



Research article

Approximate cloaking for the heat equation via transformation optics

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Abstract: In this paper, we establish approximate cloaking for the heat equation via transformation optics. We show that the degree of visibility is of the order ε in three dimensions and $|\ln \varepsilon|^{-1}$ in two dimensions, where ε is the regularization parameter. To this end, we first transform the problem in time domain into a family of problems in frequency domain by taking the Fourier transform with respect to time, and then derive appropriate estimates in the frequency domain.

Keywords: heat equation; approximate cloaking; frequency analysis

1. Introduction and statement of the results

Cloaking using transformation optics (changes of variables) was introduced by Pendry, Schurig, and Smith [30] for the Maxwell system and by Leonhardt [16] in the geometric optics setting. These authors used a singular change of variables, which blows up a point into a cloaked region. The same transformation had been used to establish (singular) non-uniqueness in Calderon's problem in [10]. To avoid using the singular structure, various regularized schemes have been proposed. One of them was suggested by Kohn, Shen, Vogelius, and Weinstein [11], where instead of a point, a small ball of radius ε is blown up to the cloaked region. Approximate cloaking for acoustic waves has been studied in the quasistatic regime [11, 26], the time harmonic regime [12, 19, 27, 20], and the time regime [28, 29], and approximate cloaking for electromagnetic waves has been studied in the time harmonic regime [4, 14, 24], see also the references therein. Finite energy solutions for the singular scheme have been studied extensively [9, 32, 33]. There are also other ways to achieve cloaking effects, such as the use of plasmonic coating [2], active exterior sources [31], complementary media [13, 22], or via localized resonance [23] (see also [17, 21]).

The goal of this paper is to investigate approximate cloaking for the the heat equation using transformation optics. Thermal cloaking via transformation optics was initiated by Guenneau, Amra, and

Venante [8]. Craster, Guenneau, Huttidurga, and Pavliotis [6] investigate the approximate cloaking for the heat equation using the approximate scheme in the spirit of [11]. They show that for the time large enough, the largeness depends on ε , the degree of visibility is of the order ε^d ($d = 2, 3$) for sources that are independent of time. Their analysis is first based on the fact that as time goes to infinity, the solutions converge to the stationary states and then uses known results on approximate cloaking in the quasistatic regime [11, 26].

In this paper, we show that approximate cloaking is achieved at any positive time and established the degree of invisibility of order ε in three dimensions and $|\ln \varepsilon|^{-1}$ in two dimensions. Our results hold for a general source that depends on both time and space variables, and our estimates depend only on the range of the materials inside the cloaked region. The degree of visibility obtained herein is optimal due to the fact that a finite time interval is considered (compare with [6]). The analysis in this paper is of frequency type via Fourier transform with respect to time. This approach is robust and can be used in different context. A technical issue is on the blow up of the fundamental solution of the Helmholtz type equations in two dimensions in the low frequency regime. We emphasize that even though our setting is in a bounded domain, we employ Fourier transform in time instead of eigenmodes decomposition. This has the advantage that one can put the non-perturbed system and the cloaking system in the same context.

We next describe the problem in more detail and state the main result. Our starting point is the regularization scheme [11] in which a transformation blows up a small ball B_ε ($0 < \varepsilon < 1/2$) instead of a point into the cloaked region B_1 in \mathbb{R}^d ($d = 2, 3$). Here and in what follows, for $r > 0$, B_r denotes the ball centered at the origin and of radius r in \mathbb{R}^d . Our assumption on the geometry of the cloaked region is mainly to simplify the notations. Concerning the transformation, we consider the map $F_\varepsilon : \mathbb{R}^d \rightarrow \mathbb{R}^d$ defined by

$$F_\varepsilon(x) = \begin{cases} x & \text{in } \mathbb{R}^d \setminus B_2, \\ \left(\frac{2-2\varepsilon}{2-\varepsilon} + \frac{|x|}{2-\varepsilon} \right) \frac{x}{|x|} & \text{in } B_2 \setminus B_\varepsilon, \\ \frac{x}{\varepsilon} & \text{in } B_\varepsilon. \end{cases} \quad (1.1)$$

In what follows, we use the standard notations

$$F_*A(y) = \frac{\nabla F(x)A(x)\nabla F^T(x)}{|\det \nabla F(x)|}, \quad F_*\rho(y) = \frac{\rho(x)}{|\det \nabla F(x)|}, \quad x = F^{-1}(y), \quad (1.2)$$

for the “pushforward” of a symmetric, matrix-valued function A , and a scalar function ρ , by the diffeomorphism F , and I denotes the identity matrix. The cloaking device in the region $B_2 \setminus B_1$ constructed from the transformation technique is given by

$$(F_{\varepsilon*}I, F_{\varepsilon*}1) \text{ in } B_2 \setminus B_1, \quad (1.3)$$

a pair of a matrix-valued function and a function that characterize the material properties in $B_2 \setminus B_1$. Physically, this is the pair of the thermal diffusivity and the mass density of the material.

