Joint Sparsity with Partially Known Support and Application to Ultrasound Imaging

Adrien Besson, *Student Member, IEEE,* Dimitris Perdios, *Student Member, IEEE,* Yves Wiaux, and Jean-Philippe Thiran, *Senior Member, IEEE,*

***Abstract*—We investigate the benefits of known partial support for the recovery of joint-sparse signals and demonstrate that it is advantageous in terms of recovery performance for both rank-blind and rank-aware algorithms. We suggest extensions of several joint-sparse recovery algorithms, e.g. simultaneous normalized iterative hard thresholding, subspace greedy methods and subspace-augmented multiple signal classification techniques. We describe a direct application of the proposed methods for compressive multiplexing of ultrasound (US) signals. The technique exploits the compressive multiplexer architecture for signal compression and relies on joint-sparsity of US signals in the frequency domain for signal reconstruction. We validate the proposed algorithms on numerical experiments and show their superiority against state-of-the-art approaches in rank-defective cases. We also demonstrate that the techniques lead to a significant increase of the image quality on *in vivo* carotid images compared to reconstruction without partially known support. The supporting code is available on https://github.com/AdriBesson/spl2018_joint_sparse.**

***Index Terms*—Compressed sensing, ultrasound, greedy algorithms, joint sparsity, MUSIC.**

I. Introduction

COMPRESSED sensing (CS) [1], [3] aims to solve a single measurement vector (SMV) problem where one would like to retrieve a k-sparse vector $\boldsymbol{x} \in \Sigma_k$ from measurements $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x} \in \mathbb{K}^m$, where $\mathbb{K}$ denotes a scalar field, e.g. $\mathbb{R}$ or $\mathbb{C}$, $\boldsymbol{A} \in \mathbb{K}^{m\times n}$, $\Sigma_k = \{\boldsymbol{x} \in \mathbb{K}^n \mid |\mathrm{supp}(\boldsymbol{x})| \leq k\}$ and $\mathrm{supp}\,(\boldsymbol{x}) = \{i \in \{1, \ldots, n\} \mid x_i \neq 0\}$.

Distributed CS extends CS to the multiple measurement vectors (MMV) problem [4], [5] whose purpose is to recover multiple sparse vectors $\boldsymbol{X} = [\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_N] \in \mathbb{K}^{n\times N}$ from measurements $\boldsymbol{Y} = \boldsymbol{A}\boldsymbol{X} \in \mathbb{K}^{m\times N}$ [6]. Under the assumption that the signals $\boldsymbol{x}_i$, $i = 1, \ldots, N$, share the same support (*JSM-2* model in [6]), the MMV problem can be written as

$$\min_{\boldsymbol{X}\in\mathbb{K}^{n\times N}} \|\boldsymbol{X}\|_{0,\mathrm{row}} \text{ subject to } \boldsymbol{Y} = \boldsymbol{A}\boldsymbol{X}, \tag{1}$$

where $\|\boldsymbol{X}\|_{0,\mathrm{row}}$ counts the number of non-zero rows of $\boldsymbol{X}$. Many techniques have been introduced to tackle the MMV problem. A first group exploits the multiple signal classification (MUSIC) algorithm, popular in array signal processing. Indeed, Feng and Bresler [7] demonstrate that $\boldsymbol{X}$ can be retrieved using a singular value decomposition of $\boldsymbol{Y}$ in the full-rank case. Extensions of the MUSIC algorithm to rank-defective cases have been proposed such as subspace-augmented MUSIC (SA-MUSIC) [8], compressive MUSIC (CS-MUSIC) [10] and semi-supervised MUSIC [22]. The second group of techniques extend algorithms developed for standard CS to the MMV case. Mixed-norm algorithms exploit extensions of ℓ_1-minimization algorithms [11], [26], [27]. Several greedy algorithms have also been extended leading to simultaneous orthogonal matching pursuit [12], [28], simultaneous normalized hard thresholding pursuit [14], [13], simultaneous compressive sampling matching pursuit [13] and simultaneous normalized iterative hard thresholding (SNIHT) [13]. In [17], Davies and Eldar introduce the rank-aware orthogonal recursive matching pursuit (RA-ORMP), a greedy method which exploits the rank information of $\boldsymbol{X}$. Lee *et al.* [8] propose the orthogonal subspace matching pursuit (OSMP), very similar to RA-ORMP.

