Appendix A
Proof of Theorem 3

The proof is close to the one of Blanchard *et al.* [1] and includes contributions from Carrillo *et al.* [2].

As a preliminary, we remind the definition of the asymmetric restricted isometry (ARIP) constants that will be used in the proof.

Definition 1 (ARIP constants [1])**.** *Consider* $\boldsymbol{A} \in \mathbb{K}^{m\times n}$*. The lower and upper ARIP constants of order k denoted as* L_k *and* U_k*, respectively, are defined as*

$$L_k = \min_{b\geq 0} b,\ \text{subject to}\ (1-b)\,\|\boldsymbol{x}\|_2^2 \leq \|\boldsymbol{A}\boldsymbol{x}\|_2^2,\ \forall \boldsymbol{x}\in\Sigma_k$$
$$U_k = \min_{b\geq 0} b,\ \text{subject to}\ (1-b)\,\|\boldsymbol{x}\|_2^2 \geq \|\boldsymbol{A}\boldsymbol{x}\|_2^2,\ \forall \boldsymbol{x}\in\Sigma_k$$

Recall that $\|\boldsymbol{X}\|_{0,\mathrm{row}} = k$ and $\mathrm{supp}\,(\boldsymbol{X}) = J$, $|J| = k$, $J = J_0 \cup J_1$, where $|J_0| = r$, and $|J_1| = k-r$. Let $\boldsymbol{V}^i = \boldsymbol{X}^i + \omega^i \boldsymbol{A}^*(\boldsymbol{Y} - \boldsymbol{A}\boldsymbol{X}^i)$. By replacing $\boldsymbol{Y}$ by its expression, we have that:

$$\boldsymbol{V}^i = \boldsymbol{X}^i + \omega^i\boldsymbol{A}^*\boldsymbol{A}(\boldsymbol{X}_{(J)} - \boldsymbol{X}) + \omega^i\boldsymbol{A}^*\tilde{\boldsymbol{E}}. \tag{13}$$

Define the update $\boldsymbol{X}^{i+1} = \boldsymbol{V}^i_{(J_0)} + \mathcal{H}_{k-r}(\boldsymbol{V}^i_{(\bar{J}_0)})$. Also define $U^i = \mathrm{supp}(\mathcal{H}_{k-r}(\boldsymbol{V}^i_{(\bar{J}_0)}))$. It can be easily checked that $|U^i| \leq k-r$, as described in [3].

Now, we can write the following inequality:

$$\begin{aligned}\|\boldsymbol{V}^i - \boldsymbol{X}^{i+1}\|_F^2 &= \|\boldsymbol{V}^i_{(J_0)} - \boldsymbol{X}^{i+1}_{(J_0)}\|_F^2 + \|\boldsymbol{V}^i_{(\bar{J}_0)} - \boldsymbol{X}^{i+1}_{(\bar{J}_0)}\|_F^2, &(14)\\ &\leq \|\boldsymbol{V}^i_{(J_0)} - \boldsymbol{X}_{(J_0)}\|_F^2 + \|\boldsymbol{V}^i_{(\bar{J}_0)} - \boldsymbol{X}_{(J_1)}\|_F^2, &(15)\\ &= \|\boldsymbol{V}^i - \boldsymbol{X}_{(J)}\|_F^2, &(16)\end{aligned}$$

since $\boldsymbol{V}^i_{(J_0)} = \boldsymbol{X}^{i+1}_{(J_0)}$ and $\boldsymbol{X}^{i+1}_{(\bar{J}_0)}$ is the best (k-r)-term approximation of $\boldsymbol{V}^i_{(\bar{J}_0)}$. Following the same reasoning as [1], we can express the following inequality:

$$\begin{aligned}\|\boldsymbol{X}_{(J)} - \boldsymbol{X}^{i+1}\|_F^2 &\leq 2\omega^i|\langle\tilde{\boldsymbol{E}}, \boldsymbol{A}(\boldsymbol{X}^{i+1} - \boldsymbol{X}_{(J)})\rangle| \\ &+ 2|\langle(\boldsymbol{I} - \omega^i\boldsymbol{A}^*_Q\boldsymbol{A}_Q)(\boldsymbol{X}^i - \boldsymbol{X}_{(J)}), (\boldsymbol{X}^{i+1} - \boldsymbol{X}_{(J)})\rangle|, \quad (17)\end{aligned}$$

where $Q = J \cup J^i \cup J^{i+1}$ has a cardinality bounded by

$$|Q| = |J_0 \cup J_1 \cup U^i \cup U^{i+1}| \leq 3k - 2r \leq ck, \tag{18}$$

where $c \in \mathbb{N}$ such that $ck \geq 3k - 2r$. Now, using Lemma 5 of [1], we can write that

$$\begin{aligned}&|\langle(\boldsymbol{I} - \omega^i\boldsymbol{A}^*_Q\boldsymbol{A}_Q)(\boldsymbol{X}^i - \boldsymbol{X}_{(J)}), (\boldsymbol{X}^{i+1} - \boldsymbol{X}_{(J)})\rangle| \\ &\qquad\leq \varphi(ck)\|\boldsymbol{X}^i - \boldsymbol{X}_{(J)}\|_F\|\boldsymbol{X}^{i+1} - \boldsymbol{X}_{(J)}\|_F \quad (19)\end{aligned}$$

where $\varphi(ck) = \frac{U_{ck}+L_{ck}}{1-L_k}$.

