SIAM J. CONTROL OPTIM.
Vol. 57, No. 2, pp. 1127–1156

OPTIMAL TIME FOR THE CONTROLLABILITY OF LINEAR HYPERBOLIC SYSTEMS IN ONE-DIMENSIONAL SPACE*

JEAN-MICHEL CORON† AND HOAI-MINH NGUYEN‡

Abstract. We are concerned about the controllability of a general linear hyperbolic system of the form $\partial_t w(t,x) = \Sigma(x)\partial_x w(t,x) + \gamma C(x)w(t,x)$ ($\gamma \in \mathbb{R}$) in one space dimension using boundary controls on one side. More precisely, we establish the optimal time for the null and exact controllability of the hyperbolic system for generic γ. We also present examples which yield that the generic requirement is necessary. In the case of constant Σ and of two positive directions, we prove that the null-controllability is attained for any time greater than the optimal time for all $\gamma \in \mathbb{R}$ and for all C which is analytic if the slowest negative direction can be alerted by *both* positive directions. We also show that the null-controllability is attained at the optimal time by a feedback law when $C \equiv 0$. Our approach is based on the backstepping method paying a special attention on the construction of the kernel and the selection of controls.

Key words. hyperbolic systems, boundary controls, backstepping, optimal time

AMS subject classifications. 35F05, 35F15, 35B37, 58G16, 93C20

DOI. 10.1137/18M1185600

1. Introduction.

Linear hyperbolic systems in one-dimensional space are frequently used in modeling of many systems such as traffic flow, heat exchangers, and fluids in open channels. The stability and boundary stabilization of these hyperbolic systems have been studied intensively in the literature; see, e.g., [3] and the references therein. In this paper, we are concerned about the optimal time for the null-controllability and exact controllability of such systems using boundary controls on one side. More precisely, we consider the system

$$\partial_t w(t,x) = \Sigma(x)\partial_x w(t,x) + \gamma C(x)w(t,x) \text{ for } (t,x) \in \mathbb{R}_+ \times (0,1). \tag{1.1}$$

Here $w = (w_1, \dots, w_n)^\mathsf{T} : \mathbb{R}_+ \times (0,1) \to \mathbb{R}^n$ ($n \geq 2$), $\gamma \in \mathbb{R}$, Σ and C are ($n \times n$) real matrix-valued functions defined in $[0,1]$. We assume that for every $x \in [0,1]$, $\Sigma(x)$ is diagonal with $m \geq 1$ distinct positive eigenvalues and $k = n - m \geq 1$ distinct negative eigenvalues. Using Riemann coordinates, one might assume that $\Sigma(x)$ is of the form

$$\Sigma(x) = \operatorname{diag}\big(-\lambda_1(x), \dots, -\lambda_k(x), \lambda_{k+1}(x), \dots, \lambda_n(x)\big), \tag{1.2}$$

where

$$-\lambda_1(x) < \cdots < -\lambda_k(x) < 0 < \lambda_{k+1}(x) < \cdots < \lambda_{k+m}(x). \tag{1.3}$$

Throughout the paper, we assume that

$$\lambda_i \text{ is Lipschitz on } [0,1] \text{ for } 1 \leq i \leq n \, (= k+m). \tag{1.4}$$

*Received by the editors May 4, 2018; accepted for publication (in revised form) February 11, 2019; published electronically March 28, 2019.
http://www.siam.org/journals/sicon/57-2/M118560.html
Funding: The authors received support from the Institute for Theoretical Studies, ETH Zürich. They are also partially supported by ANR Finite4SoS grant ANR-15-CE23-0007.
†Sorbonne Université, Université Paris-Diderot SPC, CNRS, INRIA, Laboratoire Jacques-Louis Lions, équipe Cage, Paris, France (coron@ann.jussieu.fr).
‡Ecole Polytechnique Fédérale de Lausanne, EPFL, SB MATHAA CAMA, Station 8, CH-1015 Lausanne, Switzerland (hoai-minh.nguyen@epfl.ch).

We are interested in the following type of boundary conditions and boundary controls. The boundary conditions at $x = 0$ are given by

$$(w_1, \dots, w_k)^{\mathsf{T}}(t, 0) = B(w_{k+1}, \dots, w_{k+m})^{\mathsf{T}}(t, 0) \text{ for } t \geq 0 \tag{1.5}$$

for some $(k \times m)$ real *constant* matrix B, and the boundary controls at $x = 1$ are

$$w_{k+1}(t, 1) = W_{k+1}(t), \quad \dots, \quad w_{k+m}(t, 1) = W_{k+m}(t) \text{ for } t \geq 0, \tag{1.6}$$

where $W_{k+1}, \dots, W_{k+m}$ are controls. Our goal is to obtain the optimal time for the null-controllability and exact controllability of (1.1), (1.5), and (1.6). Let us recall that the control system (1.1), (1.5), and (1.6) is null-controllable (resp., exactly controllable) at the time $T > 0$ if, for every initial data $w_0 : (0, 1) \to \mathbb{R}^n$ in $[L^2(0, 1)]^n$ (resp., for every initial data $w_0 : (0, 1) \to \mathbb{R}^n$ in $[L^2(0, 1)]^n$ and for every (final) state $w_T : (0, 1) \to \mathbb{R}^n$ in $[L^2(0, 1)]^n$), there is a control $W = (W_{k+1}, \dots, W_{k+m})^{\mathsf{T}} : (0, T) \to \mathbb{R}^m$ in $[L^2(0, T)]^m$ such that the solution of (1.1), (1.5), and (1.6) satisfying $w(0, x) = w_0(x)$ vanishes (resp., reaches w_T) at the time T: $w(T, x) = 0$ (resp., $w(T, x) = w_T(x)$). Set

$$\tau_i := \int_0^1 \frac{1}{\lambda_i(\xi)} \, d\xi \text{ for } 1 \leq i \leq n \tag{1.7}$$

and

$$T_{opt} := \begin{cases} \max\left\{\tau_1 + \tau_{m+1}, \dots, \tau_k + \tau_{m+k}, \tau_{k+1}\right\} & \text{if } m \geq k, \\ \max\left\{\tau_{k+1-m} + \tau_{k+1}, \tau_{k+2-m} + \tau_{k+2}, \dots, \tau_k + \tau_{k+m}\right\} & \text{if } m < k. \end{cases} \tag{1.8}$$

The first result in this paper, which implies in particular that one can reach the null-controllability of (1.1), (1.5), and (1.6) at the time T_{opt} for generic γ (and B), is the following.

THEOREM 1.1. *Assume that* (1.3) *and* (1.4) *hold. We define*

$$\mathcal{B} := \left\{B \in \mathbb{R}^{k \times m} \quad \textit{such that } (1.10) \textit{ holds for } 1 \leq i \leq \min\{k, m-1\}\right\}, \tag{1.9}$$

where

(1.10)
the $i \times i$ matrix formed from the last i columns and the last i rows of B is invertible.

Then,

1. *in the case $m = 1$, there exists a (linear) time independent feedback which yields the null-controllability at the time T_{opt};*
2. *in the case $m = 2$, if $B \in \mathcal{B}$, $B_{k1} \neq 0$, Σ is constant, and ($T_{opt} = \tau_k + \tau_{k+2} = \tau_{k-1} + \tau_{k+1}$ if $k \geq 2$ and $T_{opt} = \tau_1 + \tau_3 = \tau_2$ if $k = 1$), then there exists a nonzero constant matrix C such that the system is* not *null-controllable at the time T_{opt};*
3. *in the case $m \geq 2$, we have* (i) *for each $B \in \mathcal{B}$, outside a discrete set of γ in $\mathbb{R}$, the control system* (1.1), (1.5), *and* (1.6) *is null-controllable at the time T_{opt}, and* (ii) *for each γ outside a discrete set in $\mathbb{R}$, outside a set of zero measure of B in $\mathcal{B}$, the control system* (1.1), (1.5), *and* (1.6) *is null-controllable at the time T_{opt}.*

Theorem 1.1 is proved in section 4. The optimality of T_{opt} is shown in Proposition 1.6 for $C \equiv 0$ (see also Remark 4.4).

Remark 1.2. In Proposition 5.1, we present a null-controllability result, which holds for all γ and $B \in \mathcal{B}$, for a time which is larger than T_{opt} but smaller than T_2 defined in (1.14) for $m \geq 2$.

Concerning the exact controllability, we have the following theorem, whose proof is just a straightforward modification of the one of Theorem 1.1 (see Remark 4.3).

THEOREM 1.3. *Assume that* $m \geq k \geq 1$, (1.3) *and* (1.4) *hold. Define*

$$\mathcal{B}_e := \big\{B \in \mathbb{R}^{k\times m} \quad \textit{such that } (1.10) \textit{ holds for } 1 \leq i \leq k\big\}, \tag{1.11}$$

Then, (i) *for each* $B \in \mathcal{B}_e$, *outside a discrete set of* γ *in* $\mathbb{R}$, *the control system* (1.1), (1.5), *and* (1.6) *is exactly controllable at the time* T_{opt}, *and* (ii) *for each* γ *outside a discrete set in* $\mathbb{R}$, *outside a set of zero measure of* B *in* $\mathcal{B}_e$, *the control system* (1.1), (1.5), *and* (1.6) *is exactly controllable at the time* T_{opt}.

Remark 1.4. In the case $k = m = 1$, the result of Theorem 1.3 holds for all γ and $B \in B_e$, which was already proved in [23]. Our proof can be modified to obtain this result.

In the case where $k \geq 1$, $m = 2$, Σ is constant, $B \in \mathcal{B}$, and $B_{k1} \neq 0$, we show that the system is *null-controllable* for any time greater than T_{opt} for all $\gamma \in \mathbb{R}$ and C analytic. More precisely, we have the following.

THEOREM 1.5. *Let* $k \geq 1$, $m = 2$, *and* $T > T_{opt}$. *Assume that* (1.3) *holds,* $B \in \mathcal{B}$ *and* $B_{k1} \neq 0$, Σ *is constant, and* C *is analytic on* $[0, L]$,[1] *where*

$$L = \frac{\rho_k}{\rho_k - 1} \quad \textit{with} \quad \rho_k = \begin{cases} \frac{\lambda_{k+2}}{\lambda_{k+1}} & \textit{if } k = 1, \\ \min\big\{ \min_{1\leq j<i\leq k} \frac{\lambda_j}{\lambda_i}, \frac{\lambda_{k+2}}{\lambda_{k+1}}\big\} & \textit{if } k \geq 2. \end{cases} \tag{1.12}$$

Then the system is null-controllable at the time T. *Similarly, if in addition* $m \geq k$ *and* $B \in \mathcal{B}_e$, *then the system is exactly controllable at the time* T.

Theorem 1.5 is proved in section 6.

In the case $C \equiv 0$, we can prove that T_{opt} is the optimal time for the null-controllability of the considered system via a linear time independent feedback law. More precisely, we have the following.

PROPOSITION 1.6. *Assume that* $C \equiv 0$ *and* (1.10) *holds for* $1 \leq i \leq \min\{k, m-1\}$. *There exists a* linear time independent feedback *which yields the null-controllability at the time* T_{opt}. *Assume in addition that* (1.10) *holds for* $i = \min\{k, m\}$; *then, for any* $T < T_{opt}$, *there exists an initial datum such that* $u(T, \cdot) \not\equiv 0$ *for every control.*

Proposition 1.6 is proved in section 7.

We now briefly describe the method used in the proofs. Our approach relies on backstepping due to Miroslav Krstic and his coauthors (see also Remark 1.7). More precisely, we make the following change of variables:

$$u(t, x) = w(t, x) - \int_0^x K(x, y) w(t, y)\, dy$$

[1]This means that C is analytic in a neighborhood of $[0, L]$.

for some kernel $K : \mathcal{T} = \big\{(x, y) \in (0,1)^2; 0 < y < x\big\} \to \mathbb{R}^n$. The idea is to choose K in such a way that the controllability of the target system of u is easier to investigate. In our case, K is chosen so that (2.6) holds with $K(x, 0)$ having appropriate properties; see in particular (2.9) and (2.11).

The use of the backstepping method to obtain the null-controllability for hyperbolic systems in one dimension was initiated in [10] for the case $m = k = 1$. This approach has been developed later on for a more general hyperbolic system in [14, 1, 7]. In [10], the optimal time T_{opt} is obtained for the case $m = k = 1$. In [14], the authors considered the case where Σ is constant. They obtained the null-controllability for the time

$$T_1 := \tau_k + \sum_{l=1}^{m} \tau_{k+l}. \tag{1.13}$$

It was later shown in [1, 7] that one can reach the null-controllability at the time

$$T_2 := \tau_k + \tau_{k+1}. \tag{1.14}$$

In [14, 1, 7], one does not require any conditions on B and the optimal time in this case is T_2. With the convention (1.3), it is clear that

$$T_{opt} \le T_2 \le T_1$$

and

$$T_2 < T_1 \text{ if } m > 1 \quad \text{ and } \quad T_{opt} < T_2 \text{ if } m > 1 \text{ or } k > 1.$$

When $C \equiv 0$, Hu [13] established the exact controllability for quasi-linear systems, i.e., $A = A(u)$, in the case $m \ge k$ for the time

$$T_3 := \max\{\tau_{k+1}, \tau_k + \tau_{m+1}\}$$

under a condition on B, which is equivalent to (1.10) with $i = k$ in our setting. It is clear in the case $m \ge k$ that

$$T_{opt} \le T_3 \le T_2, \quad T_3 = T_{opt} \text{ if } k = 1, \quad \text{ and } \quad T_{opt} < T_3 \text{ if } k > 1.$$

In the linear case, the null controllability was established for the time T_2 without any assumption on B and the exact controllability was obtained in the case $m = k$ under a condition which is different but has some similar features to condition (1.10) with $i = k$ in [23, Theorem 3.2]. In the quasi-linear case with $m \ge k$, the exact controllability was derived in [11, Theorem 3.2] (see also [12]) for $m \ge k$ and for the time T_2 under a condition which is equivalent to (1.10) with $i = k$ in our setting.

