Design of concrete structures using structural optimization based on the stress field method

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Abstract

The Elastic Plastic Stress Field method (EPSF) is a widely used tool for design and verification of structural concrete works. EPSF uses the elastic-plastic material behavior, which allows accounting for compatibility conditions and thus provides exact solutions according to limit analysis. By applying structural optimization method on the basis of EPSF, an automatic approach for the optimal design of reinforced concrete structures is developed. Optimal design is obtained by solving a minimization problem with a number of compliance constraints. The result respects the yield conditions of both materials and the deformation capacity requirements of the structure. The efficiency of the approach is illustrated by a set of examples.

1 Introduction

Optimal design approach of reinforced concrete structures is relevant to ensure efficient and reduced material consumption and also in helping architects and engineers to find tailored solutions. Due to the material response of concrete (with a different response in compression and in tension) and to the composite nature of reinforced concrete structures, the design of reinforced concrete structures is a complex task. In this frame, strut-and-tie models (STM) and stress fields (SF) are tools commonly used for its design at ultimate limit state. Both methods constitute lower-bound solutions of limit analysis, providing safe estimates of the actual strength and helping in understanding the structural response of a member.

STM was originally developed for reinforced concrete from the intuitive analogy of a truss structure [1-2] and later were shown to constitute lower bound solutions of limit analysis [3]. For a given member, multiple solutions are normally possible, leading to different reinforcement layouts and load-carrying configurations. Therefore, STM needs criteria to select the most suitable model for a given case. Application of structural optimization methods to generate suitable STM for concrete have been widely studied, focusing on the material behavior (elastic or plastic), the modelling of steel reinforcement in concrete (discrete or continuum) and different optimization criteria and constraints (e.g. [4-9]).

On the other hand, the stress field method was developed as a direct application of the theory of plasticity [10]. Its earlier applications were based on rigid plastic material laws (neglecting the tensile strength of concrete [11-16]) and showed its potential to generate lower-bound solutions, providing safe solutions for design. Recent developments of the stress field method have consisted on the implementation of the elastic-plastic behavior of concrete in compression (with no tensile strength) and the elastic-plastic behavior for steel. This approach (named as elatic-plastic stress fields, EPSF [17]) allows accounting for compatibility conditions when determining the stress field and yields to a licit mechanism at failure. Thus, since both a lower-bound and a mechanism (upper-bound) are provided, EPSF provides exact solutions according to limit analysis [17] and can be used for optimization of reinforcement design [17, 18].

An extension of EPSF is proposed in this paper, which aims at further automating the design process by combining EPSF with structural optimization method. The proposed method can be used directly to generate efficient layouts of reinforcement and thickness of concrete elements. In Section 2, the basic principles of EPSF are introduced. In Section 3 a topology optimization problem is formulated to conduct optimal design for reinforced concrete strucures at ULS. The advantages of the optimal design procedure are discussed in Section 4 by means of two numerical examples. Section 5 eventually summarizes the main contents of the work and outlines the possible extentions of this research.

2 EPSF method

2.1 Basic Princples of EPSF

The elastic-plastic stress field method (EPSF) was developed by Fernández Ruiz and Muttoni [17] to overcome some limitations of rigid-plastic stress fields (such as the direct evaluation of the concrete efficiency factor due to transverse cracking) and to generate in an automated manner suitable solutions for design. In the EPSF method, the basic aim is to find a licit stress field that satisfies both the equilibrium and the compatibility conditions and also respects the plastic yield criteria of the materials. The influence of cracking on the concrete strength is introduced by means of an effective compressive strength according to the transverse strain [19]. A finite element procedure is used to automatically develop EPSF [17].

The EPSF method has shown to be robust to predict the failure load and associated mechanisms [17, 20, 21,22]. The key element of EPSF method is the material model and their implementation with finite element method, which will be explained in Section 2.2.

