The stochastic heat equation: hitting probabilities and the probability density function of the supremum via Malliavin calculus

THÈSE N^{O} 8695 (2018)

PRÉSENTÉE LE 22 JUIN 2018

À LA FACULTÉ DES SCIENCES DE BASE
CHAIRE DE PROBABILITÉS
PROGRAMME DOCTORAL EN MATHÉMATIQUES

ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

POUR L'OBTENTION DU GRADE DE DOCTEUR ÈS SCIENCES

PAR

Fei PU

acceptée sur proposition du jury:

Prof. T. Mountford, président du jury
Prof. R. Dalang, directeur de thèse
Prof. D. Nualart, rapporteur
Prof. M. Sanz-Solé, rapporteuse
Prof. C. Hongler, rapporteur

ÉCOLE POLYTECHNIQUE
FÉDÉRALE DE LAUSANNE

Suisse
2018

Abstract

In this thesis, we study systems of linear and/or non-linear stochastic heat equations and fractional heat equations in spatial dimension 1 driven by space-time white noise. The main topic is the study of hitting probabilities for the solutions to these systems.

We first study the properties of the probability density functions of the solution to non-linear systems of stochastic fractional heat equations driven by multiplicative space-time white noise. Using the techniques of Malliavin calculus, we prove that the one-point probability density function of the solution is infinitely differentiable, uniformly bounded and positive everywhere. Moreover, a Gaussian-type upper bound on the two-point probability density function is obtained by a detailed analysis of the small eigenvalues of the Malliavin matrix. We establish an optimal lower bound on hitting probabilities for the (non-Gaussian) solution, which is as sharp as that for the Gaussian solution to a system of linear equations.

We develop a new method to study the upper bound on hitting probabilities, from the perspective of probability density functions. For the solution to the linear stochastic heat equation, we prove that the random vector, which consists of the solution and the supremum of a linear increment of the solution over a time segment, has an infinitely differentiable probability density function. We derive a formula for this density and establish a Gaussian-type upper bound. The smoothness property and Gaussian-type upper bound for the density of the supremum of the solution over a space-time rectangle touching the $t = 0$ axis are also studied. Furthermore, we extend these results to the solutions of systems of linear stochastic fractional heat equations.

For a system of linear stochastic heat equations with Dirichlet boundary conditions, we present a sufficient condition for certain sets to be hit with probability one.

Key words: hitting probabilities, stochastic (fractional) heat equation, Malliavin calculus, probability density function, Gaussian-type upper bound, supremum of a Gaussian random field, space-time white noise, capacity, Hausdorff measure.

Résumé

Dans cette thèse, nous étudions des systèmes linéaires et/ou non-linéaires d'équations de la chaleur stochastiques et d'équations de la chaleur fractionaires en dimension spatiale 1 régies par un bruit blanc en temps et en espace. Le sujet principal est l'étude de la probabilité que les solutions de ces systèmes visitent un ensemble donné.

Dans un premier temps, nous étudions les propriétés des fonctions de densité des solutions des équations de la chaleur fractionnaires stochastiques régies par un bruit blanc multiplicatif. En utilisant les techniques du calcul de Malliavin, nous prouvons que la fonction de densité de la solution est infiniment différentiable, uniformément bornée et partout positive. De plus, une borne supérieure de type gaussien est obtenue pour la densité conjointe grâce à une étude détaillée des petites valeurs propres de la matrice de Malliavin. Pour la solution (non-gaussienne), nous établissons une borne inférieure optimale sur les probabilités de visiter un ensemble donné, qui est aussi précise que celle pour la solution gaussienne d'équations linéaires.

Nous développons une nouvelle méthode pour étudier les bornes supérieures des probabilités de visiter un ensemble, basée sur les fonctions de densité. Pour la solution du système linéaire de la chaleur stochastique, nous montrons que le vecteur aléatoire, qui consiste en la solution et le supremum d'un incrément linéaire de la solution dans un intervalle de temps, a une fonction de densité infiniment différentiable. Nous donnons une formule pour cette densité et établissons une borne supérieur de type gaussien pour celle-ci. La propriété de régularité et la borne supérieure de type gaussien pour la densité du supremum de la solution dans un rectangle en temps et espace qui touche l'axe $t = 0$ sont aussi étudiées. De plus, nous étendons ces résultats à la solution de systèmes d'équations linéaires de la chaleur fractionnaires stochastiques.

Pour un système d'équations linéaires de la chaleur stochastiques avec des conditions aux bords de Dirichlet, nous présentons une condition suffisante pour que certains ensembles soient visités avec probabilité un.

Mots cléfs : probabilités de visiter un ensemble, équation de la chaleur (fractionnaire) stochastique, calcul de Malliavin, fonction de densité, borne supérieure de type gaussien, supremum d'un champ aléatoire gaussien, bruit blanc en temps et espace, capacité, mesure de Hausdorff.

Acknowledgements

First of all, I would like to express my gratitude to my supervisor, Prof. Robert C. Dalang, for giving me this opportunity to work on this thesis as his student. During the work with Prof. Robert C. Dalang, I have learned a lot from his insights on mathematics, his knowledge of mathematics, his altitude towards mathematics, his experience of research and more. I am grateful for his support and encouragement throughout my study at EPFL.

I would like to thank my parents, my sister and brother-in-law, and my niece for their support of my studies throughout these years.

I express my thanks to all the members of my thesis committee: Prof. Thomas Mountford (president of the committee), Prof. Clément Hongler, Prof. David Nualart and Prof. Marta Sanz-Solé.

The EPFL mathematics community provides a comfortable research environment. I would like to thank Prof. Clément Hongler for his two doctoral courses on lattice models and conformal field theory, in which I learned a lot on statistical mechanics. I would also like to thank Prof. Marta Sanz-Solé for her introductory course on SPDEs from which I received a systematic study in the theory of SPDEs. I would like to thank Prof. Thomas Mountford for organizing probability seminars from time to time. Besides, I would like to thank all the lecturers in the summer and winter schools that I have attended during my study. My thanks also goes to my colleagues: Jean-Benoît Rossel, Le Chen, Thomas Humeau, Carlo Ciccarella, David Candil, Hamza Ouared, Carsten Chong, Hao Xue, Zhiwen Jiang and our secretary Ariane Cordonier. In particular, I would like to thank David for the French abstract of this thesis.

I would like to thank Prof. Zenghu Li for his support and the discussions with him on my research. I would like to thank Lingyi Zheng for her kind support and sharing her life with me. I would like to thank my friends: Mátyás Barczy, Le Chen, Dadao Feng, Feng Liu, Yiqun Liu, Ziyi Liu, Wenbiao Luo, Tao Ma, Xiaochun Men, Mingfu Shao, Ping Xi, Min Ye, Xiuwei Zhang.

Lausanne, March 2018 Fei Pu

To my parents

Contents

1 Introduction

Stochastic partial differential equations (SPDEs) are a generalization of partial differential equations, with terms that correspond to random external forces. Usually, the random external force is taken to be space-time white noise. In many contexts, it is natural to consider systems of such SPDEs. The white noise may be multidimensional and there is one SPDE for each component of the solution. In this case, the components of the solution interact with each other. This thesis studies certain properties of such systems of SPDEs.

1.1 Literature review

The solutions to systems of SPDEs arise as an important class of $\mathbb{R}^d$-valued stochastic processes. Potential theory is one important topic in the study of such stochastic processes. We refer to [10, 37, 74] for potential theory for single-parameter processes and to [44] for multiparameter processes. In probabilistic potential theory, one basic question is to determine whether a stochastic process visits, or hits, a fixed deterministic set $A \subset \mathbb{R}^d$ with positive probability. We are interested in relating the hitting probabilities of the solution to various geometric quantities, such as Hausdorff measure and capacity.

Let us recall the main existing results on hitting probabilities for some classical stochastic processes. For example, the well-known theorem of Kakutani (see [44, Theorem 3.1.1, Chapter 10]) states that for a d-dimensional Brownian motion $\{B(t), t \geq 0\}$ starting at $x \in \mathbb{R}^d$ and for any compact set $A \subset \mathbb{R}^d$ with $x \notin A$,

$$\mathrm{P}\{\exists t > 0 \text{ such that } B(t) \in A\} > 0 \quad \Longleftrightarrow \quad \mathrm{Cap}_{d-2}(A) > 0,$$

where Cap_β denotes the capacity with respect to the Newtonian β-kernel. This result was extended to the Brownian sheet by Khoshnevisan and Shi [46]. Let $W = \{W_t, t \in \mathbb{R}_+^N\}$ denote an $\mathbb{R}^d$-valued Brownian sheet. They showed that for all $M > 0$ and $0 < a < b < \infty$, there exists a finite positive constant C such that for all compact subsets $A \subseteq [-M, M]^d$,

$$C^{-1}\mathrm{Cap}_{d-2N}(A) \leq \mathrm{P}\{\exists t \in [a,b]^N : W_t \in A\} \leq C\,\mathrm{Cap}_{d-2N}(A).$$

Motivated by this, Dalang and Nualart [29] studied hitting probabilities for the solution to the reduced hyperbolic SPDE on $\mathbb{R}_+^2$ (essentially equivalent to the wave equation in spatial dimension 1):

$$\frac{\partial^2 X_t^i}{\partial t_1 \partial t_2} = \sum_{j=1}^{d} \sigma_{ij}(X_t) \frac{\partial^2 W_t^j}{\partial t_1 \partial t_2} + b_i(X_t),$$

where $t = (t_1, t_2) \in \mathbb{R}_+^2$, and $X_t^i = 0$ if $t_1 t_2 = 0$, for all $1 \leq i \leq d$. There, Dalang and Nualart used Malliavin calculus to show that there exists a finite positive constant K depending on $b > a > 0$ and $M > 0$ such that for all compact subsets $A \subseteq [-M, M]^d$,

$$K^{-1}\mathrm{Cap}_{d-4}(A) \leq \mathrm{P}\{\exists t \in [a,b]^2 : X_t \in A\} \leq K\,\mathrm{Cap}_{d-4}(A).$$

In the context of systems of stochastic heat equations, consider the following system:

$$\frac{\partial u_i}{\partial t}(t,x) = \Delta u_i(t,x) + \sum_{j=1}^{d} \sigma_{ij}(u(t,x))\dot{W}^j(t,x) + b_i(u(t,x)), \tag{1.1.1}$$

for $1 \leq i \leq d$, where $(t,x) \in]0,\infty[\times [0,1]$ and $u := (u_1, \ldots, u_d)$ with Neumann boundary conditions. We set $b = (b_i), \sigma = (\sigma_{ij})$. Let $I \subset]0, T]$ and $J \subseteq [0,1]$ be two compact intervals. We are interested in the hitting probability $\mathrm{P}\{u(I \times J) \cap A \neq \emptyset\}$, where $u(I \times J)$ denotes the range of $I \times J$ under the random map $(t,x) \mapsto u(t,x)$. In the case where the noise is additive, i.e., $\sigma \equiv \mathrm{Id}$, $b \equiv 0$, Dalang, Khoshnevisan and Nualart [25] have established upper and lower bounds on hitting probabilities for the Gaussian solution. They show that there exists $c > 0$ depending on M, I, J with $M > 0$, such that, for all Borel sets $A \subseteq [-M, M]^d$,

$$c^{-1}\mathrm{Cap}_{d-6}(A) \leq \mathrm{P}\{u(I \times J) \cap A \neq \emptyset\} \leq c\,\mathscr{H}_{d-6}(A), \tag{1.1.2}$$

where $\mathscr{H}_\beta$ denotes the β-dimensional Hausdorff measure. If the noise is multiplicative, i.e., σ and b are not constants (but are sufficiently regular), then using techniques of Malliavin calculus, Dalang, Khoshnevisan and Nualart [26] have obtained upper and lower bounds on hitting probabilities for the non-Gaussian solution, analogous to, but slightly different from, (1.1.2). Indeed, they prove that there exists $c > 0$ depending on M, I, J, η with $M > 0, \eta > 0$, such that, for all Borel sets $A \subseteq [-M, M]^d$,

$$c^{-1}\mathrm{Cap}_{d+\eta-6}(A) \leq \mathrm{P}\{u(I \times J) \cap A \neq \emptyset\} \leq c\,\mathscr{H}_{d-\eta-6}(A). \tag{1.1.3}$$

Furthermore, these results have been extended to higher spatial dimensions driven by spatially homogeneous noise in Dalang, Khoshnevisan and Nualart [27].

This type of question has also been studied for systems of stochastic wave equations, in particular, in higher spatial dimensions, by Dalang and Sanz-Solé [30] and [31]. We recall some of their results. Consider the d-dimensional system of stochastic wave equations

$$\left(\frac{\partial^2}{\partial t^2} - \Delta\right) u_i(t,x) = \sum_{j=1}^{d} \sigma_{i,j}(u(t,x))\dot{M}^j(t,x) + b_i(u(t,x)), \quad (t,x) \in]0,T] \times \mathbb{R}^k, \tag{1.1.4}$$

for $1 \leq i \leq d$, where the d-dimensional driving noise $\dot{M}$ is white in time with a spatially homogeneous covariance given by the Riesz kernel $f(x) = \|x\|^{\beta}$, $0 < \beta < (2 \wedge k)$. Let I and J be two compact subsets of $]0,T]$ and $\mathbb{R}^k$, respectively. Fix $M > 0$ and $\eta > 0$. If σ is an invertible matrix with constant entries, $b \equiv 0$ and $k \in \mathbb{N}$, Dalang and Sanz-Solé [30] proved that there exists a positive constant c depending on I, J, M, β, k and d, such that, for any Borel set $A \subset [-M,M]^d$,

$$c^{-1}\mathrm{Cap}_{d-\frac{2(k+1)}{2-\beta}}(A) \leq \mathrm{P}\{u(I \times J) \cap A \neq \emptyset\} \leq c\,\mathscr{H}_{d-\frac{2(k+1)}{2-\beta}}(A). \tag{1.1.5}$$

If σ and b are not constants and satisfy some smoothness and Lipschitz conditions, Dalang and Sanz-Solé [31] have established, for $k \in \{1,2,3\}$, that there exists a positive constant c depending on I, J, M, β, k, d and η, such that, for any Borel set $A \subset [-M,M]^d$,

$$c^{-1}\mathrm{Cap}_{d\left(1+\frac{4d}{2-\beta}\right)+\eta-\frac{2(k+1)}{2-\beta}}(A) \leq \mathrm{P}\{u(I \times J) \cap A \neq \emptyset\} \leq c\,\mathscr{H}_{d-\eta-\frac{2(k+1)}{2-\beta}}(A) \tag{1.1.6}$$

(their result applies to more general covariances than those given by a Riesz kernel). These results were extended to the case of linear fractional colored noise by Clarke de la Cerda and Tudor [22]. Hitting probabilities for the solutions to systems of elliptic stochastic equations have been studied in Sanz-Solé and Viles [79]. For systems of linear stochastic fractional heat equations in spatial dimension 1 driven by space-time white noise, the question of hitting points was studied in Wu [84]. We also refer to Mueller and Tribe [62] for a (Gaussian) random string, Dalang, Mueller and Zambotti [28] for a heat equation with reflection, and Nualart and Viens [71] for a system of heat equations driven by an additive fractional Brownian motion.

We are also interested in studying the probability density function of the supremum of the solution; see Chapter 4 for the detailed motivation. The question of smoothness of the density of the supremum of a multiparameter Gaussian process dates back to the work of Florit and Nualart [39], in which they establish a general criterion (see Theorem 1.5.5) for the smoothness of the density assuming that the random vector is locally in $\mathbb{D}^{\infty}$ and apply it to show that the maximum of the Brownian sheet on a rectangle possesses an infinitely differentiable density. Moreover, this method was applied to prove that the supremum of the fractional Brownian motion has an infinitely differentiable density; see Lanjri Zadi and Nualart [55]. Some general results on the regularity of the density of the maximum of Gaussian random fields have been developed by Cirel'son [21], Pitt and Lanh [73], Weber [83], Lifshits [56, 57], Diebolt and Posse [36] and Azaïs and Wschebor [4]. We also refer to Hayashi and Kohatsu-Higa [42] and Nakatsu [63] for the smoothness of densities for diffusion processes.

1.2 Main results of the thesis

In this thesis, we study the following system of linear and/or non-linear stochastic heat equations in spatial dimension 1 driven by space-time white noise:

$$\frac{\partial u_i}{\partial t}(t,x) = \frac{\partial^2 u_i}{\partial x^2}(t,x) + \sum_{j=1}^{d} \sigma_{ij}(u(t,x))\dot{W}^j(t,x) + b_i(u(t,x)), \quad t \in \mathbb{R}_+,\ x \in U, \tag{1.2.1}$$

and its extension to the following system of stochastic fractional heat equations:

$$\frac{\partial u_i}{\partial t}(t,x) = {}_xD^\alpha u_i(t,x) + \sum_{j=1}^{d} \sigma_{ij}(u(t,x))\dot{W}^j(t,x) + b_i(u(t,x)), \quad t \in \mathbb{R}_+,\ x \in \mathbb{R}, \tag{1.2.2}$$

for $1 \le i \le d$, where $\mathbb{R}_+ := [0,\infty[$, U is equal to $[0,1]$ or $\mathbb{R}$, $1 < \alpha < 2$, $u := (u_1,\dots,u_d)$, with initial conditions $u(0,x) = u_0(x)$ for all $x \in U$, and Dirichlet or Neumann boundary conditions if $U = [0,1]$.

Our main topic is the study of hitting probabilities for the solutions to (1.2.1) and (1.2.2). As we have seen from (1.1.2) and (1.1.3), the lower and upper bounds on hitting probabilities for non-Gaussian solutions are not as sharp as those for Gaussian solutions; compare also (1.1.5) and (1.1.6) for stochastic wave equations. Our main objective was to remove the η on the left- and right-hand sides of (1.1.3).

In Chapter 2, we succeed in removing the η in the dimension of capacity in (1.1.3), and we generalize these results to solutions of systems of stochastic fractional heat equations; see our Theorem 2.1.4. The proof of the lower bound is essentially based on the analysis of the one-point and two-point joint densities of the solution. In particular, the presence of η in the dimension of capacity in (1.1.3) comes from a Gaussian-type upper bound on the joint density of $Z := (u(s,y), u(t,x) - u(s,y))$; see [26, Theorem 1.1(c)]. In Theorem 2.1.1, we manage to remove this η in the Gaussian-type upper bound on the joint density of Z, so that this becomes the best possible upper bound, as in the Gaussian case. This requires a detailed analysis of the small eigenvalues of the Malliavin matrix γ_Z of Z; see Proposition 2.5.10. We prove Proposition 2.5.10 by giving a better estimate on the Malliavin derivative of the solution; see Lemma A.3.3, which, for a certain range of parameters, is an improvement of Morien [60, Lemma 4.2]; see also Lemma A.3.2. This estimate is used in Lemma 2.5.4 to obtain a bound on the integral terms in the Malliavin derivative of u (compare with [26, Lemma 6.11]), then in Proposition 2.5.10 to bound negative moments of the smallest eigenvalue of the Malliavin matrix (compare with [26, Proposition 6.9]), and finally in Proposition 2.5.8 and Theorem 2.5.13 to bound negative moments of the Malliavin matrix (compare with [26, Proposition 6.6] and [26, Theorem 6.3]). This improves the result (1.1.3) of [26], and the method extends to systems of stochastic fractional heat equations (1.2.2) for $1 < \alpha \le 2$ with a unified proof.

In Chapter 3, we study the hitting probability of the Gaussian solution ($\sigma \equiv \mathrm{Id}$, $b \equiv 0$) satisfying (1.2.1) on $U = [0,1]$ with Dirichlet boundary conditions, from another perspective. In Theorem

3.1.1, we show that for Borel sets A satisfying $\mathrm{Cap}_{d-6}(A) > 0$,

$$\mathrm{P}\{u([0,\infty[\times[0,1]) \cap A \neq \emptyset\} = 1. \tag{1.2.3}$$

This is obtained by using the strong Markov property, a recurrence property when the solution is viewed as parameterized only by time and taking values in the space of continuous functions, and the lower bound on hitting probabilities in (1.1.2). Intuitively, the solution visits infinitely many times a ball in the space of continuous functions with a large radius, and between visits, it hits A with a probability bounded below by $\mathrm{Cap}_{d-6}(A)$ times a constant. Formally, we are able to sum up these probabilities by using the strong Markov property and obtain this probability one result.

We turn to considering the upper bound on hitting probabilities in (1.1.3) for the non-Gaussian solution, which we expect should be consistent with the result for the Gaussian solution in (1.1.2). We remark that, following the general approach for upper bounds on hitting probabilities in [25], it is sufficient to bound appropriately the probability that the solution visits a small ball within a small space-time region:

$$\mathrm{P}\left\{\inf_{(t,x)\in R^n_{k,l}} |u(t,x) - z| \leq 2^{-n}\right\}, \tag{1.2.4}$$

where $R^n_{k,l}$ is defined as in (4.1.3) (for simplicity, we consider here one single equation, i.e., $d = 1$). One possible way to estimate this probability is to study the regularity of the joint probability density function of the random vector

$$\left(u(t^n_k, x^n_l),\ \sup_{(t,x)\in R^n_{k,l}} u(t,x) - u(t^n_k, x^n_l)\right), \tag{1.2.5}$$

where the supremum of the solution appears, and to establish good bounds on this density function; see the detailed description in Section 4.1 and Theorem 4.1.1 that motivates this study.

In Chapter 4, we apply Theorem 1.5.5 of Florit and Nualart [39] to study the density of the supremum of linear and rectangular increments of the solution to the linear stochastic heat equation. In Theorem 4.2.1, we prove that the random vector

$$\left(u(s_0, y_0),\ \sup_{t\in[s_0,s_0+\delta_1]} (u(t,y_0) - u(s_0,y_0))\right) \tag{1.2.6}$$

has an infinitely differentiable density on $\mathbb{R}\times]0,\infty[$. Furthermore, in Theorem 4.2.2, we establish a Gaussian-type upper bound on this density, which provides an alternative method to study the upper bound on hitting probabilities of the solution. To achieve this, we present a formula for this density using the integration by parts formula in Proposition 4.5.6. The main technical effort to analyze this density is Proposition 4.6.2, in which we use the properties of the divergence operator to estimate the Skorohod integral appearing in the formula for the density.

Finally, in Chapter 5, we extend the results of Chapter 4 to the solution of the linear stochastic fractional heat equation. It is known that the fractional differential operator affects the Hölder continuity of the solution. The smoothness of the densities of the supremum of linear increments of the solution over a time segment and of the supremum of the solution over a space-time rectangle still hold: see Theorem 5.1.1. Moreover, in Theorems 5.1.2 and 5.1.4, we show how the corresponding Gaussian-type upper bounds on these densities depend on the degree of the fractional differential operator ${}_xD^\alpha$ in a consistent way.

1.3 Stochastic heat equation

In this section, we give a rigorous formulation for equations (1.2.1) and (1.2.2), following Walsh [81]. Let $\dot{W}^i = (\dot{W}^i(t,x))_{(t,x)\in\mathbb{R}_+\times U}$, $i = 1,\ldots,d$, be independent space-time white noises defined on a probability space $(\Omega,\mathscr{F},\mathrm{P})$. The space-time white noise $\dot{W}^i$ is a distribution-valued and centered Gaussian process with covariance

$$\mathrm{E}[\dot{W}(\varphi)\dot{W}(\psi)] = \int_0^\infty dt \int_U dx\, \varphi(t,x)\psi(t,x),$$

for $\varphi,\psi \in C_0^\infty(\mathbb{R}_+\times U)$ (the space of infinitely differentiable functions with compact support in $\mathbb{R}_+\times U$). For each $t\geq 0$, we denote by $\mathscr{B}([0,t]\times U)$ the collection of Borel sets on $[0,t]\times U$ with finite Lebesgue measure. The filtration generated by the space-time white noise is defined by

$$\mathscr{F}_t = \sigma\{\dot{W}(A),\, A\in\mathscr{B}([0,t]\times U)\}\vee\mathscr{N}, \quad t\geq 0, \tag{1.3.1}$$

where $\mathscr{N}$ is the σ-field generated by P-null sets.

We assume that for all $1\leq i,j\leq d$, the functions $b_i, \sigma_{ij}:\mathbb{R}^d\to\mathbb{R}$ are globally Lipschitz continuous. We set $b=(b_i)$, $\sigma=(\sigma_{ij})$. The fractional differential operator D^α appearing in (1.2.2) will be defined in Section 2.1. If $U=[0,1]$, we impose Neumann or Dirichlet boundary conditions on the solution.

A *mild solution* of (1.2.1) is a jointly measurable $\mathbb{R}^d$-valued process $u=\{u(t,x), t\geq 0, x\in U\}$, adapted to the filtration $(\mathscr{F}_t)_{t\geq 0}$ defined in (1.3.1), such that for $i\in\{1,\ldots,d\}$,

$$\begin{aligned} u_i(t,x) = &\int_0^t\int_U G(t-r,x,v)\sum_{j=1}^d \sigma_{ij}(u(r,v))W^j(dr,dv)\\ &+\int_0^t\int_U G(t-r,x,v)b_i(u(r,v))\,dr\,dv + \int_U G(t,x,v)u_0^i(v)\,dv, \end{aligned} \tag{1.3.2}$$

where $G(t,x,v)$ denotes the Green kernel for the heat equation. A *mild solution* of (1.2.2) is a jointly measurable $\mathbb{R}^d$-valued process $u=\{u(t,x), t\geq 0, x\in\mathbb{R}\}$, adapted to the filtration $(\mathscr{F}_t)_{t\geq 0}$

defined in (1.3.1) with $U = \mathbb{R}$, such that for $i \in \{1, \dots, d\}$,

$$u_i(t,x) = \int_0^t \int_{\mathbb{R}} G_\alpha(t-r, x-v) \sum_{j=1}^d \sigma_{ij}(u(r,v)) W^j(dr, dv) \\ + \int_0^t \int_{\mathbb{R}} G_\alpha(t-r, x-v) b_i(u(r,v)) dr dv + \int_{\mathbb{R}} G_\alpha(t, x-v) u_0^i(v) dv, \tag{1.3.3}$$

where $G_\alpha(t,x)$ denotes the Green kernel for the fractional heat equation:

$$\begin{cases} \frac{\partial}{\partial t} G(t,x) = {}_x D^\alpha G(t,x), & t > 0,\ x \in \mathbb{R}, \\ G(0,x) = \delta_0(x), \end{cases} \tag{1.3.4}$$

where δ_0 is the Dirac distribution.

If $U = [0,1]$, in the case of Neumann boundary conditions, the Green kernel $G(t,x,y)$ for the heat equation is given by

$$G(t,x,y) = \frac{1}{\sqrt{4\pi t}} \sum_{n \in \mathbb{Z}} \left(\exp\left(-\frac{(y-x-2n)^2}{4t} \right) + \exp\left(-\frac{(y+x-2n)^2}{4t} \right) \right), \tag{1.3.5}$$

or in the case of Dirichlet boundary conditions, by

$$G(t,x,y) = \frac{1}{\sqrt{4\pi t}} \sum_{n \in \mathbb{Z}} \left(\exp\left(-\frac{(y-x-2n)^2}{4t} \right) - \exp\left(-\frac{(y+x-2n)^2}{4t} \right) \right); \tag{1.3.6}$$

see [6]. If $U = \mathbb{R}$, the Green kernel $G(t,x,y)$ (denoted by $G(t,x-y)$) for the heat equation without boundary is given by

$$G(t,x) = \frac{1}{\sqrt{4\pi t}} \exp\left(-\frac{x^2}{4t} \right). \tag{1.3.7}$$

The Green kernel for the fractional heat equation is given via Fourier transform. We write $G_\alpha(t,x,v)$ as $G_\alpha(t,x-v)$ and

$$G_\alpha(t,x) = \frac{1}{2\pi} \int_{\mathbb{R}} \exp(-i\lambda x - t|\lambda|^\alpha) d\lambda. \tag{1.3.8}$$

See the Appendix for the properties of this Green kernel.

1.4 Notations for potential theory

In this section, we recall some notations concerning potential theory, from [44]. For all Borel sets $F \subseteq \mathbb{R}^d$, we define $\mathscr{P}(F)$ to be the set of all probability measures with compact support contained in F. For all integers $k \geq 1$ and $\mu \in \mathscr{P}(\mathbb{R}^k)$, we let $I_\beta(\mu)$ denote the β-*dimensional* *energy* of μ, that is,

$$I_\beta(\mu) := \iint \mathrm{K}_\beta(\|x-y\|)\mu(dx)\mu(dy),$$

where $\|x\|$ denotes the Euclidian norm of $x \in \mathbb{R}^k$,

$$\mathrm{K}_\beta(r) := \begin{cases} r^{-\beta} & \text{if } \beta > 0, \\ \log(N_0/r) & \text{if } \beta = 0, \\ 1 & \text{if } \beta < 0, \end{cases} \tag{1.4.1}$$

and N_0 is a sufficiently large constant determined according to the context. For example, in Chapter 2, its value is specified in the proof of Lemmas 2.2.3 and 2.2.4.

For all $\beta \in \mathbb{R}$, integers $k \geq 1$, and Borel sets $F \subseteq \mathbb{R}^k$, $\mathrm{Cap}_\beta(F)$ denotes the β-*dimensional capacity* of F, that is,

$$\mathrm{Cap}_\beta(F) := \left[\inf_{\mu \in \mathscr{P}(F)} I_\beta(\mu)\right]^{-1},$$

where $1/\infty := 0$. Note that if $\beta < 0$, then $\mathrm{Cap}_\beta(\cdot) \equiv 1$.

Given $\beta \geq 0$, the β-dimensional *Hausdorff measure* of F is defined by

$$\mathscr{H}_\beta(F) = \lim_{\epsilon \to 0^+} \inf \left\{ \sum_{i=1}^{\infty} (2r_i)^\beta : F \subseteq \bigcup_{i=1}^{\infty} B(x_i, r_i), \sup_{i \geq 1} r_i \leq \epsilon \right\}.$$

When $\beta < 0$, we define $\mathscr{H}_\beta(F)$ to be infinite.

Throughout this thesis, for $1 < \alpha \leq 2$, we consider the following *fractional parabolic metric*: For all $s, t \in [0, \infty[$ and $x, y \in \mathbb{R}$,

$$\Delta_\alpha((t,x);(s,y)) := |t-s|^{\frac{\alpha-1}{\alpha}} + |x-y|^{\alpha-1}. \tag{1.4.2}$$

Clearly, this is a metric on $\mathbb{R}^2$ which generates the usual Euclidean topology on $\mathbb{R}^2$. We simply write Δ instead of Δ_2 when $\alpha = 2$.

1.5 Elements of Malliavin calculus

In this section, we introduce, following Nualart [64] (see also [78]), some elements of Malliavin calculus. Let $W = \{W(h), h \in \mathscr{H}\}$ denote the isonormal Gaussian process (see [64, Definition 1.1.1]) associated with space-time white noise, where $\mathscr{H}$ is the Hilbert space $L^2([0,T] \times U, \mathbb{R}^d)$. Let $\mathscr{S}$ denote the class of smooth random variables of the form

$$G = g(W(h_1), \ldots, W(h_n)),$$

where $n \geq 1$, $g \in \mathscr{C}_p^\infty(\mathbb{R}^n)$, the set of real-valued functions g such that g and all its partial derivatives have at most polynomial growth and $h_i \in \mathscr{H}$. Given $G \in \mathscr{S}$, its derivative is defined to be the $\mathbb{R}^d$-valued stochastic process $DG = (D_{t,x}G = (D_{t,x}^{(1)}G, \dots, D_{t,x}^{(d)}G), (t,x) \in [0,T] \times U)$ given by

$$D_{t,x}G = \sum_{i=1}^n \partial_i g(W(h_1), \dots, W(h_n)) h_i(t,x).$$

More generally, we can define the derivative $D^k F$ of order k of F by setting

$$D_\alpha^k G = \sum_{i_1,\dots,i_k=1}^n \frac{\partial}{\partial x_{i_1}} \cdots \frac{\partial}{\partial x_{i_k}} g(W(h_1), \dots, W(h_n)) h_{i_1}(\alpha_1) \otimes \cdots \otimes h_{i_k}(\alpha_k),$$

where $\alpha = (\alpha_1, \dots, \alpha_k)$, $\alpha_i = (t_i, x_i), 1 \leq i \leq k$ and the notation $\otimes$ denotes the tensor product of functions.

For $p, k \geq 1$, the space $\mathbb{D}^{k,p}$ is the closure of $\mathscr{S}$ with respect to the seminorm $\|\cdot\|_{k,p}^p$ defined by

$$\|G\|_{k,p}^p = \mathrm{E}[|G|^p] + \sum_{j=1}^k \mathrm{E}\left[\|D^j G\|_{\mathscr{H}^{\otimes j}}^p\right],$$

where

$$\|D^j G\|_{\mathscr{H}^{\otimes j}}^2 = \int_0^T dt_1 \int_U dx_1 \cdots \int_0^T dt_j \int_U dx_j \left(D_{t_1,x_1}^{i_1} \cdots D_{t_j,x_j}^{i_j} G\right)^2.$$

We set $\mathbb{D}^\infty = \cap_{p \geq 1} \cap_{k \geq 1} \mathbb{D}^{k,p}$.

For any given Hilbert space V, the corresponding Sobolev space of V-valued random variables can also be introduced. More precisely, let $\mathscr{S}_V$ denote the family of V-valued smooth random variables of the form

$$G = \sum_{j=1}^n G_j v_j, \quad (v_j, G_j) \in V \times \mathscr{S}.$$

We define

$$D^k G = \sum_{j=1}^n (D^k G_j) \otimes v_j, \quad k \geq 1.$$

Then D^k is a closable operator from $\mathscr{S}_V \subset L^p(\Omega, V)$ into $L^p(\Omega, \mathscr{H}^{\otimes k} \otimes V)$ for any $p \geq 1$. For $p, k \geq 1$, a seminorm is defined on $\mathscr{S}_V$ by

$$\|G\|_{k,p,V}^p = \mathrm{E}\left[\|G\|_V^p\right] + \sum_{j=1}^k \mathrm{E}\left[\|D^j G\|_{\mathscr{H}^{\otimes j} \otimes V}^p\right].$$

We denote by $\mathbb{D}^{k,p}(V)$ the closure of $\mathscr{S}_V$ with respect to the seminorm $\|\cdot\|_{k,p,V}^p$. We set $\mathbb{D}^\infty(V) =$

$\cap_{p\geq 1}\cap_{k\geq 1}\mathbb{D}^{k,p}(V)$.

The derivative operator D on $L^2(\Omega)$ has an adjoint, termed the Skorohod integral and denoted by δ, which is an unbounded and closed operator on $L^2(\Omega,\mathscr{H})$; see [64, Section 1.3]. Its domain, denoted by Dom δ, is the set of elements $u \in L^2(\Omega,\mathscr{H})$ such that there exists a constant c such that $|\mathrm{E}[\langle DG, u\rangle_{\mathscr{H}}]| \leq c\|G\|_{0,2}$, for any $G \in \mathbb{D}^{1,2}$. If $u \in \mathrm{Dom}\,\delta$, then $\delta(u)$ is the element of $L^2(\Omega)$ characterized by the following duality relation:

$$\mathrm{E}[G\delta(u)] = \mathrm{E}\left[\sum_{i=1}^{d}\int_0^T\int_U D_{t,x}^{(j)}G\, u_j(t,x)\,dt\,dx\right], \quad \text{for all } G \in \mathbb{D}^{1,2}.$$

A first application of Malliavin calculus is the following global criterion for existence and smoothness of densities of probability laws.

Theorem 1.5.1 ([64, Proposition 2.1.5] or [78, Theorem 5.2]). *Let $F = (F^1,\dots,F^d)$ be an $\mathbb{R}^d$-valued random vector satisfying the following two conditions:*

(i) $F \in (\mathbb{D}^{\infty})^d$;

(ii) The Malliavin matrix of F defined by $\gamma_F = (\langle DF^i, DF^j\rangle_{\mathscr{H}})_{1\leq i,j\leq d}$ is invertible a.s. and $(\det\gamma_F)^{-1} \in L^p(\Omega)$ for all $p \geq 1$.

Then the probability law of F has an infinitely differentiable density function.

A random vector F that satisfies conditions (i) and (ii) of Theorem 1.5.1 is said to be *nondegenerate*. For a nondegenerate random vector, the following integration by parts formula plays a key role.

Proposition 1.5.2 ([65, Proposition 3.2.1] or [78, Propostion 5.4]). *Let $F = (F^1,\dots,F^d) \in (\mathbb{D}^{\infty})^d$ be a nondegenerate random vector, let $G \in \mathbb{D}^{\infty}$ and let $g \in \mathscr{C}_p^{\infty}(\mathbb{R}^d)$. Fix $k \geq 1$. Then for any multi-index $\alpha = (\alpha_1,\dots,\alpha_k) \in \{1,\dots,d\}^k$, there is an element $H_\alpha(F,G) \in \mathbb{D}^{\infty}$ such that*

$$E[(\partial_\alpha g(F)G)] = E[g(F)H_\alpha(F,G)].$$

In fact, the random variables $H_\alpha(F,G)$ are recursively given by

$$H_\alpha(F,G) = H_{(\alpha_k)}(F, H_{(\alpha_1,\dots,\alpha_{k-1})}(F,G)),$$
$$H_{(i)}(F,G) = \sum_{i=1}^{d}\delta(G(\gamma_F^{-1})_{i,j}DF^j).$$

Proposition 1.5.2 with $G = 1$ and $\alpha = (1,\dots,d)$ implies the following expression for the density of a nondegenerate random vector.

Corollary 1.5.3 ([65, Corollary 3.2.1]). *Let $F = (F^1,\dots,F^d) \in (\mathbb{D}^{\infty})^d$ be a nondegenerate random vector and let $p_F(z)$ denote the density of F. Then for every subset σ of the set of indices $\{1,\dots,d\}$,*

$$p_F(z) = (-1)^{d-|\sigma|}E[1_{\{F^i>z^i, i\in\sigma, F^i<z^i, i\notin\sigma\}}H_{(1,\dots,d)}(F,1)],$$

where $|\sigma|$ is the cardinality of σ, and, in agreement with Proposition 1.5.2,

$$H_{(1,\dots,d)}(F,1) = \delta((\gamma_F^{-1}DF)^d\delta((\gamma_F^{-1}DF)^{d-1}\delta(\cdots\delta((\gamma_F^{-1}DF)^1)\cdots))).$$

The next result gives a criterion for uniform boundedness of the density of a nondegenerate random vector.

Proposition 1.5.4 ([26, Proposition 3.4])**.** *For all $p > 1$ and $l \geq 1$, let $c_1 = c_1(p) > 0$ and $c_2 = c_2(l,p) \geq 0$ be fixed. Let $F \in (\mathbb{D}^\infty)^d$ be a nondegenerate random vector such that*

(a) $E[(\det\gamma_F)^{-p}] \leq c_1$;

(b) $E\Big[\|D^l(F^i)\|^p_{\mathscr{H}^{\otimes l}}\Big] \leq c_2, \quad i = 1,\dots,d.$

Then the density of F is uniformly bounded, and the bound does not depend on F but only on the constants $c_1(p)$ and $c_2(l,p)$.

In order to handle random vectors whose components are not in $\mathbb{D}^\infty$, we recall the following general criterion for smoothness of densities established in [39].

Theorem 1.5.5 ([39, Theorem 2.1] or [64, Theorem 2.1.4])**.** *Let $F = (F^1,\dots,F^d)$ be a random vector whose components are in $\mathbb{D}^{1,2}$. Let A be an open subset of $\mathbb{R}^d$. Suppose that there exist $\mathscr{H}$-valued random variables u_A^j, $j = 1,\dots,d$ and a $d\times d$ random matrix $\gamma_A = (\gamma_A^{i,j})$ such that*

(i) $u_A^j \in \mathbb{D}^\infty(\mathscr{H})$ *for all* $j = 1,\dots,d$,

(ii) $\gamma_A^{i,j} \in \mathbb{D}^\infty$ *for all* $i,j = 1,\dots,d$, *and* $|\det\gamma_A|^{-1} \in L^p(\Omega)$ *for all* $p \geq 1$,

(iii) $\langle DF^i, u_A^j\rangle_{\mathscr{H}} = \gamma_A^{i,j}$ *on* $\{F \in A\}$, *for all* $i,j = 1,\dots,d$.

Then the random vector possesses an infinitely differentiable density on the open set A.

A random vector F that satisfies the conditions in Theorem 1.5.5 is said to be *locally nondegenerate.*

Throughout the thesis, the letters C, c with or without index will denote generic positive constants whose values may change from line to line, unless specified otherwise.

1.6 Summary of the thesis

In this section, we summarize the main contributions of this thesis.

1.6.1 Lower bound on hitting probabilities

Following the general approach in [25], the lower bounds in (1.1.2) and (1.1.3) are a consequence of the following properties of the one-point and two-point joint probability density functions of the solution. Let $p_{t,x}(\cdot)$ and $p_{s,y;t,x}(\cdot,\cdot)$ denote the densities of $u(t,x)$ and $(u(s,y),u(t,x))$ respectively.

L1 For all $M>0$, there exists a positive and finite constant $C=C(I,J,M,d)$ such that for all $z\in[-M,M]^d$,

$$\int_I dt\int_J dx\, p_{t,x}(z)\geq C. \tag{1.6.1}$$

L2 For all $M>0$, there exists $c=c(I,J,M,d)>0$ such that for all $s,t\in I$ and $x,y\in J$ with $(t,x)\neq(s,y)$, and for every $z_1,z_2\in[-M,M]^d$,

$$p_{s,y;t,x}(z_1,z_2)\leq\frac{c}{[|t-s|^{1/2}+|x-y|]^{\beta/2}}\exp\left(-\frac{\|z_1-z_2\|^2}{c\,(|t-s|^{1/2}+|x-y|)}\right). \tag{1.6.2}$$

In the Gaussian case, i.e., $\sigma\equiv\mathrm{Id}$, $b\equiv 0$, the formulas for densities are given in terms of variance-covariance matrix. We can analyze this variance-covariance matrix to obtain the estimate in (1.6.2) with $\beta=d$.

In the case where the solution is not Gaussian, in other words, the entries of σ are not constants, we impose some regularity conditions on σ and b; see hypotheses **P1** (or the weaker hypothesis **P1'**) and **P2** in Section 2.1. Using techniques of Malliavin calculus, Dalang, Khoshnevisan and Nualart [26] showed that the estimate in (1.6.2) for non-Gaussian solutions holds with $\beta=d+\eta$ and with the constant c also depending on η.

The first contribution of this thesis is the sharpening of the estimate on the two-point density of the non-Gaussian solution so that it has the same Gaussian-type upper bound as the two-point density of the Gaussian solution. And then we extend this result to the solution of a system of stochastic fractional heat equations ($1<\alpha\leq 2$). In fact, we have established the following properties on the densities of the solution to (1.3.3).

Theorem 1.6.1 (Theorem 2.1.1)**.** *Assume* ***P1*** *and* ***P2****. Fix $T>0$ and let $I\subset]0,T]$ and $J\subset\mathbb{R}$ be two fixed non-trivial compact intervals.*

(a) The density $p_{t,x}(z)$ is a smooth function in z and is uniformly bounded over $z\in\mathbb{R}^d$, $t\in I$ and $x\in J$.

(b) For all $(t,x)\in\,]0,T]\times\mathbb{R}$ and $z\in\mathbb{R}^d$, the density $p_{t,x}(z)$ is strictly positive.

(c) There exists $c>0$ such that for all $s,t\in I$, $x,y\in J$ with $(s,y)\neq(t,x)$ and $z_1,z_2\in\mathbb{R}^d$,

$$p_{s,y;t,x}(z_1,z_2)\leq c(|t-s|^{\frac{\alpha-1}{\alpha}}+|x-y|^{\alpha-1})^{-d/2}\exp\left(-\frac{\|z_1-z_2\|^2}{c(|t-s|^{\frac{\alpha-1}{\alpha}}+|x-y|^{\alpha-1})}\right). \tag{1.6.3}$$

Moreover, we establish the following lower and upper bounds on hitting probabilities.

Theorem 1.6.2 (Theorem 2.1.4)**.** *Assume* ***P1'*** *and* ***P2****. Fix $T>0, M>0$ and $\eta>0$. Let $I\subset]0,T]$ and $J\subset\mathbb{R}$ be two fixed non-trivial compact intervals.*

(a) There exists $c>0$ depending on I,J and M such that for all compact sets $A\subseteq[-M,M]^d$,

$$\mathrm{P}\{u(I\times J)\cap A\neq\emptyset\}\geq c\,Cap_{d-\frac{2(\alpha+1)}{\alpha-1}}(A).$$

(b) *There exists $C > 0$ depending on I, J and η such that for all compact sets $A \subset \mathbb{R}^d$,*

$$\mathrm{P}\{u(I \times J) \cap A \neq \emptyset\} \leq C \mathscr{H}_{d-\eta-\frac{2(\alpha+1)}{\alpha-1}}(A).$$

In particular, in the case $\alpha = 2$, which corresponds to the equation (1.3.2), the lower bound in part (a) improves the lower bound in (1.1.3) and is best possible, in view of (1.1.2).

The estimate on the two-point density in (1.6.3) requires a detailed analysis of the behavior of the Malliavin matrix of $(u(s,y), u(t,x))$. Since we are interested in how this density blows up as $(t,x) \to (s,y)$, we have studied the density (denoted by $p_Z(\cdot,\cdot)$) of the random variable Z defined by

$$Z := (u(s,y), u(t,x) - u(s,y)). \tag{1.6.4}$$

These two densities $p_{s,y;t,x}(\cdot,\cdot)$ and $p_Z(\cdot,\cdot)$ are related by

$$p_{s,y;t,x}(z_1, z_2) = p_Z(z_1, z_2 - z_1), \quad z_1, z_2 \in \mathbb{R}^d.$$

Let γ_Z be the Malliavin matrix of Z. Note that $\gamma_Z = ((\gamma_Z)_{m,l})_{m,l=1,\dots,2d}$ is a symmetric $2d \times 2d$ random matrix with four $d \times d$ blocs of the form

$$\gamma_Z = \begin{pmatrix} \gamma_Z^{(1)} & \vdots & \gamma_Z^{(2)} \\ \cdots & \vdots & \cdots \\ \gamma_Z^{(3)} & \vdots & \gamma_Z^{(4)} \end{pmatrix}$$

where

$$\begin{aligned}
\gamma_Z^{(1)} &= \left(\langle D(u_i(s,y)), D(u_j(s,y))\rangle_{\mathscr{H}}\right)_{i,j=1,\dots,d}, \\
\gamma_Z^{(2)} &= \left(\langle D(u_i(s,y)), D(u_j(t,x) - u_j(s,y))\rangle_{\mathscr{H}}\right)_{i,j=1,\dots,d}, \\
\gamma_Z^{(3)} &= \left(\langle D(u_i(t,x) - u_i(s,y)), D(u_j(s,y))\rangle_{\mathscr{H}}\right)_{i,j=1,\dots,d}, \\
\gamma_Z^{(4)} &= \left(\langle D(u_i(t,x) - u_i(s,y)), D(u_j(t,x) - u_j(s,y))\rangle_{\mathscr{H}}\right)_{i,j=1,\dots,d}.
\end{aligned}$$

Under the hypotheses **P1'** and **P2**, Malliavin calculus provides a formula for the density $p_Z(\cdot,\cdot)$. Following the general approach in [26], the main effort is to estimate the negative moments of the determinant of γ_Z. We have obtained the following

Proposition 1.6.3 (Proposition 2.5.8)**.** *Fix $T > 0$ and let $I \subset]0,T]$ and $J \subset \mathbb{R}$ be two fixed non-trivial compact intervals. Assume **P1'** and **P2**. There exists C depending on T such that for any $(s,y), (t,x) \in I \times J$, $(s,y) \neq (t,x)$, $p > 1$,*

$$E\left[(\det \gamma_Z)^{-p}\right]^{1/p} \leq C(|t-s|^{\frac{\alpha-1}{\alpha}} + |x-y|^{\alpha-1})^{-d}. \tag{1.6.5}$$

In [26, Proposition 6.6 (a)], an extra exponent η appears in the estimate of the negative moments of the determinant of γ_Z; there, the exponent d in (1.6.5) is replaced by $d+\eta$.

The main idea for the proof of Proposition 1.6.3 is to use a perturbation argument. Indeed, for (t,x) close to (s,y), the random matrix γ_Z is close to

$$\bar{\gamma} = \begin{pmatrix} \gamma_Z^{(1)} & \vdots & 0 \\ \cdots & \vdots & \cdots \\ 0 & \vdots & 0 \end{pmatrix}.$$

We then expect that for (t,x) close to (s,y), there will be d large eigenvalues of γ_Z which will contribute a factor of order 1 to the determinant of γ_Z, and d small eigenvalues of γ_Z, that will each contribute a factor of order $|t-s|^{\frac{\alpha-1}{\alpha}} + |x-y|^{\alpha-1}$ to the determinant of γ_Z.

Let us concentrate on the small eigenvalues. Since we are in the situation of negative moments, we can use the smallest eigenvalue $\inf_{\xi\in\mathbb{R}^{2d}} \xi^T\gamma_Z\xi$ of γ_Z to control the small eigenvalues. In fact, we have proved the following

Proposition 1.6.4 (Proposition 2.5.10)**.** *Fix $T>0$. Assume **P1'** and **P2**. There exists C depending on T such that for all $s,t\in I, 0\le t-s<1, x,y\in J, (s,y)\neq(t,x)$, and $p>1$,*

$$E\left[\left(\inf_{\xi=\in\mathbb{R}^{2d}} \xi^T\gamma_Z\xi\right)^{-dp}\right] \le C(|t-s|^{\frac{\alpha-1}{\alpha}} + |x-y|^{\alpha-1})^{-pd}. \tag{1.6.6}$$

The presence of η in the previous work [26] is due to their method of proof. We address this problem by giving a better estimate on the Malliavin derivative of the solution; see the following lemma, which is an improvement of Lemma 4.2 in [60].

Lemma 1.6.5 (Lemma A.3.3)**.** *Fix $T>0, c_0>1$ and $0<\gamma_0<1$. For all $q\ge1$ there exists $C>0$ such that for all $T\ge t\ge s\ge\epsilon>0$ with $t-s>c_0\epsilon^{\gamma_0}$ and $x\in\mathbb{R}$,*

$$\sum_{k,i=1}^{d} E\left[\left(\int_{s-\epsilon}^{s} dr\int_{\mathbb{R}} dv\left(D_{r,v}^{(k)}(u_i(t,x))\right)^2\right)^q\right] \le C\epsilon^{(1-\gamma_0+\gamma_0\frac{\alpha-1}{\alpha})q}.$$

1.6.2 Probability density function of the supremum

For the upper bound on hitting probabilities of the solution to (1.3.2), it suffices to estimate appropriately the probability in (1.2.4). By the triangle inequality,

$$
\begin{aligned}
&\mathrm{P}\left\{\inf_{(t,x)\in R^n_{k,l}} |u(t,x)-z| \le 2^{-n}\right\} \\
&\quad \le \mathrm{P}\left\{|u(t^n_k,x^n_l)-z| \le 2^{-n} + \sup_{(t,x)\in R^n_{k,l}} |u(t,x)-u(t^n_k,x^n_l)|\right\}. \qquad (1.6.7)
\end{aligned}
$$

Since the supremum of the absolute value of a continuous function is equal to either the maximum of the function, or the minimum of the function times -1, the probability in (1.6.7) is approximately equal to

$$
2\cdot\mathrm{P}\left\{|u(t^n_k,x^n_l)-z| \le 2^{-n} + \sup_{(t,x)\in R^n_{k,l}} u(t,x)-u(t^n_k,x^n_l)\right\}. \qquad (1.6.8)
$$

Formally, the random variables $u(t^n_k,x^n_l)$ and $\sup_{(t,x)\in R^n_{k,l}} u(t,x)-u(t^n_k,x^n_l)$ are not independent, but from the perspective of probability density functions, we expect that the joint density (denoted by $p_n(\cdot,\cdot)$, whose existence needs to be proved) of the random vector in (1.2.5) is bounded above by the product of the marginal densities of the components (times a constant). The density of $u(t^n_k,x^n_l)$ is bounded uniformly over (t^n_k,x^n_l); see [26, Theorem 1.1(a)] and our Theorem 2.1.1(a). So the joint density of this random vector is dominated by the density of the random variable $\sup_{(t,x)\in R^n_{k,l}} u(t,x)-u(t^n_k,x^n_l)$.

To derive a satisfactory estimate for the density of $\sup_{(t,x)\in R^n_{k,l}} u(t,x)-u(t^n_k,x^n_l)$, recall that the probability density function of the maximum of Brownian motion $\max_{0\le t\le T} B(t)$ given by

$$
z \mapsto \frac{2}{\sqrt{2\pi T}} \exp\left(\frac{-z^2}{2T}\right) 1_{]0,\infty[}(z).
$$

Relating this formula to the fact that the sample paths of Brownian motion are almost $\frac{1}{2}$-Hölder continuous suggests that the joint density $p_n(\cdot,\cdot)$ should satisfy the following estimate:

$$
\begin{aligned}
p_n(z_1,z_2) &\le \frac{c}{\sqrt{(2^{-4n})^{1/2}+2^{-2n}}} \exp\left(\frac{-z_2^2}{c\left((2^{-4n})^{1/2}+2^{-2n}\right)}\right) \qquad (1.6.9)\\
&= c\,2^n \exp\left(\frac{-z_2^2}{c\,2^{-2n}}\right), \qquad \text{for all} \quad z_1\in\mathbb{R},\ z_2>0. \qquad (1.6.10)
\end{aligned}
$$

We are aiming at obtaining an estimate as in (1.6.9) for the solution of (1.3.2). In this thesis, we only consider the linear equation, i.e., $\sigma\equiv 1$, $b\equiv 0$, in which case the solution is Gaussian. And we consider the supremum of an increment in time of the solution over an interval (at a fixed spatial position), and the supremum of the solution over a space-time rectangle that touches the $t = 0$ axis.

Fix two compact intervals $I \subset]0, T]$ and $J \subset]0, 1[$ with positive length. Choose $(s_0, y_0) \in I \times J$ and small positive numbers δ_1, δ_2. Set

$$F_1 = u(s_0, y_0), \quad F_2 = \sup_{t \in [s_0, s_0+\delta_1]} (u(t, y_0) - u(s_0, y_0)), \quad F = (F_1, F_2), \tag{1.6.11}$$

and

$$M_0 = \sup_{(t,x) \in [0,\delta_1] \times [y_0, y_0+\delta_2]} u(t, x). \tag{1.6.12}$$

Define the random variables $u_A = (u_A^1, u_A^2)$, $\gamma_A = (\gamma_A^{i,j})_{1 \le i,j \le 2}$ and $u_{\bar{A}}$, $\gamma_{\bar{A}}$ as in Section 4.5. We show that the random variables F and M_0 are locally nondegenerate and therefore, by Theorem 1.5.5, they have infinitely differentiable densities. Moreover, the integration by parts formula established in the proof of Theorem 1.5.5 (see [64, Theorem 2.1.4]) leads us to a formula for the density of each random variable. Then we use the properties of the divergence operator to give a Gaussian-type upper bound on this (joint) density.

The main results concerning the upper bound on hitting probabilities in this thesis are the following theorems, in which we refer to Chapter 4 for the definition of the random variables and vectors u_A, γ_A, $u_{\bar{A}}$ and $\gamma_{\bar{A}}$.

Theorem 1.6.6 (Theorem 4.2.1(a) and Theorem 4.2.2)**.** *Assume $\sigma \equiv 1$ and $b \equiv 0$. Fix $I \times J \subset]0, T] \times]0, 1[$.*

(i) The random vector F in (1.6.11) *possesses an infinitely differentiable density on* $\mathbb{R} \times]0, \infty[$ *and the formula of the density is given by*

$$p(z_1, z_2) = E\left[1_{\{F_1 > z_1, F_2 > z_2\}} \delta\left(u_A^1 \delta\left(u_A^2 / \gamma_A^{2,2}\right)\right)\right], \qquad \text{for all } z_1 \in \mathbb{R},\ z_2 > 0. \tag{1.6.13}$$

(ii) There exists a constant $c = c(I, J)$ such that for all small $\delta_1 > 0$, and for $z_2 \ge \delta_1^{1/4}$, $z_1 \in \mathbb{R}$ and any $(s_0, y_0) \in I \times J$,

$$p(z_1, z_2) \le \frac{c}{\sqrt{\delta_1^{1/2}}} \exp\left(-\frac{z_2^2}{c\,\delta_1^{1/2}}\right) (|z_1|^{-\frac{1}{4}} \wedge 1) \exp(-\frac{z_1^2}{c}). \tag{1.6.14}$$

The estimate in (1.6.14) is only valid for $z_2 \ge \delta_1^{1/4}$. But this is sufficient to obtain the upper bound on hitting probabilities: see Section 4.1.

Theorem 1.6.7 (Theorem 4.2.1(b) and Theorem 4.2.5)**.** *Assume $\sigma \equiv 1$ and $b \equiv 0$. Fix $J \subset]0, 1[$.*

(i) The random variable M_0 in (1.6.12) *possesses an infinitely differentiable density on* $]0, \infty[$ *and the formula of the density is given by*

$$p_0(z) = E\left[1_{\{M_0 > z\}} \delta\left(u_{\bar{A}} / \gamma_{\bar{A}}\right)\right], \qquad \text{for all } z > 0. \tag{1.6.15}$$

(ii) There exists a constant $c = c(T, J)$ such that for all small $\delta_1, \delta_2 > 0$, and for $z \geq (\delta_1^{1/2} + \delta_2)^{1/2}$ and any $y_0 \in J$,

$$p_0(z) \leq \frac{c}{\sqrt{\delta_1^{1/2} + \delta_2}} \exp\left(-\frac{z^2}{c(\delta_1^{1/2} + \delta_2)}\right). \tag{1.6.16}$$

We give some explanation on the bound in (1.6.14) (the method to prove the bound in (1.6.16) is the same). The bound in (1.6.14) follows from the formula for the density in (1.6.13) by proceeding as follows. First, by Hölder's inequality, we have

$$p(z_1, z_2) \leq \mathrm{P}\{|F_1| > |z_1|\}^{1/4} \mathrm{P}\{F_2 > z_2\}^{1/4} \|\delta(\delta(u_A^2/\gamma_A^{2,2}) u_A^1)\|_{L^2(\Omega)}. \tag{1.6.17}$$

Since F_1 is a Gaussian random variable, the estimate on the tail probability $\mathrm{P}\{|F_1| > |z_1|\}$ corresponds to the factors involving the variable z_1 in (1.6.14). Using Borell's inequality, $\mathrm{P}\{F_2 > z_2\}^{1/4}$ is bounded above by the exponential factor involving the variable z_2 in (1.6.14). It remains to prove that the $L^2(\Omega)$-norm of the random variable $\delta(\delta(u_A^2/\gamma_A^{2,2}) u_A^1)$ is bounded above by $\delta_1^{-1/4}$ times a constant. We use the properties of the Skorohod integral δ to express $\delta(\delta(u_A^2/\gamma_A^{2,2}) u_A^1)$ as

$$\delta(\delta(u_A^2/\gamma_A^{2,2}) u_A^1) = T_1 + T_2 - T_3 + T_4 - T_5 + T_6,$$

where

$$T_1 = \frac{\delta(u_A^2)}{\gamma_A^{2,2}} \delta(u_A^1), \quad T_2 = \frac{\langle D\gamma_A^{2,2}, u_A^2 \rangle_{\mathscr{H}}}{(\gamma_A^{2,2})^2} \delta(u_A^1), \quad T_3 = \frac{1}{\gamma_A^{2,2}} \langle D\delta(u_A^2), u_A^1 \rangle_{\mathscr{H}},$$
$$T_4 = \frac{\delta(u_A^2)}{(\gamma_A^{2,2})^2} \langle D\gamma_A^{2,2}, u_A^1 \rangle_{\mathscr{H}}, \quad T_5 = \frac{2\langle D\gamma_A^{2,2}, u_A^2 \rangle_{\mathscr{H}}}{(\gamma_A^{2,2})^3} \langle D\gamma_A^{2,2}, u_A^1 \rangle_{\mathscr{H}},$$
$$T_6 = \frac{1}{(\gamma_A^{2,2})^2} \langle D\langle D\gamma_A^{2,2}, u_A^2 \rangle_{\mathscr{H}}, u_A^1 \rangle_{\mathscr{H}}.$$

Due to our choice of u_A and γ_A, the last three terms T_4, T_5 and T_6 vanish. To estimate the moments of the Skorohod integrals, for example $\delta(u_A^2)$, one can use the Hölder's inequality for Malliavin norms (see [64, Proposition 1.5.7]), but the upper bound is not of the correct order. To handle this problem, we use the fact that we have defined u_A^2 so that it is an adapted process; in this case, the Skorohod integral $\delta(u_A^2)$ coincides with a Walsh integral. Then we apply Burkholder's inequality to bound the moments of the three terms T_1, T_2, T_3 respectively, and each of them will give us the correct upper bound of the order $\delta_1^{-1/4}$ for all $z_2 \geq \delta_1^{1/4}$. We state the main technical effort to prove (1.6.14) as follows.

Proposition 1.6.8 (Proposition 4.6.2)**.** *(a) There exists $c_p > 0$, not depending on $(s_0, y_0) \in I \times J$, such that for all $\delta_1 > 0$ and for all $z_2 \geq \delta_1^{1/4}$,*

$$\|T_i\|_{L^p(\Omega)} \leq c_p \delta_1^{-1/4}, \quad \textit{for } i \in \{1, 2, 3\}. \tag{1.6.18}$$

(b) T_4, T_5 *and* T_6 *vanish.*

This entire procedure can be extended to the case of the linear stochastic fractional heat equation (1.2.2). Let u solve linear stochastic fractional heat equation (1.2.2) with $d = 1$, $\sigma \equiv 1$, $b \equiv 0$ and let F and M_0 be defined as in (1.6.11) and (1.6.12) respectively with this u.

Theorem 1.6.9 (Theorem 5.1.1(a) and Theorem 5.1.2)**.** *Assume* $\sigma \equiv 1$ *and* $b \equiv 0$*. Fix two compact intervals* $I \subset]0, T]$ *and* $J \subset \mathbb{R}$ *with positive length. The random vector* F *has an infinitely differentiable density on* $\mathbb{R}\times]0,\infty[$*, denoted by* $(z_1, z_2) \mapsto p(z_1, z_2)$*. And there exists a constant* $c = c(I, J)$ *such that for all small* $\delta_1 > 0$*, and for* $z_2 \geq \delta_1^{(\alpha-1)/(2\alpha)}$*,* $z_1 \in \mathbb{R}$ *and any* $(s_0, y_0) \in I \times J$*,*

$$p(z_1, z_2) \leq \frac{c}{\sqrt{\delta_1^{(\alpha-1)/\alpha}}} \exp\left(-\frac{z_2^2}{c\,\delta_1^{(\alpha-1)/\alpha}}\right)(|z_1|^{-\frac{1}{4}} \wedge 1) \exp\left(-\frac{z_1^2}{c}\right).$$

Theorem 1.6.10 (Theorem 5.1.1(b) and Theorem 5.1.4)**.** *Assume* $\sigma \equiv 1$ *and* $b \equiv 0$*. Fix a compact interval* $J \subset \mathbb{R}$ *with positive length. The random variable* M_0 *has an infinitely differentiable density on* $]0,\infty[$*, denoted by* $z \mapsto p_0(z)$*. And there exists a constant* $c = c(T, J)$ *such that for all small* $\delta_1, \delta_2 > 0$*, and for* $z \geq (\delta_1^{(\alpha-1)/\alpha} + \delta_2^{\alpha-1})^{1/2}$ *and any* $y_0 \in J$*,*

$$p_0(z) \leq \frac{c}{\sqrt{\delta_1^{(\alpha-1)/\alpha} + \delta_2^{\alpha-1}}} \exp\left(-\frac{z^2}{c\,(\delta_1^{(\alpha-1)/\alpha} + \delta_2^{\alpha-1})}\right).$$

Finally, we also study the supremum of certain increments of the solution over a space-time rectangle. Define the random variable M by

$$M := \sup_{(t,x)\in[0,T]\times[0,1]} (u(t,x) - u(t,0)).$$

We apply Theorem 1.5.5 to prove the smoothness of the density of the random variable M for the solution with Neumann boundary conditions. The method is different from that for the random vector F defined in (1.6.11), and similar to the case of Brownian sheet.

Theorem 1.6.11 (Theorem 4.2.1(b))**.** *In the case of Neumann boundary conditions, the random variable* M *has an infinitely differentiable density on* $]0,\infty[$.

2 Hitting probabilities for systems of stochastic heat equations with multiplicative noise

In this chapter, we study hitting probabilities for the solution to systems of non-linear stochastic fractional heat equations. Using techniques of Malliavin calculus, we first derive the upper bound on the one-point density of the solution $u(t,x)$. Secondly, we prove the positivity of the one-point density of the solution $u(t,x)$. Furthermore, we establish the Gaussian-type upper bound on the two-point density function of $(u(t,x), u(s,y))$, which corresponds exactly to the best upper bound that is available in the case of Gaussian processes. From these results, we deduce upper and lower bounds on hitting probabilities of the process $\{u(t,x) : (t,x) \in \mathbb{R}_+ \times \mathbb{R}\}$, in terms of Hausdorff measure and Newtonian capacity, respectively.

2.1 Introduction and main results

We consider a system of non-linear stochastic fractional heat equations with vanishing initial conditions on the whole space $\mathbb{R}$, that is,

$$\frac{\partial u_i}{\partial t}(t,x) = {}_xD^\alpha u_i(t,x) + \sum_{j=1}^{d} \sigma_{ij}(u(t,x))\dot{W}^j(t,x) + b_i(u(t,x)), \tag{2.1.1}$$

for $1 \le i \le d$, $t \in [0,T]$, $x \in \mathbb{R}$, where $u := (u_1, \ldots, u_d)$, with initial conditions $u(0,x) = 0$ for all $x \in \mathbb{R}$. Here, $\dot{W} := (\dot{W}^1, \ldots, \dot{W}^d)$ is a vector of d independent space-time white noises on $[0,T] \times \mathbb{R}$ defined on a probability space $(\Omega, \mathscr{F}, \mathrm{P})$. The fractional differential operator D^α is given by

$$D^\alpha \varphi(x) = \mathscr{F}^{-1}\{-|\lambda|^\alpha \mathscr{F}\{\varphi(x);\lambda\}; x\}, \tag{2.1.2}$$

where $\mathscr{F}$ denotes the Fourier transform. The fractional differential operator D^α coincides with the fractional power of the Laplacian. When $\alpha = 2$, it is Laplacian itself. For $1 < \alpha < 2$, it can also be represented by

$$D^\alpha \varphi(x) = c_\alpha \int_{\mathbb{R}} \frac{\varphi(x+y) - \varphi(x) - y\varphi'(x)}{|y|^{1+\alpha}} dy \tag{2.1.3}$$

with certain positive constant c_α depending only on α; see [33], [34], [50] and [17]. We refer to [48] for additional equivalent definitions of D^α.

Consider the following three hypotheses on the coefficients of the system (2.1.1):

P1 The functions σ_{ij} and b_i are bounded and infinitely differentiable with bounded partial derivatives of all orders, for $1 \le i, j \le d$.

P1' The functions σ_{ij} and b_i are infinitely differentiable with bounded partial derivatives of all positive orders, and the σ_{ij} are bounded, for $1 \le i, j \le d$.

P2 The matrix σ is uniformly elliptic, that is, $\|\sigma(x)\xi\|^2 \ge \rho^2 > 0$ for some $\rho > 0$, for all $x \in \mathbb{R}^d$, $\|\xi\| = 1$.

Notice that hypothesis **P1'** is weaker than hypothesis **P1**, since in **P1'**, the functions b_i, $i = 1,\dots,d$ are not assumed to be bounded.

Recall from Section 1.3 that a *mild solution* of (2.1.1) is a jointly measurable $\mathbb{R}^d$-valued process $u = \{u(t,x), t \ge 0, x \in \mathbb{R}\}$, adapted to the filtration $(\mathscr{F}_t)_{t\ge 0}$ defined in (1.3.1), such that for $i \in \{1,\dots,d\}$,

$$\begin{aligned} u_i(t,x) = &\int_0^t \int_{\mathbb{R}} G_\alpha(t-r, x-v) \sum_{j=1}^d \sigma_{ij}(u(r,v)) W^j(dr, dv) \\ &+ \int_0^t \int_{\mathbb{R}} G_\alpha(t-r, x-v) b_i(u(r,v)) dr dv, \end{aligned} \tag{2.1.4}$$

where the Green kernel $G_\alpha(t,x)$ is given in (1.3.8), and the stochastic integral in (2.1.4) is interpreted as in [81]. In fact, to make sense of the stochastic integral in (2.1.4), the function $(r,v) \mapsto 1_{\{r<t\}} G_\alpha(t-r, x-v)$ must belongs to $L^2([0,T]\times\mathbb{R})$. This is why the requirement that $1 < \alpha \le 2$ is needed; see also [17, 34].

The problems of existence, uniqueness and Hölder continuity of the solution to non-linear stochastic fractional heat equations have been studied by many authors; see, e.g., [5, 12, 17, 34] and the references therein. Adapting these results to the case $d \ge 1$, one can show that there exists a unique process $u = \{u(t,x), t \ge 0, x \in \mathbb{R}\}$ that is a mild solution of (2.1.1), such that for any $T > 0$ and $p \ge 1$,

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}} \mathrm{E}\left[|u_i(t,x)|^p\right] < \infty, \quad i \in \{1,\dots,d\}. \tag{2.1.5}$$

Moreover, the following estimate holds for the moments of increments of the solution: for all $s, t \in [0,T]$, $x, y \in \mathbb{R}$ and $p > 1$,

$$\mathrm{E}[\|u(t,x) - u(s,y)\|^p] \le C_{T,p} (\Delta_\alpha((t,x);(s,y)))^{p/2}, \tag{2.1.6}$$

where Δ_α is the fractional parabolic metric defined in (1.4.2). We will also establish an analogous estimate on the Hölder continuity of the Malliavin derivative of the solution; see Proposition 2.5.2. We denote by $K_m = [0,m] \times [-m,m]$ and $\beta_p = 1 - \frac{2(\alpha+1)}{p(\alpha-1)}$ with $p > \frac{2(\alpha+1)}{\alpha-1}$. By Kolmogorov's continuity theorem (see [51, Theorem 1.4.1, p. 31] and [18, Proposition 4.2]), the solution u has a continuous modification which we continue to denote by u that satisfies, for all integers m and $0 \le \beta < \beta_p$,

$$\mathrm{E}\left[\left(\sup_{\substack{(t,x),(s,y)\in K_m\\(t,x)\neq(s,y)}} \frac{\|u(t,x)-u(s,y)\|}{[|t-s|^{(\alpha-1)/(2\alpha)}+|x-y|^{(\alpha-1)/2}]^\beta}\right)^p\right] < \infty. \tag{2.1.7}$$

Let $I \subset]0,T]$ and $J \subset \mathbb{R}$ be two fixed compact intervals with positive length. We choose m sufficiently large so that $I \times J \subset K_m$.

Adapting the results from [12] to the case $d \ge 1$, the $\mathbb{R}^d$-valued random vector $u(t,x) = (u_1(t,x),\ldots,u_d(t,x))$ admits a smooth probability density function, denoted by $p_{t,x}(\cdot)$ for all $(t,x) \in [0,T]\times\mathbb{R}$: see our Proposition 2.3.2. For $(s,y) \neq (t,x)$, let $p_{s,y;t,x}(\cdot,\cdot)$ denote the joint density function of the $\mathbb{R}^{2d}$-valued random vector

$$(u(s,y),u(t,x)) = (u_1(s,y),\ldots,u_d(s,y),u_1(t,x),\ldots,u_d(t,x)) \tag{2.1.8}$$

(the existence of $p_{s,y;t,x}(\cdot,\cdot)$ is a consequence of our Theorem 1.5.1, (2.3.4) and Proposition 2.5.8).

Theorem 2.1.1. *Assume* ***P1*** *and* ***P2****. Fix $T > 0$ and let $I \subset]0,T]$ and $J \subset \mathbb{R}$ be two fixed non-trivial compact intervals.*

(a) The density $p_{t,x}(z)$ is a smooth function in z and is uniformly bounded over $z \in \mathbb{R}^d$, $t \in I$ and $x \in J$.

(b) For all $(t,x) \in]0,T]\times\mathbb{R}$ and $z \in \mathbb{R}^d$, the density $p_{t,x}(z)$ is strictly positive.

(c) There exists $c > 0$ such that for all $s,t \in I$, $x,y \in J$ with $(s,y) \neq (t,x)$ and $z_1,z_2 \in \mathbb{R}^d$,

$$p_{s,y;t,x}(z_1,z_2) \le c(|t-s|^{\frac{\alpha-1}{\alpha}}+|x-y|^{\alpha-1})^{-d/2}\exp\left(-\frac{\|z_1-z_2\|^2}{c(|t-s|^{\frac{\alpha-1}{\alpha}}+|x-y|^{\alpha-1})}\right). \tag{2.1.9}$$

Remark 2.1.2. *(a) Theorem 2.1.1(a) remains valid under a slightly weaker version of* ***P1****, in which the b_i, σ_{ij} need not be bounded (but their derivatives of all positive orders are bounded).*

(b) Theorem 2.1.1(b) remains valid under ***P1'****.*

(c) With hypothesis ***P1*** *replaced by the slightly weaker version* ***P1'*** *in Theorem 2.1.1, the statements (a) and (b) remain valid and statement (c) is replaced by:*

(c') *There exists $c > 0$ such that for all $s, t \in I, x, y \in J$ with $(s, y) \neq (t, x)$, $z_1, z_2 \in \mathbb{R}^d$ and $p \geq 1$,*

$$p_{s,y;t,x}(z_1, z_2) \leq c(|t-s|^{\frac{\alpha-1}{\alpha}} + |x-y|^{\alpha-1})^{-d/2} \left[\frac{|t-s|^{\frac{\alpha-1}{\alpha}} + |x-y|^{\alpha-1}}{\|z_1 - z_2\|^2} \wedge 1\right]^{p/(4d)}. \tag{2.1.10}$$

In fact, the boundedness of the functions $b_i = 0, i = 1, \ldots, d$ in hypothesis **P1** is only used when we derive the exponential factor on the right-hand side of (2.1.9) by applying Girsanov's theorem. However, under the hypothesis **P1'**, when b_i is not bounded, Girsanov's theorem is no longer applicable. We establish (2.1.10) in Section 2.5.3 and, following [27, 31], show in Section 2.2.3 that this estimate is also sufficient for our purposes.

Remark 2.1.3. *The results of Theorem 2.1.1 and Remark 2.1.2 (as well as Theorems 2.1.4, 2.1.5 below) include the case $\alpha = 2$, that is, they apply to the solutions of the stochastic heat equations with Neumann or Dirichlet boundary conditions; see Remark 2.5.14.*

We prove the smoothness and uniform boundedness of the one-point density (Theorem 2.1.1(a)) in Section 2.3. The proof of strict positivity of the one-point density (Theorem 2.1.1(b)) is given in Section 2.4.3. We present the Gaussian-type upper bound on the two-point density (Theorem 2.1.1(c)) in Section 2.5.3.

Our main contribution is to obtain the Gaussian-type upper bound in (c), which is a significant improvement of Theorem 1.1(c) in [26]. In fact, for the stochastic heat equation, the optimal Gaussian-type upper bound holds when $t = s$, while an extra term η appears in the exponent when $t \neq s$; see Theorem 1.1 in Dalang, Khoshnevisan and Nualart [26]. We improve their result by a detailed analysis of the small eigenvalues of the Malliavin matrix of $(u(t,x), u(s,y))$ as a function of (s, y, t, x). To be more precise, we achieve this by giving a better estimate on the Malliavin derivative of the solution; see Lemma A.3.3, which is an improvement of Lemma 4.2 in Morien [60]; see the discussion in Section 1.2. We point out that the Gaussian-type upper bound for the two-point joint density of the solution plays a crucial role in the study of the lower bound on the hitting probabilities. The estimate in Theorem 2.1.1(c) leads to the optimal lower bound for the hitting probability; see Theorem 2.1.4(a) below. The upper bound in Theorem 2.1.4(b) is an extension to $1 < \alpha \leq 2$ of the corresponding result of [26, Theorem 1.2] for $\alpha = 2$.

Using Theorem 2.1.1 together with results from Dalang, Khoshnevisan and Nualart [25], we shall prove the following results for the hitting probabilities of the solution.

Theorem 2.1.4. *Assume **P1'** and **P2**. Fix $T > 0, M > 0$ and $\eta > 0$. Let $I \subset]0, T]$ and $J \subset \mathbb{R}$ be two fixed non-trivial compact intervals.*

(a) *There exists $c > 0$ depending on I, J and M such that for all compact sets $A \subseteq [-M, M]^d$,*

$$\mathrm{P}\{u(I \times J) \cap A \neq \emptyset\} \geq c\, \mathrm{Cap}_{d-\frac{2(\alpha+1)}{\alpha-1}}(A).$$

(b) *There exists $C > 0$ depending on I, J and η such that for all compact sets $A \subseteq \mathbb{R}^d$,*

$$\mathrm{P}\{u(I \times J) \cap A \neq \emptyset\} \leq C \mathscr{H}_{d - \frac{2(\alpha+1)}{\alpha-1} - \eta}(A).$$

If $\sigma \equiv \mathrm{Id}$ and $b \equiv 0$, by Theorem 7.6 in [85], the upper bound in Theorem 2.1.4(b) can be improved to the best result available for the Gaussian case.

Theorem 2.1.5. *Denote by v the solution of (2.1.1) with $\sigma \equiv \mathrm{Id}$ and $b \equiv 0$. Fix $T > 0$. Let $I \subset]0, T]$ and $J \subset \mathbb{R}$ be two fixed non-trivial compact intervals. There exists $C > 0$ depending on I and J such that for all compact sets $A \subseteq \mathbb{R}^d$,*

$$\mathrm{P}\{v(I \times J) \cap A \neq \emptyset\} \leq C \mathscr{H}_{d - \frac{2(\alpha+1)}{\alpha-1}}(A).$$

These two theorems are proved in the next section.

2.2 Proof of Theorems 2.1.4 and 2.1.5 (assuming Theorem 2.1.1)

In this section, we give the proof of Theorems 2.1.4 and 2.1.5 (assuming Theorem 2.1.1). The organization of the proof is similar to Section 2 of [27].

2.2.1 Proof of Theorem 2.1.4(b)

We start by proving Theorem 2.1.4(b). For all positive integers n, set

$$t_k^n := k 2^{-\frac{2n\alpha}{\alpha-1}}, \quad x_l^n := l 2^{-\frac{2n}{\alpha-1}}$$

and

$$I_k^n = [t_k^n, t_{k+1}^n], \quad J_l^n = [x_l^n, x_{l+1}^n], \quad R_{k,l}^n = I_k^n \times J_l^n. \tag{2.2.1}$$

By (2.1.7) we have

$$\mathrm{E}\left[\sup_{(t,x) \in R_{k,l}^n} \|u(t,x) - u(t_k^n, x_l^n)\|^p\right] \leq C 2^{-n\beta p}, \tag{2.2.2}$$

where β is chosen as in (2.1.7).

Lemma 2.2.1. *Fix $\eta > 0$. There exists $c > 0$ such that for all $z \in \mathbb{R}^d$, n large and $R_{k,l}^n \subset I \times J$,*

$$P\left\{u(R_{k,l}^n) \cap B(z, 2^{-n}) \neq \emptyset\right\} \leq c 2^{-n(d-\eta)}. \tag{2.2.3}$$

Proof. The proof follows along the same lines as [25, Theorem 3.3] by Theorem 2.1.1(a) and

(2.2.2). We give its details for reader's convenience. First, by the triangle inequality,

$$\begin{aligned}
\mathrm{P}\left\{u(R^n_{k,l})\cap B(z,2^{-n})\neq\emptyset\right\} &\leq \mathrm{P}\left\{\|u(t^n_k,x^n_l)-z\|\leq 2^{-n}+\sup_{(t,x)\in R^n_{k,l}}\|u(t,x)-u(t^n_k,x^n_l)\|\right\}\\
&\leq \mathrm{P}\left\{\|u(t^n_k,x^n_l)-z\|\leq 2^{-n}+2^{-n(d-\eta)/d}\right\}\\
&\quad+\mathrm{P}\left\{\sup_{(t,x)\in R^n_{k,l}}\|u(t,x)-u(t^n_k,x^n_l)\|\geq 2^{-n(d-\eta)/d}\right\}.
\end{aligned}$$

Using Theorem 2.1.1(a), the first probability above is bounded above by $\tilde{c}2^{-n(d-\eta)}$. We apply Markov's inequality to the second term above, and by (2.2.2), for all $p\geq 1$,

$$\begin{aligned}
&\mathrm{P}\left\{u(R^n_{k,l})\cap B(z,2^{-n})\neq\emptyset\right\}\\
&\quad\leq \tilde{c}2^{-n(d-\eta)}+2^{np(d-\eta)/d}\mathrm{E}\left[\sup_{(t,x)\in R^n_{k,l}}\|u(t,x)-u(t^n_k,x^n_l)\|^p\right]\\
&\quad\leq \tilde{c}2^{-n(d-\eta)}+\tilde{c}2^{np(d-\eta)/d}2^{-np\beta}\\
&\quad= \tilde{c}2^{-n(d-\eta)}(1+2^{np((d-\eta)/p+(d-\eta)/d-\beta)})\\
&\quad\leq c2^{-n(d-\eta)},
\end{aligned}$$

where the last inequality holds because we can choose p large enough and then β close to 1 so that $(d-\eta)/p+(d-\eta)/d-\beta\leq 0$. □

Proof of Theorem 2.1.4(b). Fix $\epsilon\in\,]0,1[$ and $n\in\mathbb{N}$ such that $2^{-n-1}<\epsilon\leq 2^{-n}$, and write

$$\mathrm{P}\{u(I\times J)\cap B(z,\epsilon)\neq\emptyset\}\leq\sum_{(k,l):R^n_{k,l}\cap I\times J\neq\emptyset}\mathrm{P}\left\{u(R^n_{k,l})\cap B(z,2^{-n})\neq\emptyset\right\}.$$

The number of pairs (k,l) involved in the sum is at most $2^{\frac{2n(\alpha+1)}{\alpha-1}}$ times a constant. Lemma 2.2.1 implies that for all $z\in A$, $\eta>0$ and large n,

$$\begin{aligned}
\mathrm{P}\{u(I\times J)\cap B(z,\epsilon)\neq\emptyset\}&\leq\tilde{C}2^{-n(d-\eta)}2^{\frac{2n(\alpha+1)}{\alpha-1}}\\
&\leq C\epsilon^{d-\frac{2(\alpha+1)}{\alpha-1}-\eta}.
\end{aligned}\tag{2.2.4}$$

Note that C does not depend on (n,ϵ). Therefore, (2.2.4) is valid for all $\epsilon\in\,]0,1[$.

Now we use a *covering argument*: Choose $\tilde{\epsilon}\in]0,1[$ and let $\{B_i\}_{i=1}\infty$ be a sequence of open balls in $\mathbb{R}^d$ with respective radii $r_i\in\,]0,\tilde{\epsilon}[$ such that

$$A\subseteq\cup_{i=1}^{\infty}B_i\quad\text{and}\quad\sum_{i=1}^{\infty}(2r_i)^{d-\frac{2(\alpha+1)}{\alpha-1}-\eta}\leq\mathscr{H}_{d-\frac{2(\alpha+1)}{\alpha-1}-\eta}(A)+\tilde{\epsilon}.\tag{2.2.5}$$

Because $\mathrm{P}\{u(I\times J)\cap A\neq\emptyset\}$ is at most $\sum_{i=1}^{\infty}\mathrm{P}\{u(I\times J)\cap B_i\neq\emptyset\}$, the bounds in (2.2.4) and

(2.2.5) together imply that

$$\mathrm{P}\{u(I\times J)\cap A\neq\emptyset\}\leq C\left(\mathscr{H}_{d-\frac{2(\alpha+1)}{\alpha-1}-\eta}(A)+\tilde{\epsilon}\right). \tag{2.2.6}$$

Let $\tilde{\epsilon}\to 0^+$ to conclude. □

2.2.2 Proof of Theorem 2.1.5

In the case $b\equiv 1$ and $\sigma\equiv I_d$, the components of $v=(v_1,\dots,v_d)$ are independent and identically distributed.

Proposition 2.2.2. *For any $0<t_0<T$, $p\geq 1$ and K a compact set, there exists $c_1=c_1(p,t_0,K)>0$ such that for any $t_0\leq s\leq t\leq T, x,y\in K$,*

$$E\left[|v_1(t,x)-v_1(s,y)|^p\right]\geq c_1\left(|t-s|^{\frac{\alpha-1}{\alpha}}+|x-y|^{\alpha-1}\right)^{p/2}. \tag{2.2.7}$$

Proof. The proof is similar to that of Proposition 2.1 of [27]. Since v is Gaussian, it is equivalent to prove (2.2.7) for $p=2$. By Ito's isometry, we have

$$\begin{aligned}\mathrm{E}\left[|v_1(t,x)-v_1(s,y)|^2\right]&=\int_s^t\int_{\mathbb{R}}G_\alpha^2(t-r,x-v)\,dv\,dr\\&\quad+\int_0^s\int_{\mathbb{R}}(G_\alpha(t-r,x-v)-G_\alpha(s-r,y-v))^2\,dv\,dr\\&:=I_1+I_2.\end{aligned} \tag{2.2.8}$$

Case 1: $t-s\geq|x-y|^\alpha$. In this case, by the semi-group property of the Green kernel (A.6) and the scaling property of the Green kernel (A.2), we have

$$\begin{aligned}I_1+I_2\geq I_1&=\int_s^t G_\alpha(2(t-r),0)\,dr=\int_s^t(2(t-r))^{-1/\alpha}G_\alpha(1,0)\,dr\\&=c_\alpha(t-s)^{\frac{\alpha-1}{\alpha}}\geq\frac{c_\alpha}{2}\left((t-s)^{\frac{\alpha-1}{\alpha}}+|x-y|^{\alpha-1}\right).\end{aligned}$$

Case 2: $t-s\leq|x-y|^\alpha$. In this case, by the Plancherel theorem,

$$\begin{aligned}I_1+I_2\geq I_2&=\int_0^s\int_{\mathbb{R}}(G_\alpha(t-r,x-y+v)-G_\alpha(s-r,v))^2\,dv\,dr\\&=\frac{1}{2\pi}\int_0^s\int_{\mathbb{R}}\left|e^{-(s-r)|\lambda|^\alpha}-e^{-(t-r)|\lambda|^\alpha}e^{i\lambda(x-y)}\right|^2d\lambda\,dr\\&=\frac{1}{2\pi}\int_0^s\int_{\mathbb{R}}e^{-2(s-r)|\lambda|^\alpha}\left|1-e^{-(t-s)|\lambda|^\alpha}e^{i\lambda(x-y)}\right|^2d\lambda\,dr.\end{aligned}$$

We use the elementary inequality $|1-re^{i\theta}|\geq\frac{1}{2}|1-e^{i\theta}|$, valid for all $r\in[0,1]$ and $\theta\in\mathbb{R}$, to see that

$$I_2 \geq \int_{\mathbb{R}} \frac{1-e^{-2s|\lambda|^\alpha}}{8\pi|\lambda|^\alpha} \left|1-e^{i\lambda(x-y)}\right|^2 d\lambda.$$

Because $x-y \in K-K$ and K is compact, fix $C>0$ such that $|x-y| \leq C$. When $x \neq y$, we change the variable by letting $\xi = |x-y|\lambda$ and write $e_0 = (x-y)/|x-y|$ to see that the right-hand side of the above inequality is equal to

$$\begin{aligned} |x-y|^{\alpha-1} \int_{\mathbb{R}} \frac{1-e^{-2s|\xi|^\alpha/|x-y|^\alpha}}{8\pi|\xi|^\alpha} \left|1-e^{ie_0\xi}\right|^2 d\xi &\geq |x-y|^{\alpha-1} \int_{\mathbb{R}} \frac{1-e^{-2s|\xi|^\alpha/C^\alpha}}{8\pi|\xi|^\alpha} \left|1-e^{ie_0\xi}\right|^2 d\xi \\ &\geq |x-y|^{\alpha-1} \int_{\mathbb{R}} \frac{1-e^{-2t_0|\xi|^\alpha/C^\alpha}}{8\pi|\xi|^\alpha} \left|1-e^{ie_0\xi}\right|^2 d\xi. \end{aligned}$$

The integral above is a positive constant. Therefore, when $t-s \leq |x-y|^\alpha$,

$$\mathrm{E}\left[|v_1(t,x)-v_1(s,y)|^2\right] \geq c|x-y|^{\alpha-1} \geq \frac{c}{2}\left(|t-s|^{\frac{\alpha-1}{\alpha}} + |x-y|^{\alpha-1}\right).$$

Case 1 and case 2 together imply (2.2.7). □

Now we apply Theorem 7.6 in [85] to prove Theorem 2.1.5, which is similar to the proof of Theorem 1.5 of [27]. It suffices to verify Conditions (C1) and (C2) of [85, Sect. 2.4, p.158] with $N=2, H_1 = \frac{\alpha-1}{2\alpha}, H_2 = \frac{\alpha-1}{2}$.

First, we observe that $\mathrm{E}[v_1(t,x)^2] = c_\alpha t^{\frac{\alpha-1}{\alpha}}$ (see (A.4)), which implies that there are positive constants c_1, c_2 such that for all $(t,x),(s,y) \in I \times J$,

$$c_1 \leq \mathrm{E}[v_1(t,x)^2] \leq c_2. \tag{2.2.9}$$

By (2.2.7) and (2.1.6), there exist positive constants c_3, c_4 such that for all $(t,x),(s,y) \in I \times J$,

$$c_3\left(|t-s|^{\frac{\alpha-1}{\alpha}} + |x-y|^{\alpha-1}\right) \leq \mathrm{E}\left[|v_1(t,x)-v_1(s,y)|^2\right] \leq c_4\left(|t-s|^{\frac{\alpha-1}{\alpha}} + |x-y|^{\alpha-1}\right). \tag{2.2.10}$$

Hence condition C1 is satisfied by (2.2.9) and (2.2.10). Similar to the argument in the proof of Theorem 1.5 of [27], condition C2 holds by applying the fourth point of Remark 2.2 in [85], since $(t,x) \mapsto \mathrm{E}[v_1(t,x)] = c_\alpha t^{\frac{\alpha-1}{\alpha}}$ is continuous in $I \times J$ with continuous partial derivatives.

Therefore we have finished the proof of Theorem 2.1.5.

2.2.3 Proof of Theorem 2.1.4(a)

The proof is similar to that of Theorem 2.1(1) of [25]; see also [27, Sect 2.4], which requires the following two lemmas analogous to [25, Lemma 2.2(1)] and [27, Lemma 2.3].

Lemma 2.2.3. *Fix $N>0$. There exists a finite and positive constant $C_1 = C_1(I,J,d,N)$ such that* *for all* $a \in [0, N]$,

$$\int_I dt \int_I ds \int_J dx \int_J dy \frac{e^{-a^2/\Delta_\alpha((t,x);(s,y))}}{\Delta_\alpha^{d/2}((t,x);(s,y))} \leq C_1 K_{d-\frac{2(\alpha+1)}{\alpha-1}}(a). \tag{2.2.11}$$

Proof. The proof follows along the same lines as [25, Lemma 2.2(1)]. Using the change of variables $\tilde{u} = t - s$ (t fixed), $\tilde{v} = x - y$ (x fixed), we see that the integral on the left-hand side of (2.2.11) is bounded above by

$$4|I||J| \int_0^{|I|} d\tilde{u} \int_0^{|J|} d\tilde{v}\, (\tilde{u}^{\frac{\alpha-1}{\alpha}} + \tilde{v}^{\alpha-1})^{-d/2} \exp\left(-\frac{a^2}{\tilde{u}^{\frac{\alpha-1}{\alpha}} + \tilde{v}^{\alpha-1}}\right). \tag{2.2.12}$$

Another change of variables $[\tilde{u} = (ua^2)^{\alpha/(\alpha-1)}, \tilde{v} = (va^2)^{1/(\alpha-1)}]$ implies that this is equal to

$$C a^{\frac{2\alpha+2}{\alpha-1}-d} \int_0^{|I|^{(\alpha-1)/\alpha} a^{-2}} du \int_0^{|J|^{\alpha-1} a^{-2}} dv \frac{u^{1/(\alpha-1)} v^{(2-\alpha)/(\alpha-1)}}{(u+v)^{d/2}} \exp\left(-\frac{1}{u+v}\right). \tag{2.2.13}$$

We pass to polar coordinates to deduce that the preceding is bounded above by

$$C a^{\frac{2\alpha+2}{\alpha-1}-d} (I_1 + I_2(a)), \tag{2.2.14}$$

where

$$I_1 = \int_0^{\bar{K}N^{-2}} d\rho\, \rho^{\frac{2}{\alpha-1}-\frac{d}{2}} \exp(-c/\rho), \tag{2.2.15}$$

$$I_2(a) = \int_{\bar{K}N^{-2}}^{\bar{K}a^{-2}} d\rho\, \rho^{\frac{2}{\alpha-1}-\frac{d}{2}}, \tag{2.2.16}$$

where $\bar{K} = (|I|^{2(\alpha-1)/\alpha} + |J|^{2(\alpha-1)})^{1/2}$. Clearly, $I_1 \leq C < \infty$. Moreover, if $\frac{2}{\alpha-1} - \frac{d}{2} + 1 \neq 0$, i.e. $\frac{2(\alpha+1)}{\alpha-1} \neq d$, then

$$I_2(a) = \bar{K}^{(\alpha+1)/(\alpha-1)-d/2} \frac{a^{d-2(\alpha+1)/(\alpha-1)} - N^{d-2(\alpha+1)/(\alpha-1)}}{(\alpha+1)/(\alpha-1) - d/2}. \tag{2.2.17}$$

There are three separate cases to consider. (i) If $\frac{2(\alpha+1)}{\alpha-1} < d$, then $I_2(a) \leq C < \infty$ for all $a \in [0, N]$. (ii) If $\frac{2(\alpha+1)}{\alpha-1} > d$, then $I_2(a) \leq c a^{d-2(\alpha+1)/(\alpha-1)}$. (iii) If $\frac{2(\alpha+1)}{\alpha-1} = d$, then

$$I_2(a) = 2(\ln\frac{1}{a} + \ln N). \tag{2.2.18}$$

We combine these observations to conclude that the expression in (2.2.14) is bounded above by $C K_{d-\frac{2(\alpha+1)}{\alpha-1}}(a)$, provided that N_0 in (1.4.1) is sufficient large. This completes the lemma. □

Lemma 2.2.4. *For all* $N > 0$ *and* $p > 4d(\frac{d}{2} - \frac{2}{\alpha-1} - 1)$. *There exists a finite and positive constant*

$C_2 = C_2(I,J,d,N,p)$ *such that for all* $a \in [0,N]$,

$$\int_I dt \int_I ds \int_J dx \int_J dy \,(|t-s|^{\frac{\alpha-1}{\alpha}} + |x-y|^{\alpha-1})^{-d/2} \left[\frac{|t-s|^{\frac{\alpha-1}{\alpha}} + |x-y|^{\alpha-1}}{a^2} \wedge 1 \right]^{p/(4d)}$$
$$\leq C_2 K_{d-\frac{2(\alpha+1)}{\alpha-1}}(a). \tag{2.2.19}$$

Proof. Similar to the derivation of (2.2.13) by changing variables, the integral on the left-hand side of (2.2.19) is equal to

$$Ca^{\frac{2\alpha+2}{\alpha-1}-d} \int_0^{|I|^{(\alpha-1)/\alpha} a^{-2}} du \int_0^{|J|^{\alpha-1} a^{-2}} dv \frac{u^{1/(\alpha-1)} v^{(2-\alpha)/(\alpha-1)}}{(u+v)^{d/2}} [(u+v) \wedge 1]^{p/(4d)}. \tag{2.2.20}$$

Passing to the polar coordinates, this is bounded above by

$$Ca^{\frac{2\alpha+2}{\alpha-1}-d}(I_1 + I_2(a)), \tag{2.2.21}$$

where

$$I_1 = \int_0^{\bar{K}N^{-2}} d\rho\, \rho^{\frac{2}{\alpha-1}-\frac{d}{2}} \rho^{p/(4d)},$$
$$I_2(a) = \int_{\bar{K}N^{-2}}^{\bar{K}a^{-2}} d\rho\, \rho^{\frac{2}{\alpha-1}-\frac{d}{2}},$$

where the constant $\bar{K}$ is given below (2.2.16). Clearly, $I_1 \leq C < \infty$ since $\frac{2}{\alpha-1} - \frac{d}{2} + \frac{p}{4d} > -1$ by the hypothesis on p. The remainder of the proof is the same as that of Lemma 2.2.3. □

Proof of Theorem 2.1.4(a). The proof of this result follows along the same lines as the proof of [25, Theorem 2.1(1)], therefore we will only sketch the steps that differ; see also the proof of [27, Theorem 1.2(b)]. We need to replace their $\beta - 6$ by $d - \frac{2(\alpha+1)}{\alpha-1}$.

We first note that our Theorem 2.1.1(a) and (b) indicate that

$$\inf_{\|z\| \leq M} \int_I dt \int_J dx\, p_{t,x}(z) \geq C > 0, \tag{2.2.22}$$

which proves hypothesis **A1'** of [25, Theorem 2.1(1)] (see [25, Remark 2.5(a)]).

Let us now follow the proof of [25, Theorem 2.1(1)]. Define, for all $z \in \mathbb{R}^d$ and $\epsilon > 0$, $\tilde{B}(z,\epsilon) := \{y \in \mathbb{R}^d : |y - z| < \epsilon\}$, where $|z| := \max_{1 \leq j \leq d} |z_j|$, and

$$J_\epsilon(z) = \frac{1}{(2\epsilon)^d} \int_I dt \int_J dx\, \mathbf{1}_{\tilde{B}(z,\epsilon)}(u(t,x)), \tag{2.2.23}$$

as in [25, (2.28)].

Assume first that $d < \frac{2(\alpha+1)}{\alpha-1}$. Using Theorem 2.1.1(c) or Remark 2.1.2(c'), we find, instead of

[25, (2.30)],

$$\mathrm{E}[(J_\epsilon(z))^2] \le c \int_I dt \int_I ds \int_J dx \int_J dy [\Delta_\alpha((t,x);(s,y))]^{-d/2}.$$

The change of variables $u = t - s$ (t fixed), $v = x - y$ (x fixed), implies that the above integral is bounded above by

$$C \int_0^{|I|} du \int_0^{|J|} dv \left(u^{\frac{\alpha-1}{\alpha}} + v^{\alpha-1}\right)^{-d/2} \le C' \int_0^{|I|} du \, \Psi_{|J|,(\alpha-1)d/2}(u^{(\alpha-1)d/(2\alpha)}), \tag{2.2.24}$$

where Ψ is defined by

$$\Psi_{a,\nu}(\rho) := \int_0^a \frac{dx}{\rho + x^\nu},$$

for all $a, \nu, \rho > 0$, as in (2.23) of [25]. Hence, by Lemma 2.3 of [25], for all $\epsilon > 0$,

$$\mathrm{E}\left[(J_\epsilon(z))^2\right] \le C \int_0^{|I|} du \, \mathrm{K}_{1-\frac{2}{(\alpha-1)d}}(u^{(\alpha-1)d/(2\alpha)}).$$

In order to bound the above integral, we consider three different cases: (i) If $0 < d < \frac{2}{\alpha-1}$, then $1 - \frac{2}{(\alpha-1)d} < 0$ and the integral equals $|I|$. (ii) If $\frac{2}{\alpha-1} < d < \frac{2(\alpha+1)}{\alpha-1}$, then $\mathrm{K}_{1-\frac{2}{(\alpha-1)d}}(u^{(\alpha-1)d/(2\alpha)}) = u^{1/\alpha-(\alpha-1)d/(2\alpha)}$ and the integral is finite. (iii) If $d = \frac{2}{\alpha-1}$, then $\mathrm{K}_0(u^{1/\alpha}) = \log(N_0/u^{1/\alpha})$ and the integral is also finite. The remainder of the proof of Theorem 2.1.4(a) when $d < \frac{2(\alpha+1)}{\alpha-1}$ follows exactly as in [25, Theorem 2.1(1) Case 1].