Theorems 1.1 and 1.3 and Proposition 1.6 confirm that generically the optimal time to reach the null-controllability for the system in (1.1), (1.5), and (1.6) is T_{opt}. Condition $B \in \mathcal{B}$ (resp., $B \in \mathcal{B}_e$) is very natural to obtain the null-controllability (resp., exact-controllability) at T_{opt} (see section 4.2 for details) which roughly speaking allows us to use the l controls $W_{k+m-l+1}, \dots, W_{k+m}$ to control $u_{k-l+1}, \dots, u_k$ for $1 \le l \le \min\{k, m\}$ (the possibility to implement l controls corresponding to the fastest positive speeds to control l components corresponding to the lowest negative speeds).

In comparison with the previous works mentioned above, our analysis contains two new ingredients. First, after transforming the system into a new one (target system) via the backstepping method as usual, we carefully choose the control varying with respect to time so that the zero state is reachable at T_{opt}; in the previous works, the zero controls were used for the target system. Second, the boundary conditions of the kernel obtained from the backstepping approach given in this paper are different from the known ones. Our idea is to explore as much as possible the boundary conditions of the kernel to make the target system as simple as possible from the control point of view.

Remark 1.7. The backstepping method also has been used to stabilize the wave equation [16, 22, 19], the parabolic equations in [20, 21], and nonlinear parabolic equations [24]. The standard backstepping approach relies on the Volterra transform of the second kind. In some situations, more general transformations are considered as for Korteweg–de Vries equations [5], the Kuramoto–Sivashinsky equations [8], and Schrödinger's equation [6]. The use of the backstepping method to obtain the null-controllability of the heat equation is given in [9]. A concise introduction of this method applied to numerous partial differential equations can be found in [17].

The paper is organized as follows. In section 2, we apply the backstepping approach to derive the target system and the equations for the kernel. Section 3 is devoted to some properties on the control systems and the kernel. The proofs of Theorems 1.1 and 1.5 are presented in sections 4 and 6, respectively. A null-controllability result which holds for all γ and $B \in \mathcal{B}$ is given in section 5. In section 7, we present the proof of Proposition 1.6.

2. A change of variables via backstepping approach. Systems of the kernel and the target. In what follows, we assume that $\gamma = 1$, and the general case can be obtained from this case by replacing C by γC. As in [2, section 3], [10, section 4], and [15, section 3], without loss of generality, one can assume that $C_{ii}(x) = 0$ for $1 \le i \le n$. The key idea of the backstepping approach is to make the change of variables

$$u(t,x) = w(t,x) - \int_0^x K(x,y)w(t,y)\,dy \tag{2.1}$$

for some kernel $K : \mathcal{T} \to \mathbb{R}^{n\times n}$ which is chosen in such a way that the system for u is easier to control. Here

$$\mathcal{T} = \big\{(x,y) \in (0,1)^2; 0 < y < x\big\}. \tag{2.2}$$

To determine/derive the equations for K, we first compute $\partial_t u(t,x) - \Sigma(x)\partial_x u(t,x)$. Taking into account (2.1), we formally have[2]

$$\begin{aligned}
\partial_t u(t,x) &= \partial_t w(t,x) - \int_0^x K(x,y)\partial_t w(t,y)\,dy \\
&= \partial_t w(t,x) - \int_0^x \Big[K(x,y)\big(\Sigma(y)\partial_y w(t,y) + C(y)w(t,y)\big)\Big]\,dy \quad \text{(by (1.1))} \\
&= \partial_t w(t,x) - K(x,x)\Sigma(x)w(t,x) + K(x,0)\Sigma(0)w(t,0) \\
&\quad + \int_0^x \Big[\partial_y\big(K(x,y)\Sigma(y)\big)w(t,y) \\
&\qquad\quad - K(x,y)C(y)w(t,y)\Big]\,dy \quad \text{(by integrating by parts)}
\end{aligned}$$

[2]We assume here that u, w, and K are smooth enough so that the computations below make sense.

and

$$\partial_x u(t,x) = \partial_x w(t,x) - \int_0^x \partial_x K(x,y) w(t,y)\, dy - K(x,x) w(t,x).$$

It follows from (1.1) that

(2.3)

$$\begin{aligned}&\partial_t u(t,x) - \Sigma(x)\partial_x u(t,x)\\ &\quad = \Big(C(x) - K(x,x)\Sigma(x) + \Sigma(x)K(x,x)\Big) w(t,x) + K(x,0)\Sigma(0)u(t,0)\\ &\qquad + \int_0^x \Big[\partial_y K(x,y)\Sigma(y) + K(x,y)\Sigma'(y) - K(x,y)C(y) + \Sigma(x)\partial_x K(x,y)\Big] w(t,y)\, dy.\end{aligned}$$

We seek a kernel K which satisfies the two conditions

$$\partial_y K(x,y)\Sigma(y) + \Sigma(x)\partial_x K(x,y) + K(x,y)\Sigma'(y) - K(x,y)C(y) = 0 \text{ in } \mathcal{T} \tag{2.4}$$

and

$$\mathcal{C}(x) := C(x) - K(x,x)\Sigma(x) + \Sigma(x)K(x,x) = 0 \text{ for } x \in (0,1), \tag{2.5}$$

so that one formally has

$$\partial_t u(t,x) = \Sigma(x)\partial_x u(t,x) + K(x,0)\Sigma(0)u(t,0) \text{ for } (t,x) \in \mathbb{R}_+ \times (0,1). \tag{2.6}$$

In fact, such a K exists so that (2.6) holds (see Proposition 3.5). We have

(2.7) the (i,j) component of the matrix $\partial_y K(x,y)\Sigma(y) + \Sigma(x)\partial_x K(x,y)$ is

$$a_{ij}(y)\partial_y K_{ij}(x,y) + b_{ij}(x)\partial_x K_{ij}(x,y),$$

where

$$\big(a_{ij}(y), b_{ij}(x)\big) = \begin{cases} \big(-\lambda_j(y), -\lambda_i(x)\big) & \text{if } 1 \le i,j \le k,\\ \big(\lambda_j(y), -\lambda_i(x)\big) & \text{if } 1 \le i \le k < k+1 \le j \le k+m,\\ \big(\lambda_j(y), \lambda_i(x)\big) & \text{if } k+1 \le i,j \le k+m,\\ \big(-\lambda_j(y), \lambda_i(x)\big) & \text{if } 1 \le j \le k < k+1 \le i \le k+m.\end{cases} \tag{2.8}$$

We denote

$$\Gamma_1 = \big\{(x,x); x \in (0,1)\big\}, \quad \Gamma_2 = \big\{(x,0); x \in (0,1)\big\}, \quad \text{and} \quad \Gamma_3 = \big\{(1,y); y \in (0,1)\big\}.$$

Remark 2.1. By the characteristic method, it is possible to impose the following boundary conditions for K_{ij} when Σ is constant:

- on Γ_1 if $a_{ij}/b_{ij} \le 0$; see case (a) in Figure 1;
- on both Γ_1 and Γ_2 if $0 < a_{ij}/b_{ij} < 1$; see case (b) in Figure 1;
- on Γ_1 and Γ_3 if $a_{ij}/b_{ij} > 1$; see case (c) in Figure 1;
- on Γ_2 if $a_{ij}/b_{ij} = 1$; see case (d) in Figure 1.

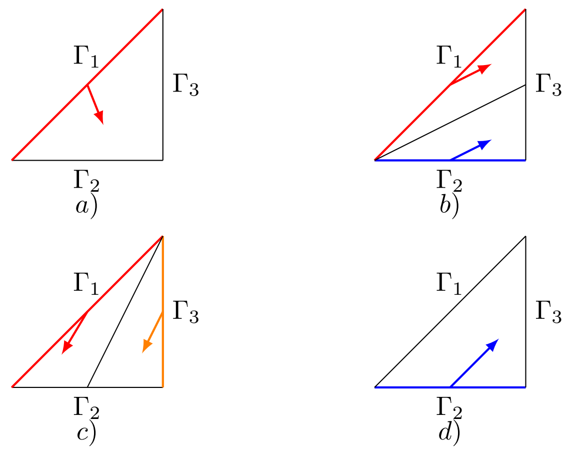

FIG. 1. *The characteristic vectors of K_{ij} in the case Σ is constant:* (a) *in the case $a_{ij}/b_{ij} < 0$ ($1 \le i \le k < k+1 \le j \le k+m$ or $1 \le j \le k < k+1 \le i \le k+m$),* (b) *in the case $0 < a_{ij}/b_{ij} < 1$ ($1 \le i < j \le k$ or $k+1 \le j < i \le k+m$),* (c) *in the case $a_{ij}/b_{ij} > 1$ ($1 \le j < i \le k$ or $k+1 \le i < j \le k+m$), and* (d) *in the case $a_{ij}/b_{ij} = 1$ ($1 \le i = j \le k+m$).*

To impose (appropriate) boundary conditions of K on Γ_2 so that the system for u is simple, we investigate the term $K(x,0)\Sigma(0)u(t,0)$. Set

$$Q := \begin{pmatrix} 0_k & B \\ 0_{m,k} & I_m \end{pmatrix}. \tag{2.9}$$

Here and in what follows, $0_{i,j}$ denotes the zero matrix of size $i \times j$, and 0_i and I_i denote the zero matrix and the identity matrix of the size $i \times i$ for i, $j \in \mathbb{N}$. Using the boundary conditions at $x = 0$ in (1.5) and the fact that $u(t,0) = w(t,0)$, we obtain

$$u(t,0) = Qu(t,0).$$

It follows that

$$K(x,0)\Sigma(0)u(t,0) = K(x,0)\Sigma(0)Qu(t,0).$$

We have, by the definition of Q in (2.9),

$$\Sigma(0)Q = \begin{pmatrix} 0_k & \Sigma_-(0)B \\ 0_{m,k} & \Sigma_+(0) \end{pmatrix}.$$

Here and in what follows, we define, for $x \in [0,1]$,

$$\Sigma_-(x) := \operatorname{diag}\big(-\lambda_1(x), \dots, -\lambda_k(x)\big) \text{ and } \Sigma_+(x) := \operatorname{diag}\big(\lambda_{k+1}(x), \dots, \lambda_{k+m}(x)\big).$$

Denote

$$K(x,0) = \begin{pmatrix} K_{--}(x) & K_{-+}(x) \\ K_{+-}(x) & K_{++}(x) \end{pmatrix},$$

where K_{--}, K_{-+}, K_{+-}, and K_{++} are matrices of size $k \times k$, $k \times m$, $m \times k$, and $m \times m$, respectively. Set

$$S(x) := K(x,0)\Sigma(0)Q. \tag{2.10}$$

We have

(2.11)
$$S(x) = \begin{pmatrix} 0_k & K_{--}(x)\Sigma_-(0)B + K_{-+}(x)\Sigma_+(0) \\ 0_{m,k} & K_{+-}(x)\Sigma_-(0)B + K_{++}(x)\Sigma_+(0) \end{pmatrix} = \begin{pmatrix} 0_k & S_{-+}(x) \\ 0_{m,k} & S_{++}(x) \end{pmatrix}.$$

We impose boundary conditions for K_{ij} on Γ_1, Γ_2, and Γ_3 as follows:

(BC_1) For (i,j) with $1 \le i \neq j \le k+m$, we impose the boundary condition for K_{ij} on Γ_1 in such a way that $\mathcal{C}_{ij}(x) = 0$ (recall that $\mathcal{C}$ is defined in (2.5)). More precisely, we have, noting that $a_{ij} \neq b_{ij}$,

$$K_{ij}(x,x) = C_{ij}(x)/\big(a_{ij}(x) - b_{ij}(x)\big) \text{ for } x \in (0,1). \tag{2.12}$$

(BC_2) Set

$$\mathcal{J} = \big\{(i,j);\ 1 \le i \le j \le k \text{ or } k+1 \le j \le i \le k+m\big\},$$

Note that if $\big(i \neq j$ and $(i,j) \in \mathcal{J}\big)$, then $0 < a_{ij}(0)/b_{i,j}(0) < 1$ and the characteristic trajectory passing $(0,0)$ is inside $\mathcal{T}$ as in case (b) in Figure 1. Using (2.7) and (2.8), we can impose the boundary condition of K_{ij} on Γ_2 with $(i,j) \in \mathcal{J}$ in such a way that, for $x \in (0,1)$,

$$K_{ij}(x,0) = 0 \text{ for } 1 \le i \le j \le k \tag{2.13}$$

and

$$(S_{++})_{pq}(x) = 0 \text{ for } 1 \le q \le p \le m. \tag{2.14}$$

These imposed conditions can be written under the form, for $(i,j) \in \mathcal{J}$,

$$K_{ij}(x,0) = \sum_{(r,s)\notin\mathcal{J}} c_{ijrs}(B) K_{rs}(x,0) \text{ for } x \in (0,1), \tag{2.15}$$

for some $c_{ijrs}(B)$ which is linear with respect to B. Indeed, (2.13) can be written under the form of (2.15) with $c_{ijrs} = 0$, and for $1 \le q \le p \le m$, $K_{p,q}$ can be written under the form of (2.15) since the (p,q) component of $S_{++} = K_{+-}(x)\Sigma_-(0)B + K_{++}(x)\Sigma_+(0)$ is 0.