Let Ω with $B_2 \Subset \Omega \subset \mathbb{R}^d$ ($d = 2, 3$)* be a bounded region for which the heat flow is considered. Suppose that the medium outside B_2 (the cloaking device and the cloaked region) is homogeneous so

*The notation $D \Subset \Omega$ means that the closure of D is a subset of Ω .

that it is characterized by the pair $(I, 1)$, and the cloaked region B_1 is characterized by a pair (a_O, ρ_O) where a_O is a matrix-valued function and ρ_O is a real function, both defined in B_1 . The medium in Ω is then given by

$$(A_c, \rho_c) = \begin{cases} (I, 1) & \text{in } \Omega \setminus B_2, \\ (F_{\varepsilon^*} I, F_{\varepsilon^*} 1) & \text{in } B_2 \setminus B_1, \\ (a_O, \rho_O) & \text{in } B_1. \end{cases} \quad (1.4)$$

In what follows, we make the usual assumption that a_O is symmetric and uniformly elliptic and ρ_O is a positive function bounded above and below by positive constants, i.e., for a.e. $x \in B_1$,

$$\Lambda^{-1}|\xi|^2 \leq \langle a_O(x)\xi, \xi \rangle \leq \Lambda|\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^d, \quad (1.5)$$

and

$$\Lambda^{-1} \leq \rho_O(x) \leq \Lambda, \quad (1.6)$$

for some $\Lambda \geq 1$. Given a function $f \in L^1((0, +\infty), L^2(\Omega))$ and an initial condition $u_0 \in L^2(\Omega)$, in the medium characterized by (A_c, ρ_c) , one obtains a unique weak solution $u_c \in L^2((0, \infty); H^1(\Omega)) \cap C([0, +\infty); L^2(\Omega))$ of the system

$$\begin{cases} \partial_t(\rho_c u_c) - \operatorname{div}(A_c \nabla u_c) = f & \text{in } (0, +\infty) \times \Omega, \\ u_c = 0 & \text{on } (0, +\infty) \times \partial\Omega, \\ u_c(t = 0, \cdot) = u_0 & \text{in } \Omega, \end{cases} \quad (1.7)$$

and in the homogeneous medium characterized by $(I, 1)$, one gets a unique weak solution $u \in L^2((0, \infty); H^1(\Omega)) \cap C([0, +\infty); L^2(\Omega))$ of the system

$$\begin{cases} \partial_t u - \Delta u = f & \text{in } (0, +\infty) \times \Omega, \\ u_c = 0 & \text{on } (0, +\infty) \times \partial\Omega, \\ u_c(t = 0, \cdot) = u_0 & \text{in } \Omega. \end{cases} \quad (1.8)$$

The approximate cloaking meaning of the scheme (1.4) is given in the following result:

Theorem 1.1. *Let $u_0 \in L^2(\Omega)$ and $f \in L^1((0, +\infty); L^2(\Omega))$ be such that $\operatorname{supp} u_0, \operatorname{supp} f(t, \cdot) \subset \Omega \setminus B_2$ for $t > 0$. Assume that u_c and u are the solution of (1.7) and (1.8) respectively. Then, for $0 < \varepsilon < 1/2$,*

$$\|u_c(t, \cdot) - u(t, \cdot)\|_{H^1(\Omega \setminus B_2)} \leq C e(\varepsilon, d) (\|f\|_{L^1((0, +\infty); L^2(\Omega))} + \|u_0\|_{L^2(\Omega)}),$$

for some positive constant C depending on Λ but independent of f, u_0 , and ε , where

$$e(\varepsilon, d) = \begin{cases} \varepsilon & \text{if } d = 3, \\ |\ln \varepsilon|^{-1} & \text{if } d = 2. \end{cases}$$

As a consequence of Theorem 1.1, $\lim_{\varepsilon \rightarrow 0} u_c(t, \cdot) = u(t, \cdot)$ in $(0, +\infty) \times (\Omega \setminus B_2)$ for all f with compact support outside $(0, +\infty) \times B_2$ and for all u_0 with compact support outside B_2 . One therefore cannot detect the difference between (A_c, ρ_c) and $(I, 1)$ as $\varepsilon \rightarrow 0$ by observation of u_c outside B_2 : cloaking is achieved for observers outside B_2 in the limit as $\varepsilon \rightarrow 0$.

We now briefly describe the idea of the proof. The starting point of the analysis is the invariance of the heat equations under a change of variables which we now state.

Lemma 1.1. *Let $d \geq 2$, $T > 0$, Ω be a bounded open subset of \mathbb{R}^d of class C^1 , and let A be an elliptic symmetric matrix-valued function, and ρ be a bounded, measurable function defined on Ω bounded above and below by positive constants. Let $F : \Omega \mapsto \Omega$ be bijective such that F and F^{-1} are Lipschitz, $\det \nabla F > c$ for a.e. $x \in \Omega$ for some $c > 0$, and $F(x) = x$ near $\partial\Omega$. Let $f \in L^1((0, T); L^2(\Omega))$ and $u_0 \in L^2(\Omega)$. Then $u \in L^2((0, T); H_0^1(\Omega)) \cap C([0, T]; L^2(\Omega))$ is the weak solution of*

$$\begin{cases} \partial_t(\rho u) - \operatorname{div}(A \nabla u) = f & \text{in } \Omega_T, \\ u = 0 & \text{on } (0, T) \times \partial\Omega, \\ u(0, \cdot) = u_0 & \text{in } \Omega, \end{cases} \quad (1.9)$$

if and only if $v(t, \cdot) := u(t, \cdot) \circ F^{-1} \in L^2((0, T); H_0^1(\Omega)) \cap C([0, T]; L^2(\Omega))$ is the weak solution of

$$\begin{cases} \partial_t(F_* \rho v) - \operatorname{div}(F_* A \nabla v) = F_* f & \text{in } \Omega_T, \\ u = 0 & \text{on } (0, T) \times \partial\Omega, \\ v(0, \cdot) = u_0 \circ F^{-1} & \text{in } \Omega. \end{cases} \quad (1.10)$$

