

1 **SYMPLECTIC MODEL-REDUCTION WITH A WEIGHTED INNER**  
2 **PRODUCT\***

3 BABAK MABOUDI AFKHAM<sup>†</sup>, ASHISH BHATT<sup>‡</sup>, BERNARD HAASDONK<sup>‡</sup>, AND JAN S.  
4 HESTHAVEN<sup>†</sup>

5 **Abstract.** In the recent years, considerable attention has been paid to preserving structures and  
6 invariants in reduced basis methods, in order to enhance the stability and robustness of the reduced  
7 system. In the context of Hamiltonian systems, symplectic model reduction seeks to construct a  
8 reduced system that preserves the symplectic symmetry of Hamiltonian systems. However, symplectic  
9 methods are based on the standard Euclidean inner products and are not suitable for problems  
10 equipped with a more general inner product. In this paper we generalize symplectic model reduction  
11 to allow for the norms and inner products that are most appropriate to the problem while preserving  
12 the symplectic symmetry of the Hamiltonian systems. To construct a reduced basis and accelerate  
13 the evaluation of nonlinear terms, a greedy generation of a symplectic basis is proposed. Furthermore,  
14 it is shown that the greedy approach yields a norm bounded reduced basis. The accuracy and the  
15 stability of this model reduction technique is illustrated through the development of reduced models  
16 for a vibrating elastic beam and the sine-Gordon equation.

17 **Key words.** Structure Preserving, Weighted MOR, Hamiltonian Systems, Greedy Reduced  
18 Basis, Symplectic DEIM

19 **AMS subject classifications.** 78M34, 34C20, 35B30, 37K05, 65P10, 37J25

20 **1. Introduction.** Reduced order models have emerged as a powerful approach  
21 to cope with increasingly complex new applications in engineering and science. These  
22 methods substantially reduce the dimensionality of the problem by constructing a  
23 reduced configuration space. Exploration of the reduced space is then possible with  
24 significant acceleration [26, 23].

25 Over the past decade, reduced basis (RB) methods have demonstrated great suc-  
26 cess in lowering of the computational costs of solving elliptic and parabolic differential  
27 equations [27, 28]. However, model order reduction (MOR) of hyperbolic problems  
28 remains a challenge. Such problems often arise from a set of conservation laws and  
29 invariants. These intrinsic structures are lost during MOR which results in a qualita-  
30 tively wrong, and sometimes unstable reduced system [3].

31 Recently, the construction of RB methods that conserve intrinsic structures has  
32 attracted attention [2, 1, 29, 18, 8, 13, 7, 37]. Structure preservation in MOR not only  
33 constructs a physically meaningful reduced system, but can also enhance the robust-  
34 ness and stability of the reduced system. In system theory, conservation of passivity  
35 can be found in the work of [38, 22]. Energy preserving and inf-sup stable methods  
36 for finite element methods (FEM) are developed in [18, 5]. Also, a conservative MOR  
37 technique for finite-volume methods is proposed in [12].

38 Moreover, the simulation of reduced models incurs solution errors and the estima-  
39 tion of this error is essential in applications of MOR [24, 40, 19]. Finding tight error  
40 bounds for a general reduced system has shown to be computationally expensive and

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<sup>†</sup>Institute of Mathematics (MATH), School of Basic Sciences (FSB), Ecole Polytechnique Fédérale de Lausanne, 1015 Lausanne, Switzerland ([babak.maboudi@epfl.ch](mailto:babak.maboudi@epfl.ch), [jan.hesthaven@epfl.ch](mailto:jan.hesthaven@epfl.ch)).

<sup>‡</sup>University of Stuttgart, IANS, Pfaffenwaldring 57, 70569 Stuttgart, Germany ([\[ashish.bhatt,haasdonk@mathematik.uni-stuttgart.de\]](mailto:[ashish.bhatt,haasdonk@mathematik.uni-stuttgart.de])).