(BC_3) For (i,j) with either $1 \le j < i \le k$ or $k+1 \le i < j \le k+m$, we impose the zero boundary condition of K_{ij} on Γ_3, i.e.,

$$K_{ij}(1,y) = 0 \text{ for } y \in (0,1). \tag{2.16}$$

(Note that in this case $a_{ij}(1)/b_{ij}(1) > 1$ and hence the characteristic trajectory passing $(1,1)$ is in $\mathcal{T}$ as in case (c) in Figure 1).

Below are the form of $S(= S^{k,m})$ when (BC_2) is taken into account for some pairs (k,m):

$$S^{2,3}(x) = \begin{pmatrix} 0 & 0 & * & * & * \\ 0 & 0 & * & * & * \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad S^{3,2}(x) = \begin{pmatrix} 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{2.17}$$

Here and in what follows, in a matrix, $*$ means that this part of that matrix can be whatever.

Remark 2.2. We here impose (2.13) on Γ_1 and (2.16) on Γ_3. These choices are just for the simplicity of presentation. We later modify these in the proof of Theorem 1.5.

3. Properties of the control systems and the kernel. In this section, we establish the well-posedness of u, w, and K and the unique determination of w from u. For notational ease, we assume that $\gamma = 1$ (except in Lemma 3.3 and its proof), and the general case follows easily. We first investigate the well-posedness of w and u under the boundary conditions and the controls considered. We consider a more general control system, for $T > 0$,

$$\begin{cases} \partial_t v(t,x) = \Sigma(x)\partial_x v(t,x) + C(x)v(t,x) + D(x)v(t,0) + f(t,x) & \text{for } (t,x) \in (0,T)\times(0,1),\\ v_-(t,0) = Bv_+(t,0) + g(t) & \text{for } t \in (0,T),\\ v_+(t,1) = \displaystyle\sum_{r=1}^{R} A_r(t)v\big(t,x_r\big) + \int_0^1 M(t,y)v(t,y)\,dy + h(t) & \text{for } t \in (0,T),\\ v(t=0,x) = v_0(x) & \text{for } x \in (0,1), \end{cases} \tag{3.1}$$

where $v_- = (v_1,\dots,v_k)^\mathsf{T}$ and $v_+ = (v_{k+1},\dots,v_{k+m})^\mathsf{T}$. Here $R \in \mathbb{N}$, $C, D : [0,1] \to \mathbb{R}^{n\times n}$, $A_r : [0,T] \to \mathbb{R}^{m\times n}$, $x_r \in [0,1]$ $(1 \le r \le R)$, $M : [0,T]\times[0,1] \to \mathbb{R}^{n\times n}$, $f \in \big[L^\infty\big((0,T)\times(0,1)\big)\big]^n$, $g \in [L^\infty(0,T)]^k$, and $h \in [L^\infty(0,T)]^m$. We make the following assumptions for this system:

$$x_r < c < 1 \text{ for some constant } c, \tag{3.2}$$

$$C, D \in [L^\infty(0,1)]^{n\times n}, \quad A_r \in [L^\infty(0,T)]^{n\times n}, \quad \text{and} \quad M \in [L^\infty\big((0,T)\times(0,1)\big)]^{n\times n}. \tag{3.3}$$

We are interested in bounded broad solutions of (3.1) whose definition is as follows. Extend λ_i in $\mathbb{R}$ by $\lambda_i(0)$ for $x < 0$ and $\lambda_i(1)$ for $x \ge 1$. For $(s,\xi) \in [0,T]\times[0,1]$, define $x_i(t,s,\xi)$ for $t \in \mathbb{R}$ by

$$\frac{d}{dt}x_i(t,s,\xi) = \lambda_i\big(x_i(t,s,\xi)\big) \text{ and } x_i(s,s,\xi) = \xi \text{ if } 1 \le i \le k \tag{3.4}$$

and

$$\frac{d}{dt}x_i(t,s,\xi) = -\lambda_i\big(x_i(t,s,\xi)\big) \text{ and } x_i(s,s,\xi) = \xi \text{ if } k+1 \le i \le k+m. \tag{3.5}$$

The following definition of broad solutions for (3.1) is used in this paper.

Definition 3.1. *A function $v = (v_1,\dots,v_{k+m}) : (0,T)\times(0,1) \to \mathbb{R}^{k+m}$ is called a broad solution of* (3.1) *if $v \in [L^\infty\big((0,T)\times(0,1)\big)]^{k+m} \cap [C\big([0,T];L^2(0,1)\big)]^{k+m} \cap [C\big([0,1];L^2(0,T)\big)]^{k+m}$ and if, for almost every $(\tau,\xi) \in (0,T)\times(0,1)$, we have*

1. *for $k+1 \le i \le k+m$,*

$$\begin{aligned} v_i(\tau,\xi) = &\int_t^\tau \sum_{j=1}^{n} \Big(C_{ij}\big(x_i(s,\tau,\xi)\big)v_j\big(s,x_i(s,\tau,\xi)\big) \\ &+ D_{ij}\big(x_i(s,\tau,\xi)\big)v_j(s,0) + f_i\big(s,x_i(s,\tau,\xi)\big)\Big)\,ds \\ &+ \sum_{r=1}^{R}\sum_{j=1}^{n} A_{r,ij}(t)v_j\big(t,x_r\big) + \int_0^1 \sum_{j=1}^{n} M_{ij}(t,x)v_j(t,x)\,dx + h(t) \end{aligned} \tag{3.6}$$

if $x_i(0,\tau,\xi) > 1$ *and* t *is such that* $x_i(t,\tau,\xi) = 1$*, and*

$$v_i(\tau,\xi) = \int_0^\tau \sum_{j=1}^n \Big(C_{ij}\big(x_i(s,\tau,\xi)\big) v_j\big(s, x_i(s,\tau,\xi)\big) + D_{ij}\big(x_i(s,\tau,\xi)\big) v_j(s,0)$$
$$+ f_i\big(s, x_i(s,\tau,\xi)\big)\Big)\, ds + v_{0,i}\big(x_i(0,\tau,\xi)\big) \tag{3.7}$$

if $x_i(0,\tau,\xi) < 1$*;*

2. *for* $1 \le i \le k$*,*

$$v_i(\tau,\xi) = \int_t^\tau \sum_{j=1}^n \Big(C_{ij}(x_i(s,\tau,\xi)) v_j(s, x_i(s,\tau,\xi)) + D_{ij}(x_i(s,\tau,\xi)) v_j(s,0)$$
$$+ f_i\big(s, x_i(s,\tau,\xi)\big)\Big)\, ds + \sum_{j=1}^m B_{ij} v_{j+k}(t,0) + g_i(t) \tag{3.8}$$

if $x_i(0,\tau,\xi) < 0$ *and* t *is such that* $x_i(t,\tau,\xi) = 0$ *where* $v_{j+k}(t,0)$ *is defined by the right-hand side (RHS) of* (3.6) *or* (3.7) *with* $(\tau,\xi) = (t,0)$*, and*

$$v_i(\tau,\xi) = \int_0^\tau \sum_{j=1}^n \Big(C_{ij}(x_i(s,\tau,\xi)) v_j(s, x_i(s,\tau,\xi)) + D_{ij}(x_i(s,\tau,\xi)) v_j(s,0)$$
$$+ f_i\big(s, x_i(s,\tau,\xi)\big)\Big)\, ds + v_{0,i}\big(x_i(0,\tau,\xi)\big) \tag{3.9}$$

if $x_i(0,\tau,\xi) > 0$*.*

Here and in what follows, v_i denotes the ith component of v, $v_{i,0}$ denotes the ith component of v_0, and $A_{r,ij}$ denotes the (i,j) component of A_r.

Classical solutions are smooth broad solutions. Conversely, smooth broad solutions are classical solutions. This is a consequence of the following lemma on the well-posedness of (3.1).

Lemma 3.2. *Let* $v_0 \in [L^\infty(0,1)]^n$*,* $f \in \big[L^\infty\big((0,T)\times(0,1)\big)\big]^n$*,* $g \in [L^\infty(0,T)]^k$*, and* $h \in [L^\infty(0,T)]^m$*, and assume* (3.2) *and* (3.3)*. Then* (3.1) *has a unique broad solution* v*.*

Proof. The proof is based on a fixed point argument. To this end, define $\mathcal{F}$ from $\mathcal{Y} := \big[L^\infty\big((0,T)\times(0,1)\big)\big]^n \cap \big[C\big([0,T]; L^2(0,1)\big)\big]^n \cap \big[C\big([0,1]; L^2(0,T)\big)\big]^n$ into itself as follows, for $v \in \mathcal{Y}$ and for $(\tau,\xi) \in (0,T)\times(0,1)$:

$$\big(\mathcal{F}(v)\big)_i(\tau,\xi) \text{ is the RHS of (3.6) or (3.7) or (3.8) or (3.9)} \quad \text{under the corresponding conditions.} \tag{3.10}$$

Set

$$\mathcal{N} := \|B\|_{L^\infty} + \|C\|_{L^\infty} + \|D\|_{L^\infty} + \|M\|_{L^\infty} + \sum_{r=1}^R \|A_r\|_{L^\infty}.$$

We claim that there exist two constants L_1, $L_2 > 1$ depending only on c, $\mathcal{N}$, and Σ such that $\mathcal{F}$ is a contraction map for the norm

$$\|v\| := \sup_{1\le i\le n} \text{ ess sup }_{(\tau,\xi)\in(0,T)\times(0,1)} e^{-L_1\tau - L_2\xi} |v_i(\tau,\xi)|. \tag{3.11}$$

We first consider the case where $\big(\mathcal{F}(v)\big)_i(\tau,\xi)$ is given by the RHS of (3.7) or (3.9). We claim that, for $v, \hat v \in \mathcal{Y}$,

$$e^{-L_1\tau - L_2\xi}\big|\big(\mathcal{F}(v)\big)_i(\tau,\xi) - \big(\mathcal{F}(\hat v)\big)_i(\tau,\xi)\big| \le \|v-\hat v\|/(10n) \tag{3.12}$$

if L_2 is large enough and L_1 is much larger than L_2. Indeed, we have, with $V = v - \hat v$,

$$\begin{aligned}\big|\big(\mathcal{F}(v)\big)_i(\tau,\xi) - \big(\mathcal{F}(\hat v)\big)_i(\tau,\xi)\big| &\le \mathcal{N}\int_0^\tau \big(|V(s,x_i(s,\tau,\xi))| + |V(s,0)|\big)\,ds\\ &\le 2\sqrt{n}\mathcal{N}L_1^{-1}\|V\|e^{\tau L_1 + L_2},\end{aligned}$$

which implies (3.12).

We next consider the case where $\big(\mathcal{F}(v)\big)_i(\tau,\xi)$ is given by the RHS of (3.6). We have

$$\begin{aligned}\big|\big(\mathcal{F}(v) - \mathcal{F}(\hat v)\big)_i(\tau,\xi)\big| &\le \mathcal{N}\bigg(\int_t^\tau \big(|V(s,x_i(s,\tau,\xi))| + |V(s,0)|\big)\,ds\\ &\qquad + |V(t,x_r)| + \int_0^1 |V(t,x)|\,dx\bigg)\\ &\le 2\sqrt{n}\mathcal{N}\big(L_1^{-1}e^{L_1\tau+L_2}\|V\| + e^{L_1 t + L_2 c}\|V\| + L_2^{-1}e^{L_1 t + L_2}\|V\|\big).\\ &\le 4\sqrt{n}\mathcal{N}\big(L_1^{-1}e^{L_1\tau+L_2}\|V\| + L_2^{-1}e^{L_1 t + L_2}\|V\|\big)\end{aligned}$$

if L_2 is large enough since $c < 1$. Since $\tau - t \ge C(1-\xi)$ for some positive constant depending only on Σ, k, and m by the definitions of x_i and t, it follows that

$$e^{-L_1\tau - L_2\xi}\big|\big(\mathcal{F}(v) - \mathcal{F}(\hat v)\big)_i(\tau,\xi)\big| \le \|V\|/2 \tag{3.13}$$

if L_2 is large and L_1 is much larger than L_2.

We finally consider the case where $\big(\mathcal{F}(v)\big)_i(\tau,\xi)$ is given by the RHS of (3.8). We have

$$\begin{aligned}\big|\big(\mathcal{F}(v) - \mathcal{F}(\hat v)\big)_i(\tau,\xi)| &\le \mathcal{N}\left(\int_t^\tau \big(|V(s,x_i(s,\tau,\xi))| + |V(s,0)|\big)\,ds + \sum_{j=k+1}^{k+m}|V_j(t,0)|\right)\\ &\le 2\sqrt{n}\mathcal{N}\left(L_1^{-1}e^{L_1\tau+L_2}\|V\| + \sum_{j=k+1}^{k+m}|V_j(t,0)|\right). \qquad (3.14)\end{aligned}$$

From (3.6) and (3.7), as in the previous cases, we have

$$\mathcal{N}e^{L_2}e^{-L_1 t}|V_j(t,0)| \le \|V\|/(10n) \quad \text{ for } k+1 \le j \le k+m$$

if L_2 is large and L_1 is much larger than L_2. We derive from (3.14) that

$$e^{-L_1\tau - L_2\xi}\big|\big(\mathcal{F}(v)\big)_i(\tau,\xi) - \big(\mathcal{F}(\hat v)\big)_i(\tau,\xi)\big| \le \|V\|/2 \tag{3.15}$$

if L_2 is large enough and L_1 is much larger than L_2.