2.2 Implemention of the effective elastic plastic material model of EPSF

The material model for steel reinforcement is a bi-linear elastic-plastic material model both in tension and compression, as shown in Fig.1 (a). With respect to the concrete, the material is considered to be elastic-perfectly plastic in compression, with an effective concrete compression strength as shown in Fig.1 (b). The effective compressive strength is determined by multypling the uniaxial compressive strength (f_c) with a softening reduction factor η_ε for transverse strain [19] and a reduction factor η_{f_c} for brittle behaviour [14], as provided in Equation (1). This material behavior of concrete corresponds to a Mohr-Coulomb yield surface with a tension cut-off and an associative flow rule (Fig.1 (c)).

To implement the aboved-mentioned material behaviour, a displacement-based nonlinear finite elment method is used. The nonlinear system is solved by the secant method. At each iteration, the global secant stiffness matrix is assembled by the following procedure:

- The strain field is firstly computed using the displacement field got from last iteration
- The principal stress in each concrete element is evaluated as a function of the principal strain by using the modified elastic plastic material model (Fig.1 (b)) and under the assumption that the principal stress direction is parallel to the strain direction, (as in Equation (2))
- The secant elastic modulus of concrete is calculated for each concrete element (Equation (3)) and used to to assemble the secant stiffness matrix.
- The nodal force of each element and the corresponding residual nodal force at each node is then calculated with this stress field (Fig.1 (f)).
- An incremental displacement field Δu_i is calculated to balance the residual nodal force. The incremental displacement is then added to form the new displacement field u_i. This procedure continues until the residual force is below a tolerance level. More detailed information about EPSF can be find in other references [17, 20, 21, 22].

The resulting EPSF contains the licit stress field and also the efficient material strength information within the design domain. The secant stiffness for a concrete element is reduced to zero in the direction of tensile strain and also partially reduced with the factor $\eta_\varepsilon$in the compression strain direction when large tensile strain occurs in the transverse direction. Structural optimization can be efficiently conducted based on these informations, which will be explained in detail in the next section.

3 Topology Optimization Problem Formulation

The aim of using structural optimization method with EPSF is to generate an optimal layout of the reinforcement steel and of concrete thickness. More precisely, the aim is to minimize the amount of material and to ensure the load bearing capcity as well as sufficient deformation capacity of the structure for a given load situation. In the scope of EPSF, this aim can be achieved by minimizing the amount of material with a global compliance constraint. The global compliance constraint is defined as the work of the external forces. Since the load is set to be constant, the global compliance is dependent on the displacement at the position of the external loads, which is an indicator of the complementary of energy of deformation of the structure and can be used as an indirect constraint on the load bearing capacity. As for the deformation capacity, it is ensured by ensuring that steel reinforcement yields before concrete crushes.

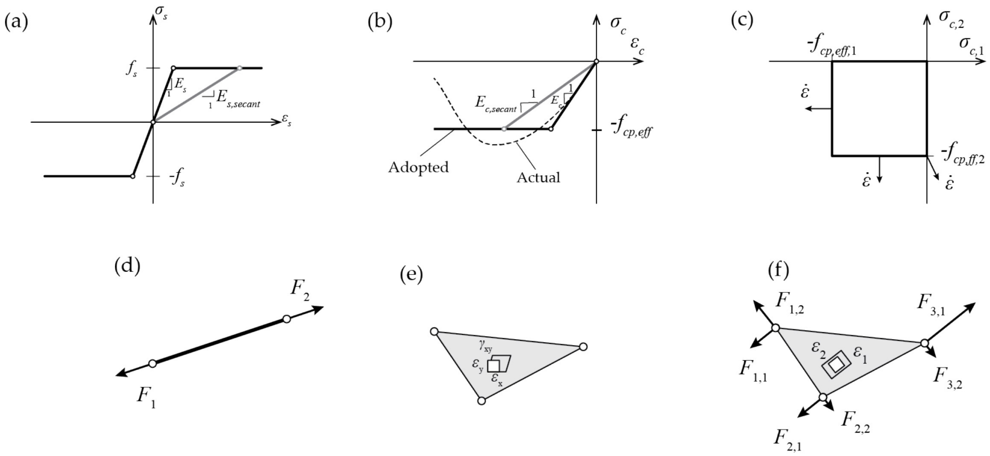

Fig. 1 Elastic-plastic stress fields [17]: Material model for (a) steel; (b) concrete and (c) Mohr-Coulomb yield surface for concrete in plane stress state with neglected tensile strength; (d) finite element for reinforcing bars and (e)(f) constant strain triangle for modelling concrete.