CS with partially known support consists in injecting a prior knowledge of the support of the unknown signal into the CS problem resulting in weaker conditions than standard CS. The concept has been introduced independently by Vaswani and Lu [23], von Borries *et al.* [44] and Khajehnejad *et al.* [45], and extended by Jacques [24]. Carrillo *et al.* [16], [15] have also suggested extensions of various greedy algorithms.

In this work, we propose to study the benefits of known partial support on the performance of joint-sparse recovery algorithms. In Section II, we present uniqueness conditions for the solution of Problem (1) in case of partially known support. We also propose extensions of several algorithms, i.e. SNIHT, RA-ORMP, OSMP and subspace augmented MUSIC methods which are validated on numerical experiments. In Section III, we show an application of the proposed algorithms to the recovery of ultrasound (US) signals from multiplexed measurements. Concluding remarks are given in Section IV.

A. Besson, D. Perdios and J.-Ph. Thiran are with the Signal Processing Laboratory 5 (LTS5), Ecole Polytechnique Fédérale de Lausanne, CH-1015, Lausanne, Switzerland. Y. Wiaux is with the Institute of Sensors, Signals and Systems, Heriot-Watt University, EH14 4AS Edinburgh, United-Kingdom. J.-Ph. Thiran is also with the Department of Radiology, University Hospital Center (CHUV) and University of Lausanne (UNIL), CH-1011, Lausanne, Switzerland. This work was supported in part by the UltrasoundToGo RTD project (no. 20NA21_145911) funded by Nano-Tera.ch with Swiss Confederation financing. This work was also supported by the Swiss SNF project number 205320_175974.

II. Joint Sparsity with Partially Known Support

A. Notation

Given a space $\mathcal{I} \subset \mathbb{K}^N$, $\mathbf{P}_{\mathcal{I}}$ and $\mathbf{P}_{\mathcal{I}}^{\perp}$ define the projectors onto $\mathcal{I}$ and its orthogonal complement $\mathcal{I}^{\perp}$. Similarly, given a set of integers $J \subset \{1, \ldots, n\}$, $\bar{J} = \{1, \ldots, n\} \setminus J$ denotes its complement and $|J|$ its cardinality. The Hermitian transpose of a matrix $\boldsymbol{X} \in \mathbb{K}^{n\times n}$ is denoted by $\boldsymbol{X}^*$. $\|\boldsymbol{X}\|_F$ is the Frobenius norm of $\boldsymbol{X}$. $\boldsymbol{X}_{J_0} \in \mathbb{K}^{n\times|J_0|}$ is the sub-matrix formed by the columns of $\boldsymbol{X}$ indexed by J_0. $\boldsymbol{X}_{(J_0)} \in \mathbb{K}^{n\times n}$ is the matrix $\boldsymbol{X}$ restricted to the rows indexed by J_0. Hence, all the rows with entries indexed by J_0 are unchanged while the others are set to 0. The space spanned by the columns of $\boldsymbol{X}$ is defined by $\mathcal{R}(\boldsymbol{X})$. The rank of $\boldsymbol{X}$ is designated by $\mathrm{rank}(\boldsymbol{X})$ and $\mathrm{spark}(\boldsymbol{X})$ defines its spark i.e. the smallest number of columns from $\boldsymbol{X}$ that are linearly dependent. We use $\mathrm{supp}(\boldsymbol{X})$ as the row-support of $\boldsymbol{X}$ and $\Sigma_k^{(n,N)}$ as the set of k-row-sparse matrices of $\mathbb{K}^{n\times N}$. We also refer the reader to the definition of the upper and lower asymmetric restricted isometry (ARIP) constants of order k [13], denoted as U_k and L_k and whose definitions are given in supplementary material of this work. In the remainder, we are interested in recovering $\boldsymbol{X} \in \Sigma_k^{(n,N)}$ such that $\mathrm{supp}(\boldsymbol{X}) = J = J_0 \cup J_1$, with $J_0 \subset \{1,\dots,n\}$ and $J_1 \subset \bar{J}_0$, from measurements $\boldsymbol{Y} = \boldsymbol{A}\boldsymbol{X}$, with $\boldsymbol{A} \in \mathbb{K}^{m\times n}$. We assume that J_0 is known *a priori*.