Combining (3.12), (3.13), and (3.15) yields, for $v, \hat v \in \mathcal{Y}$,

$$\|\mathcal{F}(v) - \mathcal{F}(\hat v)\| \le \|v - \hat v\|/2.$$

Thus $\mathcal{F}$ is a contraction mapping. By the Banach fixed-point theorem, there exists a unique $v \in \mathcal{Y}$ such that

$$\mathcal{F}(v) = v.$$

The proof is complete. ∎

Concerning K, we have the following result.

LEMMA 3.3. *Assume* (3.2) *and* (3.3). *There exists a unique broad bounded solution* $K : \mathcal{T} \to \mathbb{R}^{n\times n}$ *of system* (2.4), (2.12), (2.15), *and* (2.16). *Moreover,* $(\gamma, B) \in \mathbb{R} \times \mathbb{R}^{k\times m} \mapsto K \in [L^\infty(\mathcal{T})]^{n\times n}$ *is analytic.*

Remark 3.4. The broad solution meaning of K is understood via the characteristic approach similar to Definition 3.1. The continuity assumptions in Definition 3.1 are replaced by the assumption that $\tilde{K}(\cdot, y) \in L^2([0,1])$ is continuous w.r.t. to $y \in [0,1)$ where $\tilde{K}(x,y) = K((1-y)x, y)$ and similar facts for x and $x+y$ variables.

Proof. Using a similar approach, one can establish the existence and uniqueness of K. The real analytic with respect to each component of B can be proved by showing that K is holomorphic with respect to each component of B. In fact, for notational ease, assuming again that $\gamma = 1$, one can prove that

$$\frac{\partial K}{\partial B_{pq}} = \hat{K} \text{ in } \mathcal{T}$$

(the derivative is understood for a complex variable), where $\hat{K}$ is the bounded broad solution of (2.4),

$$\hat{K}_{ij}(x,x) = 0 \text{ for } x \in (0,1),\ 1 \le i \ne j \le k+m,$$
$$\hat{K}_{ij}(1,y) = 0 \text{ for } y \in (0,1),\ 1 \le i < j \le k \text{ or } k+1 \le j < i \le k+m$$

(which are derived from (2.12) and (2.16)), and for $(i,j) \in \mathcal{J}$,

$$\hat{K}_{ij}(x,0) = \sum_{(r,s)\notin\mathcal{J}} c_{ijrs}(B)\hat{K}_{rs}(x,0) + \sum_{(r,s)\notin\mathcal{J}} \frac{\partial c_{ijrs}(B)}{\partial B_{pq}} K_{rs}(x,0) \text{ for } x \in (0,1), \tag{3.16}$$

which is obtained from (2.15). The existence and uniqueness of $\hat{K}$ can be established as in the proof of Lemma 3.2, where the second term in the RHS of (3.16) plays a role similar to the one of g in Lemma 3.2. The details of the proof are left to the reader.

The analyticity with respect to γ can be proved by showing that K is holomorphic with respect to γ. In fact, one can prove that

$$\frac{\partial K}{\partial \gamma} = \hat{\mathbf{K}} \text{ in } \mathcal{T}$$

(the derivative is understood for a complex variable), where $\hat{\mathbf{K}}$ is the bounded broad solution of

$$\partial_y \hat{\mathbf{K}}(x,y)\Sigma(y) + \Sigma(x)\partial_x \hat{\mathbf{K}}(x,y) + \hat{\mathbf{K}}(x,y)\Sigma'(y) - \gamma\hat{\mathbf{K}}(x,y)C(y) = K(x,y)C(y) \text{ in } \mathcal{T}, \tag{3.17}$$

by (2.4),

$$\hat{\mathbf{K}}_{ij}(x,x) = \frac{C_{ij}(x)}{a_{ij}(x) - b_{ij}(x)} \text{ for } x \in (0,1),\ 1 \le i \ne j \le k+m, \tag{3.18}$$
$$\hat{\mathbf{K}}_{ij}(1,x) = 0 \text{ for } x \in (0,1),\ 1 \le i < j \le k \text{ or } k+1 \le j < i \le k+m,$$

by (2.12), and (2.16), and

$$\hat{\mathbf{K}}_{ij}(x,0) = \sum_{(r,s)\notin \mathcal{J}} c_{ijrs}(B)\hat{\mathbf{K}}_{rs}(x,0) \text{ for } x \in (0,1),\ (i,j) \in \mathcal{J} \tag{3.19}$$

by (2.15). Here $K(x,y)$ denotes the solution corresponding to fixed γ and B. Note that γ does not appear in the boundary conditions of $\hat{\mathbf{K}}$. The details are omitted. □

A connection between w and u is given in the following proposition.

PROPOSITION 3.5. *Let $w_0 \in L^\infty\big((0,1)\big)$ and let $w \in L^\infty\big((0,T)\times(0,1)\big)$. Define u_0 and u from w_0 and w by* (2.1)*, respectively, and let S be given by* (2.10)*. Assume* (3.2) *and* (3.3)*. We have that, if w is a broad solution of the system*

$$\begin{cases} \partial_t w(t,x) = \Sigma(x)\partial_x w(t,x) + C(x)w(t,x) & \textit{for } (t,x) \in (0,T)\times(0,1), \\ w_-(t,x=0) = Bw_+(t,x=0) & \textit{for } t \in (0,T), \\ u_+(t,1) = \displaystyle\sum_{r=1}^{R} A_r(t)u\big(t,x_r\big) + \int_0^1 M(t,y)u(t,y)\,dy & \textit{for } t \in (0,T), \\ w(t=0)(x) = w_0(x) & \textit{for } x \in (0,1), \end{cases} \tag{3.20}$$

then u is a broad solution of the system

$$\begin{cases} \partial_t u(t,x) = \Sigma(x)\partial_x u(t,x) + S(x)u(t,0) & \textit{for } (t,x) \in (0,T)\times(0,1), \\ u_-(t,x=0) = Bu_+(t,x=0) & \textit{for } t \in (0,T), \\ u_+(t,1) = \displaystyle\sum_{r=1}^{R} A_r(t)u\big(t,x_r\big) + \int_0^1 M(t,y)u(t,y)\,dy & \textit{for } t \in (0,T), \\ u(t=0,x) = u_0(x) & \textit{for } x \in (0,1). \end{cases} \tag{3.21}$$

Remark 3.6. In the two sides of the third condition in (3.20), u is given by (2.1). Therefore, this condition is understood as a condition on w. By Lemma 3.2, there exist a unique broad solution w of (3.20) and a unique broad solution u of (3.21).

Proof. We first assume in addition that C and Σ are smooth on $[0,1]$. Let K_n be a C^1-solution of (2.4) and (2.12) such that

$$\|K_n\|_{L^\infty(\mathcal{T})} \le M \quad \text{and} \quad K_n \to K \text{ in } L^1(\mathcal{T}), \tag{3.22}$$

where M is a positive constant independent of n. Such a K_n can be obtained by considering the solution of (2.4), (2.12), and

$$K_{n,ij}(x,0) = \sum_{(r,s)\notin\mathcal{J}} c_{ijrs}(B)K_{n,rs}(x,0) + g_n(x,0) \text{ for } x \in (0,1)$$

and

$$K_{n,ij}(1,x) = h_n(x) \text{ for } x \in (0,1)$$

instead of (2.15) and (2.16), respectively, where (g_n), (h_n) are chosen such that (g_n) and (h_n) are bounded in $L^\infty(0,1)$, $(g_n),(h_n) \to 0$ in $L^1(0,1)$, and the compatibility conditions hold for K_n at $(0,0)$ and $(1,1)$.[3] Set, for sufficiently small positive ε,

$$w_\varepsilon(t,x) = \frac{1}{2\varepsilon}\int_{t-\varepsilon}^{t+\varepsilon} w(s,x)\,ds \text{ in } (\varepsilon, T-\varepsilon)\times(0,1).$$

Then $w_\varepsilon \in W^{1,\infty}\big((\varepsilon,T-\varepsilon)\times(0,1)\big)$ and

$$\partial_t w_\varepsilon(t,x) = \Sigma(x)\partial_x w_\varepsilon(t,x) + C(x) w_\varepsilon(t,x) \text{ in } (\varepsilon,T-\varepsilon)\times(0,1).$$

Define

$$u_{n,\varepsilon}(t,x) = w_\varepsilon(t,x) - \int_0^x K_n(x,y) w_\varepsilon(t,y)\,dy \text{ in } (\varepsilon,T-\varepsilon)\times(0,1)$$

and

$$u_n(t,x) = w(t,x) - \int_0^x K_n(x,y) w(t,y)\,dy \text{ in } (0,T)\times(0,1).$$

As in (2.6), we have

$$\partial_t u_{n,\varepsilon}(t,x) = \Sigma(x)\partial_x u_{n,\varepsilon}(t,x) + K_n(x,0)\Sigma(0)u_{n,\varepsilon}(t,0) \text{ in } (\varepsilon,T-\varepsilon)\times(0,1).$$

By letting $\varepsilon \to 0$, we obtain

$$\partial_t u_n(t,x) = \Sigma(x)\partial_x u_n(t,x) + K_n(x,0)\Sigma(0)u_n(t,0) \text{ for } (t,x)\in(0,T)\times(0,1).$$

By letting $n \to +\infty$ and using (3.22), we derive that

$$\partial_t u(t,x) = \Sigma(x)\partial_x u(t,x) + K(x,0)\Sigma(0)u(t,0) \text{ for } (t,x)\in(0,T)\times(0,1).$$

This yields the first equation of (3.21). The other parts of (3.21) are clear from the definition of w_0 and w.

We next consider the general case, in which no further additional smooth assumption on Σ and C is required. The proof in the case can be derived from the previous case by approximating Σ and C by smooth functions. The details are omitted. ∎

The fact that w is uniquely determined from u is a consequence of the following standard result on the Volterra equation of the second kind whose proof is omitted; this implies in particular that $w(t,\cdot) \equiv 0$ in $(0,1)$ if $u(t,\cdot) \equiv 0$ in $(0,1)$.

LEMMA 3.7. *Let $d \in \mathbb{N}$, $\tau_1,\tau_2 \in \mathbb{R}$ be such that $\tau_1 < \tau_2$ and let $G : \big\{(t,s) : \tau_1 \le s \le t \le \tau_2\big\} \to \mathbb{R}^{d\times d}$ be bounded measurable. For every $F \in \big[L^\infty(\tau_1,\tau_2)\big]^d$, there exists a unique solution $U \in \big[L^\infty(\tau_1,\tau_2)\big]^d$ of the following equation:*

$$U(t) = F(t) + \int_{\tau_1}^t G(t,s)U(s)\,ds \quad \text{for } t \in (\tau_1,\tau_2).$$

[3] One needs to establish the stability for the L^1-norm for the system of K. This can be done as in [4].

4. Null-controllability for generic γ and B: Proof of Theorem 1.1.

4.1. Proof of part 1 of Theorem 1.1. Choose $u_{k+1}(t,1) = 0$ for $t \ge 0$. Since $S_{++} = 0$ by (2.14), we have

$$u_{k+1}(t,0) = 0 \text{ for } t \ge \tau_{k+1} \quad \text{and} \quad u_{k+1}(T_{opt},x) = 0 \text{ for } x \in (0,1).$$

This implies, by (1.5),

$$u_i(t,0) = 0 \text{ for } t \ge \tau_{k+1}, 1 \le i \le k.$$

We derive from (2.6) that

$$u_i(T_{opt},x) = 0 \text{ for } x \in (0,1), 1 \le i \le k.$$

The null-controllability at the time T_{opt} is attained for u and hence for w by Lemma 3.7.

4.2. Proof of part 3 of Theorem 1.1. We here establish part 3 of Theorem 1.1 even for $m \ge 1$. We hence assume that $m \ge 1$ in this section. Set

$$t_0 = T_{opt}, \quad t_1 = t_0 - \tau_1, \dots, \quad t_k = t_0 - \tau_k, \tag{4.1}$$

and, for $1 \le l \le k$,

$$x_{0,l} = 0 \quad \text{and} \quad x_{i,l} = x_l(t_0,t_i,0) \quad \text{for } 1 \le i \le l. \tag{4.2}$$

Recall that x_l is defined in (3.4) for $1 \le l \le k$. (See Figure 2 in the case where Σ is constant.)

In the next two sections, we deal with the cases $m \ge k$ and $m < k$, respectively.

4.2.1. On the case $m \ge k$. The idea of the proof is to derive sufficient conditions to be able to steer the control system from the initial data to 0 at the time T_{opt}. These conditions will be written under the form $U + \mathcal{K}U = \mathcal{F}$ (see (4.13)), where $\mathcal{K}$ is an analytic, compact operator with respect to λ and F depending on the initial data. We then apply the Fredholm theory to obtain the conclusion. We now proceed with the proof.