$$
\begin{aligned}
f_{cp,eff,i} &= f_c \cdot \eta_{f_c} \cdot \eta_{\varepsilon,j} \\
\eta_{f_c} &= \left(\frac{30}{f_c}\right)^{\frac{1}{3}} \leq 1.0 \\
\eta_{\varepsilon,j} &= \frac{1}{0.8 + 170\varepsilon_j} \leq 1
\end{aligned}
\tag{1}
$$

$$
\sigma_i = \sigma_i(\varepsilon_i, \varepsilon_j) = \begin{cases} 0, & 0 \leq \varepsilon_i \\ E_c \varepsilon_i, & -\frac{f_{cp,eff,i}}{E_c} < \varepsilon_i < 0 \\ f_{cp,eff,i}, & \varepsilon_i \leq -\frac{f_{cp,eff,i}}{E_c} \end{cases}
\tag{2}
$$

$$
E_{c,secant,i} = \begin{cases} 0, & \varepsilon_i = 0 \\ \frac{\sigma_i}{\varepsilon_i}, & \varepsilon_i \neq 0 \end{cases} \qquad \text{for } i=1,2\ ; j=2,1
\tag{3}
$$

3.1 Design parametrization

The topology optimization formulation uses the well-known solid isotroptic microstructure with penalty (SIMP) approach [23] to drive the result to a 0-1 distribution of material. Following the convention in EPSF, the reinforcement is modelled as 1-D bar elements and the the concrete is modelled as 2-D constant strain triangles. The cross-section area of each steel bar element and the thickness of each concrete element is defined as a physical variable, which is associated to a design variable. The relation between a physical variable and a design variable is as follows:

$$
a_{p,i} = a_{max,i} x_i^p
\tag{4}
$$

where $a_{p,i}$ is the physical variable; $a_{max,i}$ is the upper bound of the physical variable, representing the maximum allowable cross-section area of steel bar or the maximum thickness of concrete element; and x_i is the design variable, defined as $x_{min} \leq x_i \leq 1$. The minimum value is set to be a very small positive value to avoid singularity of matrix in the solving procedure; $p \geq 1$ is an integer value that is called the material penalization factor, which is used to penalize intermediate values of the design variable to drive the result to a 0-1 distribution. $p = 3$ is used in this formulation. Then the elemental stiffness matrix $K_i$can be written in the following form

$$K_i(x_i) = K_{u,i}\, a_{max,i} x_i^p \tag{5}$$

where $K_{u,i}$ represents the unit elemental stiffness matrix. In the context of EPSF, $K_{u,i}$ is computed by divide the elemental secant stiffness by the physical variable

$$K_{u,i} = K_{u,i}(E_{secant}) = \frac{1}{a_{p,i}} K_i(E_{secant}) \tag{6}$$

A density filter algorithm is also applied to the design variable to reduce the checkerboard problem and mesh-dependency issues, which will not be explained hereafter [24].

3.2 Optimization Problem Formulation

The topology optimization problem is formulated as in Eq (7), where the objective is to minimize the amount of material and the constraints are the equilibrium condition of EPSF and also the global compliance of the structure. The value of the compliance constraint C_{limit} will be discussed in detail in Section 3.3. As the strength-to-density ratio of steel is significantly higher than for concrete, if the objective of optimization is set to be the weight of the whole structure, the optimization result will tend to minimize concrere first. This is yet against the aim of achieving a layout for the ULS where the steel reinforcement yields first. To solve this issue, a high artificial density ratio $\frac{\rho_s}{\rho_c}$ is used to act as a material penalty for steel.