Assume now that $d > \frac{2(\alpha+1)}{\alpha-1}$. Define, for all $\mu \in \mathscr{P}(A)$ and $\epsilon > 0$,

$$J_\epsilon(\mu) = \frac{1}{(2\epsilon)^d} \int_{\mathbb{R}^d} \mu(dz) \int_I dt \int_J dx \, \mathbf{1}_{\tilde{B}(z,\epsilon)}(u(t,x)), \tag{2.2.25}$$

as [25, (2.35)]. Fix $\mu \in \mathscr{P}(A)$ such that

$$I_{d-\frac{2(\alpha+1)}{\alpha-1}}(\mu) \le \frac{2}{\mathrm{Cap}_{d-\frac{2(\alpha+1)}{\alpha-1}}(A)}.$$

Analogous to the proof of [25, (2.41)], we use Theorem 2.1.1(c) and Lemma 2.2.3 (or the combination of Remark 2.1.2(c') and Lemma 2.2.4), to see that for all $\epsilon > 0$

$$\mathrm{E}\left[(J_\epsilon(\mu))^2\right] \le C_2 I_{d-\frac{2(\alpha+1)}{\alpha-1}}(\mu) \le \frac{2C_2}{\mathrm{Cap}_{d-\frac{2(\alpha+1)}{\alpha-1}}(A)}.$$

The remainder of the proof of Theorem 2.1.4(a) when $d > \frac{2(\alpha+1)}{\alpha-1}$ follows as in [25, Theorem 2.1(1) Case 2].

The case $d = \frac{2(\alpha+1)}{\alpha-1}$ is proved exactly along the same lines as the proof of [25, Theorem 2.1(1) Case 3], appealing to (2.2.22), Theorem 2.1.1(c) and Lemma 2.2.3 (or the combination of

Remark 2.1.2(c') and Lemma 2.2.4). □

2.3 Existence, smoothness and uniform boundedness of the one-point density

In [12], the Malliavin differentiability and smoothness of the density of the solution to fractional SPDEs driven by spatially correlated noise was established when $d = 1$. These can also be applied to SPDEs driven by space-time white noise and the extension to $d > 1$ can easily be done by working coordinate by coordinate. In particular, for any $(t,x) \in [0,T] \times \mathbb{R}$, $i,k \in \{1,\dots,d\}$, the derivative of $u_i(t,x)$ satisfies the system of equations

$$D^{(k)}_{r,v}(u_i(t,x)) = G_\alpha(t-r,x-v)\sigma_{ik}(u(r,v)) + a_i(k,r,v,t,x), \tag{2.3.1}$$

where

$$\begin{aligned} a_i(k,r,v,t,x) = &\sum_{j=1}^{d} \int_r^t \int_{\mathbb{R}} G_\alpha(t-\theta,x-\eta) D^{(k)}_{r,v}(\sigma_{ij}(u(\theta,\eta))) W^j(d\theta,d\eta) \\ &+ \int_r^t \int_{\mathbb{R}} G_\alpha(t-\theta,x-\eta) D^{(k)}_{r,v}(b_i(u(\theta,\eta))) d\theta d\eta, \end{aligned} \tag{2.3.2}$$

if $r < t$ and $D^{(k)}_{r,v}(u_i(t,x)) = 0$ when $r > t$. Moreover, for any $p > 1, m \geq 1$ and $i \in \{1,\dots,d\}$, the order m derivatives satisfies

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}} \mathrm{E}\left[\left\| D^m(u_i(t,x))\right\|^p_{\mathscr{H}^{\otimes m}}\right] < \infty, \tag{2.3.3}$$

and by iterating the calculation which leads to (2.3.1), we see that D^m also satisfies the system of stochastic partial differential equations which are analogous to the equations in Proposition 4.1 of [25]; see also [66, (6.29)]. In particular, for all $(t,x) \in [0,T] \times \mathbb{R}$,

$$u(t,x) \in (\mathbb{D}^\infty)^d. \tag{2.3.4}$$

Our objective in this section is to prove Theorem 2.1.1(a) by using Proposition 1.5.4. The next result proves property (a) in Proposition 1.5.4 when F is replaced by $u(t,x)$.

Proposition 2.3.1. *Fix $T > 0$ and assume hypotheses **P1'** and **P2**. Then, for any $p \geq 1$,*

$$E\left[(\det \gamma_{u(t,x)})^{-p}\right]$$

is uniformly bounded over (t,x) in any closed non-trivial rectangle $I \times J \subset]0,T] \times \mathbb{R}$.

Proof. The proof follows along the same lines as [26, Proposition 4.2] by using Proposition A.2.1; see also [27, Proposition 4.1]. The main differences are the exponents appearing in the estimate. Let $(t,x) \in I \times J$ be fixed. We write

$$\det \gamma_{u(t,x)} \geq \left(\inf_{\xi \in \mathbb{R}^d} \xi^T \gamma_{u(t,x)} \xi \right)^d .$$

Let $\xi \in \mathbb{R}^d$ with $\|\xi\| = 1$ and fix $\epsilon \in]0,1[$. Using (2.3.1) and the inequality

$$(a+b)^2 \geq \frac{2}{3} a^2 - 2b^2, \tag{2.3.5}$$

valid for all $a, b \in \mathbb{R}$, we see that

$$\begin{aligned} \xi^T \gamma_{u(t,x)} \xi &= \int_0^t dr \int_{\mathbb{R}} dv \left\| \sum_{i=1}^d D_{r,v}(u_i(t,x)) \xi_i \right\|^2 \\ &\geq \int_{t(1-\epsilon)}^t dr \int_{\mathbb{R}} dv \left\| \sum_{i=1}^d D_{r,v}(u_i(t,x)) \xi_i \right\|^2 \geq \frac{2}{3} I_1 - 2 I_2, \end{aligned}$$

where

$$\begin{aligned} I_1 &= \int_{t(1-\epsilon)}^t dr \int_{\mathbb{R}} dv \sum_{k=1}^d \left(\sum_{i=1}^d G_\alpha(t-r, x-v) \sigma_{ik}(u(r,v)) \xi_i \right)^2, \\ I_2 &= \int_{t(1-\epsilon)}^t dr \int_{\mathbb{R}} dv \sum_{k=1}^d \left(\sum_{i=1}^d a_i(k,r,v,t,x) \xi_i \right)^2, \end{aligned}$$

and $a_i(k,r,v,t,x)$ is defined in (2.3.2). By hypothesis **P2** and semi-group property of the Green kernel (A.6),

$$\begin{aligned} I_1 &\geq c \int_{t(1-\epsilon)}^t \int_{\mathbb{R}} G_\alpha^2(t-r, x-v) dv dr \\ &= c \int_{t(1-\epsilon)}^t G_\alpha(2(t-r), 0) dr \\ &= \frac{c}{2} \int_0^{2t\epsilon} G_\alpha(r,0) dr = c'(2t\epsilon)^{\frac{\alpha-1}{\alpha}} \geq c'' \epsilon^{\frac{\alpha-1}{\alpha}}, \end{aligned} \tag{2.3.6}$$

where in the third equality we use (A.2) and the constants c, c' and c'' are uniform over $(t,x) \in I \times J$.

Next we apply Cauchy-Schwarz inequality to find that, for any $q \geq 1$,

$$\mathrm{E}\left[\sup_{\xi \in \mathbb{R}^d : \|\xi\| = 1} |I_2|^q \right] \leq c(\mathrm{E}[|I_{21}|^q] + \mathrm{E}[|I_{22}|^q]),$$

where

$$I_{21} = \sum_{i,j,k=1}^{d} \int_{t(1-\epsilon)}^{t} dr \int_{\mathbb{R}} dv \left(\int_{r}^{t} \int_{\mathbb{R}} G_\alpha(t-\theta, x-\eta) D_{r,v}^{(k)}(\sigma_{ij}(u(\theta,\eta))) W^j(d\theta, d\eta) \right)^2,$$

$$I_{22} = \sum_{i,k=1}^{d} \int_{t(1-\epsilon)}^{t} dr \int_{\mathbb{R}} dv \left(\int_{r}^{t} \int_{\mathbb{R}} G_\alpha(t-\theta, x-\eta) D_{r,v}^{(k)}(b_i(u(\theta,\eta))) d\theta d\eta \right)^2.$$

The estimates for the q-th moment of I_{21} and I_{22} are similar to those in [26, Proposition 4.2], so we only present the differences here. By Burkholder's inequality for martingales with values in Hilbert space (Lemma A.3.1), and using **P1'**,

$$\mathrm{E}[|I_{21}|^q] \le c \sum_{k=1}^{d} \mathrm{E}\left[\left| \int_{t(1-\epsilon)}^{t} d\theta \int_{\mathbb{R}} d\eta\, G_\alpha^2(t-\theta, x-\eta) \int_{t(1-\epsilon)}^{t\wedge\theta} dr \int_{\mathbb{R}} dv \Big(\sum_{l=1}^{d} D_{r,v}^{(k)}(u_l(\theta,\eta)) \Big)^2 \right|^q \right].$$

By Hölder's inequality with respect to the measure $G_\alpha^2(t-\theta, x-\eta) d\theta d\eta$, we see that

$$\begin{aligned}
\mathrm{E}[|I_{21}|^q] \le{}& C \sum_{k=1}^{d} \left| \int_{t(1-\epsilon)}^{t} d\theta \int_{\mathbb{R}} d\eta\, G_\alpha^2(t-\theta, x-\eta) \right|^{q-1} \\
&\times \int_{t(1-\epsilon)}^{t} d\theta \int_{\mathbb{R}} d\eta\, G_\alpha^2(t-\theta, x-\eta) \mathrm{E}\left[\left| \int_{t(1-\epsilon)}^{t\wedge\theta} dr \int_{\mathbb{R}} dv \Big(\sum_{l=1}^{d} D_{r,v}^{(k)}(u_l(\theta,\eta)) \Big)^2 \right|^q \right].
\end{aligned}$$

Lemma A.3.2 assures that

$$\begin{aligned}
\mathrm{E}[|I_{21}|^q] &\le C_T \left| \int_{t(1-\epsilon)}^{t} d\theta \int_{\mathbb{R}} d\eta\, G_\alpha^2(t-\theta, x-\eta) \right|^q \epsilon^{(\alpha-1)q/\alpha} \\
&= C_T \left| \int_{t(1-\epsilon)}^{t} G_\alpha(2(t-\theta), 0) d\theta \right|^q \epsilon^{(\alpha-1)q/\alpha} \\
&= C_T'(2t\epsilon)^{(\alpha-1)q/\alpha} \epsilon^{(\alpha-1)q/\alpha} \le C_T'' \epsilon^{2(\alpha-1)q/\alpha},
\end{aligned}$$

where in the first inequality we use the semi-group property (A.6), in the second equality we use (A.2) and the constants C_T, C_T' and C_T'' are uniform over $(t,x) \in I \times J$.

We next derive a similar bound for I_{22}. First, we use Cauchy-Schwarz inequality with respect to the measure $G_\alpha(t-\theta, x-\eta) d\theta d\eta$ to see that

$$\begin{aligned}
I_{22} &\le \sum_{i,k=1}^{d} \int_{t(1-\epsilon)}^{t} (t-r) dr \int_{\mathbb{R}} dv \int_{r}^{t} \int_{\mathbb{R}} G_\alpha(t-\theta, x-\eta) \left(D_{r,v}^{(k)}(b_i(u(\theta,\eta))) \right)^2 d\theta d\eta \\
&\le \sum_{i,k=1}^{d} t\epsilon \int_{t(1-\epsilon)}^{t} dr \int_{\mathbb{R}} dv \int_{r}^{t} \int_{\mathbb{R}} G_\alpha(t-\theta, x-\eta) \left(D_{r,v}^{(k)}(b_i(u(\theta,\eta))) \right)^2 d\theta d\eta.
\end{aligned}$$

Since the partial derivatives of b_i are bounded, by Cauchy-Schwarz inequality and Fubini's theorem,

$$\begin{aligned}
\mathrm{E}[|I_{22}|^q] &\le c \sum_{l,k=1}^{d} (t\epsilon)^q \mathrm{E}\left[\left|\int_{t(1-\epsilon)}^{t} dr \int_{\mathbb{R}} dv \int_r^t \int_{\mathbb{R}} G_\alpha(t-\theta, x-\eta) \left(D_{r,v}^{(k)}(u_l(\theta,\eta))\right)^2 d\theta d\eta\right|^q\right] \\
&= c \sum_{l,k=1}^{d} (t\epsilon)^q \mathrm{E}\left[\left|\int_{t(1-\epsilon)}^{t} d\theta \int_{\mathbb{R}} d\eta\, G_\alpha(t-\theta, x-\eta) \int_{t(1-\epsilon)}^{t\wedge\theta} dr \int_{\mathbb{R}} dv \right.\right. \\
&\qquad \left.\left. \times \left(D_{r,v}^{(k)}(u_l(\theta,\eta))\right)^2 \right|^q\right].
\end{aligned}$$

Applying Hölder's inequality with respect to the measure $G_\alpha(t-\theta, x-\eta)d\theta d\eta$,

$$\begin{aligned}
\mathrm{E}[|I_{22}|^q] &\le c \sum_{l,k=1}^{d} (t\epsilon)^q \left|\int_{t(1-\epsilon)}^{t} d\theta \int_{\mathbb{R}} d\eta\, G_\alpha(t-\theta, x-\eta)\right|^{q-1} \\
&\quad \times \int_{t(1-\epsilon)}^{t} d\theta \int_{\mathbb{R}} d\eta\, G_\alpha(t-\theta, x-\eta) \mathrm{E}\left[\left|\int_{t(1-\epsilon)}^{t\wedge\theta} dr \int_{\mathbb{R}} dv \left(D_{r,v}^{(k)}(u_l(\theta,\eta))\right)^2\right|^q\right].
\end{aligned}$$

Using Lemma A.3.2, this yields $\mathrm{E}[|I_{22}|^q] \le C_T (t\epsilon)^q (t\epsilon)^q (t\epsilon)^{(\alpha-1)q/\alpha} = C_T (t\epsilon)^{(3-1/\alpha)q}$.

Thus, we have proved that

$$\mathrm{E}\left[\sup_{\xi\in\mathbb{R}^d:\|\xi\|=1} |I_2|^q\right] \le C_T \epsilon^{2(\alpha-1)q/\alpha}, \tag{2.3.7}$$

where the constant C_T is clearly uniform over $(t,x) \in I \times J$.

Finally, we apply Proposition A.2.1 with $Z := \inf_{\|\xi\|=1}(\xi^T \gamma_{u(t,x)}\xi)$, $Y_{1,\epsilon} = Y_{2,\epsilon} = \sup_{\|\xi\|=1} I_2$, $\epsilon_0 = 1$, $\alpha_1 = \alpha_2 = (\alpha-1)/\alpha$ and $\beta_1 = \beta_2 = 2(\alpha-1)/\alpha$, to get

$$\mathrm{E}\left[(\det \gamma_{u(t,x)})^{-p}\right] \le C_T,$$

where all the constants are independent of $(t,x) \in I \times J$. □

In [12], the authors established the existence and smoothness of the density of the solution of one single stochastic fractional partial differential equation driven by spatially correlated noise. For a system of d equations driven by space-time white noise, we have the following results.

Proposition 2.3.2. *Assume **P1'** and **P2**. Fix $T > 0$ and let I and J be compact intervals as in Theorem 2.1.1. Then for any $(t,x) \in]0,T] \times \mathbb{R}$, $u(t,x)$ is a nondegenerate random vector and its density function is infinitely differentiable and uniformly bounded over $z \in \mathbb{R}^d$ and $(t,x) \in I \times J$.*

Proof. The conclusions follow from Proposition 2.3.1 and (2.3.4) together with Theorem 1.5.1 and Proposition 1.5.4. □

Proof of Theorem 2.1.1(a). This is an immediate consequence of Proposition 2.3.2. □

2.4 Strict positivity of the one-point density

The aim of this section is to prove the strict positivity of the one-point density of u stated in Theorem 2.1.1(b). We will apply a criterion of strict positivity of density introduced by Bally and Pardoux [7]. Before we give the proof of Theorem 2.1.1(b), let us review some existing literature on the strict positivity of the densities of the solutions to SPDEs.

The first related paper is by Nualart [69], in which the author extended the criterion introduced by Bally and Pardoux [7] and applied it to study the strict positivity of the densities of solutions to systems of SPDEs driven by spatially homogeneous noise that is white in time. The Gaussian-type lower bound on the density of the solution of the stochastic heat equation was established by Kohatsu-Higa [49], in which the author generalized the lower bound estimates for uniformly elliptic diffusion processes obtained by Kusuoka and Stroock [52, 53, 54]. Dalang, Khoshnevisan and Nualart [26] extended this result to the case of system of SPDEs. The Gaussian-type lower bound for the density of the solution to single spatially homogeneous SPDEs was obtained by D. Nualart and Quer-Sardanyons [67] in the case where σ is a constant, and by E. Nualart and Quer-Sardanyons [70] in the non-linear case.

Recently, Chen, Hu and Nualart [19] have studied the strict positivity of densities for non-linear stochastic fractional heat equations with measure-valued initial data and unbounded diffusion coefficient. The criteria introduced by Bally and Pardoux [7] are no longer applicable in their case and they develop a localized version. In our situation, the initial data and diffusion coefficient do not bother us and we prefer to give a classical proof of the strict positivity of the density.

For the stochastic wave equation in two spatial dimensions, the positivity of the density was studied in [14]. In the case of hyperbolic SPDEs, points of positive density were studied by Millet and Sanz-Solé [59]. We also mention that the strict positivity of the density for the stochastic Cahn-Hilliard equation was studied by Cardon-Weber [13].

2.4.1 The criterion for strict positivity of density

For $g = (g_1, \dots, g_d) \in \mathscr{H}$ and $z \in \mathbb{R}^d$, we define $\tilde{W}^j(t,x) = W^j(t,x) + z_j \int_0^t \int_{-\infty}^x g_j(s,y)\,ds\,dy$. By Girsanov's Theorem, $\tilde{W} = (\tilde{W}^1, \cdots, \tilde{W}^d)$ is a standard Brownian sheet on the probability space $(\Omega, \mathscr{F}, \tilde{\mathrm{P}})$, where

$$\frac{d\tilde{\mathrm{P}}}{d\mathrm{P}}(\omega) = J(z)(\omega),\ \omega \in \Omega,$$

where

$$J(z) = \exp\left(-\sum_{j=1}^{d} z_j \int_0^T \int_{\mathbb{R}} g_j(s,y) W^j(ds,dy) - \frac{1}{2}\sum_{j=1}^{d} z_j^2 \int_0^T \int_{\mathbb{R}} g_j^2(s,y)\,ds\,dy \right).$$

For any $(t,x) \in [0,T] \times \mathbb{R}$, let $\tilde{u}^z(t,x)$ be the solution to equation (2.1.4) with respect to the Brownian sheet $\tilde{W}$, that is, for $i = 1, \cdots, d$,

$$\begin{aligned}
\tilde{u}_i^z(t,x) &= \int_0^t \int_{\mathbb{R}} G_\alpha(t-r, x-v) \sum_{j=1}^{d} \sigma_{ij}(\tilde{u}^z(r,v)) W^j(dr,dv) \\
&\quad + \sum_{j=1}^{d} z_j \int_0^t \int_{\mathbb{R}} G_\alpha(t-r,x-v)\sigma_{ij}(\tilde{u}^z(r,v)) g_j(r,v)\,dr\,dv \\
&\quad + \int_0^t \int_{\mathbb{R}} G_\alpha(t-r,x-v) b_i(\tilde{u}^z(r,v))\,dr\,dv.
\end{aligned}$$

Then the law of u under P coincides with the law of $\tilde{u}^z$ under $\tilde{\mathrm{P}}$.

Given a sequence $\{g_n\}_{n\geq 1}$ in $\mathscr{H}$ and $z \in \mathbb{R}^d$, let $\tilde{u}_n^z(t,x)$ be the solution to equation (2.1.4) with respect to the Brownian sheet $\tilde{W}_n$, where

$$\tilde{W}_n^j(t,x) = W^j(t,x) + z_j \int_0^t \int_{-\infty}^{x} g_{nj}(s,y)\,ds\,dy.$$

That is, $\tilde{u}_n^z(t,x)$ satisfies

$$\begin{aligned}
\tilde{u}_{ni}^z(t,x) &= \int_0^t \int_{\mathbb{R}} G_\alpha(t-r, x-v) \sum_{j=1}^{d} \sigma_{ij}(\tilde{u}_n^z(r,v)) W^j(dr,dv) \\
&\quad + \sum_{j=1}^{d} z_j \int_0^t \int_{\mathbb{R}} G_\alpha(t-r,x-v)\sigma_{ij}(\tilde{u}_n^z(r,v)) g_{nj}(r,v)\,dr\,dv \\
&\quad + \int_0^t \int_{\mathbb{R}} G_\alpha(t-r,x-v) b_i(\tilde{u}_n^z(r,v))\,dr\,dv. \qquad (2.4.1)
\end{aligned}$$

Define the $d \times d$ matrix

$$\varphi_n^z(t,x) = \left(\varphi_{n,i,j}^z(t,x)\right)_{i,j} := \left(\frac{\partial}{\partial z_j} \tilde{u}_{ni}^z(t,x)\right)_{i,j}$$

and the Hessian matrix (which is a tensor of order 3) of the random vector $\tilde{u}_n^z(t,x)$,

$$\psi_n^z(t,x) = \left(\psi_{n,i,j,m}^z(t,x)\right)_{i,j,m} := \left(\frac{\partial^2}{\partial z_j \partial z_m} \tilde{u}_{n,i}^z(t,x)\right)_{i,j,m}.$$

We still use the notation $\|\cdot\|$ to denote the norm of an $n \times n$ matrix A defined as

$$\|A\| = \sup_{\xi \in \mathbb{R}^n, \|\xi\|=1} \|A\xi\|.$$

We introduce the following conditions to study the strict positivity of the density $p_{t,x}(\cdot)$ of the law of $u(t,x)$. We say that $y \in \mathbb{R}^d$ satisfies $\mathbf{H_{t,x}}(y)$ if:

$\mathbf{H_{t,x}}(y)$: there exists a sequence $\{g_n\}_{n\geq 1}$ in $L^2([0,T]\times\mathbb{R},\mathbb{R}^d)$ and positive constants c_1, c_2, r_0 and δ such that
(i) $\limsup_{n\to\infty} \mathrm{P}\left\{(\|u(t,x)-y\| \leq r)\cap(\det\varphi_n^0(t,x)\geq c_1)\right\} > 0$, for all $r \in]0, r_0]$.
(ii) $\lim_{n\to\infty}\mathrm{P}\left\{\sup_{\|z\|\leq\delta}(\|\varphi_n^z(t,x)\| + \|\psi_n^z(t,x)\|)\leq c_2\right\} = 1$.

Assume that σ and b satisfy the conditions **P1'** and **P2**.

The following theorem is an extension of the criterion by Bally and Pardoux [7] to systems of equations; see also [69, Theorem 5.7]. We give a self-contained proof for reader's convenience.

Theorem 2.4.1. *Let $(t,x) \in]0,T]\times\mathbb{R}$ and $y\in\mathbb{R}^d$ be such that $\mathbf{H_{t,x}}(y)$ holds true. Then $p_{t,x}(y) > 0$. Moreover, if $\mathbf{H_{t,x}}(y)$ holds on Supp$(P_{u(t,x)})$, then $p_{t,x}(\cdot)$ is a strictly positive function on $\mathbb{R}^d$.*

Proof. We start proving the first statement of this theorem. Let $y_0\in\mathbb{R}^d$ satisfy $\mathbf{H_{t,x}}(y_0)$.

Let R and α be the constants in Lemma A.4.1 determined by δ and β with $\beta = (1/c_1)\vee c_2$. Choose and fix r with $0 < r < \alpha$. We define $\Phi_n(z) = \tilde{u}_n^z(t,x)$ and

$$\begin{aligned}\Lambda_n &= \{\|u(t,x)-y_0\|\leq r \cap (\det\varphi_n^0(t,x)\geq 1/\beta)\}\\ &\quad\cap\left\{\sup_{\|z\|\leq\delta}(\|\varphi_n^z(t,x)\|+\|\psi_n^z(t,x)\|)\leq\beta\right\}\end{aligned}$$

It follows from (i), (ii) that there exists $n\in\mathbb{N}$ such that

(iii) $\mathrm{P}(\Lambda_n) > 0$.

From now on, n will also be fixed, such that (iii) holds. By Lemma A.4.1 for all $\omega\in\Lambda_n$, the mapping $z\mapsto\Phi_n(z,\omega)$ is a diffeomorphism between an open neighborhood $V_n(\omega)$ of 0 in $\mathbb{R}^d$ contained in the ball $B(0,R)$, and the ball $B(u(t,x)(\omega),\alpha)$.

From Girsanov's Theorem, for each $z\in\mathbb{R}^d$ and any $f\in\mathscr{B}_b(\mathbb{R}^d,\mathbb{R}_+)$ (the set of nonnegative bounded Borel functions),

$$\mathrm{E}[f(u(t,x))] = \mathrm{E}[f(\tilde{u}_n^z(t,x))J_n(z)] = \mathrm{E}[f(\Phi_n(z))J_n(z)], \tag{2.4.2}$$

where

$$J_n(z) = \exp\left(-\sum_{j=1}^d z_j\int_0^T\int_{\mathbb{R}} g_{nj}(s,y)W^j(ds,dy) - \frac{1}{2}\sum_{j=1}^d z_j^2\int_0^T\int_{\mathbb{R}} g_{nj}^2(s,y)dsdy\right).$$

Let $\psi(z) = (2\pi)^{-d/2}\exp(-\|z\|^2/2)$. From (2.4.2), we know that $\mathrm{E}[f(\Phi_n(z))J_n(z)]$ does not depend on z, so

$$
\begin{aligned}
\mathrm{E}[f(u(t,x))] &= \int_{\mathbb{R}^d} \psi(z)\mathrm{E}[f(\Phi_n(z))J_n(z)]dz \\
&\geq \mathrm{E}\left[\int_{\mathbb{R}^d} \psi(z)f(\Phi_n(z))J_n(z)dz; \Lambda_n\right] \\
&\geq \mathrm{E}\left[\int_{V_n} \psi(z)f(\Phi_n(z))J_n(z)dz; \Lambda_n\right] \\
&= \mathrm{E}\left[\int_{B(u(t,x),\alpha)} f(v)\left(\frac{\psi(z)J_n(z)}{\det\varphi_n^z(t,x)}\right)\bigg|_{z=\Phi_n^{-1}(v)} dv; \Lambda_n\right] \qquad (2.4.3) \\
&\geq \int_{\mathbb{R}^d} f(v)\theta_n(v)dv, \qquad (2.4.4)
\end{aligned}
$$

where

$$
\theta_n(v) = \mathrm{E}\left[h(\|u(t,x)-v\|)\min\left\{1_{B(u(t,x),\alpha)}(v)\left(\frac{\psi(z)J_n(z)}{\det\varphi_n^z(t,x)}\right)\bigg|_{z=\Phi_n^{-1}(v)}, 1\right\}; \Lambda_n\right],
$$

$h:\mathbb{R}_+ \mapsto [0,1]$ is continuous and satisfies $1_{[0,r]} \leq h \leq 1_{[0,(r+\alpha)/2]}$. In the equality (2.4.3), for all $\omega \in \Lambda_n$ the determinant $\det\varphi_n^z(t,x)(\omega)\big|_{z=\Phi_n^{-1}(v)}$ with $v \in B(u(t,x)(\omega),\alpha)$ is positive due to (A.23). Using the fact that $\Lambda_n \subset \{\|u(t,x)-y_0\| \leq r\}$, we know that $h(\|u(t,x)-y_0\|) = 1$ on Λ_n. Together with (iii), we have $\theta_n(y_0) > 0$.

By the definition of the function h, almost surely the function

$$
v \mapsto h(\|u(t,x)-v\|)\min\left\{1_{B(u(t,x),\alpha)}(v)\left(\frac{\psi(z)J_n(z)}{\det\varphi_n^z(t,x)}\right)\bigg|_{z=\Phi_n^{-1}(v)}, 1\right\}
$$

is equal to 0 for v with $\|v-u(t,x)\| > 2(r+\alpha)/3$ and equal to

$$
h(\|u(t,x)-v\|)\min\left\{\left(\frac{\psi(z)J_n(z)}{\det\varphi_n^z(t,x)}\right)\bigg|_{z=\Phi_n^{-1}(v)}, 1\right\}
$$

for $v \in B(u(t,x), 3(r+\alpha)/4)$. Hence it is a.s. continuous and bounded by 1. By Lebesgue's dominated convergence theorem, θ_n is continuous. Let $f(v) = \frac{1}{\epsilon^d}1_{B(y_0,\epsilon)}(v)$. Then (2.4.4) becomes

$$
\frac{1}{\epsilon^d}\int_{B(y_0,\epsilon)} p_{t,x}(v)dv \geq \frac{1}{\epsilon^d}\int_{B(y_0,\epsilon)} \theta_n(v)dv.
$$

Since $v \mapsto p_{t,x}(v)$ is continuous, letting $\epsilon \to 0$, we have $p_{t,x}(y_0) \geq \theta_n(y_0) > 0$.

For the second statement, it suffices to deduce that $\mathrm{Supp}(\mathrm{P}_{u(t,x)}) = \mathbb{R}^d$ from the first statement of this theorem. Suppose that $\mathrm{Supp}(\mathrm{P}_{u(t,x)}) \subsetneqq \mathbb{R}^d$. Then we can find $x_1 \in \mathbb{R}^d$ such that $x_1 \notin \mathrm{Supp}(\mathrm{P}_{u(t,x)})$. Choose $x_2 \in \mathrm{Supp}(\mathrm{P}_{u(t,x)})$ and we can find a continuous curve $\{x(\lambda), \lambda \in [0,1]\}$ with $x(0) = x_2$ and $x(1) = x_1$. Denote $\lambda_* = \sup\{\lambda : x(\lambda) \in \mathrm{Supp}(\mathrm{P}_{u(t,x)})\}$. Since $x_1 = x(1) \notin \mathrm{Supp}(\mathrm{P}_{u(t,x)})$ and the complement of $\mathrm{Supp}(\mathrm{P}_{u(t,x)})$ is an open set, it follows that $\lambda_* < 1$. Then we have a sequence $\lambda_n \uparrow \lambda_*$ such that $x(\lambda_n) \in \mathrm{Supp}(\mathrm{P}_{u(t,x)})$ and we also know that $x(\lambda) \not\in \mathrm{Supp}(\mathrm{P}_{u(t,x)})$ for $\lambda_* < \lambda \leq 1$. This means that $x(\lambda_*)$ is on the boundary of $\mathrm{Supp}(\mathrm{P}_{u(t,x)})$, and since this set is closed, we conclude that $x(\lambda_*) \in \mathrm{Supp}(\mathrm{P}_{u(t,x)})$. By the hypothesis that $\mathbf{H_{t,x}}(y)$ holds on $\mathrm{Supp}(\mathrm{P}_{u(t,x)})$ and the first statement, we know that $p_{t,x}(x(\lambda_*)) > 0$. Since the density function is continuous, it implies that $x(\lambda_*)$ is in the interior of $\mathrm{Supp}(\mathrm{P}_{u(t,x)})$, which contradicts with the fact that $x(\lambda_*)$ is on the boundary of $\mathrm{Supp}(\mathrm{P}_{u(t,x)})$. □

2.4.2 Finite uniform moments of $\varphi^z_{n,i,j}$ and uniform L^p-continuity of $\tilde{u}^z_n$ and φ^z_n

In this section, we give some preliminary computations needed in the proof of Theorem 2.1.1(b).

Consider the sequence $\{g_n = (g_{n1}, \dots, g_{nd})\}_{n\geq 1}$ in $\mathscr{H}$ defined by

$$g_{nj}(r,v) = v_n^{-1} 1_{[t-2^{-n},t]}(r) G_\alpha(t-r, x-v), \; n \geq 1, \; j = 1, \dots, d, \tag{2.4.5}$$

where

$$v_n := \int_0^{2^{-n}} \int_{\mathbb{R}} G^2_\alpha(r,v)\, dv\, dr = \frac{\alpha}{\alpha-1} 2^{-1/\alpha} G_\alpha(1,0) 2^{-n\frac{\alpha-1}{\alpha}}$$

by the scaling property of the Green kernel (A.2).

Taking the derivative with respect to z_j in the both sides of (2.4.1), we have

$$\begin{aligned}
\varphi^z_{n,i,j}(t,x) = &\int_0^t \int_{\mathbb{R}} G_\alpha(t-r, x-v) \sum_{m,l=1}^d \partial_m \sigma_{il}(\tilde{u}^z_n(r,v)) \varphi^z_{n,m,j}(r,v) W^l(dr,dv) \\
&+ \int_0^t \int_{\mathbb{R}} G_\alpha(t-r,x-v) \sigma_{ij}(\tilde{u}^z_n(r,v)) g_{nj}(r,v)\, dr\, dv \\
&+ \sum_{m,l=1}^d z_l \int_0^t \int_{\mathbb{R}} G_\alpha(t-r,x-v) \partial_m \sigma_{il}(\tilde{u}^z_n(r,v)) \varphi^z_{n,m,j}(r,v) g_{nl}(r,v)\, dr\, dv \\
&+ \sum_{m=1}^d \int_0^t \int_{\mathbb{R}} G_\alpha(t-r,x-v) \partial_m b_i(\tilde{u}^z_n(r,v)) \varphi^z_{n,m,j}(r,v)\, dr\, dv.
\end{aligned} \tag{2.4.6}$$

On the other hand, similar to (2.3.1), the Malliavin derivative $D^{(j)}_{r,v}(\tilde{u}^z_{n,i}(t,x))$ satisfies

$$
\begin{aligned}
D^{(j)}_{r,v}(\tilde{u}^z_{n,i}(t,x)) = &\int_0^t \int_{\mathbb{R}} G_\alpha(t-s,x-y) \sum_{m,l=1}^d \partial_m \sigma_{il}(\tilde{u}^z_n(s,y)) D^{(j)}_{r,v}(\tilde{u}^z_{n,m}(s,y)) W^l(ds,dy) \\
&+ G_\alpha(t-r,x-v)\sigma_{ij}(\tilde{u}^z_n(r,v)) \\
&+ \sum_{m,l=1}^d z_l \int_0^t \int_{\mathbb{R}} G_\alpha(t-s,x-y)\partial_m \sigma_{il}(\tilde{u}^z_n(s,y)) \\
&\qquad\qquad \times D^{(j)}_{r,v}(\tilde{u}^z_{n,m}(s,y)) g_{nl}(s,y)\,ds\,dy \\
&+ \sum_{m=1}^d \int_0^t \int_{\mathbb{R}} G_\alpha(t-s,x-y)\partial_m b_i(\tilde{u}^z_n(s,y)) D^{(j)}_{r,v}(\tilde{u}^z_{n,m}(s,y))\,ds\,dy.
\end{aligned}
$$

Comparing the above two equations we have

$$
\varphi^z_{n,i,j}(t,x) = \int_0^t \int_{\mathbb{R}} D^{(j)}_{r,v}(\tilde{u}^z_{n,i}(t,x)) g_{nj}(r,v)\,dv\,dr,\ 1 \le i,j \le d. \tag{2.4.7}
$$

Since

$$
\sup_{(t,x)\in[0,T]\times\mathbb{R}} \mathrm{E}\left[\left(\int_0^t \int_{\mathbb{R}} [D^{(j)}_{r,v}(\tilde{u}^z_{n,i}(t,x))]^2\,dv\,dr\right)^{p/2}\right] < \infty
$$

for any $p>1$ (see for instance (2.3.3)), by the Cauchy-Schwarz inequality we have

$$
\sup_{(s,y)\in[0,T]\times\mathbb{R}} \mathrm{E}[|\varphi^z_{n,i,j}(s,y)|^p] < \infty. \tag{2.4.8}
$$

We improve (2.4.8) to a bound that is uniform in z for small z.

Lemma 2.4.2. *There exists a constant $C_{p,T}$ such that for $p>1$, any large n and small δ,*

$$
\sup_{\|z\|\le\delta}\ \sup_{(s,y)\in[0,T]\times\mathbb{R}} E[|\varphi^z_{n,i,j}(s,y)|^p] \le C_{p,T}. \tag{2.4.9}
$$

Proof. From (2.4.6) we obtain that

$$
\varphi^z_{n,i,j}(t,x) = \mathscr{A}^z_{n,i,j}(t,x) + \mathscr{B}^z_{n,i,j}(t,x) + \mathscr{C}^z_{n,i,j}(t,x) + \mathscr{D}^z_{n,i,j}(t,x), \tag{2.4.10}
$$

where

$$\mathscr{A}^z_{n,i,j}(t,x) = v_n^{-1}\int_0^{2^{-n}} dr\int_{\mathbb{R}} dv\,\sigma_{ij}(\tilde{u}^z_n(t-r,v))G^2_\alpha(r,x-v),$$

$$\mathscr{B}^z_{n,i,j}(t,x) = \int_{t-2^{-n}}^t\int_{\mathbb{R}} G_\alpha(t-s,x-y)\sum_{l,m=1}^d \partial_m\sigma_{il}(\tilde{u}^z_n(s,y))\varphi^z_{n,m,j}(s,y)W^l(ds,dy),$$

$$\begin{aligned}\mathscr{C}^z_{n,i,j}(t,x) &= \sum_{l,m=1}^d z_l\int_{t-2^{-n}}^t\int_{\mathbb{R}} G_\alpha(t-s,x-y)\partial_m\sigma_{il}(\tilde{u}^z_n(s,y))\varphi^z_{n,m,j}(s,y)g_{nl}(s,y)dyds\\ &= \sum_{l,m=1}^d z_l\int_{t-2^{-n}}^t\int_{\mathbb{R}} v_n^{-1}G^2_\alpha(t-s,x-y)\partial_m\sigma_{il}(\tilde{u}^z_n(s,y))\varphi^z_{n,m,j}(s,y)dyds,\end{aligned}$$

$$\mathscr{D}^z_{n,i,j}(t,x) = \int_{t-2^{-n}}^t\int_{\mathbb{R}}\sum_{m=1}^d \partial_m b_i(\tilde{u}^z_n(s,y))\varphi^z_{n,m,j}(s,y)G_\alpha(t-s,x-y)dyds.$$

Next we study upper bounds for the p-moments of the four terms on the right-hand side of (2.4.10). For this, we assume that $\|z\|\le\delta$ for some $\delta>0$.

First we notice that by the choice of v_n, there exists a constant K such that

$$|\mathscr{A}^z_{n,i,j}(t,x)|\le K, \tag{2.4.11}$$

for all n,i,j,z and (t,x) since the functions $\sigma_{ij}, i,j=1,\dots,d$ are bounded.

By Burkholder's inequality, for $p>1$

$$\begin{aligned}\mathrm{E}[|\mathscr{B}^z_{n,i,j}(t,x)|^p] \le c_p\mathrm{E}\Bigg[\Bigg(\int_{t-2^{-n}}^t\int_{\mathbb{R}} G^2_\alpha(t-s,x-y)\sum_{l,m=1}^d(\partial_m\sigma_{il}(\tilde{u}^z_n(s,y)))^2\\ \times(\varphi^z_{n,m,j}(s,y))^2 dsdy\Bigg)^{\frac{p}{2}}\Bigg].\end{aligned}$$

Since the partial derivatives of σ_{il} are bounded, using Hölder's inequality with respect to the measure $G^2_\alpha(t-s,x-y)dsdy$, $\mathrm{E}[|\mathscr{B}^z_{n,i,j}(t,x)|^p]$ is bounded above by

$$\begin{aligned}&c_p\mathrm{E}\left[\int_{t-2^{-n}}^t\int_{\mathbb{R}} G^2_\alpha(t-s,x-y)\sum_{m=1}^d|\varphi^z_{n,m,j}(s,y)|^p dsdy\right]\\ &\qquad\times\left(\int_{t-2^{-n}}^t\int_{\mathbb{R}} G^2_\alpha(t-s,x-y)dyds\right)^{\frac{p}{2}-1}\\ &\le c_p v_n^{\frac{p}{2}-1}\int_{t-2^{-n}}^t\int_{\mathbb{R}} G^2_\alpha(t-s,x-y)\sum_{m=1}^d\sup_{(r,v)\in[0,T]\times\mathbb{R}}\mathrm{E}[|\varphi^z_{n,m,j}(r,v)|^p]dsdy\\ &= c_p v_n^{\frac{p}{2}}\sum_{m=1}^d\sup_{(s,y)\in[0,T]\times\mathbb{R}}\mathrm{E}[|\varphi^z_{n,m,j}(s,y)|^p].\end{aligned} \tag{2.4.12}$$

By Hölder's inequality with respect to the measure $v_n^{-1}G_\alpha^2(t-s,x-y)dsdy$,

$$\begin{aligned}
\mathrm{E}[|\mathscr{C}_{n,i,j}^z(t,x)|^p] &\le c_p\delta^p \mathrm{E}\left[\int_{t-2^{-n}}^t \int_{\mathbb{R}} v_n^{-1}G_\alpha^2(t-s,x-y)\sum_{m=1}^d |\varphi_{n,m,j}^z(s,y)|^p dsdy\right] \\
&\qquad \times \left(\int_{t-2^{-n}}^t \int_{\mathbb{R}} v_n^{-1}G_\alpha^2(t-s,x-y)dyds\right)^{p-1} \\
&\le c_p\delta^p \int_{t-2^{-n}}^t \int_{\mathbb{R}} v_n^{-1}G_\alpha^2(t-s,x-y)dsdy \\
&\qquad \times \sum_{m=1}^d \sup_{(r,v)\in[0,T]\times\mathbb{R}} \mathrm{E}[|\varphi_{n,m,j}^z(s,v)|^p] \\
&= c_p\delta^p \sum_{m=1}^d \sup_{(s,y)\in[0,T]\times\mathbb{R}} \mathrm{E}[|\varphi_{n,m,j}^z(s,y)|^p]. \qquad (2.4.13)
\end{aligned}$$

Similarly, applying Hölder's inequality with respect to the measure $G_\alpha(t-s,x-y)dsdy$, we have

$$\begin{aligned}
\mathrm{E}[|\mathscr{D}_{n,i,j}^z(t,x)|^p] &\le c_p \mathrm{E}\left[\int_{t-2^{-n}}^t \int_{\mathbb{R}} G_\alpha(t-s,x-y)\sum_{m=1}^d |\varphi_{n,m,j}^z(s,y)|^p dsdy\right] \\
&\qquad \times \left(\int_{t-2^{-n}}^t \int_{\mathbb{R}} G_\alpha(t-s,x-y)dyds\right)^{p-1} \\
&\le c_p 2^{-n(p-1)} \int_{t-2^{-n}}^t \int_{\mathbb{R}} G_\alpha(t-s,x-y)dsdy \\
&\qquad \times \sum_{m=1}^d \sup_{(r,v)\in[0,T]\times\mathbb{R}} \mathrm{E}[|\varphi_{n,m,j}^z(r,v)|^p] \\
&= c_p 2^{-np} \sum_{m=1}^d \sup_{(s,y)\in[0,T]\times\mathbb{R}} \mathrm{E}[|\varphi_{n,m,j}^z(s,y)|^p]. \qquad (2.4.14)
\end{aligned}$$

Now, substituting (2.4.11), (2.4.12), (2.4.13) and (2.4.14) into (2.4.10), we obtain that for all $p>1$,

$$\sum_{m=1}^d \mathrm{E}[|\varphi_{n,m,j}^z(t,x)|^p] \le K_p + c_{p,T}(v_n^{\frac{p}{2}} + \delta^p + 2^{-np}) \sum_{m=1}^d \sup_{(s,y)\in[0,T]\times\mathbb{R}} \mathrm{E}[|\varphi_{n,m,j}^z(s,y)|^p]. \qquad (2.4.15)$$

Thus choosing n large and δ small such that $c_{p,T}(v_n^{\frac{p}{2}} + \delta^p + 2^{-np}) \le \frac{1}{2}$, we obtain from (2.4.8) and the above inequality that there exists a constant $C_{p,T}$ such that for any large n

$$\sup_{\|z\|\le\delta} \sup_{(s,y)\in[0,T]\times\mathbb{R}} \mathrm{E}[|\varphi_{n,i,j}^z(s,y)|^p] \le C_{p,T}.$$

$\square$

The following two lemmas give some estimates on the uniform L^p-continuity of $\tilde{u}_n^z$ and φ_n^z in z for small z.

Lemma 2.4.3. *For δ small, there exists $c_{p,T}$ such that for any $p \geq 2$ and z, z' with $\|z\| \leq \delta$, $\|z'\| \leq \delta$*

$$\sup_{(s,y)\in[0,T]\times\mathbb{R}} E[\|\tilde{u}_n^z(s,y)) - \tilde{u}_n^{z'}(s,y)\|^p] \leq c_{p,T}\|z-z'\|^p. \tag{2.4.16}$$

Proof. Indeed, from (2.4.1) we have, for any $p \geq 2$,

$$\begin{aligned}
&\mathrm{E}[|\tilde{u}_{ni}^z(t,x)) - \tilde{u}_{ni}^{z'}(t,x)|^p] \\
&\leq c_p \mathrm{E}\left[\left|\int_0^t \int_{\mathbb{R}} G_\alpha(t-r,x-v) \sum_{j=1}^d (\sigma_{ij}(\tilde{u}_n^z(r,v)) - \sigma_{ij}(\tilde{u}_n^{z'}(r,v)))W^j(dr,dv)\right|^p\right] \\
&\quad + c_p \sum_{j=1}^d |z_j - z_j'|^p \mathrm{E}\left[\left|\int_0^t \int_{\mathbb{R}} G_\alpha(t-r,x-v)\sigma_{ij}(\tilde{u}_n^z(r,v))g_{nj}(r,v)dvdr\right|^p\right] \\
&\quad + c_p \sum_{j=1}^d |z_j'|^p \mathrm{E}\Bigg[\Bigg|\int_0^t \int_{\mathbb{R}} G_\alpha(t-r,x-v)(\sigma_{ij}(\tilde{u}_n^z(r,v)) - \sigma_{ij}(\tilde{u}_n^{z'}(r,v))) \\
&\qquad\qquad\qquad \times g_{nj}(r,v)dvdr\Bigg|^p\Bigg] \\
&\quad + c_p \mathrm{E}\left[\left|\int_0^t \int_{\mathbb{R}} G_\alpha(t-r,x-v)(b_i(\tilde{u}_n^z(r,v)) - b_i(\tilde{u}_n^{z'}(r,v)))dvdr\right|^p\right] \\
&:= \mathscr{A}_1 + \mathscr{A}_2 + \mathscr{A}_3 + \mathscr{A}_4.
\end{aligned} \tag{2.4.17}$$

Using Burkholder's inequality, the fact that the partial derivatives of σ_{ij} are bounded and Hölder's inequality with respect to the measure $G_\alpha^2(t-r,x-v)drdv$,

$$\begin{aligned}
\mathscr{A}_1 &\leq c_p \mathrm{E}\left[\left|\int_0^t \int_{\mathbb{R}} G_\alpha^2(t-r,x-v)\|\tilde{u}_n^z(r,v) - \tilde{u}_n^{z'}(r,v)\|^2 dvdr\right|^{p/2}\right] \\
&\leq c_p \left(\int_0^t \int_{\mathbb{R}} G_\alpha^2(t-r,x-v)dvdr\right)^{p/2-1} \\
&\quad \times \int_0^t \int_{\mathbb{R}} G_\alpha^2(t-r,x-v)\mathrm{E}\left[\|\tilde{u}_n^z(r,v) - \tilde{u}_n^{z'}(r,v)\|^p\right] dvdr \\
&\leq c_p \int_0^t \int_{\mathbb{R}} G_\alpha^2(t-r,x-v) \sup_{y\in\mathbb{R}} \mathrm{E}\left[\|\tilde{u}_n^z(r,y) - \tilde{u}_n^{z'}(r,y)\|^p\right] dvdr \\
&= c_p \int_0^t (t-r)^{-1/\alpha} \sup_{y\in\mathbb{R}} \mathrm{E}\left[\|\tilde{u}_n^z(r,y) - \tilde{u}_n^{z'}(r,y)\|^p\right] dr.
\end{aligned} \tag{2.4.18}$$

Since the functions $\sigma_{ij}, i,j = 1,\ldots,d$, are bounded, by the definition of g_n, it is clear that

$$\mathscr{A}_2 \leq c_p\|z-z'\|^p. \tag{2.4.19}$$

Similarly, by Hölder's inequality with respect to the probability measure $v_n^{-1}1_{[t-2^{-n},t]}(r)G_\alpha^2(t-$

$r, x-v)drdv$,

$$\mathscr{A}_3 \le c_p \delta^p \int_{t-2^{-n}}^t dr \int_{\mathbb{R}} v_n^{-1} G_\alpha^2(t-r, x-v)\, dv \sup_{y\in\mathbb{R}} \mathrm{E}\left[\|\tilde{u}_n^z(r,y) - \tilde{u}_n^{z'}(r,y)\|^p\right]$$
$$\le c_p \delta^p \sup_{(s,y)\in[0,T]\times\mathbb{R}} \mathrm{E}\left[\|\tilde{u}_n^z(s,y) - \tilde{u}_n^{z'}(s,y)\|^p\right] \tag{2.4.20}$$

For the last term, using Hölder's inequality with respect to the measure $G_\alpha(t-r, x-v)drdv$ and the fact that the partial derivatives of b_i are bounded,

$$\mathscr{A}_4 \le c_p \int_0^t dr \int_{\mathbb{R}} dv G_\alpha(t-r, x-v) \mathrm{E}\left[\|\tilde{u}_n^z(r,v) - \tilde{u}_n^{z'}(r,v)\|^p\right]$$
$$\times \left(\int_0^t dr \int_{\mathbb{R}} dv G_\alpha(t-r, x-v)\right)^{p-1}$$
$$\le c_p T^{p-1} \int_0^t dr \sup_{y\in\mathbb{R}} \mathrm{E}\left[\|\tilde{u}_n^z(r,y) - \tilde{u}_n^{z'}(r,y)\|^p\right] \tag{2.4.21}$$

From (2.4.18), (2.4.19), (2.4.20), (2.4.21) and (2.4.17), we have obtained that

$$\sup_{y\in\mathbb{R}} \mathrm{E}\left[\|\tilde{u}_n^z(t,y) - \tilde{u}_n^{z'}(t,y)\|^p\right]$$
$$\le c_{p,T}\|z-z'\|^p + c_{p,T}\delta^p \sup_{(s,y)\in[0,T]\times\mathbb{R}} \mathrm{E}[\|\tilde{u}_n^z(s,y) - \tilde{u}_n^{z'}(s,y)\|^p]$$
$$+ c_{p,T} \int_0^t dr((t-r)^{-1/\alpha} + 1) \sup_{y\in\mathbb{R}} \mathrm{E}\left[\|\tilde{u}_n^z(r,y) - \tilde{u}_n^{z'}(r,y)\|^p\right].$$

By Gronwall's lemma (see [23, Lemma 15]), we obtain

$$\sup_{(s,y)\in[0,T]\times\mathbb{R}} \mathrm{E}[\|\tilde{u}_n^z(s,y)) - \tilde{u}_n^{z'}(s,y)\|^p]$$
$$\le c_{p,T}\|z-z'\|^p + c_{p,T}\delta^p \sup_{(s,y)\in[0,T]\times\mathbb{R}} \mathrm{E}[\|\tilde{u}_n^z(s,y) - \tilde{u}_n^{z'}(s,y)\|^p].$$

We can choose δ small enough in the above inequality with $c_{p,T}\delta^p < 1$ so that (2.4.16) holds with a different constant $c_{p,T}$. ☐

Lemma 2.4.4. *Let δ be small and n large enough so that (2.4.9) and (2.4.16) hold. Then for δ small and n large there exists $c_{p,T}$ such that for any $p > 1$ and z, z' with $\|z\| \le \delta$, $\|z'\| \le \delta$*

$$\sup_{(s,y)\in[0,T]\times\mathbb{R}} E\left[\|\varphi_n^z(s,y) - \varphi_n^{z'}(s,y)\|^p\right] \le c_{p,T}\|z-z'\|^p. \tag{2.4.22}$$

Proof. We adopt the notations $\mathscr{A}, \mathscr{B}, \mathscr{C}, \mathscr{D}$ defined in the proof of Lemma 2.4.2. We first use the Lipschitz property of σ together with (2.4.16), to obtain that

$$\sup_{(s,y)\in[0,T]\times\mathbb{R}} \mathrm{E}\left[\|\mathscr{A}_n^z(s,y) - \mathscr{A}_n^{z'}(s,y)\|^p\right] \le c_{p,T}\|z-z'\|^p. \tag{2.4.23}$$

By hypothesis **P1'** and Burkholder's inequality, we know that, for any $p > 1$,

$$
\begin{aligned}
&\mathrm{E}[|\mathscr{B}^{z}_{n,i,j}(t,x) - \mathscr{B}^{z'}_{n,i,j}(t,x)|^p] \\
&\quad \le c_p \mathrm{E}\Bigg[\Big|\int_{t-2^{-n}}^{t}\int_{\mathbb{R}} G_\alpha(t-s,x-y) \sum_{l,m=1}^{d} (\partial_m \sigma_{il}(\tilde{u}^{z}_n(s,y)) - \partial_m \sigma_{il}(\tilde{u}^{z'}_n(s,y))) \\
&\qquad\qquad \times \varphi^{z}_{n,m,j}(s,y) W^l(ds,dy)\Big|^p\Bigg] \\
&\qquad + c_p \mathrm{E}\Bigg[\Big|\int_{t-2^{-n}}^{t}\int_{\mathbb{R}} G_\alpha(t-s,x-y) \sum_{l,m=1}^{d} \partial_m \sigma_{il}(\tilde{u}^{z'}_n(s,y)) \\
&\qquad\qquad \times (\varphi^{z}_{n,m,j}(s,y) - \varphi^{z'}_{n,m,j}(s,y)) W^l(ds,dy)\Big|^p\Bigg] \\
&\quad \le c_p \mathrm{E}\Bigg[\Bigg|\int_{t-2^{-n}}^{t}\int_{\mathbb{R}} G^2_\alpha(t-s,x-y) \| \tilde{u}^{z}_n(s,y)) - \tilde{u}^{z'}_n(s,y)\|^2 \sum_{m=1}^{d} (\varphi^{z}_{n,m,j}(s,y))^2 ds dy\Bigg|^{p/2}\Bigg] \\
&\qquad + c_p \mathrm{E}\Bigg[\Bigg|\int_{t-2^{-n}}^{t}\int_{\mathbb{R}} G^2_\alpha(t-s,x-y) \sum_{m=1}^{d} (\varphi^{z}_{n,m,j}(s,y) - \varphi^{z'}_{n,m,j}(s,y))^2 ds dy\Bigg|^{p/2}\Bigg].
\end{aligned}
$$

Using Hölder's inequality with respect to the measure $G^2_\alpha(t-s,x-y)dsdy$ twice, this is bounded above by

$$
\begin{aligned}
&c_p v_n^{\frac{p}{2}-1} \int_{t-2^{-n}}^{t}\int_{\mathbb{R}} G^2_\alpha(t-s,x-y) \mathrm{E}\left[\| \tilde{u}^{z}_n(s,y)) - \tilde{u}^{z'}_n(s,y)\|^p \sum_{m=1}^{d} |\varphi^{z}_{n,m,j}(s,y)|^p\right] ds dy \\
&\quad + c_p v_n^{\frac{p}{2}-1} \int_{t-2^{-n}}^{t}\int_{\mathbb{R}} G^2_\alpha(t-s,x-y) \sum_{m=1}^{d} \mathrm{E}\left[|\varphi^{z}_{n,m,j}(s,y) - \varphi^{z'}_{n,m,j}(s,y)|^p\right] ds dy \\
&\le c_p v_n^{p/2} \sum_{m=1}^{d} \sup_{\|z\|\le\delta} \sup_{(s,y)\in[0,T]\times\mathbb{R}} \mathrm{E}\left[|\varphi^{z}_{n,m,j}(s,y)|^{2p}\right]^{1/2} \\
&\qquad\qquad \times \sup_{(s,y)\in[0,T]\times\mathbb{R}} \mathrm{E}\left[\| \tilde{u}^{z}_n(s,y)) - \tilde{u}^{z'}_n(s,y)\|^{2p}\right]^{1/2} \\
&\quad + c_p v_n^{p/2} \sum_{m=1}^{d} \sup_{(s,y)\in[0,T]\times\mathbb{R}} \mathrm{E}\left[|\varphi^{z}_{n,m,j}(s,y) - \varphi^{z'}_{n,m,j}(s,y)|^p\right],
\end{aligned}
$$

where we have used the Cauchy-Schwartz inequality. From (2.4.9), there exists a constant $C'_{p,T}$ such that

$$
\sum_{m=1}^{d} \sup_{\|z\|\le\delta} \sup_{(s,y)\in[0,T]\times\mathbb{R}} \mathrm{E}\left[|\varphi^{z}_{n,m,j}(s,y)|^{2p}\right]^{1/2} \le C'_{p,T}.
$$

Hence we obtain that

$$
\begin{aligned}
&\mathrm{E}\left[|\mathscr{B}^{z}_{n,i,j}(t,x)-\mathscr{B}^{z'}_{n,i,j}(t,x)|^{p}\right]\\
&\quad\le c_p v_n^{p/2}\sup_{(s,y)\in[0,T]\times\mathbb{R}}\mathrm{E}\left[\|\tilde{u}^{z}_n(s,y))-\tilde{u}^{z'}_n(s,y)\|^{2p}\right]^{1/2}\\
&\qquad+c_p v_n^{p/2}\sum_{m=1}^{d}\sup_{(s,y)\in[0,T]\times\mathbb{R}}\mathrm{E}\left[|\varphi^{z}_{n,m,j}(s,y)-\varphi^{z'}_{n,m,j}(s,y)|^{p}\right].
\end{aligned}
\tag{2.4.24}
$$

Similarly, using hypothesis **P1'** we have, for any $p>1$,

$$
\begin{aligned}
&\mathrm{E}[|\mathscr{C}^{z}_{n,i,j}(t,x)-\mathscr{C}^{z'}_{n,i,j}(t,x)|^{p}]\\
&\quad\le c_p\sum_{l,m=1}^{d}|z_l-z'_l|^{p}\mathrm{E}\left[\left|\int_{t-2^{-n}}^{t}\int_{\mathbb{R}}v_n^{-1}G^2_\alpha(t-s,x-y)|\varphi^{z}_{n,m,j}(s,y)|dyds\right|^{p}\right]\\
&\qquad+c_p\sum_{l,m=1}^{d}|z'_l|^{p}\mathrm{E}\Big[\Big|\int_{t-2^{-n}}^{t}ds\int_{\mathbb{R}}dy v_n^{-1}G^2_\alpha(t-s,x-y)\\
&\qquad\qquad\times\left(\partial_m\sigma_{il}(\tilde{u}^{z}_n(s,y))\varphi^{z}_{n,m,j}(s,y)-\partial_m\sigma_{il}(\tilde{u}^{z'}_n(s,y))\varphi^{z'}_{n,m,j}(s,y)\right)\Big|^{p}\Big]\\
&\quad\le c_p\sum_{l,m=1}^{d}|z_l-z'_l|^{p}\mathrm{E}\left[\left|\int_{t-2^{-n}}^{t}\int_{\mathbb{R}}v_n^{-1}G^2_\alpha(t-s,x-y)|\varphi^{z}_{n,m,j}(s,y)|dyds\right|^{p}\right]\\
&\qquad+c_p\delta^{p}\sum_{m=1}^{d}\mathrm{E}\Big[\Big|\int_{t-2^{-n}}^{t}\int_{\mathbb{R}}v_n^{-1}G^2_\alpha(t-s,x-y)\|\tilde{u}^{z}_n(s,y))-\tilde{u}^{z'}_n(s,y)\|\\
&\qquad\qquad\times|\varphi^{z}_{n,m,j}(s,y)|dyds\Big|^{p}\Big]\\
&\qquad+c_p\delta^{p}\sum_{m=1}^{d}\mathrm{E}\left[\left|\int_{t-2^{-n}}^{t}\int_{\mathbb{R}}v_n^{-1}G^2_\alpha(t-s,x-y)|\varphi^{z}_{n,m,j}(s,y)-\varphi^{z'}_{n,m,j}(s,y)|dyds\right|^{p}\right].
\end{aligned}
$$

Using Hölder's inequality with respect to the probability measure $v_n^{-1}1_{[t-2^{-n},t]}(r)G^2_\alpha(t-r,x-v)drdv$ three times and (2.4.9) twice, it is bounded above by

$$
\begin{aligned}
&c_p\|z-z'\|^{p}+c_p\delta^{p}\sup_{(s,y)\in[0,T]\times\mathbb{R}}\mathrm{E}[\|\tilde{u}^{z}_n(s,y))-\tilde{u}^{z'}_n(s,y)\|^{2p}]^{1/2}\\
&\quad+c_p\delta^{p}\sum_{m=1}^{d}\sup_{(s,y)\in[0,T]\times\mathbb{R}}\mathrm{E}\left[|\varphi^{z}_{n,m,j}(s,y)-\varphi^{z'}_{n,m,j}(s,y)|^{p}\right].
\end{aligned}
\tag{2.4.25}
$$

Furthermore, by hypothesis **P1'**, for any $p>1$,

$$
\begin{aligned}
&\mathrm{E}\left[|\mathscr{D}^{z}_{n,i,j}(t,x)-\mathscr{D}^{z'}_{n,i,j}(t,x)|^p\right]\\
&\quad\le c_p\mathrm{E}\Bigg[\Big|\int_{t-2^{-n}}^t\int_{\mathbb{R}}\sum_{m=1}^d|\partial_m b_i(\tilde u^z_n(s,y))-\partial_m b_i(\tilde u^{z'}_n(s,y))|\\
&\qquad\qquad\times|\varphi^z_{n,m,j}(s,y)|G_\alpha(t-s,x-y)dyds\Big|^p\Bigg]\\
&\qquad+c_p\mathrm{E}\Bigg[\Big|\int_{t-2^{-n}}^t\int_{\mathbb{R}}\sum_{m=1}^d|\partial_m b_i(\tilde u^{z'}_n(s,y))|\\
&\qquad\qquad\times|\varphi^z_{n,m,j}(s,y)-\varphi^z_{n,m,j}(s,y)|G_\alpha(t-s,x-y)dyds\Big|^p\Bigg]\\
&\quad\le c_p\mathrm{E}\Bigg[\Big|\int_{t-2^{-n}}^t\int_{\mathbb{R}}\|\tilde u^z_n(s,y)-\tilde u^{z'}_n(s,y)\|\sum_{m=1}^d|\varphi^z_{n,m,j}(s,y)|G_\alpha(t-s,x-y)dyds\Big|^p\Bigg]\\
&\qquad+c_p\mathrm{E}\Bigg[\Big|\int_{t-2^{-n}}^t\int_{\mathbb{R}}\sum_{m=1}^d|\varphi^z_{n,m,j}(s,y)-\varphi^z_{n,m,j}(s,y)|G_\alpha(t-s,x-y)dyds\Big|^p\Bigg].
\end{aligned}
$$

Using Hölder's inequality with respect to the measure $G_\alpha(t-s,x-y)dsdy$ twice, Cauchy-Schwarz inequality and (2.4.9), it is bounded above by

$$
\begin{aligned}
&c_p2^{-np}\sup_{(s,y)\in[0,T]\times\mathbb{R}}\mathrm{E}[\|\tilde u^z_n(s,y))-\tilde u^{z'}_n(s,y)\|^{2p}]^{1/2}\\
&\quad+c_p2^{-np}\sum_{m=1}^d\sup_{(s,y)\in[0,T]\times\mathbb{R}}\mathrm{E}\left[|\varphi^z_{n,m,j}(s,y)-\varphi^{z'}_{n,m,j}(s,y)|^p\right].
\end{aligned}\tag{2.4.26}
$$

Comparing (2.4.23), (2.4.24), (2.4.25), (2.4.26) with (2.4.10) we have

$$
\begin{aligned}
&\sup_{(s,y)\in[0,T]\times\mathbb{R}}\mathrm{E}\left[\|\varphi^z_n(s,y)-\varphi^{z'}_n(s,y)\|^p\right]\\
&\quad\le c_{p,T}\|z-z'\|^p+c_p\|z-z'\|^p\\
&\qquad+c_p(v_n^{p/2}+\delta^p+2^{-np})\sup_{(s,y)\in[0,T]\times\mathbb{R}}\mathrm{E}[\|\tilde u^z_n(s,y))-\tilde u^{z'}_n(s,y)\|^{2p}]^{1/2}\\
&\qquad+c_p(v_n^{p/2}+\delta^p+2^{-np})\sup_{(s,y)\in[0,T]\times\mathbb{R}}\mathrm{E}\left[\|\varphi^z_n(s,y)-\varphi^{z'}_n(s,y)\|^p\right].
\end{aligned}
$$

Finally we choose n large and δ small in the above inequality such that $c_p(v_n^{p/2}+\delta^p+2^{-np})<1$. Then we obtain that

$$
\begin{aligned}
&\sup_{(s,y)\in[0,T]\times\mathbb{R}}\mathrm{E}\left[\|\varphi^z_n(s,y)-\varphi^{z'}_n(s,y)\|^p\right]\\
&\quad\le c_p\|z-z'\|^p+c_p\sup_{(s,y)\in[0,T]\times\mathbb{R}}\mathrm{E}[\|\tilde u^z_n(s,y))-\tilde u^{z'}_n(s,y)\|^{2p}]^{1/2}.
\end{aligned}
$$

Therefore we have proved that

$$\sup_{(s,y)\in[0,T]\times\mathbb{R}} \mathrm{E}\left[\|\varphi_n^z(s,y)-\varphi_n^{z'}(s,y)\|^p\right] \le c_{p,T}\|z-z'\|^p.$$

□

2.4.3 Proof of Theorem 2.1.1(b)

Let $(t_0,x_0)\in]0,T]\times\mathbb{R}$ be fixed. By Theorem 2.4.1, we need to show that for any $y_0\in \mathrm{Supp}(\mathrm{P}_{u(t_0,x_0)})$, the assumptions (i) and (ii) of $\mathbf{H}_{\mathbf{t_0},\mathbf{x_0}}(y_0)$ are satisfied.

We first verify assumption (i) of $\mathbf{H}_{\mathbf{t_0},\mathbf{x_0}}(y_0)$.

From now on, we assume that n is large and δ is small so that (2.4.9), (2.4.16) and (2.4.22) are satisfied.

Let $z=0$ in (2.4.10) to get that

$$\varphi^0_{n,i,j}(t_0,x_0)=\mathscr{A}^0_{n,i,j}(t_0,x_0)+\mathscr{R}_{n,i,j}(t_0,x_0), \tag{2.4.27}$$

where $\mathscr{R}_{n,i,j}(t_0,x_0)=\mathscr{B}^0_{n,i,j}(t_0,x_0)+\mathscr{D}^0_{n,i,j}(t_0,x_0)$ satisfies that for any $p>1$,

$$\mathrm{E}[|\mathscr{R}_{n,i,j}(t_0,x_0)|^p]\le c_{p,T}(v_n^{\frac{p}{2}}+2^{-np}), \tag{2.4.28}$$

by (2.4.12), (2.4.14) and (2.4.9). We now write

$$\mathscr{A}^0_{n,i,j}(t_0,x_0)=\sigma_{ij}(u(t_0,x_0))+\mathscr{O}_{n,i,j}(t_0,x_0),$$

where

$$\mathscr{O}_{n,i,j}(t_0,x_0)=v_n^{-1}\int_0^{2^{-n}}\int_{\mathbb{R}}(\sigma_{ij}(u(t_0-r,v))-\sigma_{ij}(u(t_0,x_0)))G_\alpha^2(r,x_0-v)\,dv\,dr$$

Using Minkowski's inequality with respect to the probability measure $v_n^{-1}G_\alpha^2(r,x_0-v)\,dv\,dr$ and the Lipschitz property of σ, we have that, for any $p>1$,

$$\begin{aligned}\mathrm{E}[|\mathscr{O}_{n,i,j}(t_0,x_0)|^p]\le c_p\bigg|\int_0^{2^{-n}}\int_{\mathbb{R}}&(\mathrm{E}[|\sigma_{ij}(u(t_0-r,v))-\sigma_{ij}(u(t_0,x_0))|^p])^{\frac{1}{p}}\\ &\times v_n^{-1}G_\alpha^2(r,x_0-v)\,dv\,dr\bigg|^p\\ \le c_p\bigg|\int_0^{2^{-n}}\int_{\mathbb{R}}&(\mathrm{E}[\|u(t_0-r,v)-u(t_0,x_0)\|^p])^{\frac{1}{p}}v_n^{-1}G_\alpha^2(r,x_0-v)\,dv\,dr\bigg|^p.\end{aligned}$$

By the L^p-continuity of the solution (2.1.6), this is bounded above by

$$
\begin{aligned}
& c_p \left| \int_0^{2^{-n}} \int_{\mathbb{R}} \left(r^{\frac{\alpha-1}{2\alpha}} + |x_0 - v|^{\frac{\alpha-1}{2}} \right) v_n^{-1} G_\alpha^2(r, x_0 - v) dv dr \right|^p \\
& \le c_p \left| \int_0^{2^{-n}} \int_{\mathbb{R}} r^{\frac{\alpha-1}{2\alpha}} v_n^{-1} G_\alpha^2(r, x_0 - v) dv dr \right|^p \\
& \quad + c_p \left| \int_0^{2^{-n}} \int_{\mathbb{R}} |x_0 - v|^{\frac{\alpha-1}{2}} v_n^{-1} G_\alpha^2(r, x_0 - v) dv dr \right|^p \\
& = c_p \left| v_n^{-1} \int_0^{2^{-n}} r^{\frac{\alpha-1}{2\alpha}} G_\alpha(2r, 0) dr \right|^p + c_p \left| v_n^{-1} \int_0^{2^{-n}} \int_{\mathbb{R}} |v|^{\frac{\alpha-1}{2}} G_\alpha^2(r, v) dv dr \right|^p \\
& = c_p \left| v_n^{-1} \int_0^{2^{-n}} r^{\frac{\alpha-1}{2\alpha}} r^{-1/\alpha} dr \right|^p + c_p \left| v_n^{-1} \int_0^{2^{-n}} \int_{\mathbb{R}} |v|^{\frac{\alpha-1}{2}} r^{-2/\alpha} G_\alpha^2(1, r^{-1/\alpha} v) dv dr \right|^p \\
& = c_p v_n^{-p} 2^{-np\frac{3(\alpha-1)}{2\alpha}} + c_p \left| v_n^{-1} \int_0^{2^{-n}} \int_{\mathbb{R}} |u|^{\frac{\alpha-1}{2}} r^{\frac{\alpha-1}{2\alpha}} r^{-2/\alpha} r^{\frac{1}{\alpha}} G_\alpha^2(1, u) du dr \right|^p \\
& = c_p v_n^{-p} 2^{-np\frac{3(\alpha-1)}{2\alpha}} + c_p \left| v_n^{-1} \int_0^{2^{-n}} r^{\frac{\alpha-3}{2\alpha}} dr \int_{\mathbb{R}} |u|^{\frac{\alpha-1}{2}} G_\alpha^2(1, u) du \right|^p \\
& \le c'_p v_n^{-p} 2^{-np\frac{3(\alpha-1)}{2\alpha}} = c'_p 2^{-np\frac{\alpha-1}{2\alpha}},
\end{aligned}
\tag{2.4.29}
$$

where in the second equality we use the scaling property (A.2), in the third equality we change the variable by $u := r^{-1/\alpha} v$ and in the last inequality the integral $\int_{\mathbb{R}} |u|^{\frac{\alpha-1}{2}} G_\alpha^2(1, u) du$ is finite because of (A.5).

Now, as $y_0 \in \mathrm{Supp}(\mathrm{P}_{u(t_0, x_0)})$, there exists r_0 such that for all $0 < r \le r_0$,

$$\mathrm{P}\{u(t_0, x_0) \in B(y_0; r)\} > 0.$$

Hence, for all $0 < r \le r_0$,

$$\mathrm{P}\left\{ (\| u(t_0, x_0) - y_0 \| \le r) \cap (\det \sigma(u(t_0, x_0)) \ge 2c_1) \right\} > 0,$$

where

$$c_1 := \frac{1}{2} \left(\inf_{z \in \bar{B}(y_0; r)} \inf_{\|\xi\|=1} \|\sigma(z)\xi\|^2 \right)^d.$$

From the moment estimates (2.4.28) and (2.4.29) for $\mathscr{R}_{n,i,j}(t_0, x_0)$ and $\mathscr{O}_{n,i,j}(t_0, x_0)$, these quantities converge to 0 in L^p, so that we can choose a subsequence $\{n_k\}_{k \ge 1}$ such that for any $1 \le i, j \le d$, $\varphi^0_{n_k,i,j}(t_0, x_0)$ converges to $\sigma_{ij}(u(t_0, x_0))$ a.s. as $k \to \infty$, which by Fatou's lemma implies that

$$\limsup_{n \to \infty} \mathrm{P}\left\{ (\| u(t_0, x_0) - y_0 \| \le r) \cap (\det \varphi^0_n(t_0, x_0) \ge c_1) \right\} > 0.$$

This proves (i) of $\mathbf{H}_{\mathbf{t_0},\mathbf{x_0}}(y_0)$.

We next proceed to verify (ii) of $\mathbf{H}_{\mathbf{t_0},\mathbf{x_0}}(y_0)$.

We start proving that there exist $c > 0$ and $\delta > 0$, such that

$$\lim_{n\to\infty} \mathrm{P}\left\{ \sup_{\|z\|\le\delta} \|\varphi_n^z(t_0,x_0)\| \le c \right\} = 1. \tag{2.4.30}$$

Observe from (2.4.10) and (2.4.11) that

$$\|\varphi_n^z(t_0,x_0)\| \le K + \sup_{\|z\|\le\delta} \|\mathscr{G}_n^z(t_0,x_0)\|, \tag{2.4.31}$$

where $\mathscr{G}_n^z(t_0,x_0) = \mathscr{B}_n^z(t_0,x_0) + \mathscr{C}_n^z(t_0,x_0) + \mathscr{D}_n^z(t_0,x_0)$ satisfies that for any $p > 1$

$$\sup_{\|z\|\le\delta} \mathrm{E}[\|\mathscr{G}_n^z(t_0,x_0)\|^p] \le c_{p,T}(v_n^{\frac{p}{2}} + \delta^p + 2^{-np}),$$

by (2.4.12), (2.4.13), (2.4.14) and (2.4.9).

For $\|z\| < \delta$, let $v_{n,i,j}^z (1 \le i, j \le d)$ denote the solution of the affine equation

$$\begin{aligned} v_{n,i,j}^z(t,x) = &\sum_{l,m=1}^d z_l \int_{t-2^{-n}}^t \int_{\mathbb{R}} G_\alpha(t-s,x-y)\partial_m\sigma_{il}(\tilde{u}_n^z(s,y))v_{n,m,j}^z(s,y)g_{nl}(s,y)dyds \\ &+ \mathscr{A}_{n,i,j}^z(t,x). \end{aligned} \tag{2.4.32}$$

For each n and z, equation (2.4.32) has a unique solution by Picard iteration. Define

$$I(v_{n,i,j}^z,\omega)(t,x) = \sum_{l,m=1}^d z_l \int_{t-2^{-n}}^t \int_{\mathbb{R}} G_\alpha(t-s,x-y)\partial_m\sigma_{il}(\tilde{u}_n^z(s,y))v_{n,m,j}^z(s,y)g_{nl}(s,y)dyds.$$

Since the derivative of σ is bounded, there exists a constant c such that

$$\begin{aligned} &\sum_{i,j=1}^d \sup_{n\ge1} \sup_{\|z\|\le\delta} \sup_{\omega\in\Omega} \sup_{(t,x)\in[0,T]\times\mathbb{R}} |I(v_{n,i,j}^z,\omega)(t,x)| \\ &\quad \le c\delta \sum_{i,j=1}^d \sup_{n\ge1} \sup_{\|z\|\le\delta} \sup_{\omega\in\Omega} \sup_{(t,x)\in[0,T]\times\mathbb{R}} |v_{n,i,j}^z(t,x)|. \end{aligned} \tag{2.4.33}$$

Let $v_{n,i,j}^{z,0}(t,x) := \mathscr{A}_{n,i,j}^z(t,x)$. Then from (2.4.11) there exists a constant K such that

$$\sum_{i,j=1}^d \sup_{n\ge1} \sup_{\|z\|\le\delta} \sup_{\omega\in\Omega} \sup_{(t,x)\in[0,T]\times\mathbb{R}} |v_{n,i,j}^{z,0}(t,x)(\omega)| \le K.$$

For each integer $k \ge 0$, we define

$$v_{n,i,j}^{z,k+1}(t,x)(\omega) = I(v_{n,i,j}^{z,k},\omega)(t,x) + \mathscr{A}_{n,i,j}^z(t,x)(\omega). \tag{2.4.34}$$

Suppose that for $k \geq 0$, we have

$$\sum_{i,j=1}^{d} \sup_{n\geq 1} \sup_{\|z\|\leq\delta} \sup_{\omega\in\Omega} \sup_{(t,x)\in[0,T]\times\mathbb{R}} |v_{n,i,j}^{z,k}(t,x)(\omega)| \leq K + K\sum_{i=1}^{k}(c\delta)^i.$$

Then by (2.4.33) and (2.4.34), for $k+1$ we have

$$\sum_{i,j=1}^{d} \sup_{n\geq 1} \sup_{\|z\|\leq\delta} \sup_{\omega\in\Omega} \sup_{(t,x)\in[0,T]\times\mathbb{R}} |v_{n,i,j}^{z,k+1}(t,x)| \leq K + c\delta(K + K\sum_{i=1}^{k}(c\delta)^i = K + K\sum_{i=1}^{k+1}(c\delta)^i.$$

Choose δ small such that $c\delta < 1$. Then, by induction, we obtain that

$$\sup_{k\geq 0} \sum_{i,j=1}^{d} \sup_{n\geq 1} \sup_{\|z\|\leq\delta} \sup_{\omega\in\Omega} \sup_{(t,x)\in[0,T]\times\mathbb{R}} |v_{n,i,j}^{z,k}(t,x)(\omega)| \leq \frac{K}{1-c\delta}. \tag{2.4.35}$$

Since $v_{n,i,j}^{z,k}(t,x)$ converges to $v_{n,i,j}^{z}(t,x)$ as $k\to\infty$, from (2.4.35) there exists a constant C such that

$$\sup_{n\geq 1} \sup_{\|z\|\leq\delta} \sup_{\omega\in\Omega} \sup_{(t,x)\in[0,T]\times\mathbb{R}} |v_{n,i,j}^{z}(t,x)| \leq C. \tag{2.4.36}$$

In order to establish uniform L^p-continuity of $z \mapsto v_n^z$ for small z, we use (2.4.23) and (2.4.36) to get that, for $\|z\| \leq \delta$ and $\|z'\| \leq \delta$

$$\begin{aligned}
&\mathrm{E}\left[\left|v_{n,i,j}^{z}(t,x) - v_{n,i,j}^{z'}(t,x)\right|^p\right] \\
&\quad \leq c_p\|z-z'\|^p + c_p\delta^p \sum_{m=1}^{d} \mathrm{E}\left[\left|\int_{t-2^{-n}}^{t}\int_{\mathbb{R}} v_n^{-1} G_\alpha^2(t-s,x-y)\|\tilde{u}_n^z(s,y)) - \tilde{u}_n^{z'}(s,y)\|dyds\right|^p\right] \\
&\qquad + c_p\delta^p \sum_{m=1}^{d} \mathrm{E}\left[\left|\int_{t-2^{-n}}^{t}\int_{\mathbb{R}} v_n^{-1} G_\alpha^2(t-s,x-y)|v_{n,m,j}^{z}(s,y) - v_{n,m,j}^{z'}(s,y)|dyds\right|^p\right]
\end{aligned}$$

By Hölder's inequality with respect to the measure $v_n^{-1}G_\alpha^2(t-s,x-y)dsdy$ and (2.4.16), this is bounded above by

$$\begin{aligned}
&c_p\|z-z'\|^p + c_p'\delta^p\|z-z'\|^p \\
&\quad + c_p\delta^p \sum_{m=1}^{d} \int_0^t ds\, v_n^{-1} 1_{[0,2^{-n}]}(t-s)(t-s)^{-1/\alpha} \sup_{y\in\mathbb{R}} \mathrm{E}\left[|v_{n,m,j}^{z}(s,y) - v_{n,m,j}^{z'}(s,y)|^p\right] \\
&\leq c_p\|z-z'\|^p + c_p'\delta^p\|z-z'\|^p \\
&\quad + c_p\delta^p \sum_{m=1}^{d} \sup_{(s,y)\in[0,T]\times\mathbb{R}} \mathrm{E}\left[|v_{n,m,j}^{z}(s,y) - v_{n,m,j}^{z'}(s,y)|^p\right].
\end{aligned}$$

In the above inequality we can choose δ small enough so that $c_p\delta^p < 1$. Then for $\|z\| \leq \delta$ and

$\|z'\| \le \delta$,

$$\sup_{(s,y)\in[0,T]\times\mathbb{R}} \mathrm{E}\left[\|v_n^z(s,y) - v_n^{z'}(s,y)\|^p\right] \le c_{p,T}\|z-z'\|^p. \tag{2.4.37}$$

Now we give some estimates on the moments of the difference between φ_n^z and v_n^z. Comparing (2.4.10) and (2.4.32), together with (2.4.12), (2.4.14) and (2.4.9) we have

$$\begin{aligned}
&\mathrm{E}\left[\left|\varphi_{n,i,j}^z(t,x) - v_{n,i,j}^z(t,x)\right|^p\right] \\
&\le c_p\delta^p \sum_{m=1}^d \mathrm{E}\left[\left|\int_{t-2^{-n}}^t \int_{\mathbb{R}} v_n^{-1} G_\alpha^2(t-s,x-y)|\varphi_{n,m,j}^z(s,y) - v_{n,m,j}^z(s,y)|dyds\right|^p\right] \\
&\quad + c_{p,T}(v_n^{\frac{p}{2}} + 2^{-np}).
\end{aligned}$$

By Hölder's inequality with respect to the measure $v_n^{-1}G_\alpha^2(t-s,x-y)dsdy$, this is bounded above by

$$\begin{aligned}
&c_p\delta^p \sum_{m=1}^d \int_{t-2^{-n}}^t \int_{\mathbb{R}} v_n^{-1} G_\alpha^2(t-s,x-y)\mathrm{E}\left[|\varphi_{n,m,j}^z(s,y) - v_{n,m,j}^z(s,y)|^p\right] dyds \\
&\quad + c_{p,T}(v_n^{\frac{p}{2}} + 2^{-np}) \\
&\le c_{p,T}(v_n^{\frac{p}{2}} + 2^{-np}) + c_p\delta^p \sum_{m=1}^d \sup_{(s,y)\in[0,T]\times\mathbb{R}} \mathrm{E}\left[|\varphi_{n,m,j}^z(s,y) - v_{n,m,j}^z(s,y)|^p\right].
\end{aligned}$$

Choosing δ small enough, as we did before, we have

$$\sup_{\|z\|\le\delta} \sup_{(s,y)\in[0,T]\times\mathbb{R}} \mathrm{E}\left[\|\varphi_n^z(s,y) - v_n^z(s,y)\|^p\right] \le c_{p,T}(v_n^{\frac{p}{2}} + 2^{-np}). \tag{2.4.38}$$

For convenience, we denote $X_n^z(t_0,x_0) = \varphi_n^z(t_0,x_0) - v_n^z(t_0,x_0)$. Then (2.4.22), (2.4.37) and (2.4.38) indicate that for any $p > d$

$$\mathrm{E}\left[\|X_n^z(t_0,x_0) - X_n^{z'}(t_0,x_0)\|^p\right] \le c_{p,T}\|z-z'\|^p, \tag{2.4.39}$$

$$\sup_{\|z\|\le\delta} \mathrm{E}\left[\|X_n^z(t_0,x_0)\|^p\right] \le c_{p,T}(v_n^{\frac{p}{2}} + 2^{-np}). \tag{2.4.40}$$

Choose $0 < \theta_0 < 1 - d/p$. By (2.4.39) and the Kolmogorov's continuity theorem (see [81, Corollary 1.2]), we have

$$\sup_n \mathrm{E}\left[\sup_{z\neq z'} \frac{\|X_n^z(t_0,x_0) - X_n^{z'}(t_0,x_0)\|^p}{\|z-z'\|^{\theta_0 p}}\right] < \infty, \tag{2.4.41}$$

where the $\sup_n$ denotes the supremum over large enough n for which (2.4.9), (2.4.16) and (2.4.22) are satisfied. For any $\epsilon > 0$, we can choose $\{z_i\}_{i=1}^{k(\epsilon)} \subset \bar{B}(0,\delta)$ ($k(\epsilon)$ is of the order $\delta^d\epsilon^{-d}$)

such that for any $z \in \bar{B}(0,\delta)$, there exists $z_{i(z)} \in \{z_i\}_{i=1}^{k(\epsilon)}$ satisfying $\|z - z_{i(z)}\| \le \epsilon$. And for each z_i, from (2.4.40), we can choose a large number N such that for any $n \ge N$

$$\sup_{1\le i\le k(\epsilon)} \mathrm{E}\left[\|X_n^{z_i}(t_0,x_0)\|^p\right] \le \epsilon / k(\epsilon). \tag{2.4.42}$$

Hence, for $n \ge N$,

$$\begin{aligned}
\mathrm{E}\left[\sup_{\|z\|\le\delta} \|X_n^z(t_0,x_0)\|^p\right] &\le c_p \mathrm{E}\left[\sup_{\|z\|\le\delta} \left(\|X_n^z(t_0,x_0) - X_n^{z_{i(z)}}(t_0,x_0)\|^p + \|X_n^{z_{i(z)}}(t_0,x_0)\|^p\right)\right] \\
&\le c_p \mathrm{E}\left[\sup_{\|z\|\le\delta, z\neq z_{i(z)}} \frac{\|X_n^z(t_0,x_0) - X_n^{z_{i(z)}}(t_0,x_0)\|^p}{\|z - z_{i(z)}\|^{\theta_0 p}} \|z - z_{i(z)}\|^{\theta_0 p}\right] \\
&\quad + c_p \sum_{i=1}^{k(\epsilon)} \mathrm{E}\left[\|X_n^{z_i}(t_0,x_0)\|^p\right] \\
&\le c_p \epsilon^{\theta_0 p} \sup_n \mathrm{E}\left[\sup_{z\neq z'} \frac{\|X_n^z(t_0,x_0) - X_n^{z'}(t_0,x_0)\|^p}{\|z - z'\|^{\theta_0 p}}\right] + c_p \epsilon,
\end{aligned}$$

which implies that for any $p > d$

$$\lim_{n\to\infty} \mathrm{E}\left[\sup_{\|z\|\le\delta} \|X_n^z(t_0,x_0)\|^p\right] = 0. \tag{2.4.43}$$

From (2.4.36), we know that $\{\sup_{\|z\|\le\delta} \|X_n^z(t_0,x_0)\| \le C\} \subset \{\sup_{\|z\|\le\delta} \|\varphi_n^z(t_0,x_0)\| \le 2C\}$. And (2.4.43) implies that

$$\lim_{n\to\infty} \mathrm{P}\left\{\sup_{\|z\|\le\delta} \|X_n^z(t_0,x_0)\| > C\right\} = 0.$$

Therefore, we have proved that

$$\lim_{n\to\infty} \mathrm{P}\left\{\sup_{\|z\|\le\delta} \|\varphi_n^z(t_0,x_0)\| \le 2C\right\} = 1.$$

The verification of (ii) of $\mathbf{H_{t_0,x_0}}(y_0)$ for $\psi_n(t_0,x_0)$ is similar, so we only give the main steps. Recall that

$$\psi_{n,i,j,m}^z(t,x) = \frac{\partial^2}{\partial z_j \partial z_m} \tilde{u}_{n,i}^z(t,x).$$

Then we have that

$$\psi_{n,i,j,m}^z(t,x) = \int_0^t dr_1 \int_{\mathbb{R}} dv_1 \int_0^t dr_2 \int_{\mathbb{R}} dv_2 D_{r_2,v_2}^{(m)} D_{r_1,v_1}^{(j)} (\tilde{u}_{n,i}^z(t,x)) g_{nj}(r_1,v_1) g_{nm}(r_2,v_2),$$

where the second-order Malliavin derivative satisfies the following linear stochastic differential equation:

$$\begin{aligned}
&D^{(m)}_{r_2,v_2}D^{(j)}_{r_1,v_1}(\tilde{u}^z_{n,i}(t,x))\\
&\quad= G_\alpha(t-r_1,x-v_1)D^{(m)}_{r_2,v_2}(\sigma_{ij}(\tilde{u}^z_n(r_1,v_1))) + G_\alpha(t-r_2,x-v_2)D^{(j)}_{r_1,v_1}(\sigma_{im}(\tilde{u}^z_n(r_2,v_2)))\\
&\quad+\int_{r_1\vee r_2}^t\int_{\mathbb{R}} G_\alpha(t-s,x-y)\sum_{l=1}^d D^{(m)}_{r_2,v_2}D^{(j)}_{r_1,v_1}(\sigma_{il}(\tilde{u}^z_n(s,y)))\tilde{W}^l(ds,dy)\\
&\quad+\int_{r_1\vee r_2}^t\int_{\mathbb{R}} G_\alpha(t-s,x-y)D^{(m)}_{r_2,v_2}D^{(j)}_{r_1,v_1}(b_i(\tilde{u}^z_n(s,y)))dsdy.
\end{aligned}$$

Using the chain rule and the stochastic differential equation satisfied by the first Malliavin derivative, we can compute the different terms of $\psi^z_{n,i,j,m}(t,x)$ as we did for $\varphi^z_{n,i,j}(t,x)$, and bound their p-th moments. Finally we estimates the p-th moments of the difference $\psi^z_n(t,x)-\psi^{z'}_n(t,x)$ as we did for $\varphi^z_n(t,x)$ in order to get the desired result.

We have verified the two assumptions of $\mathbf{H_{t_0,x_0}}(y_0)$ for any $y_0\in \mathrm{Supp}(\mathrm{P}_{u(t_0,x_0)})$. Therefore, the conclusion of Theorem 2.1.1(b) follows from Theorem 2.4.1.

2.5 The Gaussian-type upper bound on the two-point density

The aim of this section is to prove Theorem 2.1.1(c). We will follow the general approach in [26, Section 6]; see also [27, Section 5].

2.5.1 Technical lemmas and propositions

In this subsection, we present several technical lemmas and propositions, which will be used for the analysis of the Malliavin matrix.

Lemma 2.5.1 ([17, Proposition 4.4])**.** *For any $s,t\in[0,T], s\le t$, and $x,y\in\mathbb{R}$, there exists a constant $C_T>0$ such that*

$$\int_0^T\int_{\mathbb{R}}(g_\alpha(r,v))^2drdv\le C_T(|t-s|^{\frac{\alpha-1}{\alpha}}+|x-y|^{\alpha-1}),$$

where

$$g_\alpha(r,v):=g^\alpha_{t,x,s,y}(r,v)=1_{\{r\le t\}}G_\alpha(t-r,x-v)-1_{\{r\le s\}}G_\alpha(s-r,y-v).$$

The following result gives an estimate on the modulus of L^p-continuity of the derivative of the increment, analogous to [26, Proposition 6.2], which is comparable to (2.1.6).

Proposition 2.5.2. *For any $p\ge 2, m\ge 1$, there exists a constant $C_{p,T}$ such that for all $s,t\in[0,T], s\le t, x,y\in\mathbb{R}$,*

$$E\left[\left\|D^m(u_i(t,x)-u_i(s,y))\right\|^p_{\mathscr{H}^{\otimes m}}\right]\le C_{p,T}(|t-s|^{\frac{\alpha-1}{\alpha}}+|x-y|^{\alpha-1})^{p/2},\quad i=1,\dots,d. \tag{2.5.1}$$

Proof. The proof is slightly different from that of [26, Proposition 6.2] since the estimate for I_3 in [26, Proposition 6.2] requires the Cauchy-Schwartz inequality, which is not applicable in our situation because the Lebesgue measure of $\mathbb{R}$ is infinite.

Assume $m = 1$. Using (2.3.1), we see that, for any $p \geq 2$,

$$\mathrm{E}\left[\|D(u_i(t,x) - u_i(s,y))\|_{\mathscr{H}}^p\right] \leq c\left(\mathrm{E}\left[|I_1|^{p/2}\right] + \mathrm{E}\left[|I_2|^{p/2}\right] + \mathrm{E}\left[|I_3|^{p/2}\right] + \mathrm{E}\left[|I_4|^{p/2}\right]\right), \tag{2.5.2}$$

where

$$\begin{aligned}
I_1 &= \sum_{k=1}^{d} \int_0^T dr \int_{\mathbb{R}} dv \left(g_\alpha(r,v)\sigma_{ik}(u(r,v))\right)^2, \\
I_2 &= \sum_{j,k=1}^{d} \int_0^T dr \int_{\mathbb{R}} dv \left(\int_0^T \int_{\mathbb{R}} g_\alpha(\theta,\eta) D_{r,v}^{(k)}(\sigma_{ij}(u(\theta,\eta))) W^j(d\theta, d\eta)\right)^2, \\
I_3 &= \sum_{k=1}^{d} \int_0^T dr \int_{\mathbb{R}} dv \left(\int_0^{t-s} \int_{\mathbb{R}} G_\alpha(t-\theta, x-\eta) D_{r,v}^{(k)}(b_i(u(\theta,\eta))) d\theta d\eta\right)^2, \\
I_4 &= \sum_{k=1}^{d} \int_0^T dr \int_{\mathbb{R}} dv \Big(\int_0^s \int_{\mathbb{R}} G_\alpha(s-\theta, y-\eta) \\
&\qquad\qquad \times D_{r,v}^{(k)}(b_i(u(t-s+\theta, x-y+\eta)) - b_i(u(\theta,\eta))) d\theta d\eta\Big)^2.
\end{aligned}$$

By hypothesis **P1'** and Lemma 2.5.1,

$$\mathrm{E}\left[|I_1|^{p/2}\right] \leq C_{p,T}(|t-s|^{\frac{\alpha-1}{\alpha}} + |x-y|^{\alpha-1})^{p/2}. \tag{2.5.3}$$

Using Burkholder's inequality for Hilbert-space-valued martingales (Lemma A.3.1) and hypothesis **P1'**, we obtain

$$\mathrm{E}\left[|I_2|^{p/2}\right] \leq c \sum_{k,l=1}^{d} \mathrm{E}\left[\left|\int_0^T d\theta \int_{\mathbb{R}} d\eta\, (g_\alpha(\theta,\eta))^2 \int_0^T dr \int_{\mathbb{R}} dv\, \Theta_{k,l}^2\right|^{p/2}\right],$$

where $\Theta_{k,l} := D_{r,v}^{(k)}(u_l(\theta,\eta))$. By Hölder's inequality with respect to the measure $(g_\alpha(\theta,\eta))^2 d\theta d\eta$, we see that this is bounded above by

$$\begin{aligned}
&c \sum_{k,l=1}^{d} \int_0^T d\theta \int_{\mathbb{R}} d\eta\, (g_\alpha(\theta,\eta))^2 \sup_{(\theta,\eta)\in[0,T]\times\mathbb{R}} \mathrm{E}\left[\left(\int_0^T dr \int_{\mathbb{R}} dv\, \Theta_{k,l}^2\right)^{p/2}\right] \\
&\quad \times \left(\int_0^T \int_{\mathbb{R}} (g_\alpha(\theta,\eta))^2 d\theta d\eta\right)^{\frac{p}{2}-1} \\
&\leq C_{p,T}(|t-s|^{\frac{\alpha-1}{\alpha}} + |x-y|^{\alpha-1})^{p/2},
\end{aligned} \tag{2.5.4}$$

where we use (2.3.3) and Lemma 2.5.1. To estimate I_3, we use Hölder's inequality with respect to the measure $G_\alpha(t-\theta, x-\eta)d\theta d\eta$ twice to get that

$$
\begin{aligned}
\mathrm{E}\left[|I_3|^{p/2}\right] &\le C_{p,T}\sum_{k,l=1}^{d}(t-s)^{p/2}\mathrm{E}\Bigg[\Big(\int_0^{t-s}d\theta\int_{\mathbb{R}}d\eta\, G_\alpha(t-\theta,x-\eta) \\
&\qquad\qquad \times\int_0^T dr\int_{\mathbb{R}}dv\,\Theta_{k,l}^2\Big)^{p/2}\Bigg] \\
&\le C_{p,T}\sum_{k,l=1}^{d}(t-s)^{p/2}\left(\int_0^{t-s}d\theta\int_{\mathbb{R}}d\eta\, G_\alpha(t-\theta,x-\eta)\right)^{\frac{p}{2}-1} \\
&\quad \times\int_0^{t-s}d\theta\int_{\mathbb{R}}d\eta\, G_\alpha(t-\theta,x-\eta)\sup_{(\theta,\eta)\in[0,T]\times\mathbb{R}}\mathrm{E}\left[\left[\int_0^T dr\int_{\mathbb{R}}dv\,\Theta_{k,l}^2\right]^{p/2}\right] \\
&\le C_{p,T}(t-s)^p, \qquad (2.5.5)
\end{aligned}
$$

where in the last inequality we use (2.3.3). Using Hölder's inequality with respect to the measure $G_\alpha(t-\theta,x-\eta)d\theta d\eta$,

$$
\begin{aligned}
I_4 \le c\sum_{k=1}^{d}\int_0^T dr\int_{\mathbb{R}}dv\int_0^s d\theta\int_{\mathbb{R}}d\eta\, G_\alpha(s-\theta,y-\eta) \\
\times\left(D_{r,v}^{(k)}\left(b_i(u(t-s+\theta,x-y+\eta))-b_i(u(\theta,\eta))\right)\right)^2
\end{aligned}
$$

We apply the chain rule to compute $D_{r,v}^{(k)}b_i(u(t-s+\theta,x-y+\eta))-D_{r,v}^{(k)}b_i(u(\theta,\eta))$, subtract and add the term $b_i'(u(t-s+\theta,x-y+\eta))D_{r,v}^{(k)}u(\theta,\eta)$. Then by hypothesis **P1'**, this is bounded above by

$$
\begin{aligned}
& c\sum_{k,l=1}^{d}\int_0^T dr\int_{\mathbb{R}}dv\int_0^s d\theta\int_{\mathbb{R}}d\eta\, G_\alpha(s-\theta,y-\eta) \\
&\qquad\qquad \times\left(D_{r,v}^{(k)}(u_l(t-s+\theta,x-y+\eta)-u_l(\theta,\eta))\right)^2 \\
&\quad + c\sum_{k,l=1}^{d}\int_0^T dr\int_{\mathbb{R}}dv\int_0^s d\theta\int_{\mathbb{R}}d\eta\, G_\alpha(s-\theta,y-\eta) \\
&\qquad\qquad \times\left(u_l(t-s+\theta,x-y+\eta)-u_l(\theta,\eta)\right)^2\Theta_{k,l}^2 \\
&:= I_{41}+I_{42}.
\end{aligned}
$$

Using Hölder's inequality with respect to the measure $G_\alpha(t-\theta,x-\eta)d\theta d\eta$ and the Cauchy-

Schwartz inequality, we have

$$
\begin{aligned}
\mathrm{E}\left[|I_{42}|^{p/2}\right] &\le c \sum_{k,l=1}^{d} \left(\int_0^s d\theta \int_{\mathbb{R}} d\eta\, G_\alpha(s-\theta, y-\eta) \right)^{\frac{p}{2}-1} \\
&\quad \times \int_0^s d\theta \int_{\mathbb{R}} d\eta\, G_\alpha(s-\theta, y-\eta) \\
&\qquad \times \mathrm{E}\left[\left| u_l(t-s+\theta, x-y+\eta) - u_l(\theta,\eta) \right|^p \left(\int_0^T dr \int_{\mathbb{R}} dv\, \Theta_{k,l}^2 \right)^{p/2} \right] \\
&\le c s^{p/2} \sum_{k,l=1}^{d} \sup_{(\theta,\eta)\in[0,T]\times\mathbb{R}} \mathrm{E}\left[\left(\int_0^T dr \int_{\mathbb{R}} dv\, \Theta_{k,l}^2 \right)^p \right]^{1/2} \\
&\quad \times \sup_{(\theta,\eta)\in[0,T]\times\mathbb{R}} \mathrm{E}\left[\left| u_l(t-s+\theta, x-y+\eta) - u_l(\theta,\eta) \right|^{2p} \right]^{1/2} \\
&\le C_{p,T} s^{p/2} (|t-s|^{\frac{\alpha-1}{\alpha}} + |x-y|^{\alpha-1})^{p/2} \qquad (2.5.6)
\end{aligned}
$$

where we use (2.3.3) and (2.1.6).

