect). There are many observations that are important to mention. Firstly, the long-term variance does not affect the risk-return tradeoff. Secondly, today's variance influences the expected returns via two channels. On the one hand, conditional variance is persistent; when it is high today, it will be high in the near future as well; on the other hand, the higher the variance today, the higher the security's risk and the lower today's price, hence, today's squared return tends to be lower too. As long as $\alpha > \beta$ the first effect dominates the second.

However, when the relation is reversed the risk-return tradeoff turns negative. Depending on the financial security and on the frequency of the returns, the ARCH parameter β is found between 0 and 0.1, while α is found between 0.7 and 0.9. With this parameterization the risk-return tradeoff appears significantly weaker than traditionally (around 30% weaker). Importantly, Ma et al. (2007) show that the GARCH parameter is overestimated when the ARCH parameter is low, which is typical for financial securities. This effect further deepens the relationship between expected returns and volatility, though testing for this channel is outside of the scope of this paper.

1.4 Data and Methodology

1.4.1 Data

The data are taken from Kenneth French's data library. For the sake of replicability I believe it is essential to use a dataset that is widely accepted. Because the factors and portfolios provided here rely on standard procedures, this dataset respects this important criterion. The data consists of different subsamples summarized in Tables 1.1 and 1.2.

Each sample consists of 25 portfolios. Each portfolio is constructed at the intersection of five portfolios formed on one characteristic (such as size) and five portfolios formed on one different characteristic (such as Book-to-Market, Momentum, etc.). While most subsamples span the period 1927-2018, others cover only the years 1963-2018. This feature is reported in columns three and four of each table. As with portfolios, all the factors (risk-free rate, market, small minus big, high minus low and momentum) come from Kenneth French's data library. Hence,

Table 1.1: Summary Statistics: Major markets (excl. U.S.)

This table shows summary statistics of the different subsamples used to test my model for all major markets except for the U.S. market. The first column indicates the market. The second explains how stocks are sorted. The third column illustrates the frequency of the returns. Finally, columns four and five describe the period of time covered by each sample.

Market	Stocks sorted by	Frequency	Start date	End date
Asia	Size and Book-to-Market	D,M,A	1991-01-01	2018-03-30
Asia	Size and Investment	D,M,A	1991-01-01	2018-03-30
Asia	Size and Profitability	D,M,A	1991-01-01	2018-03-30
Asia	Size and Momentum	D,M,A	1991-01-01	2018-03-30
Europe	Size and Book-to-Market	D,M,A	1991-01-01	2018-03-30
Europe	Size and Investment	D,M,A	1991-01-01	2018-03-30
Europe	Size and Profitability	D,M,A	1991-01-01	2018-03-30
Europe	Size and Momentum	D,M,A	1991-01-01	2018-03-30
Japan	Size and Book-to-Market	D,M,A	1991-01-01	2018-03-30
Japan	Size and Investment	D,M,A	1991-01-01	2018-03-30
Japan	Size and Profitability	D,M,A	1991-01-01	2018-03-30
Japan	Size and Momentum	D,M,A	1991-01-01	2018-03-30

the one-month Treasury bill rate proxies the risk-free rate. The construction of the other factors are explained in Fama and French (1993) and Carhart (1997).

1.4.2 Estimation Strategy

In this section, I test Proposition 1.1. My model predicts that the expected risk premium of a financial security can be decomposed into two different factors; the first is the sensitivity to the market premium, as predicted by the capital asset pricing model. However, the sensitivity of the assets' risk premium to the market premium is time-varying, which is not the case in the standard CAPM. The second factor, which is novel in the literature, results from the interaction between the last holding period returns of both the market and the assets. Hence, that proposition

Table 1.2: Summary Statistics: U.S. stocks

This table shows summary statistics of the different subsamples used to test my model for the U.S. market. The first column describes how stocks are sorted. The second illustrates the frequency of the returns. Finally, the last two columns describe the period of time covered by each sample.

Stocks sorted by	Frequency	Start date	End date
Book-to-Market and Investment	D,M,A	1926-11-03	2018-03-29
Book-to-Market and Profitability	D,M,A	1926-11-03	2018-03-29
Profitability and Investment	D,M,A	1926-11-03	2018-03-29
Size and Accruals	M,A	1963-07	2018-03
Size and Book-to-Market	D,M,A	1926-11-03	2018-03-29
Size and Investment	D,M,A	1926-11-03	2018-03-29
Size and LT Reversal	D,M,A	1926-11-03	2018-03-29
Size and Momentum	D,M,A	1926-11-03	2018-03-29
Size and Market Beta	M,A	1963-07	2018-03
Size and Net Share Issues	M,A	1963-07	2018-03
Size and Profitability	D,M,A	1926-11-03	2018-03-29
Size and Residual Variance	M,A	1963-07	2018-03
Size and ST Reversal	D,M,A	1926-11-03	2018-03-29
Size and Variance	M,A	1963-07	2018-03

can be rewritten as the following (panel) equation,

$$y_{t,i} = \beta_{t,i}\left[r_t^M - r_t^f\right] + (1-\phi)\gamma r_{t-1}^M\left[r_{t-1}^i - \beta_{t,i}r_{t-1}^M\right] + u_{t,i}. \qquad (1.12)$$

As emphasized earlier the asset beta is time dependent. In fact, because both the variance of the market premium and the correlation of the asset returns with the market are time dependent, the beta is no longer constant. Hence, I use formula 1.2 to reestimate this parameter at every point in time. Then, I rearrange the equation by shifting the market premium factor to the left-hand side. Equation (1.12) then becomes,

$$\tilde{y}_{t,i} = \gamma \times \tilde{x}_{t,i} + u_{t,i},$$

where $\tilde{y}_{t,i} := y_{t,i} - \beta_{t,i}\big[r_t^M - r_t^f\big]$, $\beta_{t,i} := \sigma_t^{i,M} \times \left[\sigma_t^{M,M}\right]^{-1}$ and $\tilde{x}_{t,i} := (1 - \phi)r_{t-1}^M \left[r_{t-1}^i - \beta_{t,i}r_{t-1}^M\right]$. This is a simple panel equation with one homogenous parameter γ. Importantly, γ provides a direct estimate of the relative risk aversion coefficient. Hence, a conclusive test in favour of my framework would be that γ is both statistically significant and greater than zero.

The first column of Tables 1.3–1.8 reports the estimate of the relative risk aversion without any control or fixed effects. The second column shows the estimate of γ while controlling for the market risk premium. The third column exhibits the estimate of γ while controlling for all three Fama-French factors (i.e. market risk premium, small minus big and high minus low). Finally, I control for Carhart four factors model (Fama-French plus momentum) in the last column. Hence, models (1)–(3) are nested in model (4), which can be rewritten as follows,

$$\tilde{y}_{t,i} = \gamma \times \tilde{x}_{t,i} + \beta_i^M Market_t + \beta_i^S SMB_t + \beta_i^H HML_t + \beta_i^C Momentum_t + u_{t,i}. \quad (1.13)$$

Estimating model (1) is trivial, however, this is not the case for models (2)–(4). In these specifications the main complication resides in the fact that the parameters for the control variables $\{\beta_i^M, \beta_i^S, \beta_i^H, \beta_i^C\}$ are allowed to be heterogenous, while the last γ is homogenous. Serlenga et al. (2001) propose an interesting approach for this type of problem. An alternative procedure is to pre-multiply each cross-sectional unit equation,

$$\underbrace{\begin{bmatrix} \tilde{y}_{i,1} \\ \tilde{y}_{i,2} \\ \vdots \\ \tilde{y}_{i,T} \end{bmatrix}}_{\mathbf{y}_i} = \underbrace{\begin{bmatrix} M_1 & S_1 & H_1 & C_1 \\ M_2 & S_2 & H_2 & C_2 \\ \vdots & & & \\ M_T & S_T & H_T & C_T \end{bmatrix}}_{\mathbf{w}} \times \begin{bmatrix} \beta_1^M \\ \beta_1^S \\ \beta_1^H \\ \beta_1^C \end{bmatrix} + \gamma \times \underbrace{\begin{bmatrix} \tilde{x}_{i,1} \\ \tilde{x}_{i,2} \\ \vdots \\ \tilde{x}_{i,T} \end{bmatrix}}_{\mathbf{x}_i} + \underbrace{\begin{bmatrix} u_{i,1} \\ u_{i,2} \\ \vdots \\ u_{i,T} \end{bmatrix}}_{\mathbf{u}_i}, \ i = 1...N$$

Table 1.3: Estimates of the relative risk aversion for the Asian market

This table summarizes my main findings for the Asian market. Columns 1–4 report estimates of the relative risk aversion computed with daily returns. Columns 5–8 report estimates of the relative risk aversion computed with monthly returns. Finally, columns 9–12 report estimates of the relative risk aversion computed with annual returns.

	(1)	(2)	(3)	(4)
Size and Book-to-Market	35.851***	31.134***	20.677**	19.884**
	(3.641)	(3.352)	(2.370)	(2.273)
Size and Investment	40.268***	32.584***	21.271***	20.898***
	(4.157)	(3.823)	(2.737)	(2.685)
Size and Momentum	21.147	18.635	13.254	4.800
	(1.547)	(1.381)	(1.023)	(0.391)
Size and Profitability	38.560***	30.723***	19.461***	18.955***
	(4.284)	(3.867)	(2.696)	(2.626)
Size and Book-to-Market	2.199	5.191*	2.194	1.888
	(0.714)	(1.829)	(1.164)	(1.042)
Size and Investment	0.827	5.422*	1.671	0.491
	(0.227)	(1.759)	(0.872)	(0.271)
Size and Momentum	8.770***	9.395***	7.929***	1.274
	(2.682)	(3.405)	(3.756)	(0.809)
Size and Profitability	3.359	6.730**	2.245	1.594
	(1.031)	(2.350)	(1.340)	(0.989)
Size and Book-to-Market	−0.521	0.779	0.211	0.433
	(−0.428)	(0.933)	(0.298)	(0.569)
Size and Investment	−1.222	−0.001	−0.512	−0.816
	(−1.140)	(−0.001)	(−0.755)	(−1.197)
Size and Momentum	−1.108	0.084	−1.105	−0.285
	(−0.934)	(0.094)	(−1.319)	(−0.387)
Size and Profitability	−1.721*	−0.194	−0.775	−0.824
	(−1.716)	(−0.305)	(−1.428)	(−1.514)
Filtered by Market Factor	no	yes	yes	yes
Filtered by SMB & HML Factors	no	no	yes	yes
Filtered by Momentum Factor	no	no	no	yes

Note: * $p < 0.1$, ** $p < 0.05$, *** $p < 0.01$. Standard errors in parentheses.

Table 1.4: Estimates of the relative risk aversion for the European market

This table summarizes my main findings for the European market. Columns 1–4 report estimates of the relative risk aversion computed with daily returns. Columns 5–8 report estimates of the relative risk aversion computed with monthly returns. Finally, columns 9–12 report estimates of the relative risk aversion computed with annual returns.

	(1)	(2)	(3)	(4)
Size and Book-to-Market	14.373***	11.763***	3.319**	3.195**
	(4.413)	(5.735)	(2.062)	(2.120)
Size and Investment	15.100***	11.946***	2.320	2.288
	(4.367)	(5.574)	(1.300)	(1.376)
Size and Momentum	10.468***	9.070***	1.118	2.309
	(3.128)	(3.572)	(0.495)	(1.340)
Size and Profitability	14.800***	11.957***	2.024	2.006
	(4.363)	(5.749)	(1.194)	(1.291)
Size and Book-to-Market	−5.124	−5.115	3.525	2.480
	(−1.069)	(−1.165)	(1.311)	(0.944)
Size and Investment	−5.856	−1.452	−3.057	−4.709*
	(−1.085)	(−0.360)	(−1.203)	(−1.945)
Size and Momentum	23.538***	20.120***	21.158***	8.273***
	(4.712)	(6.044)	(7.446)	(4.111)
Size and Profitability	5.557	4.908	5.897**	4.459*
	(1.243)	(1.251)	(2.065)	(1.663)
Size and Book-to-Market	9.669***	4.603**	3.693***	3.634***
	(3.657)	(2.125)	(2.811)	(2.730)
Size and Investment	9.870***	4.857**	2.158	1.886
	(3.834)	(2.452)	(1.536)	(1.408)
Size and Momentum	6.126	2.157	0.623	2.007
	(1.557)	(0.864)	(0.280)	(0.977)
Size and Profitability	13.824***	9.195***	5.667***	5.825***
	(6.229)	(4.809)	(3.886)	(4.003)
Filtered by Market Factor	no	yes	yes	yes
Filtered by SMB & HML Factors	no	no	yes	yes
Filtered by Momentum Factor	no	no	no	yes

Note: * $p < 0.1$, ** $p < 0.05$, *** $p < 0.01$. Standard errors in parentheses.

Table 1.5: Estimates of the relative risk aversion for the Japanese market

This table summarizes my main findings for the Japanese market. Columns 1–4 report estimates of the relative risk aversion computed with daily returns. Columns 5–8 report estimates of the relative risk aversion computed with monthly returns. Finally, columns 9–12 report estimates of the relative risk aversion computed with annual returns.

	(1)	(2)	(3)	(4)
Size and Book-to-Market	13.773***	12.856**	11.167***	9.468***
	(2.671)	(2.324)	(3.419)	(2.969)
Size and Investment	14.923***	14.053**	11.555***	10.593***
	(2.923)	(2.543)	(3.338)	(3.096)
Size and Momentum	20.543***	19.530***	16.184***	8.651***
	(4.997)	(4.731)	(5.964)	(3.829)
Size and Profitability	14.846***	14.005***	12.606***	11.104***
	(3.177)	(2.812)	(4.054)	(3.648)
Size and Book-to-Market	21.218***	19.866***	3.859**	3.255*
	(5.741)	(5.305)	(2.016)	(1.673)
Size and Investment	19.750***	19.718***	1.690	0.507
	(5.452)	(5.919)	(0.948)	(0.291)
Size and Momentum	25.453***	23.740***	10.323***	2.411
	(7.353)	(7.681)	(4.846)	(1.612)
Size and Profitability	21.074***	20.364***	2.974	2.321
	(6.080)	(6.290)	(1.634)	(1.278)
Size and Book-to-Market	−6.388***	−1.272	1.033	0.729
	(−5.877)	(−1.078)	(1.201)	(0.873)
Size and Investment	−5.313***	−2.011*	0.392	0.124
	(−5.488)	(−1.815)	(0.522)	(0.164)
Size and Momentum	−6.049***	−2.076**	0.120	−0.830
	(−7.737)	(−2.187)	(0.173)	(−1.524)
Size and Profitability	−3.997***	−1.684	−0.660	−1.056
	(−2.881)	(−1.269)	(−0.916)	(−1.509)
Filtered by Market Factor	no	yes	yes	yes
Filtered by SMB & HML Factors	no	no	yes	yes
Filtered by Momentum Factor	no	no	no	yes

Note: * p < 0.1, ** p < 0.05, *** p < 0.01. Standard errors in parentheses.

Table 1.6: Estimates of the relative risk aversion for the U.S. market with daily returns

This table summarizes my main findings for the U.S. market at the daily frequency. Each row represents a different type of sorting.

	(1)	(2)	(3)	(4)
Size and Book-to-Market	22.734***	22.920***	8.634**	8.446**
	(6.049)	(5.761)	(2.338)	(2.314)
Size and Investment	34.960***	33.585***	4.235*	4.412**
	(6.001)	(5.777)	(1.943)	(2.087)
Size and Momentum	28.555***	27.583***	12.900***	9.295***
	(9.347)	(9.397)	(5.094)	(4.131)
Size and Profitability	34.036***	31.799***	6.625***	6.670***
	(5.801)	(5.461)	(2.697)	(2.778)
Book-to-Market and Investment	34.493***	31.502***	20.970***	20.409***
	(5.473)	(5.354)	(5.059)	(4.985)
Book-to-Market and Profitability	31.877***	26.538***	20.582***	20.336***
	(5.488)	(4.993)	(4.595)	(4.548)
Profitability and Investment	22.614***	18.924***	19.010***	18.089***
	(3.858)	(3.306)	(4.124)	(4.012)
Size and LT Reversal	12.127***	11.943***	5.026**	4.827**
	(3.112)	(3.145)	(2.342)	(2.294)
Size and ST Reversal	34.774***	33.751***	23.299***	23.118***
	(10.715)	(11.180)	(6.951)	(6.908)
Filtered by Market Factor	no	yes	yes	yes
Filtered by SMB & HML Factors	no	no	yes	yes
Filtered by Momentum Factor	no	no	no	yes

Note: * $p < 0.1$, ** $p < 0.05$, *** $p < 0.01$. Standard errors in parentheses.

Table 1.7: Estimates of the relative risk aversion for the U.S. market with monthly returns

This table summarizes my main findings for the U.S. market at the monthly frequency. Each row represents a different type of sorting.

	(1)	(2)	(3)	(4)
Book-to-Market and Investment	−0.193	−0.439	−0.017	−1.204
	(−0.043)	(−0.100)	(−0.005)	(−0.362)
Book-to-Market and Profitability	−0.412	1.904	2.565	1.536
	(−0.103)	(0.543)	(0.903)	(0.551)
Profitability and Investment	−8.411**	−4.292	−5.127**	−5.636**
	(−2.483)	(−1.578)	(−2.161)	(−2.444)
Size and Accruals	4.624*	7.423***	0.991	1.084
	(1.746)	(3.299)	(0.779)	(0.843)
Size and Book-to-Market	17.270***	10.916***	1.928	1.915
	(5.821)	(4.383)	(1.120)	(1.113)
Size and Investment	4.491	6.949***	0.107	0.221
	(1.502)	(2.833)	(0.078)	(0.159)
Size and LT Reversal	20.261***	13.500***	3.603***	3.283***
	(6.207)	(5.579)	(2.891)	(2.639)
Size and Momentum	19.444***	14.042***	6.631***	4.344***
	(5.055)	(4.918)	(4.668)	(3.598)
Size and Market Beta	5.315*	7.191***	0.724	0.774
	(1.812)	(3.287)	(0.559)	(0.606)
Size and Net Share Issues	−1.393*	−1.219*	0.229	0.228
	(−1.665)	(−1.892)	(0.427)	(0.417)
Size and Profitability	4.188	6.003**	−0.663	−0.803
	(1.380)	(2.324)	(−0.386)	(−0.476)
Size and Residual Variance	7.748**	9.616***	5.709***	5.268***
	(2.185)	(3.559)	(2.774)	(2.908)
Size and ST Reversal	19.545***	14.093***	5.602***	5.478***
	(5.007)	(4.783)	(4.021)	(3.855)
Size and Variance	9.636***	10.141***	6.737***	5.941***
	(2.709)	(3.861)	(3.307)	(3.387)
Filtered by Market Factor	no	yes	yes	yes
Filtered by SMB & HML Factors	no	no	yes	yes
Filtered by Momentum Factor	no	no	no	yes

Note: * $p < 0.1$, ** $p < 0.05$, *** $p < 0.01$. Standard errors in parentheses.

Table 1.8: Estimates of the relative risk aversion for the U.S. market with annual returns

This table summarizes my main findings for the U.S. market at the annual frequency. Each row represents a different type of sorting.

	(1)	(2)	(3)	(4)
Book-to-Market and Investment	7.177***	3.643	2.178	1.995
	(2.667)	(1.473)	(1.258)	(1.180)
Book-to-Market and Profitability	9.436***	6.929**	7.035***	6.087***
	(2.823)	(2.436)	(3.082)	(2.947)
Profitability and Investment	4.166*	1.587	1.168	0.804
	(1.779)	(0.779)	(0.612)	(0.416)
Size and Accruals	10.993***	10.601***	2.053**	1.615
	(5.564)	(6.411)	(1.982)	(1.593)
Size and Book-to-Market	1.834	2.761***	0.514	0.510
	(1.579)	(2.718)	(0.835)	(0.842)
Size and Investment	7.445***	6.949***	−0.005	−0.206
	(3.261)	(3.576)	(−0.005)	(−0.208)
Size and LT Reversal	1.591	1.910**	−0.612	−0.677
	(1.396)	(2.133)	(−0.757)	(−0.830)
Size and Momentum	3.253***	3.964***	1.349**	0.536
	(3.655)	(5.541)	(2.321)	(0.919)
Size and Market Beta	9.466***	7.708***	1.935	0.488
	(4.250)	(3.879)	(1.639)	(0.474)
Size and Net Share Issues	3.102***	3.195***	2.676***	2.812***
	(4.851)	(8.118)	(6.511)	(6.773)
Size and Profitability	7.204***	6.262***	−0.863	−1.190
	(3.216)	(3.168)	(−0.710)	(−0.992)
Size and Residual Variance	11.409***	8.583***	2.791**	0.698
	(4.889)	(4.557)	(2.088)	(0.637)
Size and ST Reversal	1.153	1.715	−1.041	−1.085
	(0.855)	(1.599)	(−0.782)	(−0.809)
Size and Variance	10.929***	8.212***	2.396*	0.560
	(4.696)	(4.362)	(1.840)	(0.515)
Filtered by Market Factor	no	yes	yes	yes
Filtered by SMB & HML Factors	no	no	yes	yes
Filtered by Momentum Factor	no	no	no	yes

Note: * $p < 0.1$, ** $p < 0.05$, *** $p < 0.01$. Standard errors in parentheses.

by a $T \times T$ idempotent matrix, $Q := I_{T\times T} - \mathbf{w}(\mathbf{w}'\mathbf{w})^{-1}\mathbf{w}'$. This procedure starts by projecting both $\mathbf{y}$ and $\mathbf{x}$ onto the vector space generated by the columns of $\mathbf{w}$ and then regresses the residual of the first regression on the residual of the second. Consequently, this allows me to remove all the control variables in the right-hand side (pre-multiplying vectors M, S, H, or C by matrix Q results in a $T \times 1$ vector of zeros). Hence, each cross-sectional unit equation becomes[2],

$$\underbrace{Q\tilde{\mathbf{y}}_i}_{A_i} = \gamma \underbrace{Q\tilde{\mathbf{x}}_i}_{B_i} + \underbrace{Q\mathbf{u}_i}_{v_i}, \; i = 1...N. \tag{1.14}$$

Once that procedure is applied to all the cross-sectional unit equations the panel can be reshaped and the estimate of the relative risk aversion can be assessed by simply regressing A_i on B_i.[3]

1.4.3 Analysis of the Results

I test my model with different subsamples and at different return frequencies. Tables 1.3–1.5 shows my results for the Asian, European and Japanese markets. For each of these tables rows 1–4 show estimates computed with daily returns. Rows five to eight estimates are computed with monthly returns. Finally, the last rows show estimates which are computed with annual returns. Tables 1.6–1.8 only show estimates of γ for the U.S. market. Table 1.6 shows estimates computed with daily returns. Table 1.7 and 1.8 report estimates computed with monthly and annual returns respectively.

[2]Let us define V to be the space spanned by the columns of $\mathbf{w}$. Then, A_i corresponds to the projection of $\tilde{\mathbf{y}}_i$ onto the space orthogonal to V. Likewise A_i is the residual of the regression of $\tilde{\mathbf{y}}_i$ onto the vector of controls $\mathbf{w}$, while B_i corresponds to the projection of $\tilde{\mathbf{x}}_i$ onto the space orthogonal to V. Finally, v_i is the residual of the regression of $\mathbf{u}_i$ on $\mathbf{w}$. Thus, only the noise orthogonal to V is left.

