Appendix A
Proof of Theorem 3

The proof is close to the one of Blanchard *et al.* [?] and includes contributions from Carrillo *et al.* [?].

As a preliminary, we remind the definition of the asymmetric restricted isometry (ARIP) constants that will be used in the proof.

Definition 1 (ARIP constants [?]). *Consider $\boldsymbol{A} \in \mathbb{K}^{m\times n}$. The lower and upper ARIP constants of order k denoted as L_k and U_k, respectively, are defined as*

$$L_k = \min_{b\geq 0} b, \text{ subject to } (1-b)\,\|\boldsymbol{x}\|_2^2 \leq \|\boldsymbol{A}\boldsymbol{x}\|_2^2, \ \forall \boldsymbol{x} \in \Sigma_k$$
$$U_k = \min_{b\geq 0} b, \text{ subject to } (1-b)\,\|\boldsymbol{x}\|_2^2 \geq \|\boldsymbol{A}\boldsymbol{x}\|_2^2, \ \forall \boldsymbol{x} \in \Sigma_k$$

Recall that $\|\boldsymbol{X}\|_{0,\text{row}} = k$ and $\text{supp}\,(\boldsymbol{X}) = J$, $|J| = k$, $J = J_0 \cup J_1$, where $|J_0| = r$, and $|J_1| = k - r$. Let $\boldsymbol{V}^i = \boldsymbol{X}^i + \omega^i \boldsymbol{A}^*(\boldsymbol{Y} - \boldsymbol{A}\boldsymbol{X}^i)$. By replacing $\boldsymbol{Y}$ by its expression, we have that:

$$\boldsymbol{V}^i = \boldsymbol{X}^i + \omega^i \boldsymbol{A}^*\boldsymbol{A}(\boldsymbol{X}_{(J)} - \boldsymbol{X}) + \omega^i \boldsymbol{A}^*\tilde{\boldsymbol{E}}. \tag{13}$$

Define the update $\boldsymbol{X}^{i+1} = \boldsymbol{V}^i_{(J_0)} + \mathcal{H}_{k-r}(\boldsymbol{V}^i_{(\bar{J}_0)})$. Also define $U^i = \text{supp}(\mathcal{H}_{k-r}(\boldsymbol{V}^i_{(\bar{J}_0)}))$. It can be easily checked that $|U^i| \leq k - r$, as described in [?].

Now, we can write the following inequality:

$$\|\boldsymbol{V}^i - \boldsymbol{X}^{i+1}\|_F^2 = \|\boldsymbol{V}^i_{(J_0)} - \boldsymbol{X}^{i+1}_{(J_0)}\|_F^2 + \|\boldsymbol{V}^i_{(\bar{J}_0)} - \boldsymbol{X}^{i+1}_{(\bar{J}_0)}\|_F^2, \tag{14}$$
$$\leq \|\boldsymbol{V}^i_{(J_0)} - \boldsymbol{X}_{(J_0)}\|_F^2 + \|\boldsymbol{V}^i_{(\bar{J}_0)} - \boldsymbol{X}_{(J_1)}\|_F^2, \tag{15}$$
$$= \|\boldsymbol{V}^i - \boldsymbol{X}_{(J)}\|_F^2, \tag{16}$$

since $\boldsymbol{V}^i_{(J_0)} = \boldsymbol{X}^{i+1}_{(J_0)}$ and $\boldsymbol{X}^{i+1}_{(\bar{J}_0)}$ is the best (k-r)-term approximation of $\boldsymbol{V}^i_{(\bar{J}_0)}$. Now, by expanding (16) using the Frobenius inner product and bounding the real part of the inner product by its magnitude, the following inequality holds:

$$\|\boldsymbol{X}_{(J)} - \boldsymbol{X}^{j+1}\|_F^2 \leq 2|\langle \boldsymbol{V}^i - \boldsymbol{X}_{(J)}, \boldsymbol{X}_{(J)} - \boldsymbol{X}^{i+1}\rangle|, \tag{17}$$

where $\langle\cdot,\cdot\rangle$ denotes the Frobenius inner-product. In addition, we use (13) to write that:

$$\boldsymbol{V}^i - \boldsymbol{X}_{(J)} = (\boldsymbol{I} - \omega^i \boldsymbol{A}^*\boldsymbol{A})(\boldsymbol{X}^i - \boldsymbol{X}_{(J)}) + \omega^i \boldsymbol{A}^*\tilde{\boldsymbol{E}}, \tag{18}$$

which can be injected in (17) to deduce that

$$\|\boldsymbol{X}_{(J)} - \boldsymbol{X}^{i+1}\|_F^2 \leq 2\omega^i|\langle \tilde{\boldsymbol{E}}, \boldsymbol{A}(\boldsymbol{X}^{i+1} - \boldsymbol{X}_{(J)})\rangle| + 2|\langle(\boldsymbol{I} - \omega^i \boldsymbol{A}_Q^*\boldsymbol{A}_Q)(\boldsymbol{X}^i - \boldsymbol{X}_{(J)}), (\boldsymbol{X}^{i+1} - \boldsymbol{X}_{(J)})\rangle|, \tag{19}$$

where $Q = J \cup J^i \cup J^{i+1}$ has a cardinality bounded by

$$|Q| = |J_0 \cup J_1 \cup U^i \cup U^{i+1}| \leq 3k - 2r \leq ck, \tag{20}$$

where $c \in \mathbb{N}$ such that $ck \geq 3k - 2r$. Now, using Lemma 5 of [?], we can state that if $\omega^i \leq \frac{1}{1-L_k}$, the following inequality holds

$$|\langle(\boldsymbol{I} - \omega^i \boldsymbol{A}_Q^*\boldsymbol{A}_Q)(\boldsymbol{X}^i - \boldsymbol{X}_{(J)}), (\boldsymbol{X}^{i+1} - \boldsymbol{X}_{(J)})\rangle| \leq \varphi(ck)\|\boldsymbol{X}^i - \boldsymbol{X}_{(J)}\|_F\|\boldsymbol{X}^{i+1} - \boldsymbol{X}_{(J)}\|_F \tag{21}$$

where $\varphi(ck) = \frac{U_{ck}+L_{ck}}{1-L_k}$.

