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AN ALTERNATE STATE VECTOR FORMULATION FOR THERMO-ELASTIC DEFORMATION ANALYSIS OF SANDWICH PANELS

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1. INTRODUCTION

Mathematical models based on exact thermo-elasticity equations and their solutions act as benchmark for analytical, semi-analytical and numerical solutions based on approximate models. Analytical methods used for solving the exact thermo-elastic models for advanced layered, multi-functional panels, including sandwich construction, should have the capability to incorporate any number of layers with arbitrary material properties (structural, functional, and/or multiple fields like elastic, thermal, electro-magnetic, etc.) and thickness. Compared to other analytical methods [1], the state space method [2, 3], based on the idea of converting boundary value problems to equivalent initial value problems, has been shown to be very efficient for the analysis of layered panels, including multi-field problems [4, 5].

In linear un-coupled thermo-elasticity, the conventional procedure followed by most analytical methods is to solve the elastic and thermal and fields separately [6, 7]. The same procedure has also been implemented in the state space formulation for obtaining analytical solutions to single layer [5] and sandwich panels [8].

In this paper, an alternate state vector formulation is presented in which an augmented state vector is defined including the displacements, transverse stresses, temperature and transverse heat flux. The present formulation is a more general method with advantages like provision for fully coupled response, ability to incorporate multiple edge boundary conditions and loading simultaneously. Thus, all the state variables can be obtained simultaneously by solving a single set of vector differential equation. The method is validated using the thermo-elasticity solutions for simply-supported panels obtained using Eshelby-Stroh formalism [7]. The alternate state vector formulation is found to provide accurate numerical results. After validation, the procedure is applied for the thermo-mechanical analysis of a sandwich panel with thin aluminum face-sheets and a soft-core of divinycell H35 foam.

2. AN ALTERNATE STATE VECTOR FORMULATION FOR THERMO-ELASTICITY

The governing state vector equation and associated derived variables within the framework of the three-dimensional, linear un-coupled thermo-elasticity in the absence of body forces and internal heat sources can be obtained by combining and re-arranging the following equilibrium equations, constitutive laws and strain-displacement relations.

$$\left(\sigma_{jm,m}=0\right),\left(q_{m,m}=0\right),\left(\sigma_{jm}=C_{jmqr}\varepsilon_{qr}-\beta_{jm}T\right),\left(q_m=-\kappa_{mr}T_{,r}\right),\left(\varepsilon_{qr}=\frac{1}{2}\left(u_{q,r}+u_{r,q}\right)\right),\left(j,m=1-3\right)$$

The variables in the above equations have their usual meaning (refer [7] for further details).

The present study is however limited to two-dimensional plane strain thermo-elastic deformation analysis. The non-dimensionalized state vector and derived equations, for any layer of a multi-layered panel consisting of specially orthotropic materials, are given in Eqs. 1 - 3.

$$\xi=\frac{x}{L},\eta=\frac{z}{H},\lambda=\frac{H}{L},\ \left(u,w\right)=\left(\overline{u},\overline{w}\right)\left(T_0\alpha_0 H\right),\ \left(\sigma_z,\tau_{xz}\right)=\left(\overline{\sigma}_\eta,\overline{\tau}_{\xi\eta}\right)\left(E_0\alpha_0 T_0\right),\ \left(q\right)=\left(\overline{q}\right)\left(\frac{\alpha_0\kappa_0}{H}\right) \tag{1}$$

$$\overline{u}_{,\eta}=-\lambda\overline{w}_{,\xi}+\frac{E_0}{C_{55}}\overline{\tau}_{\xi\eta};\quad \overline{\sigma}_{\eta,\eta}=-\lambda\overline{\tau}_{\xi\eta,\xi};\quad \overline{T}_{,\eta}=\frac{-\alpha_0\kappa_0}{T_0\kappa_\eta}\overline{q}_\eta;\ \overline{q}_{\eta,\eta}=\frac{\kappa_\xi T_0\lambda^2}{\kappa_0\alpha_0}\overline{T}_{,\xi\xi};$$