B. Uniqueness of the ℓ_0-norm Minimization

In this section, we extend the uniqueness condition derived by Vaswani and Lu [23] to the MMV problem. The objective is to establish guarantees of uniqueness of the solutions in the case of MMV problems with partially known support that are weaker than the ones for standard MMV problems [4], [5]. In order to do that, we reformulate the problem with partially known support as:

$$\min_{\boldsymbol{X}\in\mathbb{K}^{n\times N}} \|\boldsymbol{X}_{(\bar{J}_0)}\|_{0,\mathrm{row}} \text{ subject to } \boldsymbol{Y} = \boldsymbol{A}\boldsymbol{X}. \tag{2}$$

The following theorem provides a guarantee for uniqueness of the solution of (2) expressed in terms of an upper bound on the row sparsity k.

Theorem 1. *The matrix $\boldsymbol{X} \in \Sigma_k^{(n,N)}$, with $\mathrm{supp}(\boldsymbol{X}) = J_0 \cup J_1$, J_0 known, is the unique solution of (2), if $\boldsymbol{Y} = \boldsymbol{A}\boldsymbol{X}$ and*

$$k < \frac{\mathrm{spark}(\boldsymbol{A}) + |J_0|}{2}. \tag{3}$$

Proof. The proof follows by contradiction. Define $\boldsymbol{X}^1, \boldsymbol{X}^2 \in \Sigma_k^{(n,N)}$ such that $\boldsymbol{X}^1 \neq \boldsymbol{X}^2$ and both are solutions of (2). Consider that the rows of $\boldsymbol{X}^1$ (resp. $\boldsymbol{X}^2$) are supported on $J_0 \cup \Delta_1$ (resp. $J_0 \cup \Delta_2$) such that $|\Delta_1| = |\Delta_2| = u \leq k - |J_0|$. The rows of $\boldsymbol{X}^1 - \boldsymbol{X}^2$ are supported on $J_0 \cup \Delta_1 \cup \Delta_2$ and the sub-matrix defined by the rows indexed by $J_0 \cup \Delta_1 \cup \Delta_2$ belongs to the null-space of $\boldsymbol{A}_{J_0\cup\Delta_1\cup\Delta_2}$. When $\mathrm{spark}(\boldsymbol{A}) > |J_0 \cup \Delta_1 \cup \Delta_2|$, $\boldsymbol{A}_{J_0\cup\Delta_1\cup\Delta_2}$ has full column rank and its null-space is trivial. If (3) holds, then $\mathrm{spark}(\boldsymbol{A}) > 2k - |J_0| \geq k - 2u \geq |J_0 \cup \Delta_1 \cup \Delta_2|$ and $\boldsymbol{X}^1 = \boldsymbol{X}^2$.

□

Theorem 1 is an extension to the MMV problem of Proposition 1 of [23] and the upper bound is the same as for the SMV problem. At this point, it would be beneficial to combine the information on $\mathrm{rank}(\boldsymbol{Y})$ and the partially known support to relax the uniqueness condition provided for rank-aware algorithms [4], [7], [17]. We first remind the following lemma.

Lemma 1 (Theorem 2.4 of [4])**.** *The matrix $\boldsymbol{X} \in \Sigma_k^{(n,N)}$ is the unique solution of (1), if $\boldsymbol{Y} = \boldsymbol{A}\boldsymbol{X}$ and*

$$k < \frac{\mathrm{spark}(\boldsymbol{A}) + \mathrm{rank}(\boldsymbol{Y}) - 1}{2}. \tag{4}$$

We can now state the main claim of the section.