Recall that F_* is defined in (1.2). In this paper, we use the following standard definition of weak solutions:

Definition 1.1. *Let $d \geq 2$ and $T > 0$. We say a function*

$$u \in L^2((0, T); H_0^1(\Omega)) \cap C([0, T]; L^2(\Omega))$$

is a weak solution to (1.9) if $u(0, \cdot) = u_0$ in Ω and u satisfies

$$\frac{d}{dt} \int_{\Omega} \rho u(t, \cdot) \varphi + \int_{\Omega} A \nabla u(t, \cdot) \nabla \varphi = \int_{\Omega} f(t, \cdot) \varphi \quad \text{in } (0, T), \quad (1.11)$$

in the distributional sense for all $\varphi \in H_0^1(\Omega)$.

The existence and uniqueness of weak solutions are standard, see, e.g., [1] (in fact, in [1], f is assumed in $L^2((0, T); L^2(\Omega))$, however, the conclusion holds also for $f \in L^1((0, T); L^2(\Omega))$ with a similar proof, see, e.g., [25]). The proof of Lemma 1.1 is similar to that of the Helmholtz equation, see, e.g., [12] (see also [6] for a parabolic version).

We now return to the idea of the proof of Theorem 1.1. Set

$$u_{\varepsilon}(t, \cdot) = u_{\varepsilon}(t, \cdot) \circ F_{\varepsilon}^{-1} \quad \text{for } t \in (0, +\infty).$$

Then u_{ε} is the unique solution of the system

$$\begin{cases} \partial_t(\rho_{\varepsilon} u_{\varepsilon}) - \operatorname{div}(A_{\varepsilon} \nabla u_{\varepsilon}) = f & \text{in } (0, +\infty) \times \Omega, \\ u_{\varepsilon} = 0 & \text{on } (0, +\infty) \times \partial\Omega, \\ u_{\varepsilon}(t = 0, \cdot) = u_0 & \text{in } \Omega, \end{cases} \quad (1.12)$$

where

$$(A_{\varepsilon}, \rho_{\varepsilon}) = \begin{cases} (I, 1) & \text{in } \Omega \setminus B_{\varepsilon}, \\ (\varepsilon^{2-d} a_O(\cdot/\varepsilon), \varepsilon^{-d} \rho_O(\cdot/\varepsilon)) & \text{in } B_{\varepsilon}. \end{cases} \quad (1.13)$$

Moreover,

$$u_c - u = u_\varepsilon - u \text{ in } (0, +\infty) \times (\Omega \setminus B_2).$$

In comparing the coefficients of the systems verified by u and u_ε , the analysis can be derived from the study of the effect of a small inclusion B_ε . The case in which *finite* isotropic materials contain inside the small inclusion was investigated in [3] (see also [5] for a related context). The analysis in [3] partly involved the polarization tensor information and took the advantage of the fact that the coefficients inside the small inclusion are finite. In the cloaking context, Craster et al. [6] derived an estimate of the order ε^d for a time larger than a threshold one. Their analysis is based on long time behavior of solutions to parabolic equations and estimates for the degree of visibility of the conducting problem, see [11, 26], hence the threshold time goes to infinity as $\varepsilon \rightarrow 0$.

In this paper, to overcome the blow up of the coefficients inside the small inclusion and to achieve the cloaking effect at any positive time, we follow the approach of Nguyen and Vogelius in [28]. The idea is to derive appropriate estimates for the effect of small inclusions in the time domain from the ones in the frequency domain using the Fourier transform with respect to time. Due to the dissipative nature of the heat equation, the problem in the frequency for the heat equation is more stable than the one corresponding to the acoustic waves, see, e.g., [27, 28], and the analysis is somehow easier to handle in the high frequency regime. After using a standard blow-up argument, a technical point in the analysis is to obtain an estimate for the solutions of the equation $\Delta v + i\omega\varepsilon^2 v = 0$ in $\mathbb{R}^d \setminus B_1$ ($\omega > 0$) at the distance of the order $1/\varepsilon$ in which the dependence on ε and ω are explicit (see Lemma 2.2). Due to the blow up of the fundamental solution in two dimensions, the analysis requires new ideas. We emphasize that even though our setting is in a bounded domain with zero Dirichlet boundary condition, we employ Fourier transform in time instead of eigenmodes decomposition as in [6] to put both systems of u_ε and u in the same context.