41 often impractical. Therefore, when one is interested in a cheap surrogate for the error  
 42 or when the conserved quantity is an output of the system, it becomes imperative to  
 43 preserve system structures in the reduced model.

44 In the context of Lagrangian and Hamiltonian systems, recent works provide a  
 45 promising approach to the construction of robust and stable reduced systems. Carl-  
 46 berg, Tuminaro, and Boggs [14] suggest that a reduced order model of a Lagrangian  
 47 system be identified by an approximate Lagrangian on a reduced order configuration  
 48 space. This allows the reduced system to inherit the geometric structure of the orig-  
 49 inal system. A similar approach has been adopted in the work of Peng and Mohseni  
 50 [37] and in the work of Maboudi Afkham and Hesthaven [2] for Hamiltonian systems.  
 51 They construct a low-order symplectic linear vector space, i.e. a vector space equipped  
 52 with a symplectic 2-form, as the reduced space. Once the symplectic reduced space  
 53 is generated, a symplectic projection result in a physically meaningful reduced sys-  
 54 tem. A proper time-stepping scheme then preserves the Hamiltonian structure of  
 55 the reduced system. It is shown in [2, 37] that this approach preserves the overall  
 56 dynamics of the original system and enhances the stability of the reduced system. De-  
 57 spite the success of these method in MOR of Hamiltonian systems, these techniques  
 58 are only compatible with the Euclidean inner product. Therefore, the computational  
 59 structures that arise from a natural inner product of a problem will be lost during  
 60 MOR.

61 Weak formulations and inner-products, defined on a Hilbert space, are at the  
 62 core of the error analysis of many numerical methods for solving partial differential  
 63 equations. Therefore, it is natural to seek MOR methods that consider such features.  
 64 At the discrete level, these features often require a Euclidean vector space to be  
 65 equipped with a generalized inner product, associated with a weight matrix  $X$ . Many  
 66 works enabled conventional MOR techniques compatible with such inner products [41].  
 67 However, a MOR method that simultaneously preserves the symplectic symmetry of  
 68 Hamiltonian systems remains unknown.

69 In this paper, we seek to combine a classical MOR method with respect to a  
 70 weight matrix with the symplectic MOR. The reduced system constructed by the new  
 71 method is a generalized Hamiltonian system and the low order configuration space  
 72 associated with this system is a symplectic linear vector space with a non-standard  
 73 symplectic 2-form. It is demonstrated that the new method can be viewed as the  
 74 natural extension to [2], and therefore retains the structure preserving features, e.g.  
 75 symplecticity and stability. We also present a greedy approach for the construction  
 76 of a generalized symplectic basis for the reduced system. Structured matrices are  
 77 in general not norm bounded [30]. However, we show that the condition number of  
 78 the basis generated by the greedy method is bounded by the condition number of  
 79 the weight matrix  $X$ . Finally, to accelerate the evaluation of nonlinear terms in the  
 80 reduced system, we present a variation of the discrete empirical interpolation method  
 81 (DEIM) that preserves the symplectic structure of the reduced system.

82 What remains of this paper is organized as follows. In [section 2](#) we cover the  
 83 required background on the Hamiltonian and the generalized Hamiltonian systems.  
 84 [Section 3](#) summarizes classic MOR routine with respect to a weighted norm and the  
 85 symplectic MOR method with respect to the standard Euclidean inner product. We  
 86 introduce the symplectic MOR method with respect to a weighted inner product in  
 87 [section 4](#). [Section 5](#) illustrates the performance of the new method through a vibrating  
 88 beam and the sine-Gordon equation. We offer a few conclusive remarks in [section 6](#).

89 **2. Hamiltonian systems.** In this section we discuss the basic concepts of the  
 90 geometry of symplectic linear vector spaces and introduce Hamiltonian and General-  
 91 ized Hamiltonian systems.