Theorem 2. *The matrix $\boldsymbol{X} \in \Sigma_k^{(n,N)}$, with $\mathrm{supp}(\boldsymbol{X}) = J_0 \cup J_1$, J_0 known, is the unique solution of (2), if $\boldsymbol{Y} = \boldsymbol{A}\boldsymbol{X}$ and*

$$k < \frac{\mathrm{spark}(\boldsymbol{A}) + \mathrm{rank}(\boldsymbol{Y}_a) - 1}{2}, \tag{5}$$

where $\boldsymbol{Y}_a = [\boldsymbol{Y}, \boldsymbol{A}_{J_0}]$.

Proof. Consider $\boldsymbol{Y}_a = [\boldsymbol{Y}, \boldsymbol{A}_{J_0}]$. We define the augmented signal matrix $\boldsymbol{X}_a = [\boldsymbol{X}, \boldsymbol{I}_{J_0}]$, where $\boldsymbol{I} \in \mathbb{R}^{n\times n}$ is the identity matrix, such that $\boldsymbol{Y}_a = \boldsymbol{A}\boldsymbol{X}_a$. Define the following augmented MMV problem:

$$\min_{\boldsymbol{X}\in\mathbb{R}^{n\times(N+|J_0|)}} \|\boldsymbol{X}\|_{0,\mathrm{row}} \text{ subject to } \boldsymbol{Y}_a = \boldsymbol{A}\boldsymbol{X}. \tag{6}$$

By applying Lemma 1, $\boldsymbol{X}_a$ is the unique solution of (6) if the inequality (4) holds. Now, we show that if (6) has a unique solution, then (2) has a unique solution. Define $\boldsymbol{X}^1, \boldsymbol{X}^2 \in \Sigma_k^{(n,N)}$ such that $\boldsymbol{X}^1 \neq \boldsymbol{X}^2$ and both are solutions of (2). The augmented matrices $\boldsymbol{X}_a^1$ and $\boldsymbol{X}_a^2$ are both solutions of (6) which is impossible since $\boldsymbol{X}_a$ is unique. □

Theorem 2 can be interpreted in terms of subspace augmentation discussed in Proposition 5.4 of [8]. Indeed, the partially known support J_0 is used to augment the signal subspace $\mathcal{R}(\boldsymbol{Y})$ with basis vectors of $\mathcal{R}(\boldsymbol{A}_{J_0})$. Thus, it is advantageous when some of the basis vectors of $\mathcal{R}(\boldsymbol{A}_{J_0})$ are orthogonal to $\mathcal{R}(\boldsymbol{Y})$. Now that we have established uniqueness conditions, we propose extensions of existing joint sparse algorithms to partially known support.

C. Greedy Methods with Partially Known Support

1) RA-ORMP and OSMP: RA-ORMP and OSMP are very similar as explained in [8] and the argument detailed below may be applied to both algorithms. For conciseness, we focus on RA-ORMP in the remainder. The partially known support can be exploited in the initialization step of the RA-ORMP algorithm [17]. The idea is to consider J_0 as the initial support in the algorithm and perform the following initialization:

$$\boldsymbol{R}^0 = \mathbf{P}^{\perp}_{\mathcal{R}(\boldsymbol{A}_{J_0})}\boldsymbol{Y} \tag{7}$$

$$\boldsymbol{\Phi}' = \mathbf{P}^{\perp}_{\mathcal{R}(\boldsymbol{A}_{J_0})}\boldsymbol{A}, \tilde{\boldsymbol{\Phi}}'_n = \boldsymbol{\Phi}'_n/\|\boldsymbol{\Phi}'_n\|_2, \ \forall n \notin J_0, \tag{8}$$

where $\boldsymbol{R}^0$ is the residual and the notations $\boldsymbol{\Phi}'$ and $\tilde{\boldsymbol{\Phi}}'_n$, which account for the projected measurement matrix, are used to be consistent with [17]. RA-ORMP initalized with the above steps is denoted as RA-ORMP-PKS. The remaining steps of RA-RORMP-PKS are the same as RA-ORMP (Algorithm 3 of [17]) and aim to recover the unknown support J_1. Regarding the recovery, RA-ORMP-PKS is guaranteed to recover $\boldsymbol{X}$ from $\boldsymbol{Y}$ in the noiseless case provided that $\mathrm{rank}(\boldsymbol{Y}_a) = k$ [17].