2. Proof of the main result

To implement the analysis in the frequency domain, let us introduce the Fourier transform with respect to time t :

$$\hat{\varphi}(k, x) = \int_{\mathbb{R}} \varphi(t, x) e^{ikt} dt \text{ for } k \in \mathbb{R}, \quad (2.1)$$

for $\varphi \in L^2((-\infty, +\infty), L^2(\mathbb{R}^d))$. Extending u , u_c , u_ρ , and f by 0 for $t < 0$, and considering the Fourier with respect to time at the frequency $\omega > 0$, we obtain

$$\Delta \hat{u} + i\omega \hat{u} = -(\hat{f} + u_0) \text{ in } \Omega,$$

and

$$\operatorname{div}(A_\varepsilon \nabla \hat{u}_\varepsilon) + i\omega \rho_\varepsilon \hat{u}_\varepsilon = -(\hat{f} + u_0) \text{ in } \Omega,$$

where

$$(A_\varepsilon, \rho_\varepsilon) = \begin{cases} (I, 1) & \text{in } \Omega \setminus B_\varepsilon, \\ (\varepsilon^{2-d} a_O(\cdot/\varepsilon), \varepsilon^{-d} \rho_O(\cdot/\varepsilon)) & \text{in } B_\varepsilon. \end{cases}$$

The main ingredient in the proof of Theorem 1.1 is the following:

Proposition 2.1. Let $\omega > 0$, $0 < \varepsilon < 1/2$, and let $g \in L^2(\Omega)$ with $\text{supp } g \subset \Omega \setminus B_2$. Assume that $v, v_\varepsilon \in H^1(\Omega)$ are respectively the unique solution of the systems

$$\begin{cases} \Delta v + i\omega v = g & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases}$$

and

$$\begin{cases} \text{div}(A_\varepsilon \nabla v_\varepsilon) + i\omega \rho_\varepsilon v_\varepsilon = g & \text{in } \Omega, \\ v_\varepsilon = 0 & \text{on } \partial\Omega. \end{cases}$$

We have

$$\|v_\varepsilon - v\|_{H^1(\Omega \setminus B_2)} \leq C e(\varepsilon, \omega, d)(1 + \omega^{-1/2}) \|g\|_{L^2(\Omega)}, \quad (2.2)$$

for some positive constant C independent of ε , ω , and g . Here

$$e(\varepsilon, \omega, 3) = \varepsilon e^{-\omega^{1/2}/4}, \quad (2.3)$$

and

$$e(\varepsilon, \omega, 2) = \begin{cases} e^{-\omega^{1/2}/4} / |\ln \varepsilon| & \text{if } \omega \geq 1/2, \\ \ln \omega / \ln(\omega \varepsilon) & \text{if } 0 < \omega < 1/2. \end{cases} \quad (2.4)$$

The rest of this section is divided into three subsections. In the first subsection, we present several lemmas used in the proof of Proposition 2.1. The proofs of Proposition 2.1 and Theorem 1.1 are then given in the second and the third subsections, respectively.

2.1. Preliminaries

In this subsection, we state and prove several useful lemmas used in the proof of Proposition 2.1. Throughout, $D \subset B_1$ denotes a smooth, bounded, open subset of \mathbb{R}^d such that $\mathbb{R}^d \setminus D$ is connected, and ν denotes the unit normal vector field on ∂D , directed into $\mathbb{R}^d \setminus D$.

The first result is the following simple one:

Lemma 2.1. Let $d = 2, 3$, $k > 0$, and let $v \in H^1(\mathbb{R}^d \setminus D)$ be such that $\Delta v + ikv = 0$ in $\mathbb{R}^d \setminus D$. We have, for $R > 2$,

$$\|v\|_{H^1(B_R \setminus D)} \leq C_R(1 + k) \|v\|_{H^{1/2}(\partial D)}, \quad (2.5)$$

for some positive constants C_R independent of k and v .

Proof. Multiplying the equation by \bar{v} (the conjugate of v) and integrating by parts, we have

$$\int_{\mathbb{R}^d \setminus D} |\nabla v|^2 - ik \int_{\mathbb{R}^d \setminus D} |v|^2 = \int_{\partial D} \partial_\nu v \bar{v}.$$

This implies

$$\int_{\mathbb{R}^d \setminus D} |\nabla v|^2 + k \int_{\mathbb{R}^d \setminus D} |v|^2 \leq C \|\partial_\nu v\|_{H^{-1/2}(\partial D)} \|v\|_{H^{1/2}(\partial D)}. \quad (2.6)$$

Here and in what follows, C denotes a positive constant independent of v and k . Since $\Delta v = -ikv$ in $B_2 \setminus D$, by the trace theory, see, e.g., [7, Theorem 2.5], we have

$$\|\partial_\nu v\|_{H^{-1/2}(\partial D)} \leq C \left(\|\nabla v\|_{L^2(B_2 \setminus D)} + \|\Delta v\|_{L^2(B_2 \setminus D)} \right) \leq C \left(\|\nabla v\|_{L^2(B_2 \setminus D)} + k \|v\|_{L^2(B_2 \setminus D)} \right). \quad (2.7)$$

Combining (2.6) and (2.7) yields

$$\int_{\mathbb{R}^d \setminus D} |\nabla v|^2 + k \int_{\mathbb{R}^d \setminus D} |v|^2 \leq C(1+k) \|v\|_{H^{1/2}(\partial D)}^2. \quad (2.8)$$

The conclusion follows when $k \geq 1$.