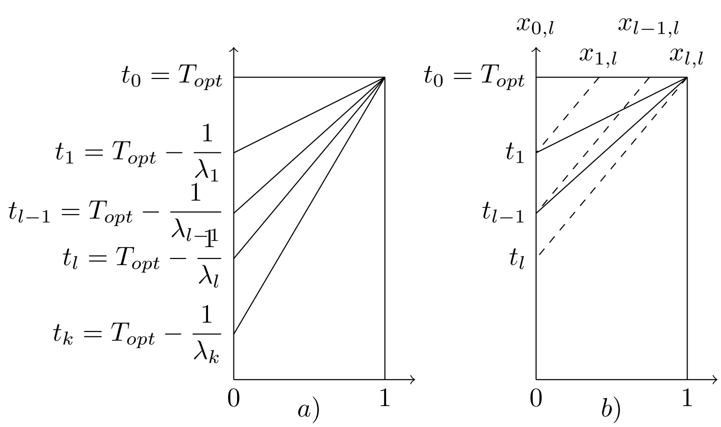

FIG. 2. *The definition of t_l is given in* (a) *and the definition of $x_{i,l}$ is given in* (b)*, where dashed lines have the same slope for constant Σ.*

We begin with deriving conditions for controls to reach the null-controllability at the time T_{opt}. First, if $m > k$, choose the control

$$u_l(t,1) = 0 \text{ for } 0 \le t \le T_{opt} - \tau_l \text{ and } k+1 \le l \le m. \tag{4.3}$$

Note that in the case $T_{opt} = \tau_l$, one does not impose any condition for u_l in (4.3). Second, choose the control, for $1 \le i \le k$,

$$u_{m+i}(t,1) = 0 \text{ for } 0 \le t < T_{opt} - \tau_i - \tau_{m+i}. \tag{4.4}$$

Note that in the case $T_{opt} = \tau_{m+i} + \tau_i$, one does not impose any condition for u_{m+i} in (4.4).

Requiring (4.3) and (4.4) is just a preparation step; other choices are possible. The main part in the construction of the controls is to choose the control $u_{m+i}(t,1)$ for $t \in (T_{opt} - \tau_i - \tau_{m+i}, T_{opt} - \tau_i)$ and for $1 \le i \le k$ such that the following k conditions hold:

(a_1)

$$u_k(T_{opt}, x) = 0 \text{ for } x \in (x_{0,k}, x_{1,k}), \quad \ldots, \quad u_1(T_{opt}, x) = 0 \text{ for } x \in (x_{0,1}, x_{1,1}).$$

(a_2)

$$u_k(T_{opt}, x) = 0 \text{ for } x \in (x_{1,k}, x_{2,k}), \quad \ldots, \quad u_2(T_{opt}, x) = 0 \text{ for } x \in (x_{1,2}, x_{2,2}).$$

...

(a_k)

$$u_k(T_{opt}, x) = 0 \text{ for } x \in (x_{k-1,k}, x_{k,k}).$$

Set

$$\mathcal{X} := L^2(t_1, t_0) \times L^2(t_2, t_0) \times \cdots \times L^2(t_k, t_0) \tag{4.5}$$

and denote

$$U_j(t) = \big(u_{m+j}(t,0), \ldots, u_{m+k}(t,0)\big)^{\mathsf{T}} \text{ for } 1 \le j \le k$$

and

$$V_j(t) = \big(u_{k+1}(t,0), \ldots, u_j(t,0)\big)^{\mathsf{T}} \text{ for } m \le j \le m+k$$

We determine

$$\big(u_{m+1}(\cdot,0), \ldots, u_{m+k}(\cdot,0)\big)^{\mathsf{T}} \in \mathcal{X}$$

via the conditions in (a_1), (a_2), ..., (a_k). Let us now find necessary and sufficient conditions on $\big(u_{m+1}(\cdot,0), \ldots, u_{m+k}(\cdot,0)\big)^{\mathsf{T}} \in \mathcal{X}$ so that $a_1)$, ..., $a_k)$ hold. These are analyzed in (b_1), ..., (b_k) below, respectively.

(b_1) From (2.6) and (2.11), using the characteristic method and the fact that $S_{ij} = 0$ for $1 \le i, j \le k$, one can write the conditions in (a_1) under the form

$$(u_1, \ldots, u_k)^{\mathsf{T}}(t,0) + \int_t^{t_0} \mathcal{L}_1(t,s)(u_{k+1}, \ldots, u_{k+m})^{\mathsf{T}}(s,0)\, ds = 0 \quad \text{ for } t_1 \le t \le t_0$$

for some $\mathcal{L}_1 \in \big[L^\infty\big((t,s); t_1 \le t \le s \le t_0\big)\big]^{k \times m}$. Using (1.10) with $i = k$ provided $m > k$, one can write the above equation under the form

(4.6)
$$U_1(t) = A_1 V_m(t) + \int_t^{t_0} G_1(t,s) V_m(s)\,ds + \int_t^{t_0} H_1(t,s) U_1(s)\,ds \quad \text{for } t_1 \le t \le t_0$$

for some $G_1 \in [L^\infty(\{(t,s);\, t_1 \le t \le s \le t_0\})]^{k\times(m-k)}$ and $H_1 \in [L^\infty(\{(t,s);\, t_1 \le t \le s \le t_0\})]^{k\times k}$ depending only on S, B, and Σ, and some matrix $A_1 \in \mathbb{R}^{k\times(m-k)}$ depending only on B. In the case $m = k$, one chooses $H_1 = 0$ (there are not A_1 and G_1 in this case by convention). Since K is analytic with respect to (γ, B), one can check that $\mathcal{L}_1$ is analytic with respect to (γ, B). In fact, $\mathcal{L}_1$ depends linearly on S and so analytically on (γ, B), and if $\gamma = 0$, then $\mathcal{L}_1 = 0$. This implies that G_1 and H_1 are analytic with respect to (γ, B). It is also clear that A_1 is analytic with respect to (γ, B) as well.

Remark 4.1. In the case $m = k$, $U_1(t) = 0$ for $t_1 \le t \le t_0$. This fact will be used to deal with the case $m < k$.

(b_2) Similar to (4.6), the condition in (a_2) is equivalent to

(4.7)
$$U_2(t) = A_2 V_{m+1}(t) + \int_t^{t_0} G_2(t,s) V_{m+1}(s)\,ds + \int_t^{t_0} H_2(t,s) U_2(s)\,ds \quad \text{for } t_2 \le t < t_1$$

for some constant matrix A_2 and some bounded functions G_2 and H_2 defined in $\big\{(s,t); t_2 \le t \le s \le t_0\big\}$ which depend only on S, B, and Σ. Moreover, A_2, G_2, and H_2 are analytic with respect to (γ, B).

...

(b_k) Similar to (4.6), the condition in (a_k) is equivalent to

(4.8)
$$U_k(t) = A_k V_{m+k-1}(t) + \int_t^{t_0} G_k(t,s) V_{m+k-1}(s)\,ds + \int_t^{t_0} H_k(t,s) U_k(s)\,ds \quad \text{for } t_k \le t < t_{k-1}$$

for some constant matrix A_k and some bounded functions G_k and H_k defined in $\big\{(s,t); t_k \le t \le s \le t_0\big\}$ which depends only on S, B, and Σ. Moreover, A_k, G_k, and H_k are analytic with respect to (γ, B).

We are next concerned about the relations between the components of $u(t,0)$. We have, by the property of S_{++} in (2.14) and the form of S in (2.11), and (4.3) and (4.4),

$$u_{m+k}(s,0) = F_{m+k}(s) \text{ for } 0 \le s \le t_k, \tag{4.9}$$

(4.10)
$$u_{m+k-1}(s,0) = F_{m+k-1}(s) + \int_0^s \mathcal{G}_{m+k-1,m+k}(\xi) u_{m+k}(\xi,0)\,d\xi \text{ for } 0 \le s \le t_{k-1},$$

$$u_{m+k-2}(s,0) = F_{m+k-2}(s) + \int_0^s \mathcal{G}_{m+k-2,m+k}(\xi) u_{m+k}(\xi,0)\,d\xi$$
$$+ \int_0^s \mathcal{G}_{m+k-2,m+k-1}(\xi) u_{m+k-1}(\xi,0)\,d\xi \text{ for } 0 \le s \le t_{k-2}, \tag{4.11}$$

...

$$u_{k+1}(s,0) = F_{k+1}(s) + \int_0^s \sum_{j=k+2}^{k+m} \mathcal{G}_{k+1,j}(\xi) u_j(\xi,0)\, d\xi \text{ for } 0 \le s \le t_1, \tag{4.12}$$

where $\mathcal{G}_{i,j}$ depends only on S and Σ and is analytic with respect to (γ, B), and F_i depends only on the initial data. Here we also use (4.3) and (4.4).

Using (4.9)–(4.12), one can write the equations in (b_1), …, (b_k) under the form

$$U + \mathcal{K}(U) = F \text{ in } \mathcal{X}, \tag{4.13}$$

where

$$U = \big(u_{m+1}(\cdot,0), \dots, u_{m+k}(\cdot,0)\big)^{\mathsf{T}}$$

and $\mathcal{K}$ is a Hilbert–Schmidt operator, therefore a compact operator, and it is analytic with respect to (γ, B).

By the theory of analytic compact theory (see, e.g., [18, Theorem 8.92]), for each $B \in \mathcal{B}$, $I + \mathcal{K}$ is invertible outside a discrete set of γ in $\mathbb{R}$ since $\|\mathcal{K}\|$ is small if γ is small.

Using this fact, since $\mathcal{B}$ has a finite number of connected components, there exists a discrete subset of $\mathbb{R}$ such that outside this set, $I + \mathcal{K}$ is invertible for almost every $B \in \mathcal{B}$ by the Fredholm theory for analytic compact operator.

Consider (γ, B) such that $I + \mathcal{K}$ is invertible. Then (4.13) has a unique solution for all F in $\mathcal{X}$. One can check that if F is bounded, then U is bounded since $\mathcal{K}U$ is bounded. To obtain the null-controllability at the time T_{opt}, in addition to the preparation step, one chooses $u_{k+m}(1,t)$ for $T_{opt} - \tau_{k+m} - \tau_k \le t \le T_{opt} - \tau_{k+m}$, …, $u_{m+1}(1,t)$ for $T_{opt} - \tau_{m+1} - \tau_1 \le t \le T_{opt} - \tau_{m+1}$ such that $\big(u_{m+1}(\cdot,0), \dots, u_{m+k}(\cdot,0)\big)^{\mathsf{T}} = U$ (this can be done by the form of S_{++}) and chooses $u_l(t,1)$ for $T_{opt} - \tau_l \le t \le T_{opt}$ and $m+1 \le l \le k+m$ in such a way that

$$u_l(T_{opt}, x) = 0 \text{ for } x \in (0,1). \tag{4.14}$$

Requirement (4.14) is again possible by the property of S_{++} in (2.14) and by the form of S in (2.11).

Remark 4.2. The above analysis shows that the existence of a bounded solution U of (4.13) implies the existence of a control to steer the system from the initial data to 0 in time T_{opt} by the characteristic method. Moreover, in the case where $m = k$ and

$$T_{opt} = \tau_1 + \tau_{m+1} = \dots = \tau_k + \tau_{m+k},$$

the existence of such a U is necessary.

Remark 4.3. We now show how to modify the proof of Theorem 1.1 in the case $m \ge k \ge 1$ to reach the exact controllability. To obtain the exact controllability with the final state v, the requirements in $(a_1), \dots, (a_k)$ become

(c_1)

$$u_k(T_{opt}, x) = v_k(x) \text{ for } x \in (x_{0,k}, x_{1,k}), \quad \dots, \quad u_1(T_{opt}, x) = v_1(x) \text{ for } x \in (x_{0,1}, x_{1,1}),$$

(c_2)

$$u_k(T_{opt}, x) = v_k(x) \text{ for } x \in (x_{1,k}, x_{2,k}), \quad \dots, \quad u_2(T_{opt}, x) = v_2(x) \text{ for } x \in (x_{1,2}, x_{2,2}),$$

…

(c_k)

$$u_k(T_{opt}, x) = v_k(x) \text{ for } x \in (x_{k-1,k}, x_{k,k}).$$

Equations (4.6), (4.7), and (4.8) then become

$$U_1(t) = J_1(t) + A_1 V_m(t) + \int_t^{t_0} G_1(t,s) V_m(s)\, ds + \int_t^{t_0} H_1(t,s) U_1(s)\, ds \quad \text{for } t_1 \le t \le t_0,$$

$$U_2(t) = J_2(t) + A_2 V_{m+1}(t) + \int_t^{t_0} G_2(t,s) V_{m+1}(s)\, ds + \int_t^{t_0} H_2(t,s) U_2(s)\, ds \qquad \text{for } t_2 \le t < t_1,$$

$$U_k(t) = J_k(t) + A_k V_{m+k-1}(t) + \int_t^{t_0} G_k(t,s) V_{m+k-1}(s)\, ds + \int_t^{t_0} H_k(t,s) U_k(s)\, ds \qquad \text{for } t_k \le t < t_{k-1}$$

for some functions $J_1, J_2, \dots, J_k$ depending on the final state v. Using (4.9)–(4.12), one can write these equations under the form

$$U + \mathcal{K}(U) = F \text{ in } \mathcal{X},$$

where F now also depends on $J_1, \dots, J_k$. The rest of the proof of the exact controllability is unchanged.