[3]Note that the estimator of the relative risk aversion, $\hat{\gamma}$, is both, unbiased and consistent as illustrated by Wawro (2009).

The analysis of Tables 1.3–1.8 reveals an interesting result. Though all specifications are reported I discuss only the last column of each table which provides the most robust estimates. While the parameters are always strongly significant at the daily frequency for the Asian, Japanese and the U.S. market, whatever the sorting applied to stocks, this is less clear for Europe. In addition, the range where the risk aversion appears is coherent with previous studies (in general below 20). The fact that the coefficients are so strongly significant at that frequency argues in favour of an anomaly, the "RiskMetrics effect", identified in most of the (main) financial markets around the world. Importantly, my model is robust even after controlling for well-established factors in the empirical asset pricing literature (i.e. the market premium, the momentum and the Fama-French factors).

Christoffersen et al. (1998) established that volatility forecastability declines quickly with the horizon; while the volatility fluctuations are highly forecastable for short horizons (such as with daily returns), the forecastability seems to vanish beyond horizons of two weeks. When choosing a portfolio a mean-variance agent balances the expected return with the risk, measured here by the conditional variance. Moreover, an investor must also decide how long they will keep their portfolio without altering its composition. Therefore, it is natural to conjecture that agents who rebalance their portfolios frequently care more about the volatility forecastability than agents who do not. If the holding period is a month almost all the predictable part of the variance is gone. Thus, the simple benchmark case derived in Section 1.3.1 (i.e. with constant variance) is expected to hold, implying that the effect predicted by my model should disappear. Contrarily, when the holding period is a day volatility is highly predictable. Hence, the model derived in Section 1.3.2 should hold. Consequently, I expect to find a "RiskMetrics" anomaly with daily data but not necessarily with lower frequency returns. This is confirmed by the data; the quantity of significant coefficients for monthly and annual returns is drastically smaller than with daily returns. It turns out that the regression coefficients are never statistically different from zero for both the Asian and the Japanese markets at the monthly and annual frequencies (with one minor exception that I consider insignificant). For the U.S. market the quantity of significant estimates decreases monotonically with the return frequency, consistent with my conjecture.

Contrastingly, the European market appears as the exception; the anomaly is observed more frequently at lower frequencies, which is puzzling. A possible explanation is that the behaviour towards risk is different across the world. This is not the first time that a study concludes that financial markets are not that integrated. Another possible interpretation could be related to the fact that the volatility is less persistent in Europe than in the other markets. Because of this feature, the risk becomes harder to forecast, which reduces the magnitude of the anomaly.

1.5 Conclusion

This paper considers an overlapping-generations economy where the agents forecast future variance with the RiskMetrics variance model. I show that this feature induces a two-factor asset pricing model in which risk premiums are determined by both the market beta and the "RiskMetrics beta"; this latter results from the interaction between the past returns of the asset and the market. I validate this prediction by estimating the model with panel regressions; importantly, my empirical strategy allows me to estimate the relative risk aversion coefficient independently from established procedures. My estimates are in line with the ones obtained with these approaches.

Furthermore, this effect is found in each of the major financial markets, except in Europe. The magnitude of the effect depends on the frequency of the returns; with high frequency data (here, daily returns) the RiskMetrics anomaly is easier to detect as compared with low frequency data (monthly or annually). The effect is robust to the major pricing factors found in the empirical asset pricing literature, which are the three Fama-French factors and the Momentum factor.

Finally, I extend my model and show that the risk-return tradeoff predicted by my model when the volatility is forecasted with a GARCH model is significantly weaker than the relation predicted by mainstream finance theories. This can provide a potential explanation as to why the relationship between expected returns and conditional volatility is, empirically, so weak.

2 Slow Arbitrage

This paper develops a dynamic model in which financially constrained agents search for markets which are subject to decreasing returns to scale. In equilibrium, agents only invest in markets with total capital below an endogenous threshold that depends on the equilibrium distribution of capital across markets. Strikingly, I show that perfect capital mobility is not necessarily the most efficient outcome, breaking down the common belief that perfect mobility is always better. Furthermore, I extend the model and allow the redistribution of wealth from buyers to sellers through taxation. With this extension I demonstrate that in the case of parameter uncertainty taxing too much will cause less damage than taxing too little, which has significant implications for fixing tax rates.

2.1 Introduction

Market efficiency depends crucially on capital mobility; in an efficient market capital quickly flows to positive net present value (NPV) projects. In this paper I show that this conventional wisdom can break down; capital mobility can have an adverse effect on welfare.

I build a model and study the equilibrium properties of capital flows through a continuum (a non-atomic measure space) of segmented markets; those are indistinguishable except for the amounts of capital invested in each. This economy is populated by risk-neutral investors all endowed with one unit of capital. These agents search markets and, upon meeting one, invest all their capital. At Poisson times markets return a profit to all investors present on top of their investment. Then, these investors return to searching. As a matter of supply and demand I assume that the profit paid to the agents in that market is strictly decreasing with the aggregate capital. Finally, I conjecture that investors cease to enter into a particular market once its aggregate capital is above a level, settled endogenously. With this model I show that an economy characterized by perfect capital mobility is not desirable.

My results suggest that the intensity at which investors search, a proxy for capital mobility, is the main driver of market liquidity; this has strong consequences on welfare. When searching is not allowed the economy performs very badly; there are no markets for liquidity and no benefits from trade, since there are no means of exchange. As searching frictions vanish investors coordinate without difficulty. In this regime the agents' optimal strategy is to spread across the entire universe of markets and, since there are no search frictions, they can do that easily. Surprisingly, the first-best outcome is not necessarily achieved here. Because their reservation value is small investors lower their sights; the level above which agents cease to enter is set higher. Then I investigate what happens when investors can set their search intensity. This is particularly useful when it comes to discussing taxation and financial incentives.

Finally, I explore a simple extension of the model; some agents (the seekers of capital) sell their own assets to cover a sudden need for cash. The buyers—the former investors—that have allocated their capital to that particular seller, pool their capital, buy the asset, consume the proceeds and return to searching. Then I allow for the redistribution of wealth from buyers to sellers through taxation. I am particularly interested in rescue packages offered by government institutions, which guarantee *toxic assets* in significant proportions. My results establish that a simple tax scheme on search technology provides a smart solution for the financing of these interventions and incites buyers to set search intensity at its first-best level. Last but not least, I show that in the case of parameter uncertainty taxing too much will always cause less damage than taxing too little.

2.1.1 Related literature

My model belongs to the literature on search and matching frictions applied to economics and finance. An important early paper on this is Diamond (1982); agents are searching for production opportunities and, upon meeting one, pay the associated cost of production or return to searching. This literature has been extended in many directions, including asset prices and allocations in OTC markets (e.g., Ricardo et al. (2011), Vayanos and Wang (2007), Vayanos and Weill (2008) and Weill (2007)). An important paper in this field of literature is Duffie et al. (2005); the authors study the consequences of search frictions in a single market. Similarly, Duffie et al. (2007) examine the impact of search and bargaining frictions on asset prices. Afonso and Lagos (2015) build a model of the market for federal funds where banks face search and bargaining frictions. More recently, Hugonnier et al. (2016) design a search and bargaining model with investors' valuations drawn from any arbitrary distribution.

My model is closely related to Duffie and Strulovici (2012). The authors design a model with which they analyze the equilibrium movements of capital between two partially segmented markets; these are indistinguishable except for the amount of capital invested in each. My framework shares some important ingredients with their model. Firstly, investors are not assigned to one particular market; rather, they can trade in all of them. Secondly, markets are distinguished by the quantity of capital invested in each. In contrast I study a continuum of markets. Additionally, while their focus is on the intermediation channel I have abstracted from that; my results put more emphasis on the consequences of capital mobility on welfare, which is complementary to their analysis.

Section 2.3 investigates the idea that assets may be sold at a discount when buyers are financially constrained (papers related to this literature include, for example, Allen and Gale (1994), Allen and Gale (2004), Allen and Gale (2005), Shleifer and Vishny (1992) and Shleifer and Vishny (1997)). This endeavour is related to the seminal work of Acharya et al. (2013). The authors argue that outside arbitrageurs do not necessarily come in and take advantage of a fire sale because they face a tradeoff; arbitrage capital entails both benefits, when assets are sold at a discount and costs, such as opportunity cost of not investing in other profitable activities. This tradeoff also shows up in my framework, nevertheless, my contribution to this literature is very different. Firstly, this investigation is an extension of my model, rather than its primary purpose. Secondly, my methodology is very different; my model is a continuous time model, while they work with a stylized three-period model. To summaries, while the importance of the dynamics of arbitrage capital is stressed in both papers, our respective approaches are completely different. Lastly, this section also refers to Kondor (2009), who studies a model of (limited) capital allocation. Interestingly he suggests that, despite the magnitude of arbitrage opportunities arbitrageurs keep some capital aside in case these opportunities become more profitable. He focuses on risk arbitrage and my model shares that feature; when capital is limited and the probability that a new (better) opportunity appears is strictly positive, keeping capital aside is optimal. Nevertheless, the mechanisms at work are quite different. In Kondor's paper arbitrageurs save some capital because, as time goes by, prices might diverge further. Contrastingly, my model is built such that the profitability of a given opportunity can only disappear over time; the arbitrage capital is generated by the large dimension of the market space.

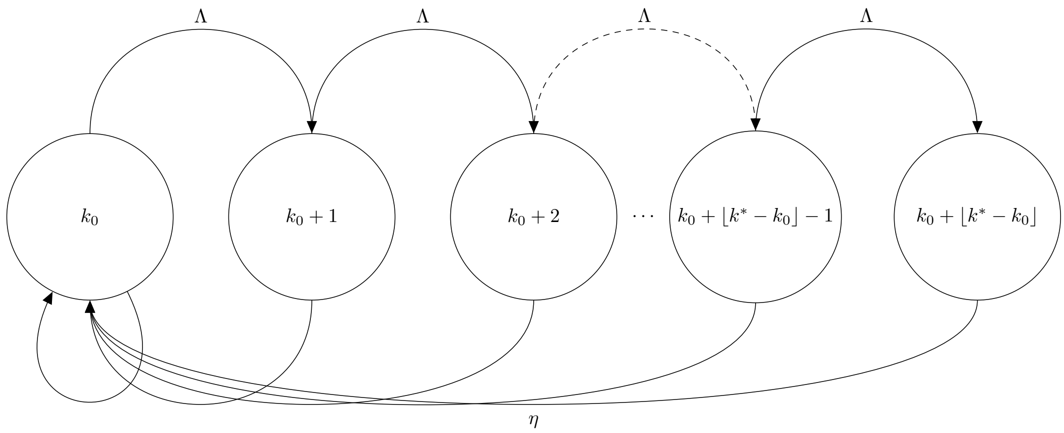
This paper is structured as follows: Section 2.2 presents the model and its main consequences, Section 2.3 applies this framework to a fire sale of assets, discussing some equilibrium properties, welfare implications and potential policy interventions. Finally, I summarize the main results and present my conclusions in Section 2.4.

2.2 Model

The economy is populated by a mass A of agents (arbitrageurs), each endowed with one dollar. There is a continuum of markets where profits are a decreasing function $\pi(k)$ of the capital invested in that market k. Agents arrive at a rate λ and decide whether to stay or keep searching further. When an agent decides to stay, they invest their dollar and wait until the profits, $\pi(k)$ are paid. The time at which a market pays off is exponentially distributed with intensity η (this is the time when the fruit matures). At that moment the agent consumes the net profit $\pi(k) - 1$ and returns to searching with their suddenly liberated dollar. In the meantime new opportunities appear at the same rate η so that the total number of opportunities remains constant. I define M as the total mass of agents which have currently invested their money in a market waiting for profits to be paid. Similarly, $A - M$ corresponds to the fraction of the total mass of arbitrageurs searching for opportunities. Then, from the law of large numbers for independent random matching, the total meeting rate at which agents find markets is $\lambda(A - M)$. For any given market traders arrive at Poisson times, thus the aggregate capital in each market follows a Poisson process. Some possible paths are shown in Figure 2.1. Therefore, if the previous capital invested in a given market was k, it jumps to

Figure 2.1: Dynamics of the aggregate capital in a single market

This figure shows the dynamics of the aggregate capital in one single market. Importantly, the aggregate capital behaves as a Poisson process (it jumps from k to $k + 1$ at exponentially distributed random times with mean $1/\Lambda$ or dies at exponentially distributed random times with mean $1/\eta$).

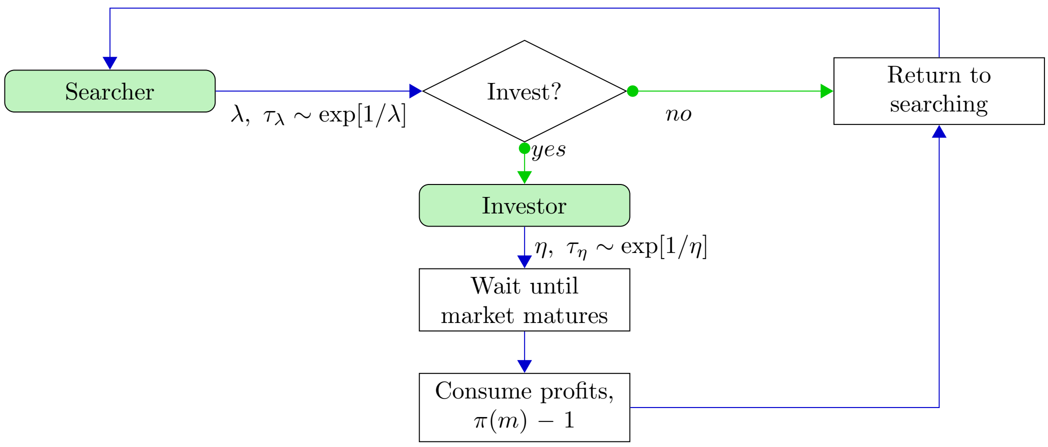


$k + 1$. Finally, I assume that when new opportunities appear they are immediately filled with some initial capital drawn from an arbitrary exogenous distribution F.

I consider an equilibrium where agents enter only when $k + 1 \leq k^*$. Whenever an agent finds a market with capital larger than $k^* - 1$ they immediately return to searching without investing. This mechanism is summarized in Figure 2.2. The stationary cross-sectional distribution of masses across markets is given by $p(k)$ to

Figure 2.2: The main mechanisms of the model

This figure underlines the main mechanisms of my model. An agent can be in two different states. If the agent is searching, they meet a market with intensity λ. When they find a market they must decide, on the basis of the capital already invested in that market, whether they will invest or not. If they do not invest they return to searching. Otherwise, they invest their unique dollar and wait until that market pays a profit. When this event occurs they consume their respective profit and return to searching.



be determined in equilibrium. Letting $p_t(k)$ be the probability distribution function at an instant t, then the measure of markets with a capital not larger than k is given by,

$$\Phi_t(k) \equiv \int_0^k p_t(x)\, \mathrm{d}x.$$

This equation is a piecewise function. From 0 to 1 there are no interactions between search and initial capital heterogeneity since an agent would not invest less than one dollar. Accordingly, the dynamics of the cross-sectional distribution for $k \in [0, 1]$ satisfies,

$$\dot{\Phi}_t(k) = \eta[F(k) - \Phi_t(k)] - \Lambda\Phi_t(k), \tag{2.1}$$

where the dot stands for the time derivative. From 1 to $k^* - 1$, searching interacts with capital heterogeneity. It is worth mentioning that a market in that range could already have experienced multiple rounds of financing. However, agents finding that market would still invest as long as $k \leq k^* - 1$. Accordingly, the dynamics of the cross-sectional distribution for $k \in]1, k^* - 1]$ satisfies,

$$\dot{\Phi}_t(k) = \eta[F(k) - \Phi_t(k)] - \Lambda\Phi_t(k) + \Lambda\Phi_t(k-1). \tag{2.2}$$

The dynamics of the cross-sectional distribution for this range is very similar to equation (2.1); the only difference is regarding how searching interacts with capital heterogeneity. When a meeting occurs the agent invests their capital and a capital of k results from adding 1 to $k - 1$. On the contrary, no one would invest if the market's aggregate capital was larger than $k^* - 1$. This implies that the dynamics of the cross-sectional distribution for $k \in]k^* - 1, k^*]$ satisfies,

$$\dot{\Phi}_t(k) = \eta[F(k) - \Phi_t(k)] - \Lambda\Phi_t(k^* - 1) + \Lambda\Phi_t(k-1). \tag{2.3}$$

All markets born with an initial capital larger than k^* are located between k^* and ∞. The first term on the right-hand side of equation (2.1) reflects that at every instant, a measure η of new markets appear, with initial capital drawn from $F(k)$. Meanwhile, there is a measure $\eta\Phi_t(k)$ of markets with capital $m \leq k$ which matures. Then, the total meeting rate is $\Lambda \equiv \lambda(A - M_t)$. Finally, the total mass of agents who have their capital in a market is given by M_t which must satisfy the following differential equation,

$$\dot{M}_t = \lambda(A - M_t)\Phi_t(k^* - 1) - \eta M_t \tag{2.4}$$

The first term on the right-hand side of (2.4) reflects the rate at which agents meet markets. Since agents invest in a market only if $k \leq k^* - 1$, the total meeting rate must be adjusted by the probability of meeting non-saturated markets, $\Phi_t(k^* - 1)$. Then ηM_t is the measure of agents that return to searching.

2.2.1 Stationary Distribution

The stationary cross-sectional distribution results from setting the time derivative equal to zero. Since the shape of the distribution depends on the functional form of $F(k)$ I discuss two different approaches to solve for the model's equilibrium. The first method is very general; I discretize the state (capital) space and solve for the equilibrium recursively. This procedure has the advantage of not requiring setting the functional form of $F(k)$ and $\pi(k)$. However, this approach can not provide a closed-form solution for the cross-sectional distribution. The second technic is based on the theory of difference equations; while this method requires to parameterize both $F(k)$ and $\pi(k)$ the cross-sectional solution can be computed in closed-form.

Capital takes value on the set $\Gamma \equiv [0, 1/N, \cdots, k^u]$, where $N \in \mathbb{N}^*$ (the case $N \to \infty$ reduces to the continous case) and k^u is chosen large enough to ensure k^* belongs to Γ (for that, I use $k^u = \pi^{-1}(1)$). While the investment capacity of agents is constant (i.e. one dollar), the initial capital in all existing markets is drawn from the distribution $F(k)$ and is distributed over the whole set Γ. Therefore, the stationary cross-sectional distribution is characterized by a recursive system, which can be rewritten as a system of equations, $A \times \mathbf{x} = \mathbf{y}$, where $A \in \mathbb{R}^{N \times k^* + 1, N \times k^* + 1}$ and, $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{N \times k^* + 1, 1}$. In particular, I define $A_{i,j}$ as the i, j element of the matrix A. Then, all elements in the diagonal equal one. Any element satisfying $A_{i+N,i}$, $i \in (1, \cdots, N \times (k^* - 1) + 1)$ equals $-\Lambda/(\eta + \Lambda)$. Any element satisfying $A_{i+N,i}$, $i \in (N \times (k^* - 1) + 2, \cdots, N \times k^*)$ equals $-\Lambda/\eta$. Finally, any element satisfying $A_{i,N\times(k^*-1)+1}$, $i \in (N\times(k^*-1)+2, \cdots, N\times k^*)$ equals Λ/η. All the other elements are zero. Concerning the vector $\mathbf{y}$, all elements $i \in (1, \cdots, N\times(k^*-1)+1)$ take the value $\eta/(\eta+\Lambda)\times F((i-1)/N)$, while all the others are simply $F((i-1)/N)$. The vector $\mathbf{x}$ is the stationary cross-sectional distribution function. Accordingly, all elements equal $\Phi((i-1)/N)$. Eventually, since matrix A is diagonally dominant it can be inverted, resulting in $\Phi = A^{-1}\times\mathbf{y}$.

The second approach uses the fact that the cross-sectional distribution satisfies a first order linear difference equation. Accordingly, the equation characterizing the stationary distribution must be,

$$\Phi(k) = C\times\left(\frac{\Lambda}{\eta+\Lambda}\right)^k + g(k),\ 1\leq k\leq k^*-1,$$

where the first part is the general solution (C is a constant to be defined) and $g(k)$ is the particular solution (which depends on the functional form of $F(k)$)[1]. For example, assuming that the initial capital is uniformly distributed over the range $[0, k^u]$, $F(k) := \frac{k}{k^u}$ results in the following stationary distribution,

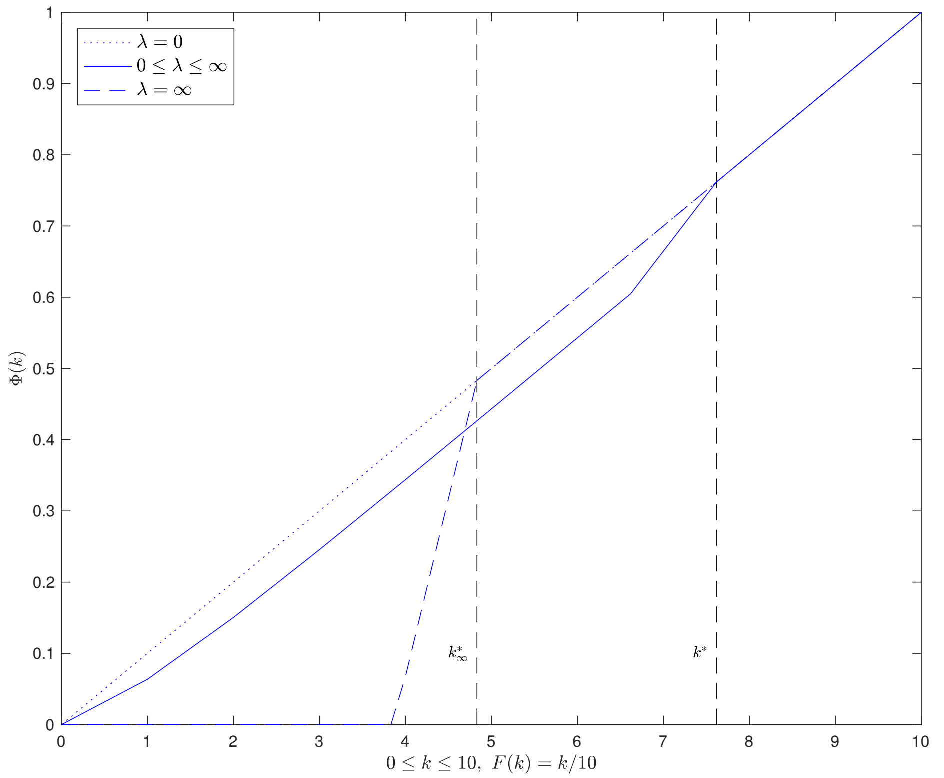
$$\Phi(k) = \begin{cases} \frac{1}{k^u}\left(\frac{\Lambda}{\eta+\Lambda}\right)^{\lfloor k\rfloor}\left[\frac{\Lambda}{\eta} - \frac{\Lambda}{\eta+\Lambda}(k-\lfloor k\rfloor)\right] + \frac{1}{k^u}\left[k - \frac{\Lambda}{\eta}\right] & 0\leq k\leq k^*-1, \\ \frac{k}{k^u} + \frac{\Lambda}{\eta}[\Phi(k-1) - \Phi(k^*-1)] & k^*-1\leq k\leq k^*, \\ \frac{k}{k^u} & k\geq k^*. \end{cases}$$

Where $\lfloor\cdot\rfloor$ is the floor function. Figure 2.3 shows different cross-sectional distributions. The first piece of this equation describes the cross-sectional distribution

[1]For difference equations, the treatment of the boundary condition is different from that of differential equations. For example, let us assume that the initial capital of a market is k_0 and k_τ is the aggregate capital invested in this market when it pays off. Importantly, k_τ belongs to the set $\{k_0, k_0+1, \cdots, k_0+m\}$ where m is a natural number such that $k^*-1-k_0 \leq m \leq k^*-k_0$. By varying k_0 one modifies the entire set. Hence, it must be keeped in mind that there is one constant for each path (one path being the set defined previously). This implies that one must compute the constant for any $k \in [0, 1[$.