Next, consider the case $0 < k < 1$. In the case where $d = 3$, the conclusion is a direct consequence of (2.8) and the Hardy inequality (see, e.g., [18, Lemma 2.5.7]):

$$\int_{\mathbb{R}^3 \setminus D} \frac{|v|^2}{|x|^2} \leq C \int_{\mathbb{R}^3 \setminus D} |\nabla v|^2. \quad (2.9)$$

We next consider the case where $d = 2$. One just needs to show

$$\int_{B_R \setminus D} |v|^2 \leq C \|v\|_{H^{1/2}(\partial D)}^2. \quad (2.10)$$

By the Hardy inequality (see, e.g., [18, Lemma 2.5.7]),

$$\int_{\mathbb{R}^2 \setminus D} \frac{|v|^2}{|x|^2 \ln(2+|x|)^2} \leq C \left(\int_{\mathbb{R}^2 \setminus D} |\nabla v|^2 + \int_{B_2 \setminus D} |v|^2 \right), \quad (2.11)$$

it suffices to prove (2.10) for $R = 2$ by contradiction. Suppose that there exists a sequence $(k_n) \rightarrow 0$ and a sequence $(v_n) \in H^1(\mathbb{R}^2 \setminus D)$ such that

$$\Delta v_n + ik_n v_n = 0 \text{ in } \mathbb{R}^2 \setminus D, \quad \|v_n\|_{L^2(B_2 \setminus D)} = 1, \quad \text{and} \quad \lim_{n \rightarrow +\infty} \|v_n\|_{H^{1/2}(\partial D)} = 0.$$

Denote

$$W^1(\mathbb{R}^2 \setminus D) = \left\{ u \in L^1_{\text{loc}}(\mathbb{R}^2 \setminus D); \frac{u(x)}{\ln(2+|x|)\sqrt{1+|x|^2}} \in L^2(\mathbb{R}^2 \setminus D) \text{ and } \nabla u \in L^2(\mathbb{R}^2 \setminus D) \right\}.$$

By (2.8) and (2.11), one might assume that v_n converges to v weakly in $H^1_{\text{loc}}(\mathbb{R}^2 \setminus D)$ and strongly in $L^2(B_2 \setminus D)$. Moreover, $v \in W^1(\mathbb{R}^2 \setminus D)$ and v satisfies

$$\Delta v = 0 \text{ in } \mathbb{R}^2 \setminus D, \quad v = 0 \text{ on } \partial D, \quad (2.12)$$

and

$$\|v\|_{L^2(B_2 \setminus D)} = 1. \quad (2.13)$$

From (2.12), we have $v = 0$ in $\mathbb{R}^2 \setminus D$ (see, e.g., [18]) which contradicts (2.13). The proof is complete. \square

We also have

Lemma 2.2. *Let $d = 2, 3$, $\omega > 0$, $0 < \varepsilon < 1/2$, and let $v \in H^1(\mathbb{R}^d \setminus D)$ be a solution of $\Delta v + i\omega\varepsilon^2 v = 0$ in $\mathbb{R}^d \setminus D$. We have, for $3/2 < |x| < R$,*

$$|v(x/\varepsilon)| \leq C\varepsilon(\varepsilon, \omega, d) \|v\|_{H^{1/2}(\partial D)}, \quad (2.14)$$

for some positive constant $C = C_R$ independent of ε , ω and v .

Recall that $e(\varepsilon, \omega, d)$ is given in (2.3) and (2.4).

Proof. By the trace theory and the regularity theory of elliptic equations, we have

$$\|v\|_{L^2(\partial B_2)} + \|\nabla v\|_{L^2(\partial B_2)} \leq C\|v\|_{H^2(B_{5/2} \setminus B_{3/2})} \leq C(1 + \omega^{1/2}\varepsilon)\|v\|_{H^1(B_3 \setminus B_1)}. \quad (2.15)$$

It follows from Lemma 2.1 that

$$\|v\|_{L^2(\partial B_2)} + \|\nabla v\|_{L^2(\partial B_2)} \leq C(1 + \omega^{3/2})\|v\|_{H^{1/2}(\partial D)}. \quad (2.16)$$

Here and in what follows in this proof, C denotes a positive constant depending only on R and D .

The representation formula gives

$$v(x) = \int_{\partial B_2} \left(G_\ell(x, y) \partial_r v(y) - \partial_{r_y} G_\ell(x, y) v(y) \right) dy \text{ for } x \in \mathbb{R}^d \setminus \bar{B}_2, \quad (2.17)$$

where $\ell = e^{i\pi/4}\varepsilon\omega^{1/2}$, and, for $x \neq y$,

$$G_\ell(x, y) = \frac{e^{i\ell|x-y|}}{4\pi|x-y|} \text{ if } d = 3 \quad \text{and} \quad G_\ell(x, y) = \frac{i}{4} H_0^{(1)}(\ell|x-y|) \text{ if } d = 2.$$

Here $H_0^{(1)}$ is the Hankel function of the first kind of order 0. Recall, see, e.g., [15, Chapter 5], that

$$H_0^{(1)}(z) = \frac{2i}{\pi} \ln \frac{|z|}{2} + 1 + \frac{2i\gamma}{\pi} + O(|z|^2 \log |z|) \quad \text{as } z \rightarrow 0, z \notin (-\infty, 0], \quad (2.18)$$

and

$$H_0^{(1)}(z) = \sqrt{\frac{2}{\pi z}} e^{i(z + \frac{\pi}{4})} (1 + O(|z|^{-1})) \quad z \rightarrow \infty, z \notin (-\infty, 0]. \quad (2.19)$$