2) SNIHT: SNIHT proposes an extension of the iterative hard thresholding (IHT) algorithm [18] to the MMV problem [13]. Based on the extension of IHT to partially known support [15], we suggest SNIHT-PKS in which steps $7-9$ of SNIHT (Algorithm 1 in [13]) are replaced by the following:

$$\boldsymbol{X}^i = \mathcal{H}^{J_0}_{k-|J_0|}(\boldsymbol{X}^{i-1} + \omega(\boldsymbol{A}^*\boldsymbol{R}^{i-1})), \tag{9}$$

where $\boldsymbol{R}^{i-1}$ are the residuals at iteration $i-1$, $\omega \in \mathbb{R}$ and the non-linear operator $\mathcal{H}_k^J(\cdot)$ is defined for an index set $J \subset \{1,\dots,n\}$ as

$$\mathcal{H}_k^J(\boldsymbol{X}) = \boldsymbol{X}_{(J)} + \mathcal{H}_k(\boldsymbol{X}_{(\bar{J})}), \tag{10}$$

where $\mathcal{H}_k(\boldsymbol{X})$ is the hard-thresholding operator which selects the k rows of $\boldsymbol{X}$ with largest ℓ_2-norm. We state the main result of this section, which provides an upper bound on the discrepancy between the output of SNIHT-PKS and the optimal row-sparse approximation of the solution.

Theorem 3 (Simultaneous Sparse Approximation with Partially Known Support)**.** *Consider that* $\boldsymbol{Y} = \boldsymbol{A}\boldsymbol{X}_{(J_0 \cup J_1)} + \tilde{\boldsymbol{E}}$*. If* $\boldsymbol{A}$ *satisfies the following ARIP conditions:* $2U_{ck} + 2L_{ck} + L_k < 1$ *where* $c \in \mathbb{N}$ *is such that* $ck \geq 3k - 2|J_0|$*, then the error of SNIHT-PKS at iteration* i *is bounded by:*

$$\|\boldsymbol{X}^i - \boldsymbol{X}_{(J)}\|_F \leq \alpha^i \|\boldsymbol{X}_{(J)}\|_F + \frac{\beta}{1-\alpha}\|\tilde{\boldsymbol{E}}\|_F, \tag{11}$$

where $\alpha = 2\frac{U_{ck}+L_{ck}}{1-L_k}$*,* $\beta = 2\frac{\sqrt{1-U_{dk}}}{1-L_k}$ *and* $d \in \mathbb{N}$ *is such that* $dk \geq 2k - |J_0|$*.*

The proof is given in the supplementary material of the proposed work. It can be seen that the results are closed to the one obtained by Carillo *et al.* [15] for the SMV case, in which the matrix $\boldsymbol{A}$ has to meet the RIP property of order $3k - 2|J_0|$. In addition, the ARIP conditions provided by Theorem 3 are weaker than the ones of SNIHT, which can be translated into fewer measurements necessary to fulfill (11). However, compared to the bound established in Theorem 1, SNIHT-PKS requires $\boldsymbol{A}$ to be ck-RIP which is stronger than $\operatorname{spark}(\boldsymbol{A}) > 2k - |J_0|$.