Denote

$$
\varphi(h,z,\theta) := \sup_{\eta\in\mathbb{R}} \sum_{k,l=1}^{d} \mathrm{E}\Big[\Big(\int_0^T \int_{\mathbb{R}} \Big(D_{r,v}^{(k)}(u_l(h+\theta, z+\eta) - u_l(\theta,\eta)) \Big)^2 dr dv \Big)^{\frac{p}{2}} \Big].
$$

By Hölder's inequality,

$$
\begin{aligned}
\mathrm{E}\left[|I_{41}|^{p/2}\right] &\le c \sum_{k,l=1}^{d} \int_0^s \left(\int_{\mathbb{R}} G_\alpha(s-\theta, y-\eta) d\theta d\eta \right)^{\frac{p}{2}-1} \\
&\quad \times \int_0^s d\theta \int_{\mathbb{R}} d\eta\, G_\alpha(s-\theta, y-\eta) \\
&\qquad \times \mathrm{E}\Big[\Big(\int_0^T \int_{\mathbb{R}} (D_{r,v}^{(k)}(u_l(t-s+\theta, x-y+\eta) - u_l(\theta,\eta)))^2 dr dv \Big)^{\frac{p}{2}} \Big] \\
&\le C_{p,T} \int_0^s \varphi(t-s, x-y, \theta) d\theta. \qquad (2.5.7)
\end{aligned}
$$

Denote $h = t-s$ and $z = x-y$. From (2.5.2)–(2.5.7), we conclude that for all $h \ge 0$, $z \in \mathbb{R}$, $s \in [0,T]$ $y \in \mathbb{R}$ and $1 \le i \le d$,

$$
\mathrm{E}\left[\left\| D(u_i(h+s, z+y) - u_i(s,y)) \right\|_{\mathscr{H}}^p \right] \le C_{p,T}(|h|^{\frac{\alpha-1}{\alpha}} + |z|^{\alpha-1})^{p/2} + C_{p,T} \int_0^s \varphi(h,z,\theta) d\theta.
$$

Taking the supremum over $y \in \mathbb{R}$ on the left-hand side of the above inequality, we obtain that for all $h \ge 0$, $z \in \mathbb{R}$ and $s \in [0,T]$,

$$
\varphi(h,z,s) \le C_{p,T}(|h|^{\frac{\alpha-1}{\alpha}} + |z|^{\alpha-1})^{p/2} + C_{p,T} \int_0^s \varphi(h,z,\theta) d\theta.
$$

By Gronwall's lemma (see [76, p.543]), we obtain that

$$\sup_{s\in[0,T]} \varphi(h,z,s) \le C_{p,T}(|h|^{\frac{\alpha-1}{\alpha}} + |z|^{\alpha-1})^{p/2},$$

which implies (2.5.1) with $m=1$.

The case $m>1$ follows along the same lines by using (2.3.3) and the stochastic partial differential equations satisfied by the iterated derivatives (see for example [26, Proposition 4.1]). $\square$

The following lemma is another version of [26, Lemma 6.11].

Lemma 2.5.3. *Assume* ***P1'****. Fix* $T>0, q\ge 1$. *There exists a constant* $c=c(q,T)\in]0,\infty[$ *such that for every* $0<2\epsilon \le s\le t\le T$ *and* $x\in\mathbb{R}$,

$$E\left[\left(\sum_{k=1}^{d}\int_{s-\epsilon}^{s} dr \int_{\mathbb{R}} dv \sum_{i=1}^{d} a_i^2(k,r,v,t,x)\right)^q\right] \le c(t-s+\epsilon)^{(\alpha-1)q/\alpha}\epsilon^{(\alpha-1)q/\alpha}.$$

Proof. The proof follows the same lines as [26, Lemma 6.11]. Define

$$A := \sum_{k=1}^{d}\int_{s-\epsilon}^{s} dr \int_{\mathbb{R}} dv \sum_{i=1}^{d} a_i^2(k,r,v,t,x).$$

From (2.3.2), we write

$$\mathrm{E}\left[|A|^q\right] \le c\left(\mathrm{E}\left[|A_1|^q\right] + \mathrm{E}\left[|A_2|^q\right]\right),$$

where

$$A_1 := \sum_{i,j,k=1}^{d}\int_{s-\epsilon}^{s} dr \int_{\mathbb{R}} dv \left|\int_r^t \int_{\mathbb{R}} G_\alpha(t-\theta,x-\eta) D_{r,v}^{(k)}(\sigma_{ij}(u(\theta,\eta)))W^j(d\theta,d\eta)\right|^2,$$

$$A_2 := \sum_{i,k=1}^{d}\int_{s-\epsilon}^{s} dr \int_{\mathbb{R}} dv \left|\int_r^t \int_{\mathbb{R}} G_\alpha(t-\theta,x-\eta) D_{r,v}^{(k)}(b_i(u(\theta,\eta)))d\theta d\eta\right|^2.$$

We bound the q-th moment of A_1 and A_2 separately.

As regards A_1, thanks to hypothesis **P1'**, we apply Burkholder's inequality for Hilbert-space-valued martingales (Lemma A.3.1) to find that

$$\mathrm{E}\left[|A_1|^q\right] \le c\sum_{k,l=1}^{d} \mathrm{E}\left[\left|\int_{s-\epsilon}^{t} d\theta \int_{\mathbb{R}} d\eta \int_{s-\epsilon}^{s} dr \int_{\mathbb{R}} dv\, \Theta_{k,l}^2\right|^q\right], \tag{2.5.8}$$

where

$$\Theta_{k,l} := \mathbf{1}_{\{\theta>r\}} G_\alpha(t-\theta,x-\eta) D_{r,v}^{(k)}(u_l(\theta,\eta)).$$

We apply Hölder's inequality with respect to the measure $G_\alpha^2(t-\theta, x-\eta)d\theta d\eta$ to find that

$$\begin{aligned}\mathrm{E}\left[|A_1|^q\right] \le c \sum_{k,l=1}^{d} \left(\int_{s-\epsilon}^{t} d\theta \int_{\mathbb{R}} d\eta\, G_\alpha^2(t-\theta, x-\eta)\right)^{q-1} \\ \times \int_{s-\epsilon}^{t} d\theta \int_{\mathbb{R}} d\eta\, G_\alpha^2(t-\theta, x-\eta) \mathrm{E}\left[\left|\int_{s-\epsilon}^{s\wedge\theta} dr \int_{\mathbb{R}} dv \left(D_{r,v}^{(k)}(u_l(\theta,\eta))\right)^2\right|^q\right].\end{aligned}$$

Since $2\epsilon \le s$, we have for $\theta \in [s-\epsilon, t]$, $s-\epsilon \ge s\wedge\theta-\epsilon \ge 0$. Hence, by Lemma A.3.2,

$$\begin{aligned}\mathrm{E}\left[\left|\int_{s-\epsilon}^{s\wedge\theta} dr \int_{\mathbb{R}} dv \left(D_{r,v}^{(k)}(u_l(\theta,\eta))\right)^2\right|^q\right] &\le \mathrm{E}\left[\left|\int_{s\wedge\theta-\epsilon}^{s\wedge\theta} dr \int_{\mathbb{R}} dv \left(D_{r,v}^{(k)}(u_l(\theta,\eta))\right)^2\right|^q\right] \\ &\le c\epsilon^{\frac{\alpha-1}{\alpha}q}, \end{aligned} \tag{2.5.9}$$

where $c \in]0,\infty[$ does not depend on $(\theta, \eta, s, t, \epsilon, x)$. Therefore,

$$\begin{aligned}\mathrm{E}\left[|A_1|^q\right] &\le c\left(\int_{s-\epsilon}^{t} d\theta \int_{\mathbb{R}} d\eta G_\alpha^2(t-\theta, x-\eta)\right)^q \epsilon^{\frac{\alpha-1}{\alpha}q} \\ &= c(t-s+\epsilon)^{\frac{\alpha-1}{\alpha}q}\epsilon^{\frac{\alpha-1}{\alpha}q}, \end{aligned} \tag{2.5.10}$$

where the calculation in the equality is due to (A.4).

Next we derive a similar bound for A_2. By the Cauchy-Schwartz inequality with respect to the measure $G_\alpha(t-\theta, x-\eta)d\theta d\eta$,

$$\begin{aligned}A_2 &\le \sum_{i,k=1}^{d} \int_{s-\epsilon}^{s} dr \int_{\mathbb{R}} dv\, (t-r) \int_r^t \int_{\mathbb{R}} G_\alpha(t-\theta, x-\eta) \left(D_{r,v}^{(k)}(b_i(u(\theta,\eta)))\right)^2 d\theta d\eta \\ &\le \sum_{i,k=1}^{d} (t-s+\epsilon) \int_{s-\epsilon}^{s} dr \int_{\mathbb{R}} dv \int_r^t \int_{\mathbb{R}} G_\alpha(t-\theta, x-\eta) \left(D_{r,v}^{(k)}(b_i(u(\theta,\eta)))\right)^2 d\theta d\eta.\end{aligned}$$

By hypothesis **P1'** and Fubini's theorem,

$$\begin{aligned}\mathrm{E}\left[|A_2|^q\right] &\le c(t-s+\epsilon)^q \sum_{k,l=1}^{d} \mathrm{E}\Bigg[\Bigg|\int_{s-\epsilon}^{s} dr \int_{\mathbb{R}} dv \int_r^t d\theta \int_{\mathbb{R}} d\eta\, G_\alpha(t-\theta, x-\eta) \\ &\qquad \times \left(D_{r,v}^{(k)}(u_l(\theta,\eta))\right)^2\Bigg|^q\Bigg] \\ &= c(t-s+\epsilon)^q \sum_{k,l=1}^{d} \mathrm{E}\Bigg[\Bigg|\int_{s-\epsilon}^{t} d\theta \int_{\mathbb{R}} d\eta\, G_\alpha(t-\theta, x-\eta) \\ &\qquad \times \int_{s-\epsilon}^{s\wedge\theta} dr \int_{\mathbb{R}} dv \left(D_{r,v}^{(k)}(u_l(\theta,\eta))\right)^2\Bigg|^q\Bigg]. \end{aligned} \tag{2.5.11}$$

We apply Hölder's inequality with respect to the measure $G_\alpha(t-\theta, x-\eta)d\theta d\eta$ to find that

$$\begin{aligned}
\mathrm{E}\left[|A_2|^q\right] &\leq c(t-s+\epsilon)^q \sum_{k,l=1}^{d} \left|\int_{s-\epsilon}^{s} d\theta \int_{\mathbb{R}} d\eta\, G_\alpha(t-\theta, x-\eta)\right|^{q-1} \\
&\quad \times \int_{s-\epsilon}^{s} d\theta \int_{\mathbb{R}} d\eta\, G_\alpha(t-\theta, x-\eta) \mathrm{E}\left[\left|\int_{s-\epsilon}^{s\wedge\theta} dr \int_{\mathbb{R}} dv \left(D_{r,v}^{(k)}(u_l(\theta,\eta))\right)^2\right|^q\right] \\
&\leq c(t-s+\epsilon)^q \left|\int_{s-\epsilon}^{s} d\theta \int_{\mathbb{R}} d\eta\, G_\alpha(t-\theta, x-\eta)\right|^q \epsilon^{\frac{\alpha-1}{\alpha}q} \\
&= c(t-s+\epsilon)^q \epsilon^q \epsilon^{\frac{\alpha-1}{\alpha}q}, \qquad (2.5.12)
\end{aligned}$$

where in the second inequality we use (2.5.9). Therefore (2.5.10) and (2.5.12) imply the result. □

The following lemma is an improvement of Lemma 2.5.3 by using Lemma A.3.3. As we mentioned in Section 1.2, this is a key element in our improvement of the lower bound in (1.1.3).

Lemma 2.5.4. *Assume **P1'**. Fix $T > 0, c_0 > 1$ and $0 < \gamma_0 < 1$. For all $q \geq 1$, there exists a constant $c = c(c_0, q, T) \in]0, \infty[$ such that for every $0 < 2\epsilon \leq s \leq t \leq T$ with $t - s > c_0\epsilon^{\gamma_0}$ and $x \in \mathbb{R}$,*

$$E\left[\left(\sum_{k=1}^{d} \int_{s-\epsilon}^{s} dr \int_{\mathbb{R}} dv \sum_{i=1}^{d} a_i^2(k,r,v,t,x)\right)^q\right] \leq c\epsilon^{\min((1+\gamma_0)\frac{\alpha-1}{\alpha}, 1-\gamma_0+\gamma_0\frac{\alpha-1}{\alpha})q}.$$

Proof. We use again the notations from the proof of Lemma 2.5.3. From (2.5.8) and (2.5.11), we have

$$\begin{aligned}
\mathrm{E}\left[|A_1|^q\right] &\leq c \sum_{k,l=1}^{d} \mathrm{E}\left[\left|\int_{s-\epsilon}^{t} d\theta \int_{\mathbb{R}} d\eta\, G_\alpha^2(t-\theta, x-\eta) \int_{s-\epsilon}^{s\wedge\theta} dr \int_{\mathbb{R}} dv \left(D_{r,v}^{(k)}(u_l(\theta,\eta))\right)^2\right|^q\right] \\
&\leq A_{11} + A_{12} + A_{13},
\end{aligned}$$

with

$$\begin{aligned}
A_{11} &:= c \sum_{k,l=1}^{d} \mathrm{E}\left[\left|\int_{s-\epsilon}^{s} d\theta \int_{\mathbb{R}} d\eta\, G_\alpha^2(t-\theta, x-\eta) \int_{s-\epsilon}^{s\wedge\theta} dr \int_{\mathbb{R}} dv \left(D_{r,v}^{(k)}(u_l(\theta,\eta))\right)^2\right|^q\right], \\
A_{12} &:= c \sum_{k,l=1}^{d} \mathrm{E}\left[\left|\int_{s}^{s+c_0\epsilon^{\gamma_0}} d\theta \int_{\mathbb{R}} d\eta\, G_\alpha^2(t-\theta, x-\eta) \int_{s-\epsilon}^{s\wedge\theta} dr \int_{\mathbb{R}} dv \left(D_{r,v}^{(k)}(u_l(\theta,\eta))\right)^2\right|^q\right], \\
A_{13} &:= c \sum_{k,l=1}^{d} \mathrm{E}\left[\left|\int_{s+c_0\epsilon^{\gamma_0}}^{t} d\theta \int_{\mathbb{R}} d\eta\, G_\alpha^2(t-\theta, x-\eta) \int_{s-\epsilon}^{s} dr \int_{\mathbb{R}} dv \left(D_{r,v}^{(k)}(u_l(\theta,\eta))\right)^2\right|^q\right],
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{E}\left[|A_2|^q\right] &\leq c \sum_{k,l=1}^{d} \mathrm{E}\left[\left|\int_{s-\epsilon}^{t} d\theta \int_{\mathbb{R}} d\eta\, G_\alpha(t-\theta, x-\eta) \int_{s-\epsilon}^{s\wedge\theta} dr \int_{\mathbb{R}} dv \left(D_{r,v}^{(k)}(u_l(\theta,\eta))\right)^2 \right|^q\right] \\
&\leq A_{21} + A_{22} + A_{23},
\end{aligned}$$

with

$$\begin{aligned}
A_{21} &:= c \sum_{k,l=1}^{d} \mathrm{E}\left[\left|\int_{s-\epsilon}^{s} d\theta \int_{\mathbb{R}} d\eta\, G_\alpha(t-\theta, x-\eta) \int_{s-\epsilon}^{s\wedge\theta} dr \int_{\mathbb{R}} dv \left(D_{r,v}^{(k)}(u_l(\theta,\eta))\right)^2 \right|^q\right], \\
A_{22} &:= c \sum_{k,l=1}^{d} \mathrm{E}\left[\left|\int_{s}^{s+c_0\epsilon^{\gamma_0}} d\theta \int_{\mathbb{R}} d\eta\, G_\alpha(t-\theta, x-\eta) \int_{s-\epsilon}^{s\wedge\theta} dr \int_{\mathbb{R}} dv \left(D_{r,v}^{(k)}(u_l(\theta,\eta))\right)^2 \right|^q\right], \\
A_{23} &:= c \sum_{k,l=1}^{d} \mathrm{E}\left[\left|\int_{s+c_0\epsilon^{\gamma_0}}^{t} d\theta \int_{\mathbb{R}} d\eta\, G_\alpha(t-\theta, x-\eta) \int_{s-\epsilon}^{s} dr \int_{\mathbb{R}} dv \left(D_{r,v}^{(k)}(u_l(\theta,\eta))\right)^2 \right|^q\right].
\end{aligned}$$

We first bound $\mathrm{E}\left[|A_1|^q\right]$. We apply Hölder's inequality with respect to the measure $G_\alpha^2(t-\theta, x-\eta)d\theta d\eta$ to find that

$$\begin{aligned}
\mathrm{E}\left[|A_{11}|^q\right] \leq c \sum_{k,l=1}^{d} &\left(\int_{s-\epsilon}^{s} d\theta \int_{\mathbb{R}} d\eta\, G_\alpha^2(t-\theta, x-\eta)\right)^{q-1} \\
&\times \int_{s-\epsilon}^{s} d\theta \int_{\mathbb{R}} d\eta\, G_\alpha^2(t-\theta, x-\eta) \mathrm{E}\left[\left|\int_{s-\epsilon}^{\theta} dr \int_{\mathbb{R}} dv \left(D_{r,v}^{(k)}(u_l(\theta,\eta))\right)^2\right|^q\right].
\end{aligned}$$

For $\theta \in [s-\epsilon, s]$, we have $s-\epsilon \geq \theta-\epsilon \geq 0$. Hence by Lemma A.3.2,

$$\begin{aligned}
\mathrm{E}\left[\left|\int_{s-\epsilon}^{\theta} dr \int_{\mathbb{R}} dv \left(D_{r,v}^{(k)}(u_l(\theta,\eta))\right)^2\right|^q\right] &\leq \mathrm{E}\left[\left|\int_{\theta-\epsilon}^{\theta} dr \int_{\mathbb{R}} dv \left(D_{r,v}^{(k)}(u_l(\theta,\eta))\right)^2\right|^q\right] \\
&\leq c\epsilon^{\frac{\alpha-1}{\alpha}q}, \qquad (2.5.13)
\end{aligned}$$

where $c \in]0,\infty[$ does not depend on $(\theta, \eta, s, t, \epsilon, x)$. Therefore, by (A.4),

$$\begin{aligned}
\mathrm{E}\left[|A_{11}|^q\right] &\leq c\epsilon^{\frac{\alpha-1}{\alpha}q} \left(\int_{s-\epsilon}^{s} d\theta \int_{\mathbb{R}} d\eta\, G_\alpha^2(t-\theta, x-\eta)\right)^q \\
&= c\epsilon^{\frac{\alpha-1}{\alpha}q} \left((t-s+\epsilon)^{(\alpha-1)/\alpha} - (t-s)^{(\alpha-1)/\alpha}\right)^q \\
&\leq c\epsilon^{\frac{\alpha-1}{\alpha}q} \epsilon^{(1-\gamma_0+\gamma_0\frac{\alpha-1}{\alpha})q}, \qquad (2.5.14)
\end{aligned}$$

where, in the last inequality, we perform the same calculation as in (A.18) under the assumption $t-s > c_0\epsilon^{\gamma_0}$. Again, we apply Hölder's inequality with respect to the measure

$G_\alpha^2(t-\theta, x-\eta)d\theta d\eta$ to find that

$$\begin{aligned}
\mathrm{E}\left[|A_{12}|^q\right] \le c \sum_{k,l=1}^{d} &\left(\int_s^{s+c_0\epsilon^{\gamma_0}} d\theta \int_{\mathbb{R}} d\eta\, G_\alpha^2(t-\theta, x-\eta)\right)^{q-1} \\
&\times \int_s^{s+c_0\epsilon^{\gamma_0}} d\theta \int_{\mathbb{R}} d\eta\, G_\alpha^2(t-\theta, x-\eta) \mathrm{E}\left[\left|\int_{s-\epsilon}^{s} dr \int_{\mathbb{R}} dv \left(D_{r,v}^{(k)}(u_l(\theta,\eta))\right)^2\right|^q\right].
\end{aligned}$$

Lemma A.3.2 implies that

$$\sum_{k,l=1}^{d} \mathrm{E}\left[\left|\int_{s-\epsilon}^{s} dr \int_{\mathbb{R}} dv \left(D_{r,v}^{(k)}(u_l(\theta,\eta))\right)^2\right|^q\right] \le c\epsilon^{\frac{\alpha-1}{\alpha}q},$$

where $c \in]0,\infty[$ does not depend on $(\theta,\eta,s,t,\epsilon,x)$. Consequently,

$$\begin{aligned}
\mathrm{E}\left[|A_{12}|^q\right] &\le c\left(\int_s^{s+c_0\epsilon^{\gamma_0}} d\theta \int_{\mathbb{R}} d\eta\, G_\alpha^2(t-\theta, x-\eta)\right)^q \epsilon^{\frac{\alpha-1}{\alpha}q} \\
&= c\left((t-s)^{\frac{\alpha-1}{\alpha}} - (t-s-c_0\epsilon^{\gamma_0})^{\frac{\alpha-1}{\alpha}}\right)^q \epsilon^{\frac{\alpha-1}{\alpha}q} \\
&\le c\left((c_0\epsilon^{\gamma_0})^{\frac{\alpha-1}{\alpha}} - (c_0\epsilon^{\gamma_0} - c_0\epsilon^{\gamma_0})^{\frac{\alpha-1}{\alpha}}\right)^q \epsilon^{\frac{\alpha-1}{\alpha}q} \\
&= c(c_0\epsilon^{\gamma_0})^{\frac{\alpha-1}{\alpha}q} \epsilon^{\frac{\alpha-1}{\alpha}q} = c'\epsilon^{(1+\gamma_0)\frac{\alpha-1}{\alpha}q},
\end{aligned} \tag{2.5.15}$$

where the second inequality is because the function $x \mapsto x^{\frac{\alpha-1}{\alpha}} - (x - c_0\epsilon^{\gamma_0})^{\frac{\alpha-1}{\alpha}}$ is decreasing on $[c_0\epsilon^{\gamma_0}, \infty[$.

For A_{13}, we have, by Hölder's inequality with respect to the measure $G_\alpha^2(t-\theta, x-\eta)d\theta d\eta$,

$$\begin{aligned}
\mathrm{E}\left[|A_{13}|^q\right] \le c \sum_{k,l=1}^{d} &\left(\int_{s+c_0\epsilon^{\gamma_0}}^{t} d\theta \int_{\mathbb{R}} d\eta\, G_\alpha^2(t-\theta, x-\eta)\right)^{q-1} \\
&\times \int_{s+c_0\epsilon^{\gamma_0}}^{t} d\theta \int_{\mathbb{R}} d\eta\, G_\alpha^2(t-\theta, x-\eta) \mathrm{E}\left[\left|\int_{s-\epsilon}^{s} dr \int_{\mathbb{R}} dv \left(D_{r,v}^{(k)}(u_l(\theta,\eta))\right)^2\right|^q\right].
\end{aligned}$$

Lemma A.3.3 implies that for any $\theta \in]s + c_0\epsilon^{\gamma_0}, t[$,

$$\sum_{k,l=1}^{d} \mathrm{E}\left[\left|\int_{s-\epsilon}^{s} dr \int_{\mathbb{R}} dv \left(D_{r,v}^{(k)}(u_l(\theta,\eta))\right)^2\right|^q\right] \le c\epsilon^{(1-\gamma_0+\gamma_0\frac{\alpha-1}{\alpha})q},$$

where $c \in]0,\infty[$ does not depend on $(\theta,\eta,s,t,\epsilon,x)$. Thus, by (A.4),

$$\begin{aligned}
\mathrm{E}\left[|A_{13}|^q\right] &\le c\left(\int_{s+c_0\epsilon^{\gamma_0}}^{t} d\theta \int_{\mathbb{R}} d\eta\, G_\alpha^2(t-\theta, x-\eta)\right)^q \epsilon^{(1-\gamma_0+\gamma_0\frac{\alpha-1}{\alpha})q} \\
&= c(t-s-c_0\epsilon^{\gamma_0})^{\frac{\alpha-1}{\alpha}q} \epsilon^{(1-\gamma_0+\gamma_0\frac{\alpha-1}{\alpha})q} \\
&\le c\epsilon^{(1-\gamma_0+\gamma_0\frac{\alpha-1}{\alpha})q}.
\end{aligned} \tag{2.5.16}$$

We proceed to derive a similar bound for $\mathrm{E}\left[|A_2|^q\right]$. We apply Hölder's inequality with respect to the measure $G_\alpha(t-\theta,x-\eta)d\theta d\eta$ to find that

$$
\begin{aligned}
\mathrm{E}\left[|A_{21}|^q\right] &\le c\sum_{k,l=1}^{d}\left|\int_{s-\epsilon}^{s}d\theta\int_{\mathbb{R}}d\eta\, G_\alpha(t-\theta,x-\eta)\right|^{q-1}\\
&\quad\times\int_{s-\epsilon}^{s}d\theta\int_{\mathbb{R}}d\eta\, G_\alpha(t-\theta,x-\eta)\mathrm{E}\left[\left|\int_{s-\epsilon}^{\theta}dr\int_{\mathbb{R}}dv\left(D_{r,v}^{(k)}(u_l(\theta,\eta))\right)^2\right|^q\right]\\
&\le c\left|\int_{s-\epsilon}^{s}d\theta\int_{\mathbb{R}}d\eta\, G_\alpha(t-\theta,x-\eta)\right|^q\epsilon^{\frac{\alpha-1}{\alpha}q}\\
&= c\epsilon^q\epsilon^{\frac{\alpha-1}{\alpha}q}=c\epsilon^{(\frac{\alpha-1}{\alpha}+1)q},
\end{aligned}
\tag{2.5.17}
$$

where in the second inequality we use (2.5.13). Similarly, we apply Hölder's inequality with respect to the measure $G_\alpha(t-\theta,x-\eta)d\theta d\eta$ to find that

$$
\begin{aligned}
\mathrm{E}\left[|A_{22}|^q\right] &\le c\sum_{k,l=1}^{d}\left|\int_{s}^{s+c_0\epsilon^{\gamma_0}}d\theta\int_{\mathbb{R}}d\eta\, G_\alpha(t-\theta,x-\eta)\right|^{q-1}\\
&\quad\times\int_{s}^{s+c_0\epsilon^{\gamma_0}}d\theta\int_{\mathbb{R}}d\eta\, G_\alpha(t-\theta,x-\eta)\mathrm{E}\left[\left|\int_{s-\epsilon}^{s}dr\int_{\mathbb{R}}dv\left(D_{r,v}^{(k)}(u_l(\theta,\eta))\right)^2\right|^q\right]\\
&\le c\left|\int_{s}^{s+c_0\epsilon^{\gamma_0}}d\theta\int_{\mathbb{R}}d\eta\, G_\alpha(t-\theta,x-\eta)\right|^q\epsilon^{\frac{\alpha-1}{\alpha}q}\\
&= c(c_0\epsilon^{\gamma_0})^q\epsilon^{\frac{\alpha-1}{\alpha}q}=c\epsilon^{(\frac{\alpha-1}{\alpha}+\gamma_0)q},
\end{aligned}
\tag{2.5.18}
$$

where in the second inequality we use Lemma A.3.2. For the last term, we use Hölder's inequality with respect to the measure $G_\alpha(t-\theta,x-\eta)d\theta d\eta$ to see that

$$
\begin{aligned}
\mathrm{E}\left[|A_{23}|^q\right] &\le c\sum_{k,l=1}^{d}\left|\int_{s+c_0\epsilon^{\gamma_0}}^{t}d\theta\int_{\mathbb{R}}d\eta\, G_\alpha(t-\theta,x-\eta)\right|^{q-1}\\
&\quad\times\int_{s+c_0\epsilon^{\gamma_0}}^{t}d\theta\int_{\mathbb{R}}d\eta\, G_\alpha(t-\theta,x-\eta)\mathrm{E}\left[\left|\int_{s-\epsilon}^{s}dr\int_{\mathbb{R}}dv\left(D_{r,v}^{(k)}(u_l(\theta,\eta))\right)^2\right|^q\right]\\
&\le c\left|\int_{s+c_0\epsilon^{\gamma_0}}^{t}d\theta\int_{\mathbb{R}}d\eta\, G_\alpha(t-\theta,x-\eta)\right|^q\epsilon^{(1-\gamma_0+\gamma_0\frac{\alpha-1}{\alpha})q}\\
&= c(t-s-c_0\epsilon^{\gamma_0})^q\epsilon^{(1-\gamma_0+\gamma_0\frac{\alpha-1}{\alpha})q}\\
&\le c\epsilon^{(1-\gamma_0+\gamma_0\frac{\alpha-1}{\alpha})q},
\end{aligned}
\tag{2.5.19}
$$

where in the second inequality we use Lemma A.3.3.

Finally, from (2.5.14), (2.5.15), (2.5.16), (2.5.17), (2.5.18) and (2.5.19), together with the choice of γ_0, we obtain the desired result. □

Remark 2.5.5. *The result of Lemma 2.5.4 is also true for solutions of stochastic heat equations with Neumann or Dirichlet boundary conditions since we can still apply the result of Lemma A.3.3; see Remark A.3.4.*

2.5.2 Study of the Malliavin matrix

Let $T > 0$ be fixed. For $s, t \in [0, T]$, $s \leq t$, and $x, y \in \mathbb{R}$ consider the $2d$-dimensional random vector

$$Z := (u(s, y), u(t, x) - u(s, y)). \tag{2.5.20}$$

Let γ_Z be the Malliavin matrix of Z. Note that $\gamma_Z = ((\gamma_Z)_{m,l})_{m,l=1,\dots,2d}$ is a symmetric $2d \times 2d$ random matrix with four $d \times d$ blocs of the form

$$\gamma_Z = \begin{pmatrix} \gamma_Z^{(1)} & \vdots & \gamma_Z^{(2)} \\ \cdots & \vdots & \cdots \\ \gamma_Z^{(3)} & \vdots & \gamma_Z^{(4)} \end{pmatrix}$$

where

$$\begin{aligned}
\gamma_Z^{(1)} &= \left(\langle D(u_i(s,y)), D(u_j(s,y))\rangle_{\mathscr{H}}\right)_{i,j=1,\dots,d}, \\
\gamma_Z^{(2)} &= \left(\langle D(u_i(s,y)), D(u_j(t,x) - u_j(s,y))\rangle_{\mathscr{H}}\right)_{i,j=1,\dots,d}, \\
\gamma_Z^{(3)} &= \left(\langle D(u_i(t,x) - u_i(s,y)), D(u_j(s,y))\rangle_{\mathscr{H}}\right)_{i,j=1,\dots,d}, \\
\gamma_Z^{(4)} &= \left(\langle D(u_i(t,x) - u_i(s,y)), D(u_j(t,x) - u_j(s,y))\rangle_{\mathscr{H}}\right)_{i,j=1,\dots,d}.
\end{aligned}$$

We let (**1**) denote the couples of $\{1,\dots,d\} \times \{1,\dots,d\}$, (**2**) denote the couples of $\{1,\dots,d\} \times \{d+1,\dots,2d\}$, (**3**) denote the couples of $\{d+1,\dots,2d\} \times \{1,\dots,d\}$ and (**4**) denote the couples of $\{d+1,\dots,2d\} \times \{d+1,\dots,2d\}$.

The next two results follow exactly along the same lines as [26, Propositions 6.5 and 6.7] using (2.3.3) and Proposition 2.5.2, with Δ replaced by Δ_α. We omit the proofs.

Proposition 2.5.6. *Fix $T > 0$ and let I and J be compact intervals as in Theorem 2.1.1. Let A_Z denote the cofactor matrix of γ_Z. Assuming **P1'**, for any $(s, y), (t, x) \in I \times J$, $(s, y) \neq (t, x)$, $p > 1$,*

$$E\left[|(A_Z)_{m,l}|^p\right]^{1/p} \leq \begin{cases} c_{p,T}(|t-s|^{\frac{\alpha-1}{\alpha}} + |x-y|^{\alpha-1})^d & \text{if } (m,l) \in (\mathbf{1}), \\ c_{p,T}(|t-s|^{\frac{\alpha-1}{\alpha}} + |x-y|^{\alpha-1})^{d-\frac{1}{2}} & \text{if } (m,l) \in (\mathbf{2}) \text{ or } (\mathbf{3}), \\ c_{p,T}(|t-s|^{\frac{\alpha-1}{\alpha}} + |x-y|^{\alpha-1})^{d-1} & \text{if } (m,l) \in (\mathbf{4}). \end{cases}$$

Proposition 2.5.7. *Fix $T > 0$ and let I and J be compact intervals as in Theorem 2.1.1. Assuming **P1'**, for any $(s, y), (t, x) \in I \times J$, $(s, y) \neq (t, x)$, $p > 1$,*

$$E\left[\|D^k(\gamma_Z)_{m,l}\|^p\right]^{1/p} \leq \begin{cases} c_{k,p,T} & \text{if } (m,l) \in (\mathbf{1}), \\ c_{k,p,T}(|t-s|^{\frac{\alpha-1}{\alpha}} + |x-y|^{\alpha-1})^{\frac{1}{2}} & \text{if } (m,l) \in (\mathbf{2}) \text{ or } (\mathbf{3}), \\ c_{k,p,T}(|t-s|^{\frac{\alpha-1}{\alpha}} + |x-y|^{\alpha-1}) & \text{if } (m,l) \in (\mathbf{4}). \end{cases}$$

The main technical effort in this subsection is the proof of the following proposition, which improves [26, Proposition 6.6(a)] and is why the η can be removed in the lower bound on hitting probabilities.

Proposition 2.5.8. *Fix $T>0$ and let I and J be compact intervals as in Theorem 2.1.1. Assume **P1'** and **P2**. There exists C depending on T such that for any $(s,y),(t,x)\in I\times J, (s,y)\neq(t,x), p>1$,*

$$E\big[(\det\gamma_Z)^{-p}\big]^{1/p}\le C(|t-s|^{\frac{\alpha-1}{\alpha}}+|x-y|^{\alpha-1})^{-d}. \tag{2.5.21}$$

Proof. The proof has the same structure as that of [26, Proposition 6.6]; see also [27, Propositon 5.5]. We write

$$\det\gamma_Z=\prod_{i=1}^{2d}(\xi^i)^T\gamma_Z\xi^i, \tag{2.5.22}$$

where $\xi=\{\xi^1,\dots,\xi^{2d}\}$ is an orthogonal basis of $\mathbb{R}^{2d}$ consisting of eigenvectors of γ_Z.

We now carry out the perturbation argument of [26, Proposition 6.6]. Let $\mathbf{0}\in\mathbb{R}^d$ and consider the spaces $E_1=\{(\lambda,\mathbf{0}):\lambda\in\mathbb{R}^d\}$ and $E_2=\{(\mathbf{0},\mu):\mu\in\mathbb{R}^d\}$. Each ξ^i can be written

$$\xi^i=(\lambda^i,\mu^i)=\beta_i(\tilde\lambda^i,\mathbf{0})+\sqrt{1-\beta_i^2}(\mathbf{0},\tilde\mu^i), \tag{2.5.23}$$

where $\lambda^i,\mu^i\in\mathbf{R}^d$, $(\tilde\lambda^i,\mathbf{0})\in E_1$, $(\mathbf{0},\tilde\mu^i)\in E_2$, with $\|\tilde\lambda^i\|=\|\tilde\mu^i\|=1$ and $0\le\beta_i\le1$. In particular, $\|\xi^i\|^2=\|\lambda^i\|^2+\|\mu^i\|^2=1$.

For a fixed small β_0, the result of [26, Lemma 6.8] gives us at least d eigenvectors $\xi^1,\dots,\xi^d$ satisfying $\beta_i\ge\beta_0, i=1,\dots,d$, which we say have a "large projection on E_1". We will show that these will contribute a factor of order 1 to the product in (2.5.22). The at most d other eigenvectors will each contribute a factor of order $|t-s|^{\frac{\alpha-1}{\alpha}}+|x-y|^{\alpha-1}$, which we say have a "small projection on E_1".

Hence, by [26, Lemma 6.8] and Cauchy-Schwarz inequality, we can write

$$\begin{aligned}\mathrm{E}\big[(\det\gamma_Z)^{-p}\big]^{1/p}\le&\sum_{K\subset\{1,\dots,2d\},|K|=d}\left(\mathrm{E}\left[\mathbf{1}_{A_K}\left(\prod_{i\in K}(\xi^i)^T\gamma_Z\xi^i\right)^{-2p}\right]\right)^{1/(2p)}\\&\times\left(\mathrm{E}\left[\left(\inf_{\substack{\xi=(\lambda,\mu)\in\mathbb{R}^{2d}:\\ \|\lambda\|^2+\|\mu\|^2=1}}\xi^T\gamma_Z\xi\right)^{-2dp}\right]\right)^{1/(2p)},\end{aligned} \tag{2.5.24}$$

where $A_K=\cap_{i\in K}\{\beta_i\ge\beta_0\}$.

With this, Propositions 2.5.10 and 2.5.11 below will conclude the proof of Proposition 2.5.8. □

Remark 2.5.9. *As a consequence of Remark 2.5.12, we see that the result of Proposition 2.5.8 is also true for the solutions of stochastic heat equations with Neumann or Dirichlet boundary conditions.*

Proposition 2.5.10. *Fix $T > 0$. Assume **P1'** and **P2**. There exists C depending on T such that for all $s, t \in I, 0 \le t - s < 1, x, y \in J, (s, y) \neq (t, x)$, and $p > 1$,*

$$E\left[\left(\inf_{\substack{\xi = (\lambda,\mu) \in \mathbb{R}^{2d}:\\ \|\lambda\|^2 + \|\mu\|^2 = 1}} \xi^T \gamma_Z \xi\right)^{-2dp}\right] \le C(|t-s|^{\frac{\alpha-1}{\alpha}} + |x-y|^{\alpha-1})^{-2dp}. \tag{2.5.25}$$

We are going to apply Lemma 2.5.4 to prove this proposition. This is a significant improvement over the proof of [26, Proposition 6.9] in which an extra exponent η appears.

Proposition 2.5.11. *Assume **P1'** and **P2**. Fix $T > 0$ and $p > 1$. Then there exists $C = C(p, T)$ such that for all $s, t \in I$ with $t \ge s, x, y \in J, (s, y) \neq (t, x)$,*

$$E\left[\mathbf{1}_{A_K}\left(\prod_{i\in K} (\xi^i)^T \gamma_Z \xi^i\right)^{-p}\right] \le C, \tag{2.5.26}$$

where A_K is defined just below (2.5.24).

Proof of Proposition 2.5.10. Since γ_Z is a matrix of inner products, we can write

$$\xi^T \gamma_Z \xi = \sum_{k=1}^d \int_0^T dr \int_{\mathbb{R}} dv \Big(\sum_{i=1}^d \big(\lambda_i D_{r,v}^{(k)}(u_i(s,y)) + \mu_i (D_{r,v}^{(k)}(u_i(t,x)) - D_{r,v}^{(k)}(u_i(s,y)))\big)\Big)^2.$$

From here on, the proof is divided into two cases.

Case 1. In the first case, we assume that $t - s > 0$ and $|x - y|^\alpha \le t - s$. Choose and fix an $\epsilon \in]0, \delta(t-s)[$, where $0 < \delta < 1$ is small but fixed; its specific value will be decided on later (see the description above (2.5.29)). Then we may write

$$\xi^T \gamma_Z \xi \ge J_1 + J_2,$$

where

$$J_1 := \sum_{k=1}^d \int_{s-\epsilon}^s dr \int_{\mathbb{R}} dv \left(\sum_{i=1}^d (\lambda_i - \mu_i)[G_\alpha(s-r, y-v)\sigma_{ik}(u(r,v)) + a_i(k, r, v, s, y)] + W \right)^2,$$

$$J_2 := \sum_{k=1}^d \int_{t-\epsilon}^t dr \int_{\mathbb{R}} dv \, W^2,$$

$a_i(k,r,v,s,y)$ is defined in (2.3.2) and

$$W := \sum_{i=1}^{d} [\mu_i G_\alpha(t-r, x-v)\sigma_{ik}(u(r,v)) + \mu_i a_i(k,r,v,t,x)].$$

Sub-case A: $\epsilon \leq \delta(t-s)^{1/\gamma_0}$ with $0 < \gamma_0 < 1$. In this case, by the elementary inequality (2.3.5),

$$J_2 \geq \hat{Y}_{1,\epsilon} - Y_{1,\epsilon},$$

where

$$\hat{Y}_{1,\epsilon} := \frac{2}{3}\sum_{k=1}^{d}\int_{t-\epsilon}^{t} dr \int_{\mathbb{R}} dv \left(\sum_{i=1}^{d} \mu_i \sigma_{ik}(u(r,v))\right)^2 G_\alpha^2(t-r, x-v),$$
$$Y_{1,\epsilon} := 2 \sup_{\|\mu\|\leq 1} \sum_{k=1}^{d}\int_{t-\epsilon}^{t} dr \int_{\mathbb{R}} dv \left(\sum_{i=1}^{d} \mu_i a_i(k,r,v,t,x)\right)^2.$$

In agreement with hypothesis **P2** and by (A.4),

$$\begin{aligned}\hat{Y}_{1,\epsilon} &\geq c\|\mu\|^2 \int_{t-\epsilon}^{t} dr \int_{\mathbb{R}} dv\, G_\alpha^2(t-r, x-v) \\ &= c\|\mu\|^2 \epsilon^{\frac{\alpha-1}{\alpha}}.\end{aligned}$$

Next we apply Lemma 2.5.3 [with $s := t$] to find that $\mathrm{E}\left[|Y_{1,\epsilon}|^q\right] \leq c\epsilon^{\frac{2\alpha-2}{\alpha}q}$, for any $q \geq 1$.

For J_1, we find that

$$J_1 \geq \hat{Y}_{2,\epsilon} - Y_{2,\epsilon},$$

where

$$\hat{Y}_{2,\epsilon} := \frac{2}{3}\sum_{k=1}^{d}\int_{s-\epsilon}^{s} dr \int_{\mathbb{R}} dv \left(\sum_{i=1}^{d} (\lambda_i - \mu_i)\sigma_{ik}(u(r,v))\right)^2 G_\alpha^2(s-r, y-v),$$

and

$$Y_{2,\epsilon} := 6(W_1 + W_2 + W_3),$$

where

$$W_1 := \sup_{\|\xi\|=1} \sum_{k=1}^{d} \int_{s-\epsilon}^{s} dr \int_{\mathbb{R}} dv \left(\sum_{i=1}^{d} \mu_i G_\alpha(t-r, x-v) \sigma_{ik}(u(r,v)) \right)^2,$$
$$W_2 := \sup_{\|\xi\|=1} \sum_{k=1}^{d} \int_{s-\epsilon}^{s} dr \int_{\mathbb{R}} dv \left(\sum_{i=1}^{d} (\lambda_i - \mu_i) a_i(k, r, v, s, y) \right)^2,$$
$$W_3 := \sup_{\|\xi\|=1} \sum_{k=1}^{d} \int_{s-\epsilon}^{s} dr \int_{\mathbb{R}} dv \left(\sum_{i=1}^{d} \mu_i a_i(k, r, v, t, x) \right)^2.$$

Hypothesis **P2** implies that $\hat{Y}_{2,\epsilon} \geq c\|\lambda-\mu\|\epsilon^{\frac{\alpha-1}{\alpha}}$. Next, we apply the Cauchy-Schwartz inequality to find that, for any $q \geq 1$,

$$\mathrm{E}\left[|W_1|^q\right] \leq \sup_{\|\xi\|=1} \|\mu\|^{2q} \times \mathrm{E}\left[\left| \sum_{k=1}^{d} \int_{s-\epsilon}^{s} dr \int_{\mathbb{R}} dv \sum_{i=1}^{d} (\sigma_{ik}(u(r,v)))^2 G_\alpha^2(t-r, x-v) \right|^q \right].$$

Thanks to hypothesis **P1'** and (A.4), this is bounded above by

$$c \left| \int_{s-\epsilon}^{s} dr \int_{\mathbb{R}} dv\, G_\alpha^2(t-r, x-v) \right|^q = c((t-s+\epsilon)^{\frac{\alpha-1}{\alpha}} - (t-s)^{\frac{\alpha-1}{\alpha}})^q.$$

Since the function $x \mapsto (x+\epsilon)^{\frac{\alpha-1}{\alpha}} - x^{\frac{\alpha-1}{\alpha}}$ is decreasing on $[0,\infty[$, under the assumption of this sub-case, this is bounded above by

$$\begin{aligned} c\left((\delta^{-\gamma_0}\epsilon^{\gamma_0}+\epsilon)^{\frac{\alpha-1}{\alpha}} - (\delta^{-\gamma_0}\epsilon^{\gamma_0})^{\frac{\alpha-1}{\alpha}}\right)^q &= c(\delta^{-\gamma_0}\epsilon^{\gamma_0})^{\frac{\alpha-1}{\alpha}q}\left((1+\delta^{\gamma_0}\epsilon^{1-\gamma_0})^{\frac{\alpha-1}{\alpha}} - 1\right)^q \\ &\leq c(\delta^{-\gamma_0}\epsilon^{\gamma_0})^{\frac{\alpha-1}{\alpha}q}(\delta^{\gamma_0}\epsilon^{1-\gamma_0}(\alpha-1)/\alpha)^q \\ &= c\epsilon^{(1-\gamma_0+\gamma_0\frac{\alpha-1}{\alpha})q}, \end{aligned}$$

where we use the inequality $(1+x)^{\frac{\alpha-1}{\alpha}} - 1 \leq \frac{\alpha-1}{\alpha}x$, for all $x \geq 0$.

In order to bound the q-th moment of W_2, we use the Cauchy-Schwarz inequality to write

$$\begin{aligned} \mathrm{E}\left[|W_2|^q\right] &\leq \sup_{\|\xi\|=1} \|\lambda-\mu\|^{2q} \times \mathrm{E}\left[\left| \sum_{k=1}^{d} \int_{s-\epsilon}^{s} dr \int_{\mathbb{R}} dv \sum_{i=1}^{d} a_i^2(k, r, v, s, y) \right|^q \right] \\ &\leq \mathrm{E}\left[\left| \sum_{k=1}^{d} \int_{s-\epsilon}^{s} dr \int_{\mathbb{R}} dv \sum_{i=1}^{d} a_i^2(k, r, v, s, y) \right|^q \right]. \end{aligned}$$

We apply Lemma 2.5.3 [with $t := s$] to find that $\mathrm{E}\left[|W_2|^q\right] \leq c\epsilon^{\frac{2\alpha-2}{\alpha}q}$.

Furthermore, under the assumption of this sub-case, by Lemma 2.5.4 we find that, for any

$q \geq 1$,

$$\mathrm{E}\left[|W_3|^q\right] \leq \sup_{\|\xi\|=1} \|\mu\|^{2q} \times \mathrm{E}\left[\left|\sum_{k=1}^{d} \int_{s-\epsilon}^{s} dr \int_{\mathbb{R}} dv \sum_{i=1}^{d} a_i^2(k,r,v,t,x)\right|^q\right]$$
$$\leq c\epsilon^{\min((1+\gamma_0)\frac{\alpha-1}{\alpha},1-\gamma_0+\gamma_0\frac{\alpha-1}{\alpha})q}.$$

The preceding bounds for W_1, W_2 and W_3 prove, in conjunction, that

$$\mathrm{E}\left[|Y_{2,\epsilon}|^q\right] \leq c\epsilon^{\min((1+\gamma_0)\frac{\alpha-1}{\alpha},1-\gamma_0+\gamma_0\frac{\alpha-1}{\alpha})q}.$$

Thus we have

$$\begin{aligned} J_1 + J_2 &\geq \hat{Y}_{1,\epsilon} + \hat{Y}_{2,\epsilon} - Y_{1,\epsilon} - Y_{2,\epsilon} \\ &\geq c(\|\mu\|^2 + \|\lambda-\mu\|^2)\epsilon^{\frac{\alpha-1}{\alpha}} - Y_{1,\epsilon} - Y_{2,\epsilon} \\ &\geq c\epsilon^{\frac{\alpha-1}{\alpha}} - Y_\epsilon, \end{aligned} \tag{2.5.27}$$

where $Y_\epsilon := Y_{1,\epsilon} + Y_{2,\epsilon}$ satisfies

$$\mathrm{E}\left[|Y_\epsilon|^q\right] \leq c\epsilon^{\min((1+\gamma_0)\frac{\alpha-1}{\alpha},1-\gamma_0+\gamma_0\frac{\alpha-1}{\alpha})q}. \tag{2.5.28}$$

Sub-case B: $\delta(t-s)^{1/\gamma_0} < \epsilon < \delta(t-s)$. In this case, we are going to give a different estimate on J_1.

$$J_1 \geq \tilde{Y}_\epsilon - 4(W_2 + W_3),$$

where

$$\tilde{Y}_\epsilon := \frac{2}{3}\sum_{k=1}^{d}\int_{s-\epsilon}^{s} dr \int_{\mathbb{R}} dv \left(\sum_{i=1}^{d}[(\lambda_i-\mu_i)G_\alpha(s-r,y-v)+\mu_i G_\alpha(t-r,x-v)]\sigma_{ik}(u(r,v))\right)^2.$$

Using the inequality

$$(a+b)^2 \geq a^2 + b^2 - 2|ab|,$$

we see that

$$\tilde{Y}_\epsilon \geq \hat{Y}_{2,\epsilon} - |B_1^{(3)}|,$$

where

$$B_1^{(3)} := \frac{4}{3}\sum_{k=1}^{d}\int_{s-\epsilon}^{s} dr \int_{\mathbb{R}} dv \left(\sum_{i=1}^{d}(\lambda_i-\mu_i)G_\alpha(s-r,y-v)\sigma_{ik}(u(r,v))\right) \\ \times\left(\sum_{i=1}^{d}\mu_i G_\alpha(t-r,x-v)\sigma_{ik}(u(r,v))\right).$$

Hypothesis **P1’** assures us that

$$\begin{aligned}\left|B_1^{(3)}\right| &\le c\int_{s-\epsilon}^{s} dr \int_{\mathbb{R}} dv\, G_\alpha(s-r,y-v)G_\alpha(t-r,x-v)\\ &= c\int_{s-\epsilon}^{s} dr\, G_\alpha(t+s-2r,x-y) = c\int_{0}^{\epsilon} dr\, G_\alpha(t-s+2r,x-y),\end{aligned}$$

where in the first equality we use the semi-group property of the Green kernel (A.6). Since for any $t>0$, the function $x\mapsto G_\alpha(t,x)$ attains its maximum at 0, this is bounded above by

$$\begin{aligned}c\int_0^{\epsilon} dr\, G_\alpha(t-s+2r,0) &= c'\int_0^{\epsilon} dr(t-s+2r)^{-\frac{1}{\alpha}}\\ &= c'((t-s+2\epsilon)^{\frac{\alpha-1}{\alpha}}-(t-s)^{\frac{\alpha-1}{\alpha}})\\ &= c'\epsilon^{\frac{\alpha-1}{\alpha}}((\frac{t-s}{\epsilon}+2)^{\frac{\alpha-1}{\alpha}}-(\frac{t-s}{\epsilon})^{\frac{\alpha-1}{\alpha}})\\ &\le c'\epsilon^{\frac{\alpha-1}{\alpha}}((1/\delta+2)^{\frac{\alpha-1}{\alpha}}-(1/\delta)^{\frac{\alpha-1}{\alpha}}),\end{aligned}$$

where the first equality is because of the scaling property of Green kernel (A.2) and in the inequality we use the assumption $\epsilon<\delta(t-s)$ and the fact that the function $x\mapsto(x+2)^{\frac{\alpha-1}{\alpha}}-x^{\frac{\alpha-1}{\alpha}}$ is decreasing on $[0,\infty[$. Hence we have

$$\begin{aligned}J_1+J_2 &\ge \hat{Y}_{1,\epsilon}+\hat{Y}_{2,\epsilon}-\left|B_1^{(3)}\right|-4W_2-4W_3-Y_{1,\epsilon}\\ &\ge c(\|\mu\|^2+\|\lambda-\mu\|^2)\epsilon^{\frac{\alpha-1}{\alpha}}-c'\epsilon^{\frac{\alpha-1}{\alpha}}((1/\delta+2)^{\frac{\alpha-1}{\alpha}}-(1/\delta)^{\frac{\alpha-1}{\alpha}})-4W_2-4W_3-Y_{1,\epsilon}\\ &\ge c_0\epsilon^{\frac{\alpha-1}{\alpha}}-c'\epsilon^{\frac{\alpha-1}{\alpha}}((1/\delta+2)^{\frac{\alpha-1}{\alpha}}-(1/\delta)^{\frac{\alpha-1}{\alpha}})-4W_2-4W_3-Y_{1,\epsilon}\end{aligned}$$

We can choose δ small so that $c_0>c'((1/\delta+2)^{\frac{\alpha-1}{\alpha}}-(1/\delta)^{\frac{\alpha-1}{\alpha}})$ and therefore,

$$J_1+J_2\ge c\epsilon^{\frac{\alpha-1}{\alpha}}-4W_2-4W_3-Y_{1,\epsilon}. \tag{2.5.29}$$

In this sub-case,

$$\begin{aligned}\mathrm{E}\left[|W_2|^q\right] &\le \sup_{\|\xi\|=1}\|\lambda-\mu\|^{2q}\times\mathrm{E}\left[\left|\sum_{k=1}^{d}\int_{s-\epsilon}^{s} dr\int_{\mathbb{R}} dv\sum_{i=1}^{d}a_i^2(k,r,v,s,y)\right|^q\right]\\ &\le \mathrm{E}\left[\left|\sum_{k=1}^{d}\int_{s-\epsilon}^{s} dr\int_{\mathbb{R}} dv\sum_{i=1}^{d}a_i^2(k,r,v,s,y)\right|^q\right].\end{aligned}$$

We apply Lemma 2.5.3 to find that $\mathrm{E}\left[|W_2|^q\right]\le c\epsilon^{\frac{2\alpha-2}{\alpha}q}$. Similarly, we find using Lemma 2.5.3

and the assumption $\delta(t-s)^{1/\gamma_0} < \epsilon$ that

$$\begin{aligned}
\mathrm{E}\left[|W_3|^q\right] &\le \sup_{\|\xi\|=1} \|\mu\|^{2q} \times \mathrm{E}\left[\left|\sum_{k=1}^d \int_{s-\epsilon}^s dr \int_{\mathbb{R}} dv \sum_{i=1}^d a_i^2(k,r,v,t,x)\right|^q\right] \\
&\le c(t-s+\epsilon)^{\frac{\alpha-1}{\alpha}q} \epsilon^{\frac{\alpha-1}{\alpha}q} \\
&\le c(\delta^{-\gamma_0}\epsilon^{\gamma_0}+\epsilon)^{\frac{\alpha-1}{\alpha}q} \epsilon^{\frac{\alpha-1}{\alpha}q} \\
&\le c\epsilon^{(1+\gamma_0)\frac{\alpha-1}{\alpha}q}.
\end{aligned}$$

Combine (2.5.27) and (2.5.29), we have for $\epsilon \in]0, \delta(t-s)[$

$$\inf_{\|\xi\|=1} \xi^T \gamma_Z \xi \ge c\epsilon^{\frac{\alpha-1}{\alpha}} - \tilde{Z}_\epsilon, \tag{2.5.30}$$

where

$$\tilde{Z}_\epsilon := Y_\epsilon 1_{\{\epsilon \le \delta(t-s)^{1/\gamma_0}\}} + 4(W_2 + W_3 + Y_{1,\epsilon}) 1_{\{\delta(t-s)^{1/\gamma_0} < \epsilon < \delta(t-s)\}}$$

and for all $q \ge 1$,

$$\mathrm{E}\left[|Y_\epsilon 1_{\{\epsilon \le \delta(t-s)^{1/\gamma_0}\}}|^q\right] \le c\epsilon^{\min((1+\gamma_0)\frac{\alpha-1}{\alpha}, 1-\gamma_0+\gamma_0\frac{\alpha-1}{\alpha})q}, \tag{2.5.31}$$

and

$$\mathrm{E}\left[|4(W_2 + W_3 + Y_{1,\epsilon}) 1_{\{\delta(t-s)^{1/\gamma_0} < \epsilon < \delta(t-s)\}}|^q\right] \le c\epsilon^{(1+\gamma_0)\frac{\alpha-1}{\alpha}q}. \tag{2.5.32}$$

We use Proposition A.2.1 to find that

$$\begin{aligned}
\mathrm{E}\left[\left(\inf_{\|\xi\|=1} \xi^T \gamma_Z \xi\right)^{-2pd}\right] &\le c\,(\delta(t-s))^{-2pd\frac{\alpha-1}{\alpha}} \\
&= c'(t-s)^{-2pd\frac{\alpha-1}{\alpha}} \\
&\le \tilde{c}\left[|t-s|^{\frac{\alpha-1}{\alpha}} + |x-y|^{\alpha-1}\right]^{-2pd},
\end{aligned}$$

whence follows the result in the case that $|x-y|^\alpha \le t-s < 1$.

Case 2. Now we work on the second case where $|x-y| > 0$ and $|x-y|^\alpha \ge t-s \ge 0$. Let $\epsilon > 0$ be such that $(1+\beta)\epsilon^{\frac{1}{\alpha}} < \frac{1}{2}|x-y|$, where $\beta > 0$ is large but fixed; its specific value will be decided on later (see the explanation for (2.5.45) and (2.5.46)). Then

$$\xi^T \gamma_Z \xi \ge I_1 + I_2,$$

where

$$I_1 := \sum_{k=1}^{d} \int_{s-\epsilon}^{s} dr \int_{\mathbb{R}} dv (\varphi_1 + \varphi_2)^2,$$
$$I_2 := \sum_{k=1}^{d} \int_{(t-\epsilon)\vee s}^{t} dr \int_{\mathbb{R}} dv \varphi_2^2,$$

and

$$\varphi_1 := \sum_{i=1}^{d} (\lambda_i - \mu_i)[G_\alpha(s-r, y-v)\sigma_{ik}(u(r,v)) + a_i(k,r,v,s,y)],$$
$$\varphi_2 := \sum_{i=1}^{d} [\mu_i G_\alpha(t-r, x-v)\sigma_{ik}(u(r,v)) + \mu_i a_i(k,r,v,t,x)].$$

From here on, Case 2 is divided into two further sub-cases.

Sub-Case A. Suppose, in addition, that $\epsilon \geq \delta(t-s)$, where δ is chosen as in case 1. In this case, we are going to prove that

$$\inf_{\|\xi\|=1} \xi^T \gamma_Z \xi \geq c\epsilon^{\frac{\alpha-1}{\alpha}} - Z_{1,\epsilon}, \tag{2.5.33}$$

where for all $q \geq 1$,

$$\mathrm{E}\left[|Z_{1,\epsilon}|^q\right] \leq c(q)\epsilon^{\frac{2\alpha-2}{\alpha}q}. \tag{2.5.34}$$

Indeed, by the elementary inequality (2.3.5) we find that

$$I_1 \geq \frac{2}{3}\tilde{A}_1 - B_1^{(1)} - B_1^{(2)},$$

where

$$\tilde{A}_1 := \sum_{k=1}^{d} \int_{s-\epsilon}^{s} dr \int_{\mathbb{R}} dv \left(\sum_{i=1}^{d} [(\lambda_i - \mu_i)G_\alpha(s-r, y-v) + \mu_i G_\alpha(t-r, x-v)]\sigma_{ik}(u(r,v)) \right)^2,$$
$$B_1^{(1)} := 4\|\lambda - \mu\|^2 \sum_{k=1}^{d} \int_{s-\epsilon}^{s} dr \int_{\mathbb{R}} dv \sum_{i=1}^{d} a_i^2(k,r,v,s,y), \tag{2.5.35}$$
$$B_1^{(2)} := 4\|\mu\|^2 \sum_{k=1}^{d} \int_{s-\epsilon}^{s} dr \int_{\mathbb{R}} dv \sum_{i=1}^{d} a_i^2(k,r,v,t,x). \tag{2.5.36}$$

Using the inequality

$$(a+b)^2 \geq a^2 + b^2 - 2|ab|,$$

we see that

$$\tilde{A}_1 \geq A_1 + A_2 - |B_1^{(3)}|,$$

where

$$A_1 := \sum_{k=1}^{d} \int_{s-\epsilon}^{s} dr \int_{\mathbb{R}} dv \left(\sum_{i=1}^{d} (\lambda_i - \mu_i) G_\alpha(s-r, y-v) \sigma_{ik}(u(r,v)) \right)^2,$$

$$A_2 := \sum_{k=1}^{d} \int_{s-\epsilon}^{s} dr \int_{\mathbb{R}} dv \left(\sum_{i=1}^{d} \mu_i G_\alpha(t-r, x-v) \sigma_{ik}(u(r,v)) \right)^2,$$

$$B_1^{(3)} := 2 \sum_{k=1}^{d} \int_{s-\epsilon}^{s} dr \int_{\mathbb{R}} dv \left(\sum_{i=1}^{d} (\lambda_i - \mu_i) G_\alpha(s-r, y-v) \sigma_{ik}(u(r,v)) \right) \times \left(\sum_{i=1}^{d} \mu_i G_\alpha(t-r, x-v) \sigma_{ik}(u(r,v)) \right).$$

We can combine terms to find that

$$I_1 \geq \frac{2}{3}(A_1 + A_2) - (B_1^{(1)} + B_1^{(2)} + |B_1^{(3)}|).$$

Moreover, we appeal to the elementary inequality (2.3.5) to find that

$$I_2 \geq \frac{2}{3} A_3 - B_2,$$

where

$$A_3 := \sum_{k=1}^{d} \int_{(t-\epsilon)\vee s}^{t} dr \int_{\mathbb{R}} dv \left(\sum_{i=1}^{d} \mu_i G_\alpha(t-r, x-v) \sigma_{ik}(u(r,v) \right)^2,$$

$$B_2 := 2 \sum_{k=1}^{d} \int_{(t-\epsilon)\vee s}^{t} dr \int_{\mathbb{R}} dv \left(\sum_{i=1}^{d} \mu_i a_i(k, r, v, t, x) \right)^2. \tag{2.5.37}$$

By hypothesis **P2** and using (A.4) three times,

$$\begin{aligned}
A_1 + A_2 + A_3 \geq \rho^2 \Big(&\|\lambda - \mu\|^2 \int_{s-\epsilon}^{s} dr \int_{\mathbb{R}} dv \, G_\alpha^2(s-r, y-v) \\
&+ \|\mu\|^2 \int_{s-\epsilon}^{s} dr \int_{\mathbb{R}} dv \, G_\alpha^2(t-r, x-v) \\
&+ \|\mu\|^2 \int_{(t-\epsilon)\vee s}^{t} dr \int_{\mathbb{R}} dv \, G_\alpha^2(t-r, x-v) \Big) \\
= c\rho^2 \Big(&\|\lambda - \mu\|^2 \epsilon^{\frac{\alpha-1}{\alpha}} + \|\mu\|^2 \Big((t-s+\epsilon)^{\frac{\alpha-1}{\alpha}} - (t-s)^{\frac{\alpha-1}{\alpha}} + (t - ((t-\epsilon) \vee s))^{\frac{\alpha-1}{\alpha}} \Big) \Big) \\
= c\rho^2 \Big(&\|\lambda - \mu\|^2 \epsilon^{\frac{\alpha-1}{\alpha}} + \|\mu\|^2 \Big((t-s+\epsilon)^{\frac{\alpha-1}{\alpha}} - (t-s)^{\frac{\alpha-1}{\alpha}} + ((t-s) \wedge \epsilon)^{\frac{\alpha-1}{\alpha}} \Big) \Big) \\
= c\rho^2 \epsilon^{\frac{\alpha-1}{\alpha}} \Big(&\|\lambda - \mu\|^2 + \|\mu\|^2 \Big((\frac{t-s}{\epsilon} + 1)^{\frac{\alpha-1}{\alpha}} - (\frac{t-s}{\epsilon})^{\frac{\alpha-1}{\alpha}} + ((\frac{t-s}{\epsilon}) \wedge 1)^{\frac{\alpha-1}{\alpha}} \Big) \Big).
\end{aligned}$$

Denote $\zeta(x) := (x+1)^{\frac{\alpha-1}{\alpha}} - x^{\frac{\alpha-1}{\alpha}} + (x\wedge 1)^{\frac{\alpha-1}{\alpha}}$, $x\in[0,\infty[$. Then it is obvious to see that

$$\hat{c}_0 := \min_{0\le x<\infty} \zeta(x) > 0. \tag{2.5.38}$$

Thus we have

$$\begin{aligned} A_1 + A_2 + A_3 &\ge c\rho^2 \epsilon^{\frac{\alpha-1}{\alpha}} \left(\|\lambda-\mu\|^2 + c_0\|\mu\|^2\right) \\ &\ge c\epsilon^{\frac{\alpha-1}{\alpha}}. \end{aligned}$$

We are aiming for (2.5.33), and propose to bound the absolute moments of $B_1^{(i)}$, $i=1,2,3$ and B_2, separately. According to Lemma 2.5.3 with $s=t$,

$$\mathrm{E}\left[\sup_{\|\xi\|=1} |B_2|^q\right] \le c(q)\epsilon^{\frac{2\alpha-2}{\alpha}q}. \tag{2.5.39}$$

Next we bound the absolute moments of $B_1^{(i)}$, $i=1,2,3$. Using Lemma 2.5.3, with $t=s$, we find that for all $q\ge 1$,

$$\mathrm{E}\left[\sup_{\|\xi\|=1} |B_1^{(1)}|^q\right] \le c\epsilon^{\frac{2\alpha-2}{\alpha}q}. \tag{2.5.40}$$

In the same way, we see that

$$\mathrm{E}\left[\sup_{\|\xi\|=1} |B_1^{(2)}|^q\right] \le c(t-s+\epsilon)^{\frac{\alpha-1}{\alpha}q}\epsilon^{\frac{\alpha-1}{\alpha}q}. \tag{2.5.41}$$

Since we are in the sub-case A where $t-s\le\delta^{-1}\epsilon$, we obtain the following:

$$\mathrm{E}\left[\sup_{\|\xi\|=1} |B_1^{(2)}|^q\right] \le c\epsilon^{\frac{2\alpha-2}{\alpha}q}. \tag{2.5.42}$$

We can combine (2.5.40) and (2.5.42) as follows:

$$\mathrm{E}\left[\sup_{\|\xi\|=1} \left(B_1^{(1)} + B_1^{(2)}\right)^q\right] \le c(q)\epsilon^{\frac{2\alpha-2}{\alpha}q}. \tag{2.5.43}$$

Finally, we turn to bounding the absolute moments of $B_1^{(3)}$. Hypothesis **P1'** assures us that

$$\begin{aligned} \left|B_1^{(3)}\right| &\le c\int_{s-\epsilon}^{s} dr \int_{\mathbb{R}} dv\, G_\alpha(s-r,y-v)G_\alpha(t-r,x-v) \\ &= c\int_{s-\epsilon}^{s} dr\, G_\alpha(t+s-2r,x-y) = c\int_0^{\epsilon} dr\, G_\alpha(t-s+2r,x-y), \end{aligned}$$

thanks to the semi-group property. When $\alpha=2$, we can follow the arguments on page 414 of

[26] to find that

$$\left|B_1^{(3)}\right| \leq c\epsilon^{1/2}\Psi(\beta), \quad \text{where} \quad \Psi(\beta) := \beta\int_0^{6/\beta^2} z^{-1/2}e^{-1/z}dz. \tag{2.5.44}$$

Thus,

$$\begin{aligned}\inf_{\|\xi\|=1} \xi^T\gamma_Z\xi &\geq \frac{2}{3}(A_1+A_2+A_3) - \left(B_1^{(1)} + B_1^{(2)} + \left|B_1^{(3)}\right| + B_2\right)\\ &\geq c_1\epsilon^{1/2} - c_2\Psi(\beta)\epsilon^{1/2} - Z_{1,\epsilon},\end{aligned}$$

where $Z_{1,\epsilon} := B_1^{(1)} + B_1^{(2)} + B_2$ satisfies $\mathrm{E}[|Z_{1,\epsilon}|^q] \leq c_1(q)\epsilon^{\frac{2\alpha-2}{\alpha}q}$. Because $\lim_{\nu\to\infty}\Psi(\nu) = 0$, we can choose β so large that $c_2\Psi(\beta) \leq c_1/4$ for the c_1 and c_2 of the preceding displayed equation. This yields,

$$\inf_{\|\xi\|=1} \xi^T\gamma_Z\xi \geq c\epsilon^{1/2} - Z_{1,\epsilon}. \tag{2.5.45}$$

When $1 < \alpha < 2$, by the scaling property (A.2), and (A.5), we have

$$\begin{aligned}\left|B_1^{(3)}\right| &\leq c\int_0^\epsilon dr(t-s+2r)^{-1/\alpha}G_\alpha(1,(x-y)(t-s+2r)^{-1/\alpha})\\ &\leq cK_\alpha\int_0^\epsilon \frac{(t-s+2r)^{-1/\alpha}}{1+\left|(x-y)(t-s+2r)^{-1/\alpha}\right|^{1+\alpha}}dr\\ &\leq cK_\alpha\int_0^\epsilon \frac{(t-s+2r)^{-1/\alpha}}{\left|(x-y)(t-s+2r)^{-1/\alpha}\right|^{1+\alpha}}dr\\ &= cK_\alpha|x-y|^{-1-\alpha}\int_0^\epsilon (t-s+2r)dr = cK_\alpha|x-y|^{-1-\alpha}[(t-s)\epsilon+\epsilon^2].\end{aligned}$$

Since $t-s \leq |x-y|^\alpha$, this is bounded above by

$$\begin{aligned}&cK_\alpha(|x-y|^{-1}\epsilon + |x-y|^{-1-\alpha}\epsilon^2)\\ &\quad\leq cK_\alpha\left(\frac{1}{(1+\beta)}\epsilon^{\frac{\alpha-1}{\alpha}} + \frac{1}{(1+\beta)^{1+\alpha}}\epsilon^{2-(1+\alpha)/\alpha}\right)\\ &\quad= cK_\alpha\left(\frac{1}{(1+\beta)} + \frac{1}{(1+\beta)^{1+\alpha}}\right)\epsilon^{\frac{\alpha-1}{\alpha}}.\end{aligned}$$

Therefore, for $1 < \alpha \leq 2$, we can choose and fix β large enough so that

$$\inf_{\|\xi\|=1} \xi^T\gamma_Z\xi \geq c\epsilon^{\frac{\alpha-1}{\alpha}} - Z_{1,\epsilon}, \tag{2.5.46}$$

where for all $q \geq 1$,

$$\mathrm{E}\left[|Z_{1,\epsilon}|^q\right] \leq c(q)\epsilon^{\frac{2\alpha-2}{\alpha}q},$$

as in (2.5.33) and (2.5.34).

Sub-case B. In this final (sub-) case we suppose that $\epsilon < \delta(t-s) \le \delta|x-y|^\alpha$. Choose and fix $0 < \epsilon < \delta(t-s)$. During the course of our proof of Case 1, we established the following:

$$\inf_{\|\xi\|=1} \xi^T \gamma_Z \xi \ge c\epsilon^{\frac{\alpha-1}{\alpha}} - \tilde{Z}_\epsilon, \tag{2.5.47}$$

where, for all $q \ge 1$,

$$\mathrm{E}\left[|\tilde{Z}_\epsilon|^q\right] \le c\epsilon^{\min((1+\gamma_0)\frac{\alpha-1}{\alpha}, 1-\gamma_0+\gamma_0\frac{\alpha-1}{\alpha})q}$$

(see (2.5.31) and (2.5.32)).

Combine Sub-Cases A and B, and, in particular, (2.5.33) and (2.5.47), to find that for all $0 < \epsilon < 2^{-\alpha}(1+\beta)^{-\alpha}|x-y|^\alpha$,

$$\inf_{\|\xi\|=1} \xi^T \gamma_Z \xi \ge c\epsilon^{\frac{\alpha-1}{\alpha}} - (\tilde{Z}_\epsilon \mathbf{1}_{\{\epsilon<\delta(t-s)\}} + Z_{1,\epsilon}\mathbf{1}_{\{t-s\le\delta^{-1}\epsilon\}}).$$

Because of this and (2.5.34), Proposition A.2.1 implies that

$$\begin{aligned}
\mathrm{E}\left[\left(\inf_{\|\xi\|=1} \xi^T \gamma_Z \xi\right)^{-2pd}\right] &\le c|x-y|^{\alpha(-2dp)(\frac{\alpha-1}{\alpha})} \\
&\le c\left(|x-y|^\alpha + |t-s|\right)^{(\frac{\alpha-1}{\alpha})(-2dp)} \\
&\le c\left(|t-s|^{\frac{\alpha-1}{\alpha}} + |x-y|^{\alpha-1}\right)^{-2dp}.
\end{aligned}$$

This completes the proof of Proposition 2.5.10. □

Remark 2.5.12. *From the proof of Proposition 2.5.10, we see that* (2.5.25) *is also valid for the solutions of stochastic heat equations with Neumann or Dirichlet boundary conditions, since we can still apply the result of Lemma 2.5.4; see Remark 2.5.5.*

Proof of Proposition 2.5.11. The proof follows along the same lines as those of [26, Proposition 6.13].

Let $0 < \epsilon < s \le t$. We fix $i_0 \in \{1, \dots, 2d\}$ and write $\tilde{\lambda}^{i_0} = (\tilde{\lambda}_1^{i_0}, \dots, \tilde{\lambda}_d^{i_0})$ and $\tilde{\mu}^{i_0} = (\tilde{\mu}_1^{i_0}, \dots, \tilde{\mu}_d^{i_0})$. We look at $(\xi^{i_0})^T \gamma_Z \xi^{i_0}$ on the event $\{\beta_{i_0} \ge \beta_0\}$. As in the proof of Proposition 2.5.10 and using the notation from (2.5.23), this is bounded below by

$$\begin{aligned}
&\sum_{k=1}^{d}\int_{s-\epsilon}^{s} dr \int_{\mathbb{R}} dv \Bigg(\sum_{i=1}^{d} \Big[\Big(\beta_{i_0} \tilde{\lambda}_i^{i_0} G_\alpha(s-r, y-v) \\
&\quad + \tilde{\mu}_i^{i_0} \sqrt{1-\beta_{i_0}^2} (G_\alpha(t-r, x-v) - G_\alpha(s-r, y-v)) \Big) \sigma_{ik}(u(r,v)) \\
&\quad + \beta_{i_0} \tilde{\lambda}_i^{i_0} a_i(k, r, v, s, y) \\
&\quad + \tilde{\mu}_i^{i_0} \sqrt{1-\beta_{i_0}^2} (a_i(k, r, v, t, x) - a_i(k, r, v, s, y)) \Big] \Bigg)^2 \\
&\quad + \sum_{k=1}^{d} \int_{s \vee (t-\epsilon)}^{t} dr \int_{\mathbb{R}} dv \Bigg(\sum_{i=1}^{d} \Big[\tilde{\mu}_i^{i_0} \sqrt{1-\beta_{i_0}^2} G_\alpha(t-r, x-v) \sigma_{ik}(u(r,v)) \\
&\quad + \tilde{\mu}_i^{i_0} \sqrt{1-\beta_{i_0}^2} a_i(k, r, v, t, x) \Big] \Bigg)^2 .
\end{aligned} \tag{2.5.48}$$

We seek lower bounds for this expression for $0 < \epsilon < \epsilon_0$ where $\epsilon_0 \in]0, \frac{1}{2}[$ is fixed. In the remainder of this proof, we will use the generic notation $\beta, \tilde{\lambda}$ and $\tilde{\mu}$ for the realizations $\beta_{i_0}(\omega), \tilde{\lambda}^{i_0}(\omega)$, and $\tilde{\mu}^{i_0}(\omega)$. The proof follows the structure of [26, Proposition 6.13].

Case 1 $t-s \le \epsilon$. Then, by the elementary inequality (2.3.5), the expression in (2.5.48) is bounded below by

$$\frac{2}{3}(f_1(s,t,\epsilon,\beta,\tilde{\lambda},\tilde{\mu},x,y) + f_2(s,t,\epsilon,\beta,\tilde{\lambda},\tilde{\mu},x,y)) - 2I_\epsilon,$$

where, from hypothesis **P2**,

$$\begin{aligned}
f_1 &\ge c\rho^2 \int_{s-\epsilon}^{s} dr \int_{\mathbb{R}} dv \Big\| \beta\tilde{\lambda} G_\alpha(s-r, y-v) \\
&\qquad + \sqrt{1-\beta^2}\tilde{\mu}(G_\alpha(t-r, x-v) - G_\alpha(s-r, y-v)) \Big\|^2,
\end{aligned} \tag{2.5.49}$$

$$f_2 \ge c\rho^2 \int_{s\vee(t-\epsilon)}^{t} dr \int_{\mathbb{R}} dv \left\| \tilde{\mu}\sqrt{1-\beta^2} G_\alpha(t-r, x-v) \right\|^2 \tag{2.5.50}$$

and $I_\epsilon = 3(I_{1,\epsilon} + I_{2,\epsilon} + I_{3,\epsilon})$, where

$$\begin{aligned}
I_{1,\epsilon} &:= \sum_{k=1}^{d} \int_{s-\epsilon}^{s} dr \int_{\mathbb{R}} dv \left(\sum_{i=1}^{d} \left[\beta\tilde{\lambda}_i - \tilde{\mu}_i \sqrt{1-\beta^2} \right] a_i(k, r, v, s, y) \right)^2, \\
I_{2,\epsilon} &:= \sum_{k=1}^{d} \int_{s-\epsilon}^{s} dr \int_{\mathbb{R}} dv \left(\sum_{i=1}^{d} \tilde{\mu}_i \sqrt{1-\beta^2} a_i(k, r, v, t, x) \right)^2, \\
I_{3,\epsilon} &:= \sum_{k=1}^{d} \int_{t-\epsilon}^{t} dr \int_{\mathbb{R}} dv \left(\sum_{i=1}^{d} \tilde{\mu}_i \sqrt{1-\beta^2} a_i(k, r, v, t, x) \right)^2.
\end{aligned}$$

There are obvious similarities between the term $I_{1,\epsilon}$ and $B_1^{(1)}$ in (2.5.35). However, we must keep in mind that $\beta, \tilde{\lambda}$ and $\tilde{\mu}$ are the realizations of $\beta_{i_0}, \tilde{\lambda}^{i_0}$, and $\tilde{\mu}^{i_0}$. Therefore,

$$\begin{aligned} I_{1,\epsilon} &:= \sum_{k=1}^{d} \int_{s-\epsilon}^{s} dr \int_{\mathbb{R}} dv \left(\sum_{i=1}^{d} \left[\beta\tilde{\lambda}_i - \tilde{\mu}_i \sqrt{1-\beta^2} \right] a_i(k,r,v,s,y) \right)^2 \\ &\leq C \sum_{k=1}^{d} \int_{s-\epsilon}^{s} dr \int_{\mathbb{R}} dv \sum_{i=1}^{d} a_i^2(k,r,v,s,y). \end{aligned}$$

Then, we apply the same method that was used to bound $\mathrm{E}[|B_1^{(1)}|^q]$ to deduce that $\mathrm{E}[|I_{1,\epsilon}|^q] \leq c(q)\epsilon^{\frac{2\alpha-2}{\alpha}q}$. Similarly, since $I_{2,\epsilon}$ is similar to $B_1^{(2)}$ from (2.5.36) and $t-s \leq \epsilon$, we see using (2.5.41) that $\mathrm{E}[|I_{2,\epsilon}|^q] \leq c(q)\epsilon^{\frac{2\alpha-2}{\alpha}q}$. Finally, using the similarity between $I_{3,\epsilon}$ and B_2 in (2.5.37), we see that $\mathrm{E}[|I_{3,\epsilon}|^q] \leq c(q)\epsilon^{\frac{2\alpha-2}{\alpha}q}$.

We claim that for every $\beta_0 > 0$, there exists $\epsilon_0 > 0$ and $c_0 > 0$ such that

$$f_1 + f_2 \geq c_0\epsilon^{\frac{\alpha-1}{\alpha}} \text{ for all } \beta \in [\beta_0, 1],\ \epsilon \in]0, \epsilon_0],\ s,t \in [0,T],\ x,y \in \mathbb{R}. \tag{2.5.51}$$

Using this for the β_0 from [26, Lemma 6.8] with α_0 replace by β_0, this will imply in particular that for $\epsilon \geq t-s$,

$$(\xi^{i_0})^T \gamma_Z \xi^{i_0} \geq c_0\epsilon^{\frac{\alpha-1}{\alpha}} - 2I_\epsilon, \tag{2.5.52}$$

where $\mathrm{E}[|I_\epsilon|^q] \leq c(q)\epsilon^{\frac{2\alpha-2}{\alpha}q}$.

Let $g_1(s,t,\epsilon,\beta,\tilde{\lambda},\tilde{\mu},x,y)$ and $g_2(s,t,\epsilon,\beta,\tilde{\lambda},\tilde{\mu},x,y)$ be defined by the same expressions as the right-hand sides of (2.5.49) and (2.5.50).

Observe that $g_1 \geq 0, g_2 \geq 0$, and if $g_1 = 0$, then for all $r \in [s-\epsilon, s[$ and $v \in \mathbb{R}$,

$$\left\| \beta\tilde{\lambda}G_\alpha(s-r,y-v) + \sqrt{1-\beta^2}\tilde{\mu}(G_\alpha(t-r,x-v) - G_\alpha(s-r,y-v)) \right\| = 0. \tag{2.5.53}$$

If, in addition, $\tilde{\lambda} = \tilde{\mu}$, then we get that for all $v \in \mathbb{R}$,

$$\left(\beta - \sqrt{1-\beta^2}\right) G_\alpha(s-r,y-v) + \sqrt{1-\beta^2}G_\alpha(t-r,x-v) = 0.$$

We take Fourier transforms to deduce from this that for all $\xi \in \mathbb{R}$,

$$\left(\beta - \sqrt{1-\beta^2}\right) e^{i\xi y} = -\sqrt{1-\beta^2}e^{i\xi x}e^{(s-t)|\xi|^\alpha}.$$

If $x = y$, then it follows that $s = t$ and $\beta - \sqrt{1-\beta^2} = -\sqrt{1-\beta^2}$, that is, $\beta = 0$. Hence,

$$\text{if } \beta \neq 0, x = y \text{ and } \tilde{\lambda} = \tilde{\mu}, \text{ then } g_1 > 0. \tag{2.5.54}$$

We shall make use of this observation shortly.

Because $\|\tilde{\lambda}\| = \|\tilde{\mu}\| = 1$, f_1 is bounded below by

$$\begin{aligned}
&c\rho^2 \int_{s-\epsilon}^{s} dr \int_{\mathbb{R}} dv \Big(\beta^2 G_\alpha^2(s-r, y-v)) + \left(1-\beta^2\right)(G_\alpha(t-r, x-v) - G_\alpha(s-r, y-v))^2 \\
&\qquad\qquad + 2\beta\sqrt{1-\beta^2}\, G_\alpha(s-r, y-v))(G_\alpha(t-r, x-v) - G_\alpha(s-r, y-v))(\tilde{\lambda}\cdot\tilde{\mu})\Big) \\
&= c\rho^2 \int_{s-\epsilon}^{s} dr \int_{\mathbb{R}} dv \Big(\Big(\beta - \sqrt{1-\beta^2}\Big)^2 G_\alpha^2(s-r, y-v) + \left(1-\beta^2\right) G_\alpha^2(t-r, x-v) \\
&\qquad\qquad + 2\Big(\beta - \sqrt{1-\beta^2}\Big)\sqrt{1-\beta^2}\, G_\alpha(s-r, y-v) G_\alpha(t-r, x-v) \\
&\qquad\qquad + 2\beta\sqrt{1-\beta^2}\, G_\alpha(s-r, y-v))(G_\alpha(t-r, x-v) - G_\alpha(s-r, y-v))(\tilde{\lambda}\cdot\tilde{\mu} - 1)\Big).
\end{aligned}$$

By the semi-group property (A.6), we set $h := t-s$ and change the variables to obtain the following bound:

$$\begin{aligned}
f_1 \geq\, & c\rho^2 \int_0^{\epsilon} dr \Big(\Big(\beta - \sqrt{1-\beta^2}\Big)^2 G_\alpha(2r, 0) + \left(1-\beta^2\right) G_\alpha(2h+2r, 0) \\
& + 2\Big(\beta - \sqrt{1-\beta^2}\Big)\sqrt{1-\beta^2}\, G_\alpha(h+2r, x-y) \\
& + 2\beta\sqrt{1-\beta^2}(G_\alpha(h+2r, x-y) - G_\alpha(2r, 0))(\tilde{\lambda}\cdot\tilde{\mu} - 1)\Big).
\end{aligned}$$

Since by the scaling property of Green kernel (A.2) and Lemma A.1.1(i),

$$\begin{aligned}
G_\alpha(h+2r, x-y) &= (h+2r)^{-1/\alpha} G_\alpha(1, (h+2r)^{-1/\alpha}(x-y)) \\
&\leq (h+2r)^{-1/\alpha} G_\alpha(1, 0) \\
&\leq (2r)^{-1/\alpha} G_\alpha(1, 0) = G_\alpha(2r, 0),
\end{aligned}$$

together with $\tilde{\lambda}\cdot\tilde{\mu} - 1 \leq 0$ it implies that

$$f_1 \geq c\rho^2 \hat{g}_1,$$

where

$$\begin{aligned}
\hat{g}_1 &:= \hat{g}_1(h, \epsilon, \beta, x, y) \\
&= \int_0^{\epsilon} dr \Big(\Big(\beta - \sqrt{1-\beta^2}\Big)^2 G_\alpha(2r, 0) + \left(1-\beta^2\right) G_\alpha(2h+2r, 0) \\
&\quad + 2\Big(\beta - \sqrt{1-\beta^2}\Big)\sqrt{1-\beta^2}\, G_\alpha(h+2r, x-y)\Big).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\hat{g}_1 = &\int_0^{\epsilon} dr \Big(\Big(\beta - \sqrt{1-\beta^2}\Big)^2 r^{-\frac{1}{\alpha}} 2^{-\frac{1}{\alpha}} G_\alpha(1, 0) + \left(1-\beta^2\right)(h+r)^{-\frac{1}{\alpha}} 2^{-\frac{1}{\alpha}} G_\alpha(1, 0) \\
&\quad + 2\Big(\beta - \sqrt{1-\beta^2}\Big)\sqrt{1-\beta^2}\, G_\alpha(h+2r, x-y)\Big).
\end{aligned}$$

On the other hand,

$$f_2 \geq c\rho^2 \int_0^{\epsilon\wedge(t-s)} dr(1-\beta^2)G_\alpha(2r,0) = c\rho^2 \hat{g}_2,$$

where

$$\hat{g}_2 := \int_0^{\epsilon\wedge h} dr(1-\beta^2)G_\alpha(2r,0) = (1-\beta^2)\frac{\alpha}{\alpha-1}2^{-\frac{1}{\alpha}}G_\alpha(1,0)(\epsilon\wedge h)^{\frac{\alpha-1}{\alpha}}.$$

Finally, we conclude that

$$\begin{aligned} f_1+f_2 &\geq c\rho^2(\hat{g}_1+\hat{g}_2) \\ &= c\rho^2\Bigg(\frac{\alpha}{\alpha-1}2^{-\frac{1}{\alpha}}G_\alpha(1,0)\Bigg(\left(\beta-\sqrt{1-\beta^2}\right)^2\epsilon^{\frac{\alpha-1}{\alpha}} \\ &\qquad + \left(1-\beta^2\right)\left((h+\epsilon)^{\frac{\alpha-1}{\alpha}} - h^{\frac{\alpha-1}{\alpha}} + (\epsilon\wedge h)^{\frac{\alpha-1}{\alpha}}\right)\Bigg) \\ &\qquad + 2\left(\beta-\sqrt{1-\beta^2}\right)\sqrt{1-\beta^2}\int_0^\epsilon dr\, G_\alpha(h+2r,x-y)\Bigg). \end{aligned} \tag{2.5.55}$$

Now we consider two different sub-cases.

Sub-case (i). Suppose $\beta-\sqrt{1-\beta^2}\geq 0$, that is, $\beta\geq 2^{-1/2}$. Then

$$\epsilon^{-\frac{\alpha-1}{\alpha}}(\hat{g}_1+\hat{g}_2)\geq \phi_1\left(\beta,\frac{h}{\epsilon}\right),$$

where

$$\phi_1(\beta,z) := \frac{\alpha}{\alpha-1}2^{-\frac{1}{\alpha}}G_\alpha(1,0)\left(\left(\beta-\sqrt{1-\beta^2}\right)^2+\left(1-\beta^2\right)\left((z+1)^{\frac{\alpha-1}{\alpha}}-z^{\frac{\alpha-1}{\alpha}}+(z\wedge 1)^{\frac{\alpha-1}{\alpha}}\right)\right).$$

Clearly,

$$\begin{aligned} \inf_{\beta\geq 2^{-1/2}}\inf_{z\geq 0}\phi_1(\beta,z) &\geq \inf_{\beta\geq 2^{-1/2}}\frac{\alpha}{\alpha-1}2^{-\frac{1}{\alpha}}G_\alpha(1,0)\left(\left(\beta-\sqrt{1-\beta^2}\right)^2+\hat{c}_0\left(1-\beta^2\right)\right) \\ &> \phi_0 > 0, \end{aligned}$$

where the value of $\hat{c}_0$ is specified in (2.5.38). Thus,

$$\inf_{\beta\geq 2^{-1/2},h\geq 0,0<\epsilon\leq\epsilon_0}\epsilon^{-\frac{\alpha-1}{\alpha}}(\hat{g}_1+\hat{g}_2)>0.$$

Sub-case (ii). Now we consider the case where $\beta-\sqrt{1-\beta^2}<0$, that is, $\beta<2^{-1/2}$. In this case, from (2.5.55), we see that

$$\epsilon^{-\frac{\alpha-1}{\alpha}}(\hat{g}_1+\hat{g}_2)\geq \psi_1\left(\beta,\frac{h}{\epsilon}\right),$$

where

$$\psi_1(\beta,z) := \frac{\alpha}{\alpha-1}2^{-\frac{1}{\alpha}}G_\alpha(1,0)\Big(\Big(\beta-\sqrt{1-\beta^2}\Big)^2+\Big(1-\beta^2\Big)\Big((z+1)^{\frac{\alpha-1}{\alpha}}-z^{\frac{\alpha-1}{\alpha}}+(z\wedge 1)^{\frac{\alpha-1}{\alpha}}\Big)$$
$$-2\Big(\sqrt{1-\beta^2}-\beta\Big)\sqrt{1-\beta^2}\Big(\Big(\frac{z}{2}+1\Big)^{\frac{\alpha-1}{\alpha}}-\Big(\frac{z}{2}\Big)^{\frac{\alpha-1}{\alpha}}\Big)\Big).$$

Note that $\psi_1(\beta,z)>0$ if $\beta\neq 0$: this corresponds to the observation made in (2.5.54). Denote $c_\alpha := \frac{\alpha}{\alpha-1}2^{-\frac{1}{\alpha}}G_\alpha(1,0)$. For $z\geq 1$, we have

$$\psi_1(\beta,z) \geq c_\alpha\left[\Big(\beta-\sqrt{1-\beta^2}\Big)^2+\Big(1-\beta^2\Big)-2\Big(\sqrt{1-\beta^2}-\beta\Big)\sqrt{1-\beta^2}\Big(\Big(\frac{3}{2}\Big)^{\frac{\alpha-1}{\alpha}}-\Big(\frac{1}{2}\Big)^{\frac{\alpha-1}{\alpha}}\Big)\right]$$
$$\geq c_\alpha\left(1-\Big(\frac{3}{2}\Big)^{\frac{\alpha-1}{\alpha}}+\Big(\frac{1}{2}\Big)^{\frac{\alpha-1}{\alpha}}\right)\left[\Big(\beta-\sqrt{1-\beta^2}\Big)^2+\Big(1-\beta^2\Big)\right]\geq \bar{c}_0,$$

where in the second inequality we use the elementary inequality $2ab\leq a^2+b^2$. Then

$$\inf_{\beta\in[\beta_0,2^{-1/2}]}\inf_{z\geq 0}\psi_1(\beta,z)\geq \min\left\{\bar{c}_0,\inf_{\beta\in[\beta_0,2^{-1/2}]}\inf_{z\in[0,1]}\psi_1(\beta,z)\right\}$$
$$\geq c_{\beta_0}>0.$$

This concludes the proof of the claim (2.5.51).

Case 2 $t-s>\epsilon$. In accord with (2.5.48), we are interested in

$$\inf_{1\geq\beta\geq\beta_0}(\xi^{i_0})^T\gamma_Z\xi^{i_0} := \min(E_{1,\epsilon},E_{2,\epsilon}),$$

where

$$E_{1,\epsilon} := \inf_{\beta_0\leq\beta\leq\sqrt{1-\epsilon^\eta}}(\xi^{i_0})^T\gamma_Z\xi^{i_0},$$
$$E_{2,\epsilon} := \inf_{\sqrt{1-\epsilon^\eta}\leq\beta\leq 1}(\xi^{i_0})^T\gamma_Z\xi^{i_0}.$$

Clearly,

$$E_{1,\epsilon}\geq\frac{2}{3}f_2-2I_{3,\epsilon}.$$

Since $\beta\leq\sqrt{1-\epsilon^\eta}$ is equivalent to $\sqrt{1-\beta^2}\geq\epsilon^{\eta/2}$, we use hypothesis **P2** to deduce that

$$f_2\geq c\rho^2\epsilon^\eta dr\int_{t-\epsilon}^t\int_{\mathbb{R}}dv\,G_\alpha^2(t-r,x-v)=c'\rho^2\epsilon^{\frac{\alpha-1}{\alpha}+\eta}$$

Therefore,

$$E_{1,\epsilon}\geq c'\rho^2\epsilon^{\frac{\alpha-1}{\alpha}+\eta}-2I_{3,\epsilon},$$

and we have seen that $I_{3,\epsilon}$ has the desirable property $\mathrm{E}[|I_{3,\epsilon}|^q] \le c(q)\epsilon^{\frac{2\alpha-2}{\alpha}q}$.

In order to estimate $E_{2,\epsilon}$, we observe using (2.5.48) that

$$E_{2,\epsilon} \ge \frac{2}{3}\tilde{f}_1 - \tilde{J}_{1,\epsilon} - \tilde{J}_{2,\epsilon} - \tilde{J}_{3,\epsilon} - \tilde{J}_{4,\epsilon},$$

where

$$\begin{aligned}
\tilde{f}_1 &\ge \beta^2 \sum_{k=1}^d \int_{s-\epsilon}^s dr \int_{\mathbb{R}} dv \left(\sum_{i=1}^d \tilde{\lambda}_i \sigma_{ik}(u(r,v)) \right)^2 G_\alpha^2(s-r, y-v),\\
\tilde{J}_{1,\epsilon} &= 8\left(1-\beta^2\right) \sum_{k=1}^d \int_{s-\epsilon}^s dr \int_{\mathbb{R}} dv \left(\sum_{i=1}^d \tilde{\mu}_i \sigma_{ik}(u(r,v)) \right)^2 G_\alpha^2(t-r, x-v),\\
\tilde{J}_{2,\epsilon} &= 8\left(1-\beta^2\right) \sum_{k=1}^d \int_{s-\epsilon}^s dr \int_{\mathbb{R}} dv \left(\sum_{i=1}^d \tilde{\mu}_i \sigma_{ik}(u(r,v)) \right)^2 G_\alpha^2(s-r, y-v),\\
\tilde{J}_{3,\epsilon} &= 8 \sum_{k=1}^d \int_{s-\epsilon}^s dr \int_{\mathbb{R}} dv \left(\sum_{i=1}^d \left(\beta\tilde{\lambda}_i - \tilde{\mu}_i \sqrt{1-\beta^2} \right) a_i(k,r,v,s,y) \right)^2,\\
\tilde{J}_{4,\epsilon} &= 8\left(1-\beta^2\right) \sum_{k=1}^d \int_{s-\epsilon}^s dr \int_{\mathbb{R}} dv \left(\sum_{i=1}^d \tilde{\mu}_i a_i(k,r,v,t,x) \right)^2,
\end{aligned}$$

Because $\beta^2 \ge 1-\epsilon^\eta$ and $\epsilon \le \epsilon_0 \le \frac{1}{2}$, hypothesis **P2** implies that $\tilde{f}_1 \ge c\epsilon^{\frac{\alpha-1}{\alpha}}$. On the other hand, since $1-\beta^2 \le \epsilon^\eta$, we can use hypothesis **P1'** and (A.4) to see that

$$\begin{aligned}
\mathrm{E}\left[\left|\tilde{J}_{1,\epsilon}\right|^q\right] &\le c(q)\epsilon^{q\eta} \left(\int_{s-\epsilon}^s dr \int_{\mathbb{R}} dv\, G_\alpha^2(t-r, x-v) \right)^q\\
&= c(q)\epsilon^{q\eta}((t-s+\epsilon)^{\frac{\alpha-1}{\alpha}} - (t-s)^{\frac{\alpha-1}{\alpha}})^q\\
&\le c(q)\epsilon^{q\eta}((\epsilon+\epsilon)^{\frac{\alpha-1}{\alpha}} - \epsilon^{\frac{\alpha-1}{\alpha}})^q = c(q)\epsilon^{(\frac{\alpha-1}{\alpha}+\eta)q},
\end{aligned}$$

where the second inequality is due to the fact that the function $x \mapsto (x+\epsilon)^{\frac{\alpha-1}{\alpha}} - x^{\frac{\alpha-1}{\alpha}}$ is decreasing on $[0,\infty[$. Similarly, we have $\mathrm{E}[|\tilde{J}_{2,\epsilon}|^q] \le c(q)\epsilon^{(\frac{\alpha-1}{\alpha}+\eta)q}$. The term $\tilde{J}_{3,\epsilon}$ is equal to $8I_{1,\epsilon}$, so $\mathrm{E}[|\tilde{J}_{3,\epsilon}|^q] \le c\epsilon^{\frac{2\alpha-2}{\alpha}q}$, and $\tilde{J}_{4,\epsilon}$ is similar to $B_1^{(2)}$ from (2.5.36), so we find using (2.5.42) that

$$\mathrm{E}\left[\left|\tilde{J}_{4,\epsilon}\right|^q\right] \le c\epsilon^{q\eta}(t-s+\epsilon)^{\frac{\alpha-1}{\alpha}q}\epsilon^{\frac{\alpha-1}{\alpha}q} \le c\epsilon^{(\frac{\alpha-1}{\alpha}+\eta)q}.$$

We conclude that when $t-s>\epsilon$, then $E_{2,\epsilon} \ge c\epsilon^{\frac{\alpha-1}{\alpha}} - \tilde{J}_\epsilon$, where $\mathrm{E}[|\tilde{J}_\epsilon|^q] \le c(q)\epsilon^{(\frac{\alpha-1}{\alpha}+\eta)q}$. Therefore, when $t-s>\epsilon$,

$$1_{\{\beta_{i_0}\ge\beta_0\}}(\xi^{i_0})^T \gamma_Z \xi^{i_0} \ge 1_{\{\beta_{i_0}\ge\beta_0\}} \min\left(c\rho^2\epsilon^{\frac{\alpha-1}{\alpha}+\eta} - 2I_{3,\epsilon}, c\epsilon^{\frac{\alpha-1}{\alpha}} - \tilde{J}_\epsilon \right).$$

Putting together the results of Case 1 and Case 2, we see that for $0 < \epsilon \le \epsilon_0$,

$$1_{\{\beta_{i_0} \ge \beta_0\}} (\xi^{i_0})^T \gamma_Z \xi^{i_0} \ge 1_{\{\beta_{i_0} \ge \beta_0\}} Z,$$

where

$$Z = \min\left(c\rho^2 \epsilon^{\frac{\alpha-1}{\alpha}+\eta} - 2I_{3,\epsilon}, c\epsilon^{\frac{\alpha-1}{\alpha}} - 2I_\epsilon \mathbf{1}_{\{\epsilon \ge t-s\}} - \tilde{J}_\epsilon \mathbf{1}_{\{\epsilon < t-s\}} \right).$$

Note that all the constants are independent of i_0. Taking into account the bounds on moments of $I_{3,\epsilon}$, I_ϵ and $\tilde{J}_\epsilon$, and then using Proposition A.2.1, we deduce that for all $p \ge 1$, there is $C > 0$ such that

$$\mathrm{E}\left[1_{\{\beta_{i_0} \ge \beta_0\}} \left((\xi^{i_0})^T \gamma_Z \xi^{i_0} \right)^{-p} \right] \le \mathrm{E}\left[1_{\{\beta_{i_0} \ge \beta_0\}} Z^{-p} \right] \le \mathrm{E}[Z^{-p}] \le C.$$

Since this applies to any $p \ge 1$, we can use Hölder's inequality to deduce (2.5.26). This proves Proposition 2.5.11. □

2.5.3 Proof of Theorem 2.1.1(c) and Remark 2.1.2(c')

Fix two compact intervals I and J as in Theorem 2.1.1. Let $(s, y), (t, x) \in I \times J$, $s \le t$, $(s, y) \ne (t, x)$, and $z_1, z_2 \in \mathbb{R}^d$. Let Z be as in (2.5.20) and let p_Z be the density of Z. Then

$$p_{s,y;t,x}(z_1, z_2) = p_Z(z_1, z_2 - z_1).$$

Apply Corollary 1.5.3 with $\sigma = \{i \in \{1, \dots, d\} : z_2^i - z_1^i \ge 0\}$ and Hölder's inequality to see that

$$\begin{aligned} p_Z(z_1, z_2 - z_1) \le & \prod_{i=1}^d \left(\mathrm{P}\left\{ |u_i(t,x) - u_i(s,y)| > |z_1^i - z_2^i| \right\} \right)^{\frac{1}{2d}} \\ & \times \| H_{(1,\dots,2d)(Z,1)} \|_{0,2}. \end{aligned} \tag{2.5.56}$$

Therefore, in order to prove the desired results of Theorem 2.1.1(c) and Remark 2.1.2(c'), it suffices to prove that:

$$\| H_{(1,\dots,2d)(Z,1)} \|_{0,2} \le c_T (|t-s|^{\frac{\alpha-1}{\alpha}} + |x-y|^{\alpha-1})^{-d/2}, \tag{2.5.57}$$

and

$$\prod_{i=1}^d \left(\mathrm{P}\left\{ |u_i(t,x) - u_i(s,y)| > |z_1^i - z_2^i| \right\} \right)^{\frac{1}{2d}} \le c \exp\left(- \frac{\|z_1 - z_2\|^2}{c_T (|t-s|^{\frac{\alpha-1}{\alpha}} + |x-y|^{\alpha-1})} \right) \tag{2.5.58}$$

under the hypothesis **P1**, and

$$\prod_{i=1}^d \left(\mathrm{P}\left\{ |u_i(t,x) - u_i(s,y)| > |z_1^i - z_2^i| \right\} \right)^{\frac{1}{2d}} \le c \left[\frac{|t-s|^{\frac{\alpha-1}{\alpha}} + |x-y|^{\alpha-1}}{\|z_1 - z_2\|^2} \wedge 1 \right]^{p/(4d)} \tag{2.5.59}$$

under the hypothesis **P1'**.