$$\overline{w}_{,\eta}=-\frac{C_{13}\lambda}{C_{33}}\overline{u}_{,\xi}+\frac{E_0}{C_{33}}\overline{\sigma}_\eta+\frac{\beta_\eta}{C_{33}\alpha_0}\overline{T};\ \overline{\tau}_{\xi\eta,\eta}=\left(\frac{C_{13}^2}{C_{33}}-C_{11}\right)\frac{\lambda^2}{E_0}\overline{u}_{,\xi\xi}-\frac{C_{13}\lambda}{C_{33}}\overline{q}_{\eta,\xi}+\left(\beta_\xi-\frac{C_{13}}{C_{33}}\beta_\eta\right)\frac{\lambda}{E_0\alpha_0}\overline{T}_{,\xi} \tag{2}$$

$$\overline{\sigma}_\xi=\left(C_{11}-\frac{C_{13}^2}{C_{33}}\right)\frac{\lambda}{E_0}\overline{u}_{,\xi}+\frac{C_{13}}{C_{33}}\overline{q}_\eta-\left(\beta_\xi-\frac{C_{13}}{C_{33}}\beta_\eta\right)\frac{1}{E_0\alpha_0}\overline{T} \tag{3}$$

The in-plane and transverse coordinates in physical and normalized coordinate systems are denoted, respectively, by the variables (x, z) and (ξ,η). The displacements and stress variables with and without over-line, respectively, denote those in the normalized and physical coordinate systems. The variables E_0, T_0, α_0, κ_0 denote, respectively, the reference stiffness constant, temperature, coefficient of thermal expansion and thermal conductivity; whereas H, L are the total thickness and length of the panel.

Exact solutions are obtained for a panel with simply-supported edge boundary conditions ($SS: \bar{w}=0, \bar{\sigma}_{\xi}=0$) by expanding the state variables into a trigonometric series (Eq. 4), where 'c' and 's' denote $\cos(m\pi\xi)$ and $\sin(m\pi\xi)$ respectively and satisfying both the governing state vector equations (Eq. 2) and SS edge boundary conditions and loads on lateral faces. On substituting the expanded state variables into the governing equation, a first order ordinary vector differential equation with constant coefficients is obtained for each layer of the panel (Eq. 5).

$$\begin{bmatrix} \bar{u} & \bar{\sigma}_{\eta} & \bar{T} & \bar{q}_{\eta} & \bar{w} & \bar{\tau}_{\xi\eta} \end{bmatrix}^T = \sum_{m=1}^{\infty} \begin{bmatrix} U_m c & Z_m s & T_m s & Q_m s & W_m s & R_m c \end{bmatrix} \tag{4}$$

$$\delta_{m,\eta} = [K]\delta_m, \quad \delta_m(\eta) = \begin{bmatrix} U_m & Z_m & T_m & Q_m & W_m & R_m \end{bmatrix}^T, \; [K] = \begin{bmatrix} [0] & [A] \\ [B] & [0] \end{bmatrix} \tag{5}$$

$$[A] = \begin{bmatrix} 0 & -\lambda m\pi & \dfrac{E_0}{C_{55}} \\ 0 & 0 & \lambda m\pi \\ -\dfrac{\alpha_0\kappa_0}{T_0\kappa_{\eta}} & 0 & 0 \end{bmatrix} [B] = \begin{bmatrix} 0 & 0 & -\dfrac{\kappa_{\xi}T_0(\lambda m\pi)^2}{\kappa_0\alpha_0} \\ \dfrac{C_{13}}{C_{33}}(\lambda m\pi) & \dfrac{E_0}{C_{33}} & \dfrac{\beta_{\eta}}{C_{33}\alpha_0} \\ \left(C_{11} - \dfrac{C_{13}^2}{C_{33}}\right)\dfrac{(\lambda m\pi)^2}{E_0} & -\dfrac{C_{13}}{C_{33}}(\lambda m\pi) & \left(\beta_{\xi} - \dfrac{C_{13}}{C_{33}}\beta_{\eta}\right)\dfrac{\lambda m\pi}{E_0\alpha_0} \end{bmatrix}$$