In addition, we can bound the first term of (17) as:

$$|\langle\tilde{\boldsymbol{E}}, \boldsymbol{A}(\boldsymbol{X}^{i+1} - \boldsymbol{X}_{(J)})\rangle| \leq \sqrt{1+U_{dk}}\|\tilde{\boldsymbol{E}}\|_F\|\boldsymbol{X}^{i+1} - \boldsymbol{X}_{(J)}\|_F, \tag{20}$$

since $\mathrm{supp}(\boldsymbol{X}^{i+1} - \boldsymbol{X}_{(J)}) = J \cup U^{i+1}$ has its cardinality bounded by $2k - r \leq dk$, with $d \in \mathbb{N}$.

With (17), (19), (20) and Lemma 2 of [1], we can write

$$\|\boldsymbol{X}_{(J)} - \boldsymbol{X}^{i+1}\|_F \leq \alpha^i\|\boldsymbol{X}_{(J)}\|_F + \frac{\beta}{1-\alpha}\|\tilde{\boldsymbol{E}}\|_F, \tag{21}$$

where $\alpha = 2\varphi(ck) < 1$ and $\beta = 2\frac{\sqrt{1+U_{dk}}}{1+L_k}$ since $\omega^i \leq \frac{1}{1+L_k}$.

Appendix B
Empirical Validation of Theorem 2

We propose an empirical validation of Theorem 2 using MUSIC and MUSIC-PKS algorithms.

The signal matrix $\boldsymbol{X} \in \mathbb{R}^{n\times N}$ is designed with $n = 64$, $N = 128$, $\mathrm{supp}\,(\boldsymbol{X}) = J_0 \cup J_1$, such that $|J_0| = |J_1| = 8$ and J_0 is known *a priori*.

We consider a Gaussian random measurement matrix $\boldsymbol{A} \in \mathbb{R}^{m\times n}$, with $\boldsymbol{A}_{i,j} \sim \mathcal{N}(0,1)$, such that $\|\boldsymbol{A}_i\|_2 = 1$ and $\mathrm{rank}\,(\boldsymbol{A}) = m \Leftrightarrow \mathrm{spark}\,(\boldsymbol{A}) = m+1$. The measurements are computed as $\boldsymbol{Y} = \boldsymbol{A}\boldsymbol{X}$.

In a first experiment, we force $\mathrm{rank}\left(\boldsymbol{X}_{(J_0)}\right) = 1$ and $\mathrm{rank}\left(\boldsymbol{X}_{(J_1)}\right) = |J_1|$ such that $\mathrm{rank}\,(\boldsymbol{Y}) = |J_1| + 1$ when $m > k$. We are in a rank-defective case in which MUSIC procedure fails. However, when $m > k$, $\mathrm{rank}\left(\left[\boldsymbol{Y}, \boldsymbol{A}_{J_0}\right]\right) = k$ and we are in the ideal case where $\mathcal{R}\left(\boldsymbol{A}_{J_0}\right)$ augments the signal subspace $\mathcal{R}\,(\boldsymbol{Y})$ such that MUSIC-PKS succeeds.

In a second experiment, we force $\mathrm{rank}\left(\boldsymbol{X}_{(J_0)}\right) = |J_0|$ and $\mathrm{rank}\left(\boldsymbol{X}_{(J_1)}\right) = 1$ in such a way that we are in the worst case scenario for MUSIC-PKS since $\mathcal{R}\left(\boldsymbol{A}_{J_0}\right) \subset \mathcal{R}\,(\boldsymbol{Y})$. In this case, MUSIC-PKS does not perform better than MUSIC.

Fig 4 displays the average recovery probability, computed as the rate of successful support recovery over 1000 random trials of the algorithms.

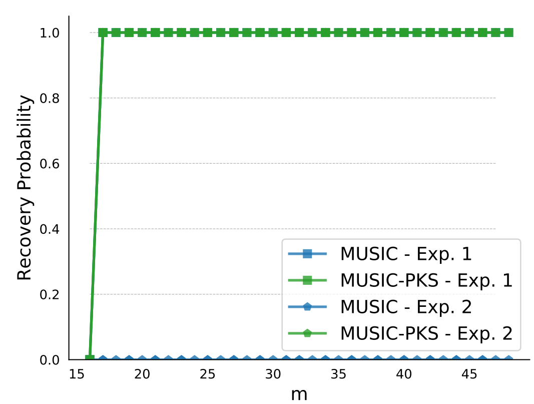

Fig. 4. Recovery probability of MUSIC and MUSIC-PKS when $\mathrm{rank}(\boldsymbol{X}_{(J_0)}) = |J_0|$ (Exp. 1) and when $\mathrm{rank}(\boldsymbol{X}_{(J_0)}) = 1$ (Exp. 2).

For the first experiment, we observe that MUSIC-PKS recovers the support of the signal for $m \geq k+1 = 17$ which exactly corresponds to the case where the augmented matrix has full rank, as stated in Theorem 2. Concerning the second experiment, both MUSIC and MUSIC-PKS fail as expected.

References

[1] J. D. Blanchard, M. Cermak, D. Hanle, and Y. Jing, “Greedy algorithms for joint sparse recovery,” *IEEE Trans. Signal Process.*, vol. 62, no. 7, pp. 1694–1704, 2014.

[2] R. E. Carrillo, L. F. Polania, and K. E. Barner, “Iterative hard thresholding for compressed sensing with partially known support,” in *2011 IEEE Int. Conf. Acoust. Speech Signal Process.*, 2011, pp. 4028–4031.

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