Remark 4.4. In the case where $m = k$ and

$$T_{opt} = \tau_1 + \tau_{m+1} = \cdots = \tau_k + \tau_{m+k},$$

the above analysis also gives the optimality of T_{opt} for all γ such that $I + \mathcal{K}$ is invertible. Indeed, assume that there exists $T < T_{opt}$ such that one can steer an arbitrary state $u(0,\cdot)$ to 0 at the time T. Without loss of generality, one might assume that $T_{opt} - T$ is small. To simplify the notation, we assume that Σ is constant. As mentioned in Remark 4.2, a necessary condition to have control is the existence of a solution $U \in \mathcal{X}$ of

$$U + \mathcal{K}U = G, \tag{4.15}$$

where G now depends on $u_i(0,x)$ for $1 \le i \le k$ and $x \in (0,1)$ and $u_i(0,x)$ for $k+1 \le i \le k+m$ and $x \in (0, 1-s_i)$ with $s_i = (T_{opt} - T)/\lambda_i$ by (4.9)–(4.12). However, for $t \in (1/\lambda_{k+m} - (T_{opt} - T), 1/\lambda_{k+m})$,

$$u_{m+k}(t, 0) = u_{k+m}(0, \lambda_{k+m} t), \tag{4.16}$$

the LHS of (4.16) is uniquely determined by G from (4.15), and the RHS of (4.16) can be chosen independently of G. This yields a contradiction.

4.2.2. On the case $m < k$. Set

$$\hat{u}(t,x) = \big(u_{k-m+1}, \dots u_{k+m}\big)^{\mathsf{T}}(t,x) \text{ in } (0,T) \times (0,1),$$

$$\hat{\Sigma}(x) = \text{ diag } (-\lambda_{k-m+1}, \dots, -\lambda_k, \lambda_{k+1}, \dots, \lambda_{m+k})(x) \text{ in } (0,1),$$

and denote

$\hat{S}(x)$ the $2m \times 2m$ matrix formed from the last $2m$ columns and the last $2m$ rows of S(x),

and

$$\hat{B} \text{ the } m \times m \text{ matrix formed from the last } m \text{ rows of } B.$$

Then $\hat{u}$ is a bounded broad solution of the system

$$\partial_t \hat{u}(t,x) = \hat{\Sigma}(x)\partial_x \hat{u}(t,x) + \hat{S}(x)\hat{u}(t,0) \tag{4.17}$$

with the boundary condition at 0 given by $(\hat{u}_1, \dots, \hat{u}_m)(t,0)^{\mathsf{T}} = \hat{B}(\hat{u}_{m+1}, \dots, \hat{u}_{2m})(t,0)^{\mathsf{T}}$. Set

$$\hat{T}_{opt} := \max\{\tau_{k+m} + \tau_k, \dots, \tau_{k+1} + \tau_{k+1-m}\} = T_{opt}.$$

Consider the pair $(\gamma, \hat{B})$ such that the control constructed in section 4.2.1 for $\hat{u}$ exists. Then, for this control,

$$\hat{u}(T_{opt}, x) = 0 \text{ for } x \in (0,1). \tag{4.18}$$

As observed in Remark 4.1, one has

$$(\hat{u}_{m+1}, \dots, \hat{u}_{2m})^{\mathsf{T}}(t,0) = 0 \text{ for } t \in [T_{opt} - \tau_{k-m+1}, T_{opt}].$$

This yields

$$(u_1, \dots, u_{k-m})^{\mathsf{T}}(T_{opt}, x) = 0 \text{ for } x \in (0,1) \tag{4.19}$$

by the form of S given in (2.11).

Combining (4.18) and (4.19) yields the null-controllability at the time T_{opt}.

4.3. Proof of part 2 of Theorem 1.1. Fix $\alpha \neq 0$ and $\beta \neq 0$, and consider

$$C^{3,3} = \begin{pmatrix} 0 & 0 & \alpha(\lambda_{k+2} + \lambda_k) \\ 0 & 0 & \beta(\lambda_{k+2} - \lambda_{k+1}) \\ 0 & 0 & 0 \end{pmatrix} \text{ and} \tag{4.20}$$

$$C(x) = C^{k+2,k+2} := \begin{pmatrix} 0_{k-1,k-1} & 0_{k-1,3} \\ 0_{3,k-1} & C^{3,3} \end{pmatrix} \text{ for } k \geq 1.$$

Set

(4.21)

$$K^{3,3} = \begin{pmatrix} 0 & 0 & \alpha \\ 0 & 0 & \beta \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad K(x) = K^{k+2,k+2} := \begin{pmatrix} 0_{k-1,k-1} & 0_{k-1,3} \\ 0_{3,k-1} & K^{3,3} \end{pmatrix} \text{ for } k \geq 1.$$

One can check that $KC = 0_{k+2}$ and K is a solution of (2.4) by noting that Σ is constant. Moreover, (2.5) holds by the choice of K and C. We have, by (2.11), that

(4.22)

$$S(x) = S^{k+2,k+2} = \begin{pmatrix} 0_{k-1,k-1} & 0_{k-1,3} \\ 0_{3,k-1} & S^{3,3} \end{pmatrix}, \text{ where } S^{3,3} = \begin{pmatrix} 0 & 0 & \lambda_{k+2}\alpha \\ 0 & 0 & \lambda_{k+2}\beta \\ 0 & 0 & 0 \end{pmatrix}.$$

In what follows, for simplicity of notation, we only consider the case $k = 2$. The other cases can be established similarly. Suppose that

$$u_2(t, 0) = au_3(t, 0) + bu_4(t, 0) \text{ for } t \geq 0. \tag{4.23}$$

Then $a \neq 0$ by condition (1.10). To obtain the null-controllability at the time T_{opt}, one has, from condition (a_1),

$$u_3(t, 0) = u_4(t, 0) = 0 \text{ for } t \in (t_1, t_0),$$

and hence from condition (a_2) and (4.22),

$$au_3(t, 0) + bu_4(t, 0) + \int_t^{t_1} \lambda_4 \alpha u_4(s, 0)\, ds = 0 \text{ for } t \in (t_2, t_1).$$

Since, by (2.6), (2.10), and (4.22),

$$u_3(t, 0) = \int_{t_2}^{t} \lambda_4 \beta u_4(s, 0)\, ds + f(t) \text{ for } t \in (t_2, t_1),$$

for some f depending on the initial data. By taking $\beta = \alpha/a$ $(a \neq 0)$, we have

$$bu_4(t, 0) + \int_{t_2}^{t_1} \lambda_4 \alpha u_4(s, 0)\, ds = -af(t) \text{ for } t \in (t_2, t_1). \tag{4.24}$$

By choosing α such that $b + \lambda_4 \alpha (t_1 - t_2) = 0$, and integrating (4.24) from t_2 to t_1, it follows, since $a \neq 0$, that

$$\int_{t_2}^{t_1} f(t)\, dt = 0. \tag{4.25}$$

This is impossible for an arbitrary initial data, for example, if $u_4(0, \cdot) = 0$, then $f(t) = u_3(0, \lambda_3 t)$ and an appropriate choice of $u_3(0, \cdot)$ yields that (4.25) does not hold. In other words, the system is not null-controllable at the time T_{opt}.

5. A null-controllability result for all $\boldsymbol{\gamma}$ and $\boldsymbol{B \in \mathcal{B}}$. A slight modification of the proof of part 3 of Theorem 1.1 gives the following result, where T_2 is defined in (1.14).

PROPOSITION 5.1. *Let $m \geq 2$. Assume that* (1.3) *and* (1.4) *hold and $B \in \mathcal{B}$. There exists $\delta > 0$ depending only on $C, B, \Sigma,$ and γ such that the system is null-controllable at the time $T_2 - \delta$.*

Proof. We only consider here the case $m \geq k$; the case $m < k$ can be handled as in section 4.2.2. The controls are chosen so that

$$u_{k+1}(t, 0) = 0 \text{ for } t \geq \tau_{k+1}, \quad \ldots, \quad u_{k+m-1}(t, 0) = 0 \text{ for } t \geq \tau_{k+m-1}, \tag{5.1}$$

$$u_{k+m}(t, 0) = 0 \text{ for } (t \geq \tau_{k+m} \text{ and } t \notin [\tau_{k+1} - \delta, \tau_{k+1}]), \tag{5.2}$$

and $u_{k+m}(t, 0)$ is chosen in $[\tau_{k+1} - \delta, \tau_{k+1}]$ in such a way that

$$u_k(T_2 - \delta, x) = 0 \text{ for } x \in [x^*, 1], \tag{5.3}$$

where $x^* = x_k(T_2 - \delta, \tau_{k+1}, 0)$ (see the definition of x_k in (3.4)). As in (b_k), we derive that, in $[\tau_{k+1} - \delta, \tau_{k+1}]$, (5.3) is equivalent to

$$u_{k+m}(t, 0) = \int_{\tau_{k+1}-\delta}^{\tau_{k+1}} K(t, s) u_{k+m}(s, 0)\, ds + f(t) \tag{5.4}$$

for some bounded function f defined in $[\tau_{k+1} - \delta, \tau_{k+1}]$ which now depends only on the initial data. Here $K : [\tau_{k+1} - \delta, \tau_{k+1}]^2 \to \mathbb{R}$ is a bounded function depending only on C, B, Σ, and γ. Since δ is small, one can check that the mapping $T : L^2([\tau_{k+1} - \delta, \tau_{k+1}]) \to L^2([\tau_{k+1} - \delta, \tau_{k+1}])$ which is given by

$$T(v)(t) = \int_{\tau_{k+1}-\delta}^{\tau_{k+1}} K(t, s) v(s)\, ds$$

is a contraction. By the contraction mapping theorem, (5.4) is uniquely solvable and the solution is bounded since f is bounded.

We now show how to construct such a control. Since $u_{m+k}(t, 0)$ for $0 \le t \le \tau_{m+k}$ is uniquely determined by the initial data (by (2.14)), one derives from (2.14) that $u_{m+k-1}(t, 0)$ for $0 \le t \le \tau_{m+k-1}$, ..., u_{k+1} for $0 \le t \le \tau_{k+1}$ are uniquely determined from the initial condition and the requirements on the constructive controls at $(t, 0)$. It follows from (2.14) again that

- $u_{k+m}(t, 1)$ for $t \ge 0$ is uniquely determined from $u_{m+k}(t, 0)$ for $t \ge \tau_{m+k}$,
- $u_{m+k-1}(t, 1)$ for $t \ge 0$ is uniquely determined from ($u_{m+k-1}(t, 0)$ for $t \ge \tau_{m+k-1}$ and $u_{m+k}(t, 0)$ for $t \ge 0$),
- ...,
- $u_{k+1}(t, 1)$ for $t \ge 0$ is uniquely determined from ($u_{k+1}(t, 0)$ for $t \ge \tau_{k+1}$, $u_{k+2}(t, 0)$ for $t \ge 0$, ..., $u_{m+k}(t, 0)$ for $t \ge 0$).

The existence and uniqueness of controls satisfying requirements are established.

It remains to check that the constructive controls give the null-controllability at the time $T_2 - \delta$ if δ is small enough. Indeed, by (5.1) and (5.2), we have

$$u_{k+1}(t, 0) = \cdots = u_{k+m}(t, 0) = 0 \text{ for } t \ge \tau_{k+1}. \tag{5.5}$$

Since $S_{--} = 0_k$, it follows from (1.5) that

$$u_1(T_2 - \delta, x) = \cdots = u_{k-1}(T_2 - \delta, x) = 0 \text{ for } x \in [0, 1] \tag{5.6}$$

and

$$u_k(T_2 - \delta, x) = 0 \text{ for } x \in [0, x^*],$$

which yields, by (5.3),

$$u_k(T_2 - \delta, x) = 0 \text{ for } x \in [0, 1]. \tag{5.7}$$

From (5.5) and the form of S, we also derive that

$$u_{k+m}(t, x) = \cdots = u_{k+1}(t, x) \text{ for } x \in [0, 1] \text{ and } t \ge \tau_{k+1};$$

in particular, if δ is small enough,

$$u_{k+m}(T_2 - \delta, x) = \cdots = u_{k+1}(T_2 - \delta, x) \text{ for } x \in [0, 1]. \tag{5.8}$$

The null-controllability at $T_2 - \delta$ now follows from (5.6), (5.7), and (5.8). ∎

6. On the case $m = 2$ and $B_{k1} \neq 0$: Proof of Theorem 1.5. We only establish the null-controllability result. The proof of the exact controllability can be derived similarly as in the spirit mentioned in Remark 4.3 and is omitted. Without loss of generality, one might assume that $\gamma = 1$ and $T - T_{opt}$ is small. As mentioned in Remark 2.2, the choice of K on Γ_3 in (2.16) can be "arbitrary." In this section, we modify this choice to reach some analytic property of K. The new K will be defined in $\hat{\mathcal{T}}$ which is the triangle formed by three points $(0,0)$, $(1,0)$, and (L, L), where L is defined in (1.12), this triangle contains $\mathcal{T}$. Since $B_{k1} \neq 0$, by (2.11), one can replace the condition $K_{kk} = 0$ in (2.13) by the condition

$$\left(S_{-+}\right)_{k1} = 0,$$

while the rest of (2.13) remains unchanged. The idea of the proof is to show that one can prepare $u(T - T_{opt}, \cdot)$ using the control in the time interval $[0, T - T_{opt}]$ in such a way that (4.13) is solvable. In what follows, we present a direct proof for Theorem 1.5.