The gradient-based method MMA [25] is used to solve the constraint minimizing optimization problem, which uses the first order sensitivity of the objective function and constraint function to approximate the original problem with a sequence of convex subproblems. The Method of sensitivity analysis is explained in Section3.4. The controlling parameters of the optimization process are listed in Table 1.

$$\begin{aligned}
&\text{Min :} \quad W = \textstyle\sum_{i_c=1}^{n_c} \rho_c x_{i_c} A_{i_c} + \sum_{i_s=1}^{n_s} \rho_s x_{i_s} l_{i_s} \\
&\text{Subjected to:} \\
&KU = \sum_{j=1}^{n_c+n_s} K_j(x_j)U = F \\
&C = FU \le C_{limit} \\
&x_{min} \le x_j \le 1
\end{aligned} \tag{7}$$

Table 1 Optimization Controling Parameters

Index	$\boldsymbol{\rho_s/\rho_c}$	p	x_{min}
Value	100	3	0.01

3.3 Suitable value of Compliance Constraint

The global compliance is the key constraint in the optimization formulation, which is decisive for the final optimization result. The compliance constraint should be high enough to allow plastic deformations to occur in the reinforcement and, on the other hand, the constraint value should not exceed a certain limit so as to preserve the load bearing capacity and to avoid severe strain loacalization. To satisfy these requirements, the compliance constraint value is selected with an "outer iteration", distinguished from the optimization iteration with a fixed compliance value, which is called the "innter iteration". The optimization process starts with a trial compliance value and then is adjusted according to the performance of the optimization solution of the inner iteration. An inner optimization iteration is started with a trial compliance constraint value. If the inner optimization completes, which means the objective function converges to a stable value, then the compliance constraint value is increased and the inner iteration starts with the new compliance constraint. If in the inner optimization iteration the EPSF solution fails to converge or any local strain exceeds a given limitation, then the compliance constraint is decreased to a more conservative value and the inner iteration restarts with the new compliance constraint value. The outer iteration is stopped and an ultimate compliance value is selected when the updating of the compliance constraint changes direction. With this final compliance constraint value, the inner iteration is conducted and the ultimate optimization result is derived. The procedure is illustrated in Fig.2. This method has been proved to be efficient in all the cases investigated. An important advantage of this method is that an indirect strain constraint is enforced in the optimization process, which significantly improves the performance of the optimization result and avoids the time consuming and inprecise treatment of local constraints.

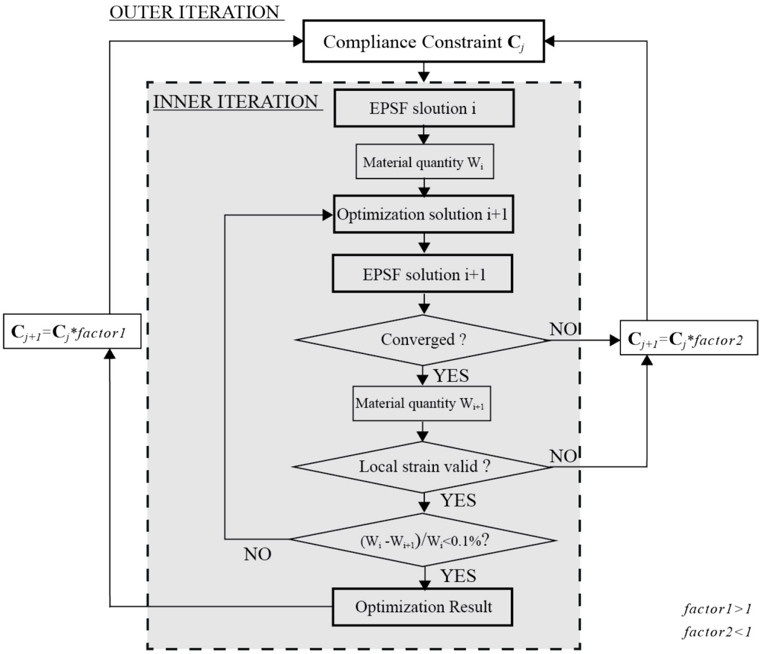

Fig. 2 Procedure for selecting a suitable compliance constraint

3.4 Sensitivity Analysis

The sensitivity analysis is related to the finite elment method used. The finite element analysis used in EPSF is in fact a transient state nonlinear problem. However, as explained before, it is a reasonable simplification to solve the nonlinear system without using an incremental approach. In this way, the problem is actually treated as a steady-state nonlinear problem. Due to this simplification, the sensitivity analysis can be conducted using the classical adjoint method [26]. The secant stiffness matrix for finite element analysis is used directly for the sensitivity analysis:

$$\frac{\partial U}{\partial x_i} = -K^{-1}\frac{\partial K}{\partial x_i}U \tag{8}$$

$$\frac{\partial K}{\partial x_i} = \frac{\partial(\sum_{j=1}^{n_c+n_s} K_j(x_j)U)}{\partial x_i} = \frac{\partial(K_{u,i}\, a_{max,i} x_i^p)}{\partial x_i} = K_{u,i} p\, a_{max,i} x_i^{(p-1)} \tag{9}$$

4 Case Study

4.1 Simply Supported Beam Case

The first case presented corresponds to a simply supported beam subjected to a concentrated load at mid-span ($F = 120$ KN). As illustrated in Fig.3 (a), the design domain of concrete has a span

$l = 3.4$ m and a height $h = 0.6$ m. The concrete thickness at starting of the optimization process is $b = 0.3$ m. Due to simmetry conditions, half of the beam is only modelled and optimized. Two beams with different stirrup spacing values are studied.

The first beam (MB1) has a stirrup spacing of 400 mm as shown in Fig.4 (a). The topology optimization result is shown in Fig.4 (b) and the resulting stress field is shown in Fig.4(c). The result contains struts with an inclination of approximately 45° with respect to the horizontal direction, arising from the intersection points between longitudinal reinforcement and stirrups, and change their direction at the location of stirrups. This result is consistent with other research results about STM for slender beams [8].

The second beam (MB2) has a stirrup spacing of 200mm as shown in Fig.5 (a). Since a smaller stirrup spacing is provided, a relatively larger design space is explored. Comparing with MB1, each strut is thinner and the most efficient ties are selected from all the available options. The optimization results are further compared in Table 2, which shows that MB2 results in significantly less concrete but almost the same amount of steel as MB1. Although there are more stirrups in MB2, there is a reduction in the cross section area of the longitudinal reinforcement near the support point. This explains why the volumn of steel is smaller in MB2.

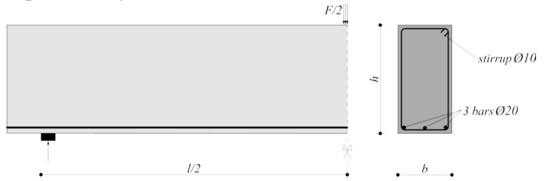

Fig. 3 Simply supported beam case: concrete geometry ($f_y = 435$ MPa $E_s = 205'000$ MPa; $f_c = 20$ MPa $E_c = 30'000$ MPa).

(a) (b) (c)

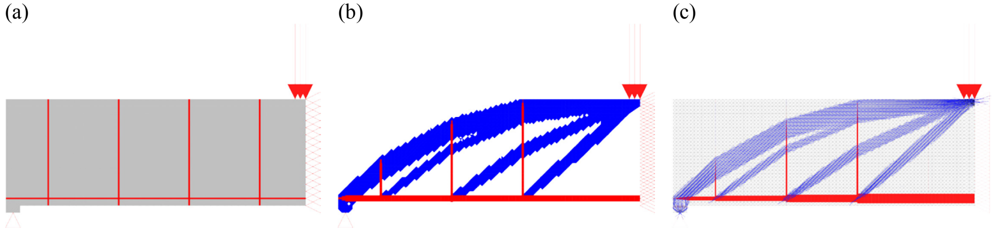

Fig. 4 MB1: (a) Input Model (b) Optimization Result and (c) Stress Field Result

(a) (b) (c)