The proof of (2.5.58) under the hypothesis **P1** is essentially the same as that of [26, (6.2)], with Δ replaced by Δ_α, by using Lemma 2.5.1, the exponential martingale inequality [64, (A.5)] and Girsanov theorem. As for the proof of (2.5.59) under the hypothesis **P1'**, it is analogous to that of [27, Theorem 1.6(b)]. We first observe that (2.5.59) holds when $\|z_1 - z_2\| = 0$, since

$$\frac{|t-s|^{\frac{\alpha-1}{\alpha}} + |x-y|^{\alpha-1}}{\|z_1 - z_2\|^2} \wedge 1 = 1,$$

for $(t,x) \neq (s,y)$. Assume now that $\|z_1 - z_2\| \neq 0$. Then there is $i \in \{1,\dots,d\}$, and we may as well assume that $i = 1$, such that $0 < |z_1^1 - z_2^1| = \max_{i=1,\dots,d} |z_1^i - z_2^i|$. Then

$$\prod_{i=1}^{d} \left(\mathrm{P}\left\{ |u_i(t,x) - u_i(s,y)| > |z_1^i - z_2^i| \right\} \right)^{\frac{1}{2d}} \leq \left(\mathrm{P}\left\{ |u_1(t,x) - u_1(s,y)| > |z_1^1 - z_2^1| \right\} \right)^{\frac{1}{2d}}.$$

Using Chebyshev's inequality and (2.1.6), we see that this is bounded above by

$$c \left[\frac{|t-s|^{\frac{\alpha-1}{\alpha}} + |x-y|^{\alpha-1}}{|z_1^1 - z_2^1|^2} \wedge 1 \right]^{p/(4d)} \leq \tilde{c} \left[\frac{|t-s|^{\frac{\alpha-1}{\alpha}} + |x-y|^{\alpha-1}}{\|z_1 - z_2\|^2} \wedge 1 \right]^{p/(4d)}.$$

We turn to proving (2.5.57), which requires the following estimate on inverse of the matrix γ_Z.

Theorem 2.5.13. *Fix $T > 0$. Assume **P1'** and **P2**. Let I and J be compact intervals as in Theorem 2.1.1. For any $(s,y),(t,x) \in I \times J$, $s \leq t$, $(s,y) \neq (t,x)$, $k \geq 0$ and $p > 1$,*

$$E\left[\|(\gamma_Z)^{-1}_{m,l}\|_{k,p} \right] \leq \begin{cases} c_{k,p,T} & \text{if } (m,l) \in (\mathbf{1}), \\ c_{k,p,T}(|t-s|^{\frac{\alpha-1}{\alpha}} + |x-y|^{\alpha-1})^{-\frac{1}{2}} & \text{if } (m,l) \in (\mathbf{2}) \text{ or } (\mathbf{3}), \\ c_{k,p,T}(|t-s|^{\frac{\alpha-1}{\alpha}} + |x-y|^{\alpha-1})^{-1} & \text{if } (m,l) \in (\mathbf{4}). \end{cases} \tag{2.5.60}$$

Proof. As in the proof of [26, Theorem 6.3], we shall use Propositions 2.5.6, 2.5.7 and 2.5.8.

When $k = 0$, the result is a consequence of the estimates of Propositions 2.5.6 and 2.5.8, using the fact that the inverse of a matrix is the inverse of its determinant multiplied by its cofactor matrix.

For $k \geq 1$, we proceed recursively as in the proof of [26, Theorem 6.3], using Proposition 2.5.7 instead of Proposition 2.5.6. □

Proof of (2.5.57). The proof is similar to that of [26, (6.3)] by using the continuity of the Skorohod integral δ (see [64, Proposition 3.2.1] and [65, (1.11) and p.131]) and Hölder's inequality for Malliavin norms (see [82, Proposition 1.10, p.50]); the main difference is that Δ is replaced by Δ_α. Comparing with the estimate in [26, (6.3)], we are able to remove the extra exponent η because of the correct estimate on the inverse of the matrix γ_Z in Theorem 2.5.13. □

Remark 2.5.14. *We conclude this chapter by remarking that* (2.5.57) *is also valid for the solutions of stochastic heat equations with Neumann or Dirichlet boundary conditions, since the result of Theorem 2.5.13 is still true in that case by applying Proposition 2.5.8; see Remark 2.5.9.*

3 Hitting probability for stochastic heat equations with additive noise

In this chapter, we study the hitting probabilities of the solution to a system of linear stochastic heat equations with Dirichlet boundary conditions. We will show that for any bounded Borel set with positive $d-6$-dimensional capacity, the solution visits this set almost surely. The strong Markov property and the recurrence property of the solution considered as a one-parameter process indexed by time are used.

3.1 Introduction and main result

In this chapter, we consider a special case of equations (1.2.1) with $\sigma \equiv \mathrm{Id}$, $b \equiv 0$ and $U = [0,1]$. That is, we consider the following system of linear stochastic partial differential equations:

$$\frac{\partial u_i}{\partial t}(t,x) = \frac{\partial^2 u_i}{\partial x^2}(t,x) + \dot{W}^i(t,x), \tag{3.1.1}$$

for $1 \le i \le d$, $t \in [0,\infty[$ and $x \in [0,1]$, where $u := (u_1, \cdots, u_d)$, with initial conditions $u(0,x) = u_0(x)$ for all $x \in [0,1]$ satisfying $u_0(\cdot) \in C([0,1],\mathbb{R}^d)$, and Dirichlet boundary conditions

$$u(t,0) = u(t,1) = 0, \quad \text{for all } t \ge 0.$$

We assume there exist d independent copies of Brownian bridge $\{B_0^i(x) : 0 \le x \le 1\}$ for $1 \le i \le d$, which are independent of the space-time white noise $\dot{W}$. Set

$$\tilde{u}_0 = \frac{\sqrt{2}}{2}(B_0^1, \cdots, B_0^d).$$

For $t \ge 0$, let $\mathscr{F}_t = \sigma\{W(s,x), \tilde{u}_0(x), s \in [0,t], x \in [0,1]\} \vee \mathscr{N}$, where $\mathscr{N}$ is the σ-field generated by P-null sets. . We say that u is a solution of (3.1.1) if u is adapted to $(\mathscr{F}_t)_{t\ge 0}$ and if for $i \in \{1, \cdots, d\}$, $t \in]0,\infty[$ and $x \in [0,1]$,

$$u_i(t,x) = \int_0^t \int_0^1 G(t-r,x,v) W^i(dr,dv) + \int_0^1 G(t,x,v) u_0^i(v) dv, \tag{3.1.2}$$

where the Green kernel $G(t,x,y)$ is given in (1.3.6) and also has the following equivalent expression

$$G(t,x,y) = \sum_{k=1}^{\infty} e^{-\pi^2 k^2 t}\phi_k(x)\phi_k(y) \tag{3.1.3}$$

with $\phi_k(x) := \sqrt{2}\sin(k\pi x)$; see, for example, [6] and [81].

For the hitting probabilities of the solution $\{u(t,x)\}_{(t,x)\in[0,\infty[\times[0,1]}$ with vanishing initial conditions, the upper and lower bounds were established by Dalang, Khoshnevisan and Nualart [25], in terms of respectively Hausdorff measure and Newtonian capacity. There, they show that there exists $c>0$ depending on M, I, J with $M>0$, and $I\subset]0,\infty[$, $J\subset]0,1[$ be non-trivial compact intervals, such that for all Borel sets $A\subseteq[-M,M]^d$,

$$c^{-1}\mathrm{Cap}_{d-6}(A) \le \mathrm{P}\{u(I\times J)\cap A \neq \emptyset\} \le c\mathscr{H}_{d-6}(A).$$

Our goal is to establish the following *probability one* result.

Theorem 3.1.1. *For any bounded Borel set $A\subseteq\mathbb{R}^d$ with positive $(d-6)$-dimensional capacity, the random field $\{u(t,x)\}_{(t,x)\in[0,\infty[\times[0,1]}$, starting with any initial value $u_0(\cdot)\in C([0,1],\mathbb{R}^d)$, visits this set A almost surely.*

We denote by $E := \{\varphi(\cdot)\in C([0,1],\mathbb{R}^d) : \varphi(0)=\varphi(1)=0\}$ equipped with the norm

$$\|\varphi(\cdot)\|_\infty := \sup_{0\le v\le 1}\sup_{1\le i\le d}|\varphi^i(v)|.$$

Denote the metric on the space E by

$$\rho(a,b) := \|a-b\|_\infty, \text{ for } a,b\in E. \tag{3.1.4}$$

Without loss of generality, we assume $\|u_0(\cdot)\|_\infty \le N$ and $A\subseteq[-N,N]^d$ for some $N>0$. As a two-parameter random field, some estimates on the probability density functions of the solution $\{u(t,x)\}_{(t,x)\in[0,\infty[\times[0,1]}$ were given in [25], from which they derived the upper and lower bounds on hitting probabilities. On the other hand, if we view the solution parameterized only by time and taking values in E, it will possess the strong Markov property. The definition of transition semigroup and construction of canonical Markov systems will be presented in Section 3.2. In Section 3.3, we show that the solution, as a Markov process, converges to an invariant distribution and therefore has a recurrence property. Intuitively, the recurrence property implies that the solution visits infinite many times A with a positive probability. In Section 3.4, we show that the lower bound on hitting probabilities still holds if the solution starts from a non-vanishing initial value, which extends the corresponding results in [25]. We finally give the proof of Theorem 3.1.1 in Section 3.5.

3.2 Strong Markov property

The strong Markov property and the invariant distribution of the solution to (3.1.1) are well known facts; see, e.g., [81] and [32]. We still give a self-contained proof for reader's convenience. We refer to [8] for the terminology of Markov processes.

As a two-parameter Gaussian process, the trajectories $(t,x) \mapsto u(t,x)$ are jointly continuous. Hence $t \mapsto u(t,\cdot)$ is continuous in E. We denote by $u_{u_0(\cdot)}(t,\cdot)$ to specify that the solution starts from $u_0(\cdot)$. And denote by $\mathscr{B}(E), \mathscr{B}_b(E)$ and $C_b(E)$ the Borel σ-field, the set of bounded Borel measurable functions and the set of bounded continuous functions on E, respectively. In what follows, we will introduce the transition semigroup associated with the process and construct the Markov system associated with the transition semigroup.

For $t \geq 0$, $u_0(\cdot) \in E$ and $\Gamma \in \mathscr{B}(E)$, we define

$$P_t(u_0(\cdot),\Gamma) := \mathrm{P}\{u_{u_0(\cdot)}(t,\cdot) \in \Gamma\}.$$

It is obvious to see that $\Gamma \mapsto P_t(u_0(\cdot),\Gamma)$ is a probability measure on $\mathscr{B}(E)$ and $P_0(u_0(\cdot),\Gamma) = 1_\Gamma(u_0(\cdot))$. Then, for $f \in \mathscr{B}_b(E)$,

$$P_t f(u_0(\cdot)) := \int_E f(u) P_t(u_0(\cdot), du) = \mathrm{E}[f(u_{u_0(\cdot)}(t,\cdot))].$$

Proposition 3.2.1. *For $t \geq 0$ and $f \in C_b(E)$, $u_0(\cdot) \mapsto P_t f(u_0(\cdot))$ is continuous on E.*

Proof. Let $(u_{0n}(\cdot))_{n\geq 1}$ be a sequence converging to $u_0(\cdot)$ in E. From equation (3.1.2) we have

$$\begin{aligned}\| u_{u_{0n}(\cdot)}(t,\cdot) - u_{u_0(\cdot)}(t,\cdot)\|_\infty &= \sup_{0\leq x\leq 1} \sup_{1\leq i\leq d} \left| \int_0^1 G(t,x,v)(u_{0n}^i(v) - u_0^i(v))dv \right| \\ &\leq \sup_{0\leq v\leq 1} \sup_{1\leq i\leq d} |u_{0n}^i(v) - u_0^i(v)| = \| u_{0n}(\cdot) - u_0(\cdot)\|_\infty,\end{aligned}$$

which implies that $\lim_{n\to\infty} P_t f(u_{0n}(\cdot)) = P_t f(u_0(\cdot))$ by the dominated convergence theorem. □

The property of $(P_t)_{t\geq 0}$ in Proposition 3.2.1 is a variant on the Feller property; see [8, p.161].

The indicator function $u_0(\cdot) \mapsto 1_\Gamma(u_0(\cdot))$ of an open set Γ can be approximated by bounded continuous function. To see this, we define a sequence of continuous functions on E by

$$f_n(a) := \min(1, n\rho(a,\Gamma^c)).$$

Then it is clear that

$$\lim_{n\to\infty} f_n(a) = 1_\Gamma(a), \quad \text{for all } a \in E,$$

which implies that

$$\lim_{n\to\infty} P_t f_n(u_0(\cdot)) = P_t(u_0(\cdot),\Gamma), \quad \text{for all } a \in E$$

by the dominated convergence theorem. Hence, we see that $u_0(\cdot) \mapsto P_t(u_0(\cdot),\Gamma)$ is measurable. Furthermore, by the monotone class theorem, $u_0(\cdot) \mapsto P_t(u_0(\cdot),\Gamma)$ is measurable for all $\Gamma \in \mathscr{B}(E)$, which implies that $(u_0(\cdot),\Gamma) \mapsto P_t(u_0(\cdot),\Gamma)$ is a transition kernel on $(E,\mathscr{B}(E))$.

We next show that $(P_t)_{t\geq 0}$ satisfies the Markovian transition semigroup property.

Proposition 3.2.2 (Markov property)**.** *For $s,t \geq 0$, $u_0(\cdot) \in E$ and $f \in \mathscr{B}_b(E)$, we have*

$$E[f(u_{u_0(\cdot)}(t+s,\cdot))|\mathscr{F}_s] = P_t f(u_{u_0(\cdot)}(s,\cdot)). \tag{3.2.1}$$

Proof. We verify the equality (3.2.1) by the following calculations. First,

$$\begin{aligned}
&\mathrm{E}[f(u_{u_0(\cdot)}(t+s,\cdot))|\mathscr{F}_s](\omega)\\
&= \mathrm{E}\left[f\left(\int_0^{t+s}\int_0^1 G(t+s-r,\cdot,v)W(dr,dv) + \int_0^1 G(t+s,\cdot,v)u_0(v)dv\right)\Big|\mathscr{F}_s\right](\omega)\\
&= \mathrm{E}\left[f\left(\int_s^{t+s}\int_0^1 G(t+s-r,\cdot,v)W(dr,dv) + \int_0^s\int_0^1 G(t+s-r,\cdot,v)W(dr,dv)\right.\right.\\
&\qquad\left.\left.+\int_0^1 G(t+s,\cdot,v)u_0(v)dv\right)\Big|\mathscr{F}_s\right](\omega).
\end{aligned}$$

Since the random variable $\int_0^s\int_0^1 G(t+s-r,\cdot,v)W(dr,dv)$ is measurable with $\mathscr{F}_s$ while $\int_s^{t+s}\int_0^1 G(t+s-r,\cdot,v)W(dr,dv)$ independent of $\mathscr{F}_s$, this is equal to

$$\begin{aligned}
&\int_\Omega \mathrm{P}(d\tilde\omega) f\left(\int_s^{t+s}\int_0^1 G(t+s-r,\cdot,v)W(dr,dv)(\tilde\omega)\right.\\
&\qquad\qquad\left.+\int_0^s\int_0^1 G(t+s-r,\cdot,v)W(dr,dv)(\omega) + \int_0^1 G(t+s,\cdot,v)u_0(v)dv\right)\\
&= \int_\Omega \mathrm{P}(d\tilde\omega) f\left(\int_0^t\int_0^1 G(t-r,\cdot,v)W(dr,dv)(\tilde\omega)\right.\\
&\qquad\qquad\left.+\int_0^s\int_0^1 G(t+s-r,\cdot,v)W(dr,dv)(\omega) + \int_0^1 G(t+s,\cdot,v)u_0(v)dv\right),
\end{aligned}$$

where the notation $\mathrm{P}(d\tilde\omega)$ means we are taking the expectation of the random variable $\tilde\omega \mapsto \int_s^{t+s}\int_0^1 G(t+s-r,\cdot,v)W(dr,dv)(\tilde\omega)$, and in the equality, we use the fact that the random variable $\int_s^{t+s}\int_0^1 G(t+s-r,\cdot,v)W(dr,dv)(\tilde\omega)$ has the same law as $\int_0^t\int_0^1 G(t-r,\cdot,v)W(dr,dv)(\tilde\omega)$.

Furthermore, by the semi-group property of G, this is equal to

$$\begin{aligned}
&\int_\Omega \mathrm{P}(d\tilde{\omega}) f\Big(\int_0^t \int_0^1 G(t-r,\cdot,v) W(dr,dv)(\tilde{\omega}) \\
&\qquad + \int_0^s \int_0^1 \Big(\int_0^1 G(t,\cdot,z) G(s-r,z,v)\,dz\Big) W(dr,dv)(\omega) \\
&\qquad + \int_0^1 \Big(\int_0^1 G(t,\cdot,z) G(s,z,v)\,dz\Big) u_0(v)\,dv\Big) \\
&= \int_\Omega \mathrm{P}(d\tilde{\omega}) f\Big(\int_0^t \int_0^1 G(t-r,\cdot,v) W(dr,dv)(\tilde{\omega}) \\
&\qquad + \int_0^1 dz\, G(t,\cdot,z)\Big(\int_0^s \int_0^1 G(s-r,z,v) W(dr,dv)(\omega) + \int_0^1 G(s,z,v) u_0(v)\,dv\Big)\Big) \\
&= \int_\Omega \mathrm{P}(d\tilde{\omega}) f\Big(\int_0^t \int_0^1 G(t-r,\cdot,v) W(dr,dv)(\tilde{\omega}) + \int_0^1 G(t,\cdot,z) u_{u_0(\cdot)}(s,z)(\omega)\,dz\Big) \\
&= P_t f(u_{u_0(\cdot)}(s,\cdot)(\omega)),
\end{aligned}$$

where the first equality holds by the stochastic Fubini theorem (see [24, Chapter 1, Theorem 5.30] or [81, Theorem 2.6]) since the condition of the stochastic Fubini theorem can be verified:

$$\begin{aligned}
&\iiiint\limits_{[0,1]\times[0,1]\times[0,s]\times[0,1]} G(t,\cdot,z)G(s-r,z,v)G(t,\cdot,z)G(s-r,z,\tilde{v})\delta_{\{v=\tilde{v}\}}\,dv\,d\tilde{v}\,dr\,dz \\
&= \iiint\limits_{[0,1]\times[0,s]\times[0,1]} G^2(t,\cdot,z)G^2(s-r,z,v)\,dv\,dr\,dz \\
&= \iint\limits_{[0,s]\times[0,1]} G^2(t,\cdot,z)G(2(s-r),z,z)\,dr\,dz \\
&\le C \iint\limits_{[0,s]\times[0,1]} G^2(t,\cdot,z)\frac{1}{\sqrt{2(s-r)}}\,dr\,dz \\
&\le C\sqrt{\frac{s}{t}} < \infty.
\end{aligned}$$

□

From (3.2.1), it is easy to derive the Markovian semigroup property:

$$\begin{aligned}
P_{t+s} f(u_0(\cdot)) &= \mathrm{E}[f(u_{u_0(\cdot)}(t+s,\cdot))] \\
&= \mathrm{E}[\mathrm{E}[f(u_{u_0(\cdot)}(t+s,\cdot))|\mathscr{F}_s]] \\
&= \mathrm{E}[P_t f(u_{u_0(\cdot)}(s,\cdot))] = P_s P_t f(u_0(\cdot)).
\end{aligned}$$

Let $\widetilde{\Omega} := C([0,\infty[,E)$ be the space of continuous functions from $[0,\infty[$ to E. For a generic element $\tilde{\omega} \in \widetilde{\Omega}$, we write $\tilde{\omega}(t,\cdot)$ to indicate the value at t, and the second variable appears since $\tilde{\omega}(t,\cdot) \in C([0,1],\mathbb{R}^d)$. Set $\tilde{u}(t,\cdot)(\tilde{\omega}) := \tilde{\omega}(t,\cdot)$, which means that $\tilde{u}(t,x)(\tilde{\omega}) = \tilde{\omega}(t,x)$, for all $x \in [0,1]$. Define

$$\widetilde{\mathscr{F}}_t^0 := \sigma\{\tilde{u}(s,\cdot) : s \le t\} \quad \text{and} \quad \widetilde{\mathscr{F}}_\infty^0 := \bigvee_{t \ge 0} \widetilde{\mathscr{F}}_t^0.$$

We define the law of the process $u = \{u_{u_0(\cdot)}(t,\cdot) : t \ge 0\}$ on $(\widetilde{\Omega}, \widetilde{\mathscr{F}}_\infty^0)$ by

$$\mathrm{P}^{u_0(\cdot)}(\tilde{u} \in A) := \mathrm{P}\{u_{u_0(\cdot)} \in A\}, \qquad \text{for } u_0(\cdot) \in E,\ A \in \widetilde{\mathscr{F}}_\infty^0, \tag{3.2.2}$$

which is determined uniquely by

$$\mathrm{P}^{u_0(\cdot)}\{\tilde{u}(t_1,\cdot) \in B_1, \dots, \tilde{u}(t_n,\cdot) \in B_n\} = \mathrm{P}\{u_{u_0(\cdot)}(t_1,\cdot) \in B_1, \dots, u_{u_0(\cdot)}(t_n,\cdot) \in B_n\} \tag{3.2.3}$$

for any $n \ge 1$, $B_1, \dots, B_n \in \mathscr{B}(E)$, and $t_1, \dots, t_n \ge 0$. We denote by $\mathrm{E}^{u_0(\cdot)}$ the corresponding expectation with respect to the probability $\mathrm{P}^{u_0(\cdot)}$. From (3.2.3), we know that $u_0(\cdot) \mapsto \mathrm{P}^{u_0(\cdot)}(A)$ is measurable for $A \in \widetilde{\mathscr{F}}_\infty^0$. Together with (3.2.1), we have, for $f \in \mathscr{B}_b(E)$,

$$\mathrm{E}^{u_0(\cdot)}[f(\tilde{u}(t+s,\cdot)) | \widetilde{\mathscr{F}}_s^0] = \mathrm{E}^{\tilde{u}(s,\cdot)}[f(\tilde{u}(t,\cdot))] = P_t f(\tilde{u}(s,\cdot)), \quad \mathrm{P}^{u_0(\cdot)} \text{ a.s.} \tag{3.2.4}$$

Let $\mathscr{N}$ be the collection of sets that are $\mathrm{P}^{u_0(\cdot)}$-null for every $u_0(\cdot) \in E$. Define

$$\widetilde{\mathscr{F}}_t := \sigma\{\widetilde{\mathscr{F}}_t^0 \cup \mathscr{N}\} \quad \text{and} \quad \widetilde{\mathscr{F}}_\infty := \bigvee_{t \ge 0} \widetilde{\mathscr{F}}_t.$$

Since the process $\{\tilde{u}(t,\cdot) : t \ge 0\}$ has the Markov property (3.2.4) and the semigroup $(P_t)_{t \ge 0}$ has the Feller property (i.e., Proposition 3.2.1), by Proposition 20.7 of [8], we know that the filtration $(\widetilde{\mathscr{F}}_t)_{t \ge 0}$ is right continuous. Furthermore we have the following strong Markov property.

Theorem 3.2.3 (Strong Markov property). *Suppose T is a finite stopping time with respect to $(\widetilde{\mathscr{F}}_t)_{t \ge 0}$ and Y is bounded and measurable with respect to $\widetilde{\mathscr{F}}_\infty$. Then*

$$E^{u_0(\cdot)}[Y \circ \theta_T | \widetilde{\mathscr{F}}_T] = E^{\tilde{u}_{u_0(\cdot)}(T,\cdot)}[Y], \tag{3.2.5}$$

where $(\theta_t)_{t \ge 0}$ is the shift operator defined by

$$\theta_t \tilde{\omega}(s,x) := \tilde{\omega}(t+s,x), \qquad \text{for } \tilde{\omega} \in \widetilde{\Omega},\ (s,x) \in [0,\infty[\times [0,1].$$

Proof. The proof is similar to that of Theorem 20.9 in [8, p.164], since only the Feller property in Proposition 3.2.1 and Markov property in (3.2.4) are needed. □

3.3 Invariant distribution

For any $t \geq 0$, and a probability measure μ on $\mathscr{B}(E)$, we set

$$P_t^* \mu(\Gamma) := \int_E P_t(u, \Gamma)\mu(du), \quad \Gamma \in \mathscr{B}(E).$$

We call μ an invariant distribution with respect to $(P_t)_{t\geq 0}$ if

$$P_t^* \mu = \mu \qquad \text{for each } t \geq 0. \tag{3.3.1}$$

Recall that the covariance of each component of the process $\{\tilde{u}_0(x) : 0 \leq x \leq 1\}$ is given by

$$\mathrm{E}[\tilde{u}_0^i(x)\tilde{u}_0^i(y)]) = \frac{1}{2}(x \wedge y - xy). \tag{3.3.2}$$

We denote the law of $\tilde{u}_0$ on E by μ.

Proposition 3.3.1. *μ is an invariant distribution with respect to the transition semigroup $(P_t)_{t\geq 0}$.*

Proof. Fix $t > 0$. We assume that $u(t,\cdot)$ starts from $\tilde{u}_0$. Then the law of $u(t,\cdot)$ on E is $P_t^*\mu$, since for $f \in \mathscr{B}_b(E)$,

$$\mathrm{E}[f(u_{\tilde{u}_0}(t,\cdot))] = \int_E \mathrm{E}[f(u_{u_0(\cdot)}(t,\cdot))]\mu(d(u_0(\cdot))) = \int_E P_t f(u_0(\cdot))\mu(d(u_0(\cdot))).$$

Clearly, the process $\{u(t,x) : 0 \leq x \leq 1\}$ is a continuous Gaussian process. So we only need to check that the component process $\{u^i(t,x) : 0 \leq x \leq 1\}$ also has the covariance given by (3.3.2).

We denote by

$$v^i(t,x) := \int_0^t \int_0^1 G(t-r,x,v)W^i(dr,dv) = \sum_{k=1}^{\infty} \phi_k(x) A_t^k, \tag{3.3.3}$$

where, from (3.1.3),

$$A_t^k := \int_0^t \int_0^1 e^{-\pi^2 k^2 (t-s)} \phi_k(v) W^i(ds,dv), \tag{3.3.4}$$

with variance

$$\mathrm{Var}(A_t^k) = \int_0^t ds \int_0^1 dv\, e^{-2\pi^2 k^2 s} \phi_k^2(v) = (1 - e^{-2\pi^2 k^2 t})/(2\pi^2 k^2).$$

Let

$$\mu^i(t,x) := \int_0^1 G(t,x,v)\tilde{u}_0^i(v)dv = \sum_{k=1}^{\infty} e^{-\pi^2 k^2 t} \phi_k(x) C^k, \tag{3.3.5}$$

where

$$C^k := \int_0^1 \phi_k(v)\tilde{u}_0^i(v)dv. \tag{3.3.6}$$

Then we have

$$\begin{aligned}
\mathrm{E}[v^i(t,x)v^i(t,y)] &= \mathrm{E}\big[\sum_{k=1}^{\infty}\phi_k(x)A_t^k\sum_{n=1}^{\infty}\phi_n(y)A_t^n\big] \\
&= \sum_{k=1}^{\infty}\frac{1-e^{-2\pi^2k^2t}}{2\pi^2k^2}\phi_k(x)\phi_k(y).
\end{aligned} \tag{3.3.7}$$

And

$$\begin{aligned}
\mathrm{E}[\mu^i(t,x)\mu^i(t,y)] &= \mathrm{E}\big[\sum_{k=1}^{\infty}e^{-\pi^2k^2t}\phi_k(x)C^k\sum_{n=1}^{\infty}e^{-\pi^2n^2t}\phi_n(y)C^n\big] \\
&= \sum_{k,n=1}^{\infty}e^{-\pi^2(k^2+n^2)t}\phi_k(x)\phi_n(y)\mathrm{E}[C^kC^n] \\
&= \sum_{k=1}^{\infty}\frac{e^{-2\pi^2k^2t}}{2\pi^2k^2}\phi_k(x)\phi_k(y),
\end{aligned} \tag{3.3.8}$$

where the last equality is based on the following identity (as a consequence of [8, (6.1)]):

$$\int_0^1\int_0^1 \sin(k\pi z)\sin(n\pi v)(z\wedge v - zv)dzdv = \begin{cases} 0 & k\neq n, \\ \frac{1}{2\pi^2k^2} & k=n. \end{cases}$$

Since $u^i(t,x) = v^i(t,x)+\mu^i(t,x)$, $v^i(t,x)$ and $\mu^i(t,x)$ are independent mutually, from equalities (3.3.7) and (3.3.8), we obtain that

$$\mathrm{E}[u^i(t,x)u^i(t,y)] = \sum_{k=1}^{\infty}\frac{1}{2\pi^2k^2}\phi_k(x)\phi_k(y) = \frac{1}{2}(x\wedge y - xy),$$

where the last equality follows from [8, (6.1)]. The proof is complete. □

We give some estimates on the moments of the increments of the solution. By [60, Lemma A.1], there exists $c>0$ such that, for all $x,y\in[0,1]$, $t\geq 0$,

$$\begin{aligned}
\mathrm{E}[(v^i(t,x)-v^i(t,y))^2] &= \int_0^t\int_0^1 (G(t-r,x,v)-G(t-r,y,v))^2dvdr \\
&\leq c|x-y|.
\end{aligned} \tag{3.3.9}$$

And for all $t \geq 1$, $x, y \in [0,1]$

$$
\begin{aligned}
|\mu^i(t,x) - \mu^i(t,y)| &= \left| \int_0^1 \sum_{k=1}^{\infty} e^{-\pi^2 k^2 t} (\phi_k(x) - \phi_k(y)) \phi_k(v) u_0^i(v) dv \right| \\
&\leq 2N \sum_{k=1}^{\infty} e^{-\pi^2 k^2 t} |\sin(k\pi x) - \sin(k\pi y)| \\
&\leq 2N\pi \sum_{k=1}^{\infty} k e^{-\pi^2 k^2} |x-y| \leq c|x-y|. \qquad (3.3.10)
\end{aligned}
$$

Lemma 3.3.2. *For all $t \geq 1$, $p \geq 1$ and δ sufficiently small, there exists $c_p > 0$ such that*

$$
E\left[\sup_{|x-y|^{\frac{1}{2}} \leq \delta} |u^i(t,x) - u^i(t,y)|^p \right] \leq c_p \delta^p \ln^p(1 + \frac{1}{\delta^4}). \qquad (3.3.11)
$$

Proof. The proof is similar to that of [25, Lemma 4.5] by applying Proposition A.1 in [25]. We define

$$
\begin{aligned}
&S := [0,1], \quad \rho(x,y) := |x-y|^{1/2}, \quad \mu(dx) := dx, \\
&\Psi(x) := e^{|x|} - 1, \quad p(x) = x, \quad f(x) := u^i(t,x), \\
&\mathscr{C}_t := \int_S \int_S \Psi\left(\frac{f(x) - f(y)}{p(\rho(x,y))} \right) \mu(dx)\mu(dy), \\
&\mathscr{C}_t^1 := \int_S \int_S \exp\left(\frac{|f(x) - f(y)|}{p(\rho(x,y))} \right) \mu(dx)\mu(dy).
\end{aligned}
$$

Then $\mathscr{C}_t \leq \mathscr{C}_t^1$ and $\mathscr{C}_t^1 \geq 1$. By (3.3.9) and (3.3.10), for all $t \geq 1$,

$$
\begin{aligned}
&\mathrm{E}[\mathscr{C}_t] \leq \mathrm{E}[\mathscr{C}_t^1] \\
&\quad = \mathrm{E}\left[\int_S dx \int_S dy \exp\left(\frac{|u^i(t,x) - u^i(t,y)|}{|x-y|^{1/2}} \right) \right] \\
&\quad = \mathrm{E}\left[\int_S dx \int_S dy \exp\left(\frac{|v^i(t,x) - v^i(t,y) + \mu^i(t,x) - \mu^i(t,y)|}{|x-y|^{1/2}} \right) \right] \\
&\quad \leq c \int_S dx \int_S dy \, \mathrm{E}\left[\exp\left(\frac{|v^i(t,x) - v^i(t,y)|}{|x-y|^{1/2}} \right) \right] \\
&\quad = c \int_S dx \int_S dy \exp\left(\frac{\mathrm{E}[|v^i(t,x) - v^i(t,y)|^2]}{2|x-y|} \right) < C,
\end{aligned}
$$

where, in the second inequality, we have used (3.3.10). Then from [25, (A.3)], we have

$$\begin{aligned}
\mathrm{E}\left[\sup_{|x-y|^{\frac{1}{2}}\le\delta}|u^i(t,x)-u^i(t,y)|^p\right] &\le 10^p\mathrm{E}\left[\left(\int_0^{2\delta}du\ln\left(1+\frac{\mathscr{C}_t}{[\mu(B_\rho(x,u/4))]^2}\right)\right)^p\right]\\
&= 10^p\mathrm{E}\left[\left(\int_0^{2\delta}du\ln\left(1+\frac{\mathscr{C}_t}{c_1u^4}\right)\right)^p\right]\\
&\le 10^p(2\delta)^{p-1}\mathrm{E}\left[\int_0^{2\delta}du\ln^p\left(1+\frac{\mathscr{C}_t^1}{c_1u^4}\right)\right]
\end{aligned}$$

Since $\mathscr{C}_t^1(\omega)\ge 1$ for all $t\ge 1$ and $\omega\in\Omega$, we can choose δ small enough such that $\mathscr{C}_t^1/(c_1u^4)\ge e^{p-1}-1$ holds for all $u\in]0,2\delta[$, $t\ge 1$ and $\omega\in\Omega$. Since the function $x\mapsto\ln^p(1+x)$ is concave on $[e^{p-1}-1,\infty[$, by Jensen's inequality, this is bounded above by

$$\begin{aligned}
10^p(2\delta)^{p-1}\int_0^{2\delta}du\ln^p\left(1+\frac{\mathrm{E}[\mathscr{C}_t^1]}{c_1u^4}\right) &\le 10^p(2\delta)^{p-1}\int_0^{2\delta}du\ln^p\left(1+\frac{C}{c_1u^4}\right)\\
&\le c10^p(2\delta)^{p-1}\delta\ln^p(1+\frac{1}{\delta^4})\\
&= c_p\delta^p\ln^p(1+\frac{1}{\delta^4}),
\end{aligned}$$

where the second inequality is due to

$$\lim_{\delta\to 0}\frac{\int_0^\delta\ln^p\left(1+1/u^4\right)du}{\delta\ln^p(1+1/\delta^4)}=1$$

by l'Hôpital's rule. This completes the proof. □

Proposition 3.3.3. *Fix $u_0(\cdot)\in E$ and let u solve* (3.1.1) *with Dirichlet boundary conditions. Then the law of $u(t,\cdot)$ converges weakly to the invariant measure μ as $t\to\infty$, or equivalently,*

$$\lim_{t\to\infty}P_tf(u_0(\cdot))=\mu(f):=\int_E f(u)\mu(du), \tag{3.3.12}$$

for any initial value $u_0(\cdot)\in E$ and $f\in C_b(E)$.

Proof. Since the components are independent, it suffices to prove that $u^i(t,\cdot)$ converges weakly to the law of $\{\frac{\sqrt{2}}{2}B_0^i(x):0\le x\le 1\}$ on $C([0,1],\mathbb{R})$, as $t\to\infty$.

We will appeal to Theorem 7.5 in [9] to prove the weak convergence. We first prove the convergence of finite dimensional distributions, i.e.,

$$(u^i(t,x_1),\dots,u^i(t,x_k)) \quad \xrightarrow{\mathrm{d}} \quad (\frac{\sqrt{2}}{2}B_0^i(x_1),\dots,\frac{\sqrt{2}}{2}B_0^i(x_k)),\quad \text{as } t\to\infty \tag{3.3.13}$$

holds for all $x_1,\dots,x_k$. The random vector $(u^i(t,x_1),\dots,u^i(t,x_k))$ is Gaussian, and its characteristic function is determined by the mean and variance/covariance which satisfy

$$\mathrm{mean}(u^i(t,x_m)) = \mu^i(t,x_m) = \sum_{k=1}^{\infty} e^{-\pi^2k^2t}\phi_k(x_m)C^k \to 0, \text{ as } t\to\infty,$$

where C_k is defined in (3.3.6), and

$$\begin{aligned}\mathrm{Cov}(u^i(t,x_m),u^i(t,x_n)) &= \mathrm{E}[v^i(t,x_m)v^i(t,x_n)]\\ &= \sum_{k=1}^{\infty}\frac{1-e^{-2\pi^2k^2t}}{2\pi^2k^2}\phi_k(x_m)\phi_k(x_n),\end{aligned}$$

converges to, as $t\to\infty$,

$$\sum_{k=1}^{\infty}\frac{1}{2\pi^2k^2}\phi_k(x_m)\phi_k(x_n) = \frac{1}{2}(x_m\wedge x_n - x_mx_n) = \frac{1}{2}\mathrm{E}[B_0^i(x_m)B_0^i(x_n)]$$

for $1\le m,n\le k$, where the first equality follows from [8, (6.1)]. This implies (3.3.13).

From (3.3.11), we use Chebyshev's inequality to obtain that for any positive ϵ,

$$\lim_{\delta\to0}\limsup_{t\to\infty}\mathrm{P}\left\{\sup_{|x-y|\le\delta}|u^i(t,x)-u^i(t,y)|\ge\epsilon\right\}=0, \tag{3.3.14}$$

which verifies the second condition for weak convergence in [9, Theorem 7.5]. Hence, we have proved the proposition. □

Remark 3.3.4. *In the case of Neumann boundary conditions, the Green kernel is given by*

$$G(t,x,y) = 1+2\sum_{k=1}^{\infty}e^{-\pi^2k^2t}\cos(k\pi x)\cos(k\pi y);$$

see [81, p. 323-326], or [25, (4.16)]. In this case, by the semi-group property of the Green kernel (A.6)*, the second moment of the solution is equal to*

$$\begin{aligned}E[u(t,x)^2] &= \int_0^t\int_0^1 G^2(t-r,x,v)\,dv\,dr = \int_0^t G(2r,x,x)\,dr\\ &= t+2\sum_{k=1}^{\infty}\cos^2(k\pi x)\int_0^t e^{-2\pi^2k^2r}\,dr\\ &= t+\sum_{k=1}^{\infty}\frac{1-e^{-2\pi^2k^2t}}{k^2\pi^2}\cos^2(k\pi x),\end{aligned}$$

which converges to ∞*, as* $t\to\infty$*. Hence, we do not expect that the law of the solution to* (3.1.1) *with Neumann boundary conditions converges to a limit as* $t\to\infty$*.*

We denote by $B(0,R)$ ($\bar{B}(0,R)$) the open (closed) ball of radius $R>0$ centered at 0 in E, $B(0,R)^c$ the complement of $B(0,R)$ and $\partial B(0,R)$ the boundary of $B(0,R)$.

From [9, (9.39)], we have

$$\mathrm{P}\left\{\sup_{0\le x\le 1}|B_0^i(x)|\le R\right\}=1+2\sum_{k=1}^{\infty}(-1)^k e^{-2k^2R^2},\quad \text{for all } R>0, \tag{3.3.15}$$

which implies that the distribution of the random variable $\sup_{0\le x\le 1}|B_0^i(x)|$ has a probability density function with respect to one-dimensional Lebesgue measure. Hence the random vector

$$\left(\sup_{0\le x\le 1}|B_0^1(x)|,\dots,\sup_{0\le x\le 1}|B_0^d(x)|\right)$$

has a density with respect to d-dimensional Lebesgue measure, which implies that

$$\mathrm{P}\left\{\sup_{1\le i\le d}\sup_{0\le x\le 1}|B_0^i(x)|=R\right\}=0. \tag{3.3.16}$$

Therefore, we have $\mu(\partial B(0,R))=0$ for any $R>0$. On the other hand, we observe that the distribution function defined by

$$F(R):=\mathrm{P}\left\{\sup_{0\le x\le 1}|B_0^i(x)|\le R\right\}=1+2\sum_{k=1}^{\infty}(-1)^k e^{-2k^2R^2}, \tag{3.3.17}$$

takes values in $]0,1[$ for all $R>0$. To see this, first from the expression of the alternating series in (3.3.17), it is clear that $F(R)<1$ for all $R>0$. To prove that the distribution function F is strictly positive, we denote

$$B_0^i(x)=B(x)-xB(1),\ \ x\in[0,1],$$

where $\{B(x):x\in[0,1]\}$ is a standard Brownian motion. Then by the triangle inequality,

$$\begin{aligned}F(R)&\ge \mathrm{P}\left\{\sup_{0\le x\le 1}|B(x)|+|B(1)|\le R\right\}\\&\ge \mathrm{P}\left\{\sup_{0\le x\le 1}|B(x)|\le R/2\right\}=H(R/2),\end{aligned}$$

where

$$H(x)=\frac{4}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}\exp\left[-\frac{(2k+1)^2\pi^2}{8x^2}\right],\quad \text{for } x>0$$

denotes the distribution function of the supremum of the absolute value of Brownian motion; see [20, p.233]. It is clear from the expression of the function H that $H(x)>0$ for all $x>0$. Hence we have proved $\mu(B(0,R))>0$.

By the equivalent statements of weak convergence (see [9, Theorem 2.1]), from (3.3.12), we obtain that

$$\lim_{t\to\infty} P_t((u_0(\cdot), B(0,R)) = \mu(B(0,R)) > 0, \quad \text{for all } \ u_0(\cdot) \in E. \tag{3.3.18}$$

Owing to Corollary 3.4.6 of [32], we get the following result on the recurrence property of the solution.

Theorem 3.3.5. *The Markov process $(\widetilde{\Omega}, \widetilde{\mathscr{F}}_t, \tilde{u}(t,\cdot), P^{u_0(\cdot)}, P_t, \theta_t)$ is recurrent with respect to $B(0,R)$ for any $R > 0$, i.e., for any $u_0(\cdot) \in E$,*

$$P^{u_0(\cdot)}\{\tilde{u}(t,\cdot) \in B(0,R),\ \textit{for an unbounded set of } t > 0\} = 1. \tag{3.3.19}$$

Remark 3.3.6. *We remark that the Markov process $(\widetilde{\Omega}, \widetilde{\mathscr{F}}_t, \tilde{u}(t,\cdot), P^{u_0(\cdot)}, P_t, \theta_t)$ is also recurrent with respect to $B(0,R)^c$ for any $R > 0$, and the proof follows similarly with that of Theorem 3.3.5.*

3.4 Lower bound on the hitting probability for solutions with a bounded initial value

We first recall the hypotheses and consequence of Theorem 2.1(1) in [25].

Theorem 3.4.1 ([25, Theorem 2.1(1)])**.** *Fix two compact intervals $I \subset]0,\infty[$ and $J \subset]0,1[$. Suppose that $\{v(t,x)\}_{(t,x)\in I\times J}$ is a two-parameter continuous random field with values in $\mathbb{R}^d$, such that $(v(t,x), v(s,y))$ has a joint probability density function $p_{t,x;s,y}(\cdot,\cdot)$, for all $s,t \in I$ and $x,y \in J$ with $(t,x) \neq (s,y)$. We denote by $p_{t,x}(\cdot)$ the density function of $v(t,x)$. Assume the following hypotheses:*

A1 *For all $M > 0$, there exists a positive and finite constant $C = C(I,J,M,d)$ such that for all $(t,x) \in I \times J$ and all $z \in [-M,M]^d$,*

$$p_{t,x}(z) \geq C. \tag{3.4.1}$$

A2 *There exists $\beta > 0$ such that for all $M > 0$, there exists $c = c(I,J,M,d) > 0$ such that for all $s,t \in I$ and $x,y \in J$ with $(t,x) \neq (s,y)$, and for every $z_1, z_2 \in [-M,M]^d$,*

$$p_{t,x;s,y}(z_1,z_2) \leq \frac{c}{[\Delta((t,x);(s,y))]^{\beta/2}} \exp\left(-\frac{\|z_1 - z_2\|^2}{c\Delta((t,x);(s,y))}\right). \tag{3.4.2}$$

Then the following statement holds.

(1) Fix $M > 0$. There exists a positive and finite constant $a = a(I,J,\beta,M,d)$ such that for all Borel sets $A \subseteq [-M,M]^d$,

$$P\{v(I\times J) \cap A \neq \emptyset\} \geq a\, Cap_{\beta-6}(A). \tag{3.4.3}$$

From now on, we assume

$$I = [1,2] \quad \text{and} \quad J = [1/4, 1/2].$$

As in (3.3.3) and (3.3.5), we still denote by, respectively, $\{v(t,x)\}_{(t,x)\in I\times J}$ the random part, and $\{\mu(t,x)\}_{(t,x)\in I\times J}$ non-random part of the solution $\{u(t,x)\}_{(t,x)\in I\times J}$ to (3.1.2) starting from $u_0(\cdot)\in \bar{B}(0,N)$ for some $N>0$.

By Proposition 4.1 of [25], the probability of visiting A for $\{v(t,x)\}_{(t,x)\in I\times J}$ has the lower bound in (3.4.3) with $\beta=d$. We claim that the lower bound also holds for $\{u(t,x)\}_{(t,x)\in I\times J}$, where the constant will depend additionally on N, but not on the specific choice of $u_0(\cdot)$.

Lemma 3.4.2. *For any $M,N>0$, there exists a finite positive constant $a=a(I,J,N,M,d)$ such that for all Borel sets $A\subseteq[-M,M]^d$, and for all $u_0(\cdot)\in\bar{B}(0,N)$,*

$$P\{u_{u_0(\cdot)}(I\times J)\cap A\neq\emptyset\}\geq a\,Cap_{d-6}(A), \tag{3.4.4}$$

and equivalently,

$$P^{u_0(\cdot)}\{\tilde{u}(I\times J)\cap A\neq\emptyset\}\geq a\,Cap_{d-6}(A) \tag{3.4.5}$$

Proof. In order to prove (3.4.4), by Theorem 3.4.1, it suffices to prove that hypotheses **A1** and **A2** are satisfied for $\{u(t,x)\}_{(t,x)\in I\times J}$, where the constants depend additionally on N, but not on the specific choice of $u_0(\cdot)$. We add the superscripts u or v to the probability density functions to indicate to which random field they correspond.

*Verification of **A1***. Fix $M>0$ and let $z\in[-M,M]^d$. Then for all $(t,x)\in I\times J$, the probability density function of $u(t,x)$ is given by

$$p^u_{t,x}(z)=\frac{1}{(2\pi\sigma^2_{t,x})^{d/2}}\exp\left(-\frac{\|z-\mu(t,x)\|^2}{2\sigma^2_{t,x}}\right), \tag{3.4.6}$$

where

$$\sigma^2_{t,x}:=\mathrm{Var}(u^i(t,x))=\int_0^t dr\int_0^1 dv\,(G(t-r,x,v))^2. \tag{3.4.7}$$

Since $(t,x)\mapsto v^i(t,x)$ is L^2 continuous by (4.11) of [25], it follows that the function $(t,x)\mapsto\sigma_{t,x}$ achieves its minimum $\rho_1>0$ and its maximum $\rho_2<\infty$ over $I\times J$. Thus

$$p^u_{t,x}(z)\geq\frac{1}{(2\pi\rho_2^2)^{d/2}}\exp\left(-\frac{-(M^2+N^2)d}{2\rho_1^2}\right),$$

which proves **A1**.

*Verification of **A2***. First, we give some estimates on the regularity of the function $(t,x)\mapsto\mu(t,x)$ on $I\times J$.

Case 1: $s=t$, $x\neq y$. From (3.3.10), there exists a constant c such that for all $t\in I$, $x,y\in J$,

$1 \leq i \leq d$,

$$|\mu^i(t,x) - \mu^i(t,y)| \leq c|x-y|. \tag{3.4.8}$$

Case 2: $x = y$, $s < t$.

$$\begin{aligned}
|\mu^i(t,x) - \mu^i(s,x)| &= \left| \int_0^1 \sum_{k=1}^{\infty} (e^{-\pi^2 k^2 t} - e^{-\pi^2 k^2 s}) \phi_k(x)\phi_k(v) u_0^i(v)\, dv \right| \\
&\leq \int_0^1 \sum_{k=1}^{\infty} e^{-\pi^2 k^2 s} |1 - e^{-\pi^2 k^2 (t-s)}| |\phi_k(x)\phi_k(v) u_0^i(v)| dv \\
&\leq 2N \sum_{k=1}^{\infty} e^{-\pi^2 k^2} |1 - e^{-\pi^2 k^2 (t-s)}|.
\end{aligned}$$

By the inequality $0 \leq 1 - e^{-x} \leq \min(x, 1)$, for all $x > 0$, this is bounded above by

$$\begin{aligned}
&2N \sum_{k=1}^{\infty} e^{-\pi^2 k^2} \min(\pi^2 k^2 (t-s), 1) \\
&= 2N(e^{-\pi^2} \min(\pi^2 (t-s), 1) + \sum_{k=2}^{\infty} e^{-\pi^2 k^2} \min(\pi^2 k^2 (t-s), 1)).
\end{aligned}$$

Using the fact that the function $x \mapsto e^{-\pi^2 x} \min(\pi^2 x(t-s), 1)$ is nonincreasing on $[1, \infty[$ for any s, t with $s < t$, this is bounded above by

$$\begin{aligned}
&2N\Big(c_1(t-s) + \int_1^{\infty} e^{-\pi^2 r^2} \min(\pi^2 r^2 (t-s), 1) dr\Big) \\
&= 2N\Big(c_1(t-s) + \int_1^{\frac{1}{\pi\sqrt{t-s}}} e^{-\pi^2 r^2} \pi^2 r^2 (t-s) dr + \int_{\frac{1}{\pi\sqrt{t-s}}}^{\infty} e^{-\pi^2 r^2} dr\Big) \\
&\leq 2N\Big(c_1(t-s) + (t-s) \int_1^{\infty} e^{-\pi^2 r^2} \pi^2 r^2 dr + \pi\sqrt{t-s} \int_{\frac{1}{\pi\sqrt{t-s}}}^{\infty} r e^{-\pi^2 r^2} dr\Big) \\
&\leq 2N(c_1(t-s) + c_2(t-s) + c_3\sqrt{t-s}\, e^{-\frac{1}{t-s}}) \\
&\leq \tilde{c}|t-s|,
\end{aligned} \tag{3.4.9}$$

where the last inequality is because that the inequality $\sqrt{x} e^{-1/x} \leq x$ is valid for all $x > 0$.

Hence, (3.4.8) and (3.4.9) together imply that there exists a constant c such that for all $s, t \in I$, $x, y \in J$, $1 \leq i \leq d$,

$$(\mu^i_{t,x} - \mu^i_{s,y})^2 \leq c((t-s)^2 + (x-y)^2). \tag{3.4.10}$$

Using the upper bound on the joint probability density function of $(v(t,x), v(s,y))$ (see of [25, (2.3) and Theorem 4.6]), the elementary equality $(a-b)^2 \geq \frac{1}{2}a^2 - b^2$ and (3.4.10), we obtain that

$$
\begin{aligned}
p^u_{t,x;s,y}(z_1,z_2) &= p^v_{t,x;s,y}(z_1-\mu(t,x), z_2-\mu(s,y)) \\
&\le \frac{c}{[\Delta((t,x);(s,y))]^{d/2}} \exp\left(-\frac{\|z_1-\mu(t,x)-z_2+\mu(s,y)\|^2}{c\Delta((t,x);(s,y))}\right) \\
&\le \frac{c}{[\Delta((t,x);(s,y))]^{d/2}} \exp\left(-\frac{\frac{1}{2}\|z_1-z_2\|^2-\|\mu(t,x)-\mu(s,y)\|^2}{c\Delta((t,x);(s,y))}\right) \\
&\le \frac{\tilde{c}}{[\Delta((t,x);(s,y))]^{d/2}} \exp\left(-\frac{\|z_1-z_2\|^2}{\tilde{c}\Delta((t,x);(s,y))}\right), \qquad (3.4.11)
\end{aligned}
$$

which proves **A2**.

Therefore, the lower bound on hitting probability in (3.4.4) follows from the result of Theorem 3.4.1. Finally, the statement (3.4.5) is a consequence of (3.4.4) and (3.2.2). □

3.5 Proof of Theorem 3.1.1

We first state a result on hitting probability for general Markov processes, which will be used to prove Theorem 3.1.1.

Proposition 3.5.1. *Let $(\Omega, \mathscr{F}_t, X(t), \theta_t, P^x)_{t\ge 0, x\in\mathscr{E}}$ be a continuous Markov system taking values on the Banach space $\mathscr{E}$, which has the strong Markov property. Fix $K > N > 0$ and $\mathscr{A} \subset \mathscr{E}$. Suppose that the process $\{X(t) : t \ge 0\}$ is recurrent with respect to $B(0,N)$ and $\bar{B}(0,K)^c$, and that there exists a positive constant $c = c(N,K,\mathscr{A})$ such that for all $x \in \bar{B}(0,N)$,*

$$P^x\{\exists t \in [0, T_1],\ s.t.\ X(t) \in \mathscr{A}\} \ge c, \tag{3.5.1}$$

where $T_1 := \inf\{t \ge 0 : \|X(t)\| > K\}$. Then for any $x \in \bar{B}(0,N)$,

$$P^x\{\exists t \ge 0,\ s.t.\ X(t) \in \mathscr{A}\} = 1. \tag{3.5.2}$$

Proof. By the recurrence property, we define inductively two sequences of finite stopping times $(T_k)_{k\ge 0}$ and $(S_k)_{k\ge 0}$, as follows. Let $T_0 = S_0 = 0$, and for $k \ge 1$,

$$T_k = \inf\{t \ge S_{k-1} : \|X(t)\| > K\}, \quad S_k = \inf\{t \ge T_k : \|X(t)\| < N\},$$

which satisfy that $T_k = S_{k-1} + T_1 \circ \theta_{S_{k-1}}$.

For each $k \geq 1$, we set $\mathscr{A}_k := \{X([S_{k-1}, T_k]) \cap \mathscr{A} \neq \emptyset\}$. By the strong Markov property,

$$\begin{aligned}
\mathrm{P}^x\{\mathscr{A}_k | \mathscr{F}_{S_{k-1}}\} &= \mathrm{P}^x\{\exists t \in [S_{k-1}, T_k],\ \text{s.t. } X(t) \in \mathscr{A} | \mathscr{F}_{S_{k-1}}\} \\
&= \mathrm{P}^x\{\exists t \in [S_{k-1}, S_{k-1} + T_1 \circ \theta_{S_{k-1}}],\ \text{s.t. } X(t) \in \mathscr{A} | \mathscr{F}_{S_{k-1}}\} \\
&= \mathrm{P}^x\{\exists t \in [0, T_1 \circ \theta_{S_{k-1}}],\ \text{s.t. } X(t + S_{k-1}) \in \mathscr{A} | \mathscr{F}_{S_{k-1}}\} \\
&= \mathrm{E}^x[1_{\{\exists t \in [0, T_1],\ \text{s.t. } X(t) \in \mathscr{A}\}} \circ \theta_{S_{k-1}} | \mathscr{F}_{S_{k-1}}] \\
&= \mathrm{E}^{X(S_{k-1})}[1_{\{\exists t \in [0, T_1],\ \text{s.t. } X(t) \in \mathscr{A}\}}] \geq c, \qquad (3.5.3)
\end{aligned}$$

where the inequality is due to (3.5.1). Therefore, for any integer $n \geq 1$,

$$\begin{aligned}
\mathrm{P}^x\{\exists t \geq 0,\ \text{s.t. } X(t) \in \mathscr{A}\} &\geq \mathrm{P}^x\left(\bigcup_{k=1}^{n} \mathscr{A}_k\right) \\
&= 1 - \mathrm{P}^x\left(\bigcap_{k=1}^{n} \mathscr{A}_k^c\right) \\
&= 1 - \mathrm{E}^x\left[\left(\prod_{k=1}^{n-1} 1_{\mathscr{A}_k^c}\right) \mathrm{P}^x\{\mathscr{A}_n^c | \mathscr{F}_{S_{n-1}}\}\right] \\
&= 1 - \mathrm{E}^x\left[\left(\prod_{k=1}^{n-1} 1_{\mathscr{A}_k^c}\right) (1 - \mathrm{P}^x\{\mathscr{A}_n | \mathscr{F}_{S_{n-1}}\})\right] \\
&\geq 1 - (1-c)\mathrm{E}^x\left[\prod_{k=1}^{n-1} 1_{\mathscr{A}_k^c}\right] \\
&\geq 1 - (1-c)^n, \qquad (3.5.4)
\end{aligned}$$

where we repeat the argument to get the last inequality. Letting $n \to \infty$, we obtain (3.5.2). □

Proof of Theorem 3.1.1. We assume that $u_0(\cdot) \in \bar{B}(0, N)$ and $A \subseteq [-N, N]^d$ for some $N > 0$, as mentioned in Section 3.1. First, we give an estimate on the following tail probability. For any $K > N$,

$$\begin{aligned}
\mathrm{P}\left\{\sup_{0 \leq t \leq 2} \|u_{u_0(\cdot)}(t, \cdot)\|_\infty \geq K\right\} &\leq \sum_{i=1}^{d} \mathrm{P}\left\{\sup_{0 \leq t \leq 2} \sup_{0 \leq x \leq 1} |u^i(t, x)| \geq K\right\} \\
&= \sum_{i=1}^{d} \mathrm{P}\left\{\sup_{0 \leq t \leq 2} \sup_{0 \leq x \leq 1} |v^i(t, x) + \mu^i(t, x)| \geq K\right\} \\
&\leq d\,\mathrm{P}\left\{\sup_{0 \leq t \leq 2} \sup_{0 \leq x \leq 1} |v^i(t, x)| \geq K - N\right\}. \qquad (3.5.5)
\end{aligned}$$

Since $(t, x) \mapsto v^i(t, x)$ is continuous almost surely, we have

$$\lim_{K \to \infty} \mathrm{P}\left\{\sup_{0 \leq t \leq 2} \sup_{0 \leq x \leq 1} |v^i(t, x)| \geq K - N\right\} = 0.$$

Then we can choose sufficiently large K, depending only on N, such that

$$\mathrm{P}\left\{\sup_{0\le t\le 2}\|u_{u_0(\cdot)}(t,\cdot)\|_\infty \ge K\right\} \le \frac{1}{2}a\,\mathrm{Cap}_{d-6}(A),$$

or, equivalently,

$$\mathrm{P}^{u_0(\cdot)}\left\{\sup_{0\le t\le 2}\|\tilde{u}(t,\cdot)\|_\infty \ge K\right\} \le \frac{1}{2}a\,\mathrm{Cap}_{d-6}(A), \tag{3.5.6}$$

uniformly for all $u_0(\cdot)$ with $\|u_0(\cdot)\|_\infty \le N$, where the constant $a = a(I,J,N,N,d)$ is specified in (3.4.4).

Define $\mathscr{A} := \{\varphi(\cdot) \in E : \exists x \in [0,1] \text{ s.t. } \varphi(x) \in A\}$. Since

$$\begin{aligned}\mathrm{P}\{u_{u_0(\cdot)}([0,\infty[\times[0,1]) \cap A \neq \emptyset\} &= \mathrm{P}^{u_0(\cdot)}\{\tilde{u}([0,\infty[\times[0,1]) \cap A \neq \emptyset\}\\ &= \mathrm{P}^{u_0(\cdot)}\{\exists t \ge 0, \text{ s.t. } \tilde{u}(t,\cdot) \cap \mathscr{A} \neq \emptyset\},\end{aligned}$$

by Proposition 3.5.1, it suffices to verify that the estimate for the hitting probability in (3.5.1) holds for the process $\{\tilde{u}(t,\cdot) : t \ge 0\}$. Indeed,

$$\begin{aligned}&\mathrm{P}^{u_0(\cdot)}\{\exists t \in [0,T_1], \text{ s.t. } \tilde{u}(t,\cdot) \in \mathscr{A}\}\\ &\quad= \mathrm{P}^{u_0(\cdot)}\{\exists (t,x) \in [0,T_1]\times[0,1], \text{ s.t. } \tilde{u}(t,x) \in A\}\\ &\quad\ge \mathrm{P}^{u_0(\cdot)}\left\{\{\exists (t,x) \in [0,T_1]\times[0,1], \text{ s.t. } \tilde{u}(t,x) \in A\} \cap \left\{\sup_{0\le t\le 2}\|\tilde{u}(t,\cdot)\|_\infty \le K\right\}\right\}\\ &\quad\ge \mathrm{P}^{u_0(\cdot)}\left\{\{\exists (t,x) \in [1,2]\times[1/4,1/2], \text{ s.t. } \tilde{u}(t,x) \in A\} \cap \left\{\sup_{0\le t\le 2}\|\tilde{u}(t,\cdot)\|_\infty \le K\right\}\right\}\\ &\quad\ge \mathrm{P}^{u_0(\cdot)}\{\exists (t,x) \in [1,2]\times[1/4,1/2], \text{ s.t. } \tilde{u}(t,x) \in A\} - \mathrm{P}^{u_0(\cdot)}\left\{\sup_{0\le t\le 2}\|\tilde{u}(t,\cdot)\|_\infty \ge K\right\}\\ &\quad\ge \frac{1}{2}a\,\mathrm{Cap}_{d-6}(A),\end{aligned} \tag{3.5.7}$$

where the last inequality follows from (3.4.5) and (3.5.6). Hence the assumption in (3.5.1) is satisfied and we complete the proof of Theorem 3.1.1. □

4 On the density of the supremum of the solution to the stochastic heat equation

In this chapter, we first develop a general criterion for an upper bound on hitting probabilities. This criterion involves a condition on the probability density function of the supremum of a random field over a rectangle or a segment. Motivated by this, we study the regularity of the probability density function of the supremum over a time interval of the solution to the linear stochastic heat equation. Using a general criterion for the smoothness of densities for locally nondegenerate random variables, we establish the smoothness of the joint density of the random vector whose components are the solution and the supremum of an increment in time of the solution over an interval (at a fixed spatial position). Applying the properties of the divergence operator, we give a Gaussian-type upper bound on this joint density, which presents a close connection with the Hölder-continuity properties of the solution. We also derive the smoothness property and a Gaussian-type upper bound for the density of the supremum of the solution over a space-time rectangle that touches the $t = 0$ axis. In the case of Neumann boundary conditions, the smoothness of the density of the supremum of rectangular increments from the origin of the solution is also proved.

4.1 Introduction and motivation

For a real-valued Gaussian random field $\{X(t) : t \in I\}$, where I is a parameter set, defined on a probability space $(\Omega, \mathscr{F}, \mathrm{P})$, the distribution function of the supremum of this random field, or the excursion probability $\mathrm{P}\{\sup_{t\in I} X(t) \geq a\}$, has been investigated extensively; see, for example, [1, 2, 72] and references therein. In general, finding a formula for the distribution function of the supremum of a stochastic process is an almost impossible task, let alone for its probability density function, which is much less studied than the probability distribution function. We first review some of the literature on the study of regularity of the probability density function of the supremum of a stochastic process.

We begin with Gaussian processes. Let $\{X(t) : t \in I\}$ be a separable, centered and real-valued Gaussian process define on the canonical probability space $(B(I), \mathscr{F}_t, \mathrm{P})$, where I is a compact set of $\mathbb{R}^n$, $B(I)$ is the space of Borel functions on I and $X(t)(\omega) := \omega_t$ for $\omega \in B(I)$. Denote $M := \sup_{t\in I} X(t)$, $F(x) := \mathrm{P}\{M \le x\}$, $F_-(x) := \mathrm{P}\{M < x\}$ and $a_0 := \inf\{x : F_-(x) > 0\}$. It is clear that M is a measurable convex function on $B(I)$. From Cirel'son [21], assuming that $M < \infty$ a.s., then F_- is continuous except possibly at a_0. And the density $F'_-(x)$ exists except perhaps on a countable set where it may have jumps downward. Moreover, if $\mathrm{E}[X(t)^2]$ does not depend on t, then F_- is continuous everywhere and F'_- is continuous except possibly at a_0 where it may have a finite jump. In the review of Cirel'son [21] in Mathematical Reviews, Dudley gives a another proof of the result that F' exists and is continuous except for downward jumps: According to Borell [11, Corollary 2.1],

$$\mathrm{P}_*(\lambda A + (1-\lambda)B) \ge \mathrm{P}(A)^\lambda \mathrm{P}(B)^{1-\lambda},$$

for A, B measurable, $0 \le \lambda \le 1$, where P_* is the inner measure of P, and then

$$\begin{aligned} F(\lambda x + (1-\lambda)y) &= \mathrm{P}\{M \le \lambda x + (1-\lambda)y\} \\ &\ge \mathrm{P}_*\{M \le \lambda x + (1-\lambda)y\} \\ &\ge \mathrm{P}_*\{\lambda\{M \le x\} + (1-\lambda)\{M \le y\}\} \\ &\ge \mathrm{P}\{M \le x\}^\lambda \mathrm{P}\{M \le y\}^{1-\lambda} = F(x)^\lambda F(y)^{1-\lambda}. \end{aligned}$$

Hence F is logarithmically concave, which shows that F' exists and is continuous except for a countable number of downward jumps.

Pitt and Lanh [73] showed, under very general conditions, that the distribution function F is absolutely continuous with a bounded density. Following the idea of Pitt and Lanh [73], Weber [83] gave an a upper bound on the probability density function of the supremum of certain Gaussian processes. Further developments have been given by many authors, among which we mention Lifshits [56, 57]. We also refer to Diebolt and Posse [36] and Azaïs and Wschebor [4, Chapter 7] and references therein for more information on the regularity of the density of the maximum of *smooth* Gaussian random fields.

In [64, Proposition 2.1.11], a criterion for the absolute continuity property of the distribution function of the supremum of a continuous process is given in terms of the Malliavin derivative of this process. Moreover, a general criterion is established for the smoothness of the probability density function for locally nondegenerate random variables; see [39, Theorem 2.1] and [64, Theorem 2.1.4].

We are interested in the properties of the probability density function of the supremum of the solutions to SPDEs. On the other hand, the density of the supremum of the solution is related to the study of upper bounds on hitting probabilities for these solutions, as we now explain.

As we have seen in Chapter 2, the upper bound on hitting probabilities for the solution to a system of non-linear stochastic (fractional) heat equations is not as sharp as that for linear stochastic (fractional) heat equations; see [25, Theorem 4.6], [26, Theorem 1.2] and our Theorems 2.1.4, 2.1.5. This is because for the non-Gaussian solution, the upper bound on the probability of visiting small balls within a small space-time region is not of the same order as for the Gaussian solution; see [25, Theorem 3.3], [25, Proposition 4.4] and our Lemma 2.2.1.

Let us study this probability of visiting small balls from another point of view. For simplicity, we denote by u the solution to one single equation (1.2.1) (i.e., $d = 1$) with vanishing initial data, that is,

$$\frac{\partial u}{\partial t}(t,x) = \frac{\partial^2 u}{\partial x^2}(t,x) + \sigma(u(t,x))\dot{W}(t,x) + b(u(t,x)), \tag{4.1.1}$$

where the coefficients σ and b satisfy the hypotheses **P1** (or **P1'**) and **P2** in Chapter 2. We would like to give an estimate on the following probability of visiting small balls:

$$\mathrm{P}\left\{\inf_{(t,x)\in R^n_{k,l}} |u(t,x) - z| \le 2^{-n}\right\}, \tag{4.1.2}$$

where

$$R^n_{k,l} := [k2^{-4n}, (k+1)2^{-4n}] \times [l2^{-2n}, (l+1)2^{-2n}]. \tag{4.1.3}$$

By the triangle inequality, this probability is bounded above by

$$\mathrm{P}\left\{|u(t^n_k, x^n_l) - z| \le 2^{-n} + \sup_{(t,x)\in R^n_{k,l}} |u(t,x) - u(t^n_k, x^n_l)|\right\}. \tag{4.1.4}$$

For the Gaussian solution (i. e., $\sigma \equiv 1$, $b \equiv 0$), Dalang, Nualart and Khoshnevisan [25] derive a formula similar to (4.1.4) by using the Gaussian property of the solution and introducing two independent random variables; see the proof of [25, Proposition 4.4]. This is not applicable in the non-Gaussian case.

Since the supremum of the absolute value of a continuous function is equal to either the maximum of this function, or the minimum of this function times -1, the probability in (4.1.4) is approximately equal to

$$2 \cdot \mathrm{P}\left\{|u(t^n_k, x^n_l) - z| \le 2^{-n} + \sup_{(t,x)\in R^n_{k,l}} u(t,x) - u(t^n_k, x^n_l)\right\}. \tag{4.1.5}$$

Even though the random variables $u(t^n_k, x^n_l)$ and $\sup_{(t,x)\in R^n_{k,l}} u(t,x) - u(t^n_k, x^n_l)$ are not independent, from the perspective of probability density functions, we expect that the joint density (denoted by $p_n(\cdot,\cdot)$, whose existence needs to be proved) of the random vector $(u(t^n_k, x^n_l), \sup_{(t,x)\in R^n_{k,l}} u(t,x) - u(t^n_k, x^n_l))$ is bounded above by a constant times the product of the marginal densities of the components. Notice that the density of $u(t^n_k, x^n_l)$ is bounded uniformly over (t^n_k, x^n_l). We expect that the joint density of the random vector $(u(t^n_k, x^n_l), \sup_{(t,x)\in R^n_{k,l}} u(t,x) - u(t^n_k, x^n_l))$ is controlled by the density of the random variable $\sup_{(t,x)\in R^n_{k,l}} u(t,x) - u(t^n_k, x^n_l)$.

To derive a satisfactory estimate for the density of $\sup_{(t,x)\in R^n_{k,l}} u(t,x) - u(t^n_k, x^n_l)$, recall that the probability density function of the maximum of Brownian motion $\max_{0\le t\le T} B(t)$ is given by

$$z \mapsto \frac{2}{\sqrt{2\pi T}} \exp\left(\frac{-z^2}{2T}\right) 1_{[0,\infty[}(z).$$

Relating this formula to the fact that the sample paths of Brownian motion are almost $\frac{1}{2}$-Hölder continuous suggests that the joint density $p_n(\cdot,\cdot)$ should satisfy the following bound:

$$\begin{aligned} p_n(z_1,z_2) &\le \frac{c}{\sqrt{(2^{-4n})^{1/2}+2^{-2n}}} \exp\left(\frac{-z_2^2}{c\left((2^{-4n})^{1/2}+2^{-2n}\right)}\right) && (4.1.6)\\ &= c2^n \exp\left(\frac{z_2^2}{c2^{-2n}}\right), \quad \text{for all } z_1 \in \mathbb{R},\, z_2 > 0. && (4.1.7)\end{aligned}$$

If we apply the Gaussian-type upper bound on the density in (4.1.7) and calculate the probability in (4.1.5), it will give us the correct upper bound on the probability of hitting small balls, as in the Gaussian case.

Motivated by the above discussion, we establish the following general criterion for an upper bound on hitting probabilities from the perspective of probability density functions, which is comparable to [25, Theorem 3.3].

Let $v = (v_1,\dots,v_d) = \{v(t,x),(t,x) \in \mathbb{R}_+ \times \mathbb{R}\}$ be a random field on $\mathbb{R}^d$ with i.i.d. components. Fix $H_1 > 0, H_2 > 0$ and $T > 0$. Let $I \subset]0,T]$ and $J \subset \mathbb{R}$ be two compact intervals.

Theorem 4.1.1. *Assume that the probability density function $p_{t,x}(z)$ of $v_1(t,x)$ is bounded uniformly over $(t,x) \in I \times J$ and $z \in \mathbb{R}$.*

(1) Suppose that there exists a constant $c = c(I,J)$ such that for all $(s_0,y_0) \in I \times J$, δ_1 and δ_2 sufficiently small, the random vectors

$$\left(v_1(s_0,y_0), \sup_{(t,x)\in[s_0,s_0+\delta_1]\times[y_0,y_0+\delta_2]} (v_1(t,x)-v_1(s_0,y_0))\right), \tag{4.1.8}$$

and

$$\left(-v_1(s_0,y_0), \sup_{(t,x)\in[s_0,s_0+\delta_1]\times[y_0,y_0+\delta_2]} (-v_1(t,x)-(-v_1(s_0,y_0)))\right) \tag{4.1.9}$$

have joint probability density functions, denoted by $p^+_{\delta_1,\delta_2}(\cdot,\cdot)$ and $p^-_{\delta_1,\delta_2}(\cdot,\cdot)$ respectively, which satisfy that

$$p^{\pm}_{\delta_1,\delta_2}(z_1,z_2) \le \frac{c}{\delta_1^{H_1}+\delta_2^{H_2}} \exp\left(\frac{-z_2^2}{c\left(\delta_1^{H_1}+\delta_2^{H_2}\right)^2}\right), \quad \textit{for all } z_1 \in \mathbb{R},\, z_2 \ge \delta_1^{H_1}+\delta_2^{H_2}. \tag{4.1.10}$$

Then there exists a constant $C = C(I, J)$ *such that for all compact sets* $A \subset \mathbb{R}^d$,

$$P\{v(I \times J) \cap A \neq \emptyset\} \leq C \mathscr{H}_{d-H_1^{-1}-H_2^{-1}}(A). \tag{4.1.11}$$

(2) Suppose that there exists a constant $c = c(I, J)$ *such that for all* $(s_0, y_0) \in I \times J$, δ_1 *sufficiently small, the random vectors*

$$\left(v_1(s_0, y_0), \sup_{t\in[s_0,s_0+\delta_1]} (v_1(t, y_0) - v_1(s_0, y_0))\right), \tag{4.1.12}$$

and

$$\left(-v_1(s_0, y_0), \sup_{t\in[s_0,s_0+\delta_1]} (-v_1(t, y_0) - (-v_1(s_0, y_0)))\right) \tag{4.1.13}$$

have joint probability density functions, denoted by $p^+_{\delta_1}(\cdot,\cdot)$ *and* $p^-_{\delta_1}(\cdot,\cdot)$ *respectively, which satisfy that*

$$p^{\pm}_{\delta_1}(z_1, z_2) \leq \frac{c}{\delta_1^{H_1}} \exp\left(\frac{-z_2^2}{c\delta_1^{2H_1}}\right), \quad \text{for all } z_1 \in \mathbb{R},\ z_2 \geq \delta_1^{H_1}. \tag{4.1.14}$$

Then there exists a constant $C = C(I, J)$ *such that for all compact sets* $A \subset \mathbb{R}^d$ *and for every* $y_0 \in J$,

$$P\{v(I \times \{y_0\}) \cap A \neq \emptyset\} \leq C \mathscr{H}_{d-H_1^{-1}}(A). \tag{4.1.15}$$

Remark 4.1.2. *(a) In order to obtain the upper bound on hitting probability in* (4.1.11)*, it is sufficient to have the estimate on the joint density in* (4.1.10) *with* $\delta_1 = \delta_2^{H_2/H_1}$*; see the proof below, in particular, the choice of* δ_1 *and* δ_2 *in* (4.1.18).

(b) In some cases, the estimates for $p^-_{\delta_1,\delta_2}$ *and* $p^-_{\delta_1}$ *are a consequence of* (4.1.10) *and* (4.1.14) *for* $p^+_{\delta_1,\delta_2}$ *and* $p^+_{\delta_1}$*, respectively. For example, let* v_1 *be the solution to* (4.1.1)*. Then* $-v_1$ *is a solution to* (4.1.1) *with the coefficients* σ *replaced by* $-\sigma(-\cdot)$ *and* b *replaced by* $-b(-\cdot)$*. Since the functions* $-\sigma(-\cdot)$ *and* $-b(-\cdot)$ *also satisfy the hypotheses* ***P1*** *(or* ***P1'****) and* ***P2****, the probability density functions of the random vectors defined in* (4.1.9) *and* (4.1.13)*, which are* $p^-_{\delta_1,\delta_2}(\cdot,\cdot)$ *and/or* $p^-_{\delta_1}(\cdot,\cdot)$*, will also satisfy the estimates in* (4.1.10) *and* (4.1.14)*, respectively, provided* $p^+_{\delta_1,\delta_2}(\cdot,\cdot)$ *and/or* $p^+_{\delta_1}(\cdot,\cdot)$ *do.*

Proof of Theorem 4.1.1. We change slightly the notations in (2.2.1) to denote, for all positive integers n,

$$t^n_k := k2^{-nH_1^{-1}}, \quad x^n_l := l2^{-nH_2^{-1}}$$

and

$$I^n_k = [t^n_k, t^n_{k+1}], \quad J^n_l = [x^n_l, x^n_{l+1}], \quad R^n_{k,l} = I^n_k \times J^n_l.$$

Following the previous discussion, we have

$$
\begin{aligned}
&\mathrm{P}\left\{\inf_{(t,x)\in R^n_{k,l}} |v_1(t,x)-z| \le 2^{-n}\right\} \\
&\le \mathrm{P}\left\{|v_1(t^n_k,x^n_l)-z| \le 2^{-n} + \sup_{(t,x)\in R^n_{k,l}} |v_1(t,x)-v_1(t^n_k,x^n_l)|\right\} \\
&\le \mathrm{P}\left\{|v_1(t^n_k,x^n_l)-z| \le 2^{-n} + \sup_{(t,x)\in R^n_{k,l}} v_1(t,x)-v_1(t^n_k,x^n_l)\right\} \\
&\quad + \mathrm{P}\left\{|v_1(t^n_k,x^n_l)-z| \le 2^{-n} + \sup_{(t,x)\in R^n_{k,l}} (-v_1(t,x)+v_1(t^n_k,x^n_l))\right\} \\
&= \mathrm{P}\left\{|v_1(t^n_k,x^n_l)-z| \le 2^{-n} + \sup_{(t,x)\in R^n_{k,l}} v_1(t,x)-v_1(t^n_k,x^n_l)\right\} \\
&\quad + \mathrm{P}\left\{|-v_1(t^n_k,x^n_l)+z| \le 2^{-n} + \sup_{(t,x)\in R^n_{k,l}} (-v_1(t,x)-(-v_1(t^n_k,x^n_l)))\right\}.
\end{aligned}
\tag{4.1.16}
$$

We will show, using (4.1.10) for $p^+_{\delta_1,\delta_2}(\cdot,\cdot)$, that the first probability on the right-hand side of (4.1.16) is bounded by 2^{-n} times a constant, and the estimate for the second one is similar by using (4.1.10) for $p^-_{\delta_1,\delta_2}$. In fact, this probability is bounded above by

$$
\begin{aligned}
&\mathrm{P}\Bigg(\left\{|v_1(t^n_k,x^n_l)-z| \le 2^{-n} + \sup_{(t,x)\in R^n_{k,l}} v_1(t,x)-v_1(t^n_k,x^n_l)\right\} \\
&\qquad \cap \left\{\sup_{(t,x)\in R^n_{k,l}} v_1(t,x)-v_1(t^n_k,x^n_l) \ge 2\cdot 2^{-n}\right\}\Bigg) \\
&\quad + \mathrm{P}\left\{|v_1(t^n_k,x^n_l)-z| \le 3\cdot 2^{-n}\right\}.
\end{aligned}
\tag{4.1.17}
$$

For the first probability in (4.1.17), we apply the assumption (4.1.10) with

$$
\delta_1 = 2^{-nH_1^{-1}} \quad \text{and} \quad \delta_2 = 2^{-nH_2^{-1}}
\tag{4.1.18}
$$

to see that it is equal to

$$
\begin{aligned}
\int_{2\cdot 2^{-n}}^{\infty} dz_2 \int_{-z_2-2^{-n}+z}^{z_2+2^{-n}+z} dz_1\, p^+_{\delta_1,\delta_2}(z_1,z_2) &\le c\int_{2\cdot 2^{-n}}^{\infty} dz_2 \int_{-z_2-2^{-n}+z}^{z_2+2^{-n}+z} dz_1\, 2^n e^{-\frac{z_2^2}{c2^{-2n}}} \\
&= 2c2^n \int_{2\cdot 2^{-n}}^{\infty} (z_2+2^{-n}) e^{-\frac{z_2^2}{c2^{-2n}}}\, dz_2 = c' 2^{-n}
\end{aligned}
\tag{4.1.19}
$$

where the last equality holds by changing variables ($z_2 = 2^{-n}\tilde{z}_2$) to calculate the integral.

Since the density of $v_1(t^n_k,x^n_l)$ is bounded uniformly for $(t^n_k,x^n_l)\in I\times J$, the second probability in (4.1.17) is bounded above by 2^{-n} times a constant. Together with (4.1.17) and (4.1.19), we have obtained that

$$P\left\{|v_1(t_k^n,x_l^n)-z|\le 2^{-n}+\sup_{(t,x)\in R_{k,l}^n} v_1(t,x)-v_1(t_k^n,x_l^n)\right\}\le c2^{-n}. \tag{4.1.20}$$

Hence, we conclude that

$$P\left\{\inf_{(t,x)\in R_{k,l}^n}|v_1(t,x)-z|\le 2^{-n}\right\}\le c2^{-n}, \tag{4.1.21}$$

which implies

$$P\left\{\inf_{(t,x)\in R_{k,l}^n}\|v(t,x)-z\|\le 2^{-n}\right\}\le c2^{-dn}. \tag{4.1.22}$$

Now we use the estimate in (4.1.22) to prove Theorem 4.1.1(1), using the arguments in the proof of Theorem 2.1.4(b). Assume that $d-H_1^{-1}-H_2^{-1}\ge 0$, otherwise, there is nothing to prove. Fix $\epsilon\in\,]0,1[$ and $n\in\mathbb{N}$ such that $2^{-n-1}<\epsilon\le 2^{-n}$, and write

$$P\{v(I\times J)\cap B(z,\epsilon)\neq\emptyset\}\le\sum_{(k,l):R_{k,l}^n\cap I\times J\neq\emptyset}P\left\{v(R_{k,l}^n)\cap B(z,2^{-n})\neq\emptyset\right\}.$$

The number of pairs (k,l) involved in the sum is at most $2^{n(H_1^{-1}+H_2^{-1})}$ times a constant. The bound (4.1.22) implies that for all $z\in A$ and large n,

$$\begin{aligned}P\{v(I\times J)\cap B(z,\epsilon)\neq\emptyset\}&\le\tilde{C}2^{-nd}2^{n(H_1^{-1}+H_2^{-1})}\\&\le C\epsilon^{d-H_1^{-1}-H_2^{-1}}.\end{aligned} \tag{4.1.23}$$

Note that C does not depend on (n,ϵ). Therefore, (4.1.23) is valid for all $\epsilon\in\,]0,1[$.

Now we use a covering argument: Choose $\tilde{\epsilon}\in\,]0,1[$ and let $\{B_i\}_{i=1}\infty$ be a sequence of open balls in $\mathbb{R}^d$ with respective radii $r_i\in\,]0,\tilde{\epsilon}[$ such that

$$A\subseteq\cup_{i=1}^\infty B_i\quad\text{and}\quad\sum_{i=1}^\infty(2r_i)^{d-H_1^{-1}-H_2^{-1}}\le\mathscr{H}_{d-H_1^{-1}-H_2^{-1}}(A)+\tilde{\epsilon}. \tag{4.1.24}$$

Because $P\{v(I\times J)\cap A\neq\emptyset\}$ is at most $\sum_{i=1}^\infty P\{v(I\times J)\cap B_i\neq\emptyset\}$, the bounds in (4.1.23) and (4.1.24) together imply that

$$P\{v(I\times J)\cap A\neq\emptyset\}\le C\left(\mathscr{H}_{d-H_1^{-1}-H_2^{-1}}(A)+\tilde{\epsilon}\right). \tag{4.1.25}$$

Letting $\tilde{\epsilon}\to 0^+$, we obtain (4.1.11).

In order to prove Theorem 4.1.1(2), we assume that $d-H_1^{-1}\ge 0$. Similar to the derivation for

(4.1.22), using (4.1.14), we have

$$\mathrm{P}\left\{\inf_{t\in I_k^n}\|v(t,y_0)-z\|\le 2^{-n}\right\}\le c2^{-dn}, \tag{4.1.26}$$

where the constant c does not depend on $y_0 \in J$ nor n. Fix $\epsilon \in]0,1[$ and $n\in\mathbb{N}$ such that $2^{-n-1}<\epsilon\le 2^{-n}$. Then, by (4.1.26),

$$\begin{aligned}\mathrm{P}\left\{v(I\times\{y_0\})\cap B(z,\epsilon)\neq\emptyset\right\} &\le \sum_{k:I_k^n\cap I\neq\emptyset}\mathrm{P}\left\{v(I_k^n\times\{y_0\})\cap B(z,2^{-n})\neq\emptyset\right\}\\ &\le C2^{nH_1^{-1}}2^{-dn}\\ &\le \tilde{C}\epsilon^{d-H_1^{-1}}.\end{aligned}$$

Now use a covering argument, as we did to prove (1), which completes the proof of Theorem 4.1.1(2). □

4.2 Main results

We would like to verify that the assumptions in Theorem 4.1.1 hold for solutions to stochastic heat equations. *We will only consider here the linear case, where the solution is Gaussian.* We consider equation (4.1.1) with $\sigma\equiv 1$, $b\equiv 0$, that is, we consider the linear stochastic heat equation

$$\frac{\partial u}{\partial t}(t,x)=\frac{\partial^2 u}{\partial x^2}(t,x)+\dot{W}(t,x), \tag{4.2.1}$$

for $t\in[0,\infty[$ and $x\in[0,1]$, with initial condition $u(0,x)=0$ for all $x\in[0,1]$, and either Neumann or Dirichlet boundary conditions.

By definition, the solution is

$$u(t,x)=\int_0^t\int_0^1 G(t-r,x,v)W(dr,dv), \tag{4.2.2}$$

where the Green kernel $G(t,x,y)$ is given in (1.3.5) and (1.3.6), respectively.

We assume that the process $\{u(t,x):(t,x)\in[0,\infty[\times[0,1]\}$ given by (4.2.2) is the jointly continuous version (see (2.1.7)), which is almost $\frac{1}{4}$-Hölder continuous in time and almost $\frac{1}{2}$-Hölder continuous in space. In fact, for any $p\ge 1$, $(t,x),(s,y)\in[0,T]\times[0,1]$, there exists a constant $C=C(p,T)$ such that

$$\mathrm{E}[|u(t,x)-u(s,y)|^p]\le C(|t-s|^{1/2}+|x-y|)^{p/2}; \tag{4.2.3}$$

see also (2.1.6).

Choose two non-trivial compact intervals $I \subset [0,T]$ and $J \subset [0,1]$. In the case of Dirichlet boundary conditions, we assume that $J \subset]0,1[$. Choose $\delta_1 > 0$ and $(s_0, y_0) \in I \times J$. For $t \in [0,T]$, we denote

$$\bar{u}(t, y_0) = u(t, y_0) - u(s_0, y_0). \tag{4.2.4}$$

Set

$$F_1 = u(s_0, y_0), \quad F_2 = \sup_{t \in [s_0, s_0+\delta_1]} \bar{u}(t, y_0) \quad \text{and} \quad F = (F_1, F_2). \tag{4.2.5}$$

Choose $\delta_2 > 0$ such that $[y_0, y_0 + \delta_2] \subset [0,1]$; in the case of Dirichlet boundary conditions, we assume that $[y_0, y_0 + \delta_2] \subset]0,1[$ (open interval). Denote by M_0 the global supremum of u over $[0, \delta_1] \times [y_0, y_0 + \delta_2]$:

$$M_0 = \sup_{(t,x) \in [0,\delta_1] \times [y_0, y_0+\delta_2]} u(t,x). \tag{4.2.6}$$

We will also consider the random variable

$$\hat{u}(t,x) = u(t,x) - u(t,0), \tag{4.2.7}$$

and set

$$M = \sup_{(t,x) \in [0,T] \times [0,1]} \hat{u}(t,x). \tag{4.2.8}$$

Our goal is to give some estimates on the joint probability density function of F, which corresponds to (4.1.14) in Theorem 4.1.1, and on the probability density function of M_0. At the same time, we also want to know if the random variables F and M_0 have infinitely differentiable densities. Malliavin calculus is a tool to study the smoothness of random variables (see Theorem 1.5.1). It is clear that the first component of F belongs to $\mathbb{D}^\infty$. We will show that F_2 belongs to $\mathbb{D}^{1,2}$ in Lemma 4.4.4. However, we do not expect that F_2 belongs to $\mathbb{D}^\infty$. The same problem arises for M_0 and M. This means we can not apply the results in Theorem 1.5.1 and Corollary 1.5.3 directly.

Florit and Nualart [39] established a general criterion (Theorem 1.5.5) for smoothness of the density assuming that the components of the random vector only belong to $\mathbb{D}^{1,2}$. According to Theorem 1.5.5, instead of imposing nondegeneracy conditions on the Malliavin matrix, it is sufficient to assume that there exist some smooth random directions such that the derivatives of the components of the random vector along those directions form a smooth matrix whose determinant has negative moments of all orders. We will make use of these results to prove the smoothness of the densities of the random variables F, M_0 and M.

We first state the results on the smoothness of the densities of these random variables.

Theorem 4.2.1. *(a) For all $(s_0, y_0) \in]0,T] \times J$ and $\delta_1 > 0$, the random vector F has an infinitely* *differentiable density on* $\mathbb{R}\times]0,\infty[$ *and if* $y_0 \in]0,1[$, *then* $F_2 > 0$ *a.s. When* $s_0 = 0$, F_1 *vanishes identically but* F_2 *takes values in* $]0,\infty[$ *a.s. and has an infinitely differentiable density on* $]0,\infty[$.

(b) *For all* $y_0 \in [0,1]$, $\delta_1 > 0$ *and* $\delta_2 > 0$ *with* $[y_0, y_0+\delta_2] \subset [0,1]$ *(*$[y_0, y_0+\delta_2] \subset]0,1[$ *in the case of Dirichlet boundary conditions), the random variable* M_0 *takes values in* $]0,\infty[$ *and has an infinitely differentiable density on* $]0,\infty[$.

(c) *In the case of Neumann boundary conditions, the random variable* M *takes values in* $]0,\infty[$ *and has an infinitely differentiable density on* $]0,\infty[$.

Statements (a) and (b) of this theorem will be proved in Section 4.5. The method to prove statement (c) is different from that of (a) and (b). We will prove statement (c) in Section 4.8.

In the proof of Theorem 1.5.5 (see [64, Theorem 2.1.4]), the integration by parts formula provides us with a formula for the density of the random vector F, from which we are able to analyze the behavior of the density. We remark that choice of u_A in Theorem 1.5.5 is not unique. We will choose a suitable adapted process so that the Skorohod integral coincides with the Walsh integral and hence we can use Burkholder's inequality instead of Hölder's inequality for Malliavin norms (see [82, Proposition 1.10, p.50]) to estimate the moments of this stochastic integral. This will allow us to give a Gaussian-type upper bound on this density.

In order to estimate the density of F, we assume $I \times J \subset]0,T]\times]0,1[$. Assume that there are constants c_1, C_1 such that

$$0 < c_1 < \underline{I} := \inf\{s : s \in I\} \quad \text{and} \quad \bar{I} := \sup\{s : s \in I\} < C_1 < T+1. \tag{4.2.9}$$

Assume also that there are constants c_2, C_2 such that

$$0 < c_2 < \underline{J} := \inf\{y : y \in J\} \quad \text{and} \quad \bar{J} := \sup\{y : y \in J\} < C_2 < 1. \tag{4.2.10}$$

Choose $\delta_1 \in]0,1[$ small enough so that

$$s_0 + \delta_1 \in I \quad \text{and} \quad \delta_1^{1/2} < \min\left\{\underline{J} - c_2, (C_2 - \bar{J})/2\right\}; \tag{4.2.11}$$

see Figure 4.1.

Denote $(z_1, z_2) \mapsto p(z_1, z_2)$ the probability density function of random vector F with δ_1 satisfying the conditions in (4.2.11) (the existence of $p(\cdot,\cdot)$ is assured by Theorem 4.2.1(a)).

Theorem 4.2.2. *Assume* $I \times J \subset]0,T]\times]0,1[$. *There exists a positive constant* $c = c(I, J)$ *such that*

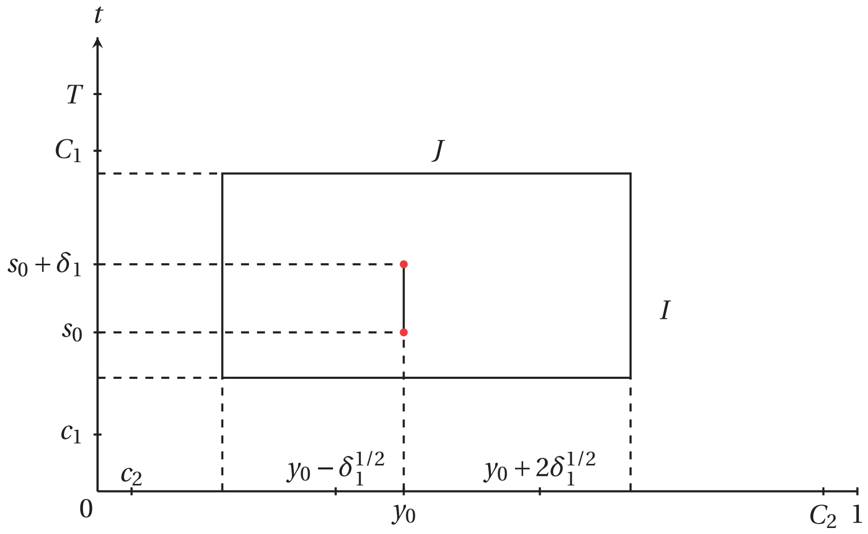

Figure 4.1 – Illustration of conditions (4.2.9)–(4.2.11)

for all $\delta_1 > 0$ *satisfying* (4.2.11), *and for all* $z_2 \geq \delta_1^{1/4}$, $z_1 \in \mathbb{R}$ *and any* $(s_0, y_0) \in I \times J$,

$$p(z_1, z_2) \leq \frac{c}{\sqrt{\delta_1^{1/2}}} \exp\left(-\frac{z_2^2}{c\delta_1^{1/2}}\right) (|z_1|^{-1/4} \wedge 1) \exp(-z_1^2/c) \tag{4.2.12}$$

$$\leq \frac{c}{\sqrt{\delta_1^{1/2}}} \exp\left(-\frac{z_2^2}{c\delta_1^{1/2}}\right). \tag{4.2.13}$$

The proof of this theorem will be presented in Section 4.6. Note that (4.2.13) is an immediate consequence of (4.2.12). As a consequence of Theorems 4.2.1 and 4.2.2, we deduce the following.

Corollary 4.2.3. *Let I and J be as above* (4.2.4). *The random variable F_2 has an infinitely differentiable density on* $]0, \infty[$, *denoted by* $z_2 \mapsto p_{F_2}(z_2)$. *Suppose that* $I \times J \subset]0, T] \times]0, 1[$. *Then there exists a positive constant* $c = c(I, J)$ *such that for all* $\delta_1 > 0$ *satisfying* (4.2.11), *and for all* $z_2 \geq \delta_1^{1/4}$ *and any* $(s_0, y_0) \in I \times J$,

$$p_{F_2}(z_2) \leq \frac{c}{\sqrt{\delta_1^{1/2}}} \exp\left(-\frac{z_2^2}{c\delta_1^{1/2}}\right). \tag{4.2.14}$$

Remark 4.2.4. *By Theorem 4.2.2 and Remark 4.1.2(b), the assumption* (4.1.14) *of Theorem 4.1.1 is satisfied for the solution to* (4.2.1) *with* $H_1 = \frac{1}{4}$. *Therefore, Theorem 4.2.2 provides an alternative proof of [25, Theorem 3.1(3)] with* $\beta = d$ *for the upper bound on hitting probabilities at a fixed spatial position.*

We will also give a Gaussian-type upper bound on the density of M_0 under the assumption

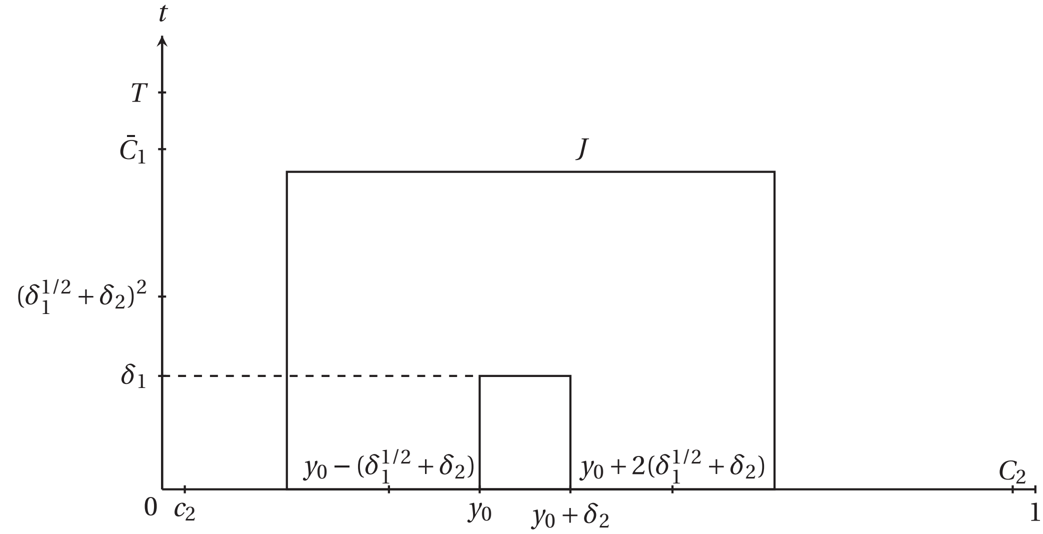

Figure 4.2 – Illustration of condition (4.2.15)

$y_0 \in J \subset]0,1[$. Choose a positive constant $\bar{C}_1$ with $\bar{C}_1 < T$. Let c_2, C_2 be chosen as in (4.2.10). Choose $\delta_1, \delta_2 \in]0,1[$ small enough so that

$$y_0+\delta_2 \in J, \quad (\delta_1^{1/2}+\delta_2)^2 < \bar{C}_1 \quad \text{and} \quad \delta_1^{1/2}+\delta_2 < \min\left\{\underline{J}-c_2, (C_2-\bar{J})/2\right\}; \tag{4.2.15}$$

see Figure 4.2.

Denote $z \mapsto p_0(z)$ the probability density function of random variable M_0 with δ_1, δ_2 satisfying the conditions in (4.2.15) (the existence of $p_0(\cdot)$ is assured by Theorem 4.2.1(b)).

Theorem 4.2.5. *Assume $J \subset]0,1[$. There exists a finite positive constant $c = c(T,J)$ such that for all δ_1, δ_2 satisfying the conditions in (4.2.15), for all $y_0 \in J$ and $z \geq (\delta_1^{1/2}+\delta_2)^{1/2}$,*

$$p_0(z) \leq \frac{c}{\sqrt{\delta_1^{1/2}+\delta_2}} \exp\left(-\frac{z^2}{c\,(\delta_1^{1/2}+\delta_2)}\right). \tag{4.2.16}$$

The proof of Theorem 4.2.5 will be presented in Section 4.7.

4.3 Preliminaries

In this section, we assume that I and J are as above (4.2.4) and we will introduce two families of random variables to control the value of the random variable F_2 and M_0, respectively. For this purpose, we will give some estimates on the rectangular increments of the solution.