We now consider the case $d = 3$. We have, for $3/2 < |x| < R$ and $y \in \partial B_2$,

$$|e^{i\ell|x/\varepsilon - y|}| \leq e^{-\frac{\sqrt{2}}{2}\omega^{1/2}|x - \varepsilon y|} \leq e^{-\omega^{1/2}|x|/3}.$$

It follows that, for $3/2 < |x| < R$ and $y \in \partial B_2$,

$$|G_\ell(x/\varepsilon, y)| \leq C\varepsilon e^{-3\omega^{1/2}/10}. \quad (2.20)$$

Similarly, one has, for $3/2 < |x| < R$ and $y \in \partial B_2$,

$$|\partial_{r_y} G_\ell(x/\varepsilon, y)| \leq C \left(\frac{\varepsilon^2 \omega^{1/2}}{|x|} + \frac{\varepsilon^2}{|x|^2} \right) e^{-\omega^{1/2}|x|/3} \leq C\varepsilon e^{-3\omega^{1/2}/10}. \quad (2.21)$$

Combining (2.17), (2.20), and (2.21) yields

$$|v(x/\varepsilon)| \leq C\varepsilon e^{-3\omega^{1/2}/10} (\|v\|_{L^2(\partial B_2)} + \|\nabla v\|_{L^2(\partial B_2)}) \text{ for } 3/2 < |x| < R.$$

We derive from (2.16) that

$$|v(x/\varepsilon)| \leq C\varepsilon e^{-\omega^{1/2}/4} \|v\|_{H^{1/2}(\partial D)} \text{ for } 3/2 < |x| < R;$$

which is the conclusion in the case $d = 3$.

We next deal with the case where $d = 2$ and $\omega > \varepsilon^{-2}/4$, which is equivalent to $|\ell| > 1/2$. From (2.19), we derive that, for $3/2 < |x| < R$ and $y \in \partial B_2$,

$$|G_\ell(x/\varepsilon, y)| \leq C\omega^{-1/4}e^{-3\omega^{1/2}/10} \quad \text{and} \quad |\partial_{r_y}G_\ell(x/\varepsilon, y)| \leq C\varepsilon\omega^{1/4}e^{-3\omega^{1/2}/10}. \quad (2.22)$$

Using (2.16) and combining (2.17) and (2.22), we obtain, since $\omega > \varepsilon^{-2}/4$,

$$|v(x/\varepsilon)| \leq C\varepsilon e^{-\omega^{1/2}/4} \|v\|_{H^{1/2}(\partial D)} \quad \text{for } 3/2 < |x| < R,$$

which gives the conclusion in this case.

We finally deal with the case where $d = 2$ and $0 < \omega < \varepsilon^{-2}/4$, which is equivalent to $|\ell| < 1/2$. From (2.17), we obtain, for $x \in \partial B_4$,

$$v(x) = \int_{\partial B_2} ([G_\ell(x, y) - G_\ell(x, 0)]\partial_r v(y) - \partial_{r_y}G_\ell(x, y)v(y)) dy + \int_{\partial B_2} G_\ell(x, 0)\partial_r v(y) dy. \quad (2.23)$$

Since $d = 2$, we have

$$\|v\|_{L^\infty(B_5 \setminus B_3)} \leq C\|v\|_{H^2(B_5 \setminus B_3)} \leq C\|v\|_{H^2(B_5 \setminus B_2)} \leq C(1 + \omega^{1/2})\|v\|_{H^1(B_6 \setminus B_1)}.$$

It follows from Lemma 2.1 and the trace theory that

$$\|v\|_{L^\infty(B_5 \setminus B_3)} + \|v\|_{L^2(\partial B_2)} + \|\nabla v\|_{L^2(\partial B_2)} \leq C(1 + \omega^{3/2})\|v\|_{H^{1/2}(\partial D)}. \quad (2.24)$$

Since, by (2.18),

$$|\nabla_y G_\ell(x, y)| \leq C \quad \text{for } x \in \partial B_4 \text{ and } y \in \partial B_2$$

and

$$|G_\ell(x, 0)| \geq C|\ln|\ell|| \quad \text{for } x \in \partial B_4,$$

we derive from (2.23) and (2.24) that

$$\left| \int_{\partial B_2} \partial_r v(y) dy \right| \leq \frac{C(1 + \omega^{3/2})}{|\ln|\ell||} \|v\|_{H^{1/2}(\partial D)}. \quad (2.25)$$

Again using (2.17), we get, for $3/2 < |x| < R$,

$$v(x/\varepsilon) = \int_{\partial B_2} ([G_\ell(x/\varepsilon, y) - G_\ell(x/\varepsilon, 0)]\partial_r v(y) - \partial_{r_y}G_\ell(x/\varepsilon, y)v(y)) dy + \int_{\partial B_2} G_\ell(x/\varepsilon, 0)\partial_r v(y) dy. \quad (2.26)$$