92 **2.1. Generalized Hamiltonian systems.** Let  $(\mathbb{R}^{2n}, \Omega)$  be a symplectic linear  
 93 vector space, with  $\mathbb{R}^{2n}$  the configuration space and  $\Omega : \mathbb{R}^{2n} \times \mathbb{R}^{2n} \rightarrow \mathbb{R}$  a closed, skew-  
 94 symmetric and non-degenerate 2-form on  $\mathbb{R}^{2n}$ . Given a smooth function  $H : \mathbb{R}^{2n} \rightarrow \mathbb{R}$ ,  
 95 the so called *Hamiltonian*, the *generalized Hamiltonian system* of evolution reads

$$96 \quad (1) \quad \begin{cases} \dot{z} = J_{2n} \nabla_z H(z), \\ z(0) = z_0. \end{cases}$$

97 Here  $z \in \mathbb{R}^{2n}$  are the configuration coordinates and  $J_{2n}$  is a constant, full-rank and  
 98 skew-symmetric  $2n \times 2n$  *structure* matrix such that  $\Omega(x, y) = x^T J_{2n} y$ , for all state  
 99 vectors  $x, y \in \mathbb{R}^{2n}$  [34]. Note that there always exists a coordinate transformation  
 100  $\tilde{z} = \mathcal{T}^{-1} z$ , with  $\mathcal{T} \in \mathbb{R}^{2n \times 2n}$ , such that  $J_{2n}$  takes the form of the *standard* symplectic  
 101 structure matrix

$$102 \quad (2) \quad \mathbb{J}_{2n} = \begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix},$$

103 in the new coordinate system [16]. Here  $0_n$  and  $I_n$  are the zero matrix and the  
 104 identity matrix of size  $n \times n$ , respectively. A central feature of Hamiltonian systems  
 105 is conservation of the Hamiltonian.

106 **THEOREM 2.1.** [34] *The Hamiltonian  $H$  is a conserved quantity of the Hamilto-*  
 107 *nian system (1) i.e.  $H(z(t)) = H(z_0)$  for all  $t \geq 0$ .*

108 Under a general coordinate transformation, the equations of evolution of a Hamil-  
 109 tonian system might not take the form (1). Indeed only transformations which pre-  
 110 serve the symplectic form, *symplectic transformations*, preserve the form of a Hamil-  
 111 tonian system [25]. Suppose that  $(\mathbb{R}^{2n}, \Omega)$  and  $(\mathbb{R}^{2k}, \Lambda)$  are two symplectic linear  
 112 vector spaces. A transformation  $\mu : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2k}$  is a symplectic transformation if

$$113 \quad (3) \quad \Omega(x, y) = \Lambda(\mu(x), \mu(y)), \quad \text{for all } x, y \in \mathbb{R}^{2n}.$$

114 In matrix notation, i.e. when we consider a set of basis vectors for  $\mathbb{R}^{2n}$  and  $\mathbb{R}^{2k}$ , a  
 115 linear symplectic transformation is of the form  $\mu(x) = Ax$  with  $A \in \mathbb{R}^{2n \times 2k}$  such that

$$116 \quad (4) \quad A^T J_{2n} A = J_{2k}.$$

117 We are interested in a class of symplectic transformations that transform a symplectic  
 118 structure  $J_{2n}$  into the standard symplectic structure  $\mathbb{J}_{2k}$ .

119 **DEFINITION 2.2.** *Let  $J_{2n} \in \mathbb{R}^{2n \times 2n}$  be a full-rank skew-symmetric structure ma-*  
 120 *trix. A matrix  $A \in \mathbb{R}^{2n \times 2k}$  is  $J_{2n}$ -symplectic if*

$$121 \quad (5) \quad A^T J_{2n} A = \mathbb{J}_{2k}.$$

122 Note that in the literature [34, 25], symplectic transformations refer to  $\mathbb{J}_{2n}$ -symplectic  
 123 matrices, in contrast to [Definition 2.2](#).