Choose an integer p_0 and $\gamma_0 \in \mathbb{R}$ such that

$$p_0 > \gamma_0 > 4. \tag{4.3.1}$$

For $r \in [s_0, s_0 + \delta_1]$, define the following family of random variables:

$$Y_r := \int_{[s_0,r]^2} \frac{(u(t,y_0) - u(s,y_0))^{2p_0}}{|t-s|^{\gamma_0/2}} \, ds dt. \tag{4.3.2}$$

By (4.2.3) and the choice of p_0, γ_0 in (4.3.1),

$$\int_{[s_0,r]^2} \frac{\mathrm{E}[(u(t,y_0) - u(s,y_0))^{2p_0}]}{|t-s|^{\gamma_0/2}} \, ds dt \le c \int_{[s_0,r]^2} \frac{|t-s|^{p_0/2}}{|t-s|^{\gamma_0/2}} \, ds dt < \infty. \tag{4.3.3}$$

Hence for all $r \in [s_0, s_0 + \delta_1]$, the random variable Y_r is finite a.s. Moreover, by Hölder's inequality and (4.2.3), for any $p \ge 1$, there exists a constant c_p, not depending on $(s_0, y_0) \in [0,T] \times [0,1]$, such that for all $r \in [s_0, s_0 + \delta_1]$,

$$\begin{aligned}
\mathrm{E}[|Y_r|^p] &\le (r - s_0)^{2(p-1)} \int_{[s_0,r]^2} \frac{\mathrm{E}[|u(t,y_0) - u(s,y_0)|^{2p_0 p}]}{|t-s|^{\gamma_0 p/2}} \, ds dt \\
&\le c_p \, (r - s_0)^{2(p-1)} \int_{[s_0,r]^2} \frac{|t-s|^{p_0 p/2}}{|t-s|^{\gamma_0 p/2}} \, ds dt \\
&\le c_p \, (r - s_0)^{2p} \delta_1^{(p_0 - \gamma_0)p/2}.
\end{aligned} \tag{4.3.4}$$

The following result shows that the family of random variables $\{Y_r : r \in [s_0, s_0 + \delta_1]\}$ can control the value of the supremum F_2.

Lemma 4.3.1. *There exists a finite positive constant c, not depending on $(s_0, y_0) \in [0,T] \times [0,1]$, such that for any $a > 0$, for all $\delta_1 > 0$ and for all $r \in [s_0, s_0 + \delta_1]$,*

$$Y_r \le R := c \, a^{2p_0} \delta_1^{-(\gamma_0 - 4)/2} \quad \Rightarrow \quad \sup_{t \in [s_0, r]} |\bar{u}(t, y_0)| \le a. \tag{4.3.5}$$

Proof. We first apply the Garsia, Rodemich, and Rumsey lemma (see Lemma A.6.1) with

$$\begin{aligned}
&S := [s_0, r], \; \rho(t,s) := |t-s|^{1/2}, \; \mu(dt) := dt, \\
&\Psi(x) := x^{2p_0}, \; p(x) := x^{\frac{\gamma_0}{2p_0}} \quad \text{and} \quad f := u(\cdot, y_0).
\end{aligned}$$

By (A.34), we deduce that for all $t, s \in [s_0, r]$,

$$
\begin{aligned}
|u(t,y_0) - u(s,y_0)| &\le 10 \int_0^{2\rho(t,s)} \frac{Y_r^{\frac{1}{2p_0}}}{[\mu(B_\rho(s,u/4))]^{1/p_0}} u^{\frac{\gamma_0}{2p_0}-1} du \\
&\le c_1\, Y_r^{\frac{1}{2p_0}} \int_0^{2\rho(t,s)} u^{-\frac{2}{p_0}} u^{\frac{\gamma_0}{2p_0}-1} du \\
&= c_2\, |t-s|^{\frac{\gamma_0-4}{4p_0}}\, Y_r^{\frac{1}{2p_0}} \\
&\le c_2\, \delta_1^{\frac{\gamma_0-4}{4p_0}}\, Y_r^{\frac{1}{2p_0}},
\end{aligned}
$$

where we have used (4.3.1); the constants c_1, c_2 do not depend on r, nor on $(s_0, y_0) \in [0, T] \times [0,1]$. Assuming $Y_r \le R$, letting $s = s_0$ in the above inequality and choosing a suitable constant in the definition of R, we obtain that

$$
\sup_{t\in[s_0,r]} |\bar{u}(t,y_0)| \le \delta_1^{(\gamma_0-4)/(4p_0)} (a^{2p_0} \delta_1^{-(\gamma_0-4)/2})^{1/(2p_0)} = a.
$$

□

We will also introduce a family of random variables to control the value of the supremum M_0. We first give an estimate on the rectangular increments of the solution.

Lemma 4.3.2. *There exists a constant C_T such that for any $\theta \in]0, \frac{1}{2}[$ and $(t,s,x,y) \in [0,T]^2 \times [0,1]^2$,*

$$
\begin{aligned}
E[(u(t,x) + u(s,y) - u(t,y) - u(s,x))^2] &\le C_T |t-s|^{\frac{1}{2}} \wedge |x-y| \\
&\le C_T |t-s|^{\frac{1}{2}-\theta} |x-y|^{2\theta}. \qquad (4.3.6)
\end{aligned}
$$

Proof. The second inequality is trivial. To prove the first inequality, on the one hand, by (4.2.3),

$$
\begin{aligned}
&\mathrm{E}[(u(t,x) + u(s,y) - u(t,y) - u(s,x))^2] \\
&\quad \le 2\mathrm{E}[(u(t,x) - u(s,x))^2] + 2\mathrm{E}[(u(s,y) - u(t,y))^2] \\
&\quad \le C_T |t-s|^{\frac{1}{2}}. \qquad (4.3.7)
\end{aligned}
$$

On the other hand, using (4.2.3) again, we have

$$
\begin{aligned}
&\mathrm{E}[(u(t,x) + u(s,y) - u(t,y) - u(s,x))^2] \\
&\quad \le 2\mathrm{E}[(u(t,x) - u(t,y))^2] + 2\mathrm{E}[(u(s,x) - u(s,y))^2] \\
&\quad \le C_T |x-y|. \qquad (4.3.8)
\end{aligned}
$$

Hence (4.3.7) and (4.3.8) establish (4.3.6). □

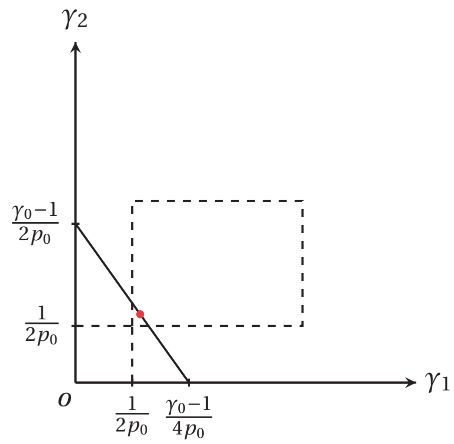

Figure 4.3 – Illustration of (4.3.11) and (4.3.12)

From now on, we fix $\theta \in]0, \frac{1}{2}[$ and set

$$\theta_1 = \frac{1}{2} - \theta, \quad \theta_2 = 2\theta. \tag{4.3.9}$$

By the isometry and Lemma 4.3.2, since $Du(t,x) = 1_{\{\cdot<t\}} G(t-\cdot, x, *)$,

$$\begin{aligned}\|D(u(t,x)+u(s,y)-u(t,y)-u(s,x))\|^2_{\mathscr{H}} &= \mathrm{E}[(u(t,x)+u(s,y)-u(t,y)-u(s,x))^2] \\ &\le C_T|t-s|^{\theta_1}|x-y|^{\theta_2},\end{aligned} \tag{4.3.10}$$

for any $(t,s,x,y) \in [0,T]^2 \times [0,1]^2$.

Let p_0 and γ_0 be defined as in (4.3.1). Let θ_1 and θ_2 be defined as in (4.3.9). We assume that p_0 is sufficiently large so that there exist γ_1, γ_2 such that

$$\frac{1}{2p_0} < \gamma_1 < \theta_1/2 - \frac{1}{2p_0}, \quad \frac{1}{2p_0} < \gamma_2 < \theta_2/2 - \frac{1}{2p_0}, \tag{4.3.11}$$

and

$$2\gamma_1 + \gamma_2 = \frac{\gamma_0 - 1}{2p_0}; \tag{4.3.12}$$

see Figure 4.3. Denote

$$\delta := \delta_1^{1/2} + \delta_2, \quad \Delta_\bullet := \delta^2 \quad \text{and} \quad \Delta_* := \delta \wedge (1-y_0). \tag{4.3.13}$$

For $r \in [0, \Delta_\bullet]$, we define

$$Y_0(r) := \int_{[0,r]^2} \frac{(u(t,y_0)-u(s,y_0))^{2p_0}}{|t-s|^{\gamma_0/2}} ds dt, \tag{4.3.14}$$

and

$$Y_1(r) := \int_{[0,r]^2} dtds \int_{[y_0,y_0+\Delta_*]^2} dxdy \frac{(u(t,x)+u(s,y)-u(t,y)-u(s,x))^{2p_0}}{|t-s|^{1+2p_0\gamma_1}|x-y|^{1+2p_0\gamma_2}}. \tag{4.3.15}$$

By Lemma 4.3.2, the choice of γ_1, γ_2 in (4.3.11) and the Gaussian property of the solution,

$$\begin{aligned}
&\int_{[0,r]^2} dtds \int_{[y_0,y_0+\Delta_*]^2} dxdy \frac{\mathrm{E}[(u(t,x)+u(s,y)-u(t,y)-u(s,x))^{2p_0}]}{|t-s|^{1+2p_0\gamma_1}|x-y|^{1+2p_0\gamma_2}} \\
&\quad \le c \int_{[0,r]^2} dtds \int_{[y_0,y_0+\Delta_*]^2} dxdy \frac{|t-s|^{p_0\theta_1}|x-y|^{p_0\theta_2}}{|t-s|^{1+2p_0\gamma_1}|x-y|^{1+2p_0\gamma_2}} < \infty.
\end{aligned}$$

Hence for all $r \in [0,\Delta_\bullet]$, the random variable $Y_1(r)$ is finite a.s. Moreover, by Hölder's inequality and (4.3.10), for any $p \ge 1$, there exists a constant c_p, not depending on $y_0 \in [0,1]$, such that for any $r \in [0,\Delta_\bullet]$,

$$\begin{aligned}
\mathrm{E}[|Y_1(r)|^p] &\le (r\Delta_*)^{2(p-1)} \int_{[0,r]^2} dtds \int_{[y_0,y_0+\Delta_*]^2} dxdy \\
&\qquad\qquad \times \frac{\mathrm{E}[|u(t,x)+u(s,y)-u(t,y)-u(s,x)|^{2p_0p}]}{|t-s|^{p(1+2p_0\gamma_1)}|x-y|^{p(1+2p_0\gamma_2)}} \\
&\le c_p\, (r\Delta_*)^{2(p-1)} \int_{[0,r]^2} dtds \int_{[y_0,y_0+\Delta_*]^2} dxdy \\
&\qquad\qquad \times \frac{|t-s|^{p_0p\theta_1}|x-y|^{p_0p\theta_2}}{|t-s|^{p(1+2p_0\gamma_1)}|x-y|^{p(1+2p_0\gamma_2)}} \\
&\le c_p\, (r\Delta_*)^{2p} \Delta_\bullet^{p(p_0\theta_1-(1+2p_0\gamma_1))} \Delta_*^{p(p_0\theta_2-(1+2p_0\gamma_2))} \\
&\le c_p\, r^{2p} \delta^{p(2p_0\theta_1-2(1+2p_0\gamma_1))} \delta^{p(p_0\theta_2-(1+2p_0\gamma_2)+2)} \\
&= c_p\, r^{2p} \delta^{p(p_0(2\theta_1+\theta_2)-2p_0(2\gamma_1+\gamma_2)-1)} \\
&= c_p\, r^{2p} \delta^{p(p_0-\gamma_0)},
\end{aligned} \tag{4.3.16}$$

where in the last inequality we use (4.3.13), and in the second equality we use (4.3.12) and the fact that $2\theta_1+\theta_2 = 1$ from the definition of θ_1, θ_2 in (4.3.9).

For $r \in [0,\Delta_\bullet]$, set

$$\bar{Y}_r := Y_0(r) + Y_1(r). \tag{4.3.17}$$

By (4.3.16) and the calculation in (4.3.4), for any $p \ge 1$, there exists a constant c_p, not depending on $y_0 \in [0,1]$, such that for any $r \in [0,\Delta_\bullet]$,

$$\mathrm{E}[|\bar{Y}_r)|^p] \le c_p\, r^{2p} \delta^{p(p_0-\gamma_0)}. \tag{4.3.18}$$

To see that the family of random variables $\{\bar{Y}_r : r \in [0,\Delta_\bullet]\}$ controls the value of the supremum of M_0, we need to use the Garsia, Rodemich, and Rumsey lemma for Banach space valued functions (see [64, Lemma A.3.1] and our Lemma A.6.2). Indeed, we will write, for $(t,x) \in$

$[0,T]\times[0,1]$,

$$u(t,x)=\check{u}(t,x)+u(t,y_0), \tag{4.3.19}$$

where

$$\check{u}(t,x)=u(t,x)-u(t,y_0). \tag{4.3.20}$$

For fixed t, $\check{u}(t,*)$ belongs to the Banach space $E_{p,\gamma}[y_0,y_0+\Delta_*]$ which we now define.

For an integer p, an arbitrary $\gamma\in]\frac{1}{2p},1[$ and a continuous function f defined on $[a,b]$, we define the Hölder seminorm

$$\|f\|_{p,\gamma}:=\left(\int_{[a,b]^2}\frac{|f(x)-f(y)|^{2p}}{|x-y|^{1+2p\gamma}}dxdy\right)^{\frac{1}{2p}}. \tag{4.3.21}$$

$E_{p,\gamma}[a,b]$ denotes the space of continuous functions vanishing at a and having a finite $\|\cdot\|_{p,\gamma}$ norm. We omit $[a,b]$ if this interval is clear from the context. Each element of $E_{p,\gamma}$ turns out to be Hölder continuous. Indeed, we apply the Garsia, Rodemich and Rumsey lemma (see Lemma A.6.2) to the real-valued function f with $\Psi(x)=x^{2p}$, $p(x)=x^{(1+2p\gamma)/(2p)}$, $d=1$ to get that there exists a constant c such that for all x, $y\in[a,b]$,

$$|f(x)-f(y)|\leq c\,|x-y|^{\gamma-\frac{1}{2p}}\,\|f\|_{p,\gamma}.$$

Moreover, as a fractional Sobolev space, $E_{p,\gamma}[a,b]$ is a separable Banach space; see [35, Proposition 4.24].

Since for any $\epsilon>0$, a.s., for any fixed t, the function $x\mapsto\check{u}(t,x)$ is $\frac{1}{2}-\epsilon$-Hölder continuous, it follows that $\check{u}(t,*)$ belongs to the Banach space $E_{p_0,\gamma_2}[y_0,y_0+\Delta_*]$ with p_0, γ_2 as defined in (4.3.1) and (4.3.11). We establish the following lemma to study the continuity of the map $t\mapsto\check{u}(t,*)$ in the Banach space $E_{p_0,\gamma_2}[y_0,y_0+\Delta_*]$.

Lemma 4.3.3. *For any $0<\xi<\theta_1/2$ and $0<\eta<\theta_2/2$, there exists a random variable C that is a.s. finite such that a.s., for all $(t,s,x,y)\in[0,T]^2\times[0,1]^2$,*

$$|u(t,x)+u(s,y)-u(t,y)-u(s,x)|\leq C|t-s|^{\xi}|x-y|^{\eta}. \tag{4.3.22}$$

Remark 4.3.4. *This property is also established in [43, Theorem 5.2].*

Proof of Lemma 4.3.3. Let $\hat{u}=\{\hat{u}(t,x):(t,x)\in[0,\infty[\times[0,1]\}$ be the random field defined in (4.2.7).

We choose p, $\bar{\gamma}_2$ such that $\xi<\theta_1/2-\frac{1}{2p}$ and $\eta+\frac{1}{2p}<\bar{\gamma}_2<\theta_2/2-\frac{1}{2p}$. Let $E_{p,\bar{\gamma}_2}[0,1]$ be the space of continuous functions defined on $[0,1]$ vanishing at 0 and having a finite $\|\cdot\|_{p,\bar{\gamma}_2}$ norm. Since a.s., for any $t\in[0,T]$, $x\mapsto\hat{u}(t,x)$ is almost $\frac{1}{2}$-Hölder continuous, we see that $\hat{u}(t,*)$ belongs to

$E_{p,\bar{\gamma}_2}$. Moreover, by (4.3.6), for any $s,\ t \in [0,T]$,

$$\begin{aligned}
\mathrm{E}[\|\hat{u}(t,*)-\hat{u}(s,*)\|_{p,\bar{\gamma}_2}^{2p}] &= \int_{[0,1]^2} \frac{\mathrm{E}[|u(t,x)+u(s,y)-u(t,y)-u(s,x)|^{2p}]}{|x-y|^{1+2p\bar{\gamma}_2}} dxdy \\
&\leq C_T|t-s|^{\theta_1 p} \int_{[0,1]^2} \frac{|x-y|^{\theta_2 p}}{|x-y|^{1+2p\bar{\gamma}_2}} dxdy \\
&\leq C_T|t-s|^{\theta_1 p}.
\end{aligned}$$

We apply the Kolmogorov continuity theorem (see [76, Theorem 2.1]) to see that the process $\{\hat{u}(t,*) : t \in [0,T]\}$ has a continuous version $\{\tilde{u}(t,*) : t \in [0,T]\}$ with values in $E_{p,\bar{\gamma}_2}$, which is $\frac{\theta_1}{2} - \frac{1}{2p} - \epsilon$-Hölder continuous for small ϵ such that $\frac{\theta_1}{2} - \frac{1}{2p} - \epsilon > \xi$, namely, there exists a random variable C, finite almost surely, such that a.s. for any $s,\ t \in [0,T]$,

$$\|\tilde{u}(t,*)-\tilde{u}(s,*)\|_{p,\bar{\gamma}_2} \leq C|t-s|^{\frac{\theta_1}{2}-\frac{1}{2p}-\epsilon}.$$

Hence we have for any $s,\ t \in [0,T]$,

$$\int_{[0,1]^2} \frac{|\tilde{u}(t,x)-\tilde{u}(s,x)-\tilde{u}(t,y)+\tilde{u}(s,y)|^{2p}}{|x-y|^{1+2p\bar{\gamma}_2}} dxdy \leq C|t-s|^{(\frac{\theta_1}{2}-\frac{1}{2p}-\epsilon)2p}.$$

We apply the Garsia, Rodemich and Rumsey lemma (see Lemma A.6.2) to the real-valued function $x \mapsto \tilde{u}(t,x)-\tilde{u}(s,x)$ with $\Psi(x) = x^{2p}, p(x) = x^{(1+2p\bar{\gamma}_2)/(2p)}, d = 1$, to get that for any $(t,s,x,y) \in [0,T]^2 \times [0,1]^2$,

$$\begin{aligned}
|\tilde{u}(t,x)-\tilde{u}(s,x)-\tilde{u}(t,y)+\tilde{u}(s,y)| &\leq C|t-s|^{\frac{\theta_1}{2}-\frac{1}{2p}-\epsilon}|x-y|^{\bar{\gamma}_2-\frac{1}{2p}} \\
&\leq C|t-s|^{\xi}|x-y|^{\eta}. \qquad (4.3.23)
\end{aligned}$$

Letting $y = 0$ in (4.3.23), we obtain

$$|\tilde{u}(t,x)-\tilde{u}(s,x)| \leq C|t-s|^{\xi}. \qquad (4.3.24)$$

Fix $(s,y) \in [0,T] \times [0,1]$. Using the triangle inequality,

$$|\tilde{u}(t,x)-\tilde{u}(s,y)| \leq |\tilde{u}(t,x)-\tilde{u}(s,x)| + |\tilde{u}(s,x)-\tilde{u}(s,y)|,$$

which converges to 0 as $(t,x) \to (s,y)$ by (4.3.24) and the fact that $x \mapsto \tilde{u}(s,x)$ is continuous since $\tilde{u}(s,*) \in E_{p,\bar{\gamma}_2}$. Therefore, a.s., $(t,x) \mapsto \tilde{u}(t,x)$ is continuous. Together with the fact that for any $t \in [0,T]$, $\mathrm{P}\{\hat{u}(t,*) = \tilde{u}(t,*)\} = 1$, we obtain that the processes $\{\hat{u}(t,x) : (t,x) \in [0,T] \times [0,1]\}$ and $\{\tilde{u}(t,x) : (t,x) \in [0,T] \times [0,1]\}$ are indistinguishable and hence (4.3.23) implies (4.3.22). □

Choose ξ, η as in Lemma 4.3.3 such that $\eta > \gamma_2 + 1/(2p_0)$, which is possible by (4.3.11). Then, by (4.3.22),

$$\begin{aligned}\|\check{u}(t,*)-\check{u}(s,*)\|_{p_0,\gamma_2}^{2p_0} &= \int_{[y_0,y_0+\Delta_*]^2}\frac{(u(t,x)+u(s,y)-u(t,y)-u(s,x))^{2p_0}}{|x-y|^{1+2p_0\gamma_2}}dxdy\\ &\le C|t-s|^{2p_0\xi}\int_{[y_0,y_0+\Delta_*]^2}|x-y|^{2p_0\eta-1-2p_0\gamma_2}dxdy\\ &\le C|t-s|^{2p_0\xi} \qquad (4.3.25)\end{aligned}$$

since $2p_0\eta-1-2p_0\gamma_2>0$, which shows that a.s., $t\mapsto\check{u}(t,*)$ is continuous in $E_{p_0,\gamma_2}[y_0,y_0+\Delta_*]$ (the space of continuous functions defined on $[y_0,y_0+\Delta_*]$ vanishing at y_0 and having a finite $\|\cdot\|_{p_0,\gamma_2}$ norm). Similarly, we can prove that a.s., $x\mapsto u(\cdot,x)$ is continuous in $E_{p_0,\gamma_1}[0,T]$ (the space of continuous functions defined on $[0,T]$ vanishing at 0 and having a finite $\|\cdot\|_{p_0,\gamma_1}$ norm), where γ_1, p_0 are defined in (4.3.11) and (4.3.1).

As a consequence of Lemma 4.3.3, we can write, for $r\in[0,\Delta_\bullet]$,

$$\begin{aligned}Y_1(r) &= \int_{[0,r]^2}dtds\int_{[y_0,y_0+\Delta_*]^2}dxdy\,\frac{(u(t,x)+u(s,y)-u(t,y)-u(s,x))^{2p_0}}{|t-s|^{1+2p_0\gamma_1}|x-y|^{1+2p_0\gamma_2}}\\ &= \int_{[0,r]^2}\frac{\|\check{u}(t,*)-\check{u}(s,*)\|_{p_0,\gamma_2}^{2p_0}}{|t-s|^{1+2p_0\gamma_1}}dtds. \qquad (4.3.26)\end{aligned}$$

We are now ready to show that the family of random variables $\{\bar{Y}_r : r\in[0,\Delta_\bullet]\}$ defined in (4.3.17) controls the value of the supremum of M_0.

Lemma 4.3.5. *There exists a finite positive constant c, not depending on $y_0\in[0,1]$, such that for any $\bar{a}>0$, $\delta_1>0$, $\delta_2>0$ and for all $r\in[0,\Delta_\bullet]$,*

$$\bar{Y}_r\le\bar{R}:=c\,\bar{a}^{2p_0}\delta^{4-\gamma_0}\quad\Rightarrow\quad\sup_{(t,x)\in[0,r]\times[y_0,y_0+\delta_2]}|u(t,x)|\le\bar{a}. \qquad (4.3.27)$$

Proof. Assuming $Y_0(r)\le\bar{R}$, similar to the proof of Lemma 4.3.1, by the Garsia, Rodemich and Rumsey lemma (see Lemma A.6.1), we deduce that for all t, $s\in[0,r]$,

$$\begin{aligned}|u(t,y_0)-u(s,y_0)| &\le c'\,|t-s|^{\frac{\gamma_0-4}{4p_0}}\,Y_0(r)^{\frac{1}{2p_0}}\\ &\le c_1\,\Delta_\bullet^{\frac{\gamma_0-4}{4p_0}}\,Y_0(r)^{\frac{1}{2p_0}}=c_1\,\delta^{\frac{\gamma_0-4}{2p_0}}\,Y_0(r)^{\frac{1}{2p_0}}, \qquad (4.3.28)\end{aligned}$$

where the constant c_1 does not depend on r, nor on $y_0\in[0,1]$. Letting $s=0$ in (4.3.28), we obtain

$$\sup_{t\in[0,r]}|u(t,y_0)|\le c_1\,\delta^{\frac{\gamma_0-4}{2p_0}}\,Y_0(r)^{\frac{1}{2p_0}}\le c_1\,\delta^{\frac{\gamma_0-4}{2p_0}}\,\bar{R}^{\frac{1}{2p_0}}. \qquad (4.3.29)$$

Hence we can choose a suitable constant c in the definition of $\bar{R}$ in (4.3.27) so that

$$\sup_{t\in[0,r]} |u(t,y_0)| \le \frac{\bar{a}}{2}. \tag{4.3.30}$$

Assuming $Y_1(r) \le \bar{R}$, from the expression of $Y_1(r)$ in (4.3.26), we first apply the Garsia, Rodemich, and Rumsey lemma (see Lemma A.6.2) to the $E_{p_0,\gamma_2}[y_0, y_0+\Delta_*]$-valued function $s \mapsto \check{u}(s,*)$ with $\Psi(x) = x^{2p_0}$, $p(x) = x^{(1+2p_0\gamma_1)/(2p_0)}$ to deduce that there exists a constant c_2 such that for all $t, s \in [0,r]$,

$$\begin{aligned}
\|\check{u}(t,*) - \check{u}(s,*)\|_{p_0,\gamma_2} &\le c'\, Y_1(r)^{\frac{1}{2p_0}} \int_0^{2|t-s|} x^{-\frac{1}{p_0}} x^{\frac{1+2p_0\gamma_1}{2p_0}-1} dx \\
&= c_2\, Y_1(r)^{\frac{1}{2p_0}} |t-s|^{\frac{2p_0\gamma_1-1}{2p_0}} \\
&\le c_2\, Y_1(r)^{\frac{1}{2p_0}} \Delta_{\bullet}^{\frac{2p_0\gamma_1-1}{2p_0}} = c_2\, Y_1(r)^{\frac{1}{2p_0}} \delta^{\frac{2(2p_0\gamma_1-1)}{2p_0}}.
\end{aligned} \tag{4.3.31}$$

Letting $s=0$, we obtain for all $t\in[0,r]$,

$$\|\check{u}(t,*)\|_{p_0,\gamma_2}^{2p_0} \le c_2\, Y_1(r)\delta^{2(2p_0\gamma_1-1)}.$$

Applying the same lemma to the real-valued function $x \mapsto \check{u}(t,x)$ (t is now fixed) with $\Psi(x) = x^{2p_0}$, $p(x) = x^{(1+2p_0\gamma_2)/(2p_0)}$, we obtain

$$|\check{u}(t,x) - \check{u}(t,y)| \le c_3\, Y_1(r)^{\frac{1}{2p_0}} \delta^{\frac{2(2p_0\gamma_1-1)}{2p_0}} |x-y|^{\frac{2p_0\gamma_2-1}{2p_0}},$$

for all $x, y \in [y_0, y_0+\Delta_*]$. Letting $y = y_0$ we obtain that for all $(t,x) \in [0,r]\times[y_0, y_0+\Delta_*]$,

$$\begin{aligned}
|u(t,x) - u(t,y_0)| &\le c_3\, Y_1(r)^{\frac{1}{2p_0}} \delta^{\frac{2(2p_0\gamma_1-1)}{2p_0}} \Delta_*^{\frac{2p_0\gamma_2-1}{2p_0}} \\
&\le c_3\, Y_1(r)^{\frac{1}{2p_0}} \delta^{\frac{2(2p_0\gamma_1-1)}{2p_0}} \delta^{\frac{2p_0\gamma_2-1}{2p_0}} \\
&= c_3\, Y_1(r)^{\frac{1}{2p_0}} \delta^{\frac{\gamma_0-4}{2p_0}},
\end{aligned}$$

where in the second inequality we use (4.3.13), and the equality is due to (4.3.12). In particular, this implies that

$$\sup_{(t,x)\in[0,r]\times[y_0,y_0+\delta_2]} |u(t,x) - u(t,y_0)| \le c_3\, Y_1(r)^{\frac{1}{2p_0}} \delta^{\frac{\gamma_0-4}{2p_0}}. \tag{4.3.32}$$

We can choose the constant c in the definition of $\bar{R}$ in (4.3.27) small so that (4.3.30) holds and

$$\sup_{(t,x)\in[0,r]\times[y_0,y_0+\delta_2]} |u(t,x) - u(t,y_0)| \le \frac{\bar{a}}{2}. \tag{4.3.33}$$

Hence, by (4.3.30), (4.3.33) and the triangle inequality, we obtain (4.3.27). □

We conclude this section by introducing a result on the uniqueness of the solution to heat equation with boundary conditions, which will be used when we check the condition (iii) of Theorem 1.5.5.

Let $f : [0,\infty[\mapsto \mathbb{R}$ be a differentiable function with continuous derivative satisfying $f(0) = 0$. Let $g \in C^{\infty}([0,1])$ satisfy the same boundary conditions as the Green kernel. We define

$$A(t,x) = \int_0^t \int_0^1 G(t-r,x,v)\left(\frac{\partial}{\partial r} - \frac{\partial}{\partial v^2}\right)(f(r)g(v))\,dv\,dr, \quad t > 0,\ x \in [0,1],$$
$$A(0,x) = 0, \quad x \in [0,1].$$

Lemma 4.3.6. *The function A is well-defined and we have $A(t,x) = f(t)g(x)$ for all $(t,x) \in [0,\infty[\times[0,1]$.*

Proof. It is clear that the function A is well-defined since both the Green kernel and the function $(r,v) \mapsto \left(\frac{\partial}{\partial r} - \frac{\partial}{\partial v^2}\right)(f(r)g(v))$ belong to $L^2([0,T]\times[0,1])$. From the definition of the function A, we see that A solves the inhomogeneous heat equation, that is, A satisfies

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x^2}\right)A(t,x) = \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x^2}\right)(f(t)g(x)), \tag{4.3.34}$$

the same boundary conditions as the Green kernel and vanishing initial condition. On the other hand, the function $f(\cdot)g(*)$ also satisfies (4.3.34) with $A(t,x)$ replaced by $f(t)g(x)$ and the same boundary and initial conditions. By the uniqueness of the solution to heat equation on bounded domains (see [38, Theorem 5, p.57]), we have $A = f(\cdot)g(*)$. $\square$

4.4 Malliavin derivatives of F_2, M_0 and M

In this section, we recall some results on the suprema F_2, M_0 and M in (4.2.5)–(4.2.8), in order to apply Theorem 1.5.5 and to prove Theorems 4.2.1, 4.2.2 and 4.2.5.

First, we state the 0-1 law for the germ σ-algebra generated by the Brownian sheet that appears in equation (4.2.1). Define $\mathscr{F}_t := \sigma\{W(s,x) : s \le t, 0 \le x \le 1\}$ and $\mathscr{F}_t^+ := \bigcap_{s>t} \mathscr{F}_s$.

Lemma 4.4.1. *For any set $B \in \mathscr{F}_0^+$, $P(B) \in \{0,1\}$.*

Proof. For any $x, y \in [0,1]$, $r \ge 0$ and $t > s \ge 0$, we know that $W(r+t,x) - W(t,x)$ is independent of $W(s,y)$. Hence $W(r+t,x) - W(t,x)$ is independent of $\mathscr{F}_s$. Furthermore we have $W(r+t,x) - W(t,x)$ is independent of $\mathscr{F}_s^+$. In particular, for any $r \ge 0$, $t > 0$ and $x \in [0,1]$, $W(r+t,x) - W(t,x)$ is independent of $\mathscr{F}_0^+$. Since

$$W(r,x) = \lim_{t\downarrow 0}(W(r+t,x) - W(t,x)),$$

we obtain that $W(r,x)$ is independent of $\mathscr{F}_0^+$. Therefore for any $r>0$, $\mathscr{F}_r$ is independent of $\mathscr{F}_0^+$, which implies $\mathscr{F}_0^+$ is independent of itself. Hence, for any $B \in \mathscr{F}_0^+$,

$$\mathrm{P}(B) = \mathrm{P}(B \cap B) = \mathrm{P}(B)^2,$$

which implies that $\mathrm{P}(B) \in \{0,1\}$. □

Lemma 4.4.2. *(a) With probability one, the sample path of the process* $\{\bar{u}(t,y_0) : t \in [s_0, s_0+\delta_1]\}$ *achieves its supremum at a unique point in* $[s_0, s_0+\delta_1]$*, denoted by* S*. If* $(s_0,y_0) \in [0,T]\times]0,1[$*, we have* $F_2 > 0$ *a.s.*

(b) With probability one, $M_0 > 0$ *and the sample path of the process* $\{u(t,x) : (t,x) \in [0,\delta_1] \times [y_0, y_0+\delta_2]\}$ *achieves its supremum at a unique point in* $]0,\delta_1] \times [y_0, y_0+\delta_2]$*, denoted by* $(\bar{S}, \bar{X})$.

(c) With probability one, $M > 0$ *and the sample path of the process* $\{\hat{u}(t,x) : (t,x) \in [0,T] \times [0,1]\}$ *achieves its supremum at a unique point in* $]0,T]\times]0,1]$*, denoted by* $(\hat{S}, \hat{X})$.

Proof. The first statement of (a) follows from [47, Lemma 2.6], since for $t,s \in [s_0, s_0+\delta_1]$ with $t \neq s$,

$$\mathrm{E}[|\bar{u}(t,y_0) - \bar{u}(s,y_0)|^2] = \mathrm{E}[|u(t,y_0) - u(s,y_0)|^2] \neq 0,$$

by Lemma A.5.3. In order to prove the second statement of (a), we denote $\{\tilde{u}(t,x) : (t,x) \in [s_0, s_0+\delta_1] \times [y_0, y_0+\delta_2]\}$ the solution to (4.2.1) on the whole space. If $s_0 = 0$, it is clear that $F_2 > 0$ a.s. by using the 0-1 law in Lemma 4.4.1; see also the proof for $M_0 > 0$ a.s. below. If $s_0 > 0$, by [45, p.23, (3.9)], we see that

$$\sup_{t \in [s_0, s_0+\delta_1]} \tilde{u}(t,y_0) - \tilde{u}(s_0,y_0) > 0 \quad \text{a.s.}$$

Since the the processes $\{u(t,x) : (t,x) \in [s_0, s_0+\delta_1] \times [y_0, y_0+\delta_2]\}$ and $\{\tilde{u}(t,x) : (t,x) \in [s_0, s_0+\delta_1] \times [y_0, y_0+\delta_2]\}$ are mutually absolute continuous by [62, Corollary 4], we conclude that

$$F_2 > 0, \ \text{a.s.}$$

We turn to proving the statement (b). Fix $x \in [y_0, y_0+\delta_2]$. It is clear that

$$\{M_0 > 0\} = \{\sup_{(t,x) \in [0,\delta_1] \times [y_0, y_0+\delta_2]} u(t,x) > 0\} \supset \limsup_{t_n \downarrow 0} \{u(t_n, x) > 0\}. \tag{4.4.1}$$

On the other hand, we know that

$$\limsup_{t_n \downarrow 0} \{u(t_n, x) > 0\} \in \mathscr{F}_0^+ \tag{4.4.2}$$

and

$$P\Big\{\limsup_{t_n\downarrow 0}\{u(t_n,x)>0\}\Big\} \geq \limsup_{t_n\downarrow 0} P\{u(t_n,x)>0\} = \frac{1}{2}, \tag{4.4.3}$$

since for every n, $u(t_n,x)$ is a centered Gaussian random variable and $P\{u(t_n,x)>0\} = \frac{1}{2}$. Hence by Lemma 4.4.1, we obtain that

$$P\Big\{\limsup_{t_n\downarrow 0}\{u(t_n,x)>0\}\Big\} = 1, \tag{4.4.4}$$

which establishes that $M_0>0$ almost surely. Furthermore, for any (t,x), $(s,y)\in\,]0,\delta_1]\times[y_0,y_0+\delta_2]$ with $(t,x)\neq(s,y)$, by Lemma A.5.3,

$$E[|u(t,x)-u(s,y)|^2]\neq 0,$$

which yields the conclusion of statement (b) by [47, Lemma 2.6].

We proceed to prove statement (c). In the case of Dirichlet boundary conditions, since $u(t,0)\equiv 0$ for any $t\geq 0$, we can repeat the proof of statement (b) to see that $M>0$ almost surely. In the case of Neumann boundary conditions, fix $t>0$. We follow [25, (4.22)] to write

$$\begin{aligned} u(t,x) &= \sqrt{2}\sum_{k=1}^{\infty}\cos(k\pi x)\xi_t^k r_k + \sum_{k=1}^{\infty}\frac{\cos(k\pi x)}{k\pi}\xi_t^k + \sqrt{t}\xi_t^0 \\ &:= R(x)+\hat{B}(x)+\sqrt{t}\xi_t^0, \end{aligned}$$

where $\{\xi_t^k\}_{k=0}^{\infty}$ is an i.i.d. sequence of standard Gaussian random variables and

$$r_k := \frac{(1-\exp(-2\pi^2k^2t))^{1/2}-1}{\sqrt{2}\pi k}.$$

We proceed to prove that almost surely $x\mapsto R(x)$ is differentiable on $[0,1]$. By Fubini's theorem, we see that

$$E\left[\sum_{k=1}^{\infty}k|\xi_t^k||r_k|\right] = \sum_{k=1}^{\infty}kE[|\xi_t^k||r_k| \leq c\sum_{k=1}^{\infty}k|r_k|<\infty,$$

where the last sum is finite because $|r_k| = O(k^{-1}\exp(-2\pi^2k^2t))$ as $k\to\infty$. Hence we have almost surely

$$\sum_{k=1}^{\infty}k|\xi_t^k||r_k|<\infty. \tag{4.4.5}$$

We denote $R_n(x) := \sum_{k=1}^{n}\cos(k\pi x)\xi_t^k r_k$. By (4.4.5), we know that, almost surely, R_n converges to R uniformly on $[0,1]$. Furthermore $R_n'(x) = \sum_{k=1}^{n}-k\pi\sin(k\pi x)\xi_t^k r_k$ and using (4.4.5) again, we see that almost surely, R_n' converges to $x\mapsto\sum_{k=1}^{\infty}-k\pi\sin(k\pi x)\xi_t^k r_k$ uniformly on $[0,1]$. Hence by [77, Theorem 7.17], we obtain that almost surely, $x\mapsto R(x)$ is differentiable, and, for

$x \in [0,1]$,

$$R'(x) = \sum_{k=1}^{\infty} -k\pi \sin(k\pi x)\xi_t^k r_k.$$

Now recall that from [81, Exercise 3.9, p.326], the standard Brownian motion $\{W(x) : x \in [0,1]\}$ has the expansion

$$W(x) = \frac{1}{\sqrt{3}}\xi_0 + \sum_{k=1}^{\infty} \frac{\sqrt{2}}{k\pi}\xi_k \cos(k\pi x),$$

where $\{\xi_k\}_{k=1}^{\infty}$ are i.i.d $N(0,1)$ (ξ_0 is also $N(0,1)$, but not independent of the other ξ_k). By the non-differentiability property of Brownian motion (see [61, Theorem 1.27]), we have

$$\limsup_{x\downarrow 0} \frac{W(x) - W(0)}{x-0} = +\infty, \quad \text{a.s.},$$

which implies

$$\limsup_{x\downarrow 0} \frac{\hat{B}(x) - \hat{B}(0)}{x-0} = +\infty \quad \text{a.s.}$$

Therefore, we have

$$\limsup_{x\downarrow 0} \frac{u(t,x) - u(t,0)}{x-0} = +\infty, \quad \text{a.s.},$$

which implies

$$\sup_{0\le x\le 1} (u(t,x) - u(t,0)) > 0 \quad \text{a.s.} \tag{4.4.6}$$

Hence $M > 0$ a.s.

Now we need to prove that the sample path of the process $\{\hat{u}(t,x) : (t,x) \in [0,T]\times[0,1]\}$ achieves its supremum uniquely on $[0,T] \times [0,1]$. Since $M > 0$ a.s., by [47, Lemma 2.6], it suffices to check that for any $(t,x),(s,y)$ in $]0,T]\times]0,1]$ ($]0,T]\times]0,1[$ in the case of Dirichlet boundary conditions) with $(t,x) \neq (s,y)$,

$$\mathrm{E}[|u(t,x) - u(t,0) - u(s,y) + u(s,0)|^2] \neq 0.$$

This is a consequence of Lemma A.5.3. Therefore, we have finished the proof. □

Remark 4.4.3. *The process $\{B_x : x \in [0,1]\}$ defined in (4.21) of [25] is not a standard Brownian motion. But we know that there exists a constant c such that for all $x, y \in [0,1]$,*

$$E[|B_x - B_y|^2] \ge c|x-y|,$$

which is sufficient for (4.25) of [25].

Lemma 4.4.4. *The random variables M_0, M and F_2 belong to $\mathbb{D}^{1,2}$ and*

$$DM_0 = 1_{\{\cdot<\bar{S}\}} G(\bar{S}-\cdot, \bar{X}, *), \tag{4.4.7}$$

$$DM = 1_{\{\cdot<\hat{S}\}} G(\hat{S}-\cdot, \hat{X}, *) - 1_{\{\cdot<\hat{S}\}} G(\hat{S}-\cdot, 0, *), \tag{4.4.8}$$

$$DF_2 = 1_{\{\cdot<S\}} G(S-\cdot, y_0, *) - 1_{\{\cdot<s_0\}} G(s_0-\cdot, y_0, *), \tag{4.4.9}$$

where the random variables $\bar{S}$, $\bar{X}$, $\hat{S}$, $\hat{X}$ and S are defined in Lemma 4.4.2.

Remark 4.4.5. *The function $(t,x) \mapsto 1_{\{\cdot<t\}} G(t-\cdot, x, *)$ (we use the notation $\cdot$ to denote the time variable and $*$ for the space variable) from $[0,T]\times[0,1]$ into $\mathscr{H}$ is continuous by the argument below (4.4.13). Therefore, $1_{\{\cdot<\bar{S}\}} G(\bar{S}-\cdot, \bar{X}, *)$ is the random element of $\mathscr{H}$ obtained by composition of the random vector $\omega \mapsto (\bar{S}(\omega), \bar{X}(\omega))$ and this continuous function.*

Proof of Lemma 4.4.4. It is similar to the proof for the Brownian sheet; see [64, Lemma 2.1.9]. We only prove (4.4.8). The proofs of (4.4.7) and (4.4.9) are similar.

Let $\{(t_k, x_k)\}_{k=1}^{\infty}$ be a dense subset of $[0,T]\times[0,1]$. Define

$$M_n := \max\{\hat{u}(t_1, x_1), \ldots, \hat{u}(t_n, x_n)\}.$$

Then M_n converges to M almost surely as $n \to \infty$. Borell's inequality (see (2.4) in [1]) implies that for any $q \geq 2$,

$$\mathrm{E}\left[\sup_{(t,x)\in[0,T]\times[0,1]} |\hat{u}(t,x)|^q\right] < \infty,$$

which indicates that M_n converges to M in $L^2(\Omega)$ as $n \to \infty$ by Lemma A.6.3. Furthermore M_n belongs to $\mathbb{D}^{1,2}$ by Proposition 1.2.4 of [64] since the function $\varphi_n : \mathbb{R}^n \to \mathbb{R}$ defined by $\varphi_n(x_1, \ldots, x_n) = \max\{x_1, \ldots, x_n\}$ is Lipschitz. We define

$$\begin{aligned} A_1^n &:= \{\hat{u}(t_1, x_1) = M_n\}, \\ A_2^n &:= \{\hat{u}(t_1, x_1) \neq M_n, \hat{u}(t_2, x_2) = M_n\}, \\ &\ \vdots \\ A_k^n &:= \{\hat{u}(t_1, x_1) \neq M_n, \ldots, \hat{u}(t_{k-1}, x_{k-1}) \neq M_n, \hat{u}(t_k, x_k) = M_n\}. \end{aligned}$$

Then it is easy to see that

$$A_k^n \cap A_m^n = \emptyset, \quad \text{if} \quad k \neq m,$$

and because almost surely the maximum is attained at a unique point, we have

$$\mathrm{P}\{\cup_{k=1}^n A_k^n\} = 1,$$

and

$$M_n = \sum_{k=1}^{n} 1_{A_k^n} \hat{u}(t_k, x_k), \quad \text{a.s.}$$

On the set A_k^n, we have $M_n - \hat{u}(t_k, x_k) = 0$ almost surely. By the local property of the operator D (see Proposition 1.3.16 in [64]), we have $D(M_n - \hat{u}(t_k, x_k)) = 0$ almost surely on the set A_k^n. Hence we have

$$\begin{aligned} DM_n &= \sum_{k=1}^{n} 1_{A_k^n} D\hat{u}(t_k, x_k) \\ &= \sum_{k=1}^{n} 1_{A_k^n} (1_{\{\cdot<t_k\}} G(t_k - \cdot, x_k, *) - 1_{\{\cdot<t_k\}} G(t_k - \cdot, 0, *)) \\ &= 1_{\{\cdot<S_n\}} G(S_n - \cdot, X_n, *) - 1_{\{\cdot<S_n\}} G(S_n - \cdot, 0, *), \end{aligned} \tag{4.4.10}$$

where (S_n, X_n) is the unique point such that $M_n = \hat{u}(S_n, X_n)$. Since for any $(t,x) \in [0,T] \times [0,1]$,

$$\|1_{\{\cdot<t\}} G(t - \cdot, x, *)\|_{\mathscr{H}}^2 = \mathrm{E}[u(t,x)^2] \leq c_T, \tag{4.4.11}$$

we have

$$\begin{aligned} \sup_{n\geq 1} \mathrm{E}[\|DM_n\|_{\mathscr{H}}^2] &\leq 2 \sup_{n\geq 1} \sum_{k=1}^{n} \mathrm{P}\{A_k^n\} (\|1_{\{\cdot<t_k\}} G(t_k - \cdot, x_k, *)\|_{\mathscr{H}}^2 + \|1_{\{\cdot<t_k\}} G(t_k - \cdot, 0, *)\|_{\mathscr{H}}^2) \\ &\leq c. \end{aligned} \tag{4.4.12}$$

Hence from Lemma 1.2.3 in [64], we know that M belongs to $\mathbb{D}^{1,2}$ and DM_n converges to M in the weak topology of $L^2(\Omega, \mathscr{H})$. In other words, for any $G \in L^2(\Omega, \mathscr{H})$,

$$\lim_{n\to\infty} \mathrm{E}[\langle DM_n, G\rangle_{\mathscr{H}}] = \mathrm{E}[\langle DM, G\rangle_{\mathscr{H}}]. \tag{4.4.13}$$

On the other hand, since for any $(t,x), (s,y) \in [0,T] \times [0,1]$, by (4.2.3),

$$\begin{aligned} \|D(u(t,x) - u(s,y))\|_{\mathscr{H}}^2 &= \|1_{\{\cdot<t\}} G(t - \cdot, x, *) - 1_{\{\cdot<s\}} G(s - \cdot, y, *)\|_{\mathscr{H}}^2 \\ &= \mathrm{E}[|u(t,x) - u(s,y)|^2] \\ &\leq C_T(|t-s|^{1/2} + |x-y|), \end{aligned} \tag{4.4.14}$$

we see that the function $(t,x) \mapsto Du(t,x) = 1_{\{\cdot<t\}} G(t - \cdot, x, *)$ from $[0,T] \times [0,1]$ into $\mathscr{H}$ is continuous. Furthermore, because the random vector (S_n, X_n) converges to $(\hat{S}, \hat{X})$ almost surely, the measurable function $\omega \mapsto 1_{\{\cdot<S_n(\omega)\}} G(S_n(\omega) - \cdot, X_n(\omega), *)$ converges to $\omega \mapsto 1_{\{\cdot<\hat{S}(\omega)\}} G(\hat{S}(\omega) - \cdot, \hat{X}(\omega), *)$ in $\mathscr{H}$ almost surely, and the measurable function $\omega \mapsto 1_{\{\cdot<S_n(\omega)\}} G(S_n(\omega) - \cdot, 0, *)$ converges to $\omega \mapsto 1_{\{\cdot<\hat{S}(\omega)\}} G(\hat{S}(\omega) - \cdot, 0, *)$ in $\mathscr{H}$ almost surely. Hence for any $G \in L^2(\Omega, \mathscr{H})$, we have

$$\lim_{n\to\infty}\langle 1_{\{\cdot<S_n\}}G(S_n-\cdot,X_n,*)-1_{\{\cdot<S_n\}}G(S_n-\cdot,0,*),G\rangle_{\mathscr{H}}$$
$$=\langle 1_{\{\cdot<\hat S\}}G(\hat S-\cdot,\hat X,*)-1_{\{\cdot<\hat S\}}G(\hat S-\cdot,0,*),G\rangle_{\mathscr{H}},\quad \text{a.s.} \tag{4.4.15}$$

By (4.4.10) and (4.4.11),

$$\begin{aligned}
&\sup_{n\geq 1}\mathrm{E}[\langle 1_{\{\cdot<S_n\}}G(S_n-\cdot,X_n,*)-1_{\{\cdot<S_n\}}G(S_n-\cdot,0,*),G\rangle^2_{\mathscr{H}}]\\
&\quad\leq 2\sup_{n\geq 1}\sum_{k=1}^n \mathrm{E}\left[1_{A_k^n}\left(\langle 1_{\{\cdot<t_k\}}G(t_k-\cdot,x_k,*),G\rangle^2_{\mathscr{H}}+\langle 1_{\{\cdot<t_k\}}G(t_k-\cdot,0,*),G\rangle^2_{\mathscr{H}}\right)\right]\\
&\quad\leq 2\sup_{n\geq 1}\sum_{k=1}^n \mathrm{E}\left[1_{A_k^n}\left(\|1_{\{\cdot<t_k\}}G(t_k-\cdot,x_k,*)\|^2_{\mathscr{H}}+\|1_{\{\cdot<t_k\}}G(t_k-\cdot,0,*)\|^2_{\mathscr{H}}\right)\|G\|^2_{\mathscr{H}}\right]\\
&\quad\leq c\,\mathrm{E}[\|G\|^2_{\mathscr{H}}]<\infty.
\end{aligned} \tag{4.4.16}$$

Hence (4.4.15), (4.4.16) and Lemma A.6.3 imply that

$$\begin{aligned}
\lim_{n\to\infty}\mathrm{E}[\langle DM_n,G\rangle_{\mathscr{H}}]&=\lim_{n\to\infty}\mathrm{E}[\langle 1_{\{\cdot<S_n\}}G(S_n-\cdot,X_n,*)-1_{\{\cdot<S_n\}}G(S_n-\cdot,0,*),G\rangle_{\mathscr{H}}]\\
&=\mathrm{E}[\langle 1_{\{\cdot<\hat S\}}G(\hat S-\cdot,\hat X,*)-1_{\{\cdot<\hat S\}}G(\hat S-\cdot,0,*),G\rangle_{\mathscr{H}}].
\end{aligned} \tag{4.4.17}$$

Comparing (4.4.13) and (4.4.17), we obtain that almost surely,

$$DM=1_{\{\cdot<\hat S\}}G(\hat S-\cdot,\hat X,*)-1_{\{\cdot<\hat S\}}G(\hat S-\cdot,0,*).$$

□

4.5 Smoothness of the densities

In this section, we suppose that I and J are as above (4.2.4) and we are going to introduce the random variables needed for Theorem 1.5.5 and prove they satisfy the conditions therein. We start by establishing the smoothness of the random variables $\{Y_r : r\in[s_0,s_0+\delta_1]\}$ and $\{\bar Y_r : r\in[0,\Delta_\bullet]\}$ defined in (4.3.2) and (4.3.17) respectively.

For simplicity of notation, we denote

$$u(1_{]s,t]\times]y,x]}):=u(t,x)+u(s,y)-u(t,y)-u(s,x),$$

and

$$Du(t,x;s,y):=D(u(t,x)+u(s,y)-u(t,y)-u(s,x))$$

for $(t,s,x,y)\in[0,T]^2\times[0,1]^2$.

Lemma 4.5.1. *(a) For any $r \in [s_0, s_0+\delta_1]$, Y_r belongs to $\mathbb{D}^\infty$ and for any integer l,*

$$D^l Y_r = \int_{[s_0,r]^2} dt ds \frac{2p_0(2p_0-1)\cdots(2p_0-l+1)}{|t-s|^{\gamma_0/2}} \times (u(t,y_0)-u(s,y_0))^{2p_0-l}(D(u(t,y_0)-u(s,y_0)))^{\otimes l}. \tag{4.5.1}$$

(b) For any $r \in [0,\Delta_\bullet]$, $\bar{Y}_r$ belongs to $\mathbb{D}^\infty$ and for any integer l,

$$\begin{aligned} D^l \bar{Y}_r = &\int_{[0,r]^2} dt ds \frac{2p_0(2p_0-1)\cdots(2p_0-l+1)}{|t-s|^{\gamma_0/2}} \\ &\times (u(t,y_0)-u(s,y_0))^{2p_0-l}(D(u(t,y_0)-u(s,y_0)))^{\otimes l} \\ &+\int_{[0,r]^2} dt ds \int_{[y_0,y_0+\Delta_*]^2} dx dy \frac{2p_0(2p_0-1)\cdots(2p_0-l+1)}{|t-s|^{1+2p_0\gamma_1}|x-y|^{1+2p_0\gamma_2}} \\ &\times u(1_{]s,t]\times]y,x]})^{2p_0-l}(Du(t,x;s,y))^{\otimes l}. \end{aligned} \tag{4.5.2}$$

Proof. We start by proving (a). We define the random function

$$h(t,s) = \begin{cases} \frac{(u(t,y_0)-u(s,y_0))^{2p_0}}{|t-s|^{\gamma_0/2}} & \text{if } t \neq s; \\ 0 & \text{otherwise.} \end{cases}$$

By the Hölder continuity of the solution (see (2.1.7)) and (4.3.1), we know that a.s., the function h is continuous and bounded on $[0,T]^2$. For $k \geq 1$, we denote $t_i = s_i = s_0 + \frac{r-s_0}{k} i$ (we omit the dependence on k for convenience) and

$$X_k = (r-s_0)^2 k^{-2} \sum_{i,j=0}^{k-1} \frac{(u(t_i,y_0)-u(s_j,y_0))^{2p_0}}{|t_i-s_j|^{\gamma_0/2}}.$$

Here we assume that

$$\frac{(u(t_i,y_0)-u(s_j,y_0))^{2p_0}}{|t_i-s_j|^{\gamma_0/2}} = 0 \quad \text{if} \quad t_i = s_j.$$

As the Riemann sum of Y_r, X_k converges to Y_r a.s. as $k \to \infty$. For any $q \geq 1$, by Hölder's inequality,

$$\begin{aligned} \mathrm{E}[|X_k|^q] &\leq ck^{-2q} k^{2(q-1)} \sum_{i,j=0}^{k-1} \frac{\mathrm{E}[(u(t_i,y_0)-u(s_j,y_0))^{2p_0 q}]}{|t_i-s_j|^{\gamma_0/2}} \\ &\leq ck^{-2} \sum_{i,j=0}^{k-1} \frac{|t_i-s_j|^{p_0 q/2}}{|t_i-s_j|^{\gamma_0 q/2}} \\ &\leq \tilde{c} k^{-2} \sum_{i,j=0}^{k-1} 1 = \tilde{c}, \end{aligned}$$

where the last inequality is due to choice of p_0, γ_0 in (4.3.1). Applying Lemma A.6.3, we see that X_k converges to Y_r in $L^q(\Omega)$ as $k\to\infty$ for any $q>1$. By the chain rule,

$$DX_k = (r-s_0)^2 k^{-2}\sum_{i,j=0}^{k-1}\frac{2p_0(u(t_i,y_0)-u(s_j,y_0))^{2p_0-1}}{|t_i-s_j|^{\gamma_0/2}}(D(u(t_i,y_0)-u(s_j,y_0))),$$

which converges almost surely to the Bochner integral

$$\int_{[s_0,r]^2} dtds\frac{2p_0(u(t,y_0)-u(s,y_0))^{2p_0-1}}{|t-s|^{\gamma_0/2}}(D(u(t,y_0)-u(s,y_0)))$$

by using the continuity of the map $(t,x)\mapsto Du(t,x)=1_{\{\cdot<t\}}G(t-\cdot,x,*)$ from $[0,T]\times[0,1]$ to $\mathscr{H}$. Furthermore, by Hölder's inequality, for any $q\geq 1$,

$$\mathrm{E}[\|DX_k\|^q_{\mathscr{H}}]\leq ck^{-2q}k^{2(q-1)}\sum_{i,j=0}^{k-1}\frac{\mathrm{E}[|u(t_i,y_0)-u(s_j,y_0)|^{q(2p_0-1)}]}{|t_i-s_j|^{\gamma_0 q/2}}\|D(u(t_i,y_0)-u(s_j,y_0))\|^q_{\mathscr{H}}.$$

By (4.2.3) and the isometry (4.4.14), this is bounded above by

$$\begin{aligned}
&\tilde{c}k^{-2q}k^{2(q-1)}\sum_{i,j=0}^{k-1}\frac{|t_i-s_j|^{(2p_0-1)q/4+q/4}}{|t_i-s_j|^{\gamma_0 q/2}}\\
&=\tilde{c}k^{-2q}k^{2(q-1)}\sum_{i,j=0}^{k-1}\frac{|t_i-s_j|^{p_0 q/2}}{|t_i-s_j|^{\gamma_0 q/2}}\\
&\leq \bar{c}k^{-2}\sum_{i,j=0}^{k-1}1=\bar{c}.
\end{aligned}$$

We apply Lemma A.6.3 again to see that DX_k converges to the Bochner integral

$$\int_{[s_0,r]^2} dtds\frac{2p_0(u(t,y_0)-u(s,y_0))^{2p_0-1}}{|t-s|^{\gamma_0/2}}(D(u(t,y_0)-u(s,y_0)))$$

in $L^q(\Omega;\mathscr{H})$ for any $q\geq 1$. Since the Malliavin derivative is closable (see [64, Proposition 1.2.1]), we obtain that

$$DY_r=\int_{[s_0,r]^2} dtds\frac{2p_0(u(t,y_0)-u(s,y_0))^{2p_0-1}}{(|t-s|^{\gamma_0/2}}(D(u(t,y_0)-u(s,y_0))) \tag{4.5.3}$$

and $Y_r\in\cap_{q\geq 1}\mathbb{D}^{1,q}$. We can repeat the above procedure to obtain that $Y_r\in\mathbb{D}^\infty$ and the equality (4.5.1).

The proof of (b) is similar. The main difference is the Malliavin derivative of the second term $Y_1(r)$ in the definition of $\bar{Y}_r$. For simplicity of notation, we only prove the smoothness of the random variable

$$Y(1)=\int_{[0,1]^4}\frac{(u(t,x)+u(s,y)-u(t,y)-u(s,x))^{2p_0}}{|t-s|^{1+2p_0\gamma_1}|x-y|^{1+2p_0\gamma_2}}dsdtdydx. \tag{4.5.4}$$

We define

$$\bar{h}(t,s,x,y) = \begin{cases} \frac{u(1_{]s,t]\times]y,x]})^{2p_0}}{|t-s|^{1+2p_0\gamma_1}|x-y|^{1+2p_0\gamma_2}} & \text{if } t \neq s \text{ and } x \neq y; \\ 0 & \text{otherwise.} \end{cases}$$

Applying Lemma 4.3.3 with $\gamma_1 + \frac{1}{2p_0} < \xi < \theta_1/2, \gamma_2 + \frac{1}{2p_0} < \eta < \theta_2/2$ so that almost surely the function $\bar{h}$ is continuous and bounded on $[0,T]^2 \times [0,1]^2$. Similar to the proof of (a), we discretize $Y(1)$ by

$$\bar{X}_k = k^{-4} \sum_{i,j,m,n=0}^{k-1} \frac{u(1_{]s_j,t_i]\times]y_n,x_m]})^{2p_0}}{|t_i - s_j|^{1+2p_0\gamma_1}|x_m - y_n|^{1+2p_0\gamma_2}},$$

where $t_i = s_i = x_i = y_i = \frac{i}{k}, i = 1,\ldots,k$ (we omit the dependence on k for convenience). Here we assume that

$$\frac{u(1_{]s_j,t_i]\times]y_n,x_m]})^{2p_0}}{|t_i - s_j|^{1+2p_0\gamma_1}|x_m - y_n|^{1+2p_0\gamma_2}} = 0, \quad \text{if } t_i = s_j \text{ or } x_m = y_n.$$

Similar to the arguments in the proof of (a), using (4.3.6), (4.3.10) and Lemma A.6.3 we have that $\bar{X}_k$ converges to $Y(1)$ in $L^q(\Omega)$ as $k \to \infty$ for any $q > 1$. Furthermore, We can prove that $D\bar{X}_k$ converges to the Bochner integral

$$\int_{[0,1]^4} \frac{2p_0 u(1_{]s,t]\times]y,x]})^{2p_0-1} Du(t,x;s,y)}{|t-s|^{1+2p_0\gamma_1}|x-y|^{1+2p_0\gamma_2}} dsdtdydx$$

in $L^q(\Omega;\mathscr{H})$ as $k \to \infty$ for any $q \geq 1$. Since the Malliavin derivative is closable, we have

$$DY(1) = \int_{[0,1]^4} \frac{2p_0 u(1_{]s,t]\times]y,x]})^{2p_0-1} Du(t,x;s,y)}{|t-s|^{1+2p_0\gamma_1}|x-y|^{1+2p_0\gamma_2}} dsdtdydx \tag{4.5.5}$$

and $Y(1) \in \cap_{q\geq 1}\mathbb{D}^{1,q}$. We repeat the above procedure to conclude that $Y(1) \in \mathbb{D}^\infty$ and for any integer l,

$$D^l Y(1) = \int_{[0,1]^4} \frac{2p_0(2p_0-1)\cdots(2p_0-l+1) u(1_{]s,t]\times]y,x]})^{2p_0-l} (Du(t,x;s,y))^{\otimes l}}{|t-s|^{1+2p_0\gamma_1}|x-y|^{1+2p_0\gamma_2}} dsdtdydx.$$

□

Moreover, we have the following estimates on moments of the Malliavin derivatives of the random variables $\{Y_r, r \in [s_0, s_0+\delta_1]\}$ and $\{\bar{Y}_r, r \in [0,\Delta_\bullet]\}$.

Lemma 4.5.2. *(a) For any $p \geq 1$, there exists a constant c_p, not depending on $(s_0, y_0) \in [0,T] \times [0,1]$, such that for all $\delta_1 > 0$ and for all $r \in [s_0, s_0+\delta_1]$,*

$$E[\|DY_r\|_{\mathscr{H}}^p] \leq c_p (r-s_0)^{2p} \delta_1^{(p_0-\gamma_0)p/2}. \tag{4.5.6}$$

(b) For any $p \geq 1$, there exists a constant c_p, not depending on $y_0 \in [0,1]$, such that for all $r \in [0, \Delta_\bullet]$,

$$E[\|D\bar{Y}_r\|_{\mathscr{H}}^p] \leq c_p\, r^{2p} \delta^{(p_0-\gamma_0)p}. \tag{4.5.7}$$

Proof. We first prove (4.5.6). By Lemma 4.5.1(a),

$$DY_r = 2p_0 \int_{[s_0,r]^2} ds\,dt \frac{(u(t,y_0)-u(s,y_0))^{2p_0-1}}{|t-s|^{\gamma_0/2}} D(u(t,y_0)-u(s,y_0)), \tag{4.5.8}$$

and for any $p \geq 1$, by Hölder's inequality,

$$\begin{aligned} \mathrm{E}[\|DY_r\|_{\mathscr{H}}^p] \leq c_p \left(\int_{[s_0,r]^2} ds\,dt\right)^{p-1} \int_{[s_0,r]^2} ds\,dt \frac{\mathrm{E}\left[|(u(t,y_0)-u(s,y_0))|^{(2p_0-1)p}\right]}{|t-s|^{\gamma_0 p/2}} \\ \times \|D(u(t,y_0)-u(s,y_0))\|_{\mathscr{H}}^p. \end{aligned} \tag{4.5.9}$$

Since

$$\|D(u(t,x)-u(s,y))\|_{\mathscr{H}}^2 = \mathrm{E}[|u(t,x)-u(s,y)|^2] \tag{4.5.10}$$

(see also the isometry (4.4.14)), by (4.2.3), we see that (4.5.9) is bounded above by

$$\begin{aligned} &c_p(r-s_0)^{2(p-1)} \int_{[s_0,r]^2} ds\,dt\,|t-s|^{p(p_0-\gamma_0)/2} \\ &\quad \leq c_p(r-s_0)^{2p} \delta_1^{(p_0-\gamma_0)p/2}, \end{aligned} \tag{4.5.11}$$

as desired.

To prove (b), it suffices to estimate the moments of $DY_1(r)$ since the estimate for the moments of $DY_0(r)$ is similar to the proof of (a). Indeed, by (4.5.2),

$$DY_1(r) = 2p_0 \int_{[0,r]^2} dt\,ds \int_{[y_0,y_0+\Delta_*]^2} dx\,dy\, \frac{u(1_{]s,t]\times]y,x]})^{2p_0-1} Du(t,x;s,y)}{|t-s|^{1+2p_0\gamma_1}|x-y|^{1+2p_0\gamma_2}},$$

and for any $p \geq 1$, by Hölder's inequality,

$$\begin{aligned}
\mathrm{E}[\|DY_1(r)\|^p_{\mathscr{H}}] &\leq c_p\,(r\Delta_*)^{2(p-1)} \int_{[0,r]^2} dtds \int_{[y_0,y_0+\Delta_*]^2} dxdy \\
&\qquad\qquad \times \frac{\mathrm{E}[|u(1_{]s,t]\times]y,x]})|^{(2p_0-1)p}]\,\|Du(t,x;s,y)\|^p_{\mathscr{H}}}{|t-s|^{(1+2p_0\gamma_1)p}|x-y|^{(1+2p_0\gamma_2)p}} \\
&\leq c_p\,(r\Delta_*)^{2(p-1)} \int_{[0,r]^2} dtds \int_{[y_0,y_0+\Delta_*]^2} dxdy \\
&\qquad\qquad \times \frac{|t-s|^{p_0p\theta_1}|x-y|^{p_0p\theta_2}}{|t-s|^{p(1+2p_0\gamma_1)}|x-y|^{p(1+2p_0\gamma_2)}} \\
&\leq c_p\,(r\Delta_*)^{2p}\Delta_\bullet^{p(p_0\theta_1-(1+2p_0\gamma_1))}\Delta_*^{p(p_0\theta_2-(1+2p_0\gamma_2))} \\
&\leq c_p\, r^{2p}\delta^{p(2p_0\theta_1-2(1+2p_0\gamma_1))}\delta^{p(p_0\theta_2-(1+2p_0\gamma_2)+2)} \\
&= c_p\, r^{2p}\delta^{p(p_0(2\theta_1+\theta_2)-2p_0(2\gamma_1+\gamma_2)-1)} = c_p\, r^{2p}\delta^{p(p_0-\gamma_0)}, \qquad (4.5.12)
\end{aligned}$$

where the in the second inequality we use (4.3.10), and the derivation of the last equality follows the same reason as that of (4.3.16).

Therefore, we have finished the proof. □

Furthermore, it is clear that for any integer i and $p \geq 1$,

$$\sup_{r\in[s_0,s_0+\delta_1]} \mathrm{E}\left[\|D^i Y_r\|^p_{\mathscr{H}^{\otimes i}}\right] < \infty, \qquad (4.5.13)$$

and

$$\sup_{r\in[0,\Delta_\bullet]} \mathrm{E}\left[\|D^i \bar{Y}_r\|^p_{\mathscr{H}^{\otimes i}}\right] < \infty. \qquad (4.5.14)$$

We proceed to introduce the random variables needed for Theorem 1.5.5 to study the smoothness of densities of the random variables F and M_0. We define the function $\psi_0 : \mathbb{R}^+ \to [0,1]$ as an infinitely differentiable function such that

$$\psi_0(x) = \begin{cases} 0 & \text{if } x > 1; \\ \psi_0(x) \in [0,1] & \text{if } x \in [\frac{1}{2}, 1], \\ 1 & \text{if } x \leq \frac{1}{2}. \end{cases} \qquad (4.5.15)$$

We first introduce the random variables needed to prove the smoothness of density of F. For $(z_1, z_2) \in \mathbb{R}\times]0,\infty[$, set

$$a = z_2/2 \quad \text{and} \quad A = \mathbb{R}\times]a,\infty[. \qquad (4.5.16)$$

Let $R = R(z_2, \delta_1)$ be defined as in Lemma 4.3.1 for the specific value of a in (4.5.16). Define

$$\psi(x) := \psi_0(x/R) \quad \text{so that} \quad \psi(x) = \begin{cases} 0 & \text{if } x > R; \\ \psi(x) \in [0,1] & \text{if } x \in [\frac{R}{2}, R], \\ 1 & \text{if } x \le \frac{R}{2} \end{cases} \tag{4.5.17}$$

and

$$\|\psi'\|_\infty := \sup_{x\in\mathbb{R}} |\psi'(x)| \le c R^{-1} \tag{4.5.18}$$

for a certain constant c not depending on z_2.

If $I \times J \subset]0, T] \times]0,1[$, let c_1, C_1, c_2, C_2 be as in (4.2.9) and (4.2.10), and $f_0 : \mathbb{R} \mapsto [0,1]$ be an infinitely differentiable function supported in $[c_1/2, (C_1 + T)/2]$ such that $f_0(t) = 1$, for all $t \in [c_1, C_1]$. Let $g_0 : \mathbb{R} \mapsto [0,1]$ be an infinitely differentiable function supported in $[c_2/2, (C_2+1)/2]$ such that $g_0(x) = 1$, for all $x \in [c_2, C_2]$. In the case of Neumann boundary conditions, if $I \subset]0, T]$ and $y_0 = 0 \in J \subset [0,1]$, we define g_0 to be an infinitely differentiable function with compact support such that $g_0(0) = 1$ and satisfies the same Neumann boundary conditions.

We define the $\mathscr{H}$-valued random variable u^1_A evaluated at (r, v) by

$$u^1_A(r, v) = \left(\frac{\partial}{\partial r} - \frac{\partial^2}{\partial v^2}\right)(f_0(r) g_0(v)). \tag{4.5.19}$$

In the case $I \times J \subset]0, T] \times]0,1[$, from the choice of the functions f_0 and g_0, we see that there exists a constant c such that for all $(s_0, y_0) \in I \times J$,

$$\|u^1_A\|_{\mathscr{H}} \le c. \tag{4.5.20}$$

Let $\phi_0 : \mathbb{R} \mapsto [0,1]$ be an infinitely differentiable function supported in $[-1,2]$ such that $\phi_0(v) = 1$, for all $v \in [0,1]$.

For $y_0 \in J \subset [0,1]$, we define ϕ_{δ_1} as an infinitely differentiable function with compact support such that $\phi_{\delta_1}(y_0) = 1$ and satisfies the same boundary conditions at 0 and 1 as the Green kernel. In particular, if $J \subset]0,1[$ and δ_1 satisfies the conditions in (4.2.11), then we choose the function ϕ_{δ_1} in the following way:

$$\phi_{\delta_1}(v) := \phi_0\left(\frac{v - y_0}{\delta_1^{1/2}}\right), \quad v \in [0,1], \tag{4.5.21}$$

so that, for some constant c,

$$|\phi'_{\delta_1}(v)| \le c\,\delta_1^{-1/2} \quad \text{and} \quad |\phi''_{\delta_1}(v)| \le c\,\delta_1^{-1}, \quad \text{for all } v \in [0,1]. \tag{4.5.22}$$

Set

$$H(r,v) := \phi_{\delta_1}(v)\int_{s_0}^{r}\psi(Y_a)da, \quad (r,v)\in[s_0,s_0+\delta_1]\times[0,1]. \tag{4.5.23}$$

We define the $\mathscr{H}$-valued random variable u_A^2 evaluated at (r,v) by

$$u_A^2(r,v) = \begin{cases} \left(\frac{\partial}{\partial r}-\frac{\partial^2}{\partial v^2}\right)H(r,v) & \text{if } (r,v)\in\,]s_0,s_0+\delta_1]\times[0,1]; \\ 0 & \text{otherwise.}\end{cases} \tag{4.5.24}$$

Finally, we define the random matrix $\gamma_A = (\gamma_A^{i,j})_{1\le i,j\le 2}$ by

$$\gamma_A = \begin{pmatrix} 1 & 0 \\ 0 & \int_{s_0}^{s_0+\delta_1}\psi(Y_r)dr \end{pmatrix}. \tag{4.5.25}$$

If $s_0=0\in I\subset[0,T]$, we only consider the random variables F_2, u_A^2 and $\gamma_A^{2,2}$ defined in (4.2.5), (4.5.24) and (4.5.25) with $s_0=0$, respectively.

We next introduce the random variables needed to prove the smoothness of density of M_0. For $z\in\,]0,\infty[$, set

$$\bar a = z/2 \quad \text{and} \quad \bar A =]\bar a,\infty[. \tag{4.5.26}$$

Let $\bar R = \bar R(z,\delta)$ be defined as in Lemma 4.3.5 for the specific value of $\bar a$ in (4.5.26). Define

$$\bar\psi(x) := \psi_0(x/\bar R) \quad \text{so that} \quad \bar\psi(x) = \begin{cases} 0 & \text{if } x>\bar R; \\ \psi(x)\in[0,1] & \text{if } x\in[\frac{\bar R}{2},\bar R], \\ 1 & \text{if } x\le\frac{\bar R}{2}\end{cases} \tag{4.5.27}$$

and

$$\|\bar\psi'\|_\infty := \sup_{x\in\mathbb{R}}|\bar\psi'(x)| \le c\,\bar R^{-1} \tag{4.5.28}$$

for a certain constant c not depending on z.

We define $\bar\phi_\delta$ as an infinitely differentiable function with compact support such that

$$\bar\phi_\delta(v) = 1, \quad \text{for all } v\in[y_0,y_0+\delta_2] \tag{4.5.29}$$

and satisfies the same boundary conditions at 0 and 1 as the Green kernel. In particular, if $J\subset\,]0,1[$ and δ_1,δ_2 satisfy the conditions in (4.2.15), we choose the function $\bar\phi_\delta$ in the following way:

$$\bar\phi_\delta(v) := \phi_0\Big(\frac{v-y_0}{\delta}\Big), \quad v\in[0,1], \tag{4.5.30}$$

where the function ϕ_0 is specified below (4.5.20), so that for some constant c,

$$|\bar{\phi}'_\delta(v)| \le c\delta^{-1} \quad \text{and} \quad |\bar{\phi}''_\delta(v)| \le c\delta^{-2}, \quad \text{for all } v \in [0,1]. \tag{4.5.31}$$

Set

$$\bar{H}(r,v) := \bar{\phi}_\delta(v)\int_0^r \bar{\psi}(\bar{Y}_a)\,da, \quad (r,v) \in [0,\Delta_\bullet] \times [0,1], \tag{4.5.32}$$

where $\{\bar{Y}_r : r \in [0,\Delta_\bullet]\}$ is defined in (4.3.17). We define the $\mathscr{H}$-valued random variable $u_{\bar{A}}$ evaluated at (r,v) by

$$u_{\bar{A}}(r,v) = \begin{cases} \left(\frac{\partial}{\partial r} - \frac{\partial^2}{\partial v^2}\right)\bar{H}(r,v) & \text{if } (r,v) \in]0,\Delta_\bullet] \times [0,1]; \\ 0 & \text{otherwise.} \end{cases} \tag{4.5.33}$$

Finally, we define the random variable

$$\gamma_{\bar{A}} = \int_0^{\Delta_\bullet} \bar{\psi}(\bar{Y}_r)\,dr. \tag{4.5.34}$$

We now prove the smoothness of these random variables, as required in Theorem 1.5.5.

Lemma 4.5.3. *For $i,j \in \{1,2\}$, $u^i_A \in \mathbb{D}^\infty(\mathscr{H})$, $\gamma^{i,j}_A \in \mathbb{D}^\infty$ and $u_{\bar{A}} \in \mathbb{D}^\infty(\mathscr{H})$, $\gamma_{\bar{A}} \in \mathbb{D}^\infty$.*

Proof. We first prove that $\gamma^{2,2}_A \in \mathbb{D}^\infty$. Similar to the proof of Lemma 4.5.1, we discretize the integral by setting

$$X_n := \frac{\delta_1}{n}\sum_{k=1}^n \psi(Y_{s_0+k\delta_1/n})$$

for $n \ge 1$. Since $r \mapsto Y_r$ is continuous, X_n converges to $\gamma^{2,2}_A$ a.s. as $n \to \infty$. By dominated convergence theorem, for any $p \ge 1$, X_n converges to $\gamma^{2,2}_A$ in $L^p(\Omega)$ as $n \to \infty$. By the chain rule, we know that $X_n \in \mathbb{D}^\infty$ and

$$DX_n = \frac{\delta_1}{n}\sum_{k=1}^n \psi'(Y_{s_0+k\delta_1/n})DY_{s_0+k\delta_1/n},$$

which converges a.s. to the Bochner integral $\int_{s_0}^{s_0+\delta_1}\psi'(Y_r)DY_r\,dr$ as $n \to \infty$ since $r \mapsto DY_r$ is continuous. For any $q \ge 1$, by Hölder's inequality,

$$\begin{aligned} \mathrm{E}[\|DX_n\|^q_{\mathscr{H}}] &\le cn^{-q}n^{q-1}\sum_{k=1}^n \mathrm{E}[\|DY_{s_0+k\delta_1/n}\|^q_{\mathscr{H}}] \\ &\le cn^{-q}n^{q-1}\sum_{k=1}^n \sup_{r\in[s_0,s_0+\delta_1]} \mathrm{E}\left[\|DY_r\|^q_{\mathscr{H}}\right] \le c, \end{aligned}$$

where the last inequality follows from (4.5.13). We apply Lemma A.6.3 to see that DX_n converges to the Bochner integral $\int_{s_0}^{s_0+\delta_1} \psi'(Y_r)DY_r dr$ in $L^q(\Omega;\mathscr{H})$ for any $q \geq 1$. Since the Malliavin derivative is closable (see [64, Proposition 1.2.1]), we obtain that

$$D\gamma_A^{2,2} = \int_{s_0}^{s_0+\delta_1} \psi'(Y_r)DY_r dr \tag{4.5.35}$$

and $\gamma_A^{2,2} \in \bigcap_{q\geq 1} \mathbb{D}^{1,q}$. In order to prove that $\gamma_A^{2,2} \in \mathbb{D}^\infty$, we can repeat this procedure and it remains to prove that for any $q, j \geq 1$,

$$\sup_{r\in[s_0,s_0+\delta_1]} \mathrm{E}[\|D^j\psi(Y_r)\|^q_{\mathscr{H}^{\otimes j}}] < \infty. \tag{4.5.36}$$

In order to prove (4.5.36), we use the Faà di Bruno formula (see formula [24.1.2] in [3]), we have

$$D^j\psi(Y_r) = \sum_{n=1}^{j} \psi^{(n)}(Y_r) \sum_{i,l_i:\sum_{i=1}^j l_i=n,\, \sum_{i=1}^j il_i=j} \bigotimes_{i=1}^{j} \frac{1}{i!}\left(\frac{D^iY_r}{l_i!}\right)^{\otimes l_i}, \tag{4.5.37}$$

where both $\bigotimes$ and $\otimes$ denote the tensor product of functions. Set

$$\Lambda_r = \psi^{(n)}(Y_r) \bigotimes_{i=1}^{j} \left(D^iY_r\right)^{\otimes l_i}. \tag{4.5.38}$$

We have

$$\|\Lambda_r\|_{\mathscr{H}^{\otimes j}} \leq c \prod_{i=1}^{j} \|D^iY_r\|^{l_i}_{\mathscr{H}^{\otimes i}}. \tag{4.5.39}$$

Then (4.5.37), (4.5.38), (4.5.39) and (4.5.13) give us (4.5.36). Hence $\gamma_A^{2,2}$ belongs to $\mathbb{D}^\infty$.

We can prove $\gamma_{\bar{A}} \in \mathbb{D}^\infty$ similarly by discretization and using (4.5.14).

We proceed to prove that $u_A^2 \in \mathbb{D}^\infty(\mathscr{H})$. By the definition of u_A^2 in (4.5.24), we can write

$$\begin{aligned} u_A^2(r,v) &= \psi(Y_r)1_{[s_0,s_0+\delta_1]}(r)\phi_{\delta_1}(v) - 1_{[s_0,s_0+\delta_1]}(r)\phi''_{\delta_1}(v)\int_{s_0}^{r}\psi(Y_a)da \\ &:= u_A^{21}(r,v) - u_A^{22}(r,v). \end{aligned} \tag{4.5.40}$$

We first prove that $u_A^{21} \in \mathbb{D}^\infty(\mathscr{H})$. For $n \geq 1$, we define

$$Y_n^1(r,v) := \sum_{k=1}^{n} \psi(Y_{s_0+\delta_1 k/n})1_{[s_0+\delta_1(k-1)/n,\, s_0+\delta_1 k/n]}(r)\phi_{\delta_1}(v).$$

For almost every $(\omega, r, v) \in \Omega\times[0,T]\times[0,1]$, $Y_n^1(r,v)$ converges to $u_A^{21}(r,v)$ as $n \to \infty$. By the dominated convergence theorem, Y_n^1 converges to u_A^{21} in $L^p(\Omega;\mathscr{H})$ for any $p \geq 1$ as $n\to\infty$. Since for any $r \in [s_0, s_0+\delta_1]$, $Y_r \in \mathbb{D}^\infty$, by (4.5.36) and chain rule, we know that for any $r \in [s_0, s_0+\delta_1]$, $\psi(Y_r) \in \mathbb{D}^\infty$.

We claim that if $Z \in \mathbb{D}^\infty$ and $h \in \mathscr{H}$, then Zh belongs to $\mathbb{D}^\infty(\mathscr{H})$. To see this, it suffices to prove that for any integer $k \ge 1$ and $p \ge 1$, Zh belongs to $\mathbb{D}^{k,p}(\mathscr{H})$. Since $Z \in \mathbb{D}^\infty$, we choose a sequence of smooth random variables $(Z_n)_{n\ge1}$ converging to Z in $\mathbb{D}^{k,p}$ as $n \to \infty$. By the definition of the norm $\|\cdot\|_{k,p,\mathscr{H}}$, $(Z_n h)_{n\ge1}$ is a Cauchy sequence in $\mathbb{D}^{k,p}(\mathscr{H})$, which converges to a limit in $\mathbb{D}^{k,p}(\mathscr{H})$, say $\tilde{Z}$. On the other hand, it is obvious to see $Z_n h$ converges to Zh in $L^2(\Omega, \mathscr{H})$ as $n \to \infty$. Hence $Zh = \tilde{Z} \in \mathbb{D}^{k,p}(\mathscr{H})$.

Applying this claim we see that Y_n^1 belongs to $\mathbb{D}^\infty(\mathscr{H})$ and

$$DY_n^1(\cdot,*) = \sum_{k=1}^n D\psi(Y_{s_0+\delta_1 k/n}) 1_{[s_0+\delta_1(k-1)/n, s_0+\delta_1 k/n]}(\cdot)\phi_{\delta_1}(*).$$

For almost every $(\omega, r, v) \in \Omega \times [0,T] \times [0,1]$, $DY_n^1(r,v)$ converges to $D\psi(Y_r)1_{[s_0,s_0+\delta_1]}(r)\phi_{\delta_1}(v)$ as $n \to \infty$ since $r \mapsto D\psi(Y_r)$ is continuous. Moreover, by Hölder's inequality, for any $q \ge 1$,

$$\begin{aligned}
\mathrm{E}\left[\int_0^T \int_0^1 \|DY_n^1(r,v)\|_{\mathscr{H}}^q dr dv\right] &= \sum_{k=1}^n \mathrm{E}\left[\int_{s_0+(k-1)\delta_1/n}^{s_0+k\delta_1/n} \|D\psi(Y_{s_0+\delta_1 k/n})\|_{\mathscr{H}}^q dr\right] \int_0^1 \phi_{\delta_1}^q(v) dv \\
&\le c \sum_{k=1}^n \int_{s_0+(k-1)\delta_1/n}^{s_0+k\delta_1/n} \sup_{r\in[s_0,s_0+\delta_1]} \mathrm{E}\left[\|DY_r\|_{\mathscr{H}}^q\right] dr \\
&\le c,
\end{aligned}$$

where the last inequality follows from (4.5.13). Applying Lemma A.6.3 (with the measure space replaced by $(\Omega \times [0,T] \times [0,1], \mathrm{P} \times \lambda^2)$, where λ^2 is the Lebesgue measure on $[0,T] \times [0,1]$), we have for any $q \ge 1$,

$$\lim_{n\to\infty} \mathrm{E}\left[\int_0^T \int_0^1 \|DY_n^1(r,v) - D\psi(Y_r)1_{[s_0,s_0+\delta_1]}(r)\phi_{\delta_1}(v)\|_{\mathscr{H}}^q dr dv\right] = 0,$$

which implies

$$\lim_{n\to\infty} \mathrm{E}\left[\left(\int_0^T \int_0^1 \|DY_n^1(r,v) - D\psi(Y_r)1_{[s_0,s_0+\delta_1]}(r)\phi_{\delta_1}(v)\|_{\mathscr{H}}^2 dr dv\right)^{q/2}\right] = 0.$$

Thus for any $q \ge 1$, $DY_n^1(\cdot,*)$ converges to $D\psi(Y_\cdot)1_{[s_0,s_0+\delta_1]}(\cdot)\phi_{\delta_1}(*)$ in $L^q(\Omega, \mathscr{H}^{\otimes 2})$ as $n \to \infty$. Since D is closable, we obtain

$$Du_A^{21}(\cdot,*) = D\psi(Y_\cdot)1_{[s_0,s_0+\delta_1]}(\cdot)\phi_{\delta_1}(*).$$

We repeat this procedure and apply (4.5.36) to conclude $u_A^{21} \in \mathbb{D}^\infty(\mathscr{H})$.

The proof for $u_A^{22} \in \mathbb{D}^\infty(\mathscr{H})$ is similar. We discretize $u_A^{22}(r,v)$ by

$$Y_n^2(r,v) := \sum_{k=1}^n \int_{s_0}^{s_0+k\delta_1/n} \psi(Y_a) da 1_{[s_0+(k-1)\delta_1/n, s_0+k\delta_1/n]}(r)\phi''_{\delta_1}(v).$$

In fact, the proof of $\gamma_A^{2,2} \in \mathbb{D}^\infty$ indicates that for any $r \in [s_0, s_0+\delta_1]$, $\int_{s_0}^r \psi(Y_a) da \in \mathbb{D}^\infty$. Hence applying the claim again, we see that $Y_n^2 \in \mathbb{D}^\infty(\mathscr{H})$. Similarly, we apply (4.5.36) again to conclude $u_A^{22} \in \mathbb{D}^\infty(\mathscr{H})$.

The proof of $u_{\bar{A}} \in \mathbb{D}^\infty(\mathscr{H})$ is similar. □

The following results gives some estimates on the $L^p(\Omega)$-norm of $(\gamma_A^{2,2})^{-1}$ and $\gamma_{\bar{A}}^{-1}$.

Lemma 4.5.4. *(a) The random variable $\gamma_A^{2,2}$ has finite negative moments of all orders. Furthermore, for any $p \geq 1$, there exists a constant c_p, not depending on $(s_0, y_0) \in I \times J$, such that for all small $\delta_1 > 0$ and for $z_2 \geq \delta_1^{1/4}$,*

$$\|(\gamma_A^{2,2})^{-1}\|_{L^p(\Omega)} \leq c_p \delta_1^{-1}. \tag{4.5.41}$$

(b) The random variable $\gamma_{\bar{A}}$ has finite negative moments of all orders. Furthermore, for any $p \geq 1$, there exists a constant c_p, not depending on $y_0 \in J$, such that for all small $\delta_1, \delta_2 > 0$ and for $z \geq (\delta_1^{1/2} + \delta_2)^{1/2}$,

$$\|\gamma_{\bar{A}}^{-1}\|_{L^p(\Omega)} \leq c_p (\delta_1^{1/2} + \delta_2)^{-2}. \tag{4.5.42}$$

Proof. We start by proving (a). By the definition of the function ψ,

$$\gamma_A^{2,2} \geq \int_{s_0}^{s_0+\delta_1} 1_{\{Y_r \leq \frac{R}{2}\}} dr := \bar{X}.$$

For $\epsilon < \delta_1$ and any $q \geq 1$, since $r \mapsto Y_r$ is increasing, we have

$$\begin{aligned} \mathrm{P}\{\bar{X} < \epsilon\} &\leq \mathrm{P}\{Y_{s_0+\epsilon} \geq R/2\} \\ &\leq (2/R)^q \mathrm{E}[|Y_{s_0+\epsilon}|^q] \leq c_q R^{-q} \epsilon^{2q} \delta_1^{(p_0-\gamma_0)q/2}, \end{aligned} \tag{4.5.43}$$

where in the second inequality we use Markov's inequality, and the last inequality is because of (4.3.4). This shows that the random variable $\gamma_A^{2,2}$ has finite negative moments of all orders by Lemma 4.4 in Chapter 3 of [24]. Moreover, for any $p \geq 1$ and $q > p/2$,

$$\begin{aligned} \mathrm{E}[\bar{X}^{-p}] &= p \int_0^\infty y^{p-1} \mathrm{P}(\bar{X}^{-1} > y) dy \\ &= p \int_0^{\delta_1^{-1}} y^{p-1} \mathrm{P}(\bar{X}^{-1} > y) dy + p \int_{\delta_1^{-1}}^\infty y^{p-1} \mathrm{P}(\bar{X}^{-1} > y) dy \\ &\leq c \delta_1^{-p} + c R^{-q} \delta_1^{(p_0-\gamma_0)q/2} \int_{\delta_1^{-1}}^\infty y^{p-1} y^{-2q} dy \\ &= c \delta_1^{-p} + c R^{-q} \delta_1^{(p_0-\gamma_0+4)q/2-p}. \end{aligned}$$

Using the definition of R in (4.3.5), this is equal to

$$c \delta_1^{-p} \left(1 + a^{-2p_0 q} \delta_1^{(\gamma_0-4)q/2} \delta_1^{(p_0-\gamma_0+4)q/2}\right).$$

Under the assumption $z_2 \geq \delta_1^{1/4}$, by (4.5.16), this is bounded above by

$$c\delta_1^{-p}\left(1+\delta_1^{-\frac{1}{4}\times 2p_0q}\delta_1^{(\gamma_0-4)q/2}\delta_1^{(p_0-\gamma_0+4)q/2}\right)=2c\delta_1^{-p},$$

which implies (4.5.41).

We proceed to prove (b). Similarly, by the definition of the function $\bar{\psi}$,

$$\gamma_{\bar{A}} \geq \int_0^{\Delta_\bullet} 1_{\{\bar{Y}_r \leq \frac{\bar{R}}{2}\}} dr := \tilde{X}.$$

For any $0<\epsilon<\Delta_\bullet$, since $r \mapsto \bar{Y}_r$ is increasing,

$$\begin{aligned} \mathrm{P}\{\tilde{X}<\epsilon\} &\leq \mathrm{P}\{\bar{Y}_\epsilon \geq \bar{R}/2\} \\ &\leq (2/\bar{R})^q \mathrm{E}[\bar{Y}_\epsilon^q] \leq c_q \bar{R}^{-q}\epsilon^{2q}\delta^{(p_0-\gamma_0)q}, \end{aligned} \tag{4.5.44}$$

where, in the last inequality, we use (4.3.18). Hence the random variable $\gamma_{\bar{A}}$ has finite negative moments of all orders. Moreover, for any $p \geq 1$ and $q > p/2$,

$$\begin{aligned} \mathrm{E}[\tilde{X}^{-p}] &= p\int_0^\infty y^{p-1}\mathrm{P}(\tilde{X}^{-1}>y)dy \\ &= p\int_0^{\Delta_\bullet^{-1}} y^{p-1}\mathrm{P}(\tilde{X}^{-1}>y)dy + p\int_{\Delta_\bullet^{-1}}^\infty y^{p-1}\mathrm{P}(\tilde{X}^{-1}>y)dy \\ &\leq c\Delta_\bullet^{-p} + c\bar{R}^{-q}\delta^{(p_0-\gamma_0)q}\int_{\Delta_\bullet^{-1}}^\infty y^{p-1}y^{-2q}dy \\ &= c\Delta_\bullet^{-p} + c\bar{R}^{-q}\delta^{(p_0-\gamma_0)q}\Delta_\bullet^{2q-p}. \end{aligned}$$

Using the definition of $\bar{R}$ in (4.3.27), this is equal to

$$c\Delta_\bullet^{-p}\left(1+\bar{a}^{-2p_0q}\delta^{-(4-\gamma_0)q}\delta^{(p_0-\gamma_0)q}\Delta_\bullet^{2q}\right).$$

Under the assumption $z \geq \delta^{1/2} = (\delta_1^{1/2}+\delta_2)^{1/2}$, by (4.5.26) and (4.3.13), this is bounded above by

$$c\Delta_\bullet^{-p}\left(1+\delta^{-\frac{1}{2}\times 2p_0q}\delta^{-(4-\gamma_0)q}\delta^{(p_0-\gamma_0)q}\Delta_\bullet^{2q}\right)=2c\Delta_\bullet^{-p},$$

which implies (4.5.42). □

Now we are ready to verify that the random variables introduced above satisfy the condition (iii) of Theorem 1.5.5.

Lemma 4.5.5. *(a) On the event* $\{F \in A\} = \{F_2 > a\}$, *we have* $\langle DF_i, u_A^j\rangle_{\mathscr{H}} = \gamma_A^{i,j}$ *for* $i,j \in \{1,2\}$.
(b) On the event $\{M_0 \in \bar{A}\} = \{M_0 > \bar{a}\}$, $\langle DM_0, u_{\bar{A}}\rangle_{\mathscr{H}} = \gamma_{\bar{A}}$.

Proof. We first prove (a). If $s_0 > 0$, by the definitions of u_A^1 in (4.5.19) and of the functions f_0, g_0, and Lemma 4.3.6, we have that

$$\begin{aligned}\langle DF_1, u_A^1\rangle_{\mathscr{H}} &= \int_0^{s_0}\int_0^1 G(s_0-r, y_0, v)\left(\frac{\partial}{\partial r}-\frac{\partial^2}{\partial v^2}\right)(f_0(r)g_0(v))\,dr\,dv \\ &= f_0(s_0)g_0(y_0) = 1 = \gamma_A^{1,1}.\end{aligned}\tag{4.5.45}$$

Second, from the definition of u_A^2 in (4.5.24), it is obvious that

$$\langle DF_1, u_A^2\rangle_{\mathscr{H}} = \int_0^{s_0}\int_0^1 G(s_0-r, y_0, v)\left(\frac{\partial}{\partial r}-\frac{\partial^2}{\partial v^2}\right)H(r,v)1_{\{s_0<r\le s_0+\delta_1\}}\,dr\,dv = 0. \tag{4.5.46}$$

By Lemmas 4.4.4 and 4.3.6,

$$\begin{aligned}\langle DF_2, u_A^1\rangle_{\mathscr{H}} &= \int_0^{S}\int_0^1 G(S-r, y_0, v)\left(\frac{\partial}{\partial r}-\frac{\partial^2}{\partial v^2}\right)(f_0(r)g_0(v))\,dr\,dv \\ &\quad -\int_0^{s_0}\int_0^1 G(s_0-r, y_0, v)\left(\frac{\partial}{\partial r}-\frac{\partial^2}{\partial v^2}\right)(f_0(r)g_0(v))\,dr\,dv \\ &= f_0(S)g_0(y_0) - f_0(s_0)g_0(y_0) = 1-1 = 0.\end{aligned}\tag{4.5.47}$$

Furthermore, by Lemma 4.3.6, for both cases $s_0>0$ and $s_0=0$,

$$\begin{aligned}\langle DF_2, u_A^2\rangle_{\mathscr{H}} &= \int_0^S dr\int_0^1 dv\, G(S-r, y_0, v)u_A^2(r,v) - \int_0^{s_0} dr\int_0^1 dv\, G(s_0-r, y_0, v)u_A^2(r,v) \\ &= \int_{s_0}^S dr\int_0^1 dv\, G(S-r, y_0, v)\left(\frac{\partial}{\partial r}-\frac{\partial^2}{\partial v^2}\right)H(r,v) - 0 \\ &= \int_0^{S-s_0} dr\int_0^1 dv\, G(S-s_0-r, y_0, v)\left(\frac{\partial}{\partial r}-\frac{\partial^2}{\partial v^2}\right)H(s_0+r,v) \\ &= H(S, y_0).\end{aligned}\tag{4.5.48}$$

Therefore,

$$\langle DF_2, u_A^2\rangle_{\mathscr{H}} = \phi_{\delta_1}(y_0)\int_{s_0}^S \psi(Y_r)\,dr = \int_{s_0}^S \psi(Y_r)\,dr \tag{4.5.49}$$

where, in the second equality, we use the fact that $\phi_{\delta_1}(y_0)=1$. Moreover, on the event $\{F\in A\} = \{F_2 > a\}$, we observe that if $r > S \ge s_0$, then $\psi(Y_r)=0$. Otherwise, we would have $\psi(Y_r)>0$, hence $Y_r \le R$ for some $r > S$, and by Lemma 4.3.1, this implies that

$$F_2 = \bar{u}(S, y_0) = \sup_{t\in[s_0,r]} \bar{u}(t, y_0) \le a < F_2,$$

which is a contradiction. Hence, on $\{F\in A\} = \{F_2>a\}$, the last integral in (4.5.49) is equal to

$$\int_{s_0}^{s_0+\delta_1}\psi(Y_r)\,dr = \gamma_A^{2,2}.$$

This completes the proof of (a).

We now prove (b). By Lemma 4.4.4,

$$\begin{aligned}
\langle DM_0, u_{\bar{A}}\rangle_{\mathscr{H}} &= \langle 1_{\{\cdot<\bar{S}\}}(G(\bar{S}-\cdot,\bar{X},*),u_{\bar{A}}\rangle_{\mathscr{H}} \\
&= \int_0^{\bar{S}}\int_0^1 G(\bar{S}-r,\bar{X},v)\left(\frac{\partial}{\partial r}-\frac{\partial^2}{\partial v^2}\right)\bar{H}(r,v)\,dv\,dr \\
&= \bar{H}(\bar{S},\bar{X}) = \bar{\phi}_\delta(\bar{X})\int_0^{\bar{S}}\bar{\psi}(\bar{Y}_r)\,dr. \qquad (4.5.50)
\end{aligned}$$

Since $\bar{X}\in[y_0,y_0+\delta_2]$, by the definition of the function $\bar{\phi}_\delta$, it implies that $\bar{\phi}_\delta(\bar{X})\equiv 1$.Hence,

$$\langle DM_0, u_{\bar{A}}\rangle_{\mathscr{H}} = \int_0^{\bar{S}}\bar{\psi}(\bar{Y}_r)\,dr.$$

On the event $\{M_0>\bar{a}\}$, for $r>\bar{S}$, we have $\bar{\psi}(\bar{Y}_r)=0$. Otherwise, we would have $\bar{\psi}(\bar{Y}_r)>0$, hence $\bar{Y}_r\le\bar{R}$ and by Lemma 4.3.5 this implies that

$$M_0 = u(\bar{S},\bar{X}) = \sup_{(t,x)\in[0,r]\times[y_0,y_0+\delta_2]} u(t,x)\le\bar{a}<M_0,$$

which is a contradiction. Therefore, on the event $\{M_0\in\bar{A}\}$,

$$\langle DM_0, u_{\bar{A}}\rangle_{\mathscr{H}} = \int_0^{\Delta_\bullet}\bar{\psi}(\bar{Y}_r)\,dr = \gamma_{\bar{A}}.$$

This proves (b). □

Proof of Theorem 4.2.1(a). The strict positivity of F_2 has been proved in Lemma 4.4.2(a). For $(s_0,y_0)\in I\times J\subset[0,T]\times[0,1]$ with $s_0>0$, by Lemmas 4.5.3, 4.5.4(a), 4.5.5(a) and Theorem 1.5.5, the random vector F has an infinitely differentiable density on $\mathbb{R}\times]z_2/2,\infty[$. Since the choice of z_2 is arbitrary, the random vector F possesses an infinitely differentiable density on $\mathbb{R}\times]0,\infty[$. Using the same argument, if $s_0=0$, then the random variable F_2 has an infinitely differentiable density on $]0,\infty[$. □

Proof of Theorem 4.2.1(b). The strict positivity of M_0 has been proved in Lemma 4.4.2(b). The proof of smoothness of the density of M_0 is similar to that of Theorem 4.2.1(a) by using Lemmas 4.5.3, 4.5.4(b), 4.5.5(b) and Theorem 1.5.5. □

We now derive the expression for the probability density functions of F and M_0 from the integration by parts formula; see [64, (2.25)]

Proposition 4.5.6. *(a) The probability density function of F at $(z_1,z_2)\in\mathbb{R}\times]0,\infty[$ is given by*

$$p(z_1,z_2) = E\left[1_{\{F_1>z_1,F_2>z_2\}}\delta\left(u_A^1\delta\left(u_A^2/\gamma_A^{2,2}\right)\right)\right]. \qquad (4.5.51)$$

(b) The probability density function of M_0 *at* $z \in]0, \infty[$ *is given by*

$$p_0(z) = E[1_{\{M_0 > z\}} \delta(u_{\bar{A}}/\gamma_{\bar{A}})]. \tag{4.5.52}$$

Proof. We first derive the formula (4.5.52). Let $\tilde{\kappa}_z : \mathbb{R} \mapsto [0,1]$ be an infinitely differentiable function such that $\tilde{\kappa}_z(x) = 0$ for all $x \le \frac{2z}{3}$ and $\tilde{\kappa}_z(x) = 1$ for all $x \ge \frac{3z}{4}$. Define $G_0 = \tilde{\kappa}(M_0)$. Consider $\bar{a}$ and $\bar{A}$ as in (4.5.26). It is clear that on the set $\{M_0 \notin \bar{A}\}$, we have $G_0 = 0$.