Since, by (2.18), for $0 < \omega < 1/2$,

$$|G_\ell(x/\varepsilon, 0)| \leq C|\ln\omega| \quad \text{and} \quad |\nabla_y G_\ell(x/\varepsilon, y)| \leq C\varepsilon \quad \text{for } 3/2 < |x| < R, y \in \partial B_2,$$

and, by (2.19), for $1/2 < \omega < \varepsilon^{-2}/4$,

$$|G_\ell(x/\varepsilon, 0)| \leq C\omega^{-1/4}e^{-3\omega^{1/2}/10} \quad \text{and} \quad |\nabla_y G_\ell(x/\varepsilon, y)| \leq C\varepsilon\omega^{1/4}e^{-3\omega^{1/2}/10} \quad \text{for } 3/2 < |x| < R, y \in \partial B_2,$$

we derive from (2.24), (2.25), and (2.26) that, for $3/2 < |x| < R$,

$$|v(x/\varepsilon)| \leq \begin{cases} \frac{C|\ln\omega|}{|\ln|\ell||} \|v\|_{H^{1/2}(\partial D)} & \text{if } 0 < \omega < 1/2, \\ \frac{C\omega^{3/2}e^{-3\omega^{1/2}/10}}{|\ln|\ell||} \|v\|_{H^{1/2}(\partial D)} & \text{if } 1/2 < \omega < \varepsilon^{-2}/4, \end{cases}$$

which yields the conclusion in the case $0 < \omega < \varepsilon^{-2}/4$. The proof is complete. \square

2.2. Proof of Proposition 2.1

In this proof, C denotes a positive constant depending only on Ω and Λ . Multiplying the equation of v_ε by \bar{v}_ε and integrating in Ω , we derive that

$$\int_{\Omega} \langle A_\varepsilon \nabla v_\varepsilon, \nabla v_\varepsilon \rangle + \omega \int_{\Omega} \rho_\varepsilon |v_\varepsilon|^2 \leq C \|g\|_{L^2(\Omega)}^2. \quad (2.27)$$

Here we used Poincaré's inequality

$$\|v_\varepsilon\|_{L^2(\Omega)} \leq C \|\nabla v_\varepsilon\|_{L^2(\Omega)}.$$

It follows from (2.27) that

$$\begin{aligned} \|v_\varepsilon(\varepsilon \cdot)\|_{H^{1/2}(\partial B_1)}^2 &\leq C \|v_\varepsilon(\varepsilon \cdot)\|_{H^1(B_1)}^2 \\ &\leq C \int_{B_\varepsilon} \frac{1}{\varepsilon^{d-2}} |\nabla v_\varepsilon|^2 + \frac{1}{\varepsilon^d} |v_\varepsilon|^2 \leq C(1 + \omega^{-1}) \|g\|_{L^2(\Omega)}^2. \end{aligned} \quad (2.28)$$

Similarly, using the equation for v and Poincaré's inequality, we get

$$\|v\|_{H^1(\Omega)} \leq C \|g\|_{L^2(\Omega)}. \quad (2.29)$$

Since $\Delta v + i\omega v = 0$ in B_2 , using Caccioppoli's inequality, we have

$$\|v\|_{H^3(B_1)} \leq C \|v\|_{H^2(B_{3/2})} \leq C \|v\|_{H^1(B_2)} \leq C \|g\|_{L^2(\Omega)}. \quad (2.30)$$

By Sobolev embedding, as $d \leq 3$,

$$\|v\|_{W^{1,\infty}(B_1)} \leq C \|v\|_{H^3(B_1)}. \quad (2.31)$$

It follows that

$$\|v(\varepsilon \cdot)\|_{H^{1/2}(\partial B_1)} \leq C \|v(\varepsilon \cdot)\|_{H^1(B_1)} \leq C \|v\|_{W^{1,\infty}(B_1)} \leq C \|g\|_{L^2(\Omega)}. \quad (2.32)$$

Set

$$w_\varepsilon = v_\varepsilon - v \text{ in } \Omega \setminus B_\varepsilon.$$

Then $w_\varepsilon \in H^1(\Omega \setminus B_\varepsilon)$ and satisfies

$$\begin{cases} \Delta w_\varepsilon + i\omega w_\varepsilon = 0 & \text{in } \Omega \setminus B_\varepsilon, \\ w_\varepsilon = v_\varepsilon - v & \text{on } \partial B_\varepsilon, \\ w_\varepsilon = 0 & \text{on } \partial \Omega. \end{cases} \quad (2.33)$$

Let $\tilde{w}_\varepsilon \in H^1(\mathbb{R}^d \setminus B_\varepsilon)$ be the unique solution of the system

$$\begin{cases} \Delta \tilde{w}_\varepsilon + i\omega \tilde{w}_\varepsilon = 0 & \text{in } \mathbb{R}^d \setminus B_\varepsilon, \\ \tilde{w}_\varepsilon = w_\varepsilon & \text{on } \partial B_\varepsilon, \end{cases} \quad (2.34)$$

and set

$$\tilde{W}_\varepsilon = \tilde{w}_\varepsilon(\varepsilon \cdot) \text{ in } \mathbb{R}^d \setminus B_1.$$