Figure 2.3: Stationary distribution when the profit function is linear

The figure bellow has the following parameters: $A = 1$, $\eta = 2$, $r = .05$, $a = 10$ and $b = 1$. The solid line shows the stationary cross-sectional distribution for an arbitrary individual search intensity satisfying $0 \leq \lambda \leq \infty$. The dotted line represents the no-search regime CDF, while the dashed line is the fast search regime.



of markets with capital ranging from 0 to $k^* - 1$. From 0 to 1 markets are not populated by agents, rather all the capital comes from the first round of financing (i.e. the capital heterogeneity). From 1 to $k^* - 1$ the searching channel interacts with the initial capital. Agents do not invest above $k^* - 1$ since they would end up in a market with $k > k^*$. For any $k \geq k^*$ the cumulative distribution function is given by $F(k)$. This is because initial capital is drawn from $F(k)$, allowing a positive measure of markets to be filled with some initial capital greater than k^*. Finally, since Λ feeds into the cross-sectional distribution of markets, which itself depends on M, one must substitute $\Phi(m)$ in equation (2.4) (when $\dot{M}_t = 0$) to solve for the endogenous stationary distribution. Because the dynamics of $\Phi_t(x)$ and M_t are nonlinear it is not guaranteed *ex ante* that a unique equilibrium exists. The following proposition treats this problematic.

Proposition 2.1 *There is only one fixed point that satisfies system* (2.1) − (2.4) *with the restriction that* M *lies between* 0 *and* A*. Hence, there is a unique cross-sectional distribution associated with each* k^**.*

2.2.2 Value Functions and Equilibrium Characterization

Bellman's principal of optimality implies that the value function for an agent in a market characterized by a capital of $k \leq k^* - 1$ satisfies,

$$V(k) = \mathbb{E}\left[e^{-r(\tau_\eta \wedge \tau_\Lambda)}\{\mathbb{1}_{\{\tau_\eta \leq \tau_\Lambda\}}[\pi(k) - 1 + V_S] + \mathbb{1}_{\{\tau_\eta > \tau_\Lambda\}}V(k+1)\right].$$

Here, τ_η denotes the time at which this specific market matures. Similarly, τ_Λ is the time at which a new agent invests in this market. An agent that has invested their capital in this market faces two different outcomes; either a new investor discovers this market, or this market pays off. Thus, the expectation is computed with respect to the distribution of the minimum between τ_η and $\tau_\Lambda := \tau_\eta \wedge \tau_\Lambda$. The indicator function $\mathbb{1}_{\{\tau_\eta \leq \tau_\Lambda\}}$ corresponds to the probability that the investment opportunity matures before another agent invests multiplied by the corresponding payoff (the agent collects the profits $[\pi(k) - 1]$ and returns to searching, with continuation utility V_S). The second part is essentially the same; $\mathbb{1}_{\{\tau_\eta > \tau_\Lambda\}}$ accounts for the probability that one investor discovers this market before it pays off. When this event occurs the aggregate capital in this market will increase by one dollar, which means that the continuation value, conditional on this event, is $V(k+1)$. After integrating both sides with respect to the distribution of $\tau_\eta \wedge \tau_\Lambda$, this equation becomes,

$$(r+\eta+\Lambda)V(k) = \eta[\pi(k) - 1 + V_S] + \Lambda V(k+1). \tag{2.5}$$

Similarly, the value function of an agent in a saturated market (i.e. where $k^* - 1 \leq k \leq k^*$) satisfies,

$$V^*(k) = \frac{\eta}{r+\eta}[\pi(k) - 1 + V_S]. \tag{2.6}$$

For these markets only the time at which these specific investment opportunities mature matters. The term $[\pi(k)-1+V_S]$ corresponds to the profit of an arbitrageur once the market matures. Similarly, I find that the value associated to searching, V_S, satisfies,

$$V_S = \mathbb{E}\left[e^{-r\tau_\lambda}\left(\int_0^{k^*-1} V(x+1)p(x)\,\mathrm{d}x + [1-\Phi(k^*-1)]V_S\right)\right].$$

Here, τ_λ denotes the time at which one particular investor finds a market. Conditional on finding a market with capital $k \leq k^* - 1$, the agent invests one dollar, resulting in the capital in that market to jump from k to $k+1$. After some algebra that equation becomes,

$$V_S = \frac{\lambda}{r+\lambda\Phi(k^*-1)} \int_0^{k^*-1} V(x+1)p(x)\,\mathrm{d}x. \tag{2.7}$$

Since equations (2.5), (2.6) and (2.7) belong to the class of linear difference equations, the approaches used to solve for the cross-sectional distribution are still applicable. Hence, the general method detailed in the previous section implies that the vector of value functions can be characterized by a recursive system which can be rewritten as a system of equations $A' \times \mathbf{x}' = \mathbf{y}'$, where $A' \in \mathbb{R}^{N\times(k^*-1)+2, N\times(k^*-1)+2}$ and $\mathbf{x}', \mathbf{y}' \in \mathbb{R}^{N\times(k^*-1)+2,1}$. In the diagonal, all elements satisfying $A'_{i,i}$, $i \in (1, (k^*-2) \times N + 1$ equal $r + \eta + \Lambda$, while all the other diagonal elements equal $r + \eta$ except the very last term, which equals $\frac{r}{\lambda} + \Phi(k^* - 1)$. Any element satisfying $A'_{i,i+N}$ equals $-\Lambda$ except the very last term (i.e. $A'_{N\times(k^*-2)+2, N\times(k^*-1)+1}$). Then, any element satisfying $A'_{N\times(k^*-1)+1,i}$, $i \in (1, \cdots, N \times (k^* - 1))$ equals $-\eta$. Finally, all elements in the last row equal $p(i)$, except the very last one, which has already been discussed. Eventually, since matrix A' is diagonally dominant it can be inverted, resulting in $V = A'^{-1} \times \mathbf{y}'$.

The second approach uses the fact that value functions satisfy a first order linear difference equation. The equation characterizing the value functions satisfies,

$$V(k) = C \times \left(\frac{r + \eta + \Lambda}{\Lambda}\right)^k + g(k), \; 1 \leq k \leq k^* - 1,$$

where the first part is the general solution (C is a constant to be defined) and $g(k)$ is the particular solution (which depends on the functional form of $\pi(k)$[2]). For instance, assuming that the profit is a linear function of the form $\pi(k) = a - bk$[3] results in the following equation,

$$V(k) = \frac{b\eta\Lambda}{(r+\eta)^2}\left(\frac{r+\eta+\Lambda}{\Lambda}\right)^{-\lfloor k^*-k \rfloor} + \frac{\eta}{r+\eta}\left[a - 1 + V_S - b\left(\frac{\Lambda}{r+\eta} + k\right)\right].$$

Figure 2.4 shows the value function when the profit is linear in the market's aggregate capital. To proceed further I substitute $V(k)$ back into V_S. Finally, the

[2]Once again, the reader must keep in mind that there is one constant for each possible path. This implies that one must compute the constant for any $k \in [k^* - 1, k^*[$.

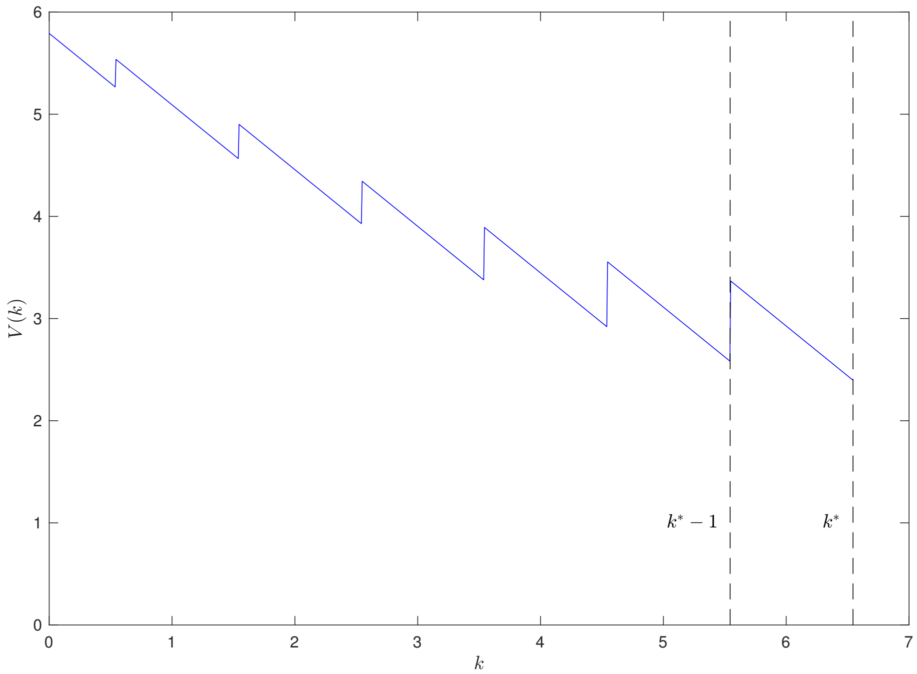
[3]Assuming this implies $k^u = (a-1)/b$.

Figure 2.4: Value function when the profit function is linear

When $\pi(k) = a - bk$, $V(k)$, the value that a buyer associates with being in a market with aggregate capital k satisfies,

$$V(k) = \frac{b\eta\Lambda}{(r+\eta)^2}\left(\frac{r+\eta+\Lambda}{\Lambda}\right)^{-\lfloor k^*-k \rfloor} + \frac{\eta}{r+\eta}\left[a - 1 + V_S - b\left(\frac{\Lambda}{r+\eta} + k\right)\right].$$

.



last variable to identify is the threshold k^*. In order to find this quantity one must remember that if an agent finds a market with current capital $k < k^* - 1$ the associated equilibrium strategy is to invest, which reveals that the value they extract from searching is strictly lower than the value they would receive by investing. If an investor finds a market with aggregate capital $k > k^* - 1$, their equilibrium strategy is not to invest, implying that there is a greater value in searching for another opportunity. Finally, if $k + 1 = k^*$, an agent is indifferent between searching or investing. Accordingly, the equilibrium k^* must satisfy the following conditions,

$$V_S \leq V(k), \ \forall k \leq k^*,$$
$$V_S > V(k), \ \forall k > k^*.$$

To put it another way, this set of conditions describes the outcome of a Nash Equilibrium. *It is not valuable for an agent to deviate from their optimal strategy given that the other agents play their own optimal strategy.* Practically, one must check each criterion for a given $k^* \in [1, k^u]$ and iterate the procedure until those conditions are met. If these criteria are satisfied for a particular configuration, then it is an equilibrium. When one uses the closed-form solution for $V(k)$ instead, the necessary and sufficient condition for a candidate to be an equilibrium is $V^*(k^*) - V_S(k^*) = 0$. The next proposition characterizes the equilibrium in general.

Proposition 2.2 *An equilibrium is characterized by the optimal threshold k^* that solves equation (2.8),*

$$\eta[\pi(k^*) - 1] - rV_S = 0, \tag{2.8}$$

once all pieces of the problem are treated simultaneously (that means equations (2.1)-(2.4) are set to zero, (2.5) and (2.6) are substituted into (2.7), which is substituted back into (2.8)).

2.2.2.1 Comparative statics

Figure 2.5 consists of four plots summarizing the main properties of the equilibrium, for a given set of parameters, when one varies the search intensity. In this figure, the profit function is simply linear, $\pi(k) = a - bk$. The upper left plot shows the shape of the aggregate search intensity (the total meeting rate Λ). The upper right plot shows how the proportion of agents who have invested their capital into a market varies with respect to λ. The shape of this function results from the interaction of two competing forces. Since λ represents the speed at which investors find markets, the proportion of agents who are searching must decrease accordingly. This is because an increase in the search intensity reflects an increase in the mass of markets found. On the other hand, because the agents find a market more easily the value function associated with searching rises. Hence, it becomes more valuable to skip some opportunities. Figure 2.5 shows a monotonic increasing function that converges asymptotically to $\bar{M} < 1$. The lower left plot illustrates how the value of searching evolves when one varies λ. This plot has to be analyzed simultaneously with the last one which shows the endogenous threshold. Both plots suggest that the agents see an increase of λ as beneficial. They can find more markets in a given time frame and reject more potential opportunities.

2.2.2.2 Model without search technology (λ^0)

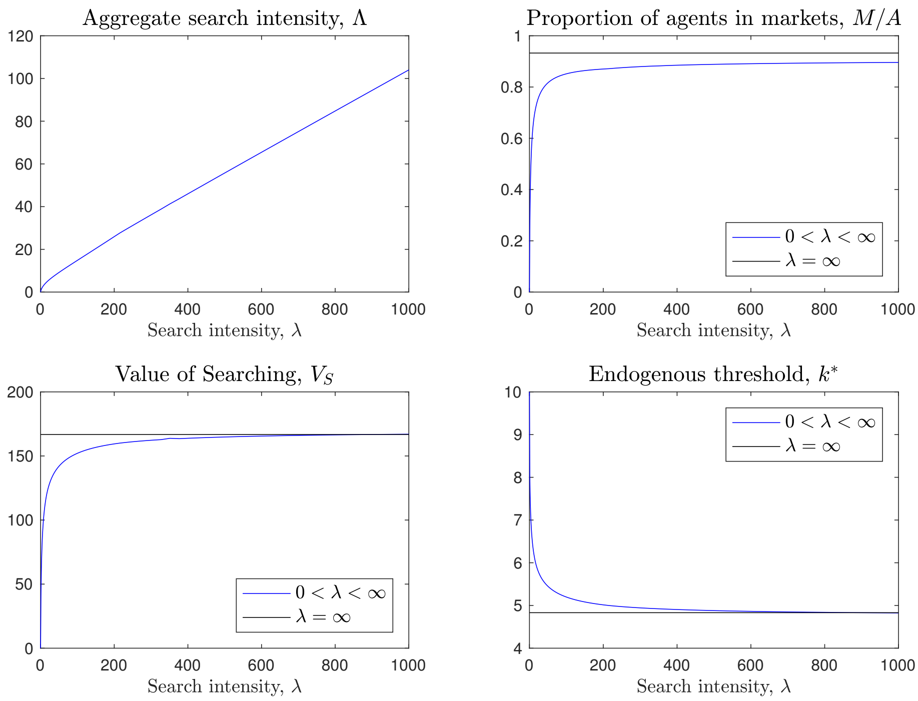
In this section I discuss the equilibrium when agents can not search (i.e. λ^0). Because searching is absent from this model, $\Phi(k) = F(k)$ and $M = 0$. Moreover, the value of searching is worth zero. Therefore, the equilibrium threshold k^* equals $k^u \equiv \pi^{-1}(1)$.

2.2.2.3 Fast Search (λ^∞)

While the previous section illustrates how the equilibrium behaves when arbitrageurs do not search, it remains only to explore its characteristics as search frictions vanish (i.e. λ^∞). In the following proposition I describe the main features of that limiting case.

Figure 2.5: The model's main statistical properties

The figures bellow have the following parameters: $A = 1$, $\eta = 2$, $r = .05$, $a = 10$ and $b = 1$. The top-left plot shows the aggregate intensity Λ. The top-right panel shows the fraction of the measure of agents, who have their capital in a market at that moment. The bottom left panel illustrates the value associated with searching. Finally, the last diagram shows the endogenous threshold k^*.



Proposition 2.3 *When the search intensity tends to infinity, $\Lambda \to \infty$ and $M \leq A$, the optimal threshold k^*_∞ is the quantity that solves equation (2.8) when V_S is characterized by the following relation,*

$$[r(A - M) + \Lambda\Phi(k^* - 1)]\, V_S = \int_0^{k^*-1} V^*(x + \lfloor k^* - x \rfloor)\Lambda p(x)\, \mathrm{d}x. \qquad (2.9)$$

Because $\Lambda \to \infty$, both $p(x)$ and $\Phi(x) \to 0$, implying that their product with the aggregate search intensity is indeterminate. The hospital theorem helps resolve this problematic; specifically, when $F(x) \sim \mathcal{U}[0, a/b]$, these quantities are respectively,

$$\begin{aligned} \Lambda p(x) = &\quad b/a \times \eta(\lfloor x \rfloor + 1), \\ \Lambda\Phi(k^* - 1) = &\quad b/a \times \eta\lfloor k^* \rfloor \left(k^* - 1 - \tfrac{1}{2}\lfloor k^* - 1 \rfloor\right). \end{aligned}$$

It remains to substitute these objects into V_S to solve for the equilibrium.

2.2.3 Is Perfect Capital Mobility Socially Efficient?

In the previous sections I have discussed two antithetical regimes; in the first one (λ^0) arbitrageurs have no search technology. Agents can not invest their capital leading to the worst possible outcome (as previously shown). This first case is contrasted by the second case (λ^∞) in which agents search at an infinite intensity. I show in this section that while λ^∞ is usually regarded as the most efficient outcome it is not always desirable. To understand why I start by drawing the cross-sectional distribution of markets for three levels of search intensity. This is represented in Figure 2.3. While the continuation value of arbitrageurs is a strictly monotonic increasing function of the search intensity as depicted in Figure 2.5, this relation is ambiguous for the agents that would draw their expected utility from the cross-sectional distribution. This follows simply from the fact that the stationary CDF for the infinite search intensity does not second-order stochastically dominate the stationary CDF of the finite intensity. I discuss welfare more formally in the following section.

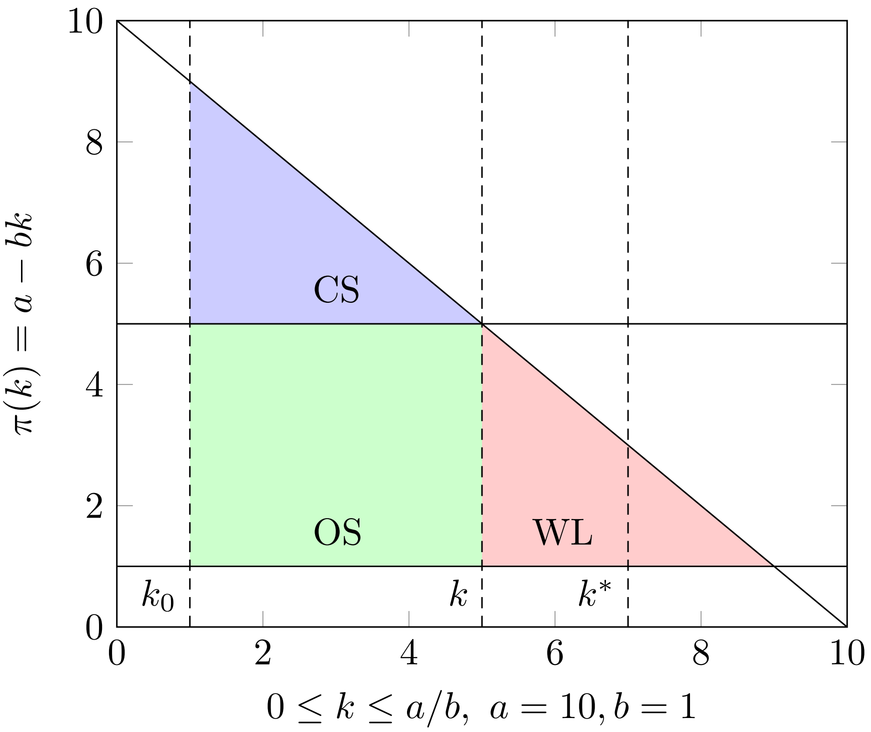
2.2.3.1 Social Surplus and Dead Weight Loss

In light of the above the profit function $\pi(k)$ can be generally considered as a demand for capital. For simplicity I review the case when this function is linear,

$\pi(k) = a - bk$. Furthermore, I assume that the distribution from which the initial capital is drawn satisfies $F(k) = \frac{a-1}{b}k$. Then, I designate the old agents as the providers of capital, while the new agents (those who provide the demand function) are the seekers of capital. Finally, k_0, which was defined as the initial capital, is considered here as demand heterogeneity (some markets saturate faster than others). Then, the consumer surplus is defined as the area between the demand

Figure 2.6: Visual representation of the strategy used to measure the welfare loss

This graph shows the offer and consumer surplus (green and blue respectively) and the welfare loss in red for $k = 5$ (the accumulated capital at exit time), $k_0 = 1$ and $k^* = 7$. The profit function is linear and capital can take values on the set $[0, a/b]$.



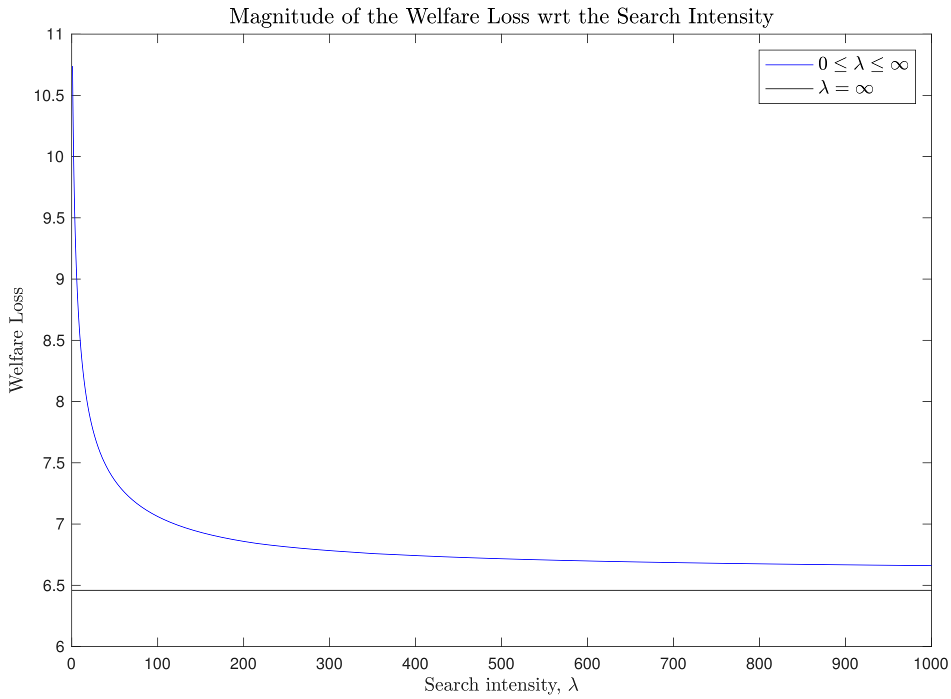
function, the price function (the horizontal line at $\pi(k) = a - bk$) and the vertical line that goes through k_0[4] weighted by the stationary CDF[5];

[4]Figure 2.6 shows the consumer and producer surplus and welfare loss.
[5]Since the capital accumulated until an exit event k is at the aggregate level, distributed according to the stationary CDF, any surplus or welfare loss must be weighted by this distribution.

$$CS(\lambda, k_0) = \int_0^{k^*} \frac{b}{2}(k - k_0)^2 p(k)\,\mathrm{d}k,\ \ k \geq k_0.$$

Figure 2.7: Welfare loss when capital is abundant

This graph shows the dead weight loss function with the following parameters: $A = 1$, $\eta = 2$, $r = .05$, $a = 10$ and $b = 1$. Here, the profit function is linear. The minimum is reached when $\lambda = \infty$.