D. MUSIC-based Methods with Partially Known Support

MUSIC-based algorithms exploit additional information provided by the signal subspace to help the recovery of $\boldsymbol{X}$ [7]. In the case of partially known support, we rely on the augmented measurement matrix $\boldsymbol{Y}_a$ rather than $\boldsymbol{Y}$ and we use the following criterion to identify $\operatorname{supp}(\boldsymbol{X})$: $\forall j \in \bar{J}_0$, $j \in J_1$ if and only if $\boldsymbol{Q}_a^* \boldsymbol{A}_j = 0$ and $\operatorname{rank}(\boldsymbol{Y}_a) = k$, where $\boldsymbol{Q}_a \in \mathbb{R}^{m \times (m-k)}$ is an orthonormal basis of $\mathcal{R}(\boldsymbol{Y}_a)^\perp$.

In the rank-defective case where $\operatorname{rank}(\boldsymbol{Y}_a) < k$, we first identify $k - \operatorname{rank}(\boldsymbol{Y})$ components of $\operatorname{supp}(\boldsymbol{X})$ using a greedy algorithm initialized with the partially known support. The remaining $\operatorname{rank}(\boldsymbol{Y})$ components either come from the signal subspace (SA-MUSIC [8]) or are identified based on a generalized MUSIC criterion (CS-MUSIC [10]). We use the partially known support to initialize the greedy algorithm since the success of subspace augmented methods relies on the successful partial support recovery of the greedy algorithm and it is known that forward selection approaches perform far better when smaller subsets have to be recovered [8].

E. Validation on Numerical Experiments

We explore the empirical performance of SA-MUSIC-PKS, RA-ORMP-PKS and SNIHT-PKS in a noiseless situation and under additive Gaussian noise with a signal-to-noise ratio of 30 dB. We consider a Gaussian random measurement matrix $\boldsymbol{A} \in \mathbb{R}^{m \times n}$, with $\boldsymbol{A}_{ij} \sim \mathcal{N}(0, 1/\sqrt{m})$ and n is fixed to 128. The signal matrix $\boldsymbol{X} \in \mathbb{R}^{n \times N}$ is built as a random matrix, with $N = k = 30$. 1000 random trials of the algorithms are run for each experiment and the recovery probability is computed as the rate of successful support recovery.

The impact of the partially known support is first analyzed by comparing the recovery probability of SA-MUSIC-PKS for a fixed rank ($s = 10$), for a number of measurements ranging between 30 and 90, when 0 %, 25 %, 50 % and 75 % of the support is known *a priori*. Then, we compare the methods with their counterpart without known support on two experiments: fixed rank ($s = 10$) for a number of measurements ranging between 30 and 90 and fixed number of measurements ($m = 51$) for a rank varying between 1 and 30. For both experiments, 75 % of the support is assumed to be known.

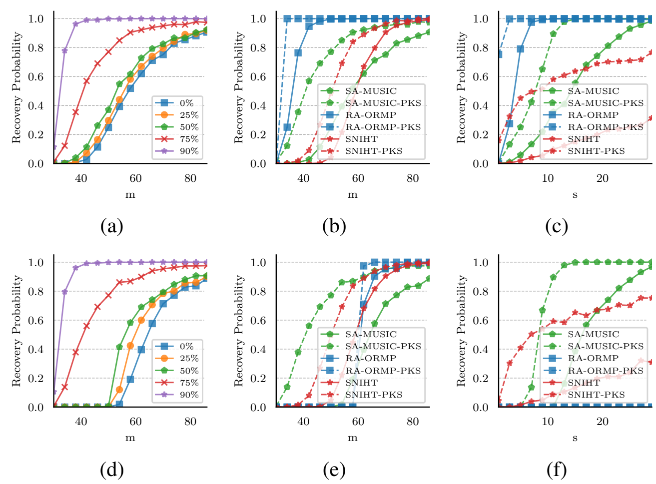


Fig. 1. (a) and (d) Recovery probability of SA-MUSIC-PKS for varying number of measurements and size of the known support. (b)-(c)-(e)-(f) Recovery probability of SA-MUSIC-PKS, RA-ORMP-PKS and SNIHT-PKS against SA-MUSIC, RA-ORMP and SNIHT for varying number of measurements ((b) and (e)) and varying ranks ((c) and (f)) in a noiseless ((b) and (c)) and in a noisy scenario ((e) and (f)).