The solution to Eq. 5 for any layer 'i' of an 'n' layered panel is given by Eq. 6; called the local transfer matrix relation for the i^{th} layer. Using these local transfer matrix relations and the interface continuity conditions between any two adjacent layers 'i' and 'i+1' (Eq. 7), the global transfer matrix for the entire panel (Eq. 8) can be obtained. Further, on relating the state vectors and load vectors at the top and bottom surface of the panel, the final set of algebraic equations to be solved can be obtained.

$$\delta_m^{(i)}(\eta) = \left[TM(\eta)\right]^{(i)} \delta_m^{(i)}(0), \left[TM(\eta)\right] = \exp\left([K]\eta\right), 0 \le \eta \le h^{(i)}; \delta_m^{(i)}\left(h^{(i)}\right) = \left[TM\left(h^{(i)}\right)\right]^{(i)} \delta_m^{(i)}(0) \tag{6}$$

$$\delta_m^{(i)}\left(h^{(i)}\right) = \delta_m^{(i+1)}(0) \tag{7}$$

$$\delta_m^{(n)}\left(h^{(n)}\right) = [TM]_G\, \delta_m^{(1)}(0), \;\; [TM]_G = [TM]_n [TM]_{n-1} \; . \; . \; . \; . \; [TM]_1, \;\; [TM]_i = \left[TM\left(h^{(i)}\right)\right]^{(i)} \tag{8}$$

3. RESULTS AND DISCUSSIONS

The alternate state vector formulation for two dimensional plane strain thermo-elastic deformation analysis of simply-supported panels is implemented in MATLAB®. Validation studies (Table 1) are conducted using analytical solutions obtained using Eshelby-Stroh formalism [7]. A very good match between the results is observed. The material and geometric parameters of the composite laminate, subjected only to a sinusoidal temperature increase of the form $T(x,H) = T_0 \sin(\pi x / L)$ on the top surface, are as follows [7]:

$$\frac{L}{H} = 5, \; \frac{(C_{11})_0}{E_0} = 1.0169, \; \frac{(C_{13})_0}{E_0} = 0.0339, \; \frac{(C_{55})_0}{E_0} = 0.05, \; \frac{(C_{33})_0}{E_0} = \frac{(C_{11})_{90}}{E_0} = \frac{(C_{33})_{90}}{E_0} = 0.1078$$

$$\frac{(C_{13})_{90}}{E_0} = 0.0278, \; \frac{(C_{55})_{90}}{E_0} = 0.02, \; \frac{(\beta_{\xi})_0}{\alpha_0 E_0} = 1.5051, \; \frac{(\beta_{\eta})_0}{\alpha_0 E_0} = \frac{(\beta_{\xi})_{90}}{\alpha_0 E_0} = \frac{(\beta_{\eta})_{90}}{\alpha_0 E_0} = 1.0102, \; \frac{(\kappa_{\xi})_0}{\alpha_0 E_0} = 100$$

$$\frac{(\kappa_{\eta})_0}{\alpha_0 E_0} = \frac{(\kappa_{\xi})_{90}}{\alpha_0 E_0} = \frac{(\kappa_{\eta})_{90}}{\alpha_0 E_0} = 1$$