Let f be a function in the space $C_0^\infty(\mathbb{R})$ of infinitely differentiable functions with compact support. Set $\varphi(x) = \int_{-\infty}^x f(y)dy$. On $\{M_0 \in \bar{A}\}$, by the chain rule of Malliavin derivative (see [64, Proposition 1.2.3]) and Lemma 4.5.5(b), we have

$$\langle D\varphi(M_0), u_{\bar{A}} \rangle_{\mathscr{H}} = \varphi'(M_0) \langle DM_0, u_{\bar{A}} \rangle_{\mathscr{H}} = \varphi'(M_0)\gamma_{\bar{A}}.$$

Hence,

$$\varphi'(M_0) = \langle D\varphi(M_0), u_{\bar{A}}/\gamma_{\bar{A}} \rangle_{\mathscr{H}}.$$

Since $G_0 = 0$ on the set $\{M_0 \notin \bar{A}\}$, we obtain

$$G_0\varphi'(M_0) = G_0 \langle D\varphi(M_0), u_{\bar{A}}/\gamma_{\bar{A}} \rangle_{\mathscr{H}}.$$

Taking expectations on both sides of the above equation and using the duality relationship between the derivative and the divergence operators we get

$$\mathrm{E}[G_0\varphi'(M_0)] = \mathrm{E}[\varphi(M_0)\delta(G_0 u_{\bar{A}}/\gamma_{\bar{A}})]. \tag{4.5.53}$$

Using the fact that

$$\varphi(M_0) = \int_{-\infty}^{M_0} \varphi'(y)dy \tag{4.5.54}$$

and Fubini's theorem, we obtain that

$$\mathrm{E}[G_0\varphi'(M_0)] = \int_{\mathbb{R}} \varphi'(y)\mathrm{E}[1_{\{M_0 > y\}}\delta(G_0 u_{\bar{A}}/\gamma_{\bar{A}})]dy, \tag{4.5.55}$$

and equivalently,

$$\mathrm{E}[G_0 f(M_0)] = \int_{\mathbb{R}} f(y)\mathrm{E}[1_{\{M_0 > y\}}\delta(G_0 u_{\bar{A}}/\gamma_{\bar{A}})]dy. \tag{4.5.56}$$

Since $G_0 = 1$ on the set $\{M_0 \ge \frac{3z}{4}\}$, this implies that for any $y \in]\frac{3z}{4}, \infty[$, the density function of M_0 at y is given by

$$p_0(y) = \mathrm{E}[1_{\{M_0 > y\}}\delta(G_0 u_{\bar{A}}/\gamma_{\bar{A}})].$$

In particular,

$$p_0(z) = \mathrm{E}[1_{\{M_0>z\}}\delta(G_0 u_{\bar{A}}/\gamma_{\bar{A}})].$$

Since $G_0 = 1$ on the set $\{M_0 > z\}$, by the local property of δ (see [64, Proposition 1.3.15]), we obtain

$$p_0(z) = \mathrm{E}[1_{\{M_0>z\}}\delta(u_{\bar{A}}/\gamma_{\bar{A}})].$$

We now derive the formula (4.5.51). Let $\kappa_{z_2} : \mathbb{R} \mapsto [0,1]$ be an infinitely differentiable function such that $\kappa_{z_2}(x) = 0$ for all $x \le \frac{2z_2}{3}$ and $\kappa_{z_2}(x) = 1$ for all $x \ge \frac{3z_2}{4}$. Define $\bar{\kappa}(y_1, y_2) = \kappa_{z_2}(y_2)$ and $G = \bar{\kappa}(F)$. Consider a and A as in (4.5.16). It is clear that on the set $\{F \not\in A\}$, we have $G = 0$.

Let g be a function in the space $C_0^\infty(\mathbb{R}^2)$ of infinitely differentiable functions with compact support. Set

$$\varphi(x_1, x_2) = \int_{-\infty}^{x_1}\int_{-\infty}^{x_2} g(y_1, y_2)dy_1dy_2. \tag{4.5.57}$$

On $\{F \in A\}$, by the chain rule of Malliavin derivative (see [64, Proposition 1.2.3]) and Lemma 4.5.5(a), we have

$$\langle D\partial_1\varphi(F), u_A^j\rangle_{\mathcal{H}} = \sum_{i=1}^{2}\partial_{1i}\varphi(F)\langle DF_i, u_A^j\rangle_{\mathcal{H}} = \sum_{i=1}^{2}\partial_{1i}\varphi(F)\gamma_A^{i,j},$$

where the notation ∂_{1i} means we take the partial derivative with respect to the first variable and then take the partial derivative with respect to the ith variable. Consequently,

$$\partial_{12}\varphi(F) = \sum_{k=1}^{2}\langle D\partial_1\varphi(F), u_A^k\rangle_{\mathcal{H}}(\gamma_A^{-1})^{k,2}.$$

Since $G = 0$ on the set $\{F \not\in A\}$, we obtain

$$G\partial_{12}\varphi(F) = \sum_{k=1}^{2} G\langle D\partial_1\varphi(F), u_A^k\rangle_{\mathcal{H}}(\gamma_A^{-1})^{k,2}.$$

Taking expectations on both sides of the above equation and using the duality relationship between the derivative and the divergence operators we get

$$\mathrm{E}[G\partial_{12}\varphi(F)] = \mathrm{E}[\partial_1\varphi(F)\delta(\sum_{k=1}^{2} Gu_A^k(\gamma_A^{-1})^{k,2})]. \tag{4.5.58}$$

We denote

$$\bar{G} = \delta(\sum_{k=1}^{2} Gu_A^k(\gamma_A^{-1})^{k,2}).$$

Since $G=0$ on the set $\{F \not\in A\}$, the local property of δ (see [64, Proposition 1.3.15]) implies that $\bar{G}=0$ on the set $\{F \not\in A\}$. On the other hand, on $\{F \in A\}$, by the chain rule and Lemma 4.5.5,

$$\langle D\varphi(F), u_A^j\rangle_{\mathscr{H}} = \sum_{i=1}^{2} \partial_i \varphi(F)\langle DF_i, u_A^j\rangle_{\mathscr{H}} = \sum_{i=1}^{2} \partial_i\varphi(F)\gamma_A^{i,j},$$

which implies that on $\{F \in A\}$,

$$\partial_1\varphi(F) = \sum_{n=1}^{2} \langle D\varphi(F), u_A^n\rangle_{\mathscr{H}}(\gamma_A^{-1})^{n,1}.$$

Multiplying both sides of the above equality by $\bar{G}$, we obtain

$$\bar{G}\partial_1\varphi(F) = \sum_{n=1}^{2} \bar{G}\langle D\varphi(F), u_A^n\rangle_{\mathscr{H}}(\gamma_A^{-1})^{n,1}. \tag{4.5.59}$$

We substitute (4.5.59) into (4.5.58) and we obtain

$$\begin{aligned}
\mathrm{E}[G\partial_{12}\varphi(F)] &= \mathrm{E}\left[\sum_{n=1}^{2} \bar{G}\langle D\varphi(F), u_A^n\rangle_{\mathscr{H}}(\gamma_A^{-1})^{n,1}\right] \\
&= \mathrm{E}\left[\varphi(F)\delta\left(\sum_{n=1}^{2} \bar{G}u_A^n(\gamma_A^{-1})^{n,1}\right)\right] \\
&= \mathrm{E}\left[\varphi(F)\delta\left(\sum_{n=1}^{2} \delta\left(\sum_{k=1}^{2} Gu_A^k(\gamma_A^{-1})^{k,2}\right) u_A^n(\gamma_A^{-1})^{n,1}\right)\right].
\end{aligned}$$

Since $(\gamma_A^{-1})^{1,1}=1$ and $(\gamma_A^{-1})^{1,2}=(\gamma_A^{-1})^{2,1}=0$ by (4.5.25), this is equal to

$$\mathrm{E}\left[\varphi(F)\delta(\delta(Gu_A^2(\gamma_A^{-1})^{2,2})u_A^1(\gamma_A^{-1})^{1,1})\right] = \mathrm{E}[\varphi(F)\delta(\delta(Gu_A^2/\gamma_A^{2,2})u_A^1)].$$

Using the fact that

$$\varphi(F) = \int_{-\infty}^{F_1}\int_{-\infty}^{F_2} \partial_{12}\varphi(y_1,y_2)\,dy_1\,dy_2 \tag{4.5.60}$$

and Fubini's theorem, we obtain that

$$\mathrm{E}[G\partial_{12}\varphi(F)] = \int_{\mathbb{R}^2} \partial_{12}\varphi(y_1,y_2)\mathrm{E}[1_{\{F_1>y_1,F_2>y_2\}}\delta(\delta(Gu_A^2/\gamma_A^{2,2})u_A^1)]\,dy_1\,dy_2, \tag{4.5.61}$$

and equivalently,

$$\mathrm{E}[Gg(F)] = \int_{\mathbb{R}^2} g(y_1,y_2)\mathrm{E}[1_{\{F_1>y_1,F_2>y_2\}}\delta(\delta(Gu_A^2/\gamma_A^{2,2})u_A^1)]\,dy_1\,dy_2. \tag{4.5.62}$$

Since $G=1$ on the set $\{F \in \mathbb{R}\times[\frac{3z_2}{4},\infty[\}$, this implies that for any $(y_1,y_2)\in\mathbb{R}\times]\frac{3z_2}{4},\infty[$, the density function of F at (y_1, y_2) is given by

$$p(y_1, y_2) = \mathrm{E}[1_{\{F_1>y_1, F_2>y_2\}}\delta(\delta(Gu_A^2/\gamma_A^{2,2})u_A^1)].$$

In particular,

$$p(z_1, z_2) = \mathrm{E}[1_{\{F_1>z_1, F_2>z_2\}}\delta(\delta(Gu_A^2/\gamma_A^{2,2})u_A^1)].$$

Since $G = 1$ on the set $\{F_2 > z_2\}$, by the local property of δ (see [64, Proposition 1.3.15]), we obtain

$$p(z_1, z_2) = \mathrm{E}[1_{\{F_1>z_1, F_2>z_2\}}\delta(\delta(u_A^2/\gamma_A^{2,2})u_A^1)].$$

□

Remark 4.5.7. *In the proof of Proposition 4.5.6, if we use the fact that*

$$\varphi(F) = -\int_{F_1}^{+\infty}\int_{-\infty}^{F_2}\partial_{12}\varphi(y_1, y_2)\,dy_2\,dy_1$$

instead of (4.5.60)*, we obtain another formula for the joint density:*

$$p(z_1, z_2) = -E[1_{\{F_1<z_1, F_2>z_2\}}\delta(\delta(u_A^2/\gamma_A^{2,2})u_A^1)]. \tag{4.5.63}$$

4.6 Gaussian-type upper bound on the density of F

In this section, we fix $I \times J \subset]0, T] \times]0, 1[$ and assume that δ_1 satisfies the conditions in (4.2.11). We derive an estimate on the density of F from the formula obtained in the previous section. This estimate will prove Theorem 4.2.2.

First, from (4.5.51) and applying Hölder's inequality, for $z_1 \geq 0$,

$$p(z_1, z_2) \leq \mathrm{P}\{F_1 > z_1\}^{1/4}\mathrm{P}\{F_2 > z_2\}^{1/4}\|\delta(\delta(u_A^2/\gamma_A^{2,2})u_A^1)\|_{L^2(\Omega)}. \tag{4.6.1}$$

On the other hand, if $z_1 < 0$, applying Hölder's inequality to (4.5.63), we have

$$p(z_1, z_2) \leq \mathrm{P}\{F_1 < z_1\}^{1/4}\mathrm{P}\{F_2 > z_2\}^{1/4}\|\delta(\delta(u_A^2/\gamma_A^{2,2})u_A^1)\|_{L^2(\Omega)}. \tag{4.6.2}$$

Combining (4.6.1) and (4.6.2), we obtain that, for all $(z_1, z_2) \in \mathbb{R}\times]0, \infty[$,

$$p(z_1, z_2) \leq \mathrm{P}\{|F_1| > |z_1|\}^{1/4}\mathrm{P}\{F_2 > z_2\}^{1/4}\|\delta(\delta(u_A^2/\gamma_A^{2,2})u_A^1)\|_{L^2(\Omega)}. \tag{4.6.3}$$

In what follows, we use the properties of the Skorohod integral δ to express $\delta(\delta(u_A^2/\gamma_A^{2,2})u_A^1)$.

Lemma 4.6.1.

$$\delta(\delta(u_A^2/\gamma_A^{2,2})u_A^1) = T_1 + T_2 - T_3 + T_4 - T_5 + T_6, \tag{4.6.4}$$

where

$$T_1 = \frac{\delta(u_A^2)}{\gamma_A^{2,2}}\delta(u_A^1), \quad T_2 = \frac{\langle D\gamma_A^{2,2}, u_A^2\rangle_{\mathscr{H}}}{(\gamma_A^{2,2})^2}\delta(u_A^1), \quad T_3 = \frac{1}{\gamma_A^{2,2}}\langle D\delta(u_A^2), u_A^1\rangle_{\mathscr{H}}, \tag{4.6.5}$$

$$T_4 = \frac{\delta(u_A^2)}{(\gamma_A^{2,2})^2}\langle D\gamma_A^{2,2}, u_A^1\rangle_{\mathscr{H}}, \quad T_5 = \frac{2\langle D\gamma_A^{2,2}, u_A^2\rangle_{\mathscr{H}}}{(\gamma_A^{2,2})^3}\langle D\gamma_A^{2,2}, u_A^1\rangle_{\mathscr{H}}, \tag{4.6.6}$$

$$T_6 = \frac{1}{(\gamma_A^{2,2})^2}\langle D\langle D\gamma_A^{2,2}, u_A^2\rangle_{\mathscr{H}}, u_A^1\rangle_{\mathscr{H}}. \tag{4.6.7}$$

Proof. First, by [64, (1.48)],

$$\delta(\delta(u_A^2/\gamma_A^{2,2})u_A^1) = \delta(u_A^2/\gamma_A^{2,2})\delta(u_A^1) - \langle D\delta(u_A^2/\gamma_A^{2,2}), u_A^1\rangle_{\mathscr{H}}. \tag{4.6.8}$$

We use [64, (1.48)] again to write

$$\delta(u_A^2/\gamma_A^{2,2}) = \delta(u_A^2)/\gamma_A^{2,2} + \langle D\gamma_A^{2,2}, u_A^2\rangle_{\mathscr{H}}/(\gamma_A^{2,2})^2. \tag{4.6.9}$$

Hence the first term on the right-hand side of (4.6.8) is equal to

$$\delta(u_A^2/\gamma_A^{2,2})\delta(u_A^1) = \frac{\delta(u_A^2)}{\gamma_A^{2,2}}\delta(u_A^1) + \frac{\langle D\gamma_A^{2,2}, u_A^2\rangle_{\mathscr{H}}}{(\gamma_A^{2,2})^2}\delta(u_A^1). \tag{4.6.10}$$

For the second term on the right-hand side of (4.6.8), we apply (4.6.9) to obtain that

$$\begin{aligned} D\delta(u_A^2/\gamma_A^{2,2}) &= D(\delta(u_A^2)/\gamma_A^{2,2}) - D(\langle D\gamma_A^{2,2}, u_A^2\rangle_{\mathscr{H}}/(\gamma_A^{2,2})^2) \\ &= \frac{D\delta(u_A^2)}{\gamma_A^{2,2}} - \frac{\delta(u_A^2)D\gamma_A^{2,2}}{(\gamma_A^{2,2})^2} - \frac{D\langle D\gamma_A^{2,2}, u_A^2\rangle_{\mathscr{H}}}{(\gamma_A^{2,2})^2} + \frac{2\langle D\gamma_A^{2,2}, u_A^2\rangle_{\mathscr{H}} D\gamma_A^{2,2}}{(\gamma_A^{2,2})^3}. \end{aligned} \tag{4.6.11}$$

Therefore the second term on the right-hand side of (4.6.8) can be written as

$$\begin{aligned} -\langle D\delta(u_A^2/\gamma_A^{2,2}), u_A^1\rangle_{\mathscr{H}} &= -\frac{1}{\gamma_A^{2,2}}\langle D\delta(u_A^2), u_A^1\rangle_{\mathscr{H}} - \frac{2\langle D\gamma_A^{2,2}, u_A^2\rangle_{\mathscr{H}}}{(\gamma_A^{2,2})^3}\langle D\gamma_A^{2,2}, u_A^1\rangle_{\mathscr{H}} \\ &\quad + \frac{\delta(u_A^2)}{(\gamma_A^{2,2})^2}\langle D\gamma_A^{2,2}, u_A^1\rangle_{\mathscr{H}} + \frac{1}{(\gamma_A^{2,2})^2}\langle D\langle D\gamma_A^{2,2}, u_A^2\rangle_{\mathscr{H}}, u_A^1\rangle_{\mathscr{H}}. \end{aligned} \tag{4.6.12}$$

Putting (4.6.10) and (4.6.12) together, we obtain (4.6.4). □

Proposition 4.6.2. *(a) For any $p \geq 2$, there exists $c_p > 0$, not depending on $(s_0, y_0) \in I \times J$, such that for all small $\delta_1 > 0$, and for all $z_2 \geq \delta_1^{1/4}$,*

$$\|T_i\|_{L^p(\Omega)} \leq c_p \delta_1^{-1/4}, \quad \textit{for } i \in \{1,2,3\}. \tag{4.6.13}$$

(b) T_4, T_5 and T_6 vanish.

An immediate consequence of Lemma 4.6.1 and Proposition 4.6.2 is the following.

Proposition 4.6.3. *There exists a finite positive constant c, not depending on $(s_0, y_0) \in I \times J$, such that for all small $\delta_1 > 0$ and for all $z_2 \geq \delta_1^{1/4}$,*

$$\|\delta(\delta(u_A^2/\gamma_A^{2,2})u_A^1)\|_{L^2(\Omega)} \leq c\delta_1^{-1/4}. \tag{4.6.14}$$

The proof of Proposition 4.6.2 is divided into the following two subsections.

4.6.1 Proof of Proposition 4.6.2(a)

Throughout Section 4.6.1, we assume that

$$z_2 \geq \delta_1^{1/4}. \tag{4.6.15}$$

Recalling the definition of R in (4.3.5), under the assumption (4.6.15), we see from (4.5.16) that

$$\begin{aligned} R^{-1} &= c^{-1}a^{-2p_0}\delta_1^{(\gamma_0-4)/2} = c' z_2^{-2p_0}\delta_1^{(\gamma_0-4)/2} \\ &\leq c\delta_1^{(\gamma_0-p_0-4)/2}. \end{aligned} \tag{4.6.16}$$

We will make use of this in the estimates below.

We first give an estimate for the moments of T_1. In order to estimate the moments of the Skorohod integral $\delta(u_A^2)$, we extend Proposition 1.3.11 of [64] to multiparameter adapted processes, as mentioned in [64, p.45].

We denote by L_a^2 the closed subspace of $L^2(\Omega \times [0,T] \times [0,1])$ formed by those processes which are adapted to the filtration $\{\mathscr{F}_s := \sigma\{W(t,x) : t \leq s, x \in [0,1]\}, s \in [0,T]\}$.

Proposition 4.6.4. *$L_a^2 \subset Dom\, \delta$ and the operator δ restricted to L_a^2 coincides with the Walsh integral, that is, for $u \in L_a^2$*

$$\delta(u) = \int_0^T \int_0^1 u(r,v) W(dr, dv) \tag{4.6.17}$$

Proof. We follow the proof of [64, Proposition 1.3.11]. Define $Z(t,x) = Y 1_{]a,b]}(t) 1_B(x)$ where the random variable Y is square integrable and measurable with respect to $\mathscr{F}_a$ and B is a bounded interval. We first prove that (4.6.17) holds for Z. Since a square integrable random variable can be approximated by random variables in $\mathbb{D}^{1,2}$ and δ is closed, we can assume $Y \in \mathbb{D}^{1,2}$. Using [64, (1.48) and Corollary 1.2.1], we have

$$\delta(Y 1_{]a,b]}(t) 1_B(x)) = Y W(]a,b] \times B) = \int_0^T \int_0^1 Z(r,v) W(dr,dv). \tag{4.6.18}$$

Since the linear span of the random variables of the same type as Z is dense in L_a^2 (see [16, Proposition 3.1]), we can find a sequence $\{Z_n\}_{n\geq 1}$ converging to u in $L^2(\Omega \times [0,T] \times [0,1])$ and

$$\delta(Z_n) = \int_0^T \int_0^1 Z_n(r,v) W(dr,dv). \tag{4.6.19}$$

By Itô's isometry, we know that $\int_0^T \int_0^1 Z_n(r,v) W(dr,dv)$ converges to $\int_0^T \int_0^1 u(r,v) W(dr,dv)$ in $L^2(\Omega)$ as $n \to \infty$. Since δ is closed, this implies $u \in \operatorname{Dom} \delta$ along with (4.6.17). $\square$

Proposition 4.6.4 enables us to use properties of Walsh integrals to estimate the $L^p(\Omega)$-norm of $\delta(u_A^2)$, as in the following lemma.

Lemma 4.6.5. *For any $p \geq 2$, there exists a constant c_p, not depending on $(s_0, y_0) \in I \times J$, such that for all $\delta_1 > 0$,*

$$\|\delta(u_A^2)\|_{L^p(\Omega)} \leq c_p \delta_1^{3/4}. \tag{4.6.20}$$

Proof. From (4.5.40), we know that for $(r,v) \in [s_0, s_0+\delta_1] \times [0,1]$,

$$u_A^2(r,v) = \phi_{\delta_1}(v)\psi(Y_r) - \phi''_{\delta_1}(v) \int_{s_0}^r \psi(Y_a) da.$$

Since u_A^2 is adapted, by Proposition 4.6.4, we have

$$\delta(u_A^2) = \int_{s_0}^{s_0+\delta_1} \int_0^1 \phi_{\delta_1}(v)\psi(Y_r) W(dr,dv) - \int_{s_0}^{s_0+\delta_1} \int_0^1 W(dr,dv) \phi''_{\delta_1}(v) \int_{s_0}^r \psi(Y_a) da. \tag{4.6.21}$$

For the first term on the right-hand side of (4.6.21), by Burkholder's inequality, for any $p \geq 2$, since $0 \le \psi \le 1$,

$$
\begin{aligned}
&\left\| \int_{s_0}^{s_0+\delta_1} \int_0^1 \phi_{\delta_1}(v)\psi(Y_r) W(dr,dv) \right\|_{L^p(\Omega)}^p \\
&\le c_p \mathrm{E}\left[\left(\int_{s_0}^{s_0+\delta_1} \int_0^1 \phi_{\delta_1}^2(v)\psi^2(Y_r)\,dr\,dv \right)^{p/2} \right] \\
&\le c_p \delta_1^{p/2} \left(\int_0^1 \phi_{\delta_1}^2(v)\,dv \right)^{p/2} \\
&\le c_p \delta_1^{p/2} \delta_1^{p/4} = c_p \delta_1^{3p/4}.
\end{aligned}
\tag{4.6.22}
$$

For the second term on the right-hand side of (4.6.21), similarly, by Burkholder's inequality, for any $p \ge 2$, since $0 \le \psi \le 1$,

$$
\begin{aligned}
&\left\| \int_{s_0}^{s_0+\delta_1} \int_0^1 W(dr,dv)\phi''_{\delta_1}(v) \int_{s_0}^r \psi(Y_a)\,da \right\|_{L^p(\Omega)}^p \\
&\le c_p \mathrm{E}\left[\left(\int_{s_0}^{s_0+\delta_1} dr \int_0^1 dv (\phi''_{\delta_1}(v))^2 \left(\int_{s_0}^r \psi(Y_a)\,da \right)^2 \right)^{p/2} \right] \\
&\le c_P \left(\int_{s_0}^{s_0+\delta_1} (r-s_0)^2\,dr \right)^{p/2} \left(\int_0^1 (\phi''_{\delta_1}(v))^2\,dv \right)^{p/2} \\
&\le c_p \delta_1^{3p/2} \left(\int_{y_0-\delta_1^{1/2}}^{y_0+2\delta_1^{1/2}} \delta_1^{-2}\,dv \right)^{p/2} \\
&= c_p \delta_1^{3p/2} \delta_1^{-3p/4} = c_p \delta_1^{3p/4},
\end{aligned}
\tag{4.6.23}
$$

where, in the third inequality, we use (4.5.22). Hence (4.6.20) follows from (4.6.21), (4.6.22) and (4.6.23). □

By (4.5.20), for any $p \ge 1$,

$$
\|\delta(u_A^1)\|_{L^p(\Omega)} = c_p \left(\int_0^T \int_0^1 (u_A^1(r,v))^2\,dr\,dv \right)^{1/2} \le c'_p. \tag{4.6.24}
$$

From (4.5.41), (4.6.20) and (4.6.24), using Hölder's inequality, we obtain that for all $p \ge 2$

$$
\|T_1\|_{L^p(\Omega)} \le c_p \delta_1^{-1} \delta_1^{3/4} = c_p \delta_1^{-1/4}. \tag{4.6.25}
$$

This proves the statement (a) of Proposition 4.6.2 for $i = 1$.

Next, we show that the estimate in Proposition 4.6.2(a) holds for T_2.

We first use the formula (4.5.40) to give an estimate on the $\mathscr{H}$-norm of u_A^2. By definition, since

$0 \le \psi \le 1$,

$$
\begin{aligned}
\|u_A^2\|_{\mathscr{H}}^2 &\le 2\int_{s_0}^{s_0+\delta_1} dr \int_0^1 dv\, \psi(Y_r)^2 \phi_{\delta_1}^2(v) \\
&\quad + 2\int_{s_0}^{s_0+\delta_1} dr \int_0^1 dv (\phi''_{\delta_1}(v))^2 \left(\int_{s_0}^{r} \psi(Y_a)\, da\right)^2 \\
&\le 2\delta_1 \int_{y_0-\delta_1^{1/2}}^{y_0+2\delta_1^{1/2}} dv + 2c\int_{s_0}^{s_0+\delta_1} (r-s_0)^2 dr \int_{y_0-\delta_1^{1/2}}^{y_0+2\delta_1^{1/2}} \delta_1^{-2} dv \\
&= c\delta_1^{3/2} + c\delta_1^3 \delta_1^{-3/2} \\
&= 2c\delta_1^{3/2},
\end{aligned} \tag{4.6.26}
$$

where in the second inequality we use (4.5.22).

Lemma 4.6.6. *For any $p \ge 1$, there exists a constant c_p, not depending on $(s_0, y_0) \in I \times J$, such that for all $\delta_1 > 0$,*

$$
\|\langle D\gamma_A^{2,2}, u_A^2\rangle_{\mathscr{H}}\|_{L^p(\Omega)} \le c_p \delta_1^{7/4}. \tag{4.6.27}
$$

Proof. Taking the Malliavin derivative of $\gamma_A^{2,2}$, we have

$$
\langle D\gamma_A^{2,2}, u_A^2\rangle_{\mathscr{H}} = \int_{s_0}^{s_0+\delta_1} \psi'(Y_r)\langle DY_r, u_A^2\rangle_{\mathscr{H}}\, dr.
$$

By Hölder's inequality, (4.5.18) and (4.6.26), for any $p \ge 1$,

$$
\begin{aligned}
\mathrm{E}\left[|\langle D\gamma_A^{2,2}, u_A^2\rangle_{\mathscr{H}}|^p\right] &\le \|\psi'\|_\infty^p \delta_1^{p-1} \int_{s_0}^{s_0+\delta_1} \mathrm{E}[|\langle DY_r, u_A^2\rangle_{\mathscr{H}}|^p]\, dr \\
&\le c_p R^{-p} \delta_1^{p-1} \int_{s_0}^{s_0+\delta_1} \mathrm{E}[\|DY_r\|_{\mathscr{H}}^p \|u_A^2\|_{\mathscr{H}}^p]\, dr \\
&\le c_p R^{-p} \delta_1^{p-1+3p/4} \int_{s_0}^{s_0+\delta_1} \mathrm{E}[\|DY_r\|_{\mathscr{H}}^p]\, dr.
\end{aligned}
$$

Using Lemma 4.5.2(a), this is bounded above by

$$
\begin{aligned}
& c_p R^{-p} \delta_1^{p-1+3p/4} \delta_1^{(p_0-\gamma_0)p/2} \int_{s_0}^{s_0+\delta_1} (r-s_0)^{2p}\, dr \\
&\quad = c_p R^{-p} \delta_1^{p-1+3p/4} \delta_1^{(p_0-\gamma_0)p/2} \delta_1^{2p+1} \\
&\quad \le c_p \delta_1^{(\gamma_0-p_0-4)p/2} \delta_1^{p-1+3p/4} \delta_1^{(p_0-\gamma_0)p/2} \delta_1^{2p+1} \\
&\quad = c_p \delta_1^{7p/4},
\end{aligned}
$$

where, in the inequality, we use (4.6.16). □

By (4.5.41), (4.6.24) and (4.6.27), using Hölder's inequality, we obtain that for any $p \geq 1$

$$\|T_2\|_{L^p(\Omega)} \leq c_p \delta_1^{-2} \delta_1^{7/4} = c_p \delta_1^{-1/4}. \tag{4.6.28}$$

This proves the statement (a) of Proposition 4.6.2 for $i = 2$.

We proceed to give an estimate on the moments of T_3.

Using (4.6.21), we take the Malliavin derivative of $\delta(u_A^2)$ and write

$$\begin{aligned} D_{\xi,\eta}\delta(u_A^2) &= 1_{[s_0,s_0+\delta_1]}(\xi)\psi(Y_\xi)\phi_{\delta_1}(\eta) - 1_{[s_0,s_0+\delta_1]}(\xi)\phi''_{\delta_1}(\eta)\int_{s_0}^{\xi}\psi(Y_a)da \\ &\quad + \int_{s_0}^{s_0+\delta_1}\int_0^1 \phi_{\delta_1}(v)\psi'(Y_r)D_{\xi,\eta}Y_r\, W(dr,dv) \\ &\quad - \int_{s_0}^{s_0+\delta_1}\int_0^1 W(dr,dv)\phi''_{\delta_1}(v)\int_{s_0}^{r}\psi'(Y_a)D_{\xi,\eta}Y_a\, da. \end{aligned} \tag{4.6.29}$$

It is clear that the inner product of the first two terms on the right-hand side of (4.6.29) and u_A^1 is equal to $\langle u_A^2, u_A^1\rangle_{\mathscr{H}}$. By the stochastic Fubini theorem (see [24, Chapter 1, Theorem 5.30] or [81, Theorem 2.6]), we see that the inner product of the third term on the right-hand side of (4.6.29) and u_A^1 is equal to

$$\int_{s_0}^{s_0+\delta_1}\int_0^1 \phi_{\delta_1}(v)\psi'(Y_r)\langle DY_r, u_A^1\rangle_{\mathscr{H}} W(dr,dv), \tag{4.6.30}$$

since the condition of the stochastic Fubini theorem can be verified:

$$\begin{aligned} &\mathrm{E}\left[\int_{s_0}^{s_0+\delta_1} d\xi \int_0^1 d\eta\, |u_A^1(\xi,\eta)| \int_{s_0}^{s_0+\delta_1} dr \int_0^1 dv\, \phi_{\delta_1}^2(v)(\psi'(Y_r))^2(D_{\xi,\eta}Y_r)^2\right] \\ &\quad \leq c\mathrm{E}\left[\int_{s_0}^{s_0+\delta_1} d\xi \int_0^1 d\eta \int_{s_0}^{s_0+\delta_1} (D_{\xi,\eta}Y_r)^2 dr\right] \\ &\quad \leq c\delta_1 \sup_{r\in[s_0,s_0+\delta_1]} \mathrm{E}\left[\|DY_r\|_{\mathscr{H}}^2\right] < \infty, \end{aligned}$$

where the last inequality is due to (4.5.13). Similarly, the inner product of the last term on the right-hand side of (4.6.29) and u_A^1 is equal to

$$\int_{s_0}^{s_0+\delta_1}\int_0^1 W(dr,dv)\phi''_{\delta_1}(v)\int_{s_0}^{r}\psi'(Y_a)\langle DY_a, u_A^1\rangle_{\mathscr{H}} da. \tag{4.6.31}$$

Therefore, by (4.6.29), (4.6.30) and (4.6.31), we write

$$\begin{aligned}\langle D\delta(u_A^2), u_A^1\rangle_{\mathscr{H}} &= \langle u_A^2, u_A^1\rangle_{\mathscr{H}} + \int_{s_0}^{s_0+\delta_1}\int_0^1 \psi'(Y_r)\langle DY_r, u_A^1\rangle_{\mathscr{H}}\phi_{\delta_1}(v)W(dr,dv)\\ &\quad - \int_{s_0}^{s_0+\delta_1}\int_0^1 W(dr,dv)\phi''_{\delta_1,\delta_2}(v)\int_{s_0}^{r} da\,\psi'(Y_a)\langle DY_a, u_A^1\rangle_{\mathscr{H}}\\ &:= \bar{T}_{31}+\bar{T}_{32}-\bar{T}_{33}. \end{aligned} \tag{4.6.32}$$

From (4.6.26) and (4.5.20), it is easy to see that for any $p\geq 1$,

$$\|\bar{T}_{31}\|_{L^p(\Omega)} \leq c_p\delta_1^{3/4}. \tag{4.6.33}$$

By Burkholder's inequality and using (4.5.18) and (4.5.20), we have for any $p\geq 2$,

$$\begin{aligned}\mathrm{E}[|\bar{T}_{32}|^p] &\leq c_p\mathrm{E}\left[\left(\int_{s_0}^{s_0+\delta_1}\int_0^1 \psi'(Y_r)^2\langle DY_r, u_A^1\rangle_{\mathscr{H}}^2\phi_{\delta_1}^2(v)\,dr\,dv\right)^{p/2}\right]\\ &\leq c_pR^{-p}\mathrm{E}\left[\left(\int_{s_0}^{s_0+\delta_1}\|DY_r\|_{\mathscr{H}}^2dr\int_0^1\phi_{\delta_1}^2(v)dv\right)^{p/2}\right]\\ &= c_pR^{-p}\left(\int_0^1\phi_{\delta_1}^2(v)dv\right)^{p/2}\mathrm{E}\left[\left(\int_{s_0}^{s_0+\delta_1}\|DY_r\|_{\mathscr{H}}^2dr\right)^{p/2}\right]. \end{aligned}\tag{4.6.34}$$

By Hölder's inequality and (4.5.6), we see that (4.6.34) is bounded above by

$$\begin{aligned}& c_pR^{-p}\delta_1^{p/4}\delta_1^{p/2-1}\int_{s_0}^{s_0+\delta_1}\mathrm{E}[\|DY_r\|_{\mathscr{H}}^p]dr\\ &\leq c_pR^{-p}\delta_1^{p/4}\delta_1^{p/2-1}\delta_1^{(p_0-\gamma_0)p/2}\int_{s_0}^{s_0+\delta_1}(r-s_0)^{2p}dr\\ &= c_pR^{-p}\delta_1^{(2(p_0-\gamma_0)+11)p/4}\\ &\leq c_p\delta_1^{(\gamma_0-p_0-4)p/2}\delta_1^{(2(p_0-\gamma_0)+11)p/4} = c_p\delta_1^{3p/4}, \end{aligned}\tag{4.6.35}$$

where in the last inequality we use (4.6.16).

We now give an estimate on the moments of $\bar{T}_{33}$. By Burkholder's inequality and using (4.5.18) and (4.5.20), we see that for any $p\geq 2$,

$$\begin{aligned}\mathrm{E}[|\bar{T}_{33}|^p] &\leq c_p\mathrm{E}\left[\left(\int_{s_0}^{s_0+\delta_1}\int_0^1\left(\int_{s_0}^{r}\psi'(Y_a)\langle DY_a,u_A^1\rangle_{\mathscr{H}}da\right)^2(\phi''_{\delta_1}(v))^2dr\,dv\right)^{p/2}\right]\\ &\leq c_pR^{-p}\left(\int_0^1(\phi''_{\delta_1}(v))^2dv\right)^{p/2}\mathrm{E}\left[\left(\int_{s_0}^{s_0+\delta_1}\left(\int_{s_0}^{r}\|DY_a\|_{\mathscr{H}}da\right)^2dr\right)^{p/2}\right]. \end{aligned}\tag{4.6.36}$$

Using Hölder's inequality twice and (4.5.22), (4.6.36) is bounded above by

$$
\begin{aligned}
&c_p R^{-p}\delta_1^{-3p/4}\mathrm{E}\left[\left(\int_{s_0}^{s_0+\delta_1} dr(r-s_0)\int_{s_0}^{r}\|DY_a\|_{\mathscr{H}}^2 da\right)^{p/2}\right]\\
&\quad\le c_p R^{-p}\delta_1^{-3p/4}\left(\int_{s_0}^{s_0+\delta_1} dr\int_{s_0}^{r} da\right)^{p/2-1}\int_{s_0}^{s_0+\delta_1} dr(r-s_0)^{p/2}\int_{s_0}^{r}\mathrm{E}[\|DY_a\|_{\mathscr{H}}^p]da.
\end{aligned}
\tag{4.6.37}
$$

Applying the estimate in (4.5.6), (4.6.37) is bounded above by

$$
\begin{aligned}
&c_p R^{-p}\delta_1^{-3p/4}\delta_1^{p-2}\delta_1^{(p_0-\gamma_0)p/2}\int_{s_0}^{s_0+\delta_1} dr(r-s_0)^{p/2}\int_{s_0}^{r}(a-s_0)^{2p}da\\
&\quad= c_p R^{-p}\delta_1^{(2(p_0-\gamma_0)+11)p/4}\\
&\quad\le c_p\delta_1^{(\gamma_0-p_0-4)p/2}\delta_1^{(2(p_0-\gamma_0)+11)p/4} = c_p\delta_1^{3p/4},
\end{aligned}
\tag{4.6.38}
$$

where in the inequality we use (4.6.16).

Therefore, by (4.6.33),(4.6.35), (4.6.38) and (4.5.41), we have obtained that for any $p\ge 2$,

$$
\|T_3\|_{L^p(\Omega)} \le c_p\delta_1^{-1/4}. \tag{4.6.39}
$$

This proves the statement (a) of Proposition 4.6.2 for $i=3$.

Therefore, we have finished the proof of Proposition 4.6.2(a).

4.6.2 Proof of Proposition 4.6.2(b)

We are going to show that the three terms T_4, T_5 and T_6 are equal to zero. First, we apply Lemma 4.3.6 to see that for any $t,s\in[s_0,s_0+\delta_1]$,

$$
\begin{aligned}
\langle D(u(t,y_0)-u(s,y_0)),u_A^1\rangle_{\mathscr{H}} &= \int_0^T\int_0^1 (1_{\{r<t\}}G_\alpha(t-r,y_0,v)-1_{\{r<s\}}G_\alpha(s-r,y_0,v))\\
&\qquad\times\left(\frac{\partial}{\partial r}-\frac{\partial^2}{\partial v^2}\right)(f_0(r)g_0(v))drdv\\
&= f_0(t)g_0(x)-f_0(s)g_0(y_0) = 1-1=0.
\end{aligned}
\tag{4.6.40}
$$

by the definition of the functions f_0 and g_0. Furthermore, by (4.5.8) and (4.6.40), we know that for $r\in[s_0,s_0+\delta_1]$,

$$
\begin{aligned}
\langle DY_r,u_A^1\rangle_{\mathscr{H}} &= 2p_0\int_{[s_0,r]^2} dsdt\,\frac{(u(t,y_0)-u(s,y_0))^{2p_0-1}}{|t-s|^{\gamma_0/2}}\langle D(u(t,y_0)-u(s,y_0)),u_A^1\rangle_{\mathscr{H}}\\
&= 0.
\end{aligned}
\tag{4.6.41}
$$

Hence, by (4.5.35),

$$\langle D\gamma_A^{2,2}, u_A^1\rangle_{\mathscr{H}} = \int_{s_0}^{y_0+\delta_1} \psi'(Y_r)\langle DY_r, u_A^1\rangle_{\mathscr{H}}\, dr = 0, \tag{4.6.42}$$

which implies that $T_4 = T_5 = 0$.

We proceed to prove that T_6 vanishes. Similar to (4.6.40), for any $t, s \in [s_0, s_0+\delta_1]$,

$$\begin{aligned}
&\langle D(u(t,y_0) - u(s,y_0)), u_A^2\rangle_{\mathscr{H}} \\
&= \int_{s_0}^{s_0+\delta_1} dr \int_0^1 dv (1_{\{r<t\}} G(t-r, y_0, v) - 1_{\{r<s\}} G(s-r, y_0, v)) \left(\frac{\partial}{\partial r} - \frac{\partial^2}{\partial v^2}\right) H(r,v) \\
&= H(t,y_0) - H(s,y_0).
\end{aligned} \tag{4.6.43}$$

Hence, by (4.5.1), for $r \in [s_0, s_0+\delta_1]$,

$$\begin{aligned}
\langle DY_r, u_A^2\rangle_{\mathscr{H}} &= 2p_0 \int_{[s_0,r]^2} ds dt \frac{(u(t,y_0)-u(s,y_0))^{2p_0-1}}{|t-s|^{\gamma_0/2}} \langle D(u(t,y_0)-u(s,y_0)), u_A^2\rangle_{\mathscr{H}} \\
&= 2p_0 \int_{[s_0,r]^2} ds dt \frac{(u(t,y_0)-u(s,y_0))^{2p_0-1}}{|t-s|^{\gamma_0/2}} (H(t,y_0) - H(s,y_0)) \\
&= 2p_0 \int_{[s_0,r]^2} ds dt \frac{(u(t,y_0)-u(s,y_0))^{2p_0-1}}{|t-s|^{\gamma_0/2}} \int_s^t \psi(Y_a) da,
\end{aligned} \tag{4.6.44}$$

where in the last equality we use the definition of the function $(t,x) \mapsto H(t,x)$. And moreover,

$$\begin{aligned}
&\langle D\langle DY_r, u_A^2\rangle_{\mathscr{H}}, u_A^1\rangle_{\mathscr{H}} \\
&= 2p_0(2p_0-1) \int_{[s_0,r]^2} ds dt \frac{(u(t,y_0)-u(s,y_0))^{2p_0-2}}{|t-s|^{\gamma_0/2}} \\
&\qquad\qquad \times \langle D(u(t,y_0)-u(s,y_0)), u_A^1\rangle_{\mathscr{H}} \int_s^t \psi(Y_a) da \\
&\quad + 2p_0 \int_{[s_0,r]^2} ds dt \frac{(u(t,y_0)-u(s,y_0))^{2p_0-1}}{|t-s|^{\gamma_0/2}} \int_s^t \psi'(Y_a)\langle DY_a, u_A^1\rangle_{\mathscr{H}} da \\
&= 0 + 0 = 0,
\end{aligned} \tag{4.6.45}$$

where, on the right-hand side of the equality, the first term vanishes due to (4.6.40) and the second term vanishes because of (4.6.41). Therefore, by definition of $\gamma_A^{2,2}$,

$$\begin{aligned}
\langle D\langle D\gamma_A^{2,2}, u_A^2\rangle_{\mathscr{H}}, u_A^1\rangle_{\mathscr{H}} &= \left\langle D\int_{s_0}^{s_0+\delta_1} \psi'(Y_r)\langle DY_r, u_A^2\rangle_{\mathscr{H}} dr, u_A^1\right\rangle_{\mathscr{H}} \\
&= \int_{s_0}^{s_0+\delta_1} \psi''(Y_r)\langle DY_r, u_A^1\rangle_{\mathscr{H}} \langle DY_r, u_A^2\rangle_{\mathscr{H}} dr \\
&\quad + \int_{s_0}^{s_0+\delta_1} \psi'(Y_r)\langle D\langle DY_r, u_A^2\rangle_{\mathscr{H}}, u_A^1\rangle_{\mathscr{H}} dr \\
&= 0,
\end{aligned} \tag{4.6.46}$$

which implies $T_6 = 0$.

This proves the statement (b) of Proposition 4.6.2.

4.6.3 Estimates for the tail probabilities

Lemma 4.6.7. *There exists a finite positive constant c, not depending on $(s_0, y_0) \in I \times J$, such that for all $z_1 \in \mathbb{R}$,*

$$P\{|F_1| > |z_1|\} \le c\,(|z_1|^{-1} \wedge 1)e^{-z_1^2/c}, \tag{4.6.47}$$

and for all $\delta_1 > 0$ and $z_2 > 0$,

$$P\{F_2 > z_2\} \le c\,\exp\left(-\frac{z_2^2}{c\,\delta_1^{1/2}}\right). \tag{4.6.48}$$

Proof. We first bound $\mathrm{P}\{|F_1| > |z_1|\}$. Since the variance of $u(s_0, y_0)$ is bounded above and below by positive constants uniformly over $(s_0, y_0) \in I \times J$ (see [25, (4.5)]), there are constants c_1, c_2, c_3, c_4 independent of $(s_0, y_0) \in I \times J$ such that for all $z_1 \in \mathbb{R}$

$$\mathrm{P}\{|F_1| > |z_1|\} \le c_1 \int_{|z_1|}^{+\infty} e^{-y^2/c_2}\,dy \le c_3(|z_1|^{-1} \wedge 1)e^{-z_1^2/c_4}, \tag{4.6.49}$$

where the last inequality holds because for $|z_1| \ge 1$ we apply the inequality in [61, Lemma 12.9], and for $|z_1| < 1$ we use the fact that $c_1 \int_{|z_1|}^{\infty} e^{-y^2/c_2}\,dy \le c_1 \int_0^{\infty} e^{-y^2/c_2}\,dy = c' \le c_3 e^{-1/c_4} \le c_3 e^{-z_1^2/c_4}$. This proves (4.6.47).

We denote

$$\sigma^2 := \sup_{t \in [s_0, s_0+\delta_1]} \mathrm{E}[\bar{u}(t, y_0)^2].$$

From (4.2.3), we have $\sigma^2 \le C\delta_1^{1/2}$. On the other hand, by [25, (4.50)], we have

$$\begin{aligned}
\mathrm{E}[F_2] &\le \mathrm{E}\left[\sup_{t \in [s_0, s_0+\delta_1]} |u(t, y_0) - u(s_0, y_0)|\right] \\
&\le \mathrm{E}\left[\sup_{[\Delta((t,x);(s_0,y_0))]^{1/2} \le \delta_1^{1/4}} |u(t, x) - u(s_0, y_0)|\right] \\
&\le c\,\delta_1^{1/4}.
\end{aligned} \tag{4.6.50}$$

Applying Borell's inequality (see [1, (2.6)]), for all $z_2 > c\delta_1^{1/4}$ (here c is the constant in (4.6.50)),

$$\begin{aligned}
\mathrm{P}\{F_2 > z_2\} &\leq 2\exp\left(-(z_2 - \mathrm{E}[F_2])^2/(2\sigma^2)\right) \leq 2\exp\left(-(z_2 - \mathrm{E}[F_2])^2/(2C\delta_1^{1/2})\right) \\
&\leq 2\exp\left(-(2z_2^2/3 - 2\mathrm{E}[F_2]^2)/(2C\delta_1^{1/2})\right) \\
&= 2\exp\left(-z_2^2/(3C\delta_1^{1/2})\right)\exp\left(\mathrm{E}[F_2]^2/(C\delta_1^{1/2})\right) \\
&\leq 2e^{c^2/C}\exp\left(-z_2^2/(3C\delta_1^{1/2})\right) \\
&= \bar{c}\exp\left(-z_2^2/(3C\delta_1^{1/2})\right).
\end{aligned} \tag{4.6.51}$$

Since for $0 \leq z_2 \leq c\delta_1^{1/4}$,

$$\exp\left(-z_2^2/(3C\delta_1^{1/2})\right) \geq e^{-\frac{c^2}{3C}},$$

we can find a constant $\tilde{c}$ such that for all $z_2 > 0$,

$$\mathrm{P}\{F_2 > z_2\} \leq \tilde{c}\exp\left(-z_2^2/(3C\delta_1^{1/2})\right). \tag{4.6.52}$$

This proves (4.6.48). □

Finally, we prove Theorem 4.2.2.

Proof of Theorem 4.2.2. This follows from (4.6.3), (4.6.47), (4.6.48) and (4.6.14). □

4.7 Gaussian-type upper bound on the density of M_0

In this section, we assume $J \subset]0,1[$ and δ_1, δ_2 satisfy the conditions in (4.2.15).

From the formula for the probability density function of M_0 in (4.5.52), by the Cauchy-Schwartz inequality,

$$p_0(z) \leq \mathrm{P}\{M_0 > z\}^{1/2}\|\delta(u_{\bar{A}}/\gamma_{\bar{A}})\|_{L^2(\Omega)}. \tag{4.7.1}$$

Proposition 4.7.1. *(a) There exists a finite positive constant c, not depending on $y_0 \in J$, such that for all small $\delta_1, \delta_2 > 0$ and for all $z \geq (\delta_1^{1/2} + \delta_2)^{1/2}$,*

$$\|\delta(u_{\bar{A}}/\gamma_{\bar{A}})\|_{L^2(\Omega)} \leq c(\delta_1^{1/2} + \delta_2)^{-1/2}. \tag{4.7.2}$$

(b) There exists a finite positive constant c, not depending on $y_0 \in J$, such that for all $\delta_1, \delta_2 > 0$ and for all $z > 0$,

$$P\{M_0 > z\} \leq c\exp\left(-\frac{z^2}{c(\delta_1^{1/2} + \delta_2)}\right). \tag{4.7.3}$$

Proof of Theorem 4.2.5. This is an immediate consequence of (4.7.1) and Proposition 4.7.1. □

The proof of Proposition 4.7.1 is given in the following two subsections.

4.7.1 Proof of Proposition 4.7.1(a)

Throughout this section, we assume that

$$z \geq (\delta_1^{1/2} + \delta_2)^{1/2} = \delta^{1/2}. \tag{4.7.4}$$

Recalling the definition of $\bar{R}$ in (4.3.27), under the assumption (4.7.4), we see from (4.5.26) that

$$\begin{aligned}\bar{R}^{-1} &= c^{-1}\bar{a}^{-2p_0}\delta^{\gamma_0-4} = c' z^{-2p_0}\delta^{\gamma_0-4}\\ &\leq c\delta^{\gamma_0-p_0-4}.\end{aligned} \tag{4.7.5}$$

In order to prove Proposition 4.7.1(a), we need the following several lemmas. Recall the definition of $u_{\bar{A}}$ in (4.5.33).

Lemma 4.7.2. *For any $p \geq 2$, there exists a constant c_p, not depending on $y_0 \in J$, such that for all $\delta_1, \delta_2 > 0$,*

$$\|\delta(u_{\bar{A}})\|_{L^p(\Omega)} \leq c_p \delta^{3/2}. \tag{4.7.6}$$

Proof. The proof is similar to that of Lemma 4.6.5. Since $u_{\bar{A}}$ is adapted, by Proposition 4.6.4, we have

$$\delta(u_{\bar{A}}) = \int_0^{\Delta_\bullet}\int_0^1 \bar{\phi}_\delta(v)\bar{\psi}(\bar{Y}_r)W(dr,dv) - \int_0^{\Delta_\bullet}\int_0^1 W(dr,dv)\bar{\phi}''_\delta(v)\int_0^r \bar{\psi}(\bar{Y}_a)da. \tag{4.7.7}$$

For the first term on the right-hand side of (4.7.7), by Burkholder's inequality, for any $p \geq 2$, since $0 \leq \bar{\psi} \leq 1$,

$$\begin{aligned}\left\|\int_0^{\Delta_\bullet}\int_0^1 \bar{\phi}_\delta(v)\bar{\psi}(\bar{Y}_r)W(dr,dv)\right\|^p_{L^p(\Omega)} &\leq c_p \mathrm{E}\left[\left(\int_0^{\Delta_\bullet}\int_0^1 \bar{\phi}^2_\delta(v)\bar{\psi}^2(\bar{Y}_r)dr\,dv\right)^{p/2}\right]\\ &\leq c_p\Delta_\bullet^{p/2}\left(\int_0^1 \bar{\phi}^2_\delta(v)dv\right)^{p/2}\\ &\leq c_p\Delta_\bullet^{p/2}\delta^{p/2} = c_p\delta^{3p/2}.\end{aligned} \tag{4.7.8}$$

For the second term on the right-hand side of (4.7.7), similarly, by Burkholder's inequality, for any $p \geq 2$, since $0 \leq \bar{\psi} \leq 1$,

$$
\begin{aligned}
&\left\| \int_0^{\Delta_\bullet} \int_0^1 W(dr, dv)\bar{\phi}''_\delta(v) \int_0^r \bar{\psi}(\bar{Y}_a)da \right\|^p_{L^p(\Omega)} \\
&\quad \leq c_p \mathrm{E}\left[\left(\int_0^{\Delta_\bullet} dr \int_0^1 dv\, (\bar{\phi}''_\delta(v))^2 \left(\int_0^r \bar{\psi}(\bar{Y}_a)da \right)^2 \right)^{p/2} \right] \\
&\quad \leq c_P \left(\int_0^{\Delta_\bullet} r^2 dr \right)^{p/2} \left(\int_0^1 (\bar{\phi}''_\delta(v))^2 dv \right)^{p/2} \\
&\quad \leq c_p \Delta_\bullet^{3p/2} \left(\int_{y_0-\delta}^{y_0+2\delta} \delta^{-4} dv \right)^{p/2} \\
&\quad = c_p \Delta_\bullet^{3p/2} \delta^{-3p/2} = c_p \delta^{3p/2},
\end{aligned} \tag{4.7.9}
$$

where, in the third inequality, we use (4.5.31). Hence, (4.7.7), (4.7.8) and (4.7.9) prove the lemma. □

Lemma 4.7.3. *There exists a constant c, not depending on $y_0 \in J$, such that for all $\delta_1, \delta_2 > 0$,*

$$
\|u_{\bar{A}}\|_{\mathscr{H}} \leq c\delta^{3/2}. \tag{4.7.10}
$$

Proof. The proof is similar to that of (4.6.26). By the definition of $u_{\bar{A}}$,

$$
\begin{aligned}
\|u_{\bar{A}}\|^2_{\mathscr{H}} &\leq 2\int_0^{\Delta_\bullet} dr \int_0^1 dv\, \bar{\psi}(\bar{Y}_r)^2 \bar{\phi}^2_\delta(v) + 2\int_0^{\Delta_\bullet} dr \int_0^1 dv\, (\bar{\phi}''_\delta(v))^2 \left(\int_0^r \bar{\psi}(\bar{Y}_a)da \right)^2 \\
&\leq 2\Delta_\bullet \int_{y_0-\delta}^{y_0+2\delta} dv + 2c \int_0^{\Delta_\bullet} r^2 dr \int_{y_0-\delta}^{y_0+2\delta} \delta^{-4} dv \\
&= c\delta^3 + c\Delta_\bullet^3 \delta^{-3} \\
&= 2c\delta^3,
\end{aligned} \tag{4.7.11}
$$

where, in the second inequality, we use (4.5.31). □

Lemma 4.7.4. *For any $p \geq 2$, there exists a constant c_p, not depending on $y_0 \in J$, such that for all $\delta_1, \delta_2 > 0$,*

$$
\|\langle D\gamma_{\bar{A}}, u_{\bar{A}} \rangle_{\mathscr{H}}\|_{L^p(\Omega)} \leq c\delta^{7/2}. \tag{4.7.12}
$$

Proof. The proof is similar to that of Lemma 4.6.6. Taking the Malliavin derivative of $\gamma_{\bar{A}}$, we have

$$
\langle D\gamma_{\bar{A}}, u_{\bar{A}} \rangle_{\mathscr{H}} = \int_0^{\Delta_\bullet} \bar{\psi}'(\bar{Y}_r) \langle D\bar{Y}_r, u_{\bar{A}} \rangle_{\mathscr{H}} dr.
$$

By Hölder's inequality, (4.5.28) and (4.7.11), for any $p \geq 1$,

$$\begin{aligned}
\mathrm{E}\left[|\langle D\gamma_{\bar{A}}, u_{\bar{A}}\rangle_{\mathscr{H}}|^p\right] &\leq \|\bar{\psi}'\|_\infty^p \Delta_\bullet^{p-1} \int_0^{\Delta_\bullet} \mathrm{E}[|\langle D\bar{Y}_r, u_{\bar{A}}\rangle_{\mathscr{H}}|^p] dr \\
&\leq c_p \bar{R}^{-p} \Delta_\bullet^{p-1} \int_0^{\Delta_\bullet} \mathrm{E}[\|D\bar{Y}_r\|_{\mathscr{H}}^p \|u_{\bar{A}}\|_{\mathscr{H}}^p] dr \\
&\leq c_p \bar{R}^{-p} \Delta_\bullet^{p-1+3p/4} \int_0^{\Delta_\bullet} \mathrm{E}[\|D\bar{Y}_r\|_{\mathscr{H}}^p] dr.
\end{aligned}$$

Applying (4.5.7), this is bounded above by

$$\begin{aligned}
&c_p R^{-p} \Delta_\bullet^{p-1+3p/4} \delta^{(p_0-\gamma_0)q} \int_0^{\Delta_\bullet} r^{2p} dr \\
&\quad = c_p R^{-p} \Delta_\bullet^{p-1+3p/4} \delta^{(p_0-\gamma_0)p} \Delta_\bullet^{2p+1} \\
&\quad \leq c_p \delta^{(\gamma_0-p_0-4)p} \Delta_\bullet^{p-1+3p/4} \delta^{(p_0-\gamma_0)p} \Delta_\bullet^{2p+1} \\
&\quad = c_p \delta^{7p/2},
\end{aligned}$$

where, in the inequality, we use (4.7.5). □

Proof of Proposition 4.7.1(a). Using the property of Skorohod integral δ (see [64, (1.48)]),

$$\delta(u_{\bar{A}}/\gamma_{\bar{A}}) = \frac{\delta(u_{\bar{A}})}{\gamma_{\bar{A}}} + \frac{\langle D\gamma_{\bar{A}}, u_{\bar{A}}\rangle_{\mathscr{H}}}{\gamma_{\bar{A}}^2} := I_1 + I_2. \tag{4.7.13}$$

By Lemmas 4.7.2 and 4.5.4(b),

$$\|I_1\|_{L^2(\Omega)} \leq c\delta^{3/2}\delta^{-2} = c\delta^{-1/2}. \tag{4.7.14}$$

By Lemmas 4.7.4 and 4.5.4(b),

$$\|I_2\|_{L^2(\Omega)} \leq c\delta^{7/2}\delta^{-4} = c\delta^{-1/2}. \tag{4.7.15}$$

Therefore, (4.7.13), (4.7.14) and (4.7.15) establish (4.7.2). □

4.7.2 Proof of Proposition 4.7.1 (b)

Proof of Proposition 4.7.1(b). The proof is similar to that of (4.6.48). We denote

$$\sigma_0^2 := \sup_{(t,x)\in[0,\delta_1]\times[y_0,y_0+\delta_2]} \mathrm{E}[u(t,x)^2].$$

From (4.2.3), we have $\sigma_0^2 \le C(\delta_1^{1/2}+\delta_2)$. On the other hand, by [25, (4.50)], we have

$$\begin{aligned}
\mathrm{E}[M_0] &\le \mathrm{E}\left[\sup_{(t,x)\in[0,\delta_1]\times[y_0,y_0+\delta_2]} |u(t,x)|\right] \\
&\le \mathrm{E}\left[\sup_{[\Delta((t,x);(0,y_0))]^{1/2}\le(\delta_1^{1/2}+\delta_2)^{1/2}} |u(t,x)|\right] \\
&\le c\,(\delta_1^{1/2}+\delta_2)^{1/2}. \qquad (4.7.16)
\end{aligned}$$

Applying Borell's inequality (see [1, (2.6)]), for all $z > c\,(\delta_1^{1/2}+\delta_2)^{1/2}$ (here c is the constant in (4.7.16)),

$$\begin{aligned}
\mathrm{P}\{M_0 > z\} &\le 2\exp\left(-(z-\mathrm{E}[M_0])^2/(2\sigma_0^2)\right) \le 2\exp\left(-(z-\mathrm{E}[M_0])^2/(2C(\delta_1^{1/2}+\delta_2))\right) \\
&\le 2\exp\left(-(2z^2/3-2\mathrm{E}[M_0]^2)/(2C(\delta_1^{1/2}+\delta_2))\right) \\
&= 2\exp\left(-z^2/(3C(\delta_1^{1/2}+\delta_2))\right)\exp\left(\mathrm{E}[M_0]^2/(C(\delta_1^{1/2}+\delta_2))\right) \\
&\le 2e^{c^2/C}\exp\left(-z^2/(3C(\delta_1^{1/2}+\delta_2))\right) \\
&= \tilde{c}\exp\left(-z^2/(3C(\delta_1^{1/2}+\delta_2))\right). \qquad (4.7.17)
\end{aligned}$$

Since for $0 \le z \le c\,(\delta_1^{1/2}+\delta_2)^{1/2}$,

$$\exp\left(-z^2/(3C(\delta_1^{1/2}+\delta_2))\right) \ge e^{-\frac{c^2}{3C}},$$

we can find a constant $\tilde{c}$ such that for all $z > 0$,

$$\mathrm{P}\{F_2 > z\} \le \tilde{c}\exp\left(-z^2/(3C(\delta_1^{1/2}+\delta_2))\right). \qquad (4.7.18)$$

This proves (4.7.3). □

Remark 4.7.5. *The results of Theorem 4.2.1(a), (b) and Theorems 4.2.2, 4.2.5 also hold for the solution without boundary ($x \in \mathbb{R}$). This is because in the definition of the random variables H and $\bar{H}$ in (4.5.23) and* (5.3.17)*, the functions ϕ_{δ_1} and $\bar{\phi}_\delta$ are compactly supported and C^∞ and the boundary conditions do not affect the smoothness of $u_A^i, \gamma_A^{i,j}, i, j \in \{1,2\}$ and $u_{\bar{A}}, \gamma_{\bar{A}}$ in Lemma 4.5.3. And the equalities (4.5.45), (4.5.46), (4.5.47), (4.5.48) and* (4.5.50) *in the proof of Lemma 4.5.5 still hold with* $[0,1]$ *replaced by* $\mathbb{R}$*. Furthermore, the formulas and estimates in (4.5.51),* (4.5.52)*, (4.5.63), (4.6.3), (4.6.4), (4.6.49) and* (4.7.13) *are generic, no matter with or without boundary conditions. Moreover, the boundary conditions do not change the estimates in (4.5.41), (4.5.6), (4.6.26), (4.6.34), (4.6.36),* (4.7.6), (4.7.10) *and* (4.7.12)*. Furthermore, the equalities (4.6.40), (4.6.41), (4.6.44), (4.6.45) and (4.6.46) remain the same. In the end, the estimates (4.6.50) and* (4.7.16) *still hold for the solution of the equation without boundary (we can redo the proof of [25, (4.50)] line by line). This will be done in more detail in Lemma 5.4.6.*

4.8 Proof of Theorem 4.2.1(c)

The aim of this section is to prove the smoothness of the density of the random variable M defined in (4.2.8), in the case of Neumann boundary conditions. We will apply the criterion of Theorem 1.5.5 to establish this result. In other words, we will construct random variables satisfying the locally nondegeneracy conditions in Theorem 1.5.5. The approach is similar to the case of Brownian sheet (see [39]), and is slightly different from the method in Section 4.5.

Choose and fix γ_1, γ_2 and an integer p such that

$$\frac{1}{2p} < \gamma_1 < \theta_1/2 - \frac{1}{2p} \quad \text{and} \quad \frac{1}{2p} < \gamma_2 < \theta_2/2 - \frac{1}{2p}. \tag{4.8.1}$$

Recall the definition of the random variables $\{\hat{u}(t,x) : (t,x) \in [0,T] \times [0,1]\}$ in (4.2.7). By (4.3.22) we know that a.s. $t \mapsto \hat{u}(t,\cdot)$ is continuous in $E_{p,\gamma_2}[0,1]$ and $x \mapsto \hat{u}(\cdot,x)$ is continuous in $E_{p,\gamma_1}[0,T]$.

We define two families of random variables:

$$\begin{aligned} Y^1(\sigma) &:= \int_{[0,\sigma]^2} \frac{\|\hat{u}(s,*) - \hat{u}(s',*)\|_{p,\gamma_2}^{2p}}{|s-s'|^{1+2p\gamma_1}} ds ds' \\ &= \int_{[0,\sigma]^2} ds ds' \int_{[0,1]^2} dx dx' \, \frac{(u(s,x) - u(s,x') - u(s',x) + u(s',x'))^{2p}}{|s-s'|^{1+2p\gamma_1}|x-x'|^{1+2p\gamma_2}} \end{aligned} \tag{4.8.2}$$

and

$$\begin{aligned} Y^2(\tau) &:= \int_{[0,\tau]^2} \frac{\|\hat{u}(\cdot,x) - \hat{u}(\cdot,x')\|_{p,\gamma_1}^{2p}}{|x-x'|^{1+2p\gamma_2}} dx dx' \\ &= \int_{[0,\tau]^2} dx dx' \int_{[0,T]^2} ds ds' \, \frac{(u(s,x) - u(s,x') - u(s',x) + u(s',x'))^{2p}}{|s-s'|^{1+2p\gamma_1}|x-x'|^{1+2p\gamma_2}}, \end{aligned} \tag{4.8.3}$$

where $(\sigma,\tau) \in [0,T] \times [0,1]$. Set

$$Y_{\sigma,\tau} = Y^1(\sigma) + Y^2(\tau), \tag{4.8.4}$$

for $(\sigma,\tau) \in [0,T] \times [0,1]$. The following lemma is analogous to Lemma 4.3.1 and Lemma 4.3.5.

Lemma 4.8.1. *For any $a > 0$, there exists a constant R, depending on a, p, γ_1 and γ_2, such that for all $(\sigma,\tau) \in [0,T] \times [0,1]$,*

$$Y_{\sigma,\tau} \le R \quad \Rightarrow \sup_{(t,x) \in ([0,\sigma] \times [0,1]) \cup ([0,T] \times [0,\tau])} |\hat{u}(t,x)| \le a. \tag{4.8.5}$$

Proof. In order to establish this property, we first apply the Garsia, Rodemich and Rumsey lemma (see Lemma A.6.2) to the $E_{p,\gamma_2}[0,1]$-valued function $s \mapsto \hat{u}(s,*)$ with $\Psi(x) = x^{2p}$, $p(x) =$

$x^{(1+2p\gamma_1)/(2p)}$, $d=1$. From this lemma, and assuming $Y^1(\sigma)\le R$, we deduce, as in (4.3.31), that

$$\|\hat{u}(s,*)-\hat{u}(s',*)\|_{p,\gamma_2}^{2p}\le c_{p,\gamma_1}R|s-s'|^{2p\gamma_1-1}$$

for all $s,s'\in[0,\sigma]$. Hence, with $s'=0$, we get

$$\|\hat{u}(s,*)\|_{p,\gamma_2}^{2p}\le c_{p,\gamma_1}R$$

for all $s\in[0,\sigma]$. Applying the same lemma to the real-valued function $x\mapsto\hat{u}(s,x)$ (s is now fixed) with $\Psi(x)=x^{2p}$, $p(x)=x^{(1+2p\gamma_2)/(2p)}$, we obtain

$$|\hat{u}(s,x)-\hat{u}(s,x')|^{2p}\le c_{p,\gamma_2}c_{p,\gamma_1}R|x-x'|^{2p\gamma_2-1}$$

for all $x,x'\in[0,1]$. Hence, letting $x'=0$, we obtain

$$\sup_{0\le t\le\sigma,0\le x\le 1}|\hat{u}(t,x)|\le c_{p,\gamma_1}^{\frac{1}{2p}}c_{p,\gamma_2}^{\frac{1}{2p}}R^{\frac{1}{2p}}.$$

Similarly, focusing on $Y^2(\tau)$, we can prove that

$$\sup_{0\le t\le T,0\le x\le\tau}|\hat{u}(t,x)|\le c_{p,\gamma_1}^{\frac{1}{2p}}c_{p,\gamma_2}^{\frac{1}{2p}}R^{\frac{1}{2p}},$$

and it suffices to choose R in such a way that $c_{p,\gamma_1}^{\frac{1}{2p}}c_{p,\gamma_2}^{\frac{1}{2p}}R^{\frac{1}{2p}}<a$. □

We next prove the smoothness of the two families of random variables defined in (4.8.2) and (4.8.3).

Lemma 4.8.2. *For any $(\sigma,\tau)\in[0,T]\times[0,1]$, $Y^1(\sigma)$ and $Y^2(\tau)$ belong to $\mathbb{D}^\infty$. For any integer l,*

$$D^lY^1(1)=\int_{[0,1]^4}\frac{2p(2p-1)\cdots(2p-l+1)u(1_{]s,t]\times]y,x]})^{2p-l}(Du(t,x;s,y))^{\otimes l}}{|t-s|^{1+2p\gamma_1}|x-y|^{1+2p\gamma_2}}dsdtdydx. \tag{4.8.6}$$

Proof. The proof follows the same lines as that of Lemma 4.5.1(b). □

As a consequence of Lemma 4.8.2, for any $(r,v)\in[0,T]\times[0,1]$,

$$DY^1(r)=2p\int_{[0,r]^2}dtds\int_{[0,1]^2}dxdy\frac{u(1_{]s,t]\times]y,x]})^{2p-1}Du(t,x;s,y)}{|t-s|^{1+2p\gamma_1}|x-y|^{1+2p\gamma_2}}, \tag{4.8.7}$$

$$DY^2(v)=2p\int_{[0,v]^2}dxdy\int_{[0,T]^2}dsdt\frac{u(1_{]s,t]\times]y,x]})^{2p-1}Du(t,x;s,y)}{|t-s|^{1+2p\gamma_1}|x-y|^{1+2p\gamma_2}}, \tag{4.8.8}$$

and for any integer i,

$$D^i Y_{r,v} = c_i \int_{[0,r]^2} dtds \int_{[0,1]^2} dxdy \frac{u(1_{]s,t]\times]y,x]})^{2p-i}(Du(t,x;s,y))^{\otimes i}}{|t-s|^{1+2p\gamma_1}|x-y|^{1+2p\gamma_2}}$$
$$+ c_i \int_{[0,v]^2} dxdy \int_{[0,T]^2} dsdt \frac{u(1_{]s,t]\times]y,x]})^{2p-i}(Du(t,x;s,y))^{\otimes i}}{|t-s|^{1+2p\gamma_1}|x-y|^{1+2p\gamma_2}}, \tag{4.8.9}$$

Hence for any $(r,v) \in [0,T]\times[0,1]$,

$$\|D^i Y_{r,v}\|_{\mathscr{H}^{\otimes i}} \leq 2c_i \int_{[0,T]^2} dsdt \int_{[0,1]^2} dxdy \frac{|u(1_{]s,t]\times]y,x]})|^{2p-i}\|(Du(t,x;s,y))^{\otimes i}\|_{\mathscr{H}^{\otimes i}}}{|t-s|^{1+2p\gamma_1}|x-y|^{1+2p\gamma_2}}, \tag{4.8.10}$$

which implies that for any $q \geq 1$

$$\begin{aligned}
&\sup_{(r,v)\in[0,T]\times[0,1]} \mathrm{E}\left[\|D^i Y_{r,v}\|^q_{\mathscr{H}^{\otimes i}}\right] \\
&\leq c_q \int_{[0,T]^2} dsdt \int_{[0,1]^2} dxdy \frac{\mathrm{E}\left[|u(1_{]s,t]\times]y,x]})|^{q(2p-i)}\right]\|(Du(t,x;s,y))^{\otimes i}\|^q_{\mathscr{H}^{\otimes i}}}{|t-s|^{q(1+2p\gamma_1)}|x-y|^{q(1+2p\gamma_2)}} \\
&\leq c_q \int_{[0,T]^2} dsdt \int_{[0,1]^2} dxdy \frac{\|Du(t,x;s,y)\|^{q(2p-i)+qi}_{\mathscr{H}}}{|t-s|^{q(1+2p\gamma_1)}|x-y|^{q(1+2p\gamma_2)}} \\
&\leq c_q \int_{[0,T]^2} dsdt \int_{[0,1]^2} dxdy \frac{|t-s|^{pq\theta_1}|x-y|^{pq\theta_2}}{|t-s|^{q(1+2p\gamma_1)}|x-y|^{q(1+2p\gamma_2)}} \leq c_q,
\end{aligned} \tag{4.8.11}$$

where, in the second inequality, we use again the fact that a centered Gaussian random variable $X \sim N(0,\sigma^2)$ has the property $\mathrm{E}[|X|^k] = c_k\sigma^k$.

For any $a > 0$ and set $A =]a,\infty[$. Let $\psi : \mathbb{R}^+ \to [0,1]$ be the infinitely differentiable function defined in (4.5.17) where R is the constant appearing in Lemma 4.8.1 determined by a, p, γ_1, γ_2.

We define the $\mathscr{H}$-valued random variable λ_A evaluated at (r,v) by

$$\begin{aligned}
\lambda_A(r,v) &:= \left(\frac{\partial}{\partial r} - \frac{\partial^2}{\partial v^2}\right)\int_0^r \int_0^v b(1-b)\psi(Y_{a,b})dbda \\
&= \int_0^v b(1-b)\psi(Y_{r,b})db - (1-2v)\int_0^r \psi(Y_{a,v})da \\
&\quad - v(1-v)\frac{dY^2(v)}{dv}\int_0^r \psi'(Y_{a,v})da \\
&:= \lambda^1_A(r,v) - \lambda^2_A(r,v) - \lambda^3_A(r,v).
\end{aligned} \tag{4.8.12}$$

Lemma 4.8.3. *λ_A belongs to $\mathbb{D}^\infty(\mathscr{H})$.*

Proof. The proof is similar to that of Lemma 4.5.3. We start by proving λ^1_A is Malliavin differentiable. For each integer $k \geq 1$, we define the Riemann sum of λ_A^1 by

$$X_k^1(r,v) := \sum_{i,j=1}^{k} \int_0^{(i-1)/k} b(1-b)\psi(Y_{(j-1)T/k,b})db 1_{[\frac{(j-1)T}{k},\frac{jT}{k}]}(r) 1_{[\frac{i-1}{k},\frac{i}{k}]}(v).$$

As the proof of $u_A^2 \in \mathbb{D}^\infty(\mathscr{H})$ in Lemma 4.5.3, we can show that X_k^1 converges to λ_A^1 in $L^q(\Omega,\mathscr{H})$ as $k \to \infty$ for any $q \geq 1$. Moreover, by (4.8.11) and Lemma A.6.3, DX_k^1 converges to the Bochner integral $\int_0^* b(1-b)D\psi(Y_{\cdot,b})db$ in $L^q(\Omega,\mathscr{H}^{\otimes 2})$ as $k \to \infty$ for any $q \geq 1$. Since D is closable, we have

$$D\lambda_A^1 = \int_0^* b(1-b)D\psi(Y_{\cdot,b})db.$$

In order to prove $\lambda_A^1 \in \mathbb{D}^\infty(\mathscr{H})$ we can repeat this procedure and it remains to prove for any $q, j \geq 1$,

$$\sup_{k\geq 1} \mathrm{E}\left[\int_0^T \int_0^1 ||D^j X_k^1(r,v)||_{\mathscr{H}^{\otimes j}}^q dv dr\right] < \infty, \tag{4.8.13}$$

which follows from

$$\sup_{(a,b)\in[0,T]\times[0,1]} \mathrm{E}\left[||D^j\psi(Y_{a,b})||_{\mathscr{H}^{\otimes j}}^q\right] < \infty. \tag{4.8.14}$$

The proof of (4.8.14) is the same as that of (4.5.36) by using Faà di Bruno formula and (4.8.11).

Similarly, we can prove that $\lambda_A^2 \in \mathbb{D}^\infty(\mathscr{H})$ and it remains to prove that λ_A^3 belongs to $\mathbb{D}^\infty(\mathscr{H})$. For each $k \geq 1$, we denote $v_i = (i-1)/k, r_j = (j-1)T/k$. We discretize λ_A^3 by

$$\begin{aligned} Y_k(r,v) := \sum_{i,j=1}^{k} 2v_i(1-v_i)\int_0^{v_i} dx \int_{[0,T]^2} dsdt \frac{u(1_{]s,t]\times]x,v_i]})^{2p}}{|t-s|^{1+2p\gamma_1}|v_i-x|^{1+2p\gamma_2}} \\ \times \int_0^{r_j} \psi'(Y_{a,v_i})da 1_{[v_i,v_{i+1}]}(v) 1_{[r_j,r_{j+1}]}(r). \end{aligned}$$

For almost every $(\omega,r,v) \in \Omega\times[0,T]\times[0,1]$, $Y_k(r,v)$ converges to $\lambda_A^3(r,v)$ as $k \to \infty$. Using Hölder's inequality and Lemma 4.3.2, we can show that for any $q \geq 1$,

$$\sup_{k\geq 1} \mathrm{E}\left[\int_0^T \int_0^1 |Y_k(r,v)|^q dv dr\right] < \infty,$$

which implies, by Lemma A.6.3 (with the measure space replaced by $(\Omega\times[0,T]\times[0,1], \mathrm{P}\times\lambda^2)$),

$$\lim_{k\to\infty} \mathrm{E}\left[\int_0^T \int_0^1 |Y_k(r,v) - \lambda_A^3(r,v)|^q dv dr\right] = 0$$

and consequently,

$$\lim_{k\to\infty} \mathrm{E}\left[\left(\int_0^T \int_0^1 |Y_k(r,v) - \lambda_A^3(r,v)|^2 dv dr\right)^{q/2}\right] = 0.$$

Thus Y_k converges to λ_A^3 in $L^q(\Omega, \mathscr{H})$ as $k \to \infty$ for any $q \geq 1$. Moreover,

$$\begin{aligned}
DY_k(r,v) = \sum_{i,j=1}^{k} & 4pv_i(1-v_i) \int_0^{v_i} dx \int_{[0,T]^2} dsdt \frac{u(1_{]s,t]\times]x,v_i]})^{2p-1} Du(t,v_i;s,x)}{|t-s|^{1+2p\gamma_1}|v_i - x|^{1+2p\gamma_2}} \\
&\times \int_0^{r_j} \psi'(Y_{a,v_i}) da 1_{[v_i,v_{i+1}]}(v) 1_{[r_j,r_{j+1}]}(r) \\
&+ \sum_{i,j=1}^{k} 2v_i(1-v_i) \int_0^{v_i} dx \int_{[0,T]^2} dsdt \frac{u(1_{]s,t]\times]x,v_i]})^{2p}}{|t-s|^{1+2p\gamma_1}|v_i - x|^{1+2p\gamma_2}} \\
&\times \int_0^{r_j} D\psi'(Y_{a,v_i}) da 1_{[v_i,v_{i+1}]}(v) 1_{[r_j,r_{j+1}]}(r),
\end{aligned}$$

which converges to the Bochner integral

$$\begin{aligned}
Z(r,v) := 4pv(1-v) &\int_0^v dx \int_{[0,T]^2} dsdt \frac{u(1_{]s,t]\times]x,v]})^{2p-1} Du(t,v;s,x)}{|t-s|^{1+2p\gamma_1}|v-x|^{1+2p\gamma_2}} \int_0^r \psi'(Y_{a,v}) da \\
+ 2v(1-v) &\int_0^v dx \int_{[0,T]^2} dsdt \frac{u(1_{]s,t]\times]x,v]})^{2p}}{|t-s|^{1+2p\gamma_1}|v-x|^{1+2p\gamma_2}} \int_0^r D\psi'(Y_{a,v}) da
\end{aligned}$$

in $\mathscr{H}$ as $k \to \infty$ for almost every $(\omega, r, v) \in \Omega \times [0,T] \times [0,1]$. Since

$$\begin{aligned}
\|DY_k(r,v)\|_{\mathscr{H}} \leq c & \sum_{i,j=1}^{k} \int_0^1 dx \int_{[0,T]^2} dsdt \frac{|u(1_{]s,t]\times]x,v_i]})|^{2p-1} \|Du(t,v_i;s,x)\|_{\mathscr{H}}}{|t-s|^{1+2p\gamma_1}|v_i - x|^{1+2p\gamma_2}} \\
&\times 1_{[v_i,v_{i+1}]}(v) 1_{[r_j,r_{j+1}]}(r) \\
&+ c \sum_{i,j=1}^{k} \int_0^1 dx \int_{[0,T]^2} dsdt \frac{u(1_{]s,t]\times]x,v_i]})^{2p}}{|t-s|^{1+2p\gamma_1}|v_i - x|^{1+2p\gamma_2}} \\
&\times \int_0^T \|DY_{a,v_i}\|_{\mathscr{H}} da 1_{[v_i,v_{i+1}]}(v) 1_{[r_j,r_{j+1}]}(r),
\end{aligned}$$

for any $q \geq 1$,

$$
\begin{aligned}
&\mathrm{E}\left[\int_0^T \int_0^1 \|DY_k(r,v)\|_{\mathscr{H}}^q dvdr\right] \\
&\leq c_q \sum_{i,j=1}^{k} \int_0^T \int_0^1 1_{[v_i,v_{i+1}]}(v) 1_{[r_j,r_{j+1}]}(r) dvdr \\
&\quad \times \mathrm{E}\left[\left|\int_0^1 dx \int_{[0,T]^2} dsdt \frac{|u(1_{]s,t]\times]x,v_i]})|^{2p-1} \|Du(t,v_i;s,x)\|_{\mathscr{H}}}{|t-s|^{1+2p\gamma_1}|v_i-x|^{1+2p\gamma_2}}\right|^q\right] \\
&\quad + c_q \sum_{i,j=1}^{k} \int_0^T \int_0^1 1_{[v_i,v_{i+1}]}(v) 1_{[r_j,r_{j+1}]}(r) dvdr \\
&\quad \times \mathrm{E}\left[\left|\int_0^T da \int_0^1 dx \int_{[0,T]^2} dsdt \frac{u(1_{]s,t]\times]x,v_i]})^{2p} \|DY_{a,v_i}\|_{\mathscr{H}}}{|t-s|^{1+2p\gamma_1}|v_i-x|^{1+2p\gamma_2}}\right|^q\right] \\
&\leq c_q k^{-2} \sum_{i,j=1}^{k} \mathrm{E}\left[\left|\int_0^1 dx \int_{[0,T]^2} dsdt \frac{|u(1_{]s,t]\times]x,v_i]})|^{2p-1} \|Du(t,v_i;s,x)\|_{\mathscr{H}}}{|t-s|^{1+2p\gamma_1}|v_i-x|^{1+2p\gamma_2}}\right|^q\right] \\
&\quad + c_q k^{-2} \sum_{i,j=1}^{k} \mathrm{E}\left[\left|\int_0^T da \int_0^1 dx \int_{[0,T]^2} dsdt \frac{u(1_{]s,t]\times]x,v_i]})^{2p} \|DY_{a,v_i}\|_{\mathscr{H}}}{|t-s|^{1+2p\gamma_1}|v_i-x|^{1+2p\gamma_2}}\right|^q\right].
\end{aligned}
$$

Using Hölder's inequality and Cauchy-Schwarz inequality, this is bounded above by

$$
\begin{aligned}
&c_q k^{-2} \sum_{i,j=1}^{k} \int_0^1 dx \int_{[0,T]^2} dsdt \frac{\mathrm{E}\left[|u(1_{]s,t]\times]x,v_i]})|^{q(2p-1)}\right] \|Du(t,v_i;s,x)\|_{\mathscr{H}}^q}{|t-s|^{1+2p\gamma_1}|v_i-x|^{1+2p\gamma_2}} \\
&\quad + c_q k^{-2} \sum_{i,j=1}^{k} \int_0^T da \int_0^1 dx \int_{[0,T]^2} dsdt \frac{\mathrm{E}\left[u(1_{]s,t]\times]x,v_i]})^{4qp}\right]^{1/2} \mathrm{E}\left[\|DY_{a,v_i}\|_{\mathscr{H}}^{2q}\right]^{1/2}}{|t-s|^{1+2p\gamma_1}|v_i-x|^{1+2p\gamma_2}}.
\end{aligned}
$$

By Lemma 4.3.2 and (4.8.11), this is bounded above by

$$
c'_q k^{-2} \sum_{i,j=1}^{k} 1 + c'_q k^{-2} \sum_{i,j=1}^{k} 1 = 2c'_q.
$$

Applying Lemma A.6.3 (with the measure space replaced by $(\Omega \times [0,T] \times [0,1], \mathrm{P} \times \lambda^2)$), we have for any $q \geq 1$

$$
\lim_{k\to\infty} \mathrm{E}\left[\int_0^T \int_0^1 ||DY_k(r,v) - Z(r,v)||_{\mathscr{H}}^q drdv\right] = 0,
$$

which implies

$$
\lim_{k\to\infty} \mathrm{E}\left[\left(\int_0^T \int_0^1 ||DY_k(r,v) - Z(r,v)||_{\mathscr{H}}^2 drdv\right)^{q/2}\right] = 0.
$$

Thus DY_k converges to Z in $L^q(\Omega, \mathscr{H}^{\otimes 2})$ as $k \to \infty$ for any $q \geq 1$. Since D is closable, we obtain that λ_A^3 is Malliavin differentiable and $D\lambda_A^3 = Z$. We can repeat this procedure to conclude that $\lambda_A^3 \in \mathbb{D}^\infty(\mathscr{H})$. The proof is complete. $\square$

Now we are ready to prove the main result of this section.

Proof of Theorem 4.2.1(c). Fix $a > 0$ and set $A =]a, \infty[$. Define the following two random variables:

$$S_a = \inf\{t \geq 0 : \sup_{0 \leq s \leq t, 0 \leq y \leq 1} \hat{u}(s, y) > a\}$$

and

$$X_a = \inf\{x \geq 0 : \sup_{0 \leq s \leq T, 0 \leq y \leq x} \hat{u}(s, y) > a\}.$$

Note that $(S_a, X_a) \leq (\hat{S}, \hat{X})$ on the set $\{M > a\}$, where $(\hat{S}, \hat{X})$ is the point where the maximum is uniquely attained in $[0, T] \times [0, 1]$.

We claim that the random element λ_A introduced in (4.8.12) and the random variable

$$G_A = \int_0^T \int_0^1 v(1-v)\psi(Y_{r,v})\,dv\,dr$$

satisfy the conditions of Theorem 1.5.5. First, λ_A belongs to $\mathbb{D}^\infty(\mathscr{H})$ by Lemma 4.8.3. Moreover, on the set $\{M > a\}$, we have

$$\psi(Y_{r,v}) = 0 \quad \text{if} \quad (r, v) \notin [0, S_a] \times [0, X_a]. \tag{4.8.15}$$

Indeed, if $\psi(Y_{r,v}) \neq 0$, then $Y_{r,v} \leq R$ by definition of ψ and from (4.8.5) this would imply $\sup_{(t,x)\in([0,r]\times[0,1])\cup([0,T]\times[0,v])} \hat{u}(t, x) \leq a$, and, hence, $r \leq S_a$, $v \leq X_a$, which is contradictory.

Consequently, on $\{M > a\}$, by (4.4.8), we obtain

$$\begin{aligned}
\langle DM, \lambda_A\rangle_{\mathscr{H}} &= \langle 1_{\{\cdot<\hat{S}\}}(G(\hat{S}-\cdot, \hat{X}, *) - G(\hat{S}-\cdot, 0, *)), \lambda_A\rangle_{\mathscr{H}} \\
&= \int_0^{\hat{S}} \int_0^1 (G(\hat{S}-r, \hat{X}, v) - G(\hat{S}-r, 0, v))\lambda_A(r, v)\,dv\,dr \\
&= \int_0^{\hat{S}} dr \int_0^1 dv (G(\hat{S}-r, \hat{X}, v) - G(\hat{S}-r, 0, v)) \\
&\quad \times \left(\frac{\partial}{\partial r} - \frac{\partial^2}{\partial v^2}\right) \int_0^r \int_0^v b(1-b)\psi(Y_{a,b})\,db\,da.
\end{aligned}$$

Since the function $(r, v) \mapsto \int_0^r \int_0^v b(1-b)\psi(Y_{a,b})\,db\,da$ satisfies the Neumann boundary conditions, applying Lemma 4.3.6, this is equal to

$$
\begin{aligned}
&\int_0^{\hat{S}}\int_0^{\hat{X}} v(1-v)\psi(Y_{r,v})dvdr - \int_0^{\hat{S}}\int_0^{0} v(1-v)\psi(Y_{r,v})dvdr \\
&\quad = \int_0^{\hat{S}}\int_0^{\hat{X}} v(1-v)\psi(Y_{r,v})dvdr = \int_0^{T}\int_0^{1} v(1-v)\psi(Y_{r,v})dvdr = G_A,
\end{aligned}
\tag{4.8.16}
$$

where the second last equality holds from the observation in (4.8.15).

By discretization, the proof of $G_A \in \mathbb{D}^\infty$ is similar to that of $\gamma_A^{2,2} \in \mathbb{D}^\infty$, as we did in Lemma 4.5.3. So it remains to prove that G_A^{-1} has finite moments of all orders. Indeed, we have

$$
\begin{aligned}
G_A &= \int_0^T\int_0^1 v(1-v)\psi(Y_{r,v})dvdr \\
&\geq \int_0^T\int_0^1 v(1-v)1_{\{Y_{r,v}<\frac{R}{2}\}}dvdr \geq \int_0^T dr\, 1_{\{Y^1(r)<\frac{R}{4}\}}\int_0^1 dv\, v(1-v)1_{\{Y^2(v)<\frac{R}{4}\}} \\
&= \lambda^1\{r\in[0,T]: Y^1(r)<R/4\}\int_0^1 v(1-v)1_{\{Y^2(v)<\frac{R}{4}\}}dv,
\end{aligned}
\tag{4.8.17}
$$

where λ^1 denotes the one-dimensional Lebesgue measure. For any $0<\epsilon<T$, we get

$$
\begin{aligned}
&\mathrm{P}\{\lambda^1\{r\in[0,T]: Y^1(r)<R/4\}<\epsilon\} \\
&\quad \leq \mathrm{P}\{Y^1(\epsilon)\geq R/4\} = \mathrm{P}\left\{\int_{[0,\epsilon]^2}\frac{||u(s,*)-u(s',*)||_{p,\gamma_2}^{2p}}{|s-s'|^{1+2p\gamma_1}}dsds' \geq R/4\right\}.
\end{aligned}
$$

For any $q\geq 1$, by Markov's inequality, this is bounded above by

$$
\begin{aligned}
&(4/R)^q\mathrm{E}\left[\left|\int_{[0,\epsilon]^2}\frac{||\hat{u}(s,*)-\hat{u}(s',*)||_{p,\gamma_2}^{2p}}{|s-s'|^{1+2p\gamma_1}}dsds'\right|^q\right] \\
&\quad \leq (4/R)^q\epsilon^{2(q-1)}\int_{[0,\epsilon]^2}dsds'\int_{[0,1]^2}dxdx'\frac{\mathrm{E}\left[|u(1_{]s,s']\times]x,x']})|^{2pq}\right]}{|s-s'|^{q(1+2p\gamma_1)}|x-x'|^{q(1+2p\gamma_2)}} \\
&\quad \leq c_q\epsilon^{2q}
\end{aligned}
\tag{4.8.18}
$$

for some positive constant c_q. By Lemma 4.4 in Chapter 3 of [24], we know that the random variable $\lambda^1\{r\in[0,T]: Y^1(r)<R/4\}$ has finite negative moments of all orders. It remains to prove the random variable $\int_0^1 v(1-v)1_{\{Y^2(v)<\frac{R}{4}\}}dv$ also has finite negative moments of all orders. For any $0<\epsilon<1/36$, we have

$$
\begin{aligned}
&\mathrm{P}\left\{\int_0^1 v(1-v)1_{\{Y^2(v)<\frac{R}{4}\}}dv<\epsilon\right\}\\
&\le \mathrm{P}\left\{\int_{\sqrt{\epsilon}}^{1/2} v(1-v)1_{\{Y^2(v)<\frac{R}{4}\}}dv<\epsilon\right\}\le \mathrm{P}\left\{\frac{\sqrt{\epsilon}}{2}\int_{\sqrt{\epsilon}}^{1/2} 1_{\{Y^2(v)<\frac{R}{4}\}}dv<\epsilon\right\}\\
&=\mathrm{P}\left\{\int_{\sqrt{\epsilon}}^{1/2} 1_{\{Y^2(v)<\frac{R}{4}\}}dv<2\sqrt{\epsilon}\right\}=\mathrm{P}\left\{\int_0^{1/2} 1_{\{Y^2(v)<\frac{R}{4}\}}dv<2\sqrt{\epsilon}+\int_0^{\sqrt{\epsilon}} 1_{\{Y^2(v)<\frac{R}{4}\}}dv\right\}\\
&\le \mathrm{P}\left\{\int_0^{1/2} 1_{\{Y^2(v)<\frac{R}{4}\}}dv<3\sqrt{\epsilon}\right\}\le \mathrm{P}\left\{Y^2(3\sqrt{\epsilon})\ge R/4\right\}\\
&=\mathrm{P}\left\{\int_{[0,3\sqrt{\epsilon}]^2}\frac{||\hat{u}(\cdot,x)-\hat{u}(\cdot,x')||_{p,\gamma_1}^{2p}}{|x-x'|^{1+2p\gamma_2}}dxdx'\ge\frac{R}{4}\right\}.
\end{aligned}
$$

For any $q\ge 1$, by Markov's inequality, this is bounded above by

$$
\begin{aligned}
&(4/R)^q\mathrm{E}\left[\left|\int_{[0,3\sqrt{\epsilon}]^2}\frac{||\hat{u}(\cdot,x)-\hat{u}(\cdot,x')||_{p,\gamma_1}^{2p}}{|x-x'|^{1+2p\gamma_2}}dxdx'\right|^q\right]\\
&\le (4/R)^q(3T\sqrt{\epsilon})^{2(q-1)}\int_{[0,3\sqrt{\epsilon}]^2}dxdx'\int_{[0,T]^2}dsds'\frac{\mathrm{E}\left[|u(1_{]s,s']\times]x,x']})|^{2pq}\right]}{|s-s'|^{q(1+2p\gamma_1)}|x-x'|^{q(1+2p\gamma_2)}}\\
&\le c_q\epsilon^q. \qquad (4.8.19)
\end{aligned}
$$

Again, we use [24, Chapter 3, Lemma 4.4] to obtain that the random variable $\int_0^1 v(1-v)1_{\{Y^2(v)<\frac{R}{4}\}}dv$ has finite negative moments of all orders, which completes the proof of Theorem 4.2.1(c). □

Remark 4.8.4. *(a) In the case of Dirichlet boundary conditions, we do not know if M has a smooth density. This is because in the definition of the random variable λ_A in* (4.8.12), *the function* $(r,v)\mapsto\int_0^r\int_0^v b(1-b)\psi(Y_{a,b})dbda$ *satisfies the Neumann boundary conditions, while in the case of Dirichlet boundary conditions, we are not able to construct a function that satisfies the Dirichlet boundary conditions.*

(b) Even with Neumann boundary conditions, the method we use does not give a Gaussian-type upper bound on the density of M. This is because in our method the family of random variables $\{\lambda_A(r,v):(r,v)\in[0,T]\times[0,1]\}$ *defined in* (4.8.12) *is not adapted to the filtration and we cannot use Burkholder's inequality to estimate the Skorohod integral, as we did in Sections 4.6 and 4.7.*

5 Extension to the linear stochastic fractional heat equation

In this chapter, we extend some of the results of the previous chapter to the solution of a linear stochastic fractional heat equation. When the heat operator is replaced by the fractional heat operator, the Hölder continuity of the solution changes accordingly. However, the method of the proof for smoothness of the density and the estimate for the Gaussian-type upper bound on the density remain the same. This Gaussian-type upper bound highlights again the connection between the density and the Hölder continuity properties of the solution.

5.1 Introduction and main results

In this chapter, we consider a special case of equations (1.2.2) with $\alpha \in]1,2[$, $\sigma \equiv \mathrm{Id}$, $b \equiv 0$ and $d = 1$. That is, we consider the following linear stochastic fractional heat equation

$$\frac{\partial u}{\partial t}(t,x) = {}_xD^{\alpha} u(t,x) + \dot{W}(t,x), \tag{5.1.1}$$

for $t \in [0,\infty[$ and $x \in \mathbb{R}$, with initial condition $u(0,x) = 0$, for all $x \in \mathbb{R}$. The definition of the fractional differential operator D^{α} is given in (2.1.2) and (2.1.3).

By definition, the solution of (5.1.1) is

$$u(t,x) = \int_0^t \int_{\mathbb{R}} G_{\alpha}(t-r, x-v) W(dr, dv), \tag{5.1.2}$$

where $G_{\alpha}(\cdot,*)$ is the fundamental solution of the Cauchy problem

$$\frac{\partial}{\partial t} G(t,x) = {}_xD^{\alpha} G(t,x), \quad t > 0,\ x \in \mathbb{R},$$
$$G(0,x) = \delta_0(x),$$

where δ_0 is the Dirac distribution. An expression for $G_\alpha(\cdot,*)$ is

$$G_\alpha(t,x) = \frac{1}{2\pi}\int_{\mathbb{R}} \exp(-i\lambda x - t|\lambda|^\alpha)\,d\lambda. \tag{5.1.3}$$

We assume the process $\{u(t,x) : (t,x) \in [0,\infty[\times\mathbb{R}\}$ given by (5.1.2) is the jointly continuous version, which is almost $\frac{\alpha-1}{2\alpha}$-Hölder continuous in time and almost $\frac{\alpha-1}{2}$-Hölder continuous in space. In fact, there exists a constant $C = C(p,T)$ such that for any $p \geq 1$, $(t,x),(s,y) \in [0,T]\times\mathbb{R}$,

$$\mathrm{E}[|u(t,x)-u(s,y)|^p] \leq C(|t-s|^{\frac{\alpha-1}{\alpha}} + |x-y|^{\alpha-1})^{p/2}; \tag{5.1.4}$$

see (2.1.6).

We adopt the same notations as in Chapter 4. Choose two non-trivial compact intervals $I \subset [0,T]$ and $J \subset \mathbb{R}$. Choose $(s_0,y_0) \in I\times J$ and $\delta_1 > 0$. For $t \in [0,T]$, we denote

$$\bar{u}(t,y_0) = u(t,y_0) - u(s_0,y_0). \tag{5.1.5}$$

Set

$$F_1 = u(s_0,y_0), \quad F_2 = \sup_{t\in[s_0,s_0+\delta_1]} \bar{u}(t,y_0) \quad \text{and} \quad F = (F_1,F_2). \tag{5.1.6}$$

Choose $\delta_2 > 0$. Denote by M_0 the global supremum of u over $[0,\delta_1]\times[y_0,y_0+\delta_2]$:

$$M_0 = \sup_{(t,x)\in[0,\delta_1]\times[y_0,y_0+\delta_2]} u(t,x). \tag{5.1.7}$$

Similar to the Theorem 4.2.1, we establish the smoothness of the probability density functions of the random variables F and M_0.

Theorem 5.1.1. *(a) For all $(s_0,y_0) \in]0,T]\times\mathbb{R}$ and $\delta_1 > 0$, the random vector F takes values in $\mathbb{R}\times]0,\infty[$ a.s. and has an infinitely differentiable density on $\mathbb{R}\times]0,\infty[$. When $s_0 = 0$, F_1 vanishes identically but F_2 takes values in $]0,\infty[$ a.s. and has an infinitely differentiable density on $]0,\infty[$.*
(b) For all $y_0 \in \mathbb{R}$, $\delta_1 > 0$ and $\delta_2 > 0$, the random variable M_0 takes values in $]0,\infty[$ a.s. and has an infinitely differentiable density on $]0,\infty[$.

