Parameter Determination of Sensor Stochastic Models under Covariate Dependency

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Noise Characterization via Allan / Wavelet Variance

Generalized Method of Wavelet Moments (GMWM)

GMWM Extensions for Covariate Dependencies
Analyzing a Signal

Types of Signals

- Oscillator
- Accelerometer
- Gravimeter
- Gyroscope
Analyzing a Signal

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Stochastic characteristics of time series: $y_t, t \in \mathbb{N}$
Analysis of an Error Signal

State-space model

\[ y_t = \omega t + u_t + \varepsilon_t \]

\[ \varepsilon_t \overset{iid}{\sim} \mathcal{N}(0, \sigma_{WN}^2) \]

\[ u_t = \phi u_{t-1} + \eta_t \]

\[ \eta_t \overset{iid}{\sim} \mathcal{N}(0, \sigma_{AR(1)}^2) \]

\[ \omega, \phi \in \mathbb{R} \]

Noise parameters

\[ \theta = (\omega, \phi, \sigma_{AR(1)}^2, \sigma_{WN}^2) \]
Analysis of an Error Signal via Allan Variance

Drift

White Noise

AR(1)

Drift + White Noise + AR(1)

Allan Variance representation
Challenges

Existing methods:

- “Graphical” Allan Variance
  - Limited to a few models
  - (Proven as) not consistent in general
  - “Inefficient” (non-automated)
- MLE (with EM algorithm)
  - Computationally intensive
  - Diverges with “complex” models

Need of an estimator

- Complex time-series model
- Computational efficiency
- Robust to outliers
Main idea

- Usage of the Wavelet Variance (WV)
- Filter of the signal with the Wavelet Function
- Exploitation of the relationship existing between a model $\theta$ and its WV $\nu(\theta)$ (i.e. mapping $\theta \mapsto \nu(\theta)$).
- “Inverse” this mapping by minimizing some discrepancies between empirical (i.e. observed WV/AV $\hat{\nu}$) and the theoretical WV for a model $\nu(\theta)$. 
**GMWM Estimator**

**Definition**
Solution of the following optimization problem with weighting matrix $\Omega$

$$\hat{\theta} = \arg\min_{\theta \in \Theta} (\hat{\nu} - \nu(\theta))^T \Omega (\hat{\nu} - \nu(\theta))$$

**Identifiable**
$$\nu(\theta_1) = \nu(\theta_2) \quad \text{iff} \quad \theta_1 = \theta_2$$

**Consistent**
$$\hat{\theta} \xrightarrow{p} \theta_0$$

**Asymptotically Normal**
$$\sqrt{T} \left( \hat{\theta} - \theta_0 \right) \xrightarrow{p} \mathcal{N} \left( 0, \Sigma \right)$$
Haar Wavelet Variance of DATASET:
navchip (Gyro. Y) - Duration: 3.5(h) @250(Hz)
GMWM Example: incomplete model

Haar Wavelet Variance of DATASET: navchip (Gyro. Y) - Duration: 3.5(h) @250(Hz)
GMWM Example: incomplete model cont.

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Wavelet Variance $\nu [rad^2/s^2]$

Scale $\tau [s]$

- Empirical WV $\nu$
- CI($\nu$, 0.95)
- GM
- WN
- QN
- Implied WV $\nu(\hat{\theta})$
GMWM Example: complete model

Haar Wavelet Variance of DATASET:
navchip (Gyro. Y) - Duration: 3.5(h) @250(Hz)
GMWM Example: covariate influence

MEMS IMU Gyroscope rotating at $30^\circ/s$ and $360^\circ/s$
## Extension for Covariate Dependency

**Definition**
- **external process:** \( X_t, t \in \mathbb{N} \)
  - previous example: rotational speed
- **White Noise** process:
  \[
  V_t \overset{iid}{\sim} \mathcal{N}(0, \gamma^2)
  \]
- **Auto-Regressive** process of order 1:
  \[
  u_t = \phi \ u_{t-1} + \varepsilon
  \]
  \[
  \varepsilon \overset{iid}{\sim} \mathcal{N}(0, \eta^2)
  \]
Extension for Covariate Dependency