124 It is natural to expect a numerical integrator that solves (1) to also satisfy the con-  
 125 servation law expressed in [Theorem 2.1](#). Conventional numerical time integrators, e.g.  
 126 general Runge-Kutta methods, do not generally preserve the symplectic symmetry of

127 Hamiltonian systems which often result in an unphysical behavior of the solution over  
 128 long time-integration. *Poisson integrators* [25] are known to preserve the Hamiltonian  
 129 of (1). To construct a general Poisson integrator, we seek a coordinate transformation  
 130  $\mathcal{T} : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$ ,  $\tilde{z} = \mathcal{T}^{-1}z$ , such that  $J_{2n} = \mathcal{T}J_{2n}\mathcal{T}^T$ . Then, a *symplectic integrator*  
 131 can preserve the symplectic structure of the transformed system. The *Störmer-Verlet*  
 132 scheme is an example of a second order symplectic time-integrator given as

$$\begin{aligned}
 & q_{m+1/2} = q_m + \frac{\Delta t}{2} \cdot \nabla_p \tilde{H}(p_m, q_{m+1/2}), \\
 133 \quad (6) \quad & p_{m+1} = p_m - \frac{\Delta t}{2} \cdot \left( \nabla_q \tilde{H}(p_m, q_{m+1/2}) + \nabla_q \tilde{H}(p_{m+1}, q_{m+1/2}) \right), \\
 & q_{m+1} = q_{m+1/2} + \frac{\Delta t}{2} \cdot \nabla_p \tilde{H}(p_{m+1}, q_{m+1/2}).
 \end{aligned}$$

134 Here,  $\tilde{z} = (q^T, p^T)^T$ ,  $\tilde{H}(\tilde{z}) = H(\mathcal{T}^{-1}z)$ ,  $\Delta t$  denotes a uniform time step-size, and  
 135  $q_m \approx q(m\Delta t)$  and  $p_m \approx p(m\Delta t)$ ,  $m \in \mathbb{N} \cup \{0\}$ , are approximate numerical solu-  
 136 tions. Note that it is important to use a backward stable method to compute the  
 137 transformation  $\mathcal{T}$ . In this paper we use the symplectic Gaussian elimination method  
 138 with complete pivoting to compute the decomposition  $J_{2n} = \mathcal{T}J_{2n}\mathcal{T}^T$ . However, one  
 139 may use a more computationally efficient method, e.g., a Cholesky-like factorization  
 140 proposed in [9] or the isotropic Arnoldi/Lanczos methods [35]. There are a few known  
 141 numerical integrators that preserve the symplectic symmetry of a generalized Hamil-  
 142 tonian system without requiring the computation of the transformation matrix  $\mathcal{T}$  [25].  
 143 The implicit midpoint rule

$$144 \quad (7) \quad z_{m+1} = z_m + \Delta t \cdot J_{2n} \nabla_z H \left( \frac{z_{m+1} + z_m}{2} \right),$$

145 for (1) is an example of such integrators. For more on the construction and the  
 146 applications of Poisson/symplectic integrators, we refer the reader to [25, 11].

147 **3. Model order reduction.** In this section we summarize the fundamentals of  
 148 MOR and discuss the conventional approach to MOR with a weighted inner product.  
 149 We then recall the main results from [2] regarding symplectic MOR. In [section 4](#)  
 150 we shall combine the two concepts to introduce the symplectic MOR of Hamiltonian  
 151 systems with respect to a weighted inner product.

152 **3.1. Model-reduction with a weighted inner product.** Consider a dynam-  
 153 ical system of the form

$$154 \quad (8) \quad \begin{cases} \dot{x}(t) = f(t, x), \\ x(0) = x_0. \end{cases}$$

155 where  $x \in \mathbb{R}^m$  and  $f : \mathbb{R} \times \mathbb{R}^m \rightarrow \mathbb{R}^m$  is some continuous function. In this paper we  
 156 assume that the time  $t$  is the only parameter on which the solution vector  $x$  depends.  
 157 Nevertheless, it is straightforward to generalize the findings of this paper to the case  
 158 of parametric MOR, where  $x$  depends on a larger set of parameters that belong to a  
 159 closed and bounded subset.