We will prove Theorem 5.1.1 in Section 5.3. The method is the same as that in Chapter 4: we will use Theorem 1.5.5. The fractional heat operator is a non-local operator, and this makes a difference here. For example, the inner product of the random elements in the condition Theorem 1.5.5(iii) gives a formula for the solution of an inhomogeneous heat equation. For the heat equation with Neumann or Dirichlet boundary conditions, uniqueness of the solution holds, while for heat equation on the whole space, uniqueness of the solution fails in general (see for example [40, p. 145]). Fortunately, we are able to overcome this problem because of

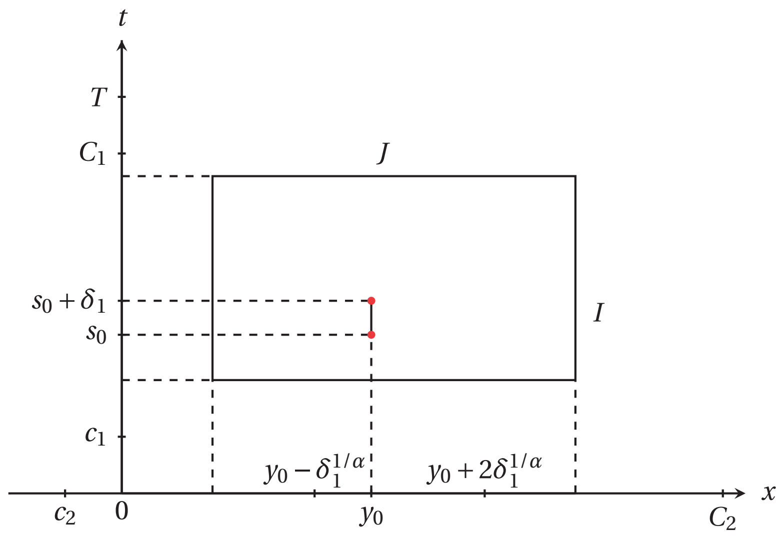

Figure 5.1 – Illustration of conditions (5.1.8)–(5.1.10)

the choice of the functions f_0, g_0 and ϕ_0 in Section 5.3.

We will also establish Gaussian-type upper bounds on the probability density functions of the random variables F and M_0.

Assume $I \times J \subset]0,T] \times \mathbb{R}$. Assume that there are constants c_1, C_1 such that

$$0 < c_1 < \underline{I} := \inf\{s : s \in I\} \quad \text{and} \quad \bar{I} := \sup\{s : s \in I\} < C_1 < T+1. \tag{5.1.8}$$

Assume also that there are constants c_2, C_2 such that

$$c_2 < \underline{J} := \inf\{y : y \in J\} \quad \text{and} \quad C_2 > \bar{J} := \sup\{y : y \in J\}. \tag{5.1.9}$$

Assume that δ_1 is small enough so that

$$s_0 + \delta_1 \in I, \quad \text{and} \quad \delta_1^{1/\alpha} < \min\left\{\underline{J} - c_2, (C_2 - \bar{J})/2\right\}; \tag{5.1.10}$$

see Figure 5.1.

Denote $(z_1, z_2) \mapsto p(z_1, z_2)$ the probability density function of the random vector F.

Theorem 5.1.2. *Assume $I \times J \subset]0,T] \times \mathbb{R}$. There exists a constant $c = c(I,J)$ such that for all*

$\delta_1 > 0$ *satisfying* (5.1.10), *for all* $z_2 \geq \delta_1^{(\alpha-1)/(2\alpha)}$, $z_1 \in \mathbb{R}$ *and any* $(s_0, y_0) \in I \times J$,

$$p(z_1, z_2) \leq \frac{c}{\sqrt{\delta_1^{(\alpha-1)/\alpha}}} \exp\left(-\frac{z_2^2}{c\,\delta_1^{(\alpha-1)/\alpha}}\right)(|z_1|^{-\frac{1}{4}} \wedge 1) \exp\left(-\frac{z_1^2}{c}\right) \tag{5.1.11}$$

$$\leq \frac{c}{\sqrt{\delta_1^{(\alpha-1)/\alpha}}} \exp\left(-\frac{z_2^2}{c\,\delta_1^{(\alpha-1)/\alpha}}\right). \tag{5.1.12}$$

We will prove Theorem 5.1.2 in Section 5.4.

Remark 5.1.3. *Note that* (5.1.11) *implies* (5.1.12) *directly. By Theorem 5.1.2(b) and Remark 4.1.2(b), the assumption* (4.1.14) *is satisfied for the solution to* (5.1.1) *with* $H_1 = \frac{\alpha-1}{2\alpha}$. *Therefore, there exists a constant* $C = C(I, J)$ *such that for all compact sets* $A \subset \mathbb{R}^d$ *and for every* $y_0 \in J$,

$$P\{v(I \times \{y_0\}) \cap A \neq \emptyset\} \leq C\,\mathscr{H}_{d-\frac{2\alpha}{\alpha-1}}(A), \tag{5.1.13}$$

where the components of the random field $v = (v_1, \ldots, v_d)$ *are independent copies of the solution* u *to* (5.1.1).

To establish the Gaussian-type upper bound on the density of the random variable M_0, we introduce some notation for simplicity. Denote

$$\delta := \delta_1^{(\alpha-1)/\alpha} + \delta_2^{\alpha-1}, \qquad \Delta_\bullet := \delta^{\frac{\alpha}{\alpha-1}}, \quad \text{and} \quad \Delta_* := \delta^{\frac{1}{\alpha-1}}. \tag{5.1.14}$$

Choose a positive constant $\bar{C}_1$ with $\bar{C}_1 < T$. Let c_2, C_2 be chosen as in (5.1.9). Assume that $\delta_1, \delta_2 \in]0,1[$ are small enough so that

$$y_0 + \delta_2 \in J, \quad \Delta_\bullet < \bar{C}_1 \quad \text{and} \quad \Delta_* < \min\left\{\underline{J} - c_2, (C_2 - \bar{J})/2\right\}; \tag{5.1.15}$$

see Figure 5.2.

Denote $z \mapsto p_0(z)$ the probability density function of random variable M_0.

Theorem 5.1.4. *Assume* $J \subset \mathbb{R}$. *There exists a finite positive constant* $c = c(T, J)$ *such that for all* δ_1, δ_2 *satisfying the conditions in* (5.1.15), *for all* $y_0 \in J$ *and* $z \geq (\delta_1^{(\alpha-1)/\alpha} + \delta_2^{\alpha-1})^{1/2}$,

$$p_0(z) \leq \frac{c}{\sqrt{\delta_1^{(\alpha-1)/\alpha} + \delta_2^{\alpha-1}}} \exp\left(-\frac{z^2}{c\,(\delta_1^{(\alpha-1)/\alpha} + \delta_2^{\alpha-1})}\right). \tag{5.1.16}$$

The proof of Theorem 5.1.4 will be presented in Section 5.5.

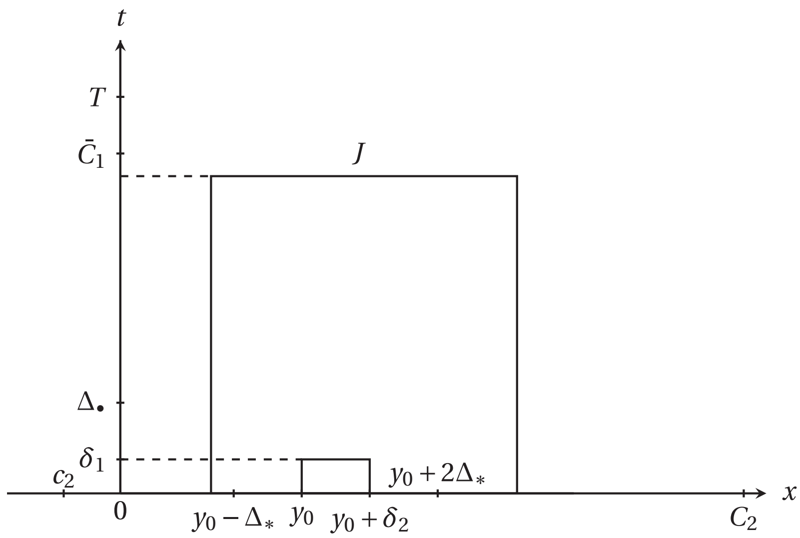

Figure 5.2 – Illustration of condition (5.1.15)

5.2 Preliminaries

In this section, we first present a result, analogous to [45, Theorem 3.3(1)], on the local behavior of the solution in time, which will be used to prove the strict positivity of F_2.

Proposition 5.2.1. *Fix $y_0 \in \mathbb{R}$. There exists a fractional Brownian motion $\{X_t : t \geq 0\}$ with Hurst index $H := \frac{\alpha-1}{2\alpha}$, such that*

$$u(t, y_0) - (\pi(\alpha-1))^{1/2}\Gamma(1/\alpha)^{-1/2}X(t), \quad t \geq 0 \tag{5.2.1}$$

defines a mean-zero Gaussian process with a version that is continuous on $[0,\infty[$ and infinitely differentiable on $]0,\infty[$. As a consequence, for all $t > 0$, and $y_0 \in \mathbb{R}$,

$$\limsup_{\epsilon\downarrow 0} \frac{u(t+\epsilon, y_0) - u(t, y_0)}{\epsilon^{(\alpha-1)/(2\alpha)}\sqrt{2\ln\ln(1/\epsilon)}} = (\pi(\alpha-1))^{-1/2}\Gamma(1/\alpha)^{1/2} \quad a.s. \tag{5.2.2}$$

Remark 5.2.2. *Similar result on the local behavior of the solution in space has been established in [41, Corollary 1.2 and Proposition 3.1]; see also [45, Theorem 3.3(2)].*

Proof of Proposition 5.2.1. The structure of this proof is similar to that of [45, Theorem 3.3(1)]. From (5.1.2), for $t, \epsilon > 0$,

$$\begin{aligned} \mathrm{E}[(u(t+\epsilon, y_0) - u(t, y_0))^2] &= \int_0^t \int_{\mathbb{R}} (G_\alpha(t+\epsilon-r, y_0-v) - G_\alpha(t-r, y_0-v))^2 dv dr \\ &\quad + \int_t^{t+\epsilon} \int_{\mathbb{R}} G_\alpha^2(t+\epsilon-r, y_0-v) dv dr \\ &:= J_1 + J_2. \end{aligned} \tag{5.2.3}$$

Using the semi-group property of the Green kernel (Lemma A.1.1(ii)), we see that

$$
\begin{aligned}
J_2 &= \int_t^{t+\epsilon} G_\alpha(2(t+\epsilon-r),0)\,dr \\
&= \int_0^{\epsilon} G_\alpha(2r,0)\,dr \\
&= \frac{\Gamma(1/\alpha)}{\pi(\alpha-1)}\cdot 2^{-1/\alpha}\epsilon^{\frac{\alpha-1}{\alpha}}.
\end{aligned} \tag{5.2.4}
$$

As for the term J_1, we have

$$
J_1 = \int_0^t \int_{\mathbb{R}} (G_\alpha(\epsilon+r,v) - G_\alpha(r,v))^2\,dv\,dr. \tag{5.2.5}
$$

Before we evaluate J_1, we first compute the following integral:

$$
\begin{aligned}
\int_0^\infty \int_{\mathbb{R}} (G_\alpha(\epsilon+r,v) - G_\alpha(r,v))^2\,dv\,dr &= \frac{1}{2\pi}\int_0^\infty dr \int_{\mathbb{R}} dv \left| e^{-(r+\epsilon)|v|^\alpha} - e^{-r|v|^\alpha} \right|^2 \\
&= \frac{1}{\pi}\int_0^\infty dr \int_0^\infty dv\, e^{-2rv^\alpha} \left| 1 - e^{-\epsilon v^\alpha} \right|^2 \\
&= \frac{1}{2\pi}\int_0^\infty \frac{\left(1-e^{-\epsilon v^\alpha}\right)^2}{v^\alpha}\,dv \\
&= \frac{1}{2\pi}\epsilon^{\frac{\alpha-1}{\alpha}} \int_0^\infty \frac{\left(1-e^{-v^\alpha}\right)^2}{v^\alpha}\,dv,
\end{aligned} \tag{5.2.6}
$$

where, in the first equality, we use the Plancherel theorem. The last integral in (5.2.6) is equal to $\frac{2\Gamma(1/\alpha)}{\alpha-1}(1-2^{-1/\alpha})$ by changing variable $[z = v^\alpha]$; see also the calculation the proof of Lemma A.1 in [45]. Therefore,

$$
J_1 = \frac{\Gamma(1/\alpha)}{\pi(\alpha-1)}(1-2^{-1/\alpha})\epsilon^{\frac{\alpha-1}{\alpha}} - \int_t^\infty \int_{\mathbb{R}} (G_\alpha(\epsilon+r,v) - G_\alpha(r,v))^2\,dv\,dr. \tag{5.2.7}
$$

Combining (5.2.3), (5.2.4) and (5.2.7), we obtain that

$$
\mathrm{E}[(u(t+\epsilon,y_0) - u(t,y_0))^2] = \frac{\Gamma(1/\alpha)}{\pi(\alpha-1)}\epsilon^{\frac{\alpha-1}{\alpha}} - \int_t^\infty \int_{\mathbb{R}} (G_\alpha(\epsilon+r,v) - G_\alpha(r,v))^2\,dv\,dr. \tag{5.2.8}
$$

In order to understand the last integral, let η denote a white noise on $\mathbb{R}$ that is independent of the space-time white noise $\dot{W}$, and consider the Gaussian process $\{T_t : t \geq 0\}$ defined by

$$
T_t := \frac{1}{\sqrt{4\pi}} \int_{\mathbb{R}} \frac{1-e^{-t|z|^\alpha}}{|z|^{\alpha/2}}\,\eta(dz), \quad t \geq 0.
$$

This is a well-defined mean-zero Wiener integral process, $T_0 = 0$, and

$$
\mathrm{Var}(T_t) = \frac{1}{4\pi}\int_{\mathbb{R}} \left(\frac{1-e^{-t|z|^\alpha}}{|z|^{\alpha/2}} \right)^2 dz < \infty \quad \text{for all } t > 0.
$$

Note that

$$
\begin{aligned}
\mathrm{E}[(T_{t+\epsilon}-T_t)^2] &= \frac{1}{4\pi}\int_{\mathbb{R}}\left(\frac{e^{-t|z|^\alpha}-e^{-(t+\epsilon)|z|^\alpha}}{|z|^{\alpha/2}}\right)^2 dz \\
&= \frac{1}{4\pi}\int_{\mathbb{R}} e^{-2t|z|^\alpha}\left(\frac{1-e^{-\epsilon|z|^\alpha}}{|z|^{\alpha/2}}\right)^2 dz \\
&= \frac{1}{2\pi}\int_t^\infty ds\, e^{-2s|z|^\alpha}\int_{\mathbb{R}} dz \left(1-e^{-\epsilon|z|^\alpha}\right)^2 \\
&= \frac{1}{2\pi}\int_t^\infty ds\int_{\mathbb{R}} dz \left|e^{-(s+\epsilon)|z|^\alpha}-e^{-s|z|^\alpha}\right|^2 \\
&= \int_t^\infty\int_{\mathbb{R}} (G_\alpha(\epsilon+s,z)-G_\alpha(s,z))^2 dz ds, \qquad (5.2.9)
\end{aligned}
$$

where, in the last inequality, we again use the Plancherel theorem. Since T and u are independent, by (5.2.8) and (5.2.9), we have

$$
\mathrm{E}[(u(t+\epsilon,y_0)+T_{t+\epsilon}-u(t,y_0)-T_t)^2] = \frac{\Gamma(1/\alpha)}{\pi(\alpha-1)}\epsilon^{\frac{\alpha-1}{\alpha}}. \qquad (5.2.10)
$$

Therefore, since $u(0,y_0)\equiv 0\equiv T_0$, we have proved that $\{X_t : t\geq 0\}$ is a fractional Brownian motion $\{X_t : t\geq 0\}$ with Hurst index $H := \frac{\alpha-1}{2\alpha}$, where

$$
X_t := (\pi(\alpha-1))^{1/2}\Gamma(1/\alpha)^{-1/2}(u(t,y_0)+T_t), \quad t\geq 0. \qquad (5.2.11)
$$

It remains to prove that the process $\{T_t : t\geq 0\}$ has a version that is continuous on $[0,\infty[$ and infinitely differentiable on $]0,\infty[$. The proof follows along the same lines as that of [45, Lemma 3.6]. We give the proof for the convenience of the reader. First, for $t, s\geq 0$,

$$
\begin{aligned}
\mathrm{E}[(T_t-T_s)^2] &= \frac{1}{4\pi}\int_{\mathbb{R}}\left(\frac{e^{-s|z|^\alpha}-e^{-t|z|^\alpha}}{|z|^{\alpha/2}}\right)^2 dz \\
&\leq \int_{\mathbb{R}}\left(\frac{1-e^{-|t-s||z|^\alpha}}{|z|^{\alpha/2}}\right)^2 dz \\
&= |t-s|^{\frac{\alpha-1}{\alpha}}\int_{\mathbb{R}}\left(\frac{1-e^{-|z|^\alpha}}{|z|^{\alpha/2}}\right)^2 dz.
\end{aligned}
$$

Applying the Kolmogorov continuity theorem, we see that T has a version that is Hölder continuous with exponent $(\alpha-1)/(2\alpha)-\epsilon$ for all small $\epsilon>0$.

Next, we consider the case that $t > 0$, and define for all $n \geq 1$,

$$
\begin{aligned}
T_t^{(n)} &:= \frac{1}{\sqrt{4\pi}} \int_{\mathbb{R}} \frac{\partial^n}{\partial t^n} \left(\frac{1 - e^{-t|z|^\alpha}}{|z|^{\alpha/2}} \right) \eta(dz) \\
&= \frac{(-1)^{n+1}}{\sqrt{4\pi}} \int_{\mathbb{R}} |z|^{\alpha n - \alpha/2} e^{-t|z|^\alpha} \eta(dz).
\end{aligned}
$$

Since the integrand belongs to $L^2(\mathbb{R})$, $\{T_t^{(n)} : t > 0\}$ is a well-defined mean-zero Gaussian process. Furthermore, for every $t, s > 0$,

$$
\begin{aligned}
\mathrm{E}\left[\left(T_t^{(n)} - T_s^{(n)} \right)^2 \right] &= \frac{1}{4\pi} \int_{\mathbb{R}} |z|^{2\alpha n - \alpha} \left| e^{-s|z|^\alpha} - e^{-t|z|^\alpha} \right|^2 dz \\
&= \frac{1}{4\pi} \int_{\mathbb{R}} |z|^{2\alpha n - \alpha} e^{-2(s \wedge t)|z|^\alpha} \left| 1 - e^{-|t-s||z|^\alpha} \right|^2 dz \\
&\leq \frac{|t-s|^2}{4\pi} \int_{\mathbb{R}} |z|^{2\alpha n + \alpha} e^{-2(s \wedge t)|z|^\alpha} dz,
\end{aligned}
$$

where, in the last inequality, we use that $1 - e^{-\theta} \leq \theta$ for all $\theta \geq 0$. It follows from the Kolmogorov continuity theorem that every $T^{(n)}$ is continuous on $]0, \infty[$ [up to a version].

If $\varphi \in C_0^\infty(]0, \infty[)$ (the space of infinitely differentiable functions with compact support on $]0, \infty[$), then we apply the stochastic Fubini theorem (see [24, Chapter 1, Theorem 5.30] or [81, Theorem 2.6]), to see that, a.s.,

$$
\begin{aligned}
\int_0^\infty T_t^{(n)} \varphi_t \, dt &= \frac{1}{\sqrt{4\pi}} \int_{\mathbb{R}} \eta(dz) \int_0^\infty dt \, \frac{\partial^n}{\partial t^n} \left(\frac{1 - e^{-t|z|^\alpha}}{|z|^{\alpha/2}} \right) \varphi_t \\
&= \frac{(-1)^n}{\sqrt{4\pi}} \int_{\mathbb{R}} \eta(dz) \int_0^\infty dt \left(\frac{1 - e^{-t|z|^\alpha}}{|z|^{\alpha/2}} \right) \frac{\partial^n}{\partial t^n} \varphi_t,
\end{aligned}
$$

thanks to integration by parts. A second appeal to the stochastic Fubini theorem yields

$$
\int_0^\infty T_t^{(n)} \varphi_t \, dt = (-1)^n \int_0^\infty T_t \frac{\partial^n}{\partial t^n} \varphi_t \, dt \quad \text{a.s.}
$$

That is, $T_t^{(n)}$ is the weak n-fold derivative of T_t for all $t > 0$. Since $T^{(n)}$ is continuous on $]0, \infty[$ for all n, this shows that in fact $T_t^{(n)}$ is a.s. the ordinary n-fold derivative of T at t. Therefore, $\{T_t : t \geq 0\}$ has a version that is infinitely differentiable on $]0, \infty[$.

Finally, (5.2.2) follows from the representation (5.2.1) and the law of the iterated logarithm for fractional Brownian motion (see [45, Theorem 2.11]). □

We next show some properties of the rectangular increments of the solution, analogous to Lemmas 4.3.2 and 4.3.3.

Lemma 5.2.3. *There exists a constant C_T such that for any $\theta \in]0, \frac{\alpha-1}{\alpha}[$ and $(t, s, x, y) \in [0, T]^2 \times$*

$\mathbb{R}^2$,

$$E[(u(t,x)+u(s,y)-u(t,y)-u(s,x))^2] \le C_T|t-s|^{\frac{\alpha-1}{\alpha}} \wedge |x-y|^{\alpha-1} \\ \le C_T|t-s|^{\frac{\alpha-1}{\alpha}-\theta}|x-y|^{\alpha\theta}. \tag{5.2.12}$$

Proof. The proof follows the same lines as the proof of Lemma 4.3.2 by using (5.1.4). □

From now on, we fix $\theta \in]0, \frac{\alpha-1}{\alpha}[$ and set

$$\theta_1 = \frac{\alpha-1}{\alpha} - \theta, \quad \theta_2 = \alpha\theta. \tag{5.2.13}$$

By the isometry and Lemma 5.2.3,

$$\|D(u(t,x)+u(s,y)-u(t,y)-u(s,x))\|^2_{\mathscr{H}} = \mathrm{E}[(u(t,x)+u(s,y)-u(t,y)-u(s,x))^2] \\ \le C_T|t-s|^{\theta_1}|x-y|^{\theta_2}, \tag{5.2.14}$$

for any $(t,s,x,y) \in [0,T]^2 \times \mathbb{R}^2$.

Lemma 5.2.4. *For any $0<\xi<\theta_1/2$ and $0<\eta<\theta_2/2$, there exists a random variable C which is a.s. finite and depends on T and the length of the interval L such that a.s., for all $(t,s,x,y) \in [0,T]^2 \times [a,b]^2$ with $[a,b] \subseteq L$,*

$$|u(t,x)+u(s,y)-u(t,y)-u(s,x)| \le C|t-s|^{\xi}|x-y|^{\eta}. \tag{5.2.15}$$

Proof. The proof follows the same lines as that of Lemma 4.3.3 with the interval $[0,1]$ replaced by $[a,b]$. For $(t,x) \in [0,\infty[\times\mathbb{R}$, we denote

$$\hat{u}(t,x) = u(t,x) - u(t,a). \tag{5.2.16}$$

We choose p, $\bar{\gamma}_2$ such that $\xi < \theta_1/2 - \frac{1}{2p}$ and $\eta + \frac{1}{2p} < \bar{\gamma}_2 < \theta_2/2 - \frac{1}{2p}$. Let $E_{p,\bar{\gamma}_2}[a,b]$ be the space of continuous functions defined on $[a,b]$ vanishing at a and having a finite $\|\cdot\|_{p,\bar{\gamma}_2}$ norm (see Section 4.3 for the definition of $E_{p,\bar{\gamma}_2}[a,b]$ and $\|\cdot\|_{p,\bar{\gamma}_2}$).

Since a.s., for any $t \in [0,T]$, $x \mapsto \hat{u}(t,x)$ is almost $\frac{\alpha-1}{2}$-Hölder continuous, we see that $\hat{u}(t,*)$ belongs to $E_{p,\bar{\gamma}_2}[a,b]$. Moreover, by (5.2.12), for any $s,\ t \in [0,T]$,

$$\mathrm{E}[\|\hat{u}(t,*)-\hat{u}(s,*)\|^{2p}_{p,\bar{\gamma}_2}] = \int_{[a,b]^2} \frac{\mathrm{E}[|u(t,x)+u(s,y)-u(t,y)-u(s,x)|^{2p}]}{|x-y|^{1+2p\bar{\gamma}_2}} dxdy \\ \le C_T|t-s|^{\theta_1 p} \int_{[a,b]^2} \frac{|x-y|^{\theta_2 p}}{|x-y|^{1+2p\bar{\gamma}_2}} dxdy \\ \le C_T|t-s|^{\theta_1 p},$$

where the constant C_T depends only on T and the length of the interval $[a,b]$. We apply the Kolmogorov continuity theorem (see [76, Theorem 2.1]) to see that the process $\{\hat{u}(t,*): t \in [0,T]\}$ has a continuous version $\{\tilde{u}(t,*): t \in [0,T]\}$ with values in $E_{p,\bar{\gamma}_2}$, which is $\frac{\theta_1}{2} - \frac{1}{2p} - \epsilon$-Hölder continuous for small ϵ such that $\frac{\theta_1}{2} - \frac{1}{2p} - \epsilon > \xi$, namely, there exists a random variable C, finite almost surely, such that a.s. for any $s,\ t \in [0,T]$,

$$\|\tilde{u}(t,*) - \tilde{u}(s,*)\|_{p,\bar{\gamma}_2} \le C|t-s|^{\frac{\theta_1}{2} - \frac{1}{2p} - \epsilon}.$$

Hence we have for any $s,\ t \in [0,T]$,

$$\int_{[a,b]^2} \frac{|\tilde{u}(t,x) - \tilde{u}(s,x) - \tilde{u}(t,y) + \tilde{u}(s,y)|^{2p}}{|x-y|^{1+2p\bar{\gamma}_2}} dxdy \le C|t-s|^{(\frac{\theta_1}{2} - \frac{1}{2p} - \epsilon)2p}.$$

We apply the Garsia, Rodemich and Rumsey lemma (see Lemma A.6.2) to the real-valued function $x \mapsto \tilde{u}(t,x) - \tilde{u}(s,x)$ with $\Psi(x) = x^{2p}$, $p(x) = x^{(1+2p\bar{\gamma}_2)/(2p)}$, $d = 1$, to get that for any $(t,s,x,y) \in [0,T]^2 \times [a,b]^2$,

$$\begin{aligned}|\tilde{u}(t,x) - \tilde{u}(s,x) - \tilde{u}(t,y) + \tilde{u}(s,y)| &\le C|t-s|^{\frac{\theta_1}{2} - \frac{1}{2p} - \epsilon}|x-y|^{\bar{\gamma}_2 - \frac{1}{2p}} \\ &\le \tilde{C}|t-s|^{\xi}|x-y|^{\eta},\end{aligned} \tag{5.2.17}$$

where $\tilde{C}$ depends on the length of the interval $[a,b]$. Letting $y = a$ in (5.2.17), we obtain

$$|\tilde{u}(t,x) - \tilde{u}(s,x)| \le C'|t-s|^{\xi}, \tag{5.2.18}$$

where C' depends on the length of the interval $[a,b]$.

Fix $(s,y) \in [0,T] \times [a,b]$. Using the triangle inequality,

$$|\tilde{u}(t,x) - \tilde{u}(s,y)| \le |\tilde{u}(t,x) - \tilde{u}(s,x)| + |\tilde{u}(s,x) - \tilde{u}(s,y)|,$$

which converges to 0 as $(t,x) \to (s,y)$ by (5.2.18) and the fact that $x \mapsto \tilde{u}(s,x)$ is continuous since $\tilde{u}(s,*) \in E_{p,\bar{\gamma}_2}$. Therefore, a.s., $(t,x) \mapsto \tilde{u}(t,x)$ is continuous. Together with the fact that for any $t \in [0,T]$, $\mathrm{P}\{\hat{u}(t,*) = \tilde{u}(t,*)\} = 1$, we obtain that the processes $\{\hat{u}(t,x) : (t,x) \in [0,T] \times [a,b]\}$ and $\{\tilde{u}(t,x) : (t,x) \in [0,T] \times [a,b]\}$ are indistinguishable and hence (5.2.17) implies (5.2.15). □

Choose an integer p_0 and $\gamma_0 \in \mathbb{R}$ such that

$$p_0 > \gamma_0 > \frac{2\alpha}{\alpha - 1}. \tag{5.2.19}$$

We assume that p_0 is sufficient large so that there exist γ_1, γ_2 such that

$$\frac{1}{2p_0} < \gamma_1 < \theta_1/2 - \frac{1}{2p_0}, \quad \frac{1}{2p_0} < \gamma_2 < \theta_2/2 - \frac{1}{2p_0}, \tag{5.2.20}$$

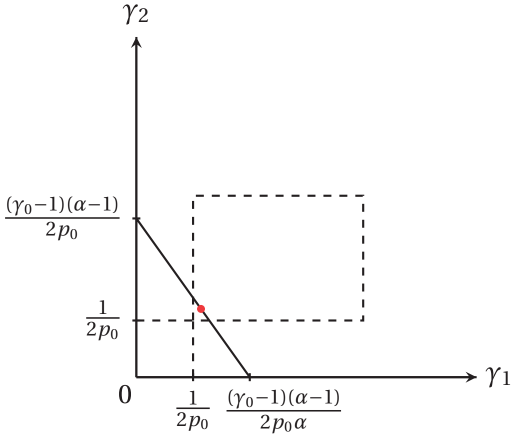

Figure 5.3 – Illustration of (5.2.20) and (5.2.21)

where θ_1, θ_2 are defined in (5.2.13), and

$$\frac{\alpha\gamma_1}{\alpha-1}+\frac{\gamma_2}{\alpha-1}=\frac{\gamma_0-1}{2p_0}; \tag{5.2.21}$$

see Figure 5.3.

We introduce the following family of random variables, which can control the value of the supremum F_2. For $r\in[s_0, s_0+\delta_1]$, we define

$$Y_r := \int_{[s_0,r]^2}\frac{(u(t,y_0)-u(s,y_0))^{2p_0}}{|t-s|^{(\alpha-1)\gamma_0/\alpha}}dsdt. \tag{5.2.22}$$

By Hölder's inequality and (5.1.4), there exists a constant c_p, not depending on $(s_0, y_0)\in[0,T]\times\mathbb{R}$, such that for any $p\geq 1$, and for all $r\in[s_0,s_0+\delta_1]$,

$$\begin{aligned}
\mathrm{E}[|Y_r|^p] &\leq (r-s_0)^{2(p-1)}\int_{[s_0,r]^2}\frac{\mathrm{E}[|u(t,y_0)-u(s,y_0)|^{2p_0p}]}{|t-s|^{(\alpha-1)\gamma_0p/\alpha}}dsdt\\
&\leq c_p\,(r-s_0)^{2(p-1)}\int_{[s_0,r]^2}\frac{|t-s|^{(\alpha-1)p_0p/\alpha}}{|t-s|^{(\alpha-1)\gamma_0p/\alpha}}dsdt\\
&\leq c_p\,(r-s_0)^{2p}\delta_1^{(\alpha-1)(p_0-\gamma_0)p/\alpha}.
\end{aligned} \tag{5.2.23}$$

Similar to Lemma 4.5.1(a), we know that $Y_r\in\mathbb{D}^\infty$, $r\in[s_0,s_0+\delta_1]$ and for any integer l,

$$\begin{aligned}
D^lY_r = \int_{[s_0,r]^2}dtds\,&\frac{2p_0(2p_0-1)\cdots(2p_0-l+1)}{|t-s|^{(\alpha-1)\gamma_0/\alpha}}\\
&\times(u(t,y_0)-u(s,y_0))^{2p_0-l}(D(u(t,y_0)-u(s,y_0)))^{\otimes l}.
\end{aligned} \tag{5.2.24}$$

Moreover, we have the following estimate on the moment of the Malliavin derivative of Y_r.

Lemma 5.2.5. *For any $p \geq 1$, there exists a constant c_p, not depending on $(s_0, y_0) \in [0,T] \times \mathbb{R}$, such that for all $\delta_1 > 0$ and for all $r \in [s_0, s_0+\delta_1]$,*

$$E[\|DY_r\|_{\mathscr{H}}^p] \leq c_p(r-s_0)^{2p}\delta_1^{(p_0-\gamma_0)(\alpha-1)p/\alpha}. \tag{5.2.25}$$

Proof. From (5.2.24), we know that

$$DY_r = 2p_0\int_{[s_0,r]^2} ds\,dt\,\frac{(u(t,x_0)-u(s,y_0))^{2p_0-1}}{|t-s|^{\gamma_0(\alpha-1)/\alpha}}D(u(t,y_0)-u(s,y_0))$$

and by Hölder's inequality,

$$\begin{aligned}\mathrm{E}[\|DY_r\|_{\mathscr{H}}^p] \leq c_p\left(\int_{[s_0,r]^2} ds\,dt\right)^{p-1}\int_{[s_0,r]^2} ds\,dt\,&\frac{\mathrm{E}\left[|(u(t,y_0)-u(s,y_0))|^{(2p_0-1)p}\right]}{|t-s|^{\gamma_0 p(\alpha-1)/\alpha}}\\ &\times\|D(u(t,y_0)-u(s,y_0))\|_{\mathscr{H}}^p.\end{aligned} \tag{5.2.26}$$

Since

$$\begin{aligned}\|D(u(t,x)-u(s,y))\|_{\mathscr{H}} &= \|u(t,x)-u(s,y)\|_{L^2(\Omega)}\\ &\leq C(|t-s|^{\frac{\alpha-1}{\alpha}}+|x-y|^{\alpha-1})^{1/2}\end{aligned} \tag{5.2.27}$$

by (5.1.4), we see that (5.2.26) is bounded above by

$$\begin{aligned}&c_p((r-s_0)^2)^{p-1}\int_{[s_0,r]^2} ds\,dt\,|t-s|^{(p_0-\gamma_0)(\alpha-1)p/\alpha}\\ &\quad\leq c_p(r-s_0)^{2p}\delta_1^{(p_0-\gamma_0)(\alpha-1)p/\alpha},\end{aligned} \tag{5.2.28}$$

which completes the proof. □

Furthermore, we have for any integer l and $q \geq 1$,

$$\sup_{r\in[s_0,s_0+\delta_1]}\mathrm{E}\left[\|D^l Y_r\|_{\mathscr{H}^{\otimes l}}^q\right] < \infty. \tag{5.2.29}$$

Lemma 5.2.6. *There exists a finite positive constant c, not depending on $(s_0, y_0) \in [0,T] \times \mathbb{R}$, such that for any $a > 0$, and for all $r \in [s_0, s_0+\delta_1]$,*

$$Y_r \leq R := c\,a^{2p_0}\delta_1^{2-\gamma_0(\alpha-1)/\alpha} \quad\Rightarrow\quad \sup_{t\in[s_0,r]}|\bar{u}(t,y_0)| \leq a. \tag{5.2.30}$$

Proof. The proof is similar to that of Lemma 4.3.1. We first apply the Garsia, Rodemich, and

Rumsey lemma (see Lemma A.6.1) with

$$S := [s_0, r],\ \rho(t,s) := |t-s|^{\frac{\alpha-1}{\alpha}},\ \mu(dt) := dt,$$
$$\Psi(x) := x^{2p_0},\ p(x) := x^{\gamma_0/(2p_0)} \quad \text{and} \quad f := u(\cdot, y_0).$$

From (A.34), and assuming $Y_r \le R$, we deduce that for all $t, s \in [s_0, r]$,

$$\begin{aligned}
|u(t,y_0) - u(s,y_0)| &\le 10 \int_0^{2\rho(t,s)} \frac{Y_r^{\frac{1}{2p_0}}}{[\mu(B_\rho(s,u/4))]^{1/p_0}} u^{\frac{\gamma_0}{2p_0}-1} du \\
&\le c_1 Y_r^{\frac{1}{2p_0}} \int_0^{2\rho(t,s)} u^{-\frac{\alpha}{(\alpha-1)p_0}} u^{\frac{\gamma_0}{2p_0}-1} du \\
&= c_2 (|t-s|^{(\alpha-1)/\alpha})^{\frac{\gamma_0}{2p_0}-\frac{\alpha}{(\alpha-1)p_0}} Y_r^{\frac{1}{2p_0}} \\
&\le c_2 \delta_1^{\frac{\gamma_0(\alpha-1)}{2p_0\alpha}-\frac{1}{p_0}} R^{\frac{1}{2p_0}},
\end{aligned} \tag{5.2.31}$$

where we have used (5.2.19); the constants c_1, c_2 do not depend on r, nor on $(s_0, y_0) \in [0,T] \times \mathbb{R}$. Letting $s = s_0$ in the above inequality and choosing a suitable constant in the definition of R, we obtain that

$$\sup_{t \in [s_0, r]} |\bar{u}(t, y_0)| \le a.$$

□

Recall the definition of the space $E_{p_0,\gamma_2}[y_0, y_0+\Delta_*]$ in Section 4.3, i.e., the space of continuous functions defined on $[y_0, y_0+\Delta_*]$ vanishing at y_0 and having a finite $\|\cdot\|_{p_0,\gamma_2}$ norm defined in (4.3.21). For $(t,x) \in [0,T] \times \mathbb{R}$, we denote

$$\check{u}(t,x) = u(t,x) - u(t,y_0). \tag{5.2.32}$$

Choose ξ, η as in Lemma 5.2.4 such that $\eta > \gamma_2 + 1/(2p_0)$, which is possible by (5.2.20). Then, by (5.2.15),

$$\begin{aligned}
\|\check{u}(t,*) - \check{u}(s,*)\|_{p_0,\gamma_2}^{2p_0} &= \int_{[y_0,y_0+\Delta_*]^2} \frac{(u(t,x)+u(s,y)-u(t,y)-u(s,x))^{2p_0}}{|x-y|^{1+2p_0\gamma_2}} dxdy \\
&\le C|t-s|^{2p_0\xi} \int_{[y_0,y_0+\Delta_*]^2} |x-y|^{2p_0\eta-1-2p_0\gamma_2} dxdy \\
&\le C|t-s|^{2p_0\xi}
\end{aligned} \tag{5.2.33}$$

since $2p_0\eta - 1 - 2p_0\gamma_2 > 0$, which implies that a.s. $t \mapsto \check{u}(t,*)$ is continuous in $E_{p_0,\gamma_2}[y_0, y_0+\Delta_*]$.

We next introduce a family of random variables, which can control the value of the supremum M_0.

For $r \in [0, \Delta_\bullet]$, we define

$$Y_0(r) := \int_{[0,r]^2} \frac{(u(t,y_0) - u(s,y_0))^{2p_0}}{|t-s|^{(\alpha-1)\gamma_0/\alpha}} dsdt, \tag{5.2.34}$$

and

$$\begin{aligned} Y_1(r) &:= \int_{[0,r]^2} \frac{\|\check{u}(t,*) - \check{u}(s,*)\|^{2p_0}_{p_0,\gamma_2}}{|t-s|^{1+2p_0\gamma_1}} dtds \\ &= \int_{[0,r]^2} dtds \int_{[y_0, y_0+\Delta_*]^2} dxdy \frac{(u(t,x) + u(s,y) - u(t,y) - u(s,x))^{2p_0}}{|t-s|^{1+2p_0\gamma_1}|x-y|^{1+2p_0\gamma_2}}. \end{aligned} \tag{5.2.35}$$

Similar to the calculation in (4.3.16), by Hölder's inequality and (5.2.12), we see that there exists a constant c_p, not depending on $y_0 \in \mathbb{R}$, such that for any $p \geq 1$ and for any $r \in [0, \Delta_\bullet]$,

$$\begin{aligned} \mathrm{E}[|Y_1(r)|^p] &\leq c_p\,(r\Delta_*)^{2p} \Delta_\bullet^{p(p_0\theta_1 - (1+2p_0\gamma_1))} \Delta_*^{p(p_0\theta_2 - (1+2p_0\gamma_2))} \\ &= c_p\, r^{2p} \delta^{p(p_0\theta_1 - (1+2p_0\gamma_1))\alpha/(\alpha-1)} \delta^{p(p_0\theta_2 - (1+2p_0\gamma_2)+2)/(\alpha-1)} \\ &= c_p\, r^{2p} \delta^{p(p_0(\frac{\alpha\theta_1}{\alpha-1} + \frac{\theta_2}{\alpha-1}) - 2p_0(\frac{\alpha\gamma_1}{\alpha-1} + \frac{\gamma_2}{\alpha-1}) - 1)} \\ &= c_p\, r^{2p} \delta^{p(p_0 - \gamma_0)}, \end{aligned} \tag{5.2.36}$$

where in the first equality we use (5.1.14), in the third equality we use (5.2.21) and the fact that $\frac{\alpha\theta_1}{\alpha-1} + \frac{\theta_2}{\alpha-1} = 1$ by the definition of θ_1, θ_2 in (5.2.13).

For $r \in [0, \Delta_\bullet]$, set

$$\bar{Y}_r := Y_0(r) + Y_1(r). \tag{5.2.37}$$

By (5.2.36) and the calculation in (5.2.23), for any $p \geq 1$, there exists a constant c_p, not depending on $y_0 \in \mathbb{R}$, such that for any $r \in [0, \Delta_\bullet]$,

$$\mathrm{E}[|\bar{Y}_r)|^p] \leq c_p\, r^{2p} \delta^{p(p_0 - \gamma_0)}. \tag{5.2.38}$$

Similar to Lemma 4.5.1(b), we know that $\bar{Y}_r \in \mathbb{D}^\infty$ for $r \in [0, \Delta_\bullet]$, and for any integer l,

$$\begin{aligned} D^l \bar{Y}_r &= D^l Y_0(r) + D^l Y_1(r) \\ &= \int_{[0,r]^2} dtds \frac{2p_0(2p_0-1)\cdots(2p_0-l+1)}{|t-s|^{(\alpha-1)\gamma_0/\alpha}} \\ &\qquad\qquad \times (u(t,y_0) - u(s,y_0))^{2p_0-l} (D(u(t,y_0) - u(s,y_0)))^{\otimes l} \\ &\quad + \int_{[0,r]^2} dtds \int_{[y_0,y_0+\Delta_*]^2} dxdy \frac{2p_0(2p_0-1)\cdots(2p_0-l+1)}{|t-s|^{1+2p_0\gamma_1}|x-y|^{1+2p_0\gamma_2}} \\ &\qquad\qquad \times u(1_{]s,t]\times]y,x]})^{2p_0-l} (Du(t,x;s,y))^{\otimes l}. \end{aligned} \tag{5.2.39}$$

We proceed to give an estimate on the moment of $D\bar{Y}_r$, analogous to (4.5.7).

Lemma 5.2.7. *For any $p \geq 1$, there exists a constant c_p, not depending on $y_0 \in \mathbb{R}$, such that for all $r \in [0, \Delta_\bullet]$,*

$$E[\|D\bar{Y}_r\|^p_{\mathcal{H}}] \leq c_p\, r^{2p} \delta^{(p_0-\gamma_0)p}. \tag{5.2.40}$$

Furthermore, for any integer l and $q \geq 1$,

$$\sup_{r\in[0,\Delta_\bullet]} E\left[\|D^l \bar{Y}_r\|^q_{\mathcal{H}^{\otimes l}}\right] < \infty. \tag{5.2.41}$$

Proof. We focus on estimate on the moment of $DY_1(r)$ since the estimate on the moment of $DY_0(r)$ is essentially the same as that of $DY(r)$ in the proof of Lemma 5.2.5. From (5.2.39),

$$DY_1(r) = 2p_0 \int_{[0,r]^2} dtds \int_{[y_0,y_0+\Delta_*]^2} dxdy \frac{u(1_{]s,t]\times]y,x]})^{2p_0-1} Du(t,x;s,y)}{|t-s|^{1+2p_0\gamma_1}|x-y|^{1+2p_0\gamma_2}},$$

and for any $p \geq 1$, by Hölder's inequality,

$$\begin{aligned}
\mathrm{E}[\|DY_1(r)\|^p_{\mathcal{H}}] &\leq c_p\,(r\Delta_*)^{2(p-1)} \int_{[0,r]^2} dtds \int_{[y_0,y_0+\Delta_*]^2} dxdy \\
&\qquad\qquad \times \frac{\mathrm{E}[|u(1_{]s,t]\times]y,x]})|^{(2p_0-1)p}]\,\|Du(t,x;s,y)\|^p_{\mathcal{H}}}{|t-s|^{(1+2p_0\gamma_1)p}|x-y|^{(1+2p_0\gamma_2)p}} \\
&\leq c_p\,(r\Delta_*)^{2(p-1)} \int_{[0,r]^2} dtds \int_{[y_0,y_0+\Delta_*]^2} dxdy \\
&\qquad\qquad \times \frac{|t-s|^{p_0p\theta_1}|x-y|^{p_0p\theta_2}}{|t-s|^{p(1+2p_0\gamma_1)}|x-y|^{p(1+2p_0\gamma_2)}} \\
&\leq c_p\,(r\Delta_*)^{2p} \Delta_\bullet^{p(p_0\theta_1-(1+2p_0\gamma_1))} \Delta_*^{p(p_0\theta_2-(1+2p_0\gamma_2))} \\
&= c_p\, r^{2p} \delta^{p(p_0\theta_1-(1+2p_0\gamma_1))\alpha/(\alpha-1)} \delta^{p(p_0\theta_2-(1+2p_0\gamma_2)+2)/(\alpha-1)} \\
&= c_p\, r^{2p} \delta^{p(p_0(\frac{\alpha\theta_1}{\alpha-1}+\frac{\theta_2}{\alpha-1})-2p_0(\frac{\alpha\gamma_1}{\alpha-1}+\frac{\gamma_2}{\alpha-1})-1)} \\
&= c_p\, r^{2p} \delta^{p(p_0-\gamma_0)},
\end{aligned} \tag{5.2.42}$$

where in the second inequality we use (5.2.14), in the first equality we use (5.1.14), in the third equality we use (5.2.21) and the fact that $\frac{\alpha\theta_1}{\alpha-1} + \frac{\theta_2}{\alpha-1} = 1$ by the definition of θ_1, θ_2 in (5.2.13).

Property (5.2.41) follows from (5.2.39) and a calculation similar to (5.2.42). □

Lemma 5.2.8. *There exists a finite positive constant c, not depending on $y_0 \in \mathbb{R}$, such that for any $\bar{a} > 0$, $\delta_1 > 0$, $\delta_2 > 0$ and for all $r \in [0, \Delta_\bullet]$,*

$$\bar{Y}_r \leq \bar{R} := c\,\bar{a}^{2p_0} \delta^{\frac{2\alpha}{\alpha-1}-\gamma_0} \quad \Rightarrow \sup_{(t,x)\in[0,r]\times[y_0,y_0+\delta_2]} |u(t,x)| \leq \bar{a}. \tag{5.2.43}$$

Proof. The proof is similar to that of Lemma 4.3.5.

Assuming $Y_0(r) \le \bar{R}$, by the Garsia, Rodemich, and Rumsey lemma (see Lemma A.6.1), we deduce, as in (5.2.31), that for all $t,\, s \in [0,r]$,

$$\begin{aligned}|u(t,y_0)-u(s,y_0)| &\le c_1\,(|t-s|^{(\alpha-1)/\alpha})^{\frac{\gamma_0}{2p_0}-\frac{\alpha}{(\alpha-1)p_0}}\, Y_0(r)^{\frac{1}{2p_0}}\\ &\le c_1\,\Delta_\bullet^{\frac{\gamma_0(\alpha-1)}{2\alpha p_0}-\frac{1}{p_0}}\, Y_0(r)^{\frac{1}{2p_0}} = c_1\,\delta^{\frac{\gamma_0}{2p_0}-\frac{\alpha}{(\alpha-1)p_0}}\, Y_0(r)^{\frac{1}{2p_0}},\end{aligned} \tag{5.2.44}$$

where the constant c_1 does not depend on r, nor on $y_0 \in \mathbb{R}$. Letting $s=0$ in (5.2.44), we obtain

$$\sup_{t\in[0,r]} |u(t,y_0)| \le c_1\,\delta^{\frac{\gamma_0}{2p_0}-\frac{\alpha}{(\alpha-1)p_0}}\, Y_0(r)^{\frac{1}{2p_0}} \le c_1\,\delta^{\frac{\gamma_0}{2p_0}-\frac{\alpha}{(\alpha-1)p_0}}\, \bar{R}^{\frac{1}{2p_0}}. \tag{5.2.45}$$

Hence we can choose a suitable constant c in the definition of $\bar{R}$ in (5.2.43) so that

$$\sup_{t\in[0,r]} |u(t,y_0)| \le \frac{\bar{a}}{2}. \tag{5.2.46}$$

Assuming $Y_1(r) \le \bar{R}$, from the expression of $Y_1(r)$ in (5.2.35), we first apply the Garsia, Rodemich, and Rumsey lemma (see Lemma A.6.2) to the $E_{p_0,\gamma_2}[y_0, y_0+\Delta_*]$-valued function $s \mapsto \check{u}(s,*)$ with $\Psi(x) = x^{2p_0}$, $p(x) = x^{(1+2p_0\gamma_1)/(2p_0)}$, $d=1$ to deduce, as in (4.3.31), that there exists a constant c_2 such that for all $t,\, s \in [0,r]$,

$$\begin{aligned}\|\check{u}(t,*)-\check{u}(s,*)\|_{p_0,\gamma_2} &\le c'\, Y_1(r)^{\frac{1}{2p_0}} \int_0^{2|t-s|} x^{-\frac{1}{p_0}}\, x^{\frac{1+2p_0\gamma_1}{2p_0}-1} dx\\ &= c_2\, Y_1(r)^{\frac{1}{2p_0}} |t-s|^{\frac{2p_0\gamma_1-1}{2p_0}}\\ &\le c_2\, Y_1(r)^{\frac{1}{2p_0}}\, \Delta_\bullet^{\frac{2p_0\gamma_1-1}{2p_0}} = c_2\, Y_1(r)^{\frac{1}{2p_0}}\, \delta^{\frac{\alpha(2p_0\gamma_1-1)}{2(\alpha-1)p_0}}.\end{aligned}$$

Letting $s=0$, we obtain for all $t \in [0,r]$,

$$\|\check{u}(t,*)\|_{p_0,\gamma_2}^{2p_0} \le c_2\, Y_1(r)\delta^{(2p_0\gamma_1-1)\alpha/(\alpha-1)}.$$

Applying the same lemma to the real-valued function $x \mapsto \check{u}(t,x)$ (t is now fixed) with $\Psi(x) = x^{2p_0}$, $p(x) = x^{(1+2p_0\gamma_2)/(2p_0)}$, we obtain

$$|\check{u}(t,x)-\check{u}(t,y)| \le c_3\, Y_1(r)^{\frac{1}{2p_0}}\, \delta^{\frac{\alpha(2p_0\gamma_1-1)}{2(\alpha-1)p_0}}\, |x-y|^{\frac{2p_0\gamma_2-1}{2p_0}}$$

for all $x,\, y \in [y_0, y_0+\Delta_*]$. Letting $y=y_0$ we obtain that for all $(t,x) \in [0,r]\times[y_0, y_0+\Delta_*]$,

$$\begin{aligned}|u(t,x)-u(t,y_0)| &\le c_3\, Y_1(r)^{\frac{1}{2p_0}}\, \delta^{\frac{\alpha(2p_0\gamma_1-1)}{2(\alpha-1)p_0}}\, \Delta_*^{\frac{2p_0\gamma_2-1}{2p_0}}\\ &= c_3\, Y_1(r)^{\frac{1}{2p_0}}\, \delta^{\frac{\alpha(2p_0\gamma_1-1)}{2(\alpha-1)p_0}}\, \delta^{\frac{2p_0\gamma_2-1}{2(\alpha-1)p_0}}\\ &= c_3\, Y_1(r)^{\frac{1}{2p_0}}\, \delta^{\frac{\gamma_0}{2p_0}-\frac{\alpha}{p_0(\alpha-1)}},\end{aligned}$$

where in the first equality we use (5.1.14), and the second equality is due to (5.2.21). In particular, this implies that

$$\sup_{(t,x)\in[0,r]\times[y_0,y_0+\delta_2]} |u(t,x)-u(t,y_0)| \le c_3\, Y_1(r)^{\frac{1}{2p_0}}\,\delta^{\frac{\gamma_0}{2p_0}-\frac{\alpha}{p_0(\alpha-1)}}. \tag{5.2.47}$$

We can choose the constant c in the definition of $\bar{R}$ in (5.2.43) small so that (5.2.46) holds and

$$\sup_{(t,x)\in[0,r]\times[y_0,y_0+\delta_2]} |u(t,x)-u(t,y_0)| \le \frac{\bar{a}}{2}. \tag{5.2.48}$$

Hence, by (5.2.46), (5.2.48) and the triangle inequality, we obtain (5.2.43). □

We conclude this section by presenting a technical result on the uniqueness of the solution to the fractional heat equation, which will allow us later on to verify the conditions in Theorem 1.5.5(iii). We first check that the fractional differential operator D^α maps $C_0^\infty(\mathbb{R})$ into the space of infinitely differentiable functions with finite moment of all orders.

Lemma 5.2.9. *For $g\in C_0^\infty(\mathbb{R})$, $D^\alpha g$ belongs to $C^\infty(\mathbb{R})\cap L^p(\mathbb{R})$, for all $p\ge 1$.*

Proof. By definition,

$$\begin{aligned}
D^\alpha g(x) &= c_\alpha \int_{\mathbb{R}} \frac{g(x+y)-g(x)-yg'(x)}{|y|^{1+\alpha}}\,dy\\
&= c_\alpha \int_{|y|\le 1} \frac{g(x+y)-g(x)-yg'(x)}{|y|^{1+\alpha}}\,dy + c_\alpha \int_{|y|>1} \frac{g(x+y)-g(x)-yg'(x)}{|y|^{1+\alpha}}\,dy\\
&:= g_1(x)+g_2(x).
\end{aligned}$$

We just prove that the function g_1 is in $C^\infty(\mathbb{R})$ and the proof for g_2 is similar. By the remainder formula in Taylor's expansion,

$$|g'(x+y)-g'(x)-yg''(x)| \le \frac{1}{2}\sup_{x\in\mathbb{R}}|g^{(3)}(x)|y^2.$$

Since $1<\alpha<2$, we have

$$\int_{|y|\le 1} \frac{y^2}{|y|^{1+\alpha}}\,dy<\infty.$$

Applying the dominated convergence theorem (see [75, Theorem 5, Chapter 5]), we can differentiate under the integral sign for the function g_1. Hence g_1 is differentiable. We can repeat this argument to conclude that g_1 is infinitely differentiable. Similarly, g_2 is also infinitely differentiable.

In order to prove that $D^\alpha g$ belongs to $L^p(\mathbb{R})$, it suffices to prove that the function $x\mapsto \int_{|y|>1}\frac{g(x+y)}{|y|^{1+\alpha}}dy$ belongs to $L^p(\mathbb{R})$ since the other parts of $D^\alpha g$ are infinitely differentiable and compactly supported. By Hölder's inequality and Fubini's theorem,

$$\begin{aligned}\int_{\mathbb{R}}\left|\int_{|y|>1}\frac{|g(x+y)|}{|y|^{1+\alpha}}dy\right|^p dx &\leq \left|\int_{|y|>1}\frac{1}{|y|^{1+\alpha}}dy\right|^{p-1}\int_{\mathbb{R}}\int_{|y|>1}\frac{|g(x+y)|^p}{|y|^{1+\alpha}}dydx\\ &= c\int_{|y|>1}dy\frac{1}{|y|^{1+\alpha}}\int_{\mathbb{R}}|g(x+y)|^p dx\\ &= c\int_{|y|>1}\frac{1}{|y|^{1+\alpha}}dy\int_{\mathbb{R}}|g(x)|^p dx<\infty,\end{aligned}$$

which completes the proof. □

Let $g \in C_0^\infty(\mathbb{R})$ and $f : [0,\infty[\mapsto \mathbb{R}$ be a differentiable function with continuous derivative satisfying $f(0) = 0$. We define

$$\begin{aligned}&A(t,x) = \int_0^t\int_{\mathbb{R}} G_\alpha(t-r,x-v)\left(\frac{\partial}{\partial r} - {}_vD^\alpha\right)(f(r)g(v))dvdr, \quad t>0,\, x\in\mathbb{R},\\ &A(0,x) = 0, \quad x\in\mathbb{R}.\end{aligned}$$

Lemma 5.2.10. *The function A is well-defined and we have $A(t,x) = f(t)g(x)$, for all $(t,x) \in [0,\infty[\times\mathbb{R}$.*

Proof. By Lemma 5.2.9, the function

$$(r,v)\mapsto\left(\frac{\partial}{\partial r} - {}_vD^\alpha\right)(f(r)g(v))$$

belongs to $L^2([0,T]\times\mathbb{R})$, and so does the Green kernel. Hence A is well-defined. Fix $t>0$. We are going to use the L^1-Fourier transform to prove this identity. For this, we first show that

$$\int_{\mathbb{R}}|A(t,x)|dx<\infty. \tag{5.2.49}$$

Indeed, by Fubini's theorem,

$$\begin{aligned}\int_{\mathbb{R}}|A(t,x)|dx &\leq \int_{\mathbb{R}}dx\int_0^t dr\int_{\mathbb{R}}dv\,G_\alpha(t-r,x-v)|f'(r)|\,|g(v)|\\ &\quad+\int_{\mathbb{R}}dx\int_0^t dr\int_{\mathbb{R}}dv\,G_\alpha(t-r,x-v)|f(r)|\,|D^\alpha g(v)|\\ &\leq c\int_0^t dr\int_{\mathbb{R}}dv|g(v)|\int_{\mathbb{R}}G_\alpha(t-r,x-v)dx\\ &\quad+c\int_0^t dr\int_{\mathbb{R}}dv|D^\alpha g(v)|\int_{\mathbb{R}}G_\alpha(t-r,x-v)dx\\ &= ct\int_{\mathbb{R}}(|g(v)|+|D^\alpha g(v)|)dv<\infty,\end{aligned}$$

where the last inequality is due to Lemma 5.2.9. Therefore, by Fubini's theorem,

$$\begin{aligned}\mathscr{F}A(t,\cdot)(\xi) &= \int_{\mathbb{R}} e^{-ix\xi}A(t,x)dx \\ &= \int_{\mathbb{R}} dx\, e^{-ix\xi}\int_0^t dr\int_{\mathbb{R}} dv\, G_\alpha(t-r,x-v)[f'(r)g(v)-f(r)D^\alpha g(v)] \\ &= \int_0^t dr\int_{\mathbb{R}} dv\left(\int_{\mathbb{R}} e^{-ix\xi}G_\alpha(t-r,x-v)dx\right)[f'(r)g(v)-f(r)D^\alpha g(v)].\end{aligned}$$

Using the Fourier transform of the Green kernel (see (5.1.3)), the above integral is equal to

$$\begin{aligned}&\int_0^t dr\int_{\mathbb{R}} dv\, e^{-iv\xi}e^{-(t-r)|\xi|^\alpha}[f'(r)g(v)-f(r)D^\alpha g(v)] \\ &\quad= \int_0^t dr\, e^{-(t-r)|\xi|^\alpha}[f'(r)\mathscr{F}g(\xi)-f(r)\mathscr{F}(D^\alpha g)(\xi)] \\ &\quad= \int_0^t dr\, e^{-(t-r)|\xi|^\alpha}[f'(r)\mathscr{F}g(\xi)+f(r)|\xi|^\alpha\mathscr{F}g(\xi)] \\ &\quad= \mathscr{F}g(\xi)\int_0^t e^{-(t-r)|\xi|^\alpha}f'(r)dr+|\xi|^\alpha\mathscr{F}g(\xi)\int_0^t e^{-(t-r)|\xi|^\alpha}f(r)dr,\end{aligned}\tag{5.2.50}$$

where in the second equality we use (2.1.2). Integrating by parts, the first integral in (5.2.50) is equal to

$$\begin{aligned}&\mathscr{F}g(\xi)(f(t)-f(0)e^{-t|\xi|^\alpha})-|\xi|^\alpha\mathscr{F}g(\xi)\int_0^t e^{-(t-r)|\xi|^\alpha}f(r)dr \\ &\quad= \mathscr{F}g(\xi)f(t)-|\xi|^\alpha\mathscr{F}g(\xi)\int_0^t e^{-(t-r)|\xi|^\alpha}f(r)dr\end{aligned}$$

since $f(0)=0$, which implies that

$$\mathscr{F}A(t,\cdot)(\xi) = f(t)\mathscr{F}g(\xi) = \mathscr{F}(f(t)g(\cdot))(\xi).\tag{5.2.51}$$

Hence for every $t>0$, $A(t,x)=f(t)g(x)$ for almost every $x\in\mathbb{R}$. On the other hand, by the Cauchy-Schwarz inequality and (4.2.3),

$$\begin{aligned}|A(t,x)-A(t,y)| &\le \int_0^t dr\int_{\mathbb{R}} dv\,|G_\alpha(t-r,x-v)-G_\alpha(t-r,y-v)| \\ &\qquad\qquad \times|f'(r)g(v)-f(r)D^\alpha g(v)| \\ &\le c\left(\int_0^t\int_{\mathbb{R}}|G_\alpha(t-r,x-v)-G_\alpha(t-r,y-v)|^2dvdr\right)^{1/2} \\ &\quad\times\left(\int_{\mathbb{R}}(|g(v)|+|D^\alpha g(v)|)^2dv\right)^{1/2} \\ &\le c|x-y|^{(\alpha-1)/2},\end{aligned}$$

which shows that for every $t>0$, the function $x\mapsto A(t,x)$ is continuous. Hence $A(t,x)=f(t)g(x)$ for all $(t,x)\in[0,\infty[\times\mathbb{R}$. □

5.3 Smoothness of the densities

It is clear that the first component of F in (5.1.6) belongs to $\mathbb{D}^\infty$. For the second component of F and the random variable M_0, we have the following result.

Lemma 5.3.1. *(a) The random variable F_2 takes values in $]0,\infty[$ a.s. Moreover, it belongs to $\mathbb{D}^{1,2}$ and*

$$DF_2 = 1_{\{\cdot<S\}} G_\alpha(S-\cdot, y_0-*) - 1_{\{\cdot<s_0\}} G_\alpha(s_0-\cdot, y_0-*), \tag{5.3.1}$$

where $S \in]s_0, s_0+\delta_1]$ is the unique point where the maximum that defines F_2 is attained.
(b) The random variable M_0 takes values in $]0,\infty[$ a.s. and belongs to $\mathbb{D}^{1,2}$, and

$$DM_0 = 1_{\{\cdot<\bar{S}\}} G_\alpha(\bar{S}-\cdot, \bar{X}-*), \tag{5.3.2}$$

where $(\bar{S}, \bar{X}) \in]0,\delta_1] \times [y_0, y_0+\delta_2]$ is the unique point where the maximum that defines M_0 is attained.

Proof. First, the strict positivity of F_2 (when $s_0 = 0$) and M_0 is a consequence of the 0-1 law; see also the arguments in the proof of Lemma 4.4.2. When $s_0 > 0$, (5.2.2) implies that $F_2 > 0$ a.s.

The proof of (5.3.1) and (5.3.2) is similar to that of Lemma 4.4.4. We have to show that the maximums F_2 and M_0 are attained at a unique point almost surely. For the random variable F_2, by [47, Lemma 2.6], it suffices to check that for any t, $s \in]0,\infty[$ with $t \neq s$,

$$\mathrm{E}[|u(t,y_0) - u(s,y_0)|^2] > 0. \tag{5.3.3}$$

Assuming $t > s$ without loss of generality, by (A.2),

$$\begin{aligned}\mathrm{E}[|u(t,y_0) - u(s,y_0)|^2] &\geq \int_s^t \int_{\mathbb{R}} G_\alpha^2(t-r, y_0-v)\, dv\, dr \\ &= \int_0^{t-s} G_\alpha(2r,0)\, dr = c_\alpha (t-s)^{\frac{\alpha-1}{\alpha}} > 0.\end{aligned}$$

Therefore, the maximum F_2 is attained at a unique point in $[s_0, s_0+\delta_1]$ almost surely. The proof of (5.3.1) is similar to that of (4.4.9).

For the random variable M_0, by [47, Lemma 2.6], it suffices to check that for any (t,x), $(s,y) \in]0,\infty[\times\mathbb{R}$ with $(t,x) \neq (s,y)$

$$\mathrm{E}[|u(t,x) - u(s,y)|^2] > 0. \tag{5.3.4}$$

If $t \neq s$, assuming $t > s$ without loss of generality, we have

$$\begin{aligned}\mathrm{E}[|u(t,x)-u(s,y)|^2] &\geq \int_s^t \int_{\mathbb{R}} G_\alpha^2(t-r,x-v)\,dv\,dr \\ &= \int_0^{t-s} G_\alpha(2r,0)\,dr = c_\alpha (t-s)^{\frac{\alpha-1}{\alpha}} > 0,\end{aligned}$$

by (A.2). If $t = s, x \neq y$, by the Plancherel theorem, we have

$$\begin{aligned}\mathrm{E}[|u(t,x)-u(t,y)|^2] &= \int_0^t \int_{\mathbb{R}} (G_\alpha(t-r,x-v) - G_\alpha(t-r,y-v))^2\,dv\,dr \\ &= \int_0^t \int_{\mathbb{R}} (G_\alpha(r,v) - G_\alpha(r,y-x+v))^2\,dv\,dr \\ &= \frac{1}{2\pi}\int_0^t \int_{\mathbb{R}} e^{-2r|\lambda|^\alpha} \left|1-e^{i\lambda(x-y)}\right|^2 d\lambda\,dr > 0,\end{aligned}$$

since the Lebesgue measure of $\{\lambda : 1-e^{i\lambda(x-y)} = 0\}$ is zero. □

We proceed to construct the random variables needed for Theorem 1.5.5. For $(z_1, z_2) \in \mathbb{R}\times]0,\infty[$, set

$$a = z_2/2 \quad \text{and} \quad A = \mathbb{R}\times]a,\infty[. \tag{5.3.5}$$

Let $\psi : \mathbb{R} \mapsto 1$ be the infinitely differentiable function defined in (4.5.17), where R is defined in Lemma 5.2.6 with a as in (5.3.5). Hence we have

$$\|\psi'\|_\infty := \sup_{x\in\mathbb{R}} |\psi'(x)| \leq c\,R^{-1} \tag{5.3.6}$$

for a certain constant c not depending on z_2.

If $I \times J \subset]0,T] \times \mathbb{R}$, let c_1, C_1, c_2, C_2 be as in (5.1.8) and (5.1.9) and $f_0 : \mathbb{R} \mapsto [0,1]$ be an infinitely differentiable function supported in $[c_1/2, (C_1+1)/2]$ such that $f_0(t) = 1$, for all $t \in [c_1, C_1]$. Let $g_0 : \mathbb{R} \mapsto [0,1]$ be an infinitely differentiable function supported in $[c_2 - 1, C_2 + 1]$ such that $g_0(x) = 1$, for all $x \in [c_2, C_2]$. We define the $\mathscr{H}$-valued random variable u_A^1 evaluated at (r, v) by

$$u_A^1(r,v) = \left(\frac{\partial}{\partial r} - {}_vD^\alpha\right)(f_0(r)g_0(v)). \tag{5.3.7}$$

By Lemma 5.2.9 and the definition of the functions f_0 and g_0, there exists constant c such that for all $(s_0, y_0) \in I \times J \subset]0,T] \times \mathbb{R}$,

$$\|u_A^1\|_{\mathscr{H}} \leq c. \tag{5.3.8}$$

We define the function ϕ_{δ_1} as

$$\phi_{\delta_1}(v) := \phi_0\left(\frac{v-y_0}{\delta_1^{1/\alpha}}\right), \quad \text{for all } v \in \mathbb{R}, \tag{5.3.9}$$

where, as below (4.5.20), $\phi_0 : \mathbb{R} \mapsto [0,1]$ is an infinitely differentiable function supported in $[-1,2]$ such that $\phi_0(v) = 1$, for all $v \in [0,1]$. Set

$$H(r,v) := \phi_{\delta_1}(v)\int_{s_0}^{r} \psi(Y_a)\,da, \quad (r,v) \in [s_0, s_0+\delta_1] \times \mathbb{R}. \tag{5.3.10}$$

We define the $\mathscr{H}$-valued random variables u_A^2 evaluated at (r,v) by

$$u_A^2(r,v) = \begin{cases} \left(\frac{\partial}{\partial r} - {}_vD^{\alpha}\right)H(r,v) & \text{if } (r,v) \in]s_0, s_0+\delta_1] \times \mathbb{R}; \\ 0 & \text{otherwise.} \end{cases} \tag{5.3.11}$$

Finally, we define the random matrix $\gamma_A = (\gamma_A^{i,j})_{1\leq i,j\leq 2}$ by

$$\gamma_A = \begin{pmatrix} 1 & 0 \\ 0 & \int_{s_0}^{s_0+\delta_1} \psi(Y_r)\,dr \end{pmatrix}. \tag{5.3.12}$$

If $s_0 = 0 \in I \subset [0,T]$, we only consider the random variables F_2, u_A^2 and $\gamma_A^{2,2}$ defined in (5.1.6), (5.3.11) and (5.3.12) with $s_0 = 0$, respectively.

We next introduce the random variables needed to prove the smoothness of density of M_0. For $z \in]0,\infty[$, set

$$\bar{a} = z/2 \quad \text{and} \quad \bar{A} =]\bar{a}, \infty[. \tag{5.3.13}$$

Let $\bar{\psi} : \mathbb{R} \mapsto 1$ be the infinitely differentiable function defined in (4.5.27), where $\bar{R}$ is defined in Lemma 5.2.8 with $\bar{a}$ as in (5.3.13). Hence we have

$$\|\bar{\psi}'\|_\infty := \sup_{x\in\mathbb{R}} |\bar{\psi}'(x)| \leq c\bar{R}^{-1} \tag{5.3.14}$$

for a certain constant c not depending on z.

We define the function $\bar{\phi}_\delta$ by

$$\bar{\phi}_\delta(v) := \phi_0\left(\frac{v-y_0}{\Delta_*}\right) = \phi_0\left(\frac{v-y_0}{\delta^{1/(\alpha-1)}}\right), \quad v \in \mathbb{R}, \tag{5.3.15}$$

where the function ϕ_0 is specified below (5.3.9), so that for some constant c,

$$|\bar{\phi}_\delta'(v)| \leq c\delta^{-\frac{1}{\alpha-1}} \quad \text{and} \quad |\bar{\phi}_\delta''(v)| \leq c\delta^{-\frac{2}{\alpha-1}}, \quad \text{for all } v \in \mathbb{R}. \tag{5.3.16}$$

Set

$$\bar{H}(r,v) := \bar{\phi}_\delta(v)\int_0^r \bar{\psi}(\bar{Y}_a)da, \quad (r,v)\in[0,\Delta_\bullet]\times[0,1], \tag{5.3.17}$$

where $\{\bar{Y}_r : r\in[0,\Delta_\bullet]\}$ is defined in (5.2.37). We define the $\mathscr{H}$-valued random variable $u_{\bar{A}}$ evaluated at (r,v) by

$$u_{\bar{A}}(r,v) = \begin{cases} \left(\frac{\partial}{\partial r} - {}_vD^\alpha\right)\bar{H}(r,v) & \text{if } (r,v)\in]0,\Delta_\bullet]\times\mathbb{R}; \\ 0 & \text{otherwise.}\end{cases} \tag{5.3.18}$$

Finally, we define the random variable

$$\gamma_{\bar{A}} = \int_0^{\Delta_\bullet} \bar{\psi}(\bar{Y}_r)dr. \tag{5.3.19}$$

The random variables introduced above are smooth, as required in Theorem 1.5.5.

Lemma 5.3.2. *For $i,j\in\{1,2\}$, $u_A^i\in\mathbb{D}^\infty(\mathscr{H})$, $\gamma_A^{i,j}\in\mathbb{D}^\infty$ and $u_{\bar{A}}\in\mathbb{D}^\infty(\mathscr{H})$, $\gamma_{\bar{A}}\in\mathbb{D}^\infty$.*

Proof. The proof is similar to that of Lemma 4.5.3. First, by discretization and (5.2.29), we see that $\gamma_A^{2,2}$ belongs to $\mathbb{D}^\infty$. On the other hand, we write

$$u_A^2(r,v) = \psi(Y_r)1_{[s_0,s_0+\delta_1]}(r)\phi_{\delta_1}(v) - 1_{[s_0,s_0+\delta_1]}(r)D^\alpha\phi_{\delta_1}(v)\int_{s_0}^r \psi(Y_a)da. \tag{5.3.20}$$

Since $v\mapsto D^\alpha\phi_{\delta_1}(v)$ belongs to $L^p(\mathbb{R})$ for all $p\geq 1$ by Lemma 5.2.9, we can use the same argument as in the proof of Lemma 4.5.3, to see that the discretization of u_A^2 converges to u_A^2 in $L^p(\Omega,\mathscr{H})$ for any $p\geq 1$, and its Malliavin derivative converges to Du_A^2 in $L^p(\Omega,\mathscr{H}^{\otimes 2})$ for any $p\geq 1$. We repeat this procedure and conclude that u_A^2 belongs to $\mathbb{D}^\infty(\mathscr{H})$.

Similarly, we can show that $u_{\bar{A}}\in\mathbb{D}^\infty(\mathscr{H})$ and $\gamma_{\bar{A}}\in\mathbb{D}^\infty$. □

In the following, we check that the random variables defined above satisfy the conditions in Theorem 1.5.5(iii).

Lemma 5.3.3. *(a) On the event $\{F\in A\}=\{F_2>a\}$, we have $\langle DF_i, u_A^j\rangle_{\mathscr{H}} = \gamma_A^{i,j}$ for $i,j\in\{1,2\}$.*
(b) On the event $\{M_0\in\bar{A}\}=\{M_0>\bar{a}\}$, $\langle DM_0, u_{\bar{A}}\rangle_{\mathscr{H}} = \gamma_{\bar{A}}$.

Proof. We start by proving (a). If $s_0>0$, by Lemma 5.2.10, we have

$$\begin{aligned}\langle DF_1, u_A^1\rangle_{\mathscr{H}} &= \int_0^{s_0}\int_{\mathbb{R}} G_\alpha(s_0-r, y_0-v)\left(\frac{\partial}{\partial r} - {}_vD^\alpha\right)(f_0(r)g_0(v))drdv \\ &= f_0(s_0)g_0(y_0) = 1 = \gamma_A^{1,1}.\end{aligned}$$

Obviously,

$$\langle DF_1, u_A^2\rangle_{\mathscr{H}} = \int_0^{s_0}\int_{\mathbb{R}} G_\alpha(s_0-r, y_0-v)\left(\frac{\partial}{\partial r} - {}_v D^\alpha\right) H(r,v)1_{\{s_0<r\le s_0+\delta_1\}}\,dr\,dv = 0.$$

Moreover, using Lemmas 5.3.1(a) and 5.2.10, we have

$$\begin{aligned}\langle DF_2, u_A^1\rangle_{\mathscr{H}} &= \int_0^{S}\int_{\mathbb{R}} G_\alpha(S-r, y_0-v)\left(\frac{\partial}{\partial r} - {}_v D^\alpha\right)(f_0(r)g_0(v))\,dr\,dv\\ &\quad -\int_0^{s_0}\int_{\mathbb{R}} G_\alpha(s_0-r, y_0-v)\left(\frac{\partial}{\partial r} - {}_v D^\alpha\right)(f_0(r)g_0(v))\,dr\,dv\\ &= f_0(S)g_0(y_0) - f_0(s_0)g_0(y_0) = 1-1=0,\end{aligned}$$

where the last equality is due to the definition of the function f_0 and g_0 since $S\in[s_0, s_0+\delta_1]$. Furthermore, for both cases $s_0>0$ and $s_0=0$,

$$\begin{aligned}\langle DF_2, u_A^2\rangle_{\mathscr{H}} &= \int_0^S dr\int_{\mathbb{R}} dv\, G_\alpha(S-r, y_0-v)u_A^2(r,v)\\ &\quad -\int_0^{s_0} dr\int_{\mathbb{R}} dv\, G_\alpha(s_0-r, y_0-v)u_A^2(r,v)\\ &= \int_{s_0}^S dr\int_{\mathbb{R}} dv G_\alpha(S-r, y_0-v)\left(\frac{\partial}{\partial r} - {}_v D^\alpha\right) H(r,v)\\ &= \int_0^{S-s_0} dr\int_{\mathbb{R}} dv G_\alpha(S-s_0-r, y_0-v)\left(\frac{\partial}{\partial r} - {}_v D^\alpha\right) H(r+s_0,v)\\ &= H(S,y_0),\end{aligned}$$

where the last equality follows from Lemma 5.2.10 by using the fact that $H(s_0, v)=0$ for all $v\in\mathbb{R}$. Therefore,

$$\langle DF_2, u_A^2\rangle_{\mathscr{H}} = \phi_{\delta_1}(y_0)\int_{s_0}^S \psi(Y_r)\,dr = \int_{s_0}^S \psi(Y_r)\,dr, \tag{5.3.21}$$

where in the second equality we use the fact $\phi_{\delta_1}(y_0)=1$. Moreover, on the event $\{F\in A\} = \{F_2>a\}$, we observe that if $r>S$, then $\psi(Y_r)=0$. Otherwise we would have $\psi(Y_r)>0$, hence $Y_r\le R$, and by Lemma 5.2.6, this would imply that

$$F_2 = \bar{u}(S, y_0) = \sup_{t\in[s_0,r]} \bar{u}(t, y_0) \le a < F_2,$$

which is a contradiction. Hence on $\{F\in A\}$, the last integral in (5.3.21) is equal to

$$\int_{s_0}^{s_0+\delta_1}\psi(Y_r)\,dr = \gamma_A^{2,2}.$$

We proceed to prove (b). By Lemma 5.3.1,

$$
\begin{aligned}
\langle DM_0, u_{\bar{A}}\rangle_{\mathscr{H}} &= \langle 1_{\{\cdot<\bar{S}\}}(G_\alpha(\bar{S}-\cdot,\bar{X}-*), u_{\bar{A}}\rangle_{\mathscr{H}} \\
&= \int_0^{\bar{S}}\int_{\mathbb{R}} G_\alpha(\bar{S}-r,\bar{X}-v)\left(\frac{\partial}{\partial r} - {}_vD^\alpha\right)\bar{H}(r,v)\,dv\,dr \\
&= \bar{H}(\bar{S},\bar{X}) = \bar{\phi}_\delta(\bar{X})\int_0^{\bar{S}}\bar{\psi}(\bar{Y}_r)\,dr. \qquad (5.3.22)
\end{aligned}
$$

Since $\bar{X}\in[y_0, y_0+\delta_2]$, by the definition of the function $\bar{\phi}_\delta$, it implies that $\bar{\phi}_\delta(\bar{X})\equiv 1$.Hence,

$$
\langle DM_0, u_{\bar{A}}\rangle_{\mathscr{H}} = \int_0^{\bar{S}}\bar{\psi}(\bar{Y}_r)\,dr.
$$

On the event $\{M_0 > \bar{a}\}$, for $r > \bar{S}$, we have $\bar{\psi}(\bar{Y}_r) = 0$. Otherwise, we would have $\bar{\psi}(\bar{Y}_r) > 0$, hence $\bar{Y}_r \le \bar{R}$ and by Lemma 5.2.8 this implies that

$$
M_0 = u(\bar{S},\bar{X}) = \sup_{(t,x)\in[0,r]\times[y_0,y_0+\delta_2]} u(t,x) \le \bar{a} < M_0,
$$

which is a contradiction. Therefore, on the event $\{M_0\in\bar{A}\}$,

$$
\langle DM_0, u_{\bar{A}}\rangle_{\mathscr{H}} = \int_0^{\Delta_\bullet}\bar{\psi}(\bar{Y}_r)\,dr = \gamma_{\bar{A}}.
$$

This completes the proof. □

The last ingredient for the smoothness of the densities of F and M_0 is the finite negative moments of $\gamma_A^{2,2}$ and $\gamma_{\bar{A}}$.

Lemma 5.3.4. *(a) The random variable $\gamma_A^{2,2}$ has finite negative moments of all orders. Furthermore, for any $p\ge 1$, there exists a constant c_p, not depending on $(s_0, y_0)\in I\times J$, such that for all $\delta_1 > 0$ and for all $z_2 \ge \delta_1^{(\alpha-1)/(2\alpha)}$,*

$$
\|(\gamma_A^{2,2})^{-1}\|_{L^p(\Omega)} \le c_p\,\delta_1^{-1}. \qquad (5.3.23)
$$

(b) The random variable $\gamma_{\bar{A}}$ has finite negative moments of all orders. Furthermore, for any $p\ge 1$, there exists a constant c_p, not depending on $y_0\in J$, such that for all small $\delta_1,\delta_2 > 0$ and for $z \ge (\delta_1^{(\alpha-1)/\alpha}+\delta_2^{\alpha-1})^{1/2}$,

$$
\|\gamma_{\bar{A}}^{-1}\|_{L^p(\Omega)} \le c_p\,(\delta_1^{(\alpha-1)/\alpha}+\delta_2^{\alpha-1})^{-\frac{\alpha}{\alpha-1}}. \qquad (5.3.24)
$$

Proof. The proof is similar to that of Lemma 4.5.4. We need to pay attention to the exponents in the calculation.

We first prove (a). By the definition of the function ψ,

$$\gamma_A^{2,2} \geq \int_{s_0}^{s_0+\delta_1} 1_{\{Y_r \leq \frac{R}{2}\}} dr := \bar{X}.$$

For $0 < \epsilon < \delta_1$ and any $q \geq 1$, since $r \mapsto Y_r$ is increasing, we have

$$\begin{aligned}
\mathrm{P}\{\bar{X} < \epsilon\} &\leq \mathrm{P}\{Y_{s_0+\epsilon} \geq R/2\} \\
&\leq (2/R)^q \mathrm{E}[|Y_{s_0+\epsilon}|^q] \leq c_q R^{-q} \epsilon^{2q} \delta_1^{(p_0-\gamma_0)(\alpha-1)q/\alpha},
\end{aligned}$$

where in the second inequality we use (5.2.23). which shows that the random variable $\gamma_A^{2,2}$ has finite negative moments of all orders by [24, Chapter 3, Lemma 4.4]. Moreover, for any $p \geq 1$ and $q > \frac{p}{2}$,

$$\begin{aligned}
\mathrm{E}[\bar{X}^{-p}] &= p \int_0^\infty y^{p-1} \mathrm{P}(\bar{X}^{-1} > y) dy \\
&= p \int_0^{1/\delta_1} y^{p-1} \mathrm{P}(\bar{X}^{-1} > y) dy + p \int_{1/\delta_1}^\infty y^{p-1} \mathrm{P}(\bar{X}^{-1} > y) dy \\
&\leq c_p \frac{1}{\delta_1^p} + c_p R^{-q} \delta_1^{(p_0-\gamma_0)(\alpha-1)q/\alpha} \int_{1/\delta_1}^\infty y^{p-1} y^{-2q} dy \\
&= c_p \delta_1^{-p} + c_p R^{-q} \delta_1^{(p_0-\gamma_0)(\alpha-1)q/\alpha} \delta_1^{2q-p} \\
&= c_p \delta_1^{-p} \left(1 + R^{-q} \delta_1^{(p_0-\gamma_0)(\alpha-1)q/\alpha} \delta_1^{2q}\right) \\
&= c_p \delta_1^{-p} \left(1 + a^{-2p_0 q} \delta_1^{\gamma_0(\alpha-1)q/\alpha - 2q} \delta_1^{(p_0-\gamma_0)(\alpha-1)q/\alpha} \delta_1^{2q}\right),
\end{aligned}$$

where the last equality uses the definition of R in (5.2.30). Using (5.3.5) and the assumption $z_2 \geq \delta_1^{(\alpha-1)/(2\alpha)}$, this is bounded above by

$$\begin{aligned}
&c_p \delta_1^{-p} \left(1 + \delta_1^{\frac{\alpha-1}{2\alpha} \times (-2p_0 q)} \delta_1^{\gamma_0(\alpha-1)q/\alpha - 2q} \delta_1^{(p_0-\gamma_0)(\alpha-1)q/\alpha} \delta_1^{2q}\right) \\
&\quad = c_p \delta_1^{-p}.
\end{aligned}$$

Therefore, we have proved (5.3.23).

We proceed to prove (b). Similarly, by the definition of the function $\bar{\psi}$,

$$\gamma_{\bar{A}} \geq \int_0^{\Delta_\bullet} 1_{\{\bar{Y}_r \leq \frac{\bar{R}}{2}\}} dr := \tilde{X}.$$

For any $0 < \epsilon < \Delta_\bullet$, since $r \mapsto \bar{Y}_r$ is increasing,

$$\begin{aligned}
\mathrm{P}\{\tilde{X} < \epsilon\} &\leq \mathrm{P}\{\bar{Y}_\epsilon \geq \bar{R}/2\} \\
&\leq (2/\bar{R})^q \mathrm{E}[\bar{Y}_\epsilon^q] \leq c_q \bar{R}^{-q} \epsilon^{2q} \delta^{(p_0-\gamma_0)q}, \qquad (5.3.25)
\end{aligned}$$

where, in the last inequality, we use (5.2.38). Hence the random variable $\gamma_{\bar{A}}$ has finite negative moments of all orders. Moreover, for any $p \geq 1$ and $q > p/2$,

$$\begin{aligned}
\mathrm{E}[\tilde{X}^{-p}] &= p\int_0^\infty y^{p-1}\mathrm{P}(\tilde{X}^{-1} > y)dy \\
&= p\int_0^{\Delta_\bullet^{-1}} y^{p-1}\mathrm{P}(\tilde{X}^{-1} > y)dy + p\int_{\Delta_\bullet^{-1}}^\infty y^{p-1}\mathrm{P}(\tilde{X}^{-1} > y)dy \\
&\leq c\Delta_\bullet^{-p} + c\bar{R}^{-q}\delta^{(p_0-\gamma_0)q}\int_{\Delta_\bullet^{-1}}^\infty y^{p-1}y^{-2q}dy \\
&= c\Delta_\bullet^{-p} + c\bar{R}^{-q}\delta^{(p_0-\gamma_0)q}\Delta_\bullet^{2q-p}.
\end{aligned}$$

Using the definition of $\bar{R}$ in (5.2.43), this is equal to

$$c\Delta_\bullet^{-p}\left(1+\bar{a}^{-2p_0q}\delta^{-(\frac{2\alpha}{\alpha-1}-\gamma_0)q}\delta^{(p_0-\gamma_0)q}\Delta_\bullet^{2q}\right).$$

Under the assumption $z \geq \delta^{1/2} = (\delta_1^{(\alpha-1)/\alpha}+\delta_2^{\alpha-1})^{1/2}$, by (5.3.13) and (5.1.14), this is bounded above by

$$c\Delta_\bullet^{-p}\left(1+\delta^{-\frac{1}{2}\times 2p_0q}\delta^{-(\frac{2\alpha}{\alpha-1}-\gamma_0)q}\delta^{(p_0-\gamma_0)q}\Delta_\bullet^{2q}\right) = 2c\Delta_\bullet^{-p},$$

which implies (5.3.24). □

Proof of Theorem 5.1.1(a). The positivity of F_2 has been proved in Lemmas 5.3.1(a). For $(s_0, y_0) \in I\times J \subset [0,T]\times\mathbb{R}$ with $s_0 > 0$, by Lemmas 5.3.1(a), 5.3.2, 5.3.3(a) and 5.3.4(a), the random vector u_A and the random matrix γ_A introduced in this section satisfy the conditions in Theorem 1.5.5. Hence the random vector F possesses an infinitely differentiable density on $\mathbb{R}\times]z_2/2,\infty[$. Since the choice of z_2 is arbitrary, it has an infinitely differentiable density on $\mathbb{R}\times]0,\infty[$. Similarly, if $s_0 = 0$, we apply these lemmas for u_A^2 and $\gamma_A^{2,2}$ and Theorem 1.5.5 to conclude that the random variable F_2 has an infinitely differentiable density on $]0,\infty[$. □

Proof of Theorem 5.1.1(b). The proof is similar to that of Theorem 5.1.1(a) by using Lemmas 5.3.1(b), 5.3.2, 5.3.3(b), 5.3.4(b) and Theorem 1.5.5. □

Proposition 5.3.5. *(a) The probability density function of F at $(z_1, z_2) \in \mathbb{R}\times]0,\infty[$ is given by*

$$p(z_1,z_2) = E\left[1_{\{F_1>z_1,F_2>z_2\}}\delta\left(u_A^1\delta\left(u_A^2/\gamma_A^{2,2}\right)\right)\right] \tag{5.3.26}$$

$$= -E[1_{\{F_1<z_1,F_2>z_2\}}\delta(\delta(u_A^2/\gamma_A^{2,2})u_A^1)]. \tag{5.3.27}$$

(b) The probability density function of M_0 at $z \in]0,\infty[$ is given by

$$p_0(z) = E[1_{\{M_0>z\}}\delta(u_{\bar{A}}/\gamma_{\bar{A}})]. \tag{5.3.28}$$

Proof. The proof is exactly the same as that of Proposition 4.5.6 and Remark 4.5.7. □

5.4 Gaussian-type upper bound on the density of F

In this section, we fix $I \times J \subset]0,T] \times \mathbb{R}$ and assume that δ_1 is small enough so that the conditions in (5.1.10) hold. Similar to the derivation of (4.6.3), we can bound the density of F by

$$p(z_1,z_2) \le \mathrm{P}\{|F_1| > |z_1|\}^{1/4}\mathrm{P}\{F_2 > z_2\}^{1/4}\|\delta(\delta(u_A^2/\gamma_A^{2,2})u_A^1)\|_{L^2(\Omega)}. \tag{5.4.1}$$

As in Lemma 4.6.1, we have the following.

Lemma 5.4.1.

$$\delta(\delta(u_A^2/\gamma_A^{2,2})u_A^1) := T_1 + T_2 - T_3 + T_4 - T_5 + T_6, \tag{5.4.2}$$

where

$$T_1 = \frac{\delta(u_A^2)}{\gamma_A^{2,2}}\delta(u_A^1), \quad T_2 = \frac{\langle D\gamma_A^{2,2}, u_A^2\rangle_{\mathscr{H}}}{(\gamma_A^{2,2})^2}\delta(u_A^1), \quad T_3 = \frac{1}{\gamma_A^{2,2}}\langle D\delta(u_A^2), u_A^1\rangle_{\mathscr{H}}, \tag{5.4.3}$$

$$T_4 = \frac{\delta(u_A^2)}{(\gamma_A^{2,2})^2}\langle D\gamma_A^{2,2}, u_A^1\rangle_{\mathscr{H}}, \quad T_5 = \frac{2\langle D\gamma_A^{2,2}, u_A^2\rangle_{\mathscr{H}}}{(\gamma_A^{2,2})^3}\langle D\gamma_A^{2,2}, u_A^1\rangle_{\mathscr{H}}, \tag{5.4.4}$$

$$T_6 = \frac{1}{(\gamma_A^{2,2})^2}\langle D\langle D\gamma_A^{2,2}, u_A^2\rangle_{\mathscr{H}}, u_A^1\rangle_{\mathscr{H}}. \tag{5.4.5}$$

Proof. The proof is identical to that of Lemma 4.6.1. □

In the remainder of the this section, we follow the same path as in Section 4.6. The main difference is that the exponents are expressed in terms of α.

Proposition 5.4.2. *(a) For any $p \ge 2$, there exists $c_p > 0$, not depending on $(s_0, y_0) \in I \times J$, such that for all $\delta_1 > 0$ and for all $z_2 \ge \delta_1^{(\alpha-1)/(2\alpha)}$,*

$$\|T_i\|_{L^p(\Omega)} \le c_p\delta_1^{(1-\alpha)/(2\alpha)}, \quad \textit{for } i \in \{1,2,3\}. \tag{5.4.6}$$

(b) T_4, T_5 and T_6 vanish.

As an immediate consequence of Lemma 5.4.1 and Proposition 5.4.2, we obtain the following.

Proposition 5.4.3. *There exists a finite positive constant c, not depending on $(s_0, y_0) \in I \times J$, such that for all $\delta_1 > 0$ and for all $z_2 \ge \delta_1^{(\alpha-1)/(2\alpha)}$,*

$$\|\delta(\delta(u_A^2/\gamma_A^{2,2})u_A^1)\|_{L^2(\Omega)} \le c\,\delta_1^{(1-\alpha)/(2\alpha)}. \tag{5.4.7}$$

The proof of Proposition 5.4.2 is divided into two subsections. Throughout the remainder of

Section 5.4, we assume that

$$z_2 \geq \delta_1^{(\alpha-1)/(2\alpha)}. \tag{5.4.8}$$

Recalling the definition of R in (5.2.30), under the assumption (5.4.8), we see from (5.3.5) that

$$\begin{aligned} R^{-1} &= c^{-1} a^{-2p_0} \delta_1^{\gamma_0(\alpha-1)/\alpha-2} = c' z_2^{-2p_0} \delta_1^{\gamma_0(\alpha-1)/\alpha-2} \\ &\leq c\,\delta_1^{(\gamma_0-p_0)(\alpha-1)/\alpha-2}. \end{aligned} \tag{5.4.9}$$

5.4.1 Proof of Proposition 5.4.2(a)

We first give an estimate on the moments of T_1.

We denote by L_a^2 the closed subspace of $L^2(\Omega \times [0,T] \times \mathbb{R})$ formed by those processes which are adapted to the filtration $\{\mathscr{F}_t := \sigma\{W(s,x) : s \leq t, x \in \mathbb{R}\}, t \in [0,T]\}$.

Lemma 5.4.4. *For any $p \geq 1$, there exists a constant c_p, not depending on $(s_0, y_0) \in I \times J$, such that for all $\delta_1 > 0$,*

$$\|\delta(u_A^2)\|_{L^p(\Omega)} \leq c_p \delta_1^{(1+\alpha)/(2\alpha)}. \tag{5.4.10}$$

Proof. From the definition of u_A^2, for $(r,v) \in [s_0, s_0+\delta_1] \times \mathbb{R}$,

$$u_A^2(r,v) = \phi_{\delta_1}(v)\psi(Y_r) - D^\alpha \phi_{\delta_1}(v) \int_{s_0}^r \psi(Y_a)\,da.$$

Since u_A^2 is adapted to $\{\mathscr{F}_t : t \in [0,T]\}$, by Proposition 4.6.4, we have

$$\delta(u_A^2) = \int_{s_0}^{s_0+\delta_1} \int_{\mathbb{R}} \phi_{\delta_1}(v)\psi(Y_r) W(dr,dv) - \int_{s_0}^{s_0+\delta_1} \int_{\mathbb{R}} W(dr,dv) D^\alpha \phi_{\delta_1}(v) \int_{s_0}^r \psi(Y_a)\,da. \tag{5.4.11}$$

Using Burkholder's inequality, for any $p \geq 1$, since $0 \leq \psi \leq 1$,

$$\begin{aligned} &\left\| \int_{s_0}^{s_0+\delta_1} \int_{\mathbb{R}} \phi_{\delta_1}(v)\psi(Y_r) W(dr,dv) \right\|_{L^p(\Omega)}^p \\ &\quad \leq c_p \mathrm{E}\left[\left(\int_{s_0}^{s_0+\delta_1} \int_{\mathbb{R}} \phi_{\delta_1}^2(v)\psi^2(Y_r)\,dv\,dr \right)^{p/2} \right] \\ &\quad \leq c_p \delta_1^{p/2} \left(\int_{\mathbb{R}} \phi_{\delta_1}^2(v)\,dv \right)^{p/2} \\ &\quad \leq c_p \delta_1^{p/2} \delta_1^{p/(2\alpha)} = c_p \delta_1^{(1+\alpha)p/(2\alpha)}. \end{aligned} \tag{5.4.12}$$

In order to estimate the second integral on the right-hand side of (5.4.11), we determine the dependence on δ_1 of the $L^2(\mathbb{R})$-norm of $D^\alpha \phi_{\delta_1}$:

$$
\begin{aligned}
\|D^\alpha \phi_{\delta_1}\|^2_{L^2(\mathbb{R})} &= c_\alpha^2 \int_{\mathbb{R}} dx \left(\int_{\mathbb{R}} \frac{\phi_{\delta_1}(x+y) - \phi_{\delta_1}(x) - y\phi'_{\delta_1}(x)}{|y|^{1+\alpha}} dy \right)^2 \\
&= c_\alpha^2 \int_{\mathbb{R}} dx \left(\int_{\mathbb{R}} \frac{\phi_0((x+y-y_0)/\delta_1^{1/\alpha}) - \phi_0((x-y_0)/\delta_1^{1/\alpha}) - y\phi_0'((x-y_0)/\delta_1^{1/\alpha})/\delta_1^{1/\alpha}}{|y|^{1+\alpha}} dy \right)^2 \\
&= c_\alpha^2 \delta_1^{1/\alpha} \int_{\mathbb{R}} d\bar{x} \left(\int_{\mathbb{R}} \frac{\phi_0(\bar{x}+\bar{y}) - \phi_0(\bar{x}) - \bar{y}\phi_0'(\bar{x})}{|\delta_1^{1/\alpha}\bar{y}|^{1+\alpha}} \delta_1^{1/\alpha} d\bar{y} \right)^2 \\
&= c_\alpha^2 \delta_1^{(1-2\alpha)/\alpha} \int_{\mathbb{R}} d\bar{x} \left(\int_{\mathbb{R}} \frac{\phi_0(\bar{x}+\bar{y}) - \phi_0(\bar{x}) - \bar{y}\phi_0'(\bar{x})}{|\bar{y}|^{1+\alpha}} d\bar{y} \right)^2 = c\delta_1^{(1-2\alpha)/\alpha}, \qquad (5.4.13)
\end{aligned}
$$

where in the third equality we change the variables by letting $x = \delta_1^{1/\alpha}\bar{x} + y_0$ and $y = \delta_1^{1/\alpha}\bar{y}$.

Now we apply (5.4.13) to estimate that, by Burkholder's inequality, for any $p \geq 2$, since $0 \leq \psi \leq 1$,

$$
\begin{aligned}
&\left\| \int_{s_0}^{s_0+\delta_1} \int_{\mathbb{R}} W(dr, dv) D^\alpha \phi_{\delta_1}(v) \int_{s_0}^{r} \psi(Y_a) da \right\|^p_{L^p(\Omega)} \\
&\leq c_p \mathrm{E}\left[\left(\int_{s_0}^{s_0+\delta_1} dr \int_{\mathbb{R}} dv (D^\alpha \phi_{\delta_1}(v))^2 \left(\int_{s_0}^{r} \psi(Y_a) da \right)^2 \right)^{p/2} \right] \\
&\leq c_p \left(\int_{s_0}^{s_0+\delta_1} (r-s_0)^2 dr \right)^{p/2} \left(\int_{\mathbb{R}} (D^\alpha \phi_{\delta_1}(v))^2 dv \right)^{p/2} \\
&\leq c_p \delta_1^{3p/2} \delta_1^{(1-2\alpha)p/(2\alpha)} = c_p \delta_1^{(1+\alpha)p/(2\alpha)}. \qquad (5.4.14)
\end{aligned}
$$

Hence (5.4.11), (5.4.12) and (5.4.14) imply (5.4.10). □

Furthermore, from (5.3.8), for any $p \geq 1$, there exist a constant c_p such that for all $(s_0, y_0) \in I \times J$,

$$
\|\delta(u_A^1)\|_{L^p(\Omega)} = c_p \|u_A^1\|_{\mathscr{H}} \leq c_p'. \qquad (5.4.15)
$$

By (5.3.23), (5.4.10) and (5.4.15), using Hölder's inequality, we obtain that for all $p \geq 2$

$$
\|T_1\|_{L^p(\Omega)} \leq c_p \delta_1^{-1} \delta_1^{(1+\alpha)/(2\alpha)} = c_p \delta_1^{(1-\alpha)/(2\alpha)}. \qquad (5.4.16)
$$

This proves the statement (a) of Proposition 5.4.2 for $i = 1$.

We next give an estimate on the moments of T_2.

We first give an estimate on the $\mathscr{H}$-norm of u_A^2. By definition,

$$\begin{aligned}
\|u_A^2\|_{\mathscr{H}}^2 &\leq 2\int_{s_0}^{s_0+\delta_1} dr \int_{\mathbb{R}} dv\, \psi(Y_r)^2 \phi_{\delta_1}^2(v) + 2\int_{s_0}^{s_0+\delta_1} dr \int_{\mathbb{R}} dv\, (D^\alpha \phi_{\delta_1}(v))^2 \left|\int_{s_0}^{r} \psi(Y_a)\, da\right|^2 \\
&\leq 2\delta_1 \int_{y_0-\delta_1^{1/\alpha}}^{y_0+2\delta_1^{1/\alpha}} dv + c\,\delta_1^{(1-2\alpha)/\alpha} \int_{s_0}^{s_0+\delta_1} (r-s_0)^2\, dr \\
&= c\,\delta_1 \delta_1^{1/\alpha} + c\,\delta_1^3 \delta_1^{(1-2\alpha)/\alpha} = 2c\,\delta_1^{(1+\alpha)/\alpha},
\end{aligned} \tag{5.4.17}$$

where, in the second inequality, we use (5.4.13).