In addition, we can bound the first term of (19) as:

$$|\langle \tilde{\boldsymbol{E}}, \boldsymbol{A}(\boldsymbol{X}^{i+1} - \boldsymbol{X}_{(J)})\rangle| \leq \sqrt{1 + U_{dk}}\|\tilde{\boldsymbol{E}}\|_F\|\boldsymbol{X}^{i+1} - \boldsymbol{X}_{(J)}\|_F, \tag{22}$$

since $\text{supp}(\boldsymbol{X}^{i+1} - \boldsymbol{X}_{(J)}) = J \cup U^{i+1}$ has its cardinality bounded by $2k - r \leq dk$, with $d \in \mathbb{N}$.

With (19), (21), (22) and Lemma 2 of [?], we can write

$$\|\boldsymbol{X}_{(J)} - \boldsymbol{X}^{i+1}\|_F \leq \alpha^i\|\boldsymbol{X}_{(J)}\|_F + \frac{\beta}{1-\alpha}\|\tilde{\boldsymbol{E}}\|_F, \tag{23}$$

where $\alpha = 2\varphi(ck) < 1$ and $\beta = 2\frac{\sqrt{1+U_{dk}}}{1-L_k}$ since $\omega^i \leq \frac{1}{1-L_k}$ (because of (21)).

Appendix B
Empirical Validation of Theorem 2

We propose an empirical validation of Theorem 2 using MUSIC and MUSIC-PKS algorithms.

The signal matrix $\boldsymbol{X} \in \mathbb{R}^{n\times N}$ is designed with $n = 64$, $N = 128$, $\text{supp}\,(\boldsymbol{X}) = J_0 \cup J_1$, such that $|J_0| = |J_1| = 8$ and J_0 is known *a priori*.

We consider a Gaussian i.i.d. measurement matrix $\boldsymbol{A} \in \mathbb{R}^{m\times n}$, with $\boldsymbol{A}_{ij} \sim \mathcal{N}(0,1)$ such that $\text{rank}\,(\boldsymbol{A}) = m$ and $\text{spark}\,(\boldsymbol{A}) = m{+}1$ with probability 1 [?], [?]. The measurements are computed as $\boldsymbol{Y} = \boldsymbol{A}\boldsymbol{X}$.

In a first experiment, we force $\text{rank}\left(\boldsymbol{X}_{(J_0)}\right) = 1$ and $\text{rank}\left(\boldsymbol{X}_{(J_1)}\right) = |J_1|$ such that $\text{rank}\,(\boldsymbol{Y}) = |J_1| + 1$ when $m > k$. We are in a rank-defective case in which MUSIC procedure fails. However, when $m > k$, $\text{rank}\left(\left[\boldsymbol{Y}, \boldsymbol{A}_{J_0}\right]\right) = k$ and we are in the ideal case where $\mathcal{R}\left(\boldsymbol{A}_{J_0}\right)$ augments the signal subspace $\mathcal{R}\,(\boldsymbol{Y})$ such that MUSIC-PKS succeeds.

In a second experiment, we force $\text{rank}\left(\boldsymbol{X}_{(J_0)}\right) = |J_0|$ and $\text{rank}\left(\boldsymbol{X}_{(J_1)}\right) = 1$ in such a way that we are in the worst case scenario for MUSIC-PKS since $\mathcal{R}\left(\boldsymbol{A}_{J_0}\right) \subset \mathcal{R}\,(\boldsymbol{Y})$. In this case, MUSIC-PKS does not perform better than MUSIC.

Fig 4 displays the average recovery probability, computed as the rate of successful support recovery over 1000 random trials of the algorithms.

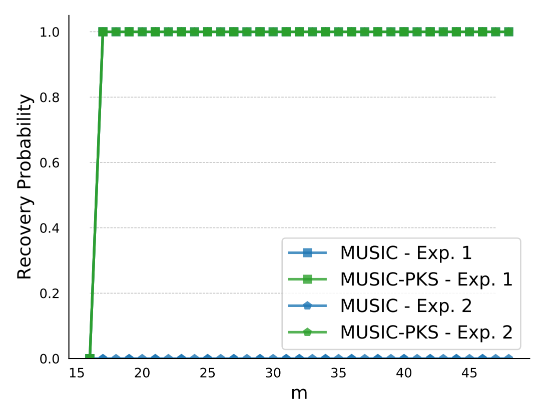

Fig. 4. Recovery probability of MUSIC and MUSIC-PKS when $\text{rank}(\boldsymbol{X}_{(J_0)}) = |J_0|$ (Exp. 1) and when $\text{rank}(\boldsymbol{X}_{(J_0)}) = 1$ (Exp. 2).

For the first experiment, we observe that MUSIC-PKS recovers the support of the signal for $m \geq k + 1 = 17$ which exactly corresponds to the case where the augmented matrix has full rank, as stated in Theorem 2. Concerning the second experiment, both MUSIC and MUSIC-PKS fail as expected.