We first consider the case $k = m$ (=2). The matrix S then has the form

$$S = \begin{pmatrix} 0 & 0 & * & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{6.1}$$

We choose a control so that $u_3(t, 0) = u_4(t, 0) = 0$ for $t \geq T - \tau_1$, $u_4(t, 0) = 0$ for $\tau_4 \leq t \leq T - \tau_2$, and $u_3(t, 0) = 0$ for $\tau_3 \leq t \leq T_{opt} - \tau_1$ (the last two choices are just a preparation step), and as in (a_2), $u_4(t, 0)$ for $t \in (T - \tau_2, T - \tau_1)$ is required to ensure that

$$u_2(T, x) = 0 \text{ for } x \in [x_{12}, 1] \tag{6.2}$$

(see (4.2) for the definition of x_{12}). One can verify that the null-controllability is attained at T for such a control if it exists. As in the proof of Proposition 5.1, it suffices to show that (6.2) is solvable for some (bounded) choice of $u_3(t, 1)$ for $t \in (T_{opt} - \tau_1 - \tau_3, T - \tau_1 - \tau_3)$. Let χ_O denote a characteristic function of a subset O of $\mathbb{R}$. By the form of S in (6.1) and the fact that $B \in \mathcal{B}$, (6.2) is equivalent to, for $t \in (T - \tau_2, T - \tau_1)$,

$$\begin{aligned} &h(t) + u_4(t, 0) + \alpha u_3(t, 0)\chi_{[T_{opt}-\tau_1, T-\tau_1]} \\ &\quad = \int_{T-\tau_2}^{t} g(t - s)u_4(s, 0)\, ds + \int_{t}^{T-\tau_1} f(s - t)u_4(s, 0)\, ds, \end{aligned} \tag{6.3}$$

where g and f are two functions depending only on K, B, and Σ, α is a nonzero constant (since $B_{21} \neq 0$), and $h(t)$ is a function now depends only on B, K, Σ, and the initial condition. Moreover, f and g are analytic by Lemma 6.2 below. Let $\mathcal{K}_1 : L^2(T - \tau_2, T - \tau_1) \to L^2(T - \tau_2, T - \tau_1)$ be defined by the RHS of (6.3). Then the adjoint operator $\mathcal{K}_1^* : L^2(T - \tau_2, T - \tau_1) \to L^2(T - \tau_2, T - \tau_1)$ is given by

$$\mathcal{K}_1^*(v) = \int_t^{T-\tau_1} g(s - t)v(s)\, ds + \int_{T-\tau_2}^{t} f(t - s)v(s)\, ds.$$

Let V be an eigenfunction of $\mathcal{K}_1^*$ with respect to the eigenvalue -1. We have, by Lemma 6.1 below,

$$V \not\equiv 0 \text{ in a neighborhood of } T - \tau_1. \tag{6.4}$$

Since the kernel of $I + \mathcal{K}_1^*$ is of finite dimension, one can prepare the state at the time $T - T_{opt}$ (i.e., $u_3(t, 1)$ for $t \in (T_{opt} - \tau_1 - \tau_3, T - \tau_1 - \tau_3)$) in such a way that the RHS of (6.3) is orthogonal to the kernel of $I + \mathcal{K}_1^*$. It follows from the Fredholm theory that (6.3) is solvable and the solution is bounded.

We next consider the case $k > m = 2$. The proof in this case follows from the previous one as in section 4.2.2. We finally consider the case $k = 1$ and $m = 2$. The matrix S then has the form

$$S = \begin{pmatrix} 0 & 0 & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{pmatrix}. \tag{6.5}$$

We choose a control such that $u_3(t, 0) = 0$ for $\tau_3 \le t \le T - \tau_1$, $u_2(t, 0) = 0$ for $\tau_2 \le t \le T_{opt}$ (a preparation step), $u_2(t, 0) = u_3(t, 0) = 0$ for $t \ge T$, and $u_3(t, 0)$ for $t \in (T - \tau_1, T)$ is required to ensure that

$$u_1(T, x) = 0 \text{ for } x \in [0, 1]. \tag{6.6}$$

One can verify that the null-controllability is attained at T for such a control if it exists. As in the proof of Proposition 5.1, it suffices to show that (6.6) is solvable. As before, (6.6) is equivalent to, for $t \in (T - \tau_1, T)$,

$$h(t) + u_3(t, 0) + \alpha u_2(t, 0)\chi_{[T_{opt}, T]} = \int_{T-\tau_1}^{t} g(t - s)u_3(s, 0)\, ds + \int_t^T f(s - t)u_3(s, 0)\, ds,$$

where g and f are two functions depending only on K, B, and Σ, α is a nonzero constant (since $B_{21} \neq 0$), and $h(t)$ is a function now depends only on B, K, Σ, and the initial condition. Moreover, f and g are analytic by Lemma 6.2 below. The proof now follows as in the case $k = m = 2$ and the details are omitted.

The following result is used in the proof of Theorem 1.5.

LEMMA 6.1. *Let $T > 0$, $f, g \in C^1([0, T])$ and let V be a continuous function defined in $[0, T]$ such that*

$$V(t) = \int_0^t f(t - s)V(s) + \int_t^T g(s - t)V(s)\, ds \text{ for } t \in [0, T].$$

Assume that g is analytic on $[0, T]$ and $V = 0$ in a neighbourhood of 0. Then $V \equiv 0$ on $[0, T]$.

Proof. It suffices to prove that V is analytic on $[0, T]$. We have

$$V'(t) = f(0)V(t) + \int_0^t f'(t - s)V(s)\, ds - g(0)V(t) - \int_t^T g'(s - t)V(s)\, ds. \tag{6.7}$$

Since $V = 0$ in a neighborhood of 0, an integration by parts gives

$$V'(t) = \int_0^t f(t - s)V'(s) + \int_t^T g(t - s)V'(s)\, ds - g(T - t)V(T). \tag{6.8}$$

By recurrence, we obtain, for $n \ge 0$,

$$V^{(n+1)}(t) = f(0)V^{(n)}(t) + \int_0^t f'(t-s)V^{(n)}(s)\,ds - g(0)V^{(n)}(t) - \int_t^T g'(s-t)V^{(n)}(s)\,ds$$
$$+ \sum_{k=0}^{n-1}(-1)^{n-k+1}g^{(n-k)}(T-t)V^{(k)}(T) \tag{6.9}$$

and

$$V^{(n+1)}(t) = \int_0^t f(t-s)V^{(n+1)}(s) + \int_t^T g(t-s)V^{(n+1)}(s)\,ds + \sum_{k=0}^{n}(-1)^{n-k+1}g^{(n-k)}(T-t)V^{(k)}(T). \tag{6.10}$$

By rescaling, without loss of generality, one might assume that

$$T = 1 \quad \text{and} \quad \|V\|_{C^1([0,T])} = 1.$$

Set

$$a_n = \|V^{(n)}\|_{L^\infty([0,T])} \text{ and } b_n = \|g^{(n)}\|_{L^\infty([0,T])} + \|g^{(n+1)}\|_{L^\infty([0,T])} + 2\|f\|_{C^1([0,T])}.$$

Using (6.9), we obtain

$$a_{n+1} \le \sum_{k=0}^{n} a_{n-k}b_k. \tag{6.11}$$

We have, by the analyticity of g,

$$b_k \le c^k k!. \tag{6.12}$$

In this proof, c denotes a constant greater than 1 and independent of k and n. It is clear that

$$\sum_{k=0}^{n} c^k k! c^{n-k}(n-k)! \le c^n(n+1)!. \tag{6.13}$$

Combining (6.11), (6.12), and (6.13) and using a recurrence argument yield

$$a_n \le c^n n!.$$

The analyticity of V now follows from the definition of a_n. The proof is complete. □

The second lemma yields the analyticity of g and f in the definition of $\mathcal{K}_1^*$ in the proof of Theorem 1.5.

LEMMA 6.2. *Let $l \ge 4$ and $\gamma_1 \le \gamma_2 \le \cdots \le \gamma_l$ be such that $\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3, \mathcal{I}_4 \ne \emptyset$, where*

$$\mathcal{I}_1 = \{i:\ \gamma_i \le 0\}, \quad \mathcal{I}_2 = \{i:\ 0 < \gamma_i < 1\}, \quad \mathcal{I}_3 = \{i:\ \gamma_i = 1\}, \quad \mathcal{I}_4 = \{i:\ \gamma_i > 1\}.$$

Denote $\hat{\mathcal{T}}$ the triangle formed by three lines $y = x$, $y = 0$, and $y = \gamma_{i_0}x - \gamma_{i_0}$ where $i_0 = \min \mathcal{I}_4$. Let $G : [0, \gamma_{i_0}/(\gamma_{i_0}-1)] \to \mathbb{R}^{n\times n}$ be analytic, and denote $\Gamma = \mathrm{diag}(\gamma_1, \ldots, \gamma_l)$ and

$$\Lambda := \{(x,y) \in \partial\hat{\mathcal{T}}; x = y\} = \{(x,x); x \in [0, \gamma_{i_0}/(\gamma_{i_0}-1)]\}.$$

Let f_i ($i \in \mathcal{I}_1 \cup \mathcal{I}_4$) be analytic functions defined in a neighborhood of Λ and let $c_{i,j} \in \mathbb{R}$ for $1 \le i, j \le l$. Assume that v is the unique broad solution of the system

$$\begin{cases} \partial_x v(x,y) + \Gamma \partial_y v(x,y) - G(y) v(x,y) = 0 \ \textit{in} \ \hat{\mathcal{T}}, \\ v_i(x,x) = f_i(x,x) \ \textit{for} \ (x,x) \in \Lambda, \ i \in \mathcal{I}_1 \cup \mathcal{I}_4, \\ v_i(x,0) = \sum_{j \in \mathcal{I}_1 \cup \mathcal{I}_4} c_{ij} v_j(x,0) \ \textit{for} \ x \in (0,1), \ i \in \mathcal{I}_2 \cup \mathcal{I}_3. \end{cases}$$

Then v is analytic in $\bar{\Delta}_{i+1} \setminus \Delta_i$ for $i \in \hat{\mathcal{I}}_2$ where $\hat{I}_2 = \{i \in \mathcal{I}_2 \text{ or } i+1 \in \mathcal{I}_2\}$ and Δ_j is the open triangle formed by three lines $y = \gamma_j x$, $y = 0$, and $y = \gamma_{i_0} x - \gamma_{i_0}$.

Proof. We first prove by recurrence that, for $k \geq 1$,

$$\|v\|_{C^k(\bar{\Delta}_{i+1} \setminus \Delta_i)} \leq C^k \|f\|_{C^k(\Gamma)} \text{ for } i \in \hat{\mathcal{I}}_2. \tag{6.14}$$

In this proof, C denotes a positive constant independent of k and f. Indeed, using the standard fixed point iteration, one can show that $v \in C^1(\bar{\Delta}_{i+1} \setminus \Delta_i)$ $(i \in \hat{\mathcal{I}}_2)$; moreover,

$$\|v\|_{C^1(\bar{\Delta}_{i+1} \setminus \Delta_i)} \leq C \|f\|_{C^1(\Gamma)} \text{ for } i \in \hat{\mathcal{I}}_2. \tag{6.15}$$

Hence (6.14) holds for $k = 1$. Assume that (6.14) is valid for some $k \geq 1$. We prove that it holds for $k+1$. Set

$$V = \partial_x v \text{ in } \hat{\mathcal{T}}.$$

We have

$$\begin{cases} \partial_x V(x,y) + \Gamma \partial_y V(x,y) - G(y) V(x,y) = 0 \text{ in } \hat{\mathcal{T}}, \\ V_i(x,x) = g_i(x,x) \text{ for } (x,y) \in \Lambda, i \in \mathcal{I}_1 \cup \mathcal{I}_4, \\ V_i = \sum_{j \in \mathcal{I}_1 \cup \mathcal{I}_4} c_{ij} V_j \text{ for } x \in (0,1), \text{ for } i \in \mathcal{I}_2 \cup \mathcal{I}_3, \end{cases}$$

where

$$g_i(x,x) = a_i G(x) v_i(x,x) + b_i \frac{d}{dx}[f_i(x,x)]$$

for some positive constant $a_i, b_i \in \mathbb{R}$ depending only on Γ. This is obtained by considering the first equation and the derivative with respect to x of the second equation in the system of v. By the recurrence, one has

$$\|u\|_{C^k(\bar{\Delta}_{i+1} \setminus \Delta_i)} \leq C^k \|g\|_{C^k(\Gamma)} \text{ for } i \in \hat{\mathcal{I}}_2.$$

Using the equation of v, one derives that

$$\|u\|_{C^{k+1}(\bar{\Delta}_{i+1} \setminus \Delta_i)} \leq C^{k+1} \|f\|_{C^{k+1}(\Gamma)} \text{ for } i \in \hat{\mathcal{I}}_2.$$

Assertion (6.14) is established.

The conclusion now follows from the analyticity of f. ∎

7. On the case $C \equiv 0$: Proof of Proposition 1.6. Note that $S \equiv 0$ since $C \equiv 0$. We first construct a time independent feedback to reach the null-controllability at the time T_{opt}. We begin with considering the case $m > k$. Condition (a_k) can be written under the form

$$u_{m+k}(t,0) = M_k (u_{k+1}, \dots, u_{m+k-1})^{\mathsf{T}}(t,0) \quad \text{for } t \in (t_k, t_{k-1}) \tag{7.1}$$

for some constant matrix M_k of size $1 \times (m-1)$ by considering (1.10) with $i = 1$. Condition (a_{k-1}) can be written under the form (7.1) and

$$u_{m+k-1}(t,0) = M_{k-1}(u_{k+1}, \dots, u_{m+k-2})^{\mathsf{T}}(t,0) \quad \text{for } t \in (t_k, t_{k-1}) \tag{7.2}$$

for some constant matrix M_{k-1} of size $1 \times (m-2)$ by applying (1.10) with $i = 2$ and using the Gaussian elimination method, etc. Finally, condition (a_1) can be written under the form (7.1), (7.2), ..., and

$$u_{m+1}(t,0) = M_1(u_{k+1}, \dots, u_m)^{\mathsf{T}}(t,0) \quad \text{for } t \in (t_k, t_{k-1}) \tag{7.3}$$

for some constant matrix M_2 of size $1 \times (m-k)$ by applying (1.10) with $i = k$ and using the Gaussian elimination method if $m > k$; this condition is replaced by the one $u_{m+1} = 0$ in the case $m = k$. The matrices $M_1, \dots, M_k$ can be obtained via the Gaussian elimination method starting with M_1 using condition (1.10) with $i = 1$, and then with M_2 using condition (1.10) with $i = 2$, ..., and finally with M_k using condition (1.10) with $i = k$.