160 Suppose that  $x$  is well approximated by a low dimensional linear subspace with  
 161 the basis matrix  $V = [v_1 | \dots | v_k] \in \mathbb{R}^{m \times k}$ ,  $v_i \in \mathbb{R}^m$  for  $i = 1, \dots, k$ . The approximate  
 162 solution to (8) in this basis reads

$$163 \quad (9) \quad x \approx Vy,$$

164 where  $y \in \mathbb{R}^k$  are the expansion coefficients of  $x$  in the basis  $V$ . Note that projection  
 165 of  $x$  onto  $\text{colspan}(V)$  depends on the inner product and the norm defined on (8). We  
 166 define the weighted inner product

$$167 \quad (10) \quad \langle x, y \rangle_X = x^T X y, \quad \text{for all } x, y \in \mathbb{R}^m,$$

168 for some symmetric and positive-definite matrix  $X \in \mathbb{R}^{m \times m}$  and refer to  $\|\cdot\|_X$  as the  
 169  $X$ -norm associated to this inner product. If we choose  $V$  to be an orthonormal basis  
 170 with respect to the  $X$ -norm ( $V^T X V = I_k$ ), then the operator

$$171 \quad (11) \quad P_{X,V}(x) = V V^T X x, \quad \text{for all } x \in \mathbb{R}^m$$

172 becomes idempotent, i.e.  $P_{X,V}$  is a projection operator onto  $\text{colspan}(V)$ .

173 Now suppose that the *snapshot matrix*  $S = [x(t_1)|x(t_2)|\dots|x(t_N)]$  is a collection  
 174 of  $N$  solutions to (8) at time instances  $t_1, \dots, t_N$ . We seek  $V$  such that it minimizes  
 175 the collective projection error of the samples onto  $\text{colspan}(V)$  which corresponds to  
 176 the minimization problem

$$177 \quad (12) \quad \begin{aligned} & \underset{V \in \mathbb{R}^{m \times k}}{\text{minimize}} && \sum_{i=1}^N \|x(t_i) - P_{X,V}(x(t_i))\|_X^2, \\ & \text{subject to} && V^T X V = I_k. \end{aligned}$$

178 Note that the solution to (12) is known as the proper orthogonal decomposition (POD)  
 179 [26, 39, 21]. Following [39] the above minimization is equivalent to

$$180 \quad (13) \quad \begin{aligned} & \underset{\tilde{V} \in \mathbb{R}^{m \times k}}{\text{minimize}} && \|\tilde{S} - \tilde{V} \tilde{V}^T \tilde{S}\|_F^2, \\ & \text{subject to} && \tilde{V}^T \tilde{V} = I_k. \end{aligned}$$

181 where  $\tilde{V} = X^{1/2} V$ ,  $\tilde{S} = X^{1/2} S$ , and  $X^{1/2}$  is the matrix square root of  $X$ . According  
 182 to the Schmidt-Mirsky-Eckart-Young theorem [33] the solution  $\tilde{V}$  to the minimization  
 183 (13) is the truncated singular value decomposition (SVD) of  $\tilde{S}$ . The basis  $V$  then is  
 184  $V = X^{-1/2} \tilde{V}$ . The reduced model of (8), using the basis  $V$  and the projection  $P_{X,V}$ ,  
 185 is

$$186 \quad (14) \quad \begin{cases} \dot{y}(t) = V^T X f(t, V y), \\ y(0) = V^T X x_0. \end{cases}$$

187 If  $k$  can be chosen such that  $k \ll m$ , then the reduced system (14) can potentially  
 188 be evaluated significantly faster than the full order system (8). Finding the matrix  
 189 square root of  $X$  can often be computationally exhaustive. In such cases, explicit use  
 190 of  $X^{1/2}$  can be avoided by finding the eigen-decomposition of the *Gramian* matrix  
 191  $G = S^T X S$  [39, 23].