On Fig. 1(a) and Fig. 1(d), we can see that SA-MUSIC-PKS is more accurate when larger part of the support is known for both the noiseless and the noisy cases, as expected. On Fig. 1(b), we observe that the methods with partially known support achieve significantly better reconstruction than their counterpart without known support in a noiseless scenario which validates the main results of Section II. On Fig. 1(e), we see that the conclusions drawn for the noiseless scenario extend to the noisy scenario for SA-MUSIC and SNIHT. Regarding RA-ORMP, we observe that the results in the noisy scenario are significantly worse than in the noiseless scenario. Indeed, $\boldsymbol{Y}$ is now full rank which perturbates the correlation step in the subspace pursuit (widely studied in the SMV problem [25]). In this case, RA-ORMP-PKS is only slightly better than RA-ORMP since the algorithm fails to recover the unknown part of the support. Figs. 1(c) and 1(f) show the benefits of partially known support in terms of the minimal value of s for perfect support recovery. As for Fig. 1(b) and 1(e), we notice that the partially known support significantly helps the recovery of the different algorithms except for RA-ORMP in the noisy case. Experiments dedicated to the empirical validation of Theorem 2 are described in the supplementary material.

III. Multiplexing of Ultrasound Signals

A. Proposed Approach

High-quality 3D US imaging necessitates a US probe of thousands of transducer elements. Connecting such a probe to the back-end system would require as many cables as the number of elements which is either unfeasible or prohibitively expensive. To address this issue, sparse array techniques have been investigated [30]. Many layouts have been designed, e.g. random sparse arrays [31], [32], Vernier arrays [33] and row-column addressed arrays [36], [34], [35]. Alternatively, micro-beamforming methods, where part of the imaging process is achieved in the head of the probe [39], and time multiplexing techniques have been investigated for both dense and sparse arrays [39], [41], [40]. While proposing a drastic reduction on the number of elements, such methods come with a degraded image quality [30].

In this section, we describe a direct application of the proposed algorithms for compressive multiplexing of US signals. More precisely, we propose to exploit the compressive multiplexer (CMUX) [19] architecture to reduce both the number of coaxial cables and the number of analog-to-digital converters (ADC) in the back-end system. We consider a US probe made of N transducer elements which receive signals as backscattered echoes from a previously insonified medium, at a rate Ω during a time T. The set of those signals is denoted as element raw-data and stored as $\boldsymbol{M} \in \mathbb{R}^{n \times N}$, where $n = T\Omega$. In the proposed architecture, described in Figure 2, we equip the head of the probe with N_c CMUX, each of which working at Ω and compressing N_t signals, with $N = N_t N_c$. Thus, one may require only $N_c \ll N$ coaxial cables connecting the probe to the back-end system and only $N_c \ll N$ ADCs. Formally, the measurements have the following form: $\boldsymbol{Y} = \mathcal{A}(\boldsymbol{M}) + \boldsymbol{E}$, where $\mathcal{A} : \mathbb{R}^{n \times N_t} \to \mathbb{R}^{n \times N_c}$ is the linear operator associated with the CMUX architecture [19] and $\boldsymbol{E} \in \mathbb{R}^{n \times N_c}$ is the noise.

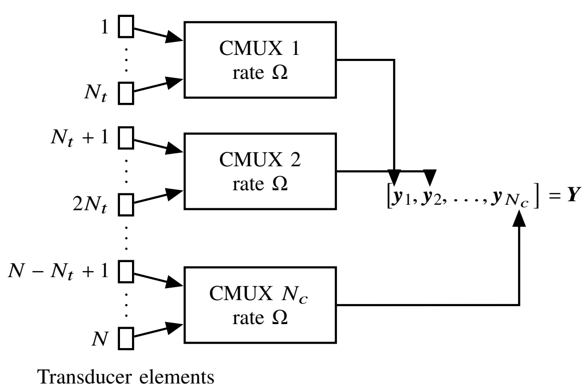


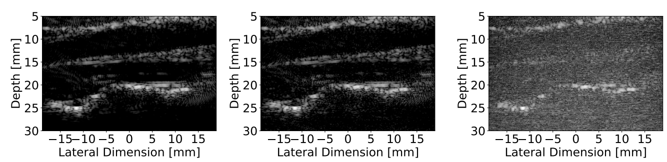
Fig. 2. Ultrasound compressive multiplexer architecture.