We now choose the following feedback law:

$$u_{m+k}(t,1) = M_k\Big(u_{k+1}\big(t, x_{k+1}(-\tau_{m+k},0,0)\big), \dots, u_{k+m-1}\big(t, x_{k+m-1}(-\tau_{m+k},0,0)\big)\Big), \tag{7.4}$$

$$u_{m+k-1}(t,1) = M_{k-1}\Big(u_{k+1}\big(t, x_{k+1}(-\tau_{m+k-1},0,0)\big), \dots, u_{k+m-2}\big(t, x_{k+m-2}(-\tau_{m+k-1},0,0)\big)\Big), \tag{7.5}$$

$$\dots$$

$$u_{m+1}(t,1) = M_1\Big(u_{k+1}\big(t, x_{k+1}(-\tau_{m+1},0,0)\big), \dots, u_m\big(t, x_{m+1}(-\tau_{m+1},0,0)\big)\Big) \tag{7.6}$$

(this condition is replaced by the one $u_{m+1}(t,1) = 0$ in the case $k = m$), and

$$u_{k+1}(t,1) = \dots = u_m(t,1) = 0. \tag{7.7}$$

Let us point out that, by Lemma 3.2, the closed-loop system of u given by $\partial_t u = \Sigma \partial_x u$ and the boundary conditions (7.4)–(7.7) is well-posed in the sense of Definition 3.1. With this law of feedback, conditions (a_k), ..., (a_1) hold. It follows that

$$u_1(T_{opt}, x) = \dots = u_k(T_{opt}, x) = 0 \text{ in } (0,1). \tag{7.8}$$

We also derive from (7.7) using the characteristic method and the fact $C = 0$ that

$$u_{k+1}(t,0) = \dots = u_m(t,0) = 0 \text{ for } t \geq \tau_{k+1}$$

and from (7.4)–(7.6) (see also (7.1)–(7.3)) that

$$u_{k+1}(t,0) = \dots = u_{k+m}(t,0) = 0 \text{ for } t \geq T_{opt}.$$

We then obtain

$$u_{k+1}(T_{opt}, x) = \dots = u_{k+m}(T_{opt}, x) = 0 \text{ for } x \in (0,1). \tag{7.9}$$

The null-controllability attained at the optimal time T_{opt} now follows from (7.8) and (7.9).

We next deal with the case $m < k$. The construction of a time independent feedback yielding a null-state at the time $t = T_{opt}$ in this case is based on the construction given in the case $m = k$ obtained previously. Set

$$\hat{u}(t,x) = \big(u_{k-m+1}, \dots u_{k+m}\big)^{\mathsf{T}}(t,x) \text{ in } (0,T) \times (0,1),$$

$$\hat{\Sigma}(x) = \operatorname{diag}(-\lambda_{k-m+1}, \dots, -\lambda_k, \lambda_{k+1}, \dots, \lambda_{m+k})(x) \text{ in } (0,1),$$

and

$$\hat{B} \text{ is the matrix formed from the last } m \text{ rows of } B.$$

Then $\hat{u}$ is a bounded broad solution of the system

$$\partial_t \hat{u}(t,x) = \hat{\Sigma}(x)\partial_x \hat{u}(t,x)$$

with the boundary condition at 0 given by $(\hat{u}_1, \dots, \hat{u}_m)(t,0)^{\mathsf{T}} = \hat{B}(\hat{u}_{m+1}, \dots, \hat{u}_{2m})(t,0)^{\mathsf{T}}$. Consider the time dependent feedback for $\hat{u}$ constructed previously. Then, as in section 4.2.2, the null-controllability is attained at T_{opt} for this feedback. The details are omitted.

We next establish the second part of Proposition 1.6 by contradiction. We only deal with the case $m \ge k$. We first consider the case $T_{opt} = \max_{1 \le i \le k}\{\tau_i + \tau_{i+m}\}$. Fix $T \in \big(\max_{1 \le i \le k} \tau_{i+m}, T_{opt}\big)$ and let $1 \le i_0 \le k$ be such that $\tau_{i_0} + \tau_{i_0+m} = T_{opt}$. Consider an initial datum u such that $u_i(t=0,x) = 0$ for $x \in (0,1)$ and for $1 \le i \ne i_0 + m \le k + m$ and $u_{i_0+m}(t=0,x) = 1$ for $x \in (0,1)$. Assume that the null-controllability is attained at T. By the convention of λ_j, one has, for some $\varepsilon > 0$ depending on Σ,

$$u_{i_0}(t,0) = u_{i_0+1}(t,0) = \cdots = u_k(t,0) = 0 \text{ for } t \in (T - \tau_{i_0}, T - \tau_{i_0} + \varepsilon).$$

As in (7.1), (7.2), and (7.3), we obtain, for $t \in (T - \tau_{i_0}, T - \tau_{i_0} + \varepsilon)$,

$$u_{m+k}(t,0) = M_k(u_{k+1}, \dots, u_{m+k-1})^{\mathsf{T}}(t,0),$$
$$u_{m+k-1}(t,0) = M_{k-1}(u_{k+1}, \dots, u_{m+k-2})^{\mathsf{T}}(t,0),$$

$\dots$

$$u_{m+i_0}(t,0) = M_{i_0}(u_{k+1}, \dots, u_{m+i_0-1})^{\mathsf{T}}(t,0). \tag{7.10}$$

Since $u_1(0,\cdot) = \cdots = u_{m+i_0-1}(0,\cdot) = 0$, it follows from (7.10) that

$$u_{m+i_0}(t,0) = 0 \text{ for } t \in (T - \tau_{i_0}, T - \tau_{i_0} + \varepsilon). \tag{7.11}$$

On the other hand, by using the characteristic method and the fact $T < \tau_{i_0} + \tau_{i_0+m}$, one has, for ε small enough,

$$u_{i_0+m}(t,0) = 1 \text{ for } t \in (T - \tau_{i_0}, T - \tau_{i_0} + \varepsilon).$$

This contradicts (7.11). The second part of Proposition 1.6 is proved in this case.

We next consider the case $T_{opt} > \max_{1 \le i \le k}\{\tau_i + \tau_{i+m}\}$. Then $T_{opt} = \tau_{k+1}$ and $m > k$. The conclusion follows by considering $u_i(0,x) = 1$ for $1 \le i \ne k+1 \le k+m$ and $u_{k+1}(0,x) = 1$.

In what follows, we present two concrete examples on the feedback form used in the context of Proposition 1.6. We first consider the case where $k = 1$, $m = 2$,

$$\Sigma_+ = \mathrm{diag}(1,2) \quad \text{ and } \quad B = (2,1).$$

One can check that (7.1) has the form

$$u_3(t,0) = -2u_2(t,0).$$

The feedback is then given by

$$u_3(t,1) = -2u_2(t,1/2) \quad \text{and} \quad u_2(t,1) = 0 \text{ for } t \geq 0.$$

We next consider the case where $k = 3$, $m = 3$,

$\Sigma_+ = \text{diag}(1,2,4)$ and the matrix formed from the last two rows of B is $\begin{pmatrix} 2 & 0 & 1 \\ -1 & -1 & 1 \end{pmatrix}$.

One can check that (7.1) has the form (by imposing the condition $u_3(t,0) = 0$)

$$u_6(t,0) = u_5(t,0) + u_4(t,0),$$

and (7.2) has the form (by imposing the condition $u_3(t,0) = u_2(t,0) = 0$)

$$u_5(t,0) = -3u_4(t,0).$$

The feedback is then given by

$$u_6(t,1) = u_5(t,1/2) + u_4(t,1/4), \quad u_5(t,1) = -3u_4(t,1/2), \quad \text{and} \quad u_4(t,1) = 0 \text{ for } t \geq 0.$$

One can verify directly that the null-controllability is reached for these feedbacks.

REFERENCES

[1] J. Auriol and F. Di Meglio, *Minimum time control of heterodirectional linear coupled hyperbolic PDEs*, Automatica J. IFAC, 71 (2016), pp. 300–307.

[2] G. Bastin and J.-M. Coron, *On boundary feedback stabilization of non-uniform linear 2×2 hyperbolic systems over a bounded interval*, Systems Control Lett., 60 (2011), pp. 900–906.

[3] G. Bastin and J.-M. Coron, *Stability and Boundary Stabilization of 1-D Hyperbolic Systems*, Progr. Nonlinear Differential Equations Appl. 88, Birkhäuser, Springer, 2016.

[4] A. Bressan, *Hyperbolic Systems of Conservation Laws: The One-Dimensional Cauchy Problem*, Oxford Lect. Ser. Math. Appl., Oxford University Press, Oxford, UK, 2000.

[5] E. Cerpa and J.-M. Coron, *Rapid stabilization for a Korteweg-de Vries equation from the left Dirichlet boundary condition*, IEEE Trans. Automat. Control, 58 (2013), pp. 1688–1695.

[6] J.-M. Coron, L. Gagnon, and M. Morancey, *Rapid stabilization of a linearized bilinear 1-D Schrödinger equation*, J. Math. Pures Appl., 115 (2018), pp. 24–73.

[7] J.-M. Coron, L. Hu, and G. Olive, *Finite-time boundary stabilization of general linear hyperbolic balance laws via Fredholm backstepping transformation*, Automatica J. IFAC, 84 (2017), pp. 95–100.

[8] J.-M. Coron and Q. Lü, *Fredholm transform and local rapid stabilization for a Kuramoto-Sivashinsky equation*, J. Differential Equations, 259 (2015), pp. 3683–3729.

[9] J.-M. Coron and H.-M. Nguyen, *Null controllability and finite time stabilization for the heat equations with variable coefficients in space in one dimension via backstepping approach*, Arch. Ration. Mech. Anal., 225 (2017), pp. 993–1023.

[10] J.-M. Coron, R. Vazquez, M. Krstic, and G. Bastin, *Local exponential H^2 stabilization of a 2×2 quasi-linear hyperbolic system using backstepping*, SIAM J. Control Optim., 51 (2013), pp. 2005–2035.

[11] T. Li, *Controllability and Observability for Quasilinear Hyperbolic Systems*, AIMS Ser. Appl. Math. 3, American Institute of Mathematical Sciences, Higher Education Press, Beijing, 2010.

[12] T. Li and B. Rao, *Local exact boundary controllability for a class of quasilinear hyperbolic systems*, Chinese Ann. Math. Ser. B, 23 (2002), pp. 209–218.

[13] L. Hu, *Sharp time estimates for exact boundary controllability of quasilinear hyperbolic systems*, SIAM J. Control Optim., 53 (2015), pp. 3383–3410.

[14] L. Hu, F. Di Meglio, R. Vazquez, and M. Krstic, *Control of homodirectional and general heterodirectional linear coupled hyperbolic PDEs*, IEEE Trans. Automat. Control, 61 (2016), pp. 3301–3314.

[15] L. Hu, R. Vazquez, F. Di Meglio, and M. Krstic, *Boundary Exponential Stabilization of 1-D Inhomogeneous Quasilinear Hyperbolic Systems*, preprint, arXiv:1512.03539, 2015.
[16] M. Krstic, B.-Z. Guo, A. Balogh, and A. Smyshlyaev, *Output-feedback stabilization of an unstable wave equation*, Automatica J. IFAC, 44 (2008), pp. 63–74.
[17] M. Krstic and A. Smyshlyaev, *Boundary Control of PDEs: A Course on Backstepping Designs*, Adv. Des. Control 16, SIAM, Philadelphia, 2008.
[18] M. Renardy and R. C. Rogers, *An Introduction to Partial Differential Equations*, 2nd ed., Texts in Appl. Math. 13, Springer-Verlag, New York, 2004.
[19] A. Smyshlyaev, E. Cerpa, and M. Krstic, *Boundary stabilization of a 1-D wave equation with in-domain antidamping*, SIAM J. Control Optim., 48 (2010), pp. 4014–4031.
[20] A. Smyshlyaev and M. Krstic, *Closed-form boundary state feedbacks for a class of 1-D partial integro-differential equations*, IEEE Trans. Automat. Control, 49 (2004), pp. 2185–2202.
[21] A. Smyshlyaev and M. Krstic, *On control design for PDEs with space-dependent diffusivity or time-dependent reactivity*, Automatica J. IFAC, 41 (2005), pp. 1601–1608.
[22] A. Smyshlyaev and M. Krstic, *Boundary control of an anti-stable wave equation with anti-damping on the uncontrolled boundary*, Systems Control Lett., 58 (2009), pp. 617–623.
[23] D. L. Russell, *Controllability and stabilizability theory for linear partial differential equations: Recent progress and open questions*, SIAM Rev., 20 (1978), pp. 639–739.
[24] R. Vazquez and M. Krstic, *Control of 1-D parabolic PDEs with Volterra nonlinearities.* I. *Design*, Automatica J. IFAC, 44 (2008), pp. 2778–2790.