Then $\widetilde{W}_\varepsilon \in H^1(\mathbb{R}^d \setminus B_1)$ is the unique solution of the system

$$\begin{cases} \Delta \widetilde{W}_\varepsilon + i\omega \varepsilon^2 \widetilde{W}_\varepsilon = 0 & \text{in } \mathbb{R}^d \setminus B_1, \\ \widetilde{W}_\varepsilon = w_\varepsilon(\varepsilon \cdot) & \text{on } \partial B_1. \end{cases} \quad (2.35)$$

Fix $r_0 > 2$ such that $\Omega \subset B_{r_0}$. By Lemma 2.2, we have, for $1 \leq |x| < r_0$, that

$$|\widetilde{W}_\varepsilon(x/\varepsilon)| \leq C e(\varepsilon, \omega, d) \|w_\varepsilon(\varepsilon \cdot)\|_{H^{1/2}(\partial B_1)},$$

which yields, for $x \in B_{r_0} \setminus B_1$, that

$$|\widetilde{w}_\varepsilon(x)| \leq C e(\varepsilon, \omega, d) \|w_\varepsilon(\varepsilon \cdot)\|_{H^{1/2}(\partial B_1)}.$$

Since $\Delta \widetilde{w}_\varepsilon + i\omega \widetilde{w}_\varepsilon = 0$ in $B_{r_0} \setminus B_1$, it follows from Caccioppoli's inequality that

$$\|\widetilde{w}_\varepsilon\|_{H^1(B_2 \setminus B_{3/2})} \leq C e(\varepsilon, \omega, d) \|w_\varepsilon(\varepsilon \cdot)\|_{H^{1/2}(\partial B_1)}. \quad (2.36)$$

Fix $\varphi \in C^2(\mathbb{R}^d)$ such that $\varphi = 1$ in $B_{3/2}$ and $\varphi = 0$ in $\mathbb{R}^d \setminus B_2$, and set

$$\chi_\varepsilon = w_\varepsilon - \varphi \widetilde{w}_\varepsilon \text{ in } \Omega \setminus B_\varepsilon.$$

Then $\chi_\varepsilon \in H_0^1(\Omega \setminus B_\varepsilon)$ and satisfies

$$\Delta \chi_\varepsilon + i\omega \chi_\varepsilon = -\Delta \varphi \widetilde{w}_\varepsilon - 2\nabla \varphi \cdot \nabla \widetilde{w}_\varepsilon \text{ in } \Omega \setminus B_\varepsilon.$$

Multiplying the equation of χ_ε by $\bar{\chi}_\varepsilon$ and integrating by parts, we obtain, by Poincaré's inequality,

$$\|\chi_\varepsilon\|_{H^1(\Omega \setminus B_\varepsilon)} \leq C \|\widetilde{w}_\varepsilon\|_{H^1(B_2 \setminus B_{3/2})}. \quad (2.37)$$

Combining (2.36) and (2.37) yields

$$\|w_\varepsilon\|_{H^1(\Omega \setminus B_2)} \leq C e(\varepsilon, \omega, d) \|w_\varepsilon(\varepsilon \cdot)\|_{H^{1/2}(\partial B_1)}. \quad (2.38)$$

The conclusion now follows from (2.28) and (2.32). \square

2.3. Proof of Theorem 1.1

Let $v_\varepsilon = u_\varepsilon - u$. Using the fact that v_ε is real, by the inversion theorem and Minkowski's inequality, we have, for $t > 0$,

$$\|v_\varepsilon(t, \cdot)\|_{L^2(\Omega \setminus B_2)} \leq C \int_0^\infty \|\hat{v}_\varepsilon(\omega, \cdot)\|_{L^2(\Omega \setminus B_2)} d\omega. \quad (2.39)$$

Using Proposition 2.1, we get

$$\begin{aligned} \int_0^\infty \|\hat{v}_\varepsilon(\omega, \cdot)\|_{L^2(\Omega \setminus B_2)} d\omega &\leq C \int_0^\infty (1 + \omega^{-1/2}) e(\varepsilon, \omega, d) \|\hat{f}(\omega) + u_0\|_{L^2(\Omega \setminus B_2)} d\omega \\ &\leq C \text{esssup}_{\omega > 0} \|\hat{f}(\omega) + u_0\|_{L^2(\Omega \setminus B_2)} \int_0^\infty (1 + \omega^{-1/2}) e(\varepsilon, \omega, d) d\omega \end{aligned}$$

$$\leq Ce(\varepsilon, d)(\|f\|_{L^1((0,+\infty);L^2(\Omega))} + \|u_0\|_{L^2(\Omega)}).$$

It follows from (2.39) that, for $t > 0$,

$$\|v_\varepsilon(t, \cdot)\|_{L^2(\Omega \setminus B_2)} \leq Ce(\varepsilon, d)(\|f\|_{L^1((0,+\infty);L^2(\Omega))} + \|u_0\|_{L^2(\Omega)}).$$

Similarly, we have, for $t > 0$,

$$\|\nabla v_\varepsilon(t, \cdot)\|_{L^2(\Omega \setminus B_2)} \leq Ce(\varepsilon, d)(\|f\|_{L^1((0,+\infty);L^2(\Omega))} + \|u_0\|_{L^2(\Omega)}).$$

The conclusion follows. □

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Conflict of interest

The authors declare no conflict of interest in this paper.

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