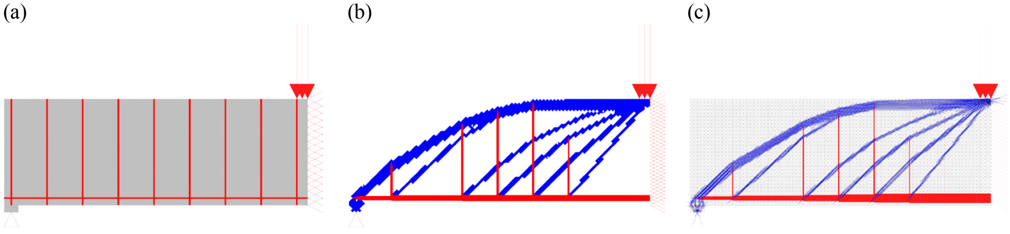

Fig. 5 MB2: (a) Input Model (b) Optimization Result and (c) Stress Field Result

Table 2 Optimization Result Comparison

Beam	Concrete Volumn(m^3)	Steel Volumn(m^3)	Global Compliance(KNm)
MB1	0.09497	0.001763	0.2376
MB2	0.04679	0.001719	0.2598

4.2 Deep Beam with Opening Case

The second case corresponds to a deep beam with an opening according to Schlaich et al [3]. This case has been used by many researchers to verify simplified design method of concrete structures. The design domain is a deep beam (length $l = 7.5$ m and height $h = 4.7$ m) with an opening (length $l_1 = 1.5$ m and height $h_1 = 1.5$ m) subjected to a concentrated load. The concrete thickness at the staring point of optimization is $b = 0.4$ m.The geometry of the reinforcement at the staring of the optimization process is shown in Fig.6 (a). The orthgonal reinforcement layout is composed of $2\emptyset 20\text{mm}$ steel bars placed at 200 mm spacing in each direction. ($f_y = 434$ MPa $E_s = 205{,}000$ MPa; $f_c = 20$ MPa $E_c = 30'000$ MPa).

The resulting topology result is shown in Fig.6 (b) and the resulting stress field is shown in Fig.6 (c). The result can be interpreted as a combination of the design proposed by Schlaich in [3] and the one proposed by Fernández Ruiz in [17]. The result is in good agreement with the reinforcement design achieved by Fernández Ruiz [17], where the contribution of horizontally distributed reinforcement and the beam below the opening are used efficiently. Yet, in the region above the upper right corner of the opening, a relatively high amount of ties are used compared with the result given by Fernández Ruiz [17] (similar result to the STM designed by Schlaich [3]). This result occurs because the global compliance constraint is used in the optimization process. The reinforcement above the opening makes a relatively high contribution to the stiffness of the whole structure and thus has a high sensitivity to the global compliance.

(a) (b) (c)

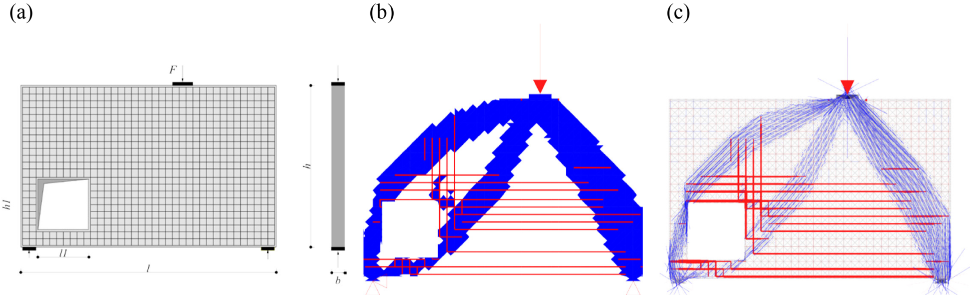

Fig. 6 Deep Beam Case: (a) Geometry Model (b) Optimization Result and (c) Stress Field Result

5 Conclusions and Outlook

This paper extends the use of the elastic-plastic stress field (EPSF) technique to the field of optimal design by applying a structural optimization method. A topology optimization problem with a simple global compliance constraint is formulated to generate optimal design. An outer iteration is used to find a suitable global compliance constraint value which ensures the load bearing capacity, allows sufficient deformation and also avoids severe strain localization. The case studies show that the proposed method is effective. The results can be used directly to optimize the reinforcement layout for ultimate limit state design considering the nonlinear behaviour of both concrete and steel.

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