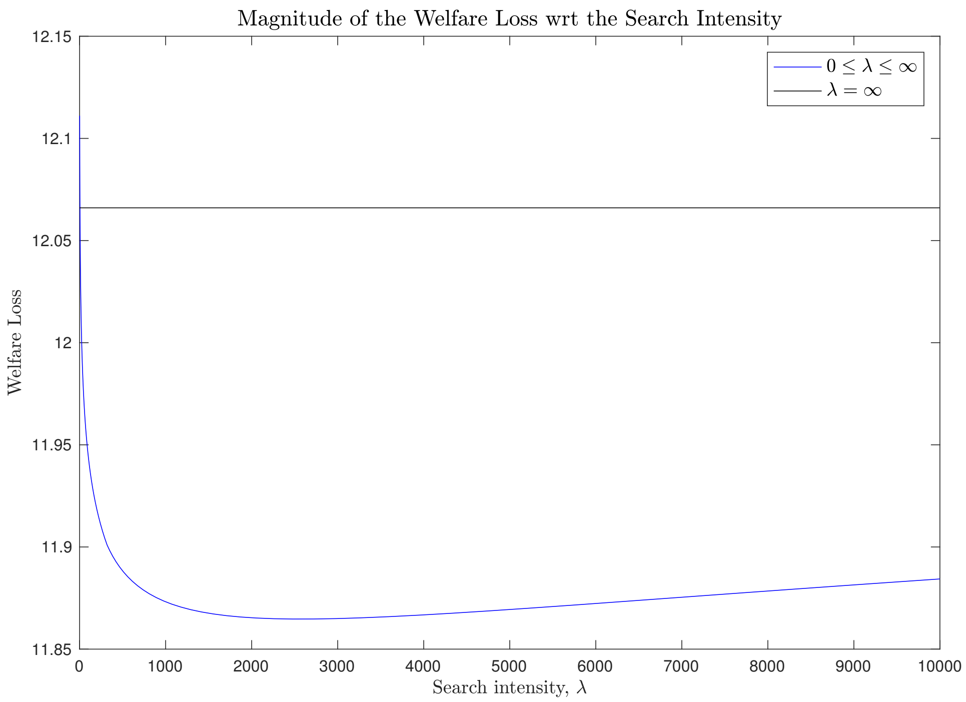


The offer surplus is defined as the area of the rectangle bounded by the vertical lines that goes through k_0 and k and the horizontal lines that goes through 1 and p.

$$OS(\lambda, k_0) = \int_0^{k^*} (k - k_0)(a - bk - 1)p(k)\,\mathrm{d}k,\ \ k \geq k_0.$$

Figure 2.8: Welfare loss when capital is scarce

This graph shows the dead weight loss function with the following parameters: $A = 0.01$, $\eta = 2$, $r = .05$, $a = 10$ and $b = 1$. Here, the profit function is linear. The minimum is reached well before $\lambda = \infty$, which is in contradiction to traditional wisdom.



Finally, the welfare loss is defined as the area between the demand function, the price function and the vertical line that goes through k. Since the dead weight loss depreciates welfare λ should be set such that this quantity is minimized.

$$WL(\lambda) = \int_0^{\frac{a-1}{b}} \frac{1}{2}(a - bk - 1)\left(\frac{a-1}{b} - k\right) p(k)\,\mathrm{d}k. \tag{2.10}$$

It is important to note that the upper boundary is beyond k^*. This follows from the fact that beyond this threshold markets are not populated (there is no offer), which is inefficient from a social planner's point of view. Since welfare loss is not affected by k_0 it is the simplest object to compute. Additionally welfare loss provides an intuitive description of the magnitude of the inefficiency in the whole economy and is shown in Figures 2.7 and 2.8. As expected, welfare loss is the highest at $\lambda = 0$, then the magnitude of the inefficiency decreases rapidly. While the loss of welfare decreases monotonically in Figure 2.7, it reaches a minimum (well before $\lambda = \infty$) in Figure 2.8. Clearly this is in contradiction with the belief that perfect capital mobility is always better. These results are summarized in the following proposition.

Proposition 2.4 *From a social planner's point of view, perfect capital mobility is not always the most desirable scenario.*

2.2.4 Endogenous Search

The following section explores which level of search intensity an agent, who seeks to maximize the value of searching, would employ. For this purpose I assume that searching is costly. Moreover, once an agent chooses the search technology they must stick with it. This makes the computations increasingly more tractable and can be interpreted as the result of the presence of large adjustment costs. Fixing a cost function $c(\lambda)$ which can be differentiated, Bellman's principal of optimality implies that the value associated with searching is given by,

$$V_S(\lambda) = \mathbb{E}\left[e^{-r\tau_\lambda}\left(\int_0^{k^*-1} V(m+1)p(m)\,\mathrm{d}m + V_S(\lambda)[1-\Phi(k^*-1)]\right) - \int_0^{\tau_\lambda} e^{-ru}c(\lambda)\,\mathrm{d}u\right].$$

The first part of the right-hand side of this equation is the standard searching value function. The second term corresponds to the present value of the costs.

Integrating this equation results in,

$$[r + \lambda\Phi(k^* - 1)]V_S(\lambda) = \lambda \left[\int_0^{k^*-1} V(m+1)p(m)\,\mathrm{d}m - c(\lambda) \right].$$

At this stage one can rely on both steady-state and non-steady state analyses. The first type of analysis regards the agents as perfectly rational; they internalize the fact that their own decisions have an impact on the equilibrium realization since they all solve the same problem. In that case an equilibrium is defined as a triplet $(k^*, \Lambda, \lambda^{**})$ that solves equations (2.4) and (2.8) and satisfies,

$$\frac{\partial V(s;\lambda)}{\partial \lambda} \equiv -\lambda \left[\int_0^{k^*-1} V(m+1)p(m)\,\mathrm{d}m - c(\lambda) \right] \Phi(k^* - 1) -$$
$$\left[\int_0^{k^*-1} V(m+1)p(m)\,\mathrm{d}m - c(\lambda) + \lambda \frac{\partial}{\partial \lambda} \left(\int_0^{k^*-1} V(m+1)p(m)\,\mathrm{d}m - c(\lambda) \right) \right] \times$$
$$[r + \lambda\Phi(k^* - 1)] = 0.$$

On the other-hand, the non-steady-state type of analysis builds on the assumption that agents believe they have no impact on the equilibrium, in particular that the cross-sectional distribution is fixed. Here the equilibrium is defined as a triplet $(m^*, \Lambda, \lambda^*)$ which simultaneously solves equations (2.4) and (2.8) and satisfies,

$$\frac{\partial V(s;\lambda)}{\partial \lambda} \equiv \left[\int_0^{k^*-1} V(m+1)p(m)\,\mathrm{d}m - c(\lambda) - \frac{\partial}{\partial \lambda} c(\lambda) \times \lambda \right] [r + \lambda\Phi(k^* - 1)] -$$
$$\lambda \left[\int_0^{k^*-1} V(m+1)p(m)\,\mathrm{d}m - c(\lambda) \right] \Phi(k^* - 1) = 0.$$

While the first approach is consistent, λ^{**} can never be solved in closed-form, which is not true for λ^*. Therefore, I discuss that case and assume a linear functional form for the searching costs ($c(\lambda) = \kappa\lambda$). Under these assumptions the optimal intensity results in,

$$\lambda^* = \frac{\sqrt{r^2 + \kappa^{-1} r \Phi(k^* - 1) \int_0^{k^*-1} V(m+1)p(m)\,\mathrm{d}m} - r}{\Phi(k^* - 1)}. \tag{2.11}$$

This suggests, empirically, that the intensity at which arbitrageurs search for opportunities is roughly proportional to $1/\sqrt{\kappa}$.

2.3 Cash-in-the-Market-Pricing

In this section I explore a simple extension of the model where some agents sell their assets to cover their sudden need for cash. Those assets are eventually split and consumed by all the respective buyers.

2.3.1 Generalities

I study an economy populated by two classes of agent. There is a unit continuum of sellers (the seekers of capital), each born with one asset which has a constant intrinsic value of a. For any of these agents a sudden need of cash occurs as the first jump of a Poisson process, of intensity η. There is one (independent) Poisson process for each seller. When a seller faces financial distress, they sell their asset, consume the proceeds and disappear. The buyers (or the liquidity providers) that have allocated their capital to that particular market, pool their capital, buy and consume the asset and return to searching. To justify this behaviour I recall that buyers never face financial distress. Therefore, they can keep the asset until the market recovers and sell it at its fair value (however, it is simpler to assume that the market recovers immediately after the buyers purchase the asset). At the same time new sellers are born at a rate of η making the measure of sellers constant over time. There is a measure A of buyers (the providers of capital). Each of them is endowed with one dollar. I define M as the measure of buyers that have allocated their capital in a market. Thus, $A - M$ is the measure of buyers that are still searching. Buyers meet a seller at a rate λ and decide whether to stay or keep searching further. From the law of large numbers for independent random matching the total meeting rate between buyers and sellers is $\Lambda \equiv \lambda(A - M)$. Given k, the quantity of capital present in a market, the buyers in that market each receive a/k. When new sellers are born, I assume that each receives some money, drawn from an arbitrary exogenous distribution. Therefore, depending on the money a seller has, they can keep a fraction of the asset and sell it when the market for that particular asset has recovered.

Once I feed these parameters into the model, the main primitives that affect the equilibrium are η, λ, r and A. I recall that they are respectively, the intensity at which a financial distress occurs and a new seller is born, the intensity at which an individual buyer searches, the rate of return of the risk-free asset and the measure of buyers. Normalizing A to one and recalling that buyers provide capital if and only if k, the quantity of capital invested in a particular market, is lower than $k^* - 1$, an endogenous threshold, an equilibrium is characterized by the usual conditions discussed in the previous section. When I further assume that the profit function is given by $\pi(k) = a/k$, which is consistent with the economy under investigation in that section, the value that a buyer associates with being in a market with aggregate capital k is provided in the following proposition.

Proposition 2.5 *Given $\pi(k) = a/k$, $V(k)$, the value that a buyer associates with* *being in a market with aggregate capital k, satisfies,*

$$\begin{aligned} V(k) = C(k + \lfloor k^* - k \rfloor) \times \xi^{k-1} + \frac{\eta}{r+\eta}(V_S - 1) + \\ \frac{a}{\Delta}\left[L(\xi^{-1}, 1, k) + \xi^k \log\left(\frac{r+\eta}{r+\eta+\Lambda}\right)\right], \end{aligned} \tag{2.12}$$

where,

$$\begin{aligned} \xi =& (r + \eta + \Lambda)/\Lambda, \\ \Delta =& (r + \eta + \Lambda)/\eta, \\ C(x) =& \frac{\eta}{r+\eta}\pi(x)\xi^{1-x} - \frac{a}{\Delta}\xi^{1-x}\left[L(\xi^{-1}, 1, x) + \xi^x \log\left(\frac{r+\eta}{r+\eta+\Lambda}\right)\right], \\ L(z, s, a)^{6} =& \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^s}. \end{aligned}$$

Solving for the equilibrium leads to important statistics which are reproduced in Figures 2.9 and 2.10. First of all, the optimal threshold, as it was the case for linear profit functions, is a monotonic decreasing function of the search intensity. As capital mobility improves sellers receive less. The welfare loss, shown in Figure 2.10, reaches a minimum which is not located in $\lambda = \infty$, breaking down the common belief that perfect capital mobility is (always) the first-best outcome.

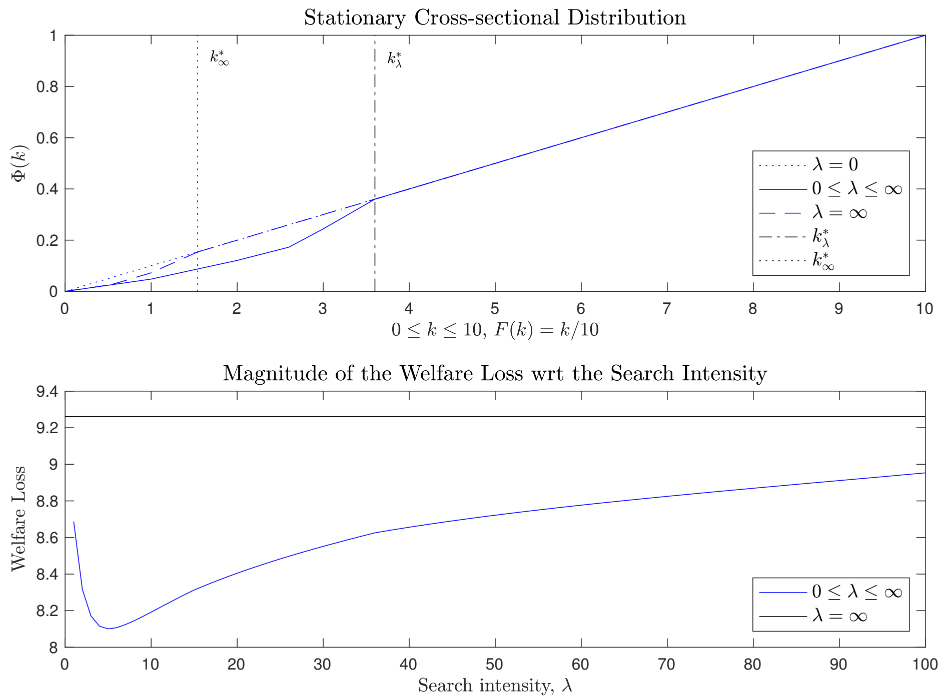
2.3.2 Government Interventions

In the following lines I extend the model and allow the redistribution of wealth from buyers to sellers through taxation. My intention is to contribute to the literature on fire sales of assets by discussing policy maker interventions, mainly

[6] This function is known as the *Lerch Transcendent* function.

Figure 2.9: Cash-in-the-market-pricing: stationary distribution and welfare loss

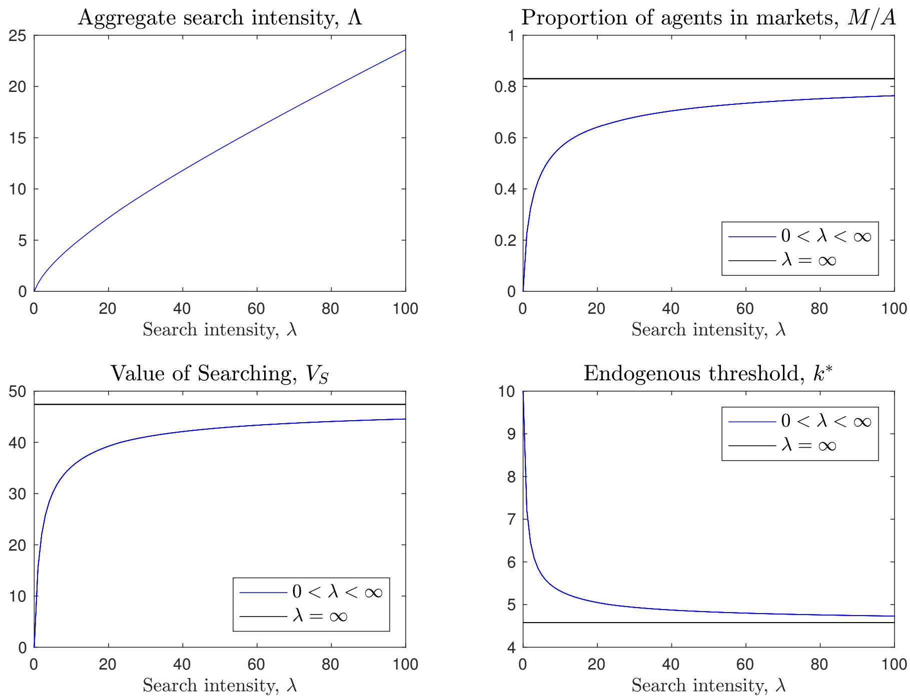
The figures bellow have the following parameters: $A = 1$, $\eta = 2$, $r = .05$, $a = 10$. The profit function is given by $\pi(k) = a/k$. In the first panel the solid blue line shows the stationary cross-sectional distribution for an arbitrary individual search intensity satisfying $0 \leq \lambda \leq \infty$. The dotted blue line represents the no-search regime CDF, while dashed-dotted blue line is the fast search regime. The second graph shows the magnitude of the welfare loss.



through the search channel. I am particularly interested in rescue packages offered by government institutions, which guarantee *toxic assets* in significant proportions. To this end I assume that the profit function generalizes to $\pi(k) = a/(b + k)$, where b is the proceed of the taxation to sellers. I investigate the situation when buyers are free to fix their search intensity, yet once they choose it they must stick with it. On top of that, policy makers tax the search process at a constant rate $c(\lambda)$, which is for simplicity assumed to be equal to $\kappa\lambda$. Accordingly, the entire proceeds of taxation equals $(A - M) \times c(\lambda)$, since only buyers who are currently searching pay

Figure 2.10: Cash-in-the-market-pricing: main statistical properties

The figures bellow have the following parameters: $A = 1$, $\eta = 2$, $r = .05$, $a = 10$. The profit function is given by $\pi(k) = a/k$. The top-left plot shows the aggregate intensity, Λ. The top-right panel shows the fraction of the measure of agents, who have their capital in a market at that moment. The bottom left panel illustrates the value associated with searching. The last diagram shows the endogenous threshold k^*.



taxes. I assume that this capital is transmitted uniformly to all sellers (except to those which have $k \geq k^*$). In consequence, every seller receives,

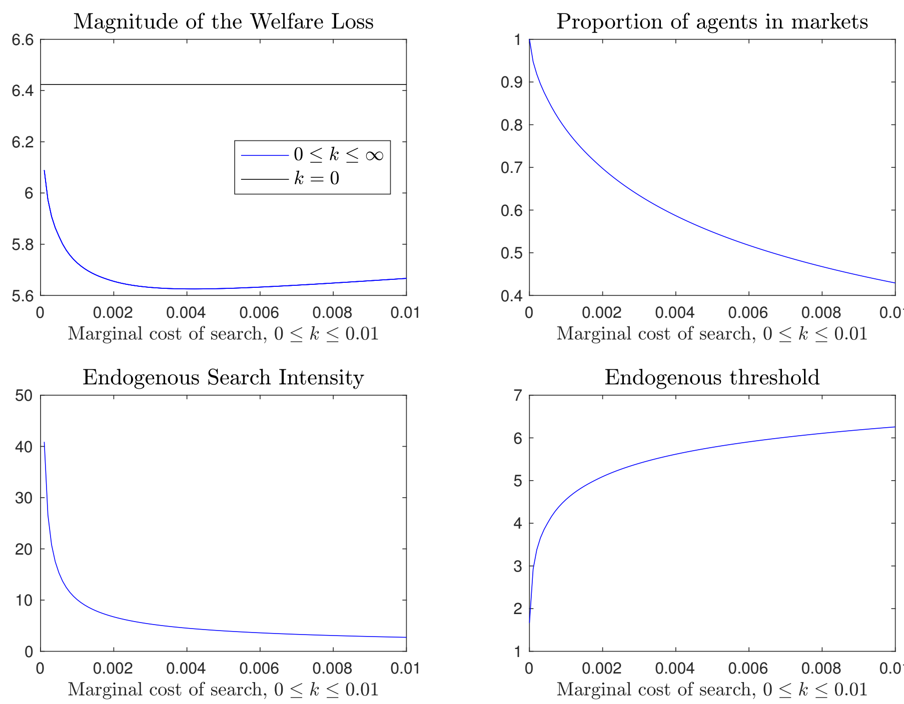
$$b \equiv \frac{\kappa\Lambda}{\eta\Phi(k^*)}.$$

The value that a buyer associates with being in a market with aggregate capital k, $V(k)$, is provided in the following proposition.

Proposition 2.6 *Given $\pi(k) = a/(b + k)$, $V(k)$, the value that a buyer associates*

Figure 2.11: Cash-in-the-market-pricing: the case with taxes

The figures bellow have the following parameters: $A = 1$, $\eta = 2$, $r = .05$, $a = 10$. The profit function is given by $\pi(k) = a/(b+k)$. This is an extension of the model that allows for the redistribution of wealth from buyers to sellers through taxation. The constant b is the proceeds of the taxation to sellers. The top-left plot shows the magnitude of the welfare loss. The top-right panel shows the fraction of the measure of agents who have their capital in a market at that moment. The bottom left panel illustrates the endogenous search intensity. The last diagram shows the endogenous threshold k^*.



with being in a market with aggregate capital k*, satisfies,*

$$V(k) = C(k + \lfloor k^* - k \rfloor) \times \xi^{k-1} + \frac{\eta}{r+\eta}(V_S - 1) + \frac{a}{\Delta}\left[L(\xi^{-1}, 1, b+k) - \xi^k L(\xi^{-1}, 1, b)\right], \tag{2.13}$$

where, ξ, Δ and $L(z,s,a)$ are defined in Proposition 2.5 and,

$$C(x) = \frac{\eta}{r+\eta}\pi(x)\xi^{1-x} - \frac{a}{\Delta}\xi^{1-x}\left[L(\xi^{-1},1,b+x) - \xi^{x}L(\xi^{-1},1,b)\right].$$

This case is shown in Figure 2.11. The top-left plot exhibits the magnitude of the welfare loss with respect to κ, the marginal cost of searching (the tax rate). When $\kappa = 0$ the individual search intensity, represented in the bottom-left plot, is infinite, corresponding to the perfect capital mobility regime. On the contrary, when $\kappa = \infty$ the regime is so costly that buyers do not search, leading to $\lambda = 0$. The endogenous threshold k^* follows the analogue pattern, which is also true for the measure of buyers searching and for those who already met a seller. While the precise level of tax rate that minimize welfare loss is not very important *per se*, the shape of the dead weight loss function provides very important implications for policy makers. Indeed, the optimal tax rate might be hard to reach in practice, mainly because of parameter uncertainty. However, the slope of the welfare loss function is significantly larger, in absolute value, for rates under the optimal tax rate compared to the slope beyond the optimal tax rate. This suggests that, under parameter uncertainty, taxing too much will cause less welfare loss than the reverse. Therefore a policy maker, under parameter uncertainty, fixing a tax rate on search technology, should always tax more rather than less.

2.4 Concluding remarks

In this paper I design a dynamic model where financially constrained agents are searching for an investment opportunity and once they find one decide whether or not they should invest. I use a simple extension of the model to show that this framework can provide interesting predictions and, more importantly, can be used to explore a wide range of economic inquiries. After conjecturing that one agent's decision to enter depends on the aggregate capital in this market, I find that the steady-state cross-sectional distribution of markets, for a given threshold, satisfies a linear difference equation and is therefore unique and stable. Furthermore, I show that the dynamic programming problem that pins down this barrier, which depends on the cross-sectional distribution, is also characterized by a linear difference equation. Eventually, the equilibrium is achieved when one simultaneously solves for the cross-sectional distribution and the threshold. This step, though it relies on numerical methods, still allows almost all the parts of the model to be expressed in closed-form.