After validation, the method is used for thermo-elastic analysis of a simply supported symmetric sandwich panel (L/H =5) made of aluminium face sheets (E = 70GPa, $\upsilon = 0.3$, α = 23x10^{-6}K^{-1}, κ =180 W/(mK), h_f / H =1/20) and a Divinycell H35 core (E=0.04GPa, $\upsilon = 0.3$, α =40x10^{-6}K^{-1}, κ =0.028W/(mK), h_c / H =18/20) under a sinusoidal temperature increase $T = T_0 \sin(\pi x / L)$ on the top surface. The numerical results are given in Table 2, where the non-dimensionalized field variables at different locations are displayed. In Table 2, the variable 'x^*' denotes location in the x-coordinate at which the field variables are maximum (that is, x^*=0 for $\tilde{u}$, $\tilde{\tau}_{xz}$ and x^*=$L/2$ for others) and '$(.)^{(f)}$', '$(.)^{(c)}$'

denote the numerical values of in-plane normal stress, respectively, in the face sheet and the core at the top face sheet-core interface.

Table 1: Comparison of displacements and stresses (non-dimensionalized as in [7]) obtained using alternate state vector formulation (Present) and Eshelby-Stroh formalism (ESF [7]) at specific locations for a simply-supported 3-ply laminate.

Variables	Present	ESF [7]
$10\tilde{u}\left(L/4,H\right)$	-1.7428	-1.743
$\tilde{w}\left(L/2,H/2\right)$	0.4049	0.405
$10\tilde{\sigma}_{xx}\left(L/2,0\right)$	-4.0829	-4.083
$10\tilde{\sigma}_{xx}\left(L/2,0\right)$	-0.7945	-0.795
$100\tilde{\tau}_{xz}\left(L/4,H/2\right)$	0.1336	0.134
$1000\tilde{\sigma}_{zz}\left(L/2,H/2\right)$	0.2795	0.279

Table 2: Displacements and stresses at specific locations (non-dimensionalized as in [7]) of a simply-supported, symmetric sandwich panel with isotropic layers under a sinusoidal temperature increase at the top surface using present formulation.

Locations \ Variables	(10x) $\tilde{u}$	$\tilde{w}$	(10^2x) $\tilde{\sigma}_{xx}$	(10^4x) $\tilde{\tau}_{xz}$	(10^5x) $\tilde{\sigma}_{zz}$
$\left(x^*,\dfrac{19H}{20}\right)$	-1.854	0.376	-1.909(f) -0.029(c)	0.414	-0.401
$\left(x^*,\dfrac{H}{2}\right)$	-0.951	0.296	-0.013	-0.173	-0.633

4. CONCLUSIONS

An alternate state vector formulation was introduced for thermo-elastic analysis of multi-layered panels. The proposed procedure successfully predicted the two-dimensional thermo-elastic deformation response of simply-supported sandwich panels.

REFERENCES

[1] C. P. Wu et al., "A review on the three-dimensional analytical approaches of multilayered and functionally graded piezoelectric plates and shells", *Comput. Mater.Continua.*, 2008; 8: 93-132.

[2] L. Y. Bahar, "A state space approach to elasticity", *J. of Franklin Inst.*, 1975; 299(1): 33-41.

[3] F. Jiarang et al., "An exact solution for the statics and dynamics of laminated thick plates with orthotropic layers", *Int. J. of Solids and Struct.,* 1990; 26(5): 655-662.

[4] Z. Zhong et al., "Three-dimensional exact analysis of a simply supported functionally gradient piezoelectric plate", *Int. J. of Solids and Struct.,* 2003; 40(20): 5335-5352.

[5] A. Alibeigloo, "Exact solution for thermo-elastic response of functionally graded rectangular plates", *Comp. Struct.*, 2010; 92: 113-121.

[6] J. N. Reddy et al., "Three-dimensional thermomechanical deformations of functionally graded rectangular plates", *Eur. J. Mech. A – Solids*, 2001; 20: 841-855.

[7] S. S. Vel et al., "Generalized plane strain thermo-elastic deformation of laminated anisotropic thick plates", *Int. J. of Solids and Struct.*, 2001; 38(8): 1395-1441.

[8] A. Alibeigloo, "Three-dimensional thermo-elasticity solution of sandwich cylindrical panel with functionally graded core", *Comp. Struct*, 2014; 107: 458-468.