**Definition**

- **external process**: $X_t, t \in \mathbb{N}$
  - ➤ previous example: rotational speed
- **White Noise** process:
  
  $$V_t | X_t \overset{iid}{\sim} \mathcal{N} (0, \gamma^2_t), \quad \gamma^2_t = g (s_1 + s_2 X_t)$$

- **Auto-Regressive** process of order 1:
  
  $$u_t = \phi \ u_{t-1} + \varepsilon$$
  
  $$\varepsilon \overset{iid}{\sim} \mathcal{N} (0, \eta^2)$$
Extension for Covariate Dependency

**Definition**

- external process: $X_t, t \in \mathbb{N}$
  - previous example: rotational speed
- *White Noise* process:
  $$V_t|X_t \overset{iid}{\sim} \mathcal{N} \left(0, \gamma_t^2\right), \quad \gamma_t^2 = g(\varsigma_1 + \varsigma_2 X_t)$$
- *Auto-Regressive* process of order 1:
  $$u_t|X_t = \phi_t u_{t-1} + \varepsilon_t, \quad \phi_t = h(\varphi_1 + \varphi_2 X_t),$$
  $$\varepsilon_t|X_t \overset{iid}{\sim} \mathcal{N} \left(0, \eta_t^2\right), \quad \eta_t^2 = k(\nu_1 + \nu_2 X_t)$$
Extension for Covariate Dependency

**Definition**

- **external process:** $X_t, t \in \mathbb{N}$
  - previous example: rotational speed
- **White Noise** process:
  
  \[ V_t|X_t \overset{iid}{\sim} \mathcal{N}(0, \gamma_t^2) \text{,} \quad \gamma_t^2 = g(\varsigma_1 + \varsigma_2 X_t) \]

- **Auto-Regressive** process of order 1:
  
  \[ u_t|X_t = \phi_t u_{t-1} + \varepsilon_t, \quad \phi_t = h(\varphi_1 + \varphi_2 X_t), \]
  
  \[ \varepsilon_t|X_t \overset{iid}{\sim} \mathcal{N}(0, \eta_t^2) \text{,} \quad \eta_t^2 = k(\upsilon_1 + \upsilon_2 X_t) \]

**Extended parameter vector**

\[
\theta = \begin{bmatrix} \varsigma^T & \varphi_1^T & \ldots & \varphi_d^T & \upsilon_1^T & \ldots & \upsilon_d^T \end{bmatrix}^T \in \Theta
\]
Dynamic GMWM estimator

\[
\hat{\theta} = \underset{\theta \in \Theta}{\text{argmin}} \frac{1}{K} \sum_{k=1}^{K} \left\| \hat{\nu}_k - \nu(\theta, c_k) \right\|_{\hat{\Omega}_k}^2
\]

\(c_k\) explains the covariate influence on the WV of bin \(k\)
Dynamic GMWM estimator

\[ \hat{\theta} = \arg\min_{\theta \in \Theta} \frac{1}{K} \sum_{k=1}^{K} \left\| \hat{\nu}_k - \nu(\theta, c_k) \right\|^2_{\hat{\Omega}_k} \]

c_k explains the covariate influence on the WV of bin k

And the properties?

Identifiable ✔

Consistent ✔

Asymptotically Normal ✔
MEMS IMU Gyroscope stochastic noise as a function of rotational speed
Conclusions

Properties

- Numerically stable
- Computationally efficient
- Covariate Dependency

Implementation
- Open-source package in statistical tool R
- Online web browser tool on ggmwm.smac-group.com
- Proofs in an upcoming publication

Application examples
- Rotational dynamics dependency: IEEE/ION PLANS 2018
- Temperature dependency: to be published with proofs
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And more...

- Implementation
  - opensource package in statistical tool $R$
  - online webbrowser tool on ggmwm.smac-group.com
- Proofs in an upcoming publication
- Application examples
  - rotational dynamics dependency: IEEE/ION PLANS 2018
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Thank you

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- https://github.com/SMAC-group/GMWM

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