With my model I show that an economy characterized by perfect capital mobility is not always desirable. When searching is not allowed, the economy performs very badly; there are no markets for liquidity and no benefits from trade, since there are no means of exchange. As searching frictions vanish, investors coordinate without difficulty. In this regime the optimal strategy of agents is to spread across the entire universe of markets and, since there are no search frictions, they can do this easily. Surprisingly, the first-best outcome is not necessarily achieved when searching frictions vanish.

Finally, I explore a simple extension of the model and discuss the fire sale of assets. After allowing for the redistribution of wealth from buyers to sellers through taxation, I show that a simple tax scheme on search technology provides an original solution for the financing of government interventions and to stimulate buyers to set search intensity at its first-best level.

2.A Proofs

Proof of Proposition 2.1. The stationary cross-sectional distribution satisfies the following difference equation,

$$\Phi(k) = \frac{\eta}{\eta + \Lambda} F(k) + \frac{\Lambda}{\eta + \Lambda} \Phi(k-1). \tag{2.14}$$

Moreover,

$$\Phi(k^* - \lfloor k^* \rfloor) = \frac{\eta}{\eta + \Lambda} F(k^* - \lfloor k^* \rfloor).$$

By iterating equation (2.14) forward $\Phi(k^* - 1)$ can be rewritten as follows,

$$\Phi(k^* - 1) = \frac{\eta}{\eta + \Lambda} \sum_{j=0}^{\lfloor k^* \rfloor - 1} \left(\frac{\Lambda}{\eta + \Lambda} \right)^j \times F(k^* - j - 1).$$

Using equation (2.4) and substituting M, this equation becomes $\mathcal{P}(\Phi(k^* - 1)) = 0$ where,

$$\begin{aligned} \mathcal{P} := \Phi(k^* - 1)[\Phi(k^* - 1) + A + \eta/\lambda]^{\lfloor k^* \rfloor} - \\ [\Phi(k^* - 1) + \eta/\lambda] \sum_{j=0}^{\lfloor k^* \rfloor - 1} A^j [\Phi(k^* - 1) + A + \eta/\lambda]^{\lfloor k^* \rfloor - j - 1} \times F(k^* - j - 1). \end{aligned} \tag{2.15}$$

Notice that $\mathcal{P}(0) < 0$. Then, let us make a change of variable and define $z = \Phi(k^* - 1) + A + \eta/\lambda$. This equation can be rewritten as follows,

$$z^{\lfloor k^* \rfloor + 1} - (A + \eta/\lambda) z^{\lfloor k^* \rfloor} - \sum_{j=0}^{\lfloor k^* \rfloor - 1} A^j z^{k-j} F(k^* - j - 1) + \sum_{j=0}^{\lfloor k^* \rfloor - 1} A^{j+1} z^{k-j-1} F(k^* - j - 1) = 0.$$

After changing the index of each sum, this equation becomes $\mathcal{P}^*(z) = 0$ where,

$$\mathcal{P}^* := z^{\lfloor k^* \rfloor + 1} - z^{\lfloor k^* \rfloor}[A + \eta/\lambda + F(k^* - 1)] + \sum_{j=1}^{\lfloor k^* \rfloor - 1} A^j z^{k-j}[F(k^* - j) - F(k^* - j - 1)] + A^k F(0). \tag{2.16}$$

Descartes' sign rule implies that this equation has at most two positive real roots. Notice further that $\mathcal{P}^*(1 + \eta/\lambda + A) > 0$ which implies that $\mathcal{P}(1) > 0$. Because $\mathcal{P}$ has at most two positive real roots and $\mathcal{P}(0) < 0 < \mathcal{P}(1)$, $\Phi^*(k^* - 1)$ is the only real root of $\mathcal{P}$ such that $0 < \mathcal{P}(\Phi^*(k^* - 1)) < 1$ (i.e. since $\Phi^*(k^* - 1)$ is the probability of the realization of an event, it must be between 0 and 1). Hence, each k^* is associated with exactly one stationary cross-sectional distribution. □

Proof of Proposition 2.2. Treated directly in Section 2.2. □

Proof of Proposition 2.3. The uniqueness of the cross-sectional distribution is guaranteed (thanks to Proposition 2.1). Hence, $\Phi(k^* - 1) = 0$ is the only real root of polynomial (2.15) when $\lambda \to \infty$. Since $\Lambda = \frac{\eta A}{\Phi(k^*-1) + \frac{\eta}{\lambda}}$, the limit of Λ when $\Phi(k^* - 1) \to 0$ and $\lambda \to \infty$ is ∞. Finally, equation (2.9) follows by multiplying equation (2.7) with $A - M$.

□

Proof of Proposition 2.4. Treated directly in Section 2.2.3.1. □

Proof of Proposition 2.5. Follows from Proposition 2.6. □

Proof of Proposition 2.6. Let us substitute equation (2.13) into equation 2.5.

Equation 2.5 can be rewritten as follows,

$$(r+\eta+\Lambda)V(k)-\Lambda V(k+1)=\eta[\pi(k)-1+V_S].$$

Firstly, notice that $C(k)=C(k+1)\ \forall k$. Hence,

$$(r+\eta+\Lambda)C(k+\lfloor k^*-k\rfloor)\times\xi^{k-1}-\Lambda C(k+1+\lfloor k^*-k-1\rfloor)\times\xi^k=0.$$

Secondly,

$$(r+\eta+\Lambda)\frac{\eta}{r+\eta}(V_S-1)-\Lambda\frac{\eta}{r+\eta}(V_S-1)=\eta(V_S-1).$$

Thirdly,

$$\frac{a}{\Delta}\xi^k L(\xi^{-1},1,b)\left[(r+\eta+\Lambda)-\Lambda\xi\right]=0.$$

Then, let us recall that $L(\xi^{-1},1,b+k)=\sum_{n=0}^{\infty}\frac{\xi^{-n}}{n+b+k}$, which implies that $L(\xi^{-1},1,b+k+1)=\xi\left[L(\xi^{-1},1,b+k)-\frac{1}{b+k}\right]$. Therefore,

$$\frac{a}{\Delta}\left[(r+\eta+\Lambda)L(\xi^{-1},1,b+k)-\Lambda L(\xi^{-1},1,b+k+1)\right]=\eta\pi(k).$$

Finally, since $k+\lfloor k^*-k\rfloor=k\ \forall k>k^*-1$,

$$V(k+\lfloor k^*-k\rfloor)=V^*(k)=\frac{\eta}{r+\eta}(\pi(k)+V_S-1),\ \forall k>k^*-1.$$

□

3 Sequential Competition and Innovation

There is not a day goes by without a passionate debate concerning Big Tech companies; while the main concern of governments seems to be related to social justice, a growing worry is associated with the consequences that large firms have on consumer welfare; nowadays these big entities engage in the serial acquisition of start-ups to preclude the entrance of competitors in their markets. Accordingly, I build a dynamic model where firms are endowed with a search technology used for designing new products that are subject to linear price-inverse demand function. Thanks to the relative tractability of my framework I compute many important statistics in closed-form. Then I show that the rise of large entities is part of a natural mechanism where giants emerge from the ashes of their predecessors. Though the actions of large technological companies impair welfare in the short-run, their consequences in the long-run seem negligible, suggesting a *laissez-faire* approach in which organizations engaging in these types of misconduct simply disappear in the long-term.

3.1 Introduction

It has been a little while since the dominance of the "Big Tech " companies, such as Facebook, Amazon, Apple, Netflix and Alphabet, seems to pose a problem for governments.[1] While the main concerns here are related to social justice with respect to traditional taxpayers and wealth redistribution across countries a growing worry is that these large entities, which are embarking on a large scale serial acquisition of start-ups, do that to preclude the entrance of competitors into their markets. Hence, the aim of this paper is twofold. Firstly, it is essential to understand whether the rise of the giants is part of a natural mechanism. Secondly, I debate the consequences that large firms have on consumer welfare and whether they really threaten the economy.

I create a dynamic model where firms are endowed with a search technology which they use for creating new products and study the firm-size distribution conditional on sequential competition. The economy is populated by a unit continuum of risk-neutral agents and a continuum of markets. Agents can either be entrepreneurs or employees. Each entity develops new technologies; upon finding a product they enter sequentially into the market for that product and compete strategically, taking into account the actions of their incumbents and the strategies of their (potential) new entrants. Despite the fact that there is no growth in my model, every product becomes obsolete eventually and is replaced by a newer version, making my model adopt the paradigm of Schumpeter's creative destruction; at Poisson times, a measure η of new markets appears, making a measure η of actual products obsolete. At this instant all of these "mature" markets pay a profit to

[1] For instance a report made public by the European Commission on the 21st of September 2017 underlines that a fair and efficient tax system in the European Union for the digital market is one of its top priorities; this report stresses the fact that technology corporations paid, on average, less than half the tax of traditional firms.

all the firms that are competing in each of these markets. The profit function depends both on the aggregate supply in that market and the individual supply. I assume that the price-inverse demand function depends linearly on the aggregate supply. Furthermore, I suppose that within a market goods are homogenous and companies all share the same marginal costs. These assumptions make the model easily tractable and the stationary cross-sectional distribution of quantities in each market solvable in closed-form.

Finally, an important feature for firms to grow (and for the firm-size distribution to be realistic) is to assume that the search technology increases (linearly) with the entities' size; large corporations have access to more talented people, benefits from customers' spillovers and have more resources to finance the research and development phase of new products. With these assumptions the firm-size process belongs to the class of birth-death processes, which makes the aggregate matching rate and the firm-size distribution solvable in closed-form. Due to the overall tractability of my model I can compute some important statistics related to the concentration within and across markets, to markups and the continuation value of firms. All these implications can be tested empirically.

My model suggests that the rise of large entities belongs to a natural mechanism; the firm-size distribution is a mixture of a power law and a distribution with exponential decay. My results indicate that the closest the search intensity is from the obsolescence rate, the largest is the aggregate intensity and the greater the consumer welfare is. Although this prediction seems puzzling, sequential competition implies that each entity behaves as a monopolist facing a residual demand curve. When the search intensity is larger, the measure of firms is smaller (because companies are larger), but consumers are better off because the average number of entities in each market is larger. Finally, I discuss the case with entry costs and show that my argument still applies.

This paper is structured as follows: Section 3.3 presents the model and its main consequences, Section 3.4 derives the main implications. Finally, I summarize the main results and present my conclusions in Section 3.5.

3.2 Related Literature

My paper is associated with the literature on Stackelberg Competition (SC), which started with von Stackelberg (1934) and includes, for example, Robson (1990), He et al. (2007) and Julien (2011). Interestingly, Anderson and Engers (1992) show that when the price-inverse demand function is linear each firm behaves as a monopolist facing a residual demand curve inherited from its predecessors. This article also demonstrates that this result can be extended to a wide class of demand functions (i.e. the exponential demand functions). In Etro (2008) the author examines a market in which a firm has a first-mover advantage over other competitors and entry in that market is endogenous; the behaviour of the leader is very different from a traditional SC model. Instead of being concerned with the reaction of competitors to its own choices it now cares about the effect of its choices on the entry decision. In particular, the leader becomes significantly more aggressive. Lastly, Julien et al. (2011) shows that under linear demand and constant marginal costs firms operate as *myopic rational agents*; the number of potential followers is deliberately ignored in the decision process and strategies are the same, irrespective of where a firm is in the hierarchy. My paper builds on these articles and provides a mechanism to make the quantity of competitors in a market endogenous. In addition, I compute in closed-form some concentration measures implied by SC.

To that extent my paper belongs to the recent field on the relationship between markups and the labour share. Notably Autor et al. (2017) is an empirical paper that relates the rise of giant firms with a fall in the aggregate labour share. The authors test several implications of that hypothesis and find evidence that some mechanisms at work favour the most productive firms, leading to an increase in product-market concentration, which translates into a reduction of the labour share. Related to this Loecker and Eeckhout (2017) take the rise in markups as given and derive a series of macroeconomic implications that should result from that. In particular, they show that the rise in markups has a negative effect on the labour share, capital share and low skilled wages. My model shows that when the economy is converging to a distribution with larger tails, markups should increase in unison with a fall in the aggregate labour share, which is consistent with these observations.

A significant part of my analysis relates to the literature on firm-size dynamics, which started with the seminal work of Hopenhayn (1992). In his paper, Hopenhayn designed a dynamic stochastic model which characterizes processes for entry and exit and for individual firms' outputs and employment. He investigates the long-run properties of the economy and in particular, size, age, profits and the value of companies. This article has been followed by a large amount of literature on firm-size distribution, which includes for instance Gabaix (2009), Luttmer (2010), Luttmer (2011) and Gabaix (2016). Most of these papers are refinements of Hopenhayn's article and generate more realistic predictions. However, the majority of these articles belong to the class of expanding variety models where markups are constant, which is far from reality. In addition these models cannot account for *creative destruction*, in which economic growth is driven by new entities replacing incumbents (i.e. Acemoglu (2008)). My framework contributes to this literature by providing a new approach for modeling firm growth. Importantly, it explicitly models markups and pricing strategies through sequential competition.

Finally, my model belongs to the literature on search and matching frictions which is based on Diamond (1982) and includes Burdett and Vishwanath (1988), Stahl (1989) and Duffie et al. (2005). The paper which is the most closely related to mine is probably Luttmer (2006). This article designs a model of search and matching between consumers and firms. Given some specific conditions (i.e. the population of consumers grows at a positive rate and the mean rate at which incumbent firms gain customers is positive) the model is able to generate a firm-size distribution which has a Pareto-like right tail which is consistent in the data. What makes my framework very different is that I do not model consumers explicitly; instead, my economy is formed by a continuum of markets which are subject to linear price-inverse demand functions. Because of this I can discuss markups which are constant in Luttmer's paper.

3.3 Model

3.3.1 The entrepreneurs' decision

Time is continuous. The economy is populated by a measure A of agents and a unit continuum of segmented markets $i \in [0, 1]$. An entrepreneur can set up a firm by finding a valuable idea (i.e. a market). The arrival of an idea is random and comes at a cost; as long as entrepreneurs pay a flow utility cost κ, they will find a market after an exponentially distributed waiting time with mean $1/\lambda$. Firms enter sequentially into market i and behave strategically, taking into account the strategies of entrants. Firm j finds a market with a search intensity $\lambda_{t,j}$. Upon finding a market firm j has to decide which quantity to supply. I assume that there are no barriers to entry, nor fixed costs. Each market is characterized by a linear inverse demand function, $P_i(Q_i) = a_i - bQ_i$, where Q_i is the total quantity produced in this market. Each firm located in market i produces a homogenous product and competes in quantities, submitting a quantity of $q_{i,j}$. Thus, the aggregate supply in market i is the sum of all individual productions, $Q_i = \sum_{j=0}^{J} q_{i,j}$. When firms choose their production level $q_{i,j}$ they must stick with it. Profits in market i are collected at the event of a Poisson jump, distributed with intensity η. Then, the market disappears. When this event occurs every firm in market i consumes their profit $\pi_j(q_j, Q_i) = q_j \times [a_i - bQ_i - c]$, where c is the marginal cost.[2] In the meantime, new markets appear at the same rate η so that the measure of markets remains constant. From the law of large numbers for independent random matching the total meeting rate is $\Lambda(t) := A \times \int_{\mathbb{R}+} \lambda(t) dF_t$. For any given market firms arrive in the event of a Poisson jump. Thus, if the previous aggregate production in the market i was Q_i it jumps to $Q_i + q^*(Q_i)$. Finally, I assume that a_i, the intercept of the demand function in market i, is distributed over $[c, \bar{a}]$, $\bar{a} > c$. This formulation is equivalent to assuming that markets are born with some *initial quantity* q_0 drawn from $F_{q_0}(x) := (\bar{a} - x)/b$ (i.e. a uniform random variable distributed over $[c/b, \bar{a}/b]$). Nevertheless, since the latter formulation is more tractable I use this approach in my calculations along with the first formulation (i.e. the heterogeneity of the intercept of the demand function). To show the equivalence of both formulations in general let us rewrite the inverse demand function as follows,

$$D(q, q_0) = \underbrace{\bar{a} - bq_0}_{:=a_i} - bq.$$

[2]Which I assume to be the same across all markets and firms.

Since a_i is linearly related to q_0, the probability distribution function (PDF) of q_0 is a translation of the PDF of a_i.

3.3.2 Market Structure

Firms compete sequentially as discussed in von Stackelberg (1934); when a firm enters into a market they take the present supply as given and choose their quantity by taking into account the strategy of potential entrants. An equilibrium, if it exists, is therefore a Subgame Perfect Equilibrium. In contrast to Etro (2008) a firm that discovers a market will always enter if it can; because there are no fixed costs, an incumbent willing to prevent the entrance of a competitor has to produce $q = (a_i - c)/b$, which makes that strategy a zero-profit investment. Furthermore, not only will the firm make a profit but they can also search faster.[3] Because at the market level firms enter randomly the profit of the i^{th} firm, conditional on the entrance of l entities, equals,

$$\pi_j(q_j, \mathbf{Q}_i) = q_j \times \underbrace{\left[a_i - b\left(\sum_{k=1}^{j-1} q_k + \sum_{k=j+1}^{I} \rho_k q_k + q_j\right) - c\right]}_{:=\text{markup in market } i},$$

where ρ_k is the probability that entity k enters. The first part is the quantity produced by firm j and the rest is the markup at the time this market pays off. Some papers, such as Anderson and Engers (1992), show that this problem can be solved for a wider class of demand functions as long as the marginal cost is constant. Hence, solving this problem by backward induction leads to the next proposition.

Proposition 3.1 (Stackelberg Competition) *The subgame perfect Nash equi-*

[3]This assumption is relaxed at the end of the paper.

librium of the Stackelberg Competition game with search frictions is entirely characterized by the following strategy; upon finding a market with production Q_{-j}*, the entrant produces,*

$$q^*_{i,j}(Q_{-j}) = \frac{a_i - c - bQ_{-j}}{2b}.$$

3.3.3 Markets Dynamics

Let us define $a = \bar{a} - c$, where a is the demand intercept, c is the marginal cost and $\bar{a}$ is the upper limit of the support of $F(x)$. The stationary cross-sectional distribution of quantity produced across all markets is given by $p(q)$, to be determined in equilibrium. Letting $p_t(q)$ be the PDF at an instant t, the measure of markets with aggregate production $Q \leq q$ is given by,

$$\Phi_t(q) := \int_0^q p_t(x)dx.$$

This equation is piecewise; from 0 to $\frac{a}{2b}$ there are no interactions between the search channel and the heterogeneity of the demand function, since a firm would never produce less than $\frac{a}{2b}$ when $Q = 0$. Accordingly, the dynamics of the cross-sectional distribution for $q \in [0, \frac{a}{2b}]$ satisfies,

$$\dot{\Phi}_t(q) = \eta[F(q) - \Phi_t(q)] - \Lambda\Phi_t(q), \tag{3.1}$$

where the dot stands for the time derivative. From $\frac{a}{2b}$ to $\frac{a}{b}$ the search channel interacts with the demand heterogeneity. A firm that finds a market with an aggregate quantity $Q \in [\frac{a}{2b}, \frac{a}{b}]$ will set its own production according to Proposition

3.1. Hence, the dynamics of the measure of markets with $Q \leq q$ is given by,

$$\dot{\Phi}(q) = \eta F(q) - (\Lambda + \eta)\Phi(q) + \Lambda\Phi(2q - a/b), \tag{3.2}$$

where the dynamic is very similar to equation (3.1), except that there is a measure $\Lambda[1 - \Phi(2q - a/b)]$ of markets which are contacted and still obey the condition $Q \leq q$. The first term on the right-hand side of equation (3.1) reflects that at every instant a measure η of new markets appear. Meanwhile $\eta\Phi_t(q)$ is the measure of markets which pay off. Upon finding a market a firm chooses its individual supply; the post-meeting aggregate production Q_i results from summing up $q^*_{i,j}(Q_{i,-j})$ with $Q_{i,-j}$. It is worth noting that because prices are defined on the positive real line, $0 < Q < a/b$. In the long term $\dot{\Phi}_t(q) = 0$ and it follows that the stationary distribution satisfies the functional equation,

$$\Phi(Q) = \frac{\eta}{\eta + \Lambda}F(Q) + \frac{\Lambda}{\eta + \Lambda}\Phi(2Q - a/b)\mathbb{1}_{\{Q \geq \frac{a}{2b}\}}. \tag{3.3}$$

This functional equation falls into the class of Schröder's Equation. Lemma 3.1 establishes how to solve an Abel Equation which clarifies how to solve Schröder's Equation.

Lemma 3.1 (Abel Equation) *Let $f(x) = a + bx$ and $h(c) = 0$. Then,*

$$h(x) = \log\left(\frac{a + x(b-1)}{a + c(b-1)}\right) / \log(b),$$

is the only function that satisfies the Abel Equation,

$$h(f(x)) = h(x) + 1.$$

For the cross-sectional distribution of markets: $c = 0$, the intercept equals $-a/b$ and the slope equals 2. Hence, Lemma 3.2 builds on Lemma 3.1 to solve the Schröder's Equation.

Lemma 3.2 (Schröder's Equation) *Let $f(x) = a + bx$ and $h(c) = 0$. Then,*

$$g(x) = s^{h(x)}, \quad h(x) := \log\left(\frac{a + x(b-1)}{a + c(b-1)}\right) / \log(b),$$

is the only function that satisfies the Schröder's Equation,

$$g(f(x)) = sg(x).$$

From that point I further assume that $F(x) = (b/a)x$, though equation 3.3 admits a unique distribution. After substituting $F(x)$ the next proposition establishes the functional form of the stationary cross-sectional (cumulative) distribution of markets with aggregate production $Q \leq q$, $\Phi(q)$.