US signals are known to have a relatively sparse spectrum [20], [21] due to the bandpass properties of transducer elements and the high sampling frequency required for delay resolution in delay-and-sum (DAS) beamforming [29]. In addition, we usually have a partial knowledge of such a spectrum which is measured by probe manufacturers. Thus, we are in a scenario where joint sparse algorithms with partially known support can be leveraged. Puts formally, we introduce the 1D discrete Fourier transform $\boldsymbol{F} \in \mathbb{C}^{n \times n}$ and the associated Fourier coefficients $\hat{\boldsymbol{M}} = \boldsymbol{F}\boldsymbol{M}$ such that $\operatorname{supp}(\hat{\boldsymbol{M}}) = J_0 \cup J_1$, where J_0 is the known part of the spectrum, $|\operatorname{supp}(\hat{\boldsymbol{M}})| \ll n$, and we solve the following joint-sparse regularization problem:

$$\min_{\hat{\boldsymbol{M}} \in \mathbb{C}^{n \times N}} \|\hat{\boldsymbol{M}}\|_{0,\text{row}} \text{ subject to } \boldsymbol{Y} = \mathcal{A}(\boldsymbol{F}^* \hat{\boldsymbol{M}}) + \boldsymbol{E}. \tag{12}$$

B. In Vivo Ultrasound Signals

US signals from *in vivo* carotids have been acquired using a Verasonics Vantage 256™ equipped with the ATL-L7-4 probe (128 el., 5.2 MHz center freq., 60 % bandwidth). The CMUX architecture is simulated off-line using Python and works at 62.5 MHz, with a multiplexing ratio 1/8. On the reconstruction side, we use SNIHT-PKS with 500 iterations. The reasons for the choice of SNIHT-PKS are the high rank-deficiency which motivates the use of rank-blind algorithms; the robustness to noise of SNIHT-PKS (Theorem 3); and the high dimensionality of the data which prevents us from using pseudo-inverses or EVD, necessary for RA-ORMP and SA-MUSIC. We assume that J_0 contains the indices of the frequencies lying between 2.9 MHz and 6 MHz which corresponds to 85 % of the signal energy. DAS beamforming is applied on the recovered US signals followed by envelope detection, normalization and log-compression. The reference B-mode image is displayed in Fig. 3(a). The images corresponding to the reconstructions with SNIHT-PKS and SNIHT are given in Fig. 3(b) and Fig. 3(c).



(a) Reference (b) SNIHT-PKS (c) SNIHT

Fig. 3. Log-compressed B-mode images (40 dB) of an *in-vivo* carotid. (a)-Reference. (b)-SNIHT-PKS (PSNR = 45 dB). (c)-SNIHT (PSNR = 28 dB).

Both visual assessment and values of the peak-signal-to-noise-ratio (PSNR), computed on the B-mode image and reported in the caption of Fig. 3, show the superior quality of the reconstruction with SNIHT-PKS compared to SNIHT.

IV. Conclusion

We investigate the recovery of jointly sparse vectors when partial support is known. We quantify the benefits of the known support in terms of a higher upper bound on the row-sparsity than standard MMV problems. We also suggest adaptations of greedy algorithms as well as MUSIC-based methods to incorporate the additional information. We apply the proposed algorithms to the recovery of ultrasound signals from compressed measurements, where the objective is to multiplex signals in order to reduce the number of coaxial cables and ADCs. By exploiting the prior knowledge on the frequency support of the signals, we demonstrate that the proposed algorithms significantly outperform the standard MMV ones.

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