Lemma 5.4.5. *For any $p \geq 1$, there exists a constant c_p, not depending on $(s_0, y_0) \in I \times J$, such that for all $\delta_1 > 0$,*

$$\|\langle D\gamma_A^{2,2}, u_A^2 \rangle_{\mathscr{H}}\|_{L^p(\Omega)} \leq c_p \delta_1^{(1+3\alpha)/(2\alpha)}. \tag{5.4.18}$$

Proof. Taking the Malliavin derivative of $\gamma_A^{2,2}$, we have

$$\langle D\gamma_A^{2,2}, u_A^2 \rangle_{\mathscr{H}} = \int_{s_0}^{s_0+\delta_1} \psi'(Y_r) \langle DY_r, u_A^2 \rangle_{\mathscr{H}}\, dr.$$

By Hölder's inequality and (5.4.17), for any $p \geq 1$,

$$\begin{aligned}
\mathrm{E}\left[|\langle D\gamma_A^{2,2}, u_A^2 \rangle_{\mathscr{H}}|^p\right] &\leq \|\psi'\|_\infty^p \delta_1^{p-1} \int_{s_0}^{s_0+\delta_1} \mathrm{E}[|\langle DY_r, u_A^2 \rangle_{\mathscr{H}}|^p]\, dr \\
&\leq c_p R^{-p} \delta_1^{p-1} \int_{s_0}^{s_0+\delta_1} \mathrm{E}[\|DY_r\|_{\mathscr{H}}^p \|u_A^2\|_{\mathscr{H}}^p]\, dr \\
&\leq c_p R^{-p} \delta_1^{p-1} \delta_1^{(1+\alpha)p/(2\alpha)} \int_{s_0}^{s_0+\delta_1} \mathrm{E}[\|DY_r\|_{\mathscr{H}}^p]\, dr.
\end{aligned}$$

Using (5.2.25), this is bounded above by

$$\begin{aligned}
&c_p R^{-p} \delta_1^{p-1} \delta_1^{(1+\alpha)p/(2\alpha)} \delta_1^{(p_0-\gamma_0)(\alpha-1)p/\alpha} \int_{s_0}^{s_0+\delta_1} (r-s_0)^{2p}\, dr \\
&\quad = c_p R^{-p} \delta_1^{p-1+2p+1+(1+\alpha)p/(2\alpha)+(p_0-\gamma_0)(\alpha-1)p/\alpha} \\
&\quad \leq c_p \delta_1^{((\gamma_0-p_0)(\alpha-1)/\alpha-2)p} \delta_1^{p-1+2p+1+(1+\alpha)p/(2\alpha)+(p_0-\gamma_0)(\alpha-1)p/\alpha} \\
&\quad = c_p \delta_1^{(1+3\alpha)p/(2\alpha)},
\end{aligned}$$

where, in the inequality, we use (5.4.9). □

By (5.3.23), (5.4.15) and (5.4.18), using Hölder's inequality, we obtain for any $p \geq 1$,

$$\|T_2\|_{L^p(\Omega)} \leq c_p\, \delta_1^{-2} \delta_1^{(1+3\alpha)/(2\alpha)} = c_p\, \delta_1^{(1-\alpha)/(2\alpha)}. \tag{5.4.19}$$

This proves the statement (a) of Proposition 5.4.2 for $i = 2$.

We proceed to estimate the moments of T_3.

Similar to the calculation in Section 4.6.1, by the properties of the derivative and divergence operator and the stochastic Fubini theorem (see [24, Chapter 1, Theorem 5.30] or [81, Theorem 2.6]), we write

$$\begin{aligned}\langle D\delta(u_A^2), u_A^1\rangle_{\mathscr{H}} &= \langle u_A^2, u_A^1\rangle_{\mathscr{H}} + \int_{s_0}^{s_0+\delta_1}\int_{\mathbb{R}} \psi'(Y_r)\langle DY_r, u_A^1\rangle_{\mathscr{H}}\phi_{\delta_1}(v)W(dr,dv)\\ &\quad - \int_{s_0}^{s_0+\delta_1}\int_{\mathbb{R}} W(dr,dv)D^\alpha\phi_{\delta_1}(v)\int_{s_0}^{r} da\,\psi'(Y_a)\langle DY_a, u_A^1\rangle_{\mathscr{H}}\\ &:= \bar{T}_{31} + \bar{T}_{32} - \bar{T}_{33}. \end{aligned} \tag{5.4.20}$$

From (5.4.17) and (5.3.8), it is easy to see that for any $p \geq 1$,

$$\|\bar{T}_{31}\|_{L^p(\Omega)} \leq c_p \delta_1^{(1+\alpha)/(2\alpha)}. \tag{5.4.21}$$

By Burkholder's inequality and using (5.3.6) and (5.3.8), we have for any $p \geq 2$,

$$\begin{aligned}\mathrm{E}[|\bar{T}_{32}|^p] &\leq c_p \mathrm{E}\left[\left(\int_{s_0}^{s_0+\delta_1}\int_{\mathbb{R}} \psi'(Y_r)^2\langle DY_r, u_A^1\rangle_{\mathscr{H}}^2\phi_{\delta_1}^2(v)\,dr\,dv\right)^{p/2}\right]\\ &\leq c_p R^{-p}\mathrm{E}\left[\left(\int_{s_0}^{s_0+\delta_1}\|DY_r\|_{\mathscr{H}}^2 dr\int_{\mathbb{R}}\phi_{\delta_1}^2(v)\,dv\right)^{p/2}\right]\\ &= c_p R^{-p}\left(\int_{\mathbb{R}}\phi_{\delta_1}^2(v)\,dv\right)^{p/2}\mathrm{E}\left[\left(\int_{s_0}^{s_0+\delta_1}\|DY_r\|_{\mathscr{H}}^2 dr\right)^{p/2}\right]. \end{aligned}\tag{5.4.22}$$

By Hölder's inequality, we see that (5.4.22) is bounded above by

$$\begin{aligned}&c_p R^{-p}\delta_1^{p/(2\alpha)}\delta_1^{p/2-1}\int_{s_0}^{s_0+\delta_1}\mathrm{E}[\|DY_r\|_{\mathscr{H}}^p]\,dr\\ &\leq c_p R^{-p}\delta_1^{p/(2\alpha)}\delta_1^{p/2-1}\delta_1^{(p_0-\gamma_0)(\alpha-1)p/\alpha}\int_{s_0}^{s_0+\delta_1}(r-s_0)^{2p}\,dr\\ &= c_p R^{-p}\delta_1^{p/(2\alpha)+5p/2+(p_0-\gamma_0)(\alpha-1)p/\alpha}\\ &\leq c_p\delta_1^{((\gamma_0-p_0)(\alpha-1)/\alpha-2)p}\delta_1^{p/(2\alpha)+5p/2+(p_0-\gamma_0)(\alpha-1)p/\alpha}\\ &= c_p\delta_1^{(1+\alpha)p/(2\alpha)}, \end{aligned}\tag{5.4.23}$$

where, in the first inequality we use (5.2.25), and in the second inequality we use (5.4.9).

We now give an estimate on the moments of $\bar{T}_{33}$. By Burkholder's inequality and using (5.3.6) and (5.3.8), we see that for any $p \geq 2$,

$$\begin{aligned} \mathrm{E}[|\bar{T}_{33}|^p] &\leq c_p \mathrm{E}\left[\left(\int_{s_0}^{s_0+\delta_1} \int_{\mathbb{R}} \left(\int_{s_0}^{r} \psi'(Y_a)\langle DY_a, u_A^1\rangle_{\mathscr{H}} da\right)^2 (D^\alpha \phi_{\delta_1}(v))^2 dr dv\right)^{p/2}\right] \\ &\leq c_p R^{-p} \left(\int_{\mathbb{R}} (D^\alpha \phi_{\delta_1}(v))^2 dv\right)^{p/2} \mathrm{E}\left[\left(\int_{s_0}^{s_0+\delta_1} \left(\int_{s_0}^{r} \|DY_a\|_{\mathscr{H}} da\right)^2 dr\right)^{p/2}\right]. \end{aligned} \tag{5.4.24}$$

By (5.4.13) and using Hölder's inequality twice, we see that (5.4.24) is bounded above by

$$\begin{aligned} & c_p R^{-p} \delta_1^{(1-2\alpha)p/(2\alpha)} \mathrm{E}\left[\left(\int_{s_0}^{s_0+\delta_1} dr (r-s_0) \int_{s_0}^{r} \|DY_a\|_{\mathscr{H}}^2 da\right)^{p/2}\right] \\ & \quad \leq c_p R^{-p} \delta_1^{(1-2\alpha)p/(2\alpha)} \left(\int_{s_0}^{s_0+\delta_1} dr \int_{s_0}^{r} da\right)^{p/2-1} \int_{s_0}^{s_0+\delta_1} dr (r-s_0)^{p/2} \int_{s_0}^{r} \mathrm{E}[\|DY_a\|_{\mathscr{H}}^p] da. \end{aligned} \tag{5.4.25}$$

Applying the estimate in (5.2.25), we obtain that (5.4.25) is bounded above by

$$\begin{aligned} & c_p R^{-p} \delta_1^{(1-2\alpha)p/(2\alpha)} \delta_1^{p-2} \delta_1^{(p_0-\gamma_0)(\alpha-1)p/\alpha} \int_{s_0}^{s_0+\delta_1} dr (r-s_0)^{p/2} \int_{s_0}^{r} (a-s_0)^{2p} da \\ & \quad = c_p R^{-p} \delta_1^{(1-2\alpha)p/(2\alpha)} \delta_1^{p-2} \delta_1^{(p_0-\gamma_0)(\alpha-1)p/\alpha} \delta_1^{2+5p/2} \\ & \quad \leq c_p \delta_1^{((\gamma_0-p_0)(\alpha-1)/\alpha-2)p} \delta_1^{(1-2\alpha)p/(2\alpha)} \delta_1^{p-2} \delta_1^{(p_0-\gamma_0)(\alpha-1)p/\alpha} \delta_1^{2+5p/2} \\ & \quad = c_p \delta_1^{(1+\alpha)p/(2\alpha)}, \end{aligned} \tag{5.4.26}$$

where, in the inequality, we use (5.4.9).

Therefore, by (5.4.21),(5.4.23), (5.4.26) and (5.3.23), we have obtained that for any $p \geq 2$,

$$\|T_3\|_{L^p(\Omega)} \leq c_p \delta_1^{-1} \delta_1^{(1+\alpha)/(2\alpha)} = c_p \delta_1^{(1-\alpha)/(2\alpha)}. \tag{5.4.27}$$

This proves the statement (a) of Proposition 5.4.2 for $i = 3$.

Therefore, we have finished the proof of Proposition 5.4.2(a).

5.4.2 Proof of Proposition 5.4.2(b)

As in Section 4.6.2, we are going to show that the three terms T_4, T_5 and T_6 are equal to zero. First, we observe that for any $t, s \in [s_0, s_0 + \delta_1]$, by Lemma 5.2.10 and the definition of the functions f_0 and g_0,

$$\begin{aligned}\langle D(u(t,y_0)-u(s,y_0)),u_A^1\rangle_{\mathscr{H}} &= \int_0^T\int_{\mathbb{R}}(1_{\{r<t\}}G_\alpha(t-r,y_0-v)-1_{\{r<s\}}G_\alpha(s-r,y_0-v))\\ &\qquad\times\left(\frac{\partial}{\partial r}-{}_vD^\alpha\right)(f_0(r)g_0(v))drdv\\ &= f_0(t)g_0(y_0)-f_0(s)g_0(y_0)=0.\end{aligned}\tag{5.4.28}$$

By (5.2.24) and (5.4.28), we know that for $r\in[s_0,s_0+\delta_1]$,

$$\begin{aligned}\langle DY_r,u_A^1\rangle_{\mathscr{H}} &= 2p_0\int_{[s_0,r]^2}dsdt\,\frac{(u(t,y_0)-u(s,y_0))^{2p_0-1}}{|t-s|^{\gamma_0(\alpha-1)/\alpha}}\langle D(u(t,y_0)-u(s,y_0)),u_A^1\rangle_{\mathscr{H}}\\ &=0.\end{aligned}\tag{5.4.29}$$

Hence,

$$\langle D\gamma_A^{2,2},u_A^1\rangle_{\mathscr{H}}=\int_{s_0}^{y_0+\delta_1}\psi'(Y_r)\langle DY_r,u_A^1\rangle_{\mathscr{H}}dr=0,\tag{5.4.30}$$

which implies $T_4=T_5=0$.

We proceed to prove that T_6 vanishes. Similar to (5.4.28), for any $t,s\in[s_0,s_0+\delta_1]$,

$$\begin{aligned}\langle D(u(t,y_0)-u(s,y_0)),u_A^2\rangle_{\mathscr{H}} &= \int_{s_0}^{s_0+\delta_1}dr\int_0^1 dv\big(1_{\{r<t\}}G(t-r,y_0-v)\\ &\qquad-1_{\{r<s\}}G(s-r,y_0-v)\big)\left(\frac{\partial}{\partial r}-{}_vD^\alpha\right)H(r,v)\\ &= H(t,y_0)-H(s,y_0)\\ &= \phi_{\delta_1}(y_0)\int_{s_0}^t\psi(Y_a)da-\phi_{\delta_1}(y_0)\int_{s_0}^s\psi(Y_a)da\\ &= \int_s^t\psi(Y_a)da,\end{aligned}\tag{5.4.31}$$

where the last equality is due to the definition of the function ϕ_{δ_1}. Hence, for $r\in[s_0,s_0+\delta_1]$,

$$\begin{aligned}\langle DY_r,u_A^2\rangle_{\mathscr{H}} &= 2p_0\int_{[s_0,r]^2}dsdt\,\frac{(u(t,y_0)-u(s,y_0))^{2p_0-1}}{|t-s|^{\gamma_0(\alpha-1)/\alpha}}\langle D(u(t,y_0)-u(s,y_0)),u_A^2\rangle_{\mathscr{H}}\\ &= 2p_0\int_{[s_0,r]^2}dsdt\,\frac{(u(t,y_0)-u(s,y_0))^{2p_0-1}}{|t-s|^{\gamma_0(\alpha-1)/\alpha}}\int_s^t\psi(Y_a)da.\end{aligned}\tag{5.4.32}$$

Moreover,

$$
\begin{aligned}
\langle D\langle DY_r, u_A^2\rangle_{\mathscr{H}}, u_A^1\rangle_{\mathscr{H}} = {} & 2p_0(2p_0-1)\int_{[s_0,r]^2} dsdt\, \frac{(u(t,y_0)-u(s,y_0))^{2p_0-2}}{|t-s|^{\gamma_0(\alpha-1)/\alpha}} \\
& \times \langle D(u(t,y_0)-u(s,y_0)), u_A^1\rangle_{\mathscr{H}} \int_s^t \psi(Y_a)da \\
& + 2p_0\int_{[s_0,r]^2} dsdt\, \frac{(u(t,y_0)-u(s,y_0))^{2p_0-1}}{|t-s|^{\gamma_0(\alpha-1)/\alpha}} \\
& \times \int_s^t \psi'(Y_a)\langle DY_a, u_A^1\rangle_{\mathscr{H}} da \\
= {} & 0+0=0, \qquad (5.4.33)
\end{aligned}
$$

where the first term vanishes due to (5.4.28) and the second term vanishes because of (5.4.29). Therefore,

$$
\begin{aligned}
\langle D\langle D\gamma_A^{2,2}, u_A^2\rangle_{\mathscr{H}}, u_A^1\rangle_{\mathscr{H}} &= \left\langle D\int_{s_0}^{s_0+\delta_1} \psi'(Y_r)\langle DY_r, u_A^2\rangle_{\mathscr{H}} dr, u_A^1\right\rangle_{\mathscr{H}} \\
&= \int_{s_0}^{s_0+\delta_1} \psi''(Y_r)\langle DY_r, u_A^1\rangle_{\mathscr{H}}\langle DY_r, u_A^2\rangle_{\mathscr{H}} dr \\
&\quad + \int_{s_0}^{s_0+\delta_1} \psi'(Y_r)\langle D\langle DY_r, u_A^2\rangle_{\mathscr{H}}, u_A^1\rangle_{\mathscr{H}} dr \\
&= 0, \qquad (5.4.34)
\end{aligned}
$$

which implies $T_6 = 0$.

This proves the statement (b) of Proposition 5.4.2.

5.4.3 Estimates for the tail probabilities

In order to bound the tail probability $\mathrm{P}\{F_2 > z_2\}$, we first give an estimate on the moments of the supremum of $|u(s,y)-u(s_0,y_0)|$, analogous to Lemma 4.5 in [25].

Lemma 5.4.6. *For $\alpha \in]1,2]$ and for all $p \geq 1$, there exists $A_{p,\alpha} > 0$ such that for all $\epsilon > 0$ and all (t,x) fixed,*

$$
E\left[\sup_{[\Delta_\alpha((t,x);(s,y))]^{1/2}\leq\epsilon} |u(t,x)-u(s,y)|^p\right] \leq A_{p,\alpha}\epsilon^p. \qquad (5.4.35)
$$

Proof. The proof is very similar to that of [25, Lemma 4.5] by applying Proposition A.1 of [25] with

$$
\begin{aligned}
&S := S_\epsilon = \{(s,y) : [\Delta_\alpha((t,x);(s,y))]^{1/2} < \epsilon\}, \quad \rho((t,x),(s,y)) := [\Delta_\alpha((t,x);(s,y))]^{1/2}, \\
&\mu(dtdx) := dtdx, \quad \Psi(x) := e^{|x|}-1, \quad p(x) := x \quad \text{and} \quad f := u.
\end{aligned}
$$

We denote

$$\mathscr{C} = \int_{S_\epsilon} dr d\bar{y} \int_{S_\epsilon} ds dy \left[\exp\left(\frac{|u(r,\bar{y}) - u(s,y)|}{(|r-s|^{\frac{\alpha-1}{\alpha}} + |\bar{y} - y|^{\alpha-1})^{1/2}} \right) - 1 \right],$$

$$\mathscr{C}^1 = \int_{S_\epsilon} dr d\bar{y} \int_{S_\epsilon} ds dy \exp\left(\frac{|u(r,\bar{y}) - u(s,y)|}{(|r-s|^{\frac{\alpha-1}{\alpha}} + |\bar{y} - y|^{\alpha-1})^{1/2}} \right).$$

Then it is obvious to see that $\mathscr{C} < \mathscr{C}^1$ and there exists a constant $c > 0$ such that for all $\omega \in \Omega$,

$$\mathscr{C}^1(\omega) \geq c\epsilon^{4(\alpha+1)/(\alpha-1)}. \tag{5.4.36}$$

Furthermore, by (5.1.4),

$$\begin{aligned} \mathrm{E}[\mathscr{C}] &\leq \mathrm{E}[\mathscr{C}^1] \\ &= \mathrm{E}\left[\int_{S_\epsilon} dr d\bar{y} \int_{S_\epsilon} ds dy \exp\left(\frac{|u(r,\bar{y}) - u(s,y)|}{(|r-s|^{\frac{\alpha-1}{\alpha}} + |\bar{y} - y|^{\alpha-1})^{1/2}} \right) \right] \\ &\leq c_0 \epsilon^{4(\alpha+1)/(\alpha-1)}. \end{aligned} \tag{5.4.37}$$

In accord with [25, Proposition A.1], and by application of Hölder's inequality,

$$\begin{aligned} &\mathrm{E}\left[\sup_{[\Delta_\alpha((t,x);(s,y))]^{1/2} \leq \epsilon} |u(t,x) - u(s,y)|^p \right] \\ &\leq 10^p \mathrm{E}\left[\left(\int_0^{2\epsilon} du \ln\left(1 + \frac{\mathscr{C}}{[\mu(B_\rho((t,x),u/4))]^2} \right) \right)^p \right] \\ &= 10^p \mathrm{E}\left[\left(\int_0^{2\epsilon} du \ln\left(1 + \frac{\mathscr{C}}{c_1 u^{4(\alpha+1)/(\alpha-1)}} \right) \right)^p \right] \\ &\leq 10^p (2\epsilon)^{p-1} \int_0^{2\epsilon} du \, \mathrm{E}\left[\ln^p \left(1 + \frac{\mathscr{C}}{c_1 u^{4(\alpha+1)/(\alpha-1)}} \right) \right] \\ &\leq 10^p (2\epsilon)^{p-1} \int_0^{2\epsilon} du \, \mathrm{E}\left[\ln^p \left(1 + \frac{\mathscr{C}^1}{c_1 u^{4(\alpha+1)/(\alpha-1)}} \right) \right] \\ &\leq 10^p (2\epsilon)^{p-1} \int_0^{2\epsilon} du \, \mathrm{E}\left[\ln^p \left(1 + \frac{c_p \mathscr{C}^1}{c_1 u^{4(\alpha+1)/(\alpha-1)}} \right) \right], \end{aligned} \tag{5.4.38}$$

where by (5.4.36) the constant $c_p > 1$ is chosen such that for all $0 < u < 2\epsilon$ and $\omega \in \Omega$,

$$\frac{c_p \mathscr{C}^1(\omega)}{c_1 u^{4(\alpha+1)/(\alpha-1)}} \geq e^{p-1} - 1. \tag{5.4.39}$$

Since the function $x \mapsto \ln^p(1+x)$ is concave on $[e^{p-1} - 1, \infty[$, we apply Jensen's inequality to bound (5.4.38) above by

$$
\begin{aligned}
&10^p(2\epsilon)^{p-1}\int_0^{2\epsilon} du \ln^p\left(1+\frac{c_p \mathrm{E}[\mathscr{C}^1]}{c_1 u^{4(1+\alpha)/(\alpha-1)}}\right)\\
&\quad \le 10^p(2\epsilon)^{p-1}\int_0^{2\epsilon} du \ln^p\left(1+\frac{c_0 c_p}{c_1}\left(\frac{\epsilon}{u}\right)^{4(\alpha+1)/(\alpha-1)}\right)\\
&\quad = 10^p 2^{p-1}\epsilon^p\int_{1/2}^{\infty} du\, u^{-2}\ln^p\left(1+\frac{c_0 c_p}{c_1}u^{4(\alpha+1)/(\alpha-1)}\right) = A_{p,\alpha}\epsilon^p, \qquad (5.4.40)
\end{aligned}
$$

as desired. □

Lemma 5.4.7. *There exists a finite positive constant c, not depending on $(s_0, y_0) \in I \times J$, such that for all $z_1 \in \mathbb{R}$,*

$$P\{|F_1| > |z_1|\} \le c\,(|z_1|^{-1} \wedge 1)e^{-z_1^2/c}, \qquad (5.4.41)$$

and for all $\delta_1 > 0$ and for all $z_2 > 0$,

$$P\{F_2 > z_2\} \le c \exp\left(-\frac{z_2^2}{c\delta_1^{(\alpha-1)/\alpha}}\right). \qquad (5.4.42)$$

Proof. The estimate for $\mathrm{P}\{|F_1| > |z_1|\}$ is similar to (4.6.47) since the variance of $u(s_0, y_0)$ is bounded above and below by positive constants uniformly for $(s_0, y_0) \in I \times J$.

We denote

$$\sigma_\alpha^2 := \sup_{t\in[s_0,s_0+\delta_1]} \mathrm{E}[\bar{u}(t, y_0)^2].$$

By (5.1.4), we have $\sigma_\alpha^2 \le C\delta_1^{(\alpha-1)/\alpha}$. On the other hand, by Lemma 5.4.6 we know that

$$
\begin{aligned}
\mathrm{E}[F_2] &\le \mathrm{E}\left[\sup_{t\in[s_0,s_0+\delta_1]} |u(t, y_0) - u(s_0, y_0)|\right]\\
&\le \mathrm{E}\left[\sup_{[\Delta_\alpha((t,x);(s_0,y_0))]^{1/2}\le \delta_1^{(\alpha-1)/(2\alpha)}} |u(t,x) - u(s_0, y_0)|\right]\\
&\le c\delta_1^{(\alpha-1)/(2\alpha)}. \qquad (5.4.43)
\end{aligned}
$$

Applying Borell's inequality (see [1, (2.6)]) for all $z_2 > c\delta_1^{(\alpha-1)/(2\alpha)}$ (here c is the constant in

(5.4.43)),

$$
\begin{aligned}
P\{F_2 > z_2\} &\le 2\exp\left(-(z_2 - E[F_2])^2/(2\sigma_\alpha^2)\right) \le 2\exp\left(-(z_2 - E[F_2])^2/(2C\delta_1^{(\alpha-1)/\alpha})\right) \\
&\le 2\exp\left(-(2z_2^2/3 - 2E[F_2]^2)(2C\delta_1^{(\alpha-1)/\alpha})\right) \\
&= 2\exp\left(-z_2^2/(3C\delta_1^{(\alpha-1)/\alpha})\right)\exp\left(E[F_2]^2/(C\delta_1^{(\alpha-1)/\alpha})\right) \\
&\le 2e^{c^2/C}\exp\left(-z_2^2/(3C\delta_1^{(\alpha-1)/\alpha})\right) \\
&= \bar{c}\exp\left(-z_2^2/(3C\delta_1^{(\alpha-1)/\alpha})\right).
\end{aligned} \tag{5.4.44}
$$

Since for $0 \le z_2 \le c\delta_1^{(\alpha-1)/(2\alpha)}$,

$$
\exp\left(-z_2^2/(3C\delta_1^{(\alpha-1)/\alpha})\right) \ge e^{-\frac{c^2}{3C}},
$$

we can find a constant $\tilde{c}$ such that for all $z_2 > 0$,

$$
P\{F_2 > z_2\} \le \tilde{c}\exp\left(-z_2^2/(3C\delta_1^{(\alpha-1)/\alpha})\right). \tag{5.4.45}
$$

This proves (5.4.42). □

Proof of Theorem 5.1.2. This follows from (5.4.41), (5.4.42), (5.4.7) and (5.4.1). □

5.5 Gaussian-type upper bound on the density of M_0

The structure of this section is similar to that of Section 4.7. In this section, we assume that δ_1, δ_2 satisfy the conditions in (5.1.15).

From the formula for the probability density function of M_0 in (5.3.28), by the Cauchy-Schwartz inequality,

$$
p_0(z) \le P\{M_0 > z\}^{1/2}\|\delta(u_{\bar{A}}/\gamma_{\bar{A}})\|_{L^2(\Omega)}. \tag{5.5.1}
$$

Proposition 5.5.1. *(a) There exists a finite positive constant c, not depending on $y_0 \in J$, such that for all small $\delta_1, \delta_2 > 0$ and for all $z \ge (\delta_1^{(\alpha-1)/\alpha} + \delta_2^{\alpha-1})^{1/2}$,*

$$
\|\delta(u_{\bar{A}}/\gamma_{\bar{A}})\|_{L^2(\Omega)} \le c(\delta_1^{(\alpha-1)/\alpha} + \delta_2^{\alpha-1})^{-1/2}. \tag{5.5.2}
$$

(b) There exists a finite positive constant c, not depending on $y_0 \in J$, such that for all $\delta_1, \delta_2 > 0$ and for all $z > 0$,

$$
P\{M_0 > z\} \le c\exp\left(-\frac{z^2}{c(\delta_1^{(\alpha-1)/\alpha} + \delta_2^{\alpha-1})}\right). \tag{5.5.3}
$$

Proof of Theorem 5.1.4. This is an immediate consequence of (5.5.1) and Proposition 5.5.1. □

The proof of Proposition 5.5.1 is given in the following two subsections.

5.5.1 Proof of Proposition 5.5.1 (a)

Throughout this section, we assume that

$$z \geq (\delta_1^{(\alpha-1)/\alpha} + \delta_2^{\alpha-1})^{1/2} = \delta^{1/2}. \tag{5.5.4}$$

Recalling the definition of $\bar{R}$ in (5.2.43), under the assumption (5.5.4), we see from (5.3.13) that

$$\begin{aligned}\bar{R}^{-1} &= c^{-1}\bar{a}^{-2p_0}\delta^{\gamma_0-\frac{2\alpha}{\alpha-1}} = c' z^{-2p_0}\delta^{\gamma_0-\frac{2\alpha}{\alpha-1}}\\ &\leq c\,\delta^{\gamma_0-p_0-\frac{2\alpha}{\alpha-1}}.\end{aligned} \tag{5.5.5}$$

In order to prove Proposition 5.5.1(a), we need the following several lemmas. Recall the definition of $u_{\bar{A}}$ in (5.3.18).

Lemma 5.5.2. *For any $p \geq 2$, there exists a constant c_p, not depending on $y_0 \in J$, such that for all $\delta_1, \delta_2 > 0$,*

$$\|\delta(u_{\bar{A}})\|_{L^p(\Omega)} \leq c_p\,\delta^{\frac{\alpha+1}{2(\alpha-1)}}. \tag{5.5.6}$$

Proof. The proof is similar to that of Lemma 5.4.4. Since $u_{\bar{A}}$ is adapted, by Proposition 4.6.4, we have

$$\delta(u_{\bar{A}}) = \int_0^{\Delta_\bullet}\int_{\mathbb{R}} \bar{\phi}_\delta(v)\bar{\psi}(\bar{Y}_r)W(dr,dv) - \int_0^{\Delta_\bullet}\int_{\mathbb{R}} W(dr,dv)D^\alpha\bar{\phi}_\delta(v)\int_0^r \bar{\psi}(\bar{Y}_a)da. \tag{5.5.7}$$

For the first term on the right-hand side of (5.5.7), by Burkholder's inequality, for any $p \geq 2$, since $0 \leq \bar{\psi} \leq 1$,

$$\begin{aligned}\left\|\int_0^{\Delta_\bullet}\int_{\mathbb{R}} \bar{\phi}_\delta(v)\bar{\psi}(\bar{Y}_r)W(dr,dv)\right\|_{L^p(\Omega)}^p &\leq c_p \mathrm{E}\left[\left(\int_0^{\Delta_\bullet}\int_{\mathbb{R}} \bar{\phi}_\delta^2(v)\bar{\psi}^2(\bar{Y}_r)drdv\right)^{p/2}\right]\\ &\leq c_p\Delta_\bullet^{p/2}\left(\int_{\mathbb{R}}\bar{\phi}_\delta^2(v)dv\right)^{p/2}\\ &\leq c_p\Delta_\bullet^{p/2}\delta^{\frac{p}{2(\alpha-1)}} = c_p\delta^{\frac{p(\alpha+1)}{2(\alpha-1)}},\end{aligned} \tag{5.5.8}$$

where in the third inequality, we have used the definition of $\bar{\phi}_\delta$ in (5.3.15). For the second term on the right-hand side of (5.5.7), we first observe that

$$\|D^\alpha\bar{\phi}_\delta\|_{L^2(\mathbb{R})}^2 = c\,\delta^{\frac{1-2\alpha}{\alpha-1}} \tag{5.5.9}$$

by the same argument as in (5.4.13), replacing $\delta_1^{1/\alpha}$ by $\delta^{1/(\alpha-1)}$.

Applying Burkholder's inequality, for any $p \geq 2$, since $0 \leq \bar{\psi} \leq 1$,

$$\begin{aligned}
&\left\| \int_0^{\Delta_\bullet} \int_{\mathbb{R}} W(dr, dv)\, D^\alpha \bar{\phi}_\delta(v) \int_0^r \bar{\psi}(\bar{Y}_a)\, da \right\|_{L^p(\Omega)}^p \\
&\leq c_p \mathrm{E}\left[\left(\int_0^{\Delta_\bullet} dr \int_{\mathbb{R}} dv\, (D^\alpha \bar{\phi}_\delta(v))^2 \left(\int_0^r \bar{\psi}(\bar{Y}_a)\, da \right)^2 \right)^{p/2} \right] \\
&\leq c_P \left(\int_0^{\Delta_\bullet} r^2\, dr \right)^{p/2} \left(\int_{\mathbb{R}} (D^\alpha \bar{\phi}_\delta(v))^2\, dv \right)^{p/2} \\
&= c_p \Delta_\bullet^{3p/2} \delta^{\frac{p(1-2\alpha)}{2(\alpha-1)}} = c_p \delta^{\frac{p(\alpha+1)}{2(\alpha-1)}},
\end{aligned} \tag{5.5.10}$$

where, in the first equality, we use (5.5.9). Hence, (5.5.7), (5.5.8) and (5.5.10) establish this lemma. □

Lemma 5.5.3. *There exists a constant c, not depending on $y_0 \in J$, such that for all $\delta_1, \delta_2 > 0$,*

$$\| u_{\bar{A}} \|_{\mathscr{H}} \leq c\, \delta^{\frac{\alpha+1}{2(\alpha-1)}}. \tag{5.5.11}$$

Proof. The proof is similar to that of (5.4.17). By the definition of $u_{\bar{A}}$,

$$\begin{aligned}
\| u_{\bar{A}} \|_{\mathscr{H}}^2 &\leq 2 \int_0^{\Delta_\bullet} dr \int_{\mathbb{R}} dv\, \bar{\psi}(\bar{Y}_r)^2 \bar{\phi}_\delta^2(v) + 2 \int_0^{\Delta_\bullet} dr \int_{\mathbb{R}} dv\, (D^\alpha \bar{\phi}_\delta(v))^2 \left(\int_0^r \bar{\psi}(\bar{Y}_a)\, da \right)^2 \\
&\leq 2\Delta_\bullet \int_{y_0 - \delta^{1/(\alpha-1)}}^{y_0 + 2\delta^{1/(\alpha-1)}} dv + 2c \int_0^{\Delta_\bullet} r^2\, dr \int_{\mathbb{R}} (D^\alpha \bar{\phi}_\delta(v))^2\, dv \\
&= c\, \delta^{\frac{\alpha+1}{\alpha-1}} + c\, \Delta_\bullet^3 \delta^{\frac{1-2\alpha}{\alpha-1}} \\
&= 2c\, \delta^{\frac{\alpha+1}{\alpha-1}},
\end{aligned} \tag{5.5.12}$$

where, in the first inequality, we use (5.5.9). □

Lemma 5.5.4. *For any $p \geq 2$, there exists a constant c_p, not depending on $y_0 \in J$, such that for all $\delta_1, \delta_2 > 0$,*

$$\| \langle D\gamma_{\bar{A}}, u_{\bar{A}} \rangle_{\mathscr{H}} \|_{L^p(\Omega)} \leq c\, \delta^{\frac{3\alpha+1}{2(\alpha-1)}}. \tag{5.5.13}$$

Proof. The proof is similar to that of Lemma 5.4.5. Taking the Malliavin derivative of $\gamma_{\bar{A}}$, we have

$$\langle D\gamma_{\bar{A}}, u_{\bar{A}} \rangle_{\mathscr{H}} = \int_0^{\Delta_\bullet} \bar{\psi}'(\bar{Y}_r) \langle D\bar{Y}_r, u_{\bar{A}} \rangle_{\mathscr{H}}\, dr.$$

By Hölder's inequality, (5.3.14) and (5.5.12), for any $p \geq 1$,

$$
\begin{aligned}
\mathrm{E}\left[|\langle D\gamma_{\bar{A}}, u_{\bar{A}}\rangle_{\mathscr{H}}|^p\right] &\leq \|\bar{\psi}'\|_\infty^p \Delta_\bullet^{p-1} \int_0^{\Delta_\bullet} \mathrm{E}[|\langle D\bar{Y}_r, u_{\bar{A}}\rangle_{\mathscr{H}}|^p]\,dr \\
&\leq c_p \bar{R}^{-p} \Delta_\bullet^{p-1} \int_0^{\Delta_\bullet} \mathrm{E}[\|D\bar{Y}_r\|_{\mathscr{H}}^p \|u_{\bar{A}}\|_{\mathscr{H}}^p]\,dr \\
&\leq c_p \bar{R}^{-p} \Delta_\bullet^{p-1} \delta^{\frac{p(\alpha+1)}{2(\alpha-1)}} \int_0^{\Delta_\bullet} \mathrm{E}[\|D\bar{Y}_r\|_{\mathscr{H}}^p]\,dr.
\end{aligned}
$$

Applying (5.2.40), this is bounded above by

$$
\begin{aligned}
& c_p R^{-p} \Delta_\bullet^{p-1} \delta^{\frac{p(\alpha+1)}{2(\alpha-1)}} \delta^{(p_0-\gamma_0)p} \int_0^{\Delta_\bullet} r^{2p}\,dr \\
&\quad = c_p R^{-p} \Delta_\bullet^{p-1} \delta^{\frac{p(\alpha+1)}{2(\alpha-1)}} \delta^{(p_0-\gamma_0)p} \Delta_\bullet^{2p+1} \\
&\quad \leq c_p \delta^{(\gamma_0-p_0-2\alpha/(\alpha-1))p} \Delta_\bullet^{p-1} \delta^{\frac{p(\alpha+1)}{2(\alpha-1)}} \delta^{(p_0-\gamma_0)p} \Delta_\bullet^{2p+1} \\
&\quad = c_p \delta^{\frac{(3\alpha+1)p}{2(\alpha-1)}},
\end{aligned}
$$

where, in the inequality, we use (5.5.5). □

Proof of Proposition 5.5.1(a). Using the property of Skorohod integral δ (see [64, (1.48)]),

$$
\delta(u_{\bar{A}}/\gamma_{\bar{A}}) = \frac{\delta(u_{\bar{A}})}{\gamma_{\bar{A}}} + \frac{\langle D\gamma_{\bar{A}}, u_{\bar{A}}\rangle_{\mathscr{H}}}{\gamma_{\bar{A}}^2} := I_1 + I_2. \tag{5.5.14}
$$

By Lemmas 5.5.2 and 5.3.4(b),

$$
\|I_1\|_{L^2(\Omega)} \leq c\,\delta^{\frac{\alpha+1}{2(\alpha-1)}} \delta^{-\frac{\alpha}{\alpha-1}} = c\,\delta^{-1/2}. \tag{5.5.15}
$$

By Lemmas 5.5.4 and 5.3.4(b),

$$
\|I_2\|_{L^2(\Omega)} \leq c\,\delta^{\frac{3\alpha+1}{2(\alpha-1)}} \delta^{-\frac{2\alpha}{\alpha-1}} = c\,\delta^{-1/2}. \tag{5.5.16}
$$

Therefore, (5.5.14), (5.5.15) and (5.5.16) establish (5.5.2). □

5.5.2 Proof of Proposition 5.5.1(b)

Proof of Proposition 5.5.1(b). The proof is similar to that of (4.7.3). We denote

$$
\sigma_0^2 := \sup_{(t,x)\in[0,\delta_1]\times[y_0,y_0+\delta_2]} \mathrm{E}[u(t,x)^2].
$$

From (5.1.4), we have $\sigma_0^2 \leq C\,(\delta_1^{(\alpha-1)/\alpha} + \delta_2^{\alpha-1})$. On the other hand, by Lemma 5.4.6, we have

$$\begin{aligned}
\mathrm{E}[M_0] &\leq \mathrm{E}\left[\sup_{(t,x)\in[0,\delta_1]\times[y_0,y_0+\delta_2]} |u(t,x)|\right] \\
&\leq \mathrm{E}\left[\sup_{[\Delta_\alpha((t,x);(0,y_0))]^{1/2}\leq(\delta_1^{(\alpha-1)/\alpha}+\delta_2^{\alpha-1})^{1/2}} |u(t,x)|\right] \\
&\leq c\,(\delta_1^{(\alpha-1)/\alpha} + \delta_2^{\alpha-1})^{1/2}. && (5.5.17)
\end{aligned}$$

Applying Borell's inequality (see [1, (2.6)]), for all $z > c\,(\delta_1^{(\alpha-1)/\alpha} + \delta_2^{\alpha-1})^{1/2}$ (here c is the constant in (5.5.17)),

$$\begin{aligned}
\mathrm{P}\{M_0 > z\} &\leq 2\exp\left(-(z-\mathrm{E}[M_0])^2/(2\sigma_0^2)\right) \leq 2\exp\left(-(z-\mathrm{E}[M_0])^2/(2C(\delta_1^{(\alpha-1)/\alpha} + \delta_2^{\alpha-1}))\right) \\
&\leq 2\exp\left(-(2z^2/3 - 2\mathrm{E}[M_0]^2)/(2C(\delta_1^{(\alpha-1)/\alpha} + \delta_2^{\alpha-1}))\right) \\
&= 2\exp\left(-z^2/(3C(\delta_1^{(\alpha-1)/\alpha} + \delta_2^{\alpha-1}))\right)\exp\left(\mathrm{E}[M_0]^2/(C(\delta_1^{(\alpha-1)/\alpha} + \delta_2^{\alpha-1}))\right) \\
&\leq 2e^{c^2/C}\exp\left(-z^2/(3C(\delta_1^{(\alpha-1)/\alpha} + \delta_2^{\alpha-1}))\right) \\
&= \tilde{c}\exp\left(-z^2/(3C(\delta_1^{(\alpha-1)/\alpha} + \delta_2^{\alpha-1}))\right). && (5.5.18)
\end{aligned}$$

Since for $0 \leq z \leq c\,(\delta_1^{(\alpha-1)/\alpha} + \delta_2^{\alpha-1})^{1/2}$,

$$\exp\left(-z^2/(3C(\delta_1^{(\alpha-1)/\alpha} + \delta_2^{\alpha-1}))\right) \geq e^{-\frac{c^2}{3C}},$$

we can find a constant $\tilde{c}$ such that for all $z > 0$,

$$\mathrm{P}\{F_2 > z\} \leq \tilde{c}\exp\left(-z^2/(3C(\delta_1^{(\alpha-1)/\alpha} + \delta_2^{\alpha-1}))\right). \tag{5.5.19}$$

This proves (5.5.3). □

A An appendix

A.1 Properties of Green kernel

In this appendix, we first present some properties of the Green kernel of the fractional heat equation (1.3.4), which are available in [17], [34], [84] and [6].

Lemma A.1.1. *For $\alpha \in]1,2]$, the Green kernel has the following properties.*

(i) $G_\alpha(t,x)$ is positive for all $(t,x) \in]0,\infty[\times\mathbb{R}$. For every fixed $t \geq 0$, the unique mode of the function $x \mapsto G_\alpha(t,x)$ is located at $x = 0$. And

$$\int_{\mathbb{R}} G_\alpha(t,x)dx = 1. \tag{A.1}$$

(ii) Scaling property:

$$G_\alpha(t,x) = t^{-1/\alpha} G_\alpha(1, t^{-1/\alpha} x). \tag{A.2}$$

In particular,

$$G_\alpha(t,0) = t^{-1/\alpha} G_\alpha(1,0) \quad \textit{with} \quad G_\alpha(1,0) = \frac{\Gamma(1/\alpha)}{\pi\alpha}, \tag{A.3}$$

where Γ is Euler's Gamma function (see [61, p.80]).

(iii) There exists a positive constant c_α depending on α such that

$$\int_a^b \int_{\mathbb{R}} G_\alpha^2(t-r, x-v)\,dv\,dr = c_\alpha\left((t-a)^{\frac{\alpha-1}{\alpha}} - (t-b)^{\frac{\alpha-1}{\alpha}}\right), \quad a \leq b \leq t. \tag{A.4}$$

(iv) For $\alpha \in]1,2[$, there exists a constant K_α such that for all $x \in \mathbb{R}$,

$$0 < G_\alpha(1,x) \leq K_\alpha(1 + |x|^{\alpha+1})^{-1}. \tag{A.5}$$

(v) Semi-group property: for any $t,s \in]0,\infty[$ and $x,y \in U$,

$$\int_U G_\alpha(t,x-v)G_\alpha(s,y-v)\,dv = G_\alpha(t+s,x,y). \tag{A.6}$$

A.2 Negative moments of random variables

The next proposition is used many times to bound negative moments of a random variable.

Proposition A.2.1 ([26, Propositon 3.5])**.** *Suppose $Z \geq 0$ is a random variable for which we can find $\epsilon_0 \in]0,1[$, processes $\{Y_{i,\epsilon}\}_{\epsilon \in]0,1[}$, and constants $c>0$ and $0 \leq \alpha_2 \leq \alpha_1$ with the property that $Z \geq \min(c\epsilon^{\alpha_1} - Y_{1,\epsilon}, c\epsilon^{\alpha_2} - Y_{2,\epsilon})$ for all $\epsilon \in]0,\epsilon_0[$. Also suppose that we can find $\beta_i > \alpha_i$ $(i=1,2)$, not depending on ϵ_0, such that*

$$C(q) := \sup_{0<\epsilon<1} \max\left(\frac{E[|Y_{1,\epsilon}|^q]}{\epsilon^{q\beta_1}}, \frac{E[|Y_{2,\epsilon}|^q]}{\epsilon^{q\beta_2}}\right) < \infty \quad \textit{for all } q \geq 1.$$

Then for all $p \geq 1$, there exists a constant $c'_p \in]0,\infty[$, not depending on ϵ_0, such that

$$E[|Z|^{-p}] \leq c'_p \epsilon_0^{-p\alpha_1}.$$

A.3 Extension of [60, Lemma 4.2]

We first recall Burkholder's inequality for Hilbert-space-valued martingales; see also [7, Eq.(4.18)] and [26, Lemma 7.6].

Lemma A.3.1 ([58, E.2. p. 212])**.** *Let $H_{s,y}$ be a predictable $L^2(([0,t]\times\mathbb{R})^m, d\alpha)$-valued process, where $m \geq 1$ and $d\alpha$ denotes Lebesgue measure. Then, for any $p \geq 1$, there exists $C>0$ such that*

$$E\left[\left|\int_{([0,t]\times\mathbb{R})^m} \left(\int_0^t \int_{\mathbb{R}} H_{s,y}(\alpha)W(ds,dy)\right)^2 d\alpha\right|^p\right] \leq CE\left[\left|\int_0^t \int_{\mathbb{R}} \left(\int_{([0,t]\times\mathbb{R})^m} H_{s,y}^2(\alpha)\,d\alpha\right) dy\,ds\right|^p\right].$$

The next result is another version of Morien [60, Lemma 4.2] for the solution of SPDE (2.1.1) without boundary.

Lemma A.3.2. *Assume **P1**. For all $q \geq 1$, $T>0$ there exists $C>0$ such that for all $T \geq t \geq s \geq \epsilon > 0$ and $x \in \mathbb{R}$,*

$$\sum_{k,i=1}^d E\left[\left(\int_{s-\epsilon}^s dr \int_{\mathbb{R}} dv \left|D_{r,v}^{(k)}(u_i(t,x))\right|^2\right)^q\right] \leq C\epsilon^{(\alpha-1)q/\alpha}.$$

Proof. The proof follows the same lines as [60, Lemma 4.2]. We define

$$H_i(t,x) := \mathrm{E}\left[\left(\int_{s-\epsilon}^{s} dr \int_{\mathbb{R}} dv \left|D^{(k)}_{r,v}(u_i(t,x))\right|^2\right)^q\right], \tag{A.7}$$

and

$$K_s(t) := \sum_{i=1}^{d} \sup_{s\le\lambda\le t} \sup_{y\in\mathbb{R}} H_i(\lambda, y) \tag{A.8}$$

which are finite by (2.3.3). Thanks to formula (2.3.1), we have

$$\begin{aligned}
H_i(t,x) &\le c\left(\int_{s-\epsilon}^{s} dr \int_{\mathbb{R}} dv\, G_\alpha^2(t-r, x-v)\right)^q \\
&\quad + c\sum_{j=1}^{d} \mathrm{E}\Bigg[\Big[\int_{s-\epsilon}^{s} dr \int_{\mathbb{R}} dv \Big(\int_r^t \int_{\mathbb{R}} G_\alpha(t-\theta, x-\eta) \\
&\qquad\qquad \times D^{(k)}_{r,v}(\sigma_{ij}(u(\theta,\eta)))W^j(d\theta, d\eta)\Big)^2\Big]^q\Bigg] \\
&\quad + c\mathrm{E}\left[\left[\int_{s-\epsilon}^{s} dr \int_{\mathbb{R}} dv \left(\int_r^t \int_{\mathbb{R}} G_\alpha(t-\theta, x-\eta) D^{(k)}_{r,v}(b_i(u(\theta,\eta)))d\theta d\eta\right)^2\right]^q\right] \\
&:= A + B + C.
\end{aligned} \tag{A.9}$$

By (A.4), we see that

$$\begin{aligned}
\int_{s-\epsilon}^{s} dr \int_{\mathbb{R}} dv\, G_\alpha^2(t-r, x-v) &= c((t-s+\epsilon)^{\frac{\alpha-1}{\alpha}} - (t-s)^{\frac{\alpha-1}{\alpha}}) \\
&\le c'\epsilon^{\frac{\alpha-1}{\alpha}},
\end{aligned} \tag{A.10}$$

since the function $x \mapsto (x+\epsilon)^{(\alpha-1)/\alpha} - x^{(\alpha-1)/\alpha}$ is decreasing on $[0,\infty[$. This implies that

$$A \le c_q \epsilon^{(\alpha-1)q/\alpha}. \tag{A.11}$$

Using Burkholder's inequality for Hilbert-space-valued martingales (Lemma A.3.1) first, and then the Cauchy-Schwarz inequality together with the fact that the partial derivatives of σ_{ij}

are bounded, we obtain

$$
\begin{aligned}
B &\le c\sum_{j=1}^{d} \mathrm{E}\left[\Big[\int_{s-\epsilon}^{t} d\theta \int_{\mathbb{R}} d\eta \int_{s-\epsilon}^{s\wedge\theta} \int_{\mathbb{R}} G_\alpha^2(t-\theta, x-\eta)\left(D_{r,v}^{(k)}(\sigma_{ij}(u(\theta,\eta)))\right)^2 dr dv\Big]^q\right] \\
&\le c\sum_{l=1}^{d} \mathrm{E}\left[\left[\int_{s-\epsilon}^{t} d\theta \int_{\mathbb{R}} d\eta \int_{s-\epsilon}^{s\wedge\theta} \int_{\mathbb{R}} G_\alpha^2(t-\theta, x-\eta)\left(D_{r,v}^{(k)}(u_l(\theta,\eta))\right)^2 dr dv\right]^q\right] \\
&= c\sum_{l=1}^{d} \mathrm{E}\left[\left[\int_{s-\epsilon}^{s} d\theta \int_{\mathbb{R}} d\eta \int_{s-\epsilon}^{s\wedge\theta} \int_{\mathbb{R}} G_\alpha^2(t-\theta, x-\eta)\left(D_{r,v}^{(k)}(u_l(\theta,\eta))\right)^2 dr dv\right]^q\right] \\
&\quad + c\sum_{l=1}^{d} \mathrm{E}\left[\left[\int_{s}^{t} d\theta \int_{\mathbb{R}} d\eta \int_{s-\epsilon}^{s\wedge\theta} \int_{\mathbb{R}} G_\alpha^2(t-\theta, x-\eta)\left(D_{r,v}^{(k)}(u_l(\theta,\eta))\right)^2 dr dv\right]^q\right] \\
&:= B_1 + B_2. \qquad \text{(A.12)}
\end{aligned}
$$

We now apply Hölder's inequality with respect to the measure $G_\alpha^2(t-\theta, x-\eta)d\theta d\eta$ to find that

$$
\begin{aligned}
B_1 &\le c\sum_{l=1}^{d} \left|\int_{s-\epsilon}^{s} d\theta \int_{\mathbb{R}} d\eta\, G_\alpha^2(t-\theta, x-\eta)\right|^{q-1} \\
&\quad \times \int_{s-\epsilon}^{s} d\theta \int_{\mathbb{R}} d\eta\, G_\alpha^2(t-\theta, x-\eta) \mathrm{E}\left[\left(\int_{s-\epsilon}^{s\wedge\theta} dr \int_{\mathbb{R}} dv \left(D_{r,v}^{(k)}(u_l(\theta,\eta))\right)^2\right)^q\right] \\
&\le c\left|\int_{s-\epsilon}^{s} d\theta \int_{\mathbb{R}} d\eta\, G_\alpha^2(t-\theta, x-\eta)\right|^{q} \\
&\quad \times \sup_{(\theta,\eta)\in[0,T]\times\mathbb{R}} \mathrm{E}\left[\left(\int_0^T dr \int_{\mathbb{R}} dv \left(D_{r,v}^{(k)}(u_l(\theta,\eta))\right)^2\right)^q\right] \\
&\le c\epsilon^{(\alpha-1)q/\alpha}, \qquad \text{(A.13)}
\end{aligned}
$$

where the last inequality follows from (A.10) and (2.3.3). Again, applying Hölder's inequality with respect to the measure $G_\alpha^2(t-\theta, x-\eta)d\theta d\eta$, we see that

$$
\begin{aligned}
B_2 &\le c\left|\int_{s}^{t} d\theta \int_{\mathbb{R}} d\eta\, G_\alpha^2(t-\theta, x-\eta)\right|^{q-1} \\
&\quad \times \int_{s}^{t} d\theta \int_{\mathbb{R}} d\eta\, G_\alpha^2(t-\theta, x-\eta) \sum_{l=1}^{d} \mathrm{E}\left[\left(\int_{s-\epsilon}^{s} dr \int_{\mathbb{R}} dv \left(D_{r,v}^{(k)}(u_l(\theta,\eta))\right)^2\right)^q\right] \\
&\le c(t-s)^{\frac{\alpha-1}{\alpha}(q-1)} \int_{s}^{t} d\theta \int_{\mathbb{R}} d\eta\, G_\alpha^2(t-\theta, x-\eta) K_s(\theta) \\
&\le c\int_{s}^{t} (t-\theta)^{-\frac{1}{\alpha}} K_s(\theta) d\theta. \qquad \text{(A.14)}
\end{aligned}
$$

We handle the third term in (A.9) in a similar way. First, by the Cauchy-Schwarz inequality with respect to the measure $G_\alpha(t-\theta,x-\eta)d\theta d\eta$, we have

$$\begin{aligned}
C &\le c\mathrm{E}\left[\left[\int_{s-\epsilon}^{s} dr\int_{\mathbb{R}} dv\int_{r}^{t}\int_{\mathbb{R}} G_\alpha(t-\theta,x-\eta)\sum_{l=1}^{d}\left(D_{r,v}^{(k)}(u_l(\theta,\eta))\right)^2 d\theta d\eta\right]^q\right]\\
&= c\mathrm{E}\left[\left[\int_{s-\epsilon}^{t} d\theta\int_{\mathbb{R}} d\eta\int_{s-\epsilon}^{s\wedge\theta} dr\int_{\mathbb{R}} dv\, G_\alpha(t-\theta,x-\eta)\sum_{l=1}^{d}\left(D_{r,v}^{(k)}(u_l(\theta,\eta))\right)^2\right]^q\right]\\
&\le c\mathrm{E}\left[\left[\int_{s-\epsilon}^{s} d\theta\int_{\mathbb{R}} d\eta\, G_\alpha(t-\theta,x-\eta)\sum_{l=1}^{d}\int_{s-\epsilon}^{s\wedge\theta} dr\int_{\mathbb{R}} dv\left(D_{r,v}^{(k)}(u_l(\theta,\eta))\right)^2\right]^q\right]\\
&\quad + c\mathrm{E}\left[\left[\int_{s}^{t} d\theta\int_{\mathbb{R}} d\eta\, G_\alpha(t-\theta,x-\eta)\sum_{l=1}^{d}\int_{s-\epsilon}^{s} dr\int_{\mathbb{R}} dv\left(D_{r,v}^{(k)}(u_l(\theta,\eta))\right)^2\right]^q\right]\\
&:= C_1 + C_2.
\end{aligned} \tag{A.15}$$

By Hölder's inequality with respect to the measure $G_\alpha(t-\theta,x-\eta)d\theta d\eta$,

$$\begin{aligned}
C_1 &\le c\left|\int_{s-\epsilon}^{s} d\theta\int_{\mathbb{R}} d\eta\, G_\alpha(t-\theta,x-\eta)\right|^{q-1}\\
&\quad\times\int_{s-\epsilon}^{s} d\theta\int_{\mathbb{R}} d\eta\, G_\alpha(t-\theta,x-\eta)\sum_{l=1}^{d}\mathrm{E}\left[\left(\int_{s-\epsilon}^{s\wedge\theta} dr\int_{\mathbb{R}} dv\left(D_{r,v}^{(k)}(u_l(\theta,\eta))\right)^2\right)^q\right]\\
&\le c\left|\int_{s-\epsilon}^{s} d\theta\int_{\mathbb{R}} d\eta\, G_\alpha(t-\theta,x-\eta)\right|^{q}\\
&\quad\times\sum_{l=1}^{d}\sup_{(\theta,\eta)\in[0,T]\times\mathbb{R}}\mathrm{E}\left[\left(\int_{0}^{T} dr\int_{\mathbb{R}} dv\left(D_{r,v}^{(k)}(u_l(\theta,\eta))\right)^2\right)^q\right]\\
&\le c\epsilon^q \le c\epsilon^{(\alpha-1)q/\alpha},
\end{aligned} \tag{A.16}$$

where in the third inequality we use (A.1) and (2.3.3). Similarly,

$$\begin{aligned}
C_2 &\le c\left|\int_{s}^{t} d\theta\int_{\mathbb{R}} d\eta\, G_\alpha(t-\theta,x-\eta)\right|^{q-1}\\
&\quad\times\int_{s}^{t} d\theta\int_{\mathbb{R}} d\eta\, G_\alpha(t-\theta,x-\eta)\sum_{l=1}^{d} E\left[\left(\int_{s-\epsilon}^{s} dr\int_{\mathbb{R}} dv\left(D_{r,v}^{(k)}(u_l(\theta,\eta))\right)^2\right)^q\right]\\
&\le c\left|\int_{s}^{t} d\theta\int_{\mathbb{R}} d\eta\, G_\alpha(t-\theta,x-\eta)\right|^{q-1}\int_{s}^{t} d\theta\int_{\mathbb{R}} d\eta\, G_\alpha(t-\theta,x-\eta)K_s(\theta)\\
&\le c\int_{s}^{t} K_s(\theta)d\theta.
\end{aligned} \tag{A.17}$$

Finally, we put (A.9) and (A.11)–(A.17) together and obtain that

$$\begin{aligned}
K_s(t) &\le c\epsilon^{(\alpha-1)q/\alpha} + c\int_{s}^{t}(1+(t-\theta)^{-\frac{1}{\alpha}})K_s(\theta)d\theta\\
&\le c\epsilon^{(\alpha-1)q/\alpha} + \overline{c}\int_{s}^{t}(t-\theta)^{-\frac{1}{\alpha}}K_s(\theta)d\theta.
\end{aligned}$$

Define $\overline{K}_s(\lambda) := K_s(\lambda + s)$. From the above inequality we have

$$\overline{K}_s(t-s) \le c\epsilon^{(\alpha-1)q/\alpha} + \overline{c}\int_0^{t-s} (t-s-\theta)^{-\frac{1}{\alpha}} \overline{K}_s(\theta) d\theta.$$

By Gronwall's lemma [23, Lemma 15], we have

$$K_s(t) = \overline{K}_s(t-s) \le c\epsilon^{(\alpha-1)q/\alpha}, \quad \text{for all } s \le t.$$

□

The following lemma is a refinement of Lemma A.3.2.

Lemma A.3.3. *Fix $T > 0, c_0 > 1$ and $0 < \gamma_0 < 1$. For all $q \ge 1$ there exists $C > 0$ such that for all $T \ge t \ge s \ge \epsilon > 0$ with $t - s > c_0\epsilon^{\gamma_0}$ and $x \in \mathbb{R}$,*

$$\sum_{k,i=1}^{d} E\left[\left(\int_{s-\epsilon}^{s} dr \int_{\mathbb{R}} dv \left(D_{r,v}^{(k)}(u_i(t,x))\right)^2\right)^q\right] \le C\epsilon^{(1-\gamma_0+\gamma_0\frac{\alpha-1}{\alpha})q}.$$

Proof. We still use the notations as in the proof of Lemma A.3.2. First, under the condition $t - s > c_0\epsilon^{\gamma_0}$, using (A.4), we have

$$\begin{aligned}
\int_{s-\epsilon}^{s} dr \int_{\mathbb{R}} dv\, G_\alpha^2(t-r, x-v) &= c((t-s+\epsilon)^{\frac{\alpha-1}{\alpha}} - (t-s)^{\frac{\alpha-1}{\alpha}}) \\
&\le c((c_0\epsilon^{\gamma_0} + \epsilon)^{\frac{\alpha-1}{\alpha}} - (c_0\epsilon^{\gamma_0})^{\frac{\alpha-1}{\alpha}}) \\
&= c(c_0\epsilon^{\gamma_0})^{\frac{\alpha-1}{\alpha}}((1 + \frac{1}{c_0}\epsilon^{1-\gamma_0})^{\frac{\alpha-1}{\alpha}} - 1) \\
&\le c(c_0\epsilon^{\gamma_0})^{\frac{\alpha-1}{\alpha}} \frac{1}{c_0}\epsilon^{1-\gamma_0}\frac{\alpha-1}{\alpha} \\
&= c\frac{\alpha-1}{\alpha} c_0^{-\frac{1}{\alpha}} \epsilon^{(1-\gamma_0+\gamma_0\frac{\alpha-1}{\alpha})},
\end{aligned} \tag{A.18}$$

where the first inequality is because the function $x \mapsto (x+\epsilon)^{\frac{\alpha-1}{\alpha}} - x^{\frac{\alpha-1}{\alpha}}$ is decreasing on $[c_0\epsilon^{\gamma_0}, \infty[$, and the second inequality is due to $(1+x)^{\frac{\alpha-1}{\alpha}} - 1 \le \frac{\alpha-1}{\alpha}x$, for all $x \ge 0$. This implies that

$$A \le c\epsilon^{(1-\gamma_0+\gamma_0\frac{\alpha-1}{\alpha})q}. \tag{A.19}$$

Using (A.18) instead of (A.10), we see that

$$B_1 \le c\epsilon^{(1-\gamma_0+\gamma_0\frac{\alpha-1}{\alpha})q}. \tag{A.20}$$

Due to the choice of γ_0 and by (A.16), we have

$$C_1 \le c\epsilon^q \le c\epsilon^{(1-\gamma_0+\gamma_0\frac{\alpha-1}{\alpha})q}. \tag{A.21}$$

The estimates for other terms remain the same as in the proof of Lemma A.3.2. Therefore, we have obtained that

$$\begin{aligned}
K_s(t) &\le c\epsilon^{(1-\gamma_0+\gamma_0\frac{\alpha-1}{\alpha})q} + c\int_s^t (1+(t-\theta)^{-\frac{1}{\alpha}})K_s(\theta)d\theta \\
&\le c\epsilon^{(1-\gamma_0+\gamma_0\frac{\alpha-1}{\alpha})q} + \overline{c}\int_s^t (t-\theta)^{-\frac{1}{\alpha}}K_s(\theta)d\theta.
\end{aligned}$$

Applying Gronwall's lemma ([23, Lemma 15]), we have

$$K_s(t) \le c\epsilon^{(1-\gamma_0+\gamma_0\frac{\alpha-1}{\alpha})q}, \quad \text{for all } s \le t.$$

□

Remark A.3.4. *The result of Lemma A.3.3 is also valid for the solutions of stochastic heat equations with Neumann or Dirichlet boundary conditions in which case $\alpha = 2$. This is because the Green kernel of heat equation with Neumann or Dirichlet boundary conditions shares similar properties with the Green kernel of fractional heat equation, which enables us to derive the same estimates as in* (A.18)*,* (A.19)*,* (A.20) *and* (A.21) *for the solutions of stochastic heat equations with Neumann or Dirichlet boundary conditions.*

A.4 Quantitative version of the inverse function theorem

The following lemma is another version of the inverse function theorem. We give its proof for reader's convenience.

Lemma A.4.1 ([7, Lemma 3.2] and [69, Lemma 5.6])**.** *For any $\beta > 1$, $\delta > 0$, there exist constants $R, \alpha > 0$ such that any mapping $\Phi : \mathbb{R}^d \mapsto \mathbb{R}^d$ satisfying*

$$|\det \Phi'(0)| \ge \frac{1}{\beta} \quad \textit{and} \quad \sup_{\|z\|\le\delta} (\|\Phi'(z)\| + \|\Phi''(z)\|) \le \beta. \tag{A.22}$$

is a diffeomorphism from a neighborhood of 0 contained in the ball $B(0,R)$ onto the ball $B(\Phi(0),\alpha)$, and

$$\inf_{\|z\|\le R} \det \Phi'(z) \ge \frac{1}{2\beta}. \tag{A.23}$$

Proof. The proof is similar to that of [77, Theorem 9.24]. First, we introduce an inequality for $d\times d$ invertible matrix A (see [80]), that is,

$$\|A^{-1}\||\det A| \le (2^d-1)\|A\|^{d-1}. \tag{A.24}$$

We apply (A.24) with $A = \Phi'(0)^{-1}$ to see that

$$\|\Phi'(0)^{-1}\| \le \frac{2^d - 1}{|\det \Phi'(0)|} \|\Phi'(0)\|^{d-1} \le (2^d - 1)\beta^d. \tag{A.25}$$

We can choose $\delta_0 = \delta_0(\beta, \delta)$ small enough so that $\delta_0 < \delta$ and for any differentiable function $f : \mathbb{R}^d \mapsto \mathbb{R}^d$ satisfying $\det f'(0) \ge 1/\beta$ and $\sup_{\|z\|\le\delta}(\|f(z)\| + \|f'(z)\|) \le \beta$, we have $\inf_{\|z\|\le\delta_0} \det f'(z) \ge \frac{1}{2\beta}$. This is possible because the mapping $z \mapsto \det f'(z)$ is differentiable and its derivative can be bounded in terms of β, and then we can apply the mean value theorem. Choose $R = \frac{1}{4(2^d-1)\beta^{d+1}} \wedge \frac{\delta_0}{2}$. Then for any function Φ satisfying (A.22), we have $\inf_{\|z\|\le R} \det \Phi'(z) \ge \frac{1}{2\beta}$.

For each function Φ satisfying (A.22) and $y \in \mathbb{R}^d$, we associate a function φ, defined by

$$\varphi(x) = x + \Phi'(0)^{-1}(y - \Phi(x)), \quad x \in \mathbb{R}^d. \tag{A.26}$$

Note that $\Phi(x) = y$ if and only if x is a fixed point of φ.

Since $\varphi'(x) = I - \Phi'(0)^{-1}\Phi'(x) = \Phi'(0)^{-1}(\Phi'(0) - \Phi'(x))$, then (A.25) and the mean value theorem imply that for any $x \in B(0, R)$,

$$\begin{aligned}
\|\varphi'(x)\| &\le \|\Phi'(0)^{-1}\| \, \|\Phi'(0) - \Phi'(x))\| \\
&\le (2^d - 1)\beta^d \sup_{\|z\|\le\delta} \|\Phi''(z)\| \|x\| \\
&\le (2^d - 1)\beta^d \beta R \le \frac{1}{2}.
\end{aligned}$$

Hence

$$\|\varphi(x_1) - \varphi(x_2)\| \le \frac{1}{2}\|x_1 - x_2\|, \quad x_1, x_2 \in B(0, R). \tag{A.27}$$

By the contraction mapping theorem, φ has at most one fixed point in $B(0, R)$, so that $\Phi(x) = y$ for at most one $x \in B(0, R)$. Thus Φ is one-to-one from $B(0, R)$ to $\Phi(B(0, R))$.

Let us prove $\Phi(B(0, R))$ is open. Pick $y_0 \in \Phi(B(0, R))$. Then $y_0 = \Phi(x_0)$ for some $x_0 \in B(0, R)$. Let B_0 be an open ball with center at x_0 and radius $r_0 > 0$ so small that its closure $\bar{B}_0$ lies in $B(0, R)$. We will show that $y \in \Phi(B(0, R))$ whenever $\|y - y_0\| < \lambda r_0$ with $\lambda = \frac{1}{2(2^d-1)\beta^d}$.

Fix y such that $\|y - y_0\| < \lambda r_0$. With φ as in (A.26),

$$\begin{aligned}
\|\varphi(x_0) - x_0\| &= \|\Phi'(0)^{-1}(y - y_0)\| < \|\Phi'(0)^{-1}\| \lambda r_0 \\
&\le (2^d - 1)\beta^d \lambda r_0 = \frac{r_0}{2},
\end{aligned}$$

where the second inequality uses (A.25). If $x \in \bar{B}_0$, it therefore follows from (A.27) that

$$\begin{aligned}\|\varphi(x) - x_0\| &\le \|\varphi(x) - \varphi(x_0)\| + \|\varphi(x_0) - x_0\| \\ &< \frac{1}{2}\|x - x_0\| + \frac{r_0}{2} \le r_0.\end{aligned}$$

Hence $\varphi(x) \in B_0$.

It follows in addition that φ is a contraction of $\bar{B}_0$ into $\bar{B}_0$. Being a closed subset of $\mathbb{R}^d$, $\bar{B}_0$ is complete. Then the contraction mapping theorem implies that φ has a fixed point $x \in \bar{B}_0$. For this $x, \Phi(x) = y$. Thus $y \in \Phi(\bar{B}_0) \subset \Phi(B(0,R))$.

Denote $\alpha = \frac{R}{4(2^d-1)\beta^d}$. We are going to repeat the above arguments to show that for any mapping Φ satisfying (A.22), $B(\Phi(0),\alpha) \subset \Phi(B(0,R))$.

Fix y such that $\|y - \Phi(0)\| < \alpha$. Then by (A.25),

$$\|\varphi(0)\| = \|\Phi'(0)^{-1}(y - \Phi(0))\| < \|\Phi'(0)^{-1}\|\alpha \le (2^d - 1)\beta^d\alpha = \frac{R}{4}.$$

If $x \in \bar{B}(0, \frac{R}{2})$, it follows from (A.27) that

$$\begin{aligned}\|\varphi(x)\| &\le \|\varphi(x) - \varphi(0)\| + \|\varphi(0)\| \\ &< \frac{1}{2}\|x\| + \frac{R}{4} \le \frac{R}{2}.\end{aligned}$$

Hence $\varphi(x) \in \bar{B}(0, \frac{R}{2})$.

Thus φ is a contraction of $\bar{B}(0, \frac{R}{2})$ into $\bar{B}(0, \frac{R}{2})$. Therefore φ has a unique fixed point $x \in \bar{B}(0, \frac{R}{2})$. Thus $y = \Phi(x) \in \Phi(\bar{B}(0, \frac{R}{2})) \subset \Phi(B(0,R))$.

Next we prove that the inverse function Φ^{-1} is differentiable from $B(\Phi(0),\alpha)$ to $\Phi^{-1}(B(\Phi(0),\alpha))$. Pick $\bar{y} \in B(\Phi(0),\alpha)$, $\bar{y} + k \in B(\Phi(0),\alpha)$. Then there exist $\bar{x} \in \Phi^{-1}(B(\Phi(0),\alpha)), \bar{x} + h \in \Phi^{-1}(B(\Phi(0),\alpha))$ so that $\bar{y} = \Phi(\bar{x})$, $\bar{y} + k = \Phi(\bar{x} + h)$. With φ as in (A.26),

$$\varphi(\bar{x} + h) - \varphi(\bar{x}) = h + \Phi'(0)^{-1}(\Phi(\bar{x}) - \Phi(\bar{x} + h)) = h - \Phi'(0)^{-1}k.$$

By (A.27), $\|h - \Phi'(0)^{-1}k\| \le \frac{1}{2}\|h\|$. Hence $\|\Phi'(0)^{-1}k\| \ge \frac{1}{2}\|h\|$, and then from (A.25),

$$\|h\| \le 2\|\Phi'(0)^{-1}\|\|k\| \le 2(2^d - 1)\beta^d\|k\|. \tag{A.28}$$

We point out that (A.28) implies that the inverse function Φ^{-1} is continuous from $B(\Phi(0),\alpha)$ to $\Phi^{-1}(B(\Phi(0),\alpha))$. By (A.23), $\Phi'(\bar{x})$ has an inverse, say $T_{\bar{x}}$. Since

$$\Phi^{-1}(\bar{y} + k) - \Phi^{-1}(\bar{y}) - T_{\bar{x}}k = h - T_{\bar{x}}k = -T_{\bar{x}}(\Phi(\bar{x} + h) - \Phi(\bar{x}) - \Phi'(\bar{x})h),$$

then inequality (A.28) implies

$$\frac{\|\Phi^{-1}(\bar{y}+k)-\Phi^{-1}(\bar{y})-T_{\bar{x}}k\|}{\|k\|} \le 2(2^d-1)\beta^d \|T_{\bar{x}}\| \frac{\|\Phi(\bar{x}+h)-\Phi(\bar{x})-\Phi'(\bar{x})h\|}{\|h\|}.$$

As $k \to 0$, (A.28) shows that $h \to 0$. The right-hand side of the above inequality thus tends to 0. Hence the same is true of the left. We have thus proved Φ^{-1} is differentiable at any $\bar{y} \in B(\Phi(0),\alpha)$ and

$$\left.\frac{d\Phi^{-1}(y)}{dy}\right|_{y=\bar{y}} = \{\Phi'(\Phi^{-1}(\bar{y}))\}^{-1}.$$

So we have proved that Φ is a diffeomorphism from $\Phi^{-1}(B(\Phi(0),\alpha)) \subset B(0,R)$ onto the ball $B(\Phi(0),\alpha)$. □

A.5 A linear independence property of the Gaussian solution

In this section, we show that for any integer n, the random variables $u(t_1,x_1),\dots,u(t_n,x_n)$ are linearly independent in $L^2(\Omega)$, where $(t_i,x_i) \in]0,\infty] \times [0,1]$ and $(t_i,x_i) \neq (t_j,x_j)$ if $i \neq j$ for $i,j \in \{1,\dots,n\}$ (in the case of Dirichlet boundary conditions we assume $x_i \in]0,1[$) are the solution of (4.2.1).

Lemma A.5.1. *For any $t>0$, $x_i \in [0,1]$ (in the case of Dirichlet boundary conditions we assume $x_i \in]0,1[$) and $x_i \neq x_j$ for $i,j \in \{1,\dots,n\}$, the covariance matrix of the Gaussian random vector $(u(t,x_1),\dots,u(t,x_n))$ is positive definite.*

Proof. It suffices to prove that the smallest eigenvalue of the covariance matrix is positive. Let $\frac{1}{4}\min_{i\neq j}|x_i-x_j|^2 > \epsilon > 0$ and $\xi \in \mathbb{R}^n$ with $\|\xi\|=1$. Then

$$\begin{aligned}\sum_{i,j=1}^n \xi_i \mathrm{Cov}(u(t,x_i),u(t,x_j))\xi_j &= \int_0^t\int_0^1 \left(\sum_{i=1}^n \xi_i G(t-r,x_i,v)\right)^2 dr\,dv\\ &\ge \sum_{j=1}^n \int_{t-\epsilon}^t \int_{x_j-\sqrt{\epsilon}}^{x_j+\sqrt{\epsilon}} \left(\sum_{i=1}^n \xi_i G(t-r,x_i,v)\right)^2 dr\,dv\\ &\ge \frac{2}{3} I_\epsilon^2(\xi) - 2I_\epsilon^1(\xi),\end{aligned}$$

where

$$I_\epsilon^2(\xi) = \sum_{j=1}^n \int_{t-\epsilon}^t \int_{x_j-\sqrt{\epsilon}}^{x_j+\sqrt{\epsilon}} \xi_j^2 G^2(t-r,x_j,v)\,dr\,dv,$$

$$I_\epsilon^1(\xi) = \sum_{j=1}^n \int_{t-\epsilon}^t \int_{x_j-\sqrt{\epsilon}}^{x_j+\sqrt{\epsilon}} \left(\sum_{i\neq j}^n \xi_i G(t-r,x_i,v)\right)^2 dr\,dv.$$

By [7, (A.3)], there exists a constant c_1 such that

$$\inf_{\|\xi\|=1} I_\epsilon^2(\xi) = \int_{t-\epsilon}^{t} \int_{x_j-\sqrt{\epsilon}}^{x_j+\sqrt{\epsilon}} G^2(t-r, x_j, v)\,dr\,dv \geq c_1\sqrt{\epsilon}. \tag{A.29}$$

Using Cauchy-Schwarz's inequality and the fact $G(t-r,x,v) \leq \frac{c}{\sqrt{2\pi(t-r)}} \exp\left(-|x-v|^2/2(t-r)\right)$ (see for example [7, (A.1)]), we have

$$\begin{aligned}
\sup_{\|\xi\|=1} I_\epsilon^1(\xi) &\leq \sum_{j=1}^{n} \sum_{i\neq j}^{n} \int_{t-\epsilon}^{t} \int_{x_j-\sqrt{\epsilon}}^{x_j+\sqrt{\epsilon}} G^2(t-r, x_i, v)\,dr\,dv \\
&\leq c \sum_{j=1}^{n} \sum_{i\neq j}^{n} \int_{t-\epsilon}^{t} \int_{x_j-\sqrt{\epsilon}}^{x_j+\sqrt{\epsilon}} \frac{1}{2\pi(t-r)} \exp\left(-|x_i-v|^2/(t-r)\right) dr\,dv \\
&\leq c \sum_{j=1}^{n} \sum_{i\neq j}^{n} \int_{t-\epsilon}^{t} \int_{x_j-\sqrt{\epsilon}}^{x_j+\sqrt{\epsilon}} \frac{1}{2\pi(t-r)} \exp\left(-\min_{i\neq j} |x_i-x_j|^2/4(t-r)\right) dr\,dv \\
&\leq c\sqrt{\epsilon} \int_0^{\epsilon} \frac{1}{r} \exp\left(-\min_{i\neq j} |x_i-x_j|^2/(4r)\right) dr.
\end{aligned} \tag{A.30}$$

Combining (A.29) and (A.30), we have

$$\inf_{\|\xi\|=1} \sum_{i,j=1}^{n} \xi_i \mathrm{Cov}(u(t,x_i), u(t,x_j))\xi_j \geq \sqrt{\epsilon}\left(c_1 - c\int_0^{\epsilon} \frac{1}{r} \exp\left(-\min_{i\neq j}|x_i-x_j|^2/(4r)\right) dr\right).$$

Since $\lim_{\epsilon\to 0} \int_0^{\epsilon} \frac{1}{r} \exp\left(-\min_{i\neq j}|x_i-x_j|^2/(4r)\right) dr = 0$, there exists a positive constant ρ_0 such that

$$\inf_{\|\xi\|=1} \sum_{i,j=1}^{n} \xi_i \mathrm{Cov}(u(t,x_i), u(t,x_j))\xi_j \geq \rho_0.$$

□

Remark A.5.2. *Since Cov*$(u(t,x_i), u(t,x_j)) = \frac{1}{2}\int_0^{2t} G(r, x_i, x_j)\,dr$, *Lemma A.5.1 is equivalent to saying that the matrix with entries* $\int_0^{2t} G(r,x_i,x_j)\,dr$ *is positive definite.*

Lemma A.5.3. *For* $(t_i, x_i) \in]0,\infty[\times[0,1]$ *(in the case of Dirichlet boundary conditions we assume* $x_i \in]0,1[$*),* $i = 1,\ldots,n$ *with* $(t_i,x_i) \neq (t_j,x_j)$ *for* $i\neq j$*, the covariance matrix of the Gaussian random vector* $(u(t_1,x_1),\ldots,u(t_n,x_n))$ *is positive definite. In particular, if* $(a_1,\ldots,a_n) \neq (0,\ldots,0)$*, then*

$$E\left[\left(\sum_{i=1}^{n} a_i u(t_i,x_i)\right)^2\right] > 0.$$

Proof. We can assume the random vector $(u(t_1,x_1),\ldots,u(t_n,x_n))$ is of the form

$$(u(t_1,x_1^1),\ldots,u(t_1,x_{n_1}^1),\ldots,u(t_k,x_1^k),\ldots,u(t_k,x_{n_k}^k))$$

such that $\sum_{i=1}^{k} n_i = n$ and $t_1 > t_2 > \cdots > t_k$. Suppose there exists $\xi \in \mathbb{R}^n$ such that

$$\sum_{i,j=1}^{n} \xi_i \mathrm{Cov}(u(t_i, x_i), u(t_j, x_j))\xi_j = 0,$$

which is equivalent to

$$\mathrm{E}\left[\left(\int_0^{t_1}\int_0^1 \sum_{i=1}^{n} \xi_i 1_{\{r<t_i\}} G(t_i - r, x_i, v) W(dr, dv)\right)^2\right] = 0. \tag{A.31}$$

By the property of Itô integral, this implies that

$$\int_{t_2}^{t_1}\int_0^1 \left(\sum_{i=1}^{n_1} \xi_i G(t_1 - r, x_i^1, v)\right)^2 dr dv = 0,$$

which means that

$$\frac{1}{2}\sum_{i,j=1}^{n_1} \xi_i \xi_j \int_0^{2(t_1-t_2)} G(r, x_i^1, x_j^1) dr = 0.$$

By Remark A.5.2, we have $\xi_1 = \cdots = \xi_{n_1} = 0$. We substitute $\xi_1 = \cdots = \xi_{n_1} = 0$ into (A.31) to obtain that

$$\mathrm{E}\left[\left(\int_0^{t_2}\int_0^1 \sum_{i=n_1+1}^{n} \xi_i 1_{\{r<t_i\}} G(t_i - r, x_i, v) W(dr, dv)\right)^2\right] = 0. \tag{A.32}$$

Using again the property of Itô integral, (A.32) implies that

$$\int_{t_3}^{t_2}\int_0^1 \left(\sum_{i=n_1+1}^{n_2} \xi_i G(t_2 - r, x_i^2, v)\right)^2 dr dv = 0,$$

which means that

$$\frac{1}{2}\sum_{i,j=n_1+1}^{n_2} \xi_i \xi_j \int_0^{2(t_2-t_3)} G(r, x_i^2, x_j^2) dr = 0.$$

By Remark A.5.2, we have $\xi_{n_1+1} = \cdots = \xi_{n_2} = 0$. We repeat this argument and conclude $\xi = 0$. □

A.6 The Garsia, Rodemich and Rumsey lemma

In this section, we present two versions of the Garsia, Rodemich and Rumsey lemma.

Lemma A.6.1 ([25, Proposition A.1])**.** *Let (S, ρ) be a metric space, μ a Radon measure on S, and $\Psi : \mathbb{R} \to \mathbb{R}_+$ an even and convex function with $\Psi(0) = 0$, $\Psi(\infty) = \infty$ and Ψ is strictly increasing on $\mathbb{R}_+$. Suppose $p : [0, \infty[\to \mathbb{R}_+$ is continuous and strictly increasing, with $p(0) = 0$. Define, for* *any continuous function $f : S \to \mathbb{R}$,*

$$\mathscr{C} := \iint \Psi\left(\frac{f(x)-f(y)}{p(\rho(x,y))}\right)\mu(dx)\mu(dy). \tag{A.33}$$

Let $B_\rho(s,r)$ denote the open ρ-ball of radius $r>0$ about $s \in S$. Then, for all $s,t \in S$,

$$|f(t)-f(s)| \le 5\int_0^{2\rho(s,t)} \left[\Psi^{-1}\left(\frac{\mathscr{C}}{[\mu(B_\rho(s,u/4))]^2}\right)+\Psi^{-1}\left(\frac{\mathscr{C}}{[\mu(B_\rho(t,u/4))]^2}\right)\right] p(du). \tag{A.34}$$

We often use the following variant for functions with values in a Banach space.

Lemma A.6.2 ([64, Lemma A.3.1])**.** *Let $p, \Psi : \mathbb{R}_+ \to \mathbb{R}_+$ be continuous and strictly increasing functions vanishing at zero and such that $\lim_{t\uparrow\infty} \Psi(t) = \infty$. Suppose that $\phi : \mathbb{R}^d \to E$ is a continuous function with values in a separable Banach space $(E, \|\cdot\|)$. Denote by B the open ball in $\mathbb{R}^d$ centered at x_0 with radius r. Then, provided*

$$\Gamma = \int_B\int_B \Psi\left(\frac{\|\phi(t)-\phi(s)\|}{p(|t-s|)}\right)ds\,dt < \infty,$$

it holds, for all $s,t \in B$,

$$\|\phi(t)-\phi(s)\| \le 8\int_0^{2|t-s|} \Psi^{-1}\left(\frac{4^{d+1}\Gamma}{\lambda_d u^{2d}}\right)p(du),$$

where λ_d is a universal constant depending only on d.

To conclude the appendix, we cite the following lemma, which is a consequence of [20, Theorems 4.5.2 and 4.5.4].

Lemma A.6.3. *Let $\{X_k\}_{k=1}^\infty$ and X be random variables taking values in some Hilbert space H such X_k converges almost surely to X as $k \to \infty$ and $\sup_{k\ge1} E[\|X_k\|_H^q] < \infty$ for some $q > 0$. Then for any $0 < r < q$, $X \in L^r(\Omega, H)$ and*

$$\lim_{k\to\infty} E[\|X_k - X\|_H^r] = 0.$$

Bibliography

[1] Adler, R.J. : An Introduction to Continuity, Extrema, and Related Topics for General Gaussian Processes. *Lect. Notes Monogr. Ser.* **12**, 1–155 (1990)

[2] Adler, R.J. and Taylor, J.: Random Fields and Geometry. Springer, New York (2007)

[3] Albramowitz, A. and Stegun, S.: Handbook of mathematical functions. Nauka, Moskow (1979)

[4] Azaïs, J. and Wschebor, M.: Level sets and extrema of random processes and fields. Wiley, Hoboken, NJ (2009)

[5] Azerad, P. and Mellouk, M.: On a stochastic partial differential equation with non-local diffusion. *Potential Anal.* **27**, 183–197 (2007)

[6] Bally, V., Millet, A. and Sanz-Solé, M.: Approximation and support theorem in Hölder norm for parabolic stochastic partial differential equation. *Ann. Probab.* **23**, 178–222 (1995)

[7] Bally, V. and Pardoux, E.: Malliavin calculus for white noise driven parabolic SPDEs. *Potential Analysis* **9**, 27–64 (1998)

[8] Bass, R.F.: Stochastic Processes. Cambridge University Press, Cambridge (2011)

[9] Billingsley, P.: Convergence of Probability Measures. Second Edition. Wiley, New York (1999)

[10] Blumenthal, R.M. and Getoor, R.K. : Markov Processes and Potential Theory. Academic Press, New York (1968)

[11] Borell, C.: Convex measures on locally convex spaces. *Ark. Mat.* **12**, 239–252 (1974)

[12] Boulanba, L., Eddahbi, M. and Mellows, M.: Fractional SPDEs driven by spatially correlated noise: existence of the solution and smoothness of its density. *Osaka J. Math.* **47**, 41–65 (2010)

[13] Cardon-Weber, C.: Cahn-Hilliard stochastic equation: strict positivity of the density. *Stoch. Stoch. Rep.* **72**, 191–227 (2002)

[14] Chaleyat-Maurel, M. and Sanz-Solé, M.: Positivity of the density for the stochastic wave equation in two spatial dimensions. *ESAIM Probab. Stat.* **7**, 89–114 (2003)

[15] Chen, L.: Moments, intermittency, and growth indices for nonlinear stochastic PDE's with rough initial conditions. PhD thesis, No. 5712, École Polytechnique Fédérale de Lausanne, 2013.

[16] Chen, L. and Dalang, R.C.: Moments and growth indices for the nonlinear stochastic heat equation with rough initial conditions. *Ann. Probab.* **43**, 3006–3051 (2015)

[17] Chen, L. and Dalang, R.C.: Moments, intermittency and growth indices for the nonlinear fractional stochastic heat equation. *Stoch PDE: Anal Comp* **3**, 360–397 (2015)

[18] Chen, L. and Dalang, R.C.: Hölder-continuity for the nonlinear stochastic heat equation with rough initial conditions. *Stoch PDE: Anal Comp* **2**, 316–352 (2014)

[19] Chen, L., Hu, Y. and Nualart, D.: Regularity and strict positivity of densities for the nonlinear stochastic heat equation (2016) arXiv:1611.03909 (2016)

[20] Chung K.L. : A Course in Probability Theory (3rd ed). Academic Press (2001)

[21] Cirel'son, B.S.: Density of the distribution of the maximum of a Gaussian process. *Teor. Verojatnost. i Primenen.* **20**, no. 4, 865–873 (Russian) (1975)

[22] Clarke de la Cerda, J. and Tudor, C. A.: Hitting times for the stochastic wave equation with fractional colored noise. *Rev. Mat. Iberoam.* **30**, no. 2 685–709 (2014)

[23] Dalang, R.C.: Extending the martingale measure stochastic integral with application to spatially homogeneous S.P.D.E's. *Electon. J. Probab.* **4**, 1–29 (1999)

[24] Dalang, R.C., Khoshnevisan, D., Mueller, C., Nualart, D., Xiao, Y. and Rassoul-Agha, F.: A Minicourse on Stochastic Partial Differential Equations. Springer-Verlag (2009)

[25] Dalang, R.C., Khoshnevisan, D. and Nualart, E.: Hitting probabilities for systems of non-linear stochastic heat equations with additive noise. *ALEA* **3**, 231–271 (2007)

[26] Dalang, R.C., Khoshnevisan, D. and Nualart, E.: Hitting probabilities for systems of non-linear stochastic heat equations with multiplicative noise. *Probab. Th. Rel. Fields* **144**, 371–424 (2009)

[27] Dalang, R.C., Khoshnevisan, D. and Nualart, E.: Hitting probabilities for systems of non-linear stochastic heat equations in spatial dimension $k \geq 1$. *Stoch PDE: Anal Comp* **1**, 94–151 (2013)

[28] Dalang, R.C., Mueller, C. and Zambotti, L.: Hitting properties of parabolic s.p.d.e.'s with reflection. *Ann. Probab.* **34**, 1423–1450 (2006)

[29] Dalang, R.C. and Nualart, E.: Potential theory for hyperbolic SPDEs. *Ann. Probab.* **32**, 2099–2148 (2004)

[30] Dalang, R.C. and Sanz-Solé, M.: Criteria for hitting probabilities with applications to systems of stochastic wave equations. *Bernoulli* **16**, 1343–1368 (2010)

[31] Dalang, R.C. and Sanz-Solé, M.: Hitting probabilities for nonlinear systems of stochastic waves. *Mem. Amer. Math. Soc.* **237** v+75 (2015)

[32] Da Prato, G. and Zabczyk, J.: Ergodicity for Infinite Dimensional Systems. Cambridge University Press, Cambridge (1996)

[33] Debbi, L.: On some properties of a high order fractional differential operator which is not in general selfadjoint. *Appl. Math. Sci* **1**, 1325–1339 (2007)

[34] Debbi, L. and Dozzi, M.: On the solutions of nonlinear stochastic fractional partial differential equations in one spatial dimension. *Stochastic Process. Appl.* **115**, 1764–1781 (2005)

[35] Demengel, F. and Demengel, G.: Functional Spaces for the Theory of Elliptic Partial Differential Equations. Springer, London (2012)

[36] Diebolt, J. and Posse, C.: On the density of the maximum of smooth Gaussian processes. *Ann. Probab.* **24**, no. 3, 1104-1129 (1996)

[37] Doob, J.: Classical Potential Theory and Its Probabilistic Counterpart. Springer-Verlag, Berlin, New York (1984)

[38] Evans, L.C.: Partial differential equations. Graduate Studies in Mathematics, vol. 19. AMS (1998)

[39] Florit, C. and Nualart, D.: A local criterion for smoothness of densities and application to the supremum of the Brownian sheet. *Statist. Probab. Lett.* **22**, 25–31 (1995)

[40] Folland, G.B.: Introduction to partial differential equations. Second edition. Princeton University Press, Princeton, NJ (1995)

[41] Foondun, M., Khoshnevisan, D. and Mahboubi, P.: Analysis of the gradient of the solution to a stochastic heat equation via fractional Brownian motion. *Stoch. Partial Differ. Equ. Anal. Comput.* **3**, 133–158 (2015)

[42] Hayashi, M. and Kohatsu-Higa, A.: Smoothness of the Distribution of the Supremum of a Multi-dimensional Diffusion Process. *Potential Anal.* **38**, 57–77 (2013)

[43] Hu, Y. and Le, K.: A multiparameter Garsia-Rodemich-Rumsey inequality and some applications. *Stochastic Process. Appl.* **123**, 3359–3377 (2013)

[44] Khoshnevisan, D.: Multiparameter Processes. Springer-Verlag, New York (2002)

[45] Khoshnevisan, D.: Analysis of Stochastic Partial Differential Equations. Published by the AMS on behalf of CBMS Regional Conference Series in Mathematics **119**, 116. Providence RI (2014)

[46] Khoshnevisan, D. and Shi, Z.: Brownian sheet and capacity. *Ann. Probab.* **27**, 1135–1159 (1999)

[47] Kim, J. and Pollard, D.: Cube root asymptotics. *Ann. Statist.* **18**, 191–219 (1990)

[48] Kwaśnicki, M.: Ten equivalent definitions of the fractional Laplace operator. *Fract. Calc. Appl. Anal.* **20**, no 1, 7–51 (2017)

[49] Kohatsu-Higa, A.: Lower bound estimates for densities of uniformly elliptic random variables on Wiener space. *Probab. Th. Rel. Fields* **126**, 421–457 (2003)

[50] Komatsu, T.: On the martingale problem for generators of stable process with perturbations. *Osaka. J. Math.* **21**, no. 1, 113–132 (1984)

[51] Kunita, H.: Stochastic flows and stochastic differential equations. Cambridge University Press, Cambridge (1990)

[52] Kusuoka, S. and Stroock, D.: Applications of the Malliavin calculus. I, in: Stochastic analysis (Katata/Kyoto, 1982), North-Holland Math. Library, North-Holland, Amsterdam, pp. 271–306 (1984)

[53] Kusuoka, S. and Stroock, D.: Applications of the Malliavin calculus. II. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **32**, 1–76 (1985)

[54] Kusuoka, S. and Stroock, D.: Applications of the Malliavin calculus. III. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **34**, 391–442 (1987)

[55] Lanjri Zadi, N. and Nualart, D.: Smoothness of the law of the supremum of the fractional Brownian motion. *Electron. Comm. Probab.* **8**, 102–111 (2003)

[56] Lifshits, M.A.: Distribution of the maximum of a Gaussian process. *Teor. Veroyatnost. i Primenen.* **31**, no. 1, 134–142 (1986)

[57] Lifshits, M.A.: Gaussian random functions, volume 322 of Mathematics and its Applications. Kluwer Academic Publisher, Dordrecht (1995)

[58] Métivier, M.: Semimartingales. de Gruyter (1982)

[59] Millet, A. and Sanz-Solé, M.: Points of positive density for the solution to a hyperbolic SPDE. *Potential Anal.* **7**, 623–659 (1997)

[60] Morien, P.-L.: The Hölder and the Besov regularity of the density for the solution of a parabolic stochastic partial differential equation. *Bernoulli* **5**, 275–298 (1999)

[61] Mörters, P. and Peres, Y. : Brownian motion. Cambridge University Press, Cambridge (2010)

[62] Mueller, C. and Tribe, R.: Hitting properties of a random string. *Electron. J. Probab.* **7**, no. 10, 29pp (2002)

[63] Nakatsu, T.: Integration by parts formulas concerning maxima of some SDEs with applications to study on density functions. *Stoch. Anal. Appl.* **34**, 293–317 (2016)

[64] Nualart, D.: The Malliavin Calculus and Related Topics (2nd ed). Springer, London (2006)

[65] Nualart, D.: Analysis on Wiener space and anticipating stochastic calculus. Ecole d'Eté de Probabilités de Saint-Flour XXV. Lect. Notes in Math., vol. 1690, pp. 123–227. Springer, Heidelberg (1998)

[66] Nualart, D. and Quer-Sardanyons, L.: Existence and smoothness of the density for spatially homogeneous SPDEs. *Potential Anal.* **27**, 281–299 (2007)

[67] Nualart, D. and Quer-Sardanyons, L.: Gaussian density estimates for solutions to quasilinear stochastic partial differential equations. *Stochastic Process. Appl.* **119**, 3914–3938 (2009)

[68] Nualart, D. and Quer-Sardanyons, L.: Optimal Gaussian density estimates for a class of stochastic equations with additive noise. *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* **14**, 25–34 (2011)

[69] Nualart, E.: On the density of systems of non-linear spatially homogeneous SPDEs. *Stoch.* **85**, 48–70 (2013)

[70] Nualart, E. and Quer-Sardanyons, L.: Gaussian estimates for the density of the non-linear stochastic heat equation in any space dimension. *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* **14**, 25–34 (2011)

[71] Nualart, E. and Viens, F.: The fractional stochastic heat equation on the circle: time regularity and potential theory. *Stochastic Process. Appl.* **119**, 1505–1540 (2009)

[72] Piterbarg, V.: Asymptotic methods in the theory of Gaussian processes and fields. *Amer. Math. Soc.* **148** (1996)

[73] Pitt, L.D. and Lanh, T.T.: Local sample path properties of Gaussian fields. *Ann. Probab.* **7**, no. 3, 477–493 (1979)

[74] Port, S.C. and Stone, C.J.: Brownian Motion and Classical Potential Theory. Academic Press, New York (1978)

[75] Roussas, G.C.: An introduction to measure-theoretic probability. Second edition. Elsevier/Academic Press, Amsterdam (2014)

[76] Revuz, D. and Yor, M. : Continuous martingales and Brownian motion (3rd ed). Springer Verlag (1999)

[77] Rudin, R.: Principles of Mathematical Analysis. Third Edition. McGraw-Hill International Editions (1976)

[78] Sanz-Solé, M.: Malliavin calculus with applications to stochastic partial differential equations. EPFL Press, Lausanne (2005)

[79] Sanz-Solé, M. and Viles, N.: Systems of stochastic Poisson equations: hitting probabilities. *Stochastic Process. Appl.* **128**, 1857–1888 (2018)

[80] Schäffer, J.J.: Norms and determinants of linear mappings. *Math. Z.* **118**, 331–339 (1970)

[81] Walsh, J.B.: An introduction to stochastic partial differential equations. Ecole d'Eté de Probabilités de Saint-Flour XIV. Lect. Notes in Math., vol. 1180, 266–437. Springer, Heidelberg (1986)

[82] Watanabe, S.: Lectures on stochastic differential equations and Malliavin calculus. In: Tata Institute of Fundamental Research Lectures on Mathematics and Physics, vol. 73. Springer, Berlin (1984)

[83] Weber, M.: Sur la densité de la distribution du maximum d'un processus gaussien. *J. Math. Kyoto Univ.* **25**, no. 3, 515–521 (1985)

[84] Wu, D.: On the solution process for a stochastic fractional partial differential equation driven by space-time white noise. *Statist. Probab. Lett.* **81**, no. 8, 1161–1172 (2011)

[85] Xiao, Y.: Sample path properties of anisotropic Gaussian random fields. In *A minicourse on stochastic partial differential equations. Lecture Notes in Math.* **1962**, 145–212. Springer, Berlin (2009)

Curriculum Vitae

I was born on January 20th, 1988 in Lile, a small town of Nanchong city, Sichuan province, China. I spent my childhood in several nearby towns in Nanchong where I attended primary school. I studied at Lile middle school and then attended Nanchong High School. Since 2006, I undertook to study mathematics at Beijing Normal University. I obtained my Bachelor degree in 2010 and Master degree in 2013. For my master thesis, I studied strong solutions of stochastic differential equations with jumps under the supervision of Prof. Zenghu Li. In September 2013, I came to EPFL as a research and teaching assistant in the Chair of Probability and I began to work with Prof. Robert C. Dalang on the present Ph.D. thesis. During my doctoral studies, I served as teaching assistant for courses in probability, analysis, linear algebra, martingales and applications. I supervised seven students' semester projects and one student's master project, and had the opportunity to attend six summer and winter schools.

Teaching and mentoring

- Problem sessions for
  - Analysis I — Spring 2017
  - Analysis III — Fall 2017
  - Analysis IV — Spring 2014, 2016
  - Linear algebra — Spring 2017
  - Martingales and applications — Spring 2015
  - Probability — Fall 2013, 2014, 2015, 2016
- Mentoring student projects
  - Markov chains: Cristof Kaufmann — Fall 2013
  - Markov decision processes and wiener processes: Morgane Ferrara — Fall 2014
  - Martingale and Brownian motion: Valentin Bandelier — Fall 2014
  - Walsh's integration theory and stochastic heat equations: David Candil (master thesis project) — Spring 2015
  - Brownian motion: Melvin Kianmanesh — Spring 2016
  - Brownian motion: Slobodan Kristic — Spring 2017
  - Stochastic differential equations: Antony Henry — Spring 2017
  - Lévy processes: Raphaël Nicolet — Fall 2017

Summer and winter schools attended

- Malliavin calculus for Lévy processes. Technische Universität München. March - April 2014
  Lecturer: Bernt Øksendal.
- 2nd Barcelona Summer School on Stochastic Analysis. Centre de Recerca Matemàtica, Bellaterra. July 2014
  Lecturers: Davar Khoshnevisan, René Schilling.
- 44th Saint Flour probability summer school. July 2014
  Lecturers: Martin Hairer, Grégory Miermont.
- Recent Breakthroughs in Singular Stochastic PDEs. Bicocca Winter School. February 2015
  Lecturers: Massimiliano Gubinelli, Lorenzo Zambotti.
- Topics in renormalisation group theory and regularity structures. University of Warwick. May 2015
  Lecturers: David Brydges, Martin Hairer, Antii Kupianinen, Gordon Slade.
- 27th Jyväskylä Summer School. August 2017
  Lecturer: David Nualart