Proposition 3.2 (Stationary Distribution) *The stationary cross-sectional (cumulative) distribution of markets with aggregate production $Q \leq q$ satisfies,*

$$\Phi(q) = \left[\frac{\Lambda}{\eta - \Lambda} - \frac{2\Lambda\eta}{(\eta+\Lambda)(\eta-\Lambda)}\frac{b}{a}g(q)\right] \times \left(\frac{\eta+\Lambda}{\Lambda}\right)^{\alpha(q)-\alpha(g(q))} + \frac{b}{a}\frac{\eta}{\eta-\Lambda}q - \frac{\Lambda}{\eta-\Lambda}, \tag{3.4}$$

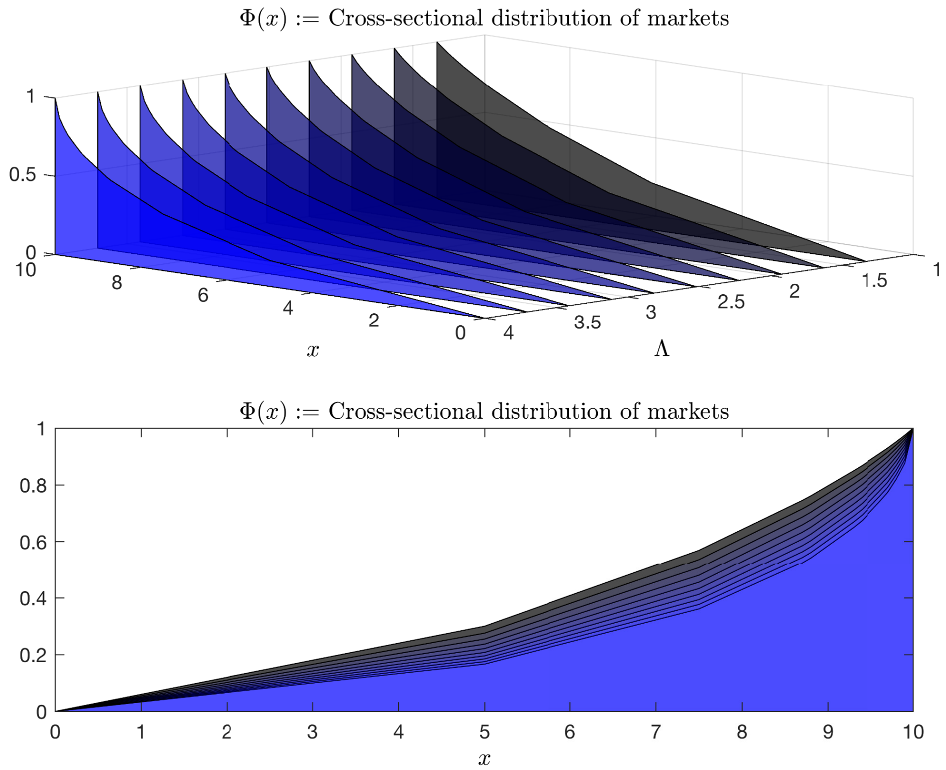
where,

$$\alpha(x) = \frac{\log\left(1 - x\frac{b}{a}\right)}{\log(2)},$$
$$g(x) = 2^{\lfloor -\alpha(x) \rfloor}\left(x - \frac{a}{b}\right) + \frac{a}{b}.$$

Figure 3.1 shows this distribution for different levels of aggregate intensity. What

Figure 3.1: Stationary cross-sectional distribution of markets

This figure shows the stationary cross-sectional distribution of markets $\Phi(x)$ with aggregate supply $Q \leq x$. Each object is computed with different levels of aggregate intensity, ranging from its lower bound $:= \frac{\eta\lambda_0 A}{\eta+2\lambda_0}$ (when $\lambda \to 0$) to its upper bound $:= \eta A$ (when $\lambda \to \eta$). It appears that the higher Λ the lower $\Phi(x)$ is. Thus, $\Phi(x;\Lambda_1)$ second order stochastically dominates $\Phi(x;\Lambda_2)$, with $\Lambda_1 > \Lambda_2$.



remains is the analysis of the stability and the uniqueness of this fixed point. Since equation 3.3 is a difference equation uniqueness is guaranteed. Then, the next proposition establishes that the stationary distribution is unique and globally stable as long as the aggregate intensity Λ only depends on time.

Proposition 3.3 (Global Stability) *When $\Lambda(t)$, the aggregate intensity, con-*

verges to a unique fixed point $\Phi_t(q)$, the measure of markets with $Q \leq q$, converges globally to a unique distribution (i.e. the one that satisfies equation (3.3)).

3.3.4 Firm-Size Dynamics

This model allows firms to grow as a function of the number of markets in which they are competing. The search intensity of firm j at time t, $\lambda_{t,j}$ is a function of the number of markets $N_{t,j}$ in which this firm is present. It is defined as follows,

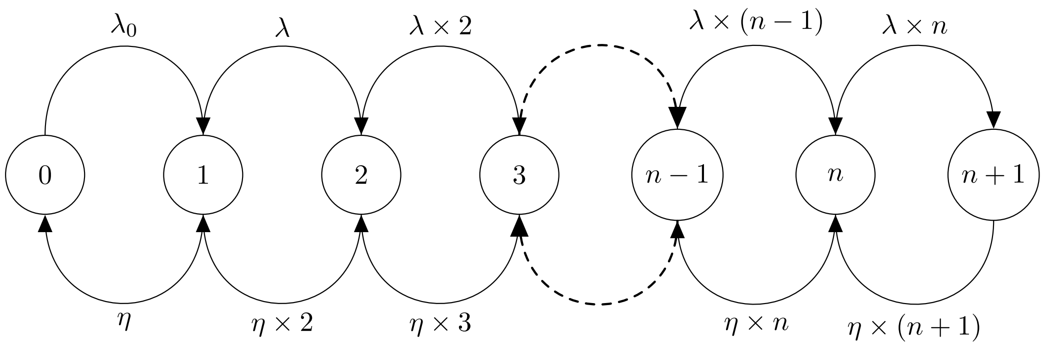
$$\lambda_{t,j} := \begin{cases} \lambda N_{t,j},\ \lambda < \eta & \text{for} \quad N_{t,j} > 0, \\ \lambda_0 & \text{for} \quad N_{t,j} = 0. \end{cases} \tag{3.5}$$

This parameterization generates spillovers on the size of a firm; the larger a firm the bigger its research and development unit and the faster its contact rate. Since each market independently pays off with intensity η, the intensity at which firm j, competing in N markets, sees one of its markets pay off is given by $\eta \times N$. This process is shown in Figure 3.2. When a firm is not competing in any market the entrepreneur is alone and searches at a rate λ_0. Since this firm cannot receive profits from markets in which it is not competing the state space is bounded above zero. Then, given that a firm is competing in one market, it must hire one employee and can search at a total rate of λ. If the market in which this firm is competing pays off before it finds a new market this firm would have to fire its employee. Hence, the entrepreneur would be alone again. Otherwise, this firm would be competing in two markets. Finally, each market pays off independently at rate η. Thus the total rate is $\eta \times N$ when a firm competes in N markets. The process that characterizes the number of markets in which a firm is competing is therefore a birth-death process (BDP). The infinitesimal generator is given by the following matrix,

$$\mathbf{Q} := \begin{pmatrix} -\lambda_0 & \lambda_0 & & & & & \\ \eta & -(\eta+\lambda) & \lambda & & & & \\ & 2\eta & -2(\eta+\lambda) & 2\lambda & & & \\ & & 3\eta & -3(\eta+\lambda) & 3\lambda & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & N\eta & -N(\eta+\lambda) & N\lambda \end{pmatrix}.$$

Figure 3.2: Firms' size process

This figure underlines the main mechanisms behind the firms' size process. Entrepreneurs start by searching for an idea at a rate λ_0. Once they find one they transform it into a final product, hire one employee and search for a new idea with a searching intensity λ. At this stage they can either find a new idea, transform it into a final product, hire a new employee and search at a new intensity $2 \times \lambda$, or see their product become obsolete. In the last case they must fire one employee. This mechanism can then be transposed for any size of firm.



The Backward Kolmogorov (BK) equation satisfies $\dot{\mathbf{P}} = \mathbf{QP}(t)$. Using the fact that $\mathbf{P}(0) = \mathbf{Q}$, the explicit solution of the BK equation can be formally rewritten as $\mathbf{P}(t) = e^{\mathbf{Q}t} = \sum_{n=0}^{\infty} \frac{\mathbf{Q}^n t^n}{n!}$. Given an initial firm-size distribution $\mathbf{F}_0$ the time t

cross-sectional distribution equals,

$$\mathbf{F}_t = \mathbf{P}(t) \times \mathbf{F}_0. \tag{3.6}$$

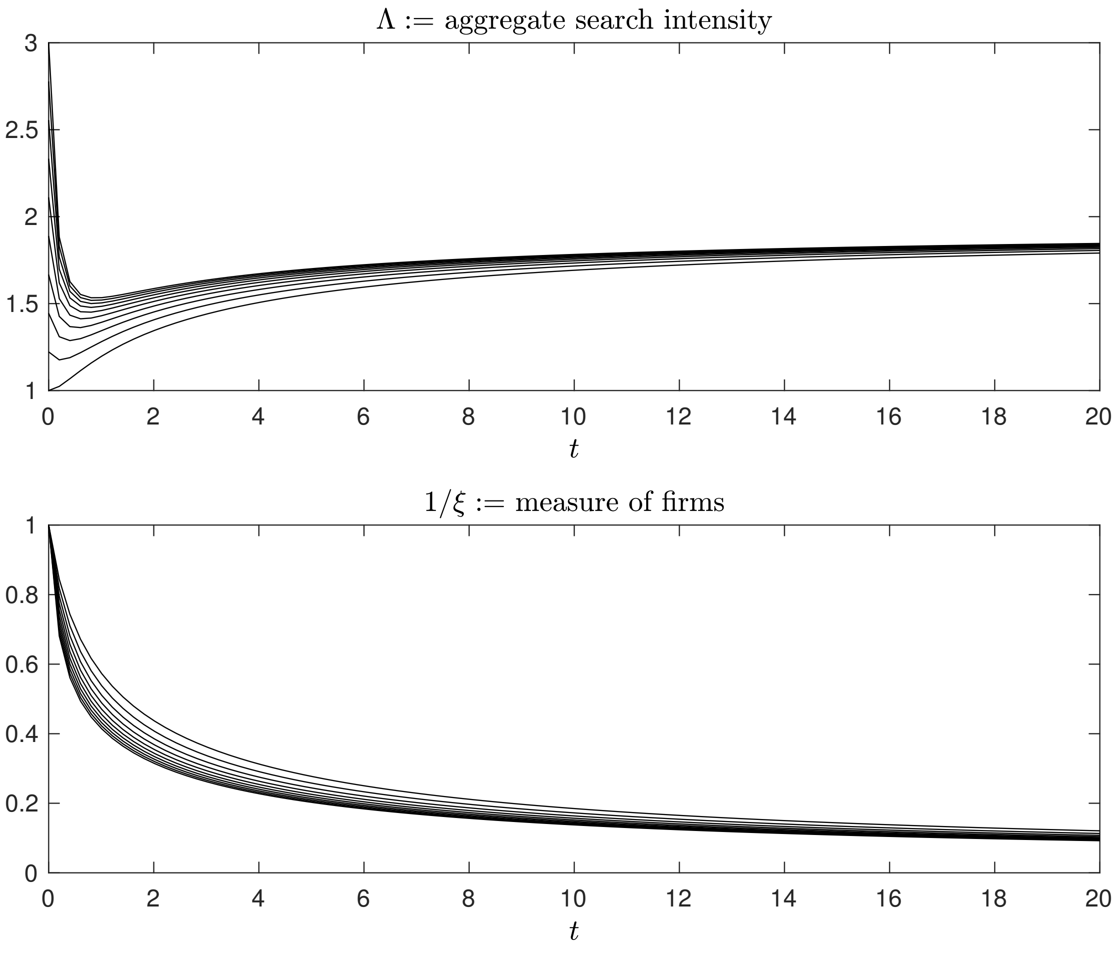
Importantly, the time t average intensity at which firms find markets is given by $\Lambda(t) = \lambda_0 f_t(0) + \lambda \sum_{i=1}^{\infty} i \times f_t(i)$, where $f_t(i)$ is element $i+1$ of vector $\mathbf{F}_t$. Consequently, the joint dynamics of firm-size and market supply for an initial distribution of firms $\mathbf{F}_0$ and markets Φ_0 satisfies equations (3.1)-(3.6). Since the firm-size process is an infinite state continuous-time Markov chain, matrix $\mathbf{Q}$ is of infinite dimensions. To avoid this problem I assume that N is bounded bellow some threshold $\bar{N}$. Hence, I replace the very last diagonal element with $-\bar{N}\eta$ instead of $-\bar{N}(\eta + \lambda)$, which makes the cross-sectional distribution sum to one. Then, when $\bar{N}$ is chosen to be large enough, this procedure allows the process to be simulated easily and does not distort it. Figure 3.3 exhibits the dynamics of the important statistics produced by the model. The top panel shows the aggregate search intensity for different levels of λ_0. The bottom panel plots the measure of firms competing in the economy as time passes. Importantly, there is a large reduction in the number of firms with time. While the average firm size increases this effect is counterbalanced by the reduction of the number of firms. This means that if entrepreneurs innovate at a faster pace than the rate at which products become obsolete ($\lambda_0 > \eta$) an economy in which entrepreneurs do not hire leads to a higher aggregate search intensity.

3.3.5 The Long-Run Economy

Using standard results in the birth-death process literature I characterize the stationary distribution in the next proposition.

Figure 3.3: The dynamics of the model's main statistical properties

The upper figure shows the dynamics of the aggregate intensity and the lower figure shows the measure of firms against time. I have assumed that when $t = 0$ the economy is populated by entrepreneurs only (i.e. $\mathbb{P}(n > 0) = 0$). This explains why the measure of firms is monotonically decreasing over time. On the other hand, as the average firm size grows the average search intensity increases. Nevertheless, this effect is counterbalanced by the reduction of the number of firms. This behaviour is shown in the upper diagram.



Proposition 3.4 *The stationary firm-size distribution is given by,*

$$\mathbf{F}_\infty = \langle f_\infty(0), f_\infty(1), f_\infty(2), \cdots \rangle,$$

where,

$$\begin{aligned} f_\infty(0) &= 1/S, \\ f_\infty(i) &= \tfrac{\lambda_0}{\eta S}\left(\tfrac{\lambda}{\eta}\right)^{i-1}\left(\tfrac{1}{i}\right),\ i>0, \\ S &= 1+\tfrac{\lambda_0}{\eta}\textstyle\sum_{i=1}^{\infty}\left(\tfrac{\lambda}{\eta}\right)^{i-1}\tfrac{1}{i} = 1-\tfrac{\lambda_0}{\lambda}\log\left(1-\tfrac{\lambda}{\eta}\right) < \infty. \end{aligned}$$

Figure 3.4 shows the firm-size distribution for different levels of λ. Thanks to Proposition 3.4 the long-run average matching intensity converges to,

$$\bar{\lambda}^\infty = \frac{\lambda_0}{S}\frac{\eta}{\eta-\lambda}.$$

Because the measure of workers is constant the measure of firms J is strictly smaller than A. In particular, the aggregate intensity must satisfy $\bar{\lambda}^\infty \times J$. Proposition 3.4 implies that the labour force required by the model has to be $\sum_{i=0}^{\infty} f_\infty(i)\times(i+1) = 1+\frac{\lambda_0}{S(\eta-\lambda)}$. However, recalling that the measure of workers is constant and equals A, the measure of firms J is given by $A\times\left(1+\frac{\lambda_0}{S(\eta-\lambda)}\right)^{-1}$, which implies that the long-term aggregate intensity satisfies,

$$\bar{\Lambda}^\infty = \frac{A\eta\lambda_0}{\lambda_0+S(\eta-\lambda)}.$$

Hence, an economy with large firms has a smaller measure of entities.

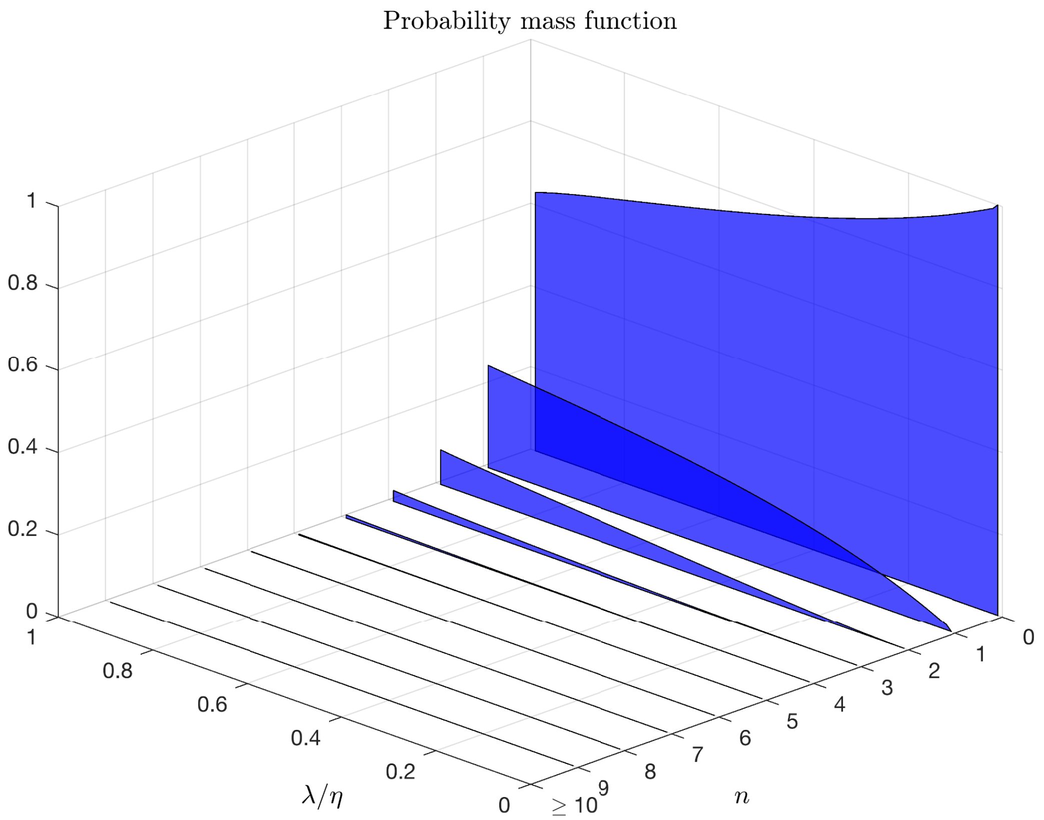
3.3.6 Value Functions in Equilibrium

Bellman's principal of optimality implies that the value function for a firm that supplies quantity x in a market with aggregate supply Q satisfies,

$$V(Q,x) = \mathbb{E}\left[e^{-r(\tau_\eta\wedge\tau_\Lambda)}\left\{\mathbb{1}_{\{\tau_\eta\le\tau_\Lambda\}}(a-bQ)x + \mathbb{1}_{\{\tau_\eta>\tau_\Lambda\}}V\left(\frac{a+bQ}{2b},x\right)\right\}\right].$$

Figure 3.4: Probability mass function: number of competitors per market

This figure shows the probability distribution function of the number of competitors per market given the search intensity λ. When this quantity is zero a match is not possible, resulting in a delta function at zero. When firms search with strictly positive intensity the number of competitors per markets rises quickly. Nevertheless, the figure makes it clear that a search intensity near zero is not a desirable outcome for consumers; competition between firms is almost not present. Even worse, most of the markets have no firms providing a supply function. As search intensity rises inefficiencies vanish.



Here, τ_η denotes the time at which this specific market matures. Similarly, τ_Λ is the time at which a new firm finds this market. A firm competing in this market faces two different events; either a new firm enters, or this market matures. Thus, the expectation is computed with respect to the distribution of the minimum between

τ_η and $\tau_\Lambda := \tau_\eta \wedge \tau_\Lambda$. Then, the indicator function $\mathbb{1}_{\{\tau_\eta \leq \tau_\Lambda\}}$ corresponds to the probability that the market matures before any other firms enter multiplied by the corresponding payoff (the agent collects the profits, $[a - bQ]x$). The second part is essentially the same; $\mathbb{1}_{\{\tau_\eta > \tau_\Lambda\}}$ accounts for the probability that one firm discovers this market before it pays off. When this event occurs the total quantity supplied in this market jumps by $q^*(Q) = \frac{a-bQ}{2b}$. After integrating both sides with respect to the distribution of $\tau_\eta \wedge \tau_\Lambda$, this equation becomes,

$$(r + \eta + \Lambda)V(Q, x) = \eta(a - bQ)x + \Lambda V\left(\frac{a + bQ}{2b}, x\right). \tag{3.7}$$

Here, I use the fact that $Q+q^*(Q) = \frac{a+bQ}{2b}$. The boundary condition is $V(a/b, x) = 0$. Equation (3.7) has a unique solution which is discussed in Proposition 3.5. Importantly, the boundary condition used here implies that only the non-homogenous part of the functional equation matters.

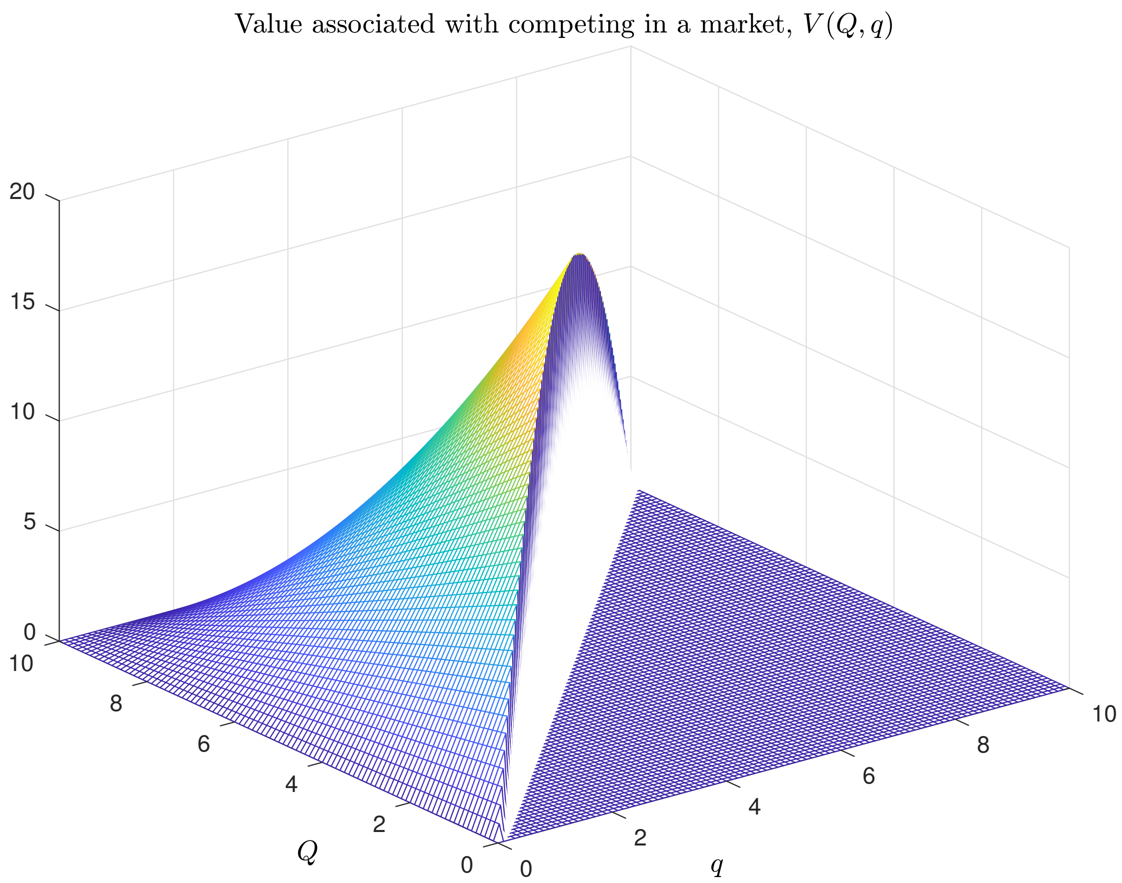
Proposition 3.5 *The value function of firm j producing $q_{i,j}$ in market i with total supply Q_i, is given by,*

$$V(Q_i, q_{i,j}) = \frac{a\eta}{r + \eta + \Lambda/2}\left[1 - \frac{b}{a}Q_i\right] \times q_{i,j}. \tag{3.8}$$

Figure 3.5 shows the continuation value against the individual and the aggregate supply. In order to find the value associated with searching V_S one must recall that the search intensity is provided by equation (3.5). In other words, the value

Figure 3.5: Value associated with competing in a market

This figure shows the value of competing in a market which depends on Q, the aggregate supply in this market and q, the individual supply.



of searching depends on the number of markets in which the firm is competing. Accordingly, it must satisfy,

$$V_S(N) = \mathbb{E}\left[e^{-r\left(\tau_{\lambda(N)} \wedge \tau_{\eta \times N}\right)}\left\{\mathbb{1}_{\{\tau_{\lambda(N)} > \tau_{\eta \times N}\}} V_S(N-1) + \mathbb{1}_{\{\tau_{\lambda(N)} < \tau_{\eta \times N}\}} V_S(N+1)\right\}\right], N \geq 1, \tag{3.9}$$

and $V_S(0) = \mathbb{E}\left[e^{-r\tau_{\lambda_0}}V_S(1)\right]$. Here, $\tau_{\lambda(N)}$ denotes the time at which firm j, competing in N markets, finds market $N+1$. Upon finding a market a firm produces the quantity $q^*(Q) = \frac{a-bQ}{2b}$, making the total production in this market jump from Q to $\frac{a+bQ}{2b}$. Otherwise, $\tau_{\eta\times N}$ denotes the time at which one market of firm j pays off. After some algebra equation (3.9) becomes,

$$\Big[r + (\lambda+\eta)\times N\Big]V_S(N) = \lambda \times NV_S(N+1) + \eta \times NV_S(N-1). \tag{3.10}$$

When $N = 0$ the value associated with searching satisfies $(r+\lambda_0)V_S(0) = \lambda_0 V_S(1)$; these firms receive the continuation value of competing in one market, discounted at a rate adjusted by the entrepreneur's search intensity. When N is large this equation is well approximated by the following second order linear difference equation with constant coefficients,

$$V_S(N+1) - \frac{\lambda+\eta}{\lambda}V_S(N) + \frac{\eta}{\lambda}V_S(N-1) = 0.$$

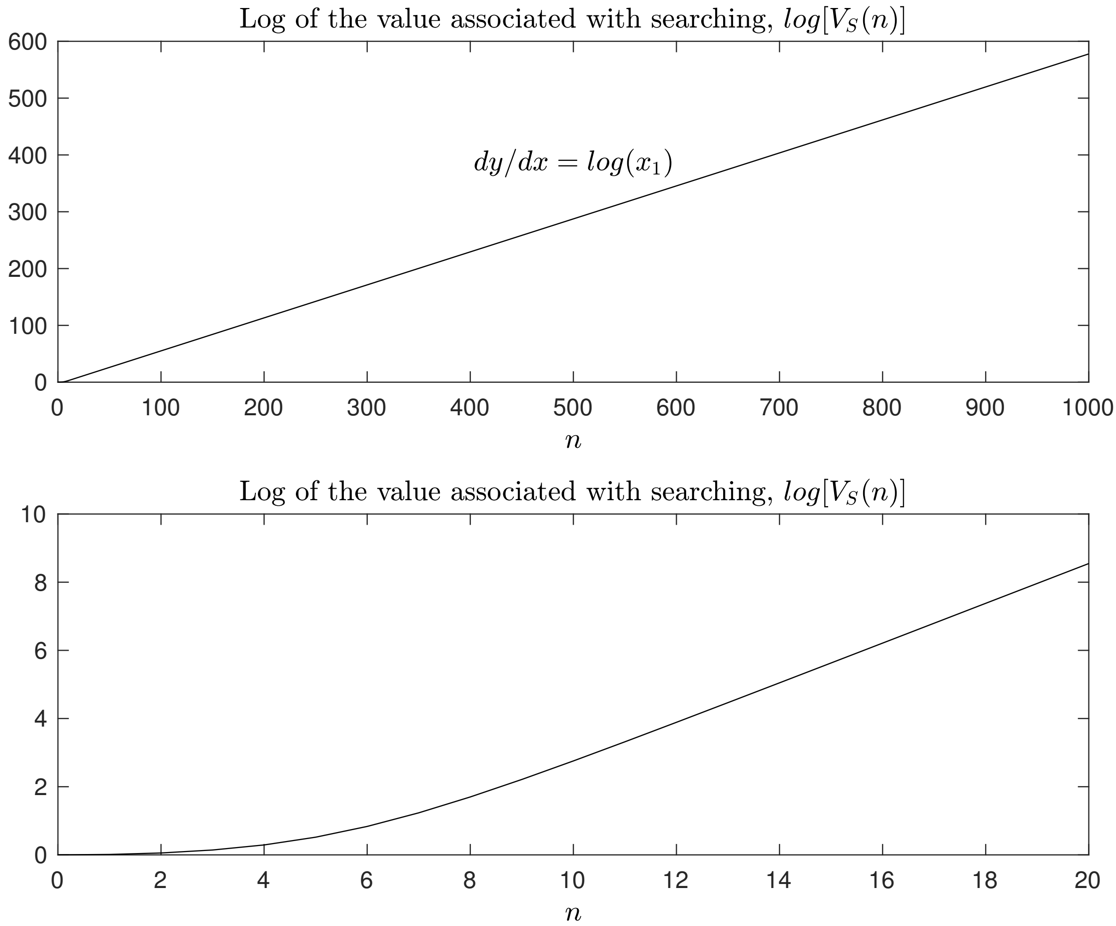
The solution of this equation is $V_S(N) = Ax_1^N + Bx_2^N$, where $x_1 = \eta/\lambda > 1$, $x_2 = 1$ and A and B are two constants. Hence, when N is large the following relation holds,

$$\log[V_S(N)] = c + \log(x_1)N,$$

where c is a constant. Figure 3.6 shows the continuation value associated with searching. The value associated with searching increases exponentially with N, which can be interpreted as the number of employees in the research and development unit. Finally, let us define $\mathbf{n}_j := \langle Q_{j,1}, Q_{j,2}, \cdots, Q_{j,N}\rangle \in \mathbb{R}^{N,1}$, the vector of the aggregate supply in each market firm j is competing in and $\mathbf{x}_j := \langle q_{j,1}, q_{j,2}, \cdots, q_{j,N}\rangle \in \mathbb{R}^{N,1}$ the vector of its individual production. Then, the

Figure 3.6: Value associated with searching with n employees

This figure shows the value associated with searching with n employees. While the approximation is not exact, I show in this paper that the log of this function is approximately linear with a slope of $\log(x_1)$, where $x_1 = \eta/\lambda$.



continuation value of a firm $FV : \mathbb{R}^{N,2} \rightarrow \mathbb{R}^{N,1}$ is given by,

$$FV(\mathbf{n}_j, \mathbf{x}_j) = V_S(N_j) + \sum_{i=1}^{N} V(Q_{j,i}, x_{j,i}).$$

3.4 Implications of the Model

Given my characterization of the steady-state equilibrium above, it is possible to derive many objects analytically. In this section I use my model as a laboratory and focus on analytical results that are important to the literature. In Section 3.4.1 I follow a market and characterize the probability distribution function of the number of firms competing in a given market. Equipped with this object I can discuss how competitors share the market and how this competition structure affects the consumer welfare. I also relate these statistics with some of the exogenous variables which are, for instance, the search intensity λ and η the rate at which products are rolled over. In Section 3.4.2 I follow a firm and discuss the stationary firm size distribution implied by my model and also discuss the consequences of having large firms in an economy.

3.4.1 Following Markets

Since every market represents consumption needs it becomes apparent that some of these will not be supplied with any products. The structure of the model makes it clear that once we know Q_0 and Q we can deduce the number of firms in that market. Nevertheless, it is more interesting to infer the probability distribution function given Q only. Thus, one can use that function *ex ante* to characterize any expected variables related to markets.

Claim 3.1 *The probability that i firms are competing in market j is given by,*

$$P(N = i) := \frac{\eta}{\Lambda}\left(\frac{\Lambda}{\eta + \Lambda}\right)^{i+1}.$$

This PDF is independent of the functional form of $F(x)$. The probability $P(N = 0)$

corresponds to the likelihood that a market has no current supplier. Similarly, the probability $P(N = 1)$ corresponds to the likelihood that customers in a market face a monopolist, etc. One can easily characterize the vector of market shares $[a_1, \cdots, a_N]$ in a particular market.

Claim 3.2 *The market share of the $i^{th} > 0$ firm is given by,*

$$a_i := \frac{1}{2 - 2^{1-N}} \left(\frac{1}{2}\right)^{i-1}.$$

A corollary of Proposition 3.1 is that the market share decreases geometrically. Nevertheless, since the shares must sum up to one, one must adjust a_1 such that this equality is satisfied. Equipped with this I can obtain a measure reflecting the market concentration; the Herfindhal index is an intuitive metric fulfilling this purpose which is defined as the sum of the squared market shares.

Claim 3.3 *The Herfindhal index in a market with $N > 0$ firms is given by,*

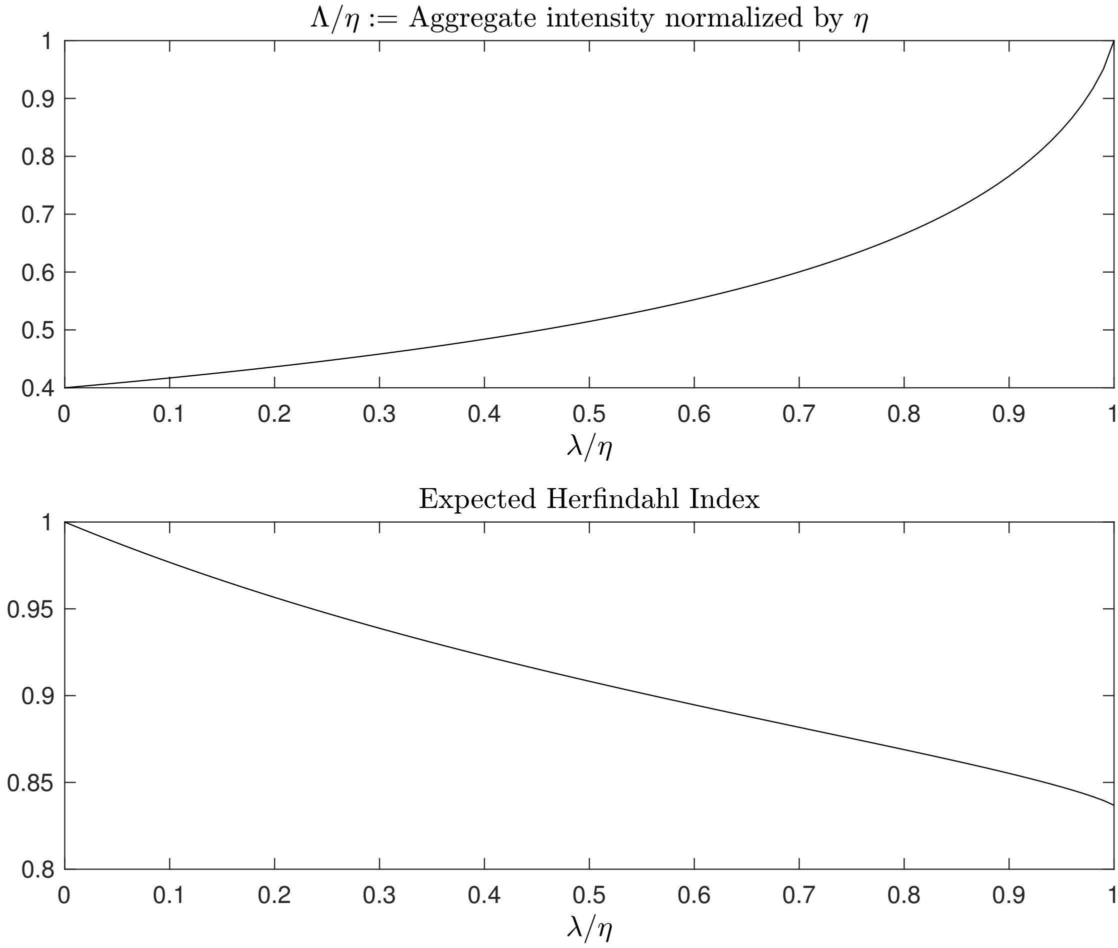
$$H(N) := a_1^2 \sum_{i=1}^{N} \left(\frac{1}{4}\right)^{i-1}.$$

Combining the probability distribution function discussed here with the cross-sectional distribution of markets I can find a new distribution reflecting, *ex ante*, the measure of markets with exactly N firms competing in it. Then, associating this with a concentration index I obtain the expected concentration across the whole market space, which is shown in Figure 3.7. Finally, I compute two additional important distributions in closed-form.

Claim 3.4 **($\Phi_M(x)$ and $\Phi_{q^*}(x)$)** *The cross-sectional distribution of markups $\Phi_t^M(x)$ is defined as $\mathbb{P}(a - bX \leq x) = \mathbb{P}\left(X \geq \frac{a-x}{b}\right) = 1 - \Phi_t\left(\frac{a-x}{b}\right)$. The cross-sectional*

Figure 3.7: The model's implied aggregate intensity and Herfindahl index

The figure shows different statistics from the model. The first panel plots the aggregate intensity, which is a strictly monotonic increasing function of λ in the range $(0, \eta)$. The lower panel shows the expected Herfindahl index, which is a measure of the competition in a market. Here I compute this quantity for all markets and weight these indexes by their respective probabilities. As might also be expected this quantity is decreasing with λ.



distribution of individual supply per market $\Phi_t^{q^*}(x)$ *is defined as* $\mathbb{P}\left(\frac{a-bX}{2b} \leq x\right) = \mathbb{P}\left(X \geq \frac{a}{b} - 2x\right) = 1 - \Phi_t\left(\frac{a}{b} - 2x\right)$.

The function $\Phi_t^M(x)$ characterizes the measure of markets where the markup m is bounded bellow x. Written differently, there is a measure $\Phi_t^M(x)$ of markets with markups $m \leq x$. It is worth noting that markups are the same for all the firms that are competing in the same market. This is the case because marginal cost is constant across all markets and for all firms. The function $\Phi_t^{q^*}(x)$ characterizes the distribution of the reaction function of firms; when a firm finds a market its aggregate quantity is distributed according to $\Phi_t(x)$. Since the optimal response of the entrant is to produce $q^*(x)$, the distribution of the reaction before knowing the exact aggregate market supply x is drawn from $\Phi_t^{q^*}(x)$.

3.4.2 Following Firms

The framework discussed in this paper enables me to model the firm-size process explicitly; Section 3.3.4 treats this problematic in greater detail. The fact that the size of firms follows a birth-death process renders possible the calculations of the firm-size distribution in closed-form. I recall that the meeting rate of a firm competing in n markets equals $\lambda \times n$. Alternatively, n can be interpreted as the number of employees working in that firm.

Claim 3.5 (Firm-size) *The probability that a firm has i employees is proportional to,*

$$\left(\frac{\lambda}{\eta}\right)^{i-1}\left(\frac{1}{i}\right),\ i > 0.$$

Importantly, the model predicts that above a critical size, which depends on how close η and λ are to each other, the behaviour of the firm-size distribution changes dramatically; the firm-size distribution is a mixture of a power law with a unit exponent (a Zipf law) when i is small and an exponential distribution when i is large. The closer λ is from η the longer the distribution behaves like a power law. Nevertheless, for a stationary distribution to exist, λ must be smaller than η. Hence, the distribution will always behave like an exponential distribution when i is large; a larger λ will simply delay the change of regime. Figure 3.8 consists of two plots; the upper panel shows the firm-size distribution for different levels of the search intensity λ. Since η is fixed, the higher λ the longer it takes for the exponential part to come into play. The second panel illustrates the firm-size distribution when $\eta \approx \lambda$; here the power law clearly dominates the exponential part. Hence, as long as λ is maintained close to η the endogenous firm-size distribution is consistent with what is seen in the data. The question then arises "When do large firms appear larger than the others?". Is it when λ is small or when $\lambda \approx \eta$? It turns out that the higher λ the larger large firms compared to average-sized firms. This relation is illustrated in Figure 3.9, which is computed by normalizing the average size of large firms (above x^{th} percentile) by the average size. For $0 < \lambda < \eta$, the probability of n being larger than x is, $\mathbb{P}(n \geq x) = \frac{\lambda_0}{\lambda S} \sum_{i=x}^{\infty} \left(\frac{\lambda}{\eta}\right)^i \frac{1}{i}$. Therefore I can find x by solving the fixed-point equation $\mathbb{P}(n \geq x) = 1 - x^{th}$. Then the average firm size of the firms that are above the x^{th} percentile equals $\frac{\lambda_0}{\lambda S} \sum_{i=x}^{\infty} \left(\frac{\lambda}{\eta}\right)^i \frac{1+1}{i}$. After some algebra the average firm size of the top firms becomes, $\left(\frac{\lambda}{\eta}\right)^x \mathcal{L}(\lambda/\eta, 1, x)$.

Claim 3.6 *The ratio of the average size of large firms normalized by the average size of all firms satisfies,*

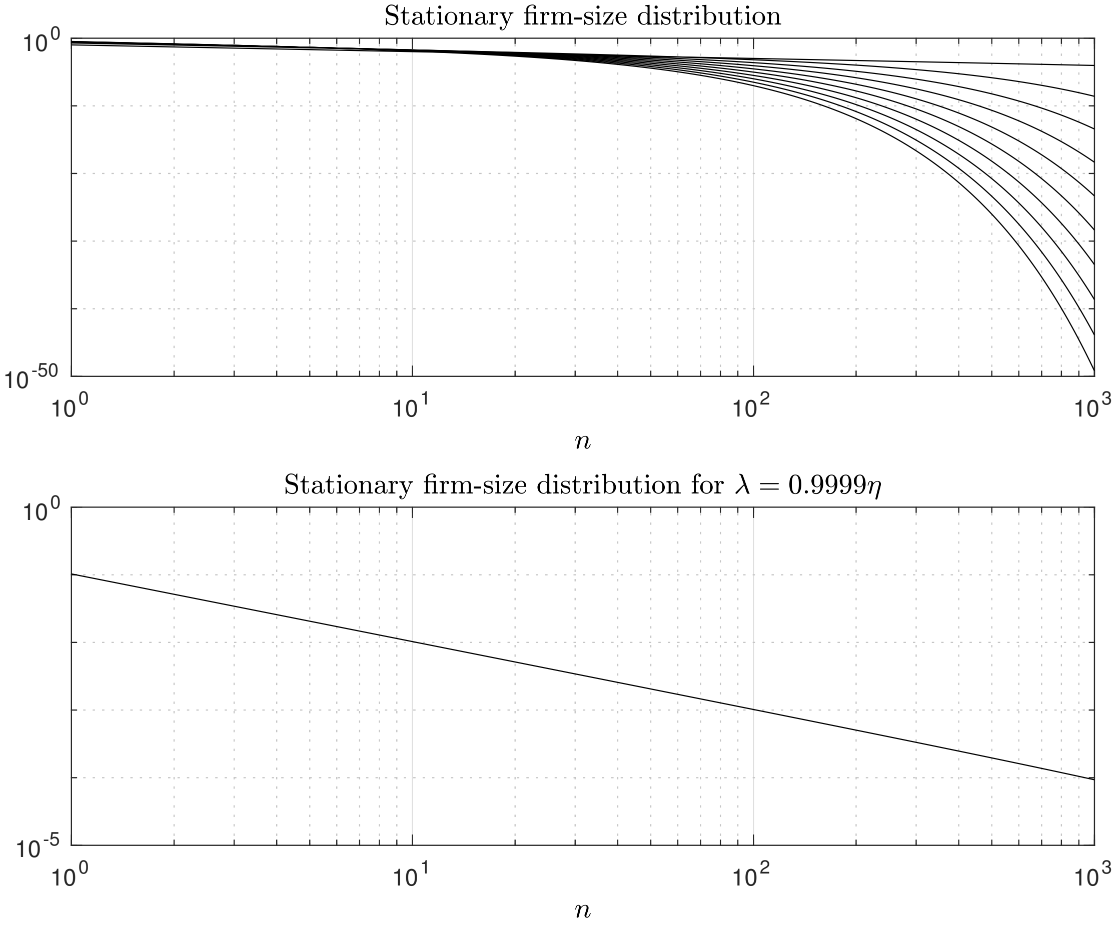
$$\frac{\eta}{\eta - \lambda} \mathcal{L}^{-1}(\lambda/\eta, 1, x).$$

where $\mathcal{L}(z, s, a) := \sum_{k=1}^{\infty} \frac{z^k}{(a+k)^s}$ *is the so-called Lerch Transcendent function and* x *satisfies* $\mathbb{P}(n \geq x) = 1 - x^{th}$.

When λ is zero there are only two types of firm; either the entrepreneur is alone, or the firm has one employee, which implies that the largest firms have at most

Figure 3.8: The model's implied stationary firm-size distribution

This figure shows the stationary firm-size distribution for different levels of λ. As discussed in this paper, the function behaves proportionally to a power law for low i and proportionally to an exponential distribution when i is large. The second plot treats the case when $\lambda = 0.9999\eta$.



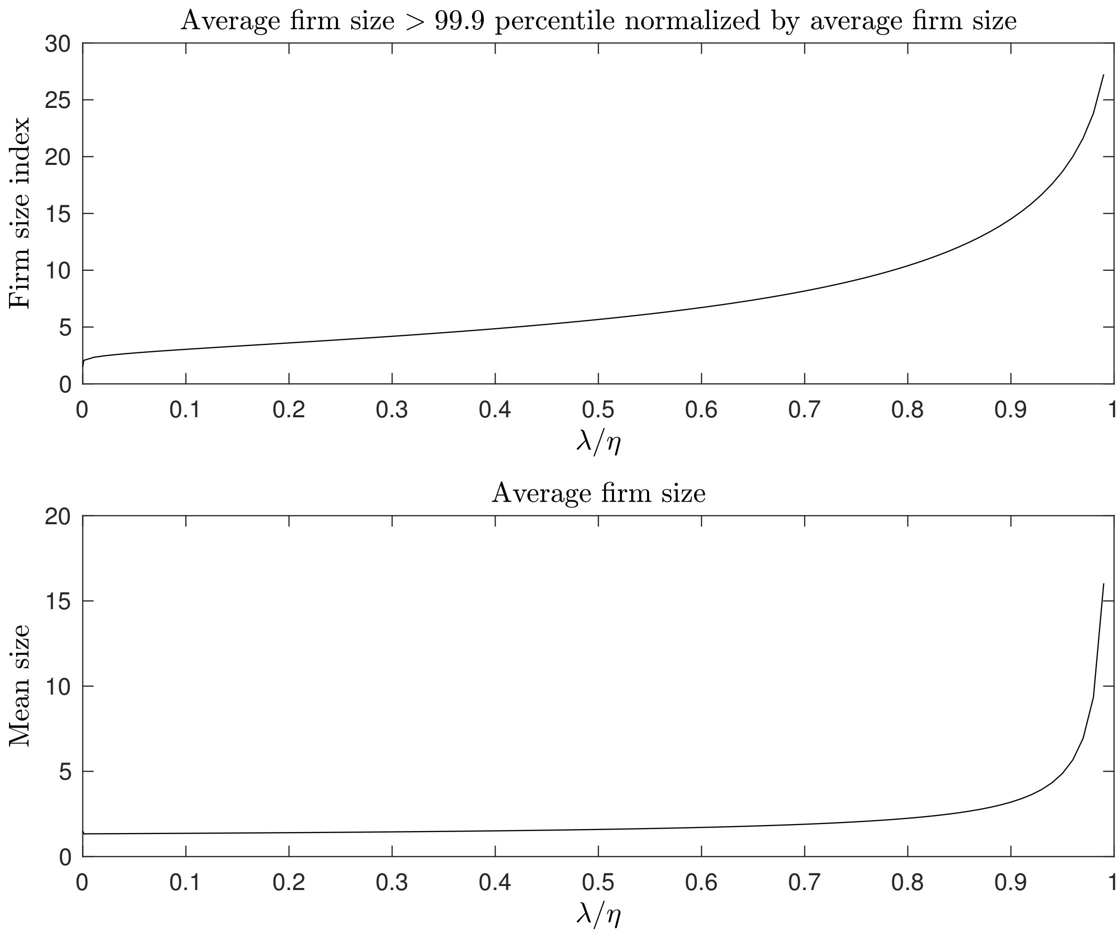
two workers. Taking the percentile high enough the average firm size above that threshold is two while the average firm size equals $\frac{\eta+2\lambda_0}{\eta+\lambda_0}$.

3.4.3 The Model With Fixed-Costs

In this paper I design a model where firms do not face fixed costs upon entering a market. This assumption makes the model highly tractable since the strategy of

Figure 3.9: The model's implied ratio of the largest firms to the average-sized firm

This figure shows the model's implied ratio of the largest firms to the average-sized firm for different levels of λ. As discussed in this paper, this function behaves proportionally to a power law for low i and proportionally to an exponential distribution when i is large. The second plot treats the case when $\lambda = 0.9999\eta$.



entrants is relatively trivial, as exhibited in Proposition 3.1. Thus, a firm invests in these projects that only have positive net present value. Though this assumption is fairly realistic in the Tech area, it is less so in other fields. Accordingly, I will discuss what occurs when this assumption is relaxed.

3.4.3.1 Generalities

When fixed costs are introduced into the model incumbents have to take into consideration the impact of their strategies on the entry decision of potential competitors. In Etro (2008) the entry decision is endogenous and the behaviour of the leader is very different from the traditional behaviour. Instead of being concerned with the reaction of competitors to its supply the leader focuses on the entry decision and becomes significantly more aggressive. This paper shows that when goods are homogeneous and marginal costs are constant the leader finds it optimal to increase output until no rivals want to enter. Here, because entrants show up only with some probabilities incumbents will behave somewhere in between these two scenarios. Moreover, I also need some conditions on the level of the fixed costs c which are discussed in Assumption 3.1.

Assumption 3.1 (Nontrivial solution) $c \leq \left(\frac{a}{4}\right)^2 \frac{1}{b}$.

In order to guarantee the existence of a nontrivial solution (when $c > 0$) I need Assumption 3.1 to hold. To see why assume it does not; then, $q^{**} := \frac{a}{b} - 2\sqrt{bc} > q^* := \frac{a}{2b}$. Thus, an incumbent that chooses the monopoly quantity is guaranteed never to face entrants. Hence, given Assumption 3.1 holds, let us further assume for illustrative purposes that at most three firms can enter in market i and the value associated with searching does not matter (the case with V_S is treated separately later). Additionally, let us call the first firm to enter A, the second B and the third C. Firm C will always act as discussed earlier and produce $q^*(Q) = \frac{a-bQ}{2b}$, $Q = q_A + q_B$. However, if $Q \geq Q^* := \frac{a}{b} - 2\sqrt{\frac{c}{b}}$, then $q^*(Q) \leq 0$. Hence, firm C chooses $q^* = 0$. Building on this B has the option to preclude the entrance of C by choosing $q^{**} := \frac{a-q_A}{b} - 2\sqrt{\frac{c}{b}}$, where q_A is the quantity produced by A. In this case the profit of firm B is $\pi_B^{**} = 2(a - bq_A)\sqrt{\frac{c}{b}} - 5c$. If B chooses not to prevent C from entering B chooses $q^* := \frac{a-bq_A}{2b}$ and makes an expected profit of $\pi_B^* := \frac{(a-bq_A)^2}{4b}\left(1-\frac{1}{2}\rho\right) - c$. Therefore, B will choose q^* if $\pi_B^* \geq \pi_B^{**}$ and q^{**} otherwise. Let us assume now that B do not preclude the entrance of C; then A can allow other firms to enter by choosing $q^* = \frac{a}{2b}$ or preclude entrants from joining with $q^{**} = \frac{a}{b} - 2\sqrt{\frac{c}{b}}$. However, the actions of B must be consistent (i.e. A chooses its quantity by presuming B will not preclude C). Lastly, in the case that B does not let C enter, A can preclude B from entering as well by choosing q^{**}. If A chooses not to do so its quantity must be consistent with the behaviour of B. Thus, A chooses $q^{***} := \frac{a}{2b} + \frac{\rho}{1-\rho}\sqrt{\frac{c}{b}} > q^*$[4]. What matters here is that, conditional on the behaviour of its followers, A will produce at least the monopolist quantity q^* (which corresponds almost one-to-one to the *no fixed-costs* model). If A precludes competitors from entering into the market consumers will immediately receive Q^*, which makes them strictly better off. Alternatively, the leader can also presume that one of its followers could enter and itself saturate the market. Here again consumers are better off since $q^{***} > q^*$.

When leaders always preclude the entrance of their followers the dynamics of the cross-sectional probability mass function (PMF) of markets satisfies[5],

$$\begin{aligned}
\dot{p}_t(0) &= \eta - (\eta + \Lambda)p_t(0), \\
\dot{p}_t(Q^*) &= \Lambda p_t(0) - \eta p_t(Q^*).
\end{aligned}$$

Hence, the stationary PMF is given by,

$$\begin{aligned}
p(0) &= \frac{\eta}{\eta+\Lambda}, \\
p(Q^*) &= \frac{\Lambda}{\eta+\Lambda}.
\end{aligned}$$

[4] It is worth noting that in a three firm model as long as $b \leq 1$ the leader will always preclude the entrance of its followers knowing that the 2^{nd} firm will exclude the 3^{rd}.

[5] Assuming no heterogeneity across markets.

Therefore, the higher the search intensity the higher the consumer surplus. Building on Sections 3.3.4 and 3.3.5, a firm finding a market has a probability $p(Q^*)$ that it is already saturated. Thus, the search intensity at which a firm finds a **non-saturated** market is $\lambda_{t,j} \times p_0$. Substituting that rate into the infinitesimal generator of the birth-death process the aggregate search intensity Λ satisfies,

$$\bar{\Lambda}^{\infty} = \frac{A\eta\lambda_0 \times p_0}{\lambda_0 \times p_0 + S(\eta - \lambda \times p_0)}, \; S = 1 - \frac{\lambda_0}{\lambda} \log\left(1 - \frac{\lambda \times p_0}{\eta}\right).$$

3.4.3.2 Impact of Searching on the Equilibrium Strategy

In this section I discuss a model where firms only produce in non-saturated markets; if joining a market is a positive NPV project, the firm enters. If the entry costs are higher than the benefits associated with joining that market the firm abstains from starting production. Alternatively, firms could take into consideration that joining a market that would not appear profitable at first can be optimal; by growing, the value associated with searching increases. Hence, the decision to enter is no longer static; it depends on the number of markets the firm is competing in. If $V_S(N) \geq V_S(N+1) - c$ the firm does not enter. Nevertheless, if the gain associated with entering that market covers the entry cost, the firm will join. When a firm uses this strategy small firms are more affected by fixed costs than large firms. Since $V_S(N)$ behaves approximately as x_1^N, any N larger than $\log\left(\frac{c}{x_1 - 1}\right) / \log(x_1)$ would make a firm of that size always willing to enter, no matter the loss that will materialize. On the other hand an entrepreneur would not adopt this strategy.

In this alternative economy the number of active firms is even smaller. This problematic is reinforced when large firms can influence the entrance costs; by setting it higher their decision to enter is not affected, while small firms are even more reluctant to enter. Hence, large firms tend to over invest while small firms are inactive. In my framework the economy is not impacted by such deadweight costs. Moreover, since the measure of active firms is larger consumers end up better off. This discussion clarifies how bad anti-competitive practices are and in particular dumping. It is worth noting that when this behaviour is banned my framework is more realistic and, because the law prohibits dumping, it is consistent to assume that such strategies cannot be part of any equilibrium outcome.

3.5 Concluding Remarks

In this paper I design a dynamic model where firms are endowed with a search technology which they use for creating new products. Firms compete sequentially in segmented markets, taking into account the strategies of potential entrants to determine their respective actions. Assuming a linear price-inverse demand function, homogenous goods and constant marginal costs make the model easily tractable and the stationary cross-sectional distribution of quantities in each market solvable in closed-form. With these assumptions the firm-size process belongs to the class of birth-death processes, which make the aggregate matching rate and the firm-size distribution solvable in closed-form as well. Building on this framework I can compute many objects of potential interest such as some measures of concentration within and across markets, the cross-sectional distribution of markups or the continuation value of firms. All these implications are discussed in great detail and can be tested empirically.

My paper tackles the problematic of large companies and, specifically, the consequences that they have on consumer welfare. My model suggests that the rise of large entities is part of a natural mechanism; the firm-size distribution will always be a mixture of a power law and a distribution with exponential decay. My results clearly advocate that the closest λ is to η, the largest is the aggregate intensity. Hence, the fatter the tails the greater the consumer welfare. At first, this result might seem surprising. Indeed, companies grow by cannibalizing others; if the measure of firms is small one can expect corporations to act as a monopolist. It turns out that under sequential competition, as modeled here, each entity behaves as a monopolist facing a residual demand curve. Therefore, they all operate like monopolists and hope no one else will join. The only credible way for incumbents to preclude the entrance of opponents is to saturate the market, which is even better for consumers. Thus, though the measure of firms is smaller, consumers are better off when the matching rate increases, because the average number of entities in each market is larger. With fixed costs, the reasoning still applies though the intuition is slightly different; consumers are better off because the measure of saturated markets increases with the aggregate search intensity.

In general, one sees entry costs as an anti-competitive toolkit that is used in order to keep opponents outside of their respective markets. While this statement is true my model shows that it is not necessarily at the expense of consumers. Indeed, with sequential competition the only way for the incumbent to preclude the entrance of rivals is to adopt a strategy that makes the reservation value associated with joining that market lower than their reservation value. Here, the incumbent has to choose a quantity that would immediately saturate the market (no one would have entered beyond that point, no matter the structure of the market).

Finally, a question raised initially was related to whether big technological companies are impairing consumer welfare. While these companies spend large amounts of capital for acquiring start-ups it is fair to see this strategy as a way to preclude the entrance of potential competitors. Though in the short-run this behaviour is indeed problematic my model suggests that the firm-size distribution and specifically the rise of large corporations is a natural mechanism. Down scaling these entities will result in other rivals taking their dominant place. More importantly, if Amazon, Facebook and Co. acquire start-ups with the aim of maintaining their relative position they can only expand at a suboptimal growth rate and will eventually be replaced by organizations that have not engaged in these types of misconduct.

3.A Proof

Proof of Proposition 3.1. Let $N \geq 1$ be the number of firms competing in i. Moreover, $Q_{i,j}$ is the quantity produced in market i before firm j enters. Assume firm j enters and knows no other firm will join that market afterwards. Then, the profit function of firm j in market i is,

$$\pi_j(q_{i,j}, Q_{i,j}) = q_{i,j}(a - b[q_{i,j} + Q_{i,j}]).$$

Firm j chooses $q_{i,j} := \operatorname{argmax} \pi_j(q_{i,j}, Q_{i,j})$. Rearranging the first-order condition (FOC), the optimum is $q^*_{i,j}(Q_{i,j}) = \frac{a-bQ_{i,j}}{2b}$. Moreover, because π_j is a second order polynomial uniqueness is guaranteed and it is sufficient to show that $\frac{\partial^2 \pi_j(q_{i,j}, Q_{i,j})}{\partial q^2_{i,j}} < 0$ to prove that $q^*_{i,j}$ is a maximum, which is true since it equals $-b < 0$. Now, take firm $j-1$. This is the very last firm which joined market i before firm j. Firm $j-1$ chooses its strategy without knowing if firm j will find that market or not. The profit function of firm $j-1$ in market i satisfies,

$$\pi_{j-1}(q_{i,j-1}, Q_{i,j-1}) = q_{i,j-1}(a - b[q_{i,j-1} + Q_{i,j-1} + \rho q_{i,j}]), \; Q_{i,j} = Q_{i,j-1} + q_{i,j-1},$$

where ρ is defined as the probability that firm j enters (in this model it equals $\Lambda/(\eta + \Lambda)$). Substituting $q^*_{i,j}(Q_{i,j})$, the profit function can be rewritten as follows,

$$\pi_{j-1}(q_{i,j-1}, Q_{i,j-1}) = q_{i,j-1}\left(1 - \frac{\rho b}{2}\right)(a - b[q_{i,j-1} + Q_{i,j-1}]),$$

Accordingly, the optimal strategy of firm $j-1$, taking into account firm j's strategy, is to produce,

$$q^*_{i,j-1}(Q_{i,j-1}) = \frac{a - bQ_{i,j-1}}{2b}.$$

Finally, take firm n which entered just before firm $n+1$. Firm n chooses its optimal quantity without knowing if firms $\geq n$ would eventually enter or not. The profit function of firm n in market i satisfies,

$$\pi_n(q_{i,n}, Q_{i,n}) = q_{i,n}(a - b[q_{i,n} + Q_{i,n} + q_{i,n+1}]).$$

Substituting $q^*_{i,n+1}(Q_{i,n})$ the profit function can be rewritten as follows,

$$\pi_n(q_{i,n}, Q_{i,n}) = q_{i,n}\left(1 - \frac{\rho b}{2}\right)(a - b[q_{i,n} + Q_{i,n}]).$$

Accordingly, the optimal strategy for firm n, taking into account the strategy of all following firms is to produce,

$$q^*_{i,n}(Q_{i,n}) = \frac{a - bQ_{i,n}}{2b}.$$

This confirms what Anderson and Engers (1992) show: when the price-inverse demand function is linear each firm behaves as a monopolist facing a residual demand curve inherited from its predecessors. However, one must also consider a strategy in which incumbents choose their supply in order to prevent the entrance of new firms. Because I assumed that there are no fixed costs, the only quantity that would make a new entrant not willing to enter is when $q = a/b$, which makes the incumbent's profit equal to zero. Thus, this strategy will never be adopted in equilibrium since it makes them worse off. □

Proof of Lemma 3.1. Let $x = \psi(y)$ and $f(x) = \psi(y + 1)$. Substituting x and $f(x)$ into $h(f(x)) = h(x) + 1$, Abel shows that this equation can be rewritten as, $psi(y + 1) = f(\psi(y))$. Using $f(x) = a + bx$ and substituting successively $y + 1$,

$y+2$, etc., the previous equation becomes,

$$\psi(y+n) = a\sum_{i=0}^{n-1} b^i + b^n\psi(y).$$

Letting $y=0$ and $\psi(0)=c$, this equation becomes $\psi(y) = a\sum_{i=0}^{y-1} b^i + b^y c$, which must be equal to $a+bx$. Taking the reciprocal of that function leads to,

$$h(x) = \frac{\log\left(\frac{a+x(b-1)}{a+c(b-1)}\right)}{\log(b)}.$$

□

Proof of Lemma 3.2. Taking the base s logarithm of $g(f(x)) = sg(x)$ on both sides, this equation becomes $h(f(x)) = h(x)+1$, which is the Abel equation discussed in Lemma 3.1. Thus, Lemma 3.1 implies $g(x) = s^{h(x)}$. □

Proof of Proposition 3.2. The homogenous part of equation 3.3 is a Schröder's Equation. Using Lemma 3.2, the solution of that equation is,

$$\Phi_G(q) = C\times\left(\frac{\eta+\Lambda}{\Lambda}\right)^{\tilde{\alpha}(q)}, \; \tilde{\alpha}(x) = \frac{\log(x-a/b)}{\log(2)} - 1.$$

Similarly, the inhomogeneous part of equation 3.3 is solved by the particular solution,

$$\Phi_P(q) = \frac{b}{a}\frac{\eta}{\eta-\Lambda}q - \frac{\Lambda}{\eta-\Lambda}.$$

The uniqueness of the Abel Equation guarantees the uniqueness of equation (3.3). The treatment of the boundary condition is slightly more complex since there is one condition for each quantity q_0 (the initial quantity in a particular market).

Thus, the path of the aggregate quantity in that market can only take values on the set $\{Q(q_0)\}_{i\in\{0,1,\cdots,i^{max}\}}$, where $Q(q_0) = \frac{(2^i-1)a+bq_0}{2^i b}$ and i^{max} are to be defined. In other words, when computing the constant one must bear in mind that there is exactly one path for each $q_0 \in [0, \frac{a}{2b}[$. For instance, take $Q^{max} \equiv \frac{b}{a} = 10$ and $Q_F = 8$. I recall that the optimal strategy is to produce $q^*(Q) = \frac{a-bQ}{2b}$. Therefore, knowing that $Q + q^*(Q)$ must equal Q', we can find $Q := 2 \times Q' - a/b$. Given i, the number of firms in the market, Q is simply,

$$Q(i) = 2^i \times Q' - (2^i - 1)\frac{a}{b}. \tag{3.11}$$

Using this one knows there are at most two firms in my example; the new firm produces $2 = q^*(Q)$, while the first firm produces $4 = q^*(Q(2))$, which leads to $q_0 = 2 > 0$. Hence, using the appropriate boundary condition the constant C must satisfy the following equation,

$$\frac{\eta}{\eta+\Lambda}\frac{b}{a}2 = C \times \left(\frac{\eta+\Lambda}{\Lambda}\right)^{\tilde{\alpha}(2)} + \frac{b}{a}\frac{\eta}{\eta-\Lambda}2 - \frac{\Lambda}{\eta-\Lambda},$$

which is the same for any market that can reduce to $q_0 = 2$ (i.e. $Q \in \{2, 6, 8\}$). I show in the following lines that it is possible to automate this procedure. Let us choose $i \geq 0$ such that i satisfies the following inequality,

$$\frac{2^i - 1}{2^i}\frac{a}{b} \leq Q \leq \frac{2^{i+1} - 1}{2^{i+1}}\frac{a}{b}.$$

Then, the maximal number of firms competing in a market with total quantity Q is i. In general, finding i with respect to Q simply requires to take the floor function of the following functional equation,

$$h(f(Q)) = h(Q) + 1, \ f(Q) = \frac{a + bQ}{2b}. \tag{3.12}$$

The unique solution of this Abel equation is $h(x) = -\frac{\log(1-xb/a)}{\log(2)} = -\alpha(x)$. Thus, equation (3.11) can be rewritten as follows $g(x) = 2^{\lfloor -\alpha(x) \rfloor} x - (2^{\lfloor -\alpha(x) \rfloor} - 1)\frac{a}{b}$. Finally, $\tilde{\alpha}(x)$ returns an imaginary part. It can be rearranged as follows,

$$\tilde{\alpha}(x) = \underbrace{\frac{\log\left(-\frac{a}{b}\right)}{\log(2)}}_{\notin \mathbb{R}} + \underbrace{\frac{\log\left(1 - x\frac{b}{a}\right)}{\log(2)} - 1}_{\equiv \alpha(x)},$$

which still returns an imaginary part. Nevertheless, once plugged into the cross-sectional distribution the complex part cancels out. □

Proof of Proposition 3.3. First of all I recall that the cross-sectional distribution is a piecewise function. For $x \in \left[0, \frac{a}{2b}\right]$, $\Phi_t(x)$ satisfies,

$$\dot{\Phi}_t(x) = \eta F(x) - (\eta + \Lambda(t))\Phi_t(x). \tag{3.13}$$

This is a first-order linear differential equation. I conjecture that Λ is a function of time only and later verify that this is true. Thus, the stability of $\Phi_t(x)$ can be treated under the assumption that $\Lambda(t) = \Lambda$ (I could alternatively write $\Lambda := \int_0^t \Lambda(u)du$). Thus, the solution of (3.13) is given by,

$$\Phi_t(x) = \frac{\eta}{\eta + \Lambda} F(x) + c(x) \exp[-(\eta + \Lambda)t],$$

where $c(x)$ is a function that depends only on x. As $t \to \infty$, $\Phi_t(x) \to \Phi(x)$, if $\eta + \Lambda > 0$, which is true by construction. Assuming now that $\frac{3a}{4b} \geq x \geq \frac{a}{2b}$, the cross-sectional distribution satisfies,

$$\dot{\Phi}_t(x) = \eta F(x) - (\eta + \Lambda(t))\Phi_t(x) + \Lambda(t)\Phi(2x - a/b). \tag{3.14}$$

Since $2x - a/b \leq \frac{a}{2b}$, $\Phi_t(2x - a/b) = \frac{\eta}{\eta+\Lambda} F(2x - a/b) + c(2x - a/b) \exp[-(\eta + \Lambda)t]$.

Therefore, using the assumption made above equation (3.14) can be rewritten as follows,

$$\dot{\Phi}_t(x) = \eta F(x) - (\eta+\Lambda)\Phi_t(x) + \Lambda \left[\frac{\eta}{\eta+\Lambda} F(2x - a/b) + c(2x - a/b)\exp[-(\eta+\Lambda)t] \right].$$

This is a first-order linear differential equation, where the non-homogenous part is a function of x only and an exponential of t, with the same exponent as in the homogenous part. Thus as $t \to \infty$, $\Phi_t(x) \to \Phi(x)$, if $\eta + \Lambda > 0$. Hence the function $\Phi_t(x)$, for $x \in \left[\frac{a}{2b}, \frac{3a}{4b}\right]$ has a unique stationary distribution, which is globally attractive. By applying the same argument iteratively, it follows that $\Phi_t(x)$, $x \in (0, a/b)$ can be rewritten as a first-order linear differential equation, where the condition regarding stability is reduced to $\eta + \Lambda > 0$. Therefore, I conclude that the cross-sectional distribution of markets has a unique fixed point which is globally stable. □

Proof of Proposition 3.4. The firm-size process is a so-called birth-death process (BDP). Using standard results coming from this area, a BDP with infinitesimal generator $\mathbf{Q}$ defined as,

$$\mathbf{Q} := \begin{pmatrix} -\lambda_0 & \lambda_0 & & & \\ \eta_1 & -(\eta_1+\lambda_1) & \lambda_1 & & \\ & \eta_2 & -(\eta_2+\lambda_2) & \lambda_2 & \\ & \ddots & \ddots & \ddots & \\ & & \eta_N & -(\eta_N+\lambda_N) & \lambda_N \end{pmatrix},$$

has a unique stationary distribution if and only if the following condition is satisfied;

$$S = 1 + \sum_{i=1}^{\infty} \frac{\prod_{j=0}^{i-1} \lambda_j}{\prod_{j=1}^{i} \eta_1} < \infty.$$

Using my generator this condition can be rewritten as,

$$S = 1 + \frac{\lambda_0}{\eta} \sum_{i=1}^{\infty} \left(\frac{\lambda}{\eta}\right)^{i-1} \frac{1}{i} = 1 - \frac{\lambda_0}{\lambda} \log\left(1 - \frac{\lambda}{\eta}\right) < \infty.$$

When this condition is satisfied it is again standard in the BDP literature that the cross-sectional distribution can be rewritten as the one presented in Proposition 3.4. □

Proof of Proposition 3.5. The homogenous part of equation 3.7 is a Schröder's Equation. Using Lemma 3.2 the solution of that equation is,

$$\Phi_G(Q) = C \times \left(\frac{r + \eta + \Lambda}{\Lambda}\right)^{-\alpha(Q)}, \ \alpha(x) = \frac{\log\left(1 - x\frac{b}{a}\right)}{\log(2)}.$$

Similarly, the inhomogeneous part of equation 3.7 is solved by the particular solution,

$$\Phi_P(Q, q) = \frac{a\eta}{r + \eta + \Lambda/2} \left[1 - \frac{b}{a} Q\right] \times q.$$

Then the treatment of the boundary condition implies that $C = 0$ for all paths and thus $\Phi(Q, q) = \Phi_P(Q, q)$. □

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