192 Besides RB methods, there exist other ways of basis generation e.g. greedy strate-  
 193 gies, the Krylov subspace method, balanced truncation, Hankel-norm approximation  
 194 etc. [4]. We refer the reader to [26, 39, 23] for further information regarding the  
 195 development and the efficiency of reduced order models.

196 **3.2. Symplectic MOR.** Conventional MOR methods, e.g. those introduced  
 197 in subsection 3.1, do not generally preserve the conservation law expressed in **The-**  
 198 **orem 2.1.** As mentioned earlier, this often results in the lack of robustness in the

199 reduced system over long time-integration. In this section we summarize the main  
 200 findings of [2] regarding symplectic model order reduction of Hamiltonian systems with  
 201 respect to the standard Euclidean inner product. Symplectic MOR aims to construct  
 202 a reduced system that conserves the geometric symmetry expressed in [Theorem 2.1](#)  
 203 which helps with the stability of the reduced system. Consider a Hamiltonian system  
 204 of the form

$$205 \quad (15) \quad \begin{cases} \dot{z}(t) = \mathbb{J}_{2n} Lz(t) + \mathbb{J}_{2n} \nabla_z f(z), \\ z(0) = z_0. \end{cases}$$

206 Here  $z \in \mathbb{R}^{2n}$  is the state vector,  $L \in \mathbb{R}^{2n \times 2n}$  is a symmetric and positive-definite  
 207 matrix and  $f : \mathbb{R}^{2n} \rightarrow \mathbb{R}$  is sufficiently smooth function. Note that the Hamiltonian  
 208 for system (15) is given by  $H(z) = \frac{1}{2} z^T Lz + f(z)$ . Suppose that the solution to (15)  
 209 is well approximated by a low dimensional symplectic subspace. Let  $A \in \mathbb{R}^{2n \times 2k}$  be a  
 210  $\mathbb{J}_{2n}$ -symplectic basis containing the basis vectors  $A = [e_1 | \dots | e_k | f_1 | \dots | f_k]$ , such that  
 211  $z \approx Ay$  with  $y \in \mathbb{R}^{2k}$  the expansion coefficients of  $z$  in this basis. Using the symplectic  
 212 inverse  $A^+ := \mathbb{J}_{2k}^T A^T \mathbb{J}_{2n}$  we can construct the reduced system

$$213 \quad (16) \quad \dot{y} = A^+ \mathbb{J}_{2n} (A^+)^T A^T L A y + A^+ \mathbb{J}_{2n} (A^+)^T \nabla_y f(Ay).$$

214 We refer the reader to [2] for the details of the derivation. It is shown in [37] that  
 215  $(A^+)^T$  is also  $\mathbb{J}_{2n}$ -symplectic, therefore  $A^+ \mathbb{J}_{2n} (A^+)^T = \mathbb{J}_{2k}$  and (16) reduces to

$$216 \quad (17) \quad \dot{y}(t) = \mathbb{J}_{2k} A^T L A y + \mathbb{J}_{2k} \nabla_y f(Ay).$$

217 This system is a Hamiltonian system with the Hamiltonian  $\mathcal{H}(y) = \frac{1}{2} y^T A^T L A y +$   
 218  $f(Ay)$ . To reduce the complexity of evaluating the nonlinear term in (17), we may  
 219 apply the discrete empirical interpolation method (DEIM) [6, 15, 42]. Assuming that  
 220  $\nabla_z f(z)$  lies near a low dimensional subspace with a basis matrix  $U \in \mathbb{R}^{2n \times r}$  the DEIM  
 221 approximation reads

$$222 \quad (18) \quad \nabla_z f(z) \approx U (\mathcal{P}^T U)^{-1} \mathcal{P}^T \nabla_z f(z).$$

223 Here  $\mathcal{P} \in \mathbb{R}^{2n \times r}$  is the interpolating index matrix [15]. For a general choice of  $U$  the  
 224 approximation in (18) destroys the Hamiltonian structure, if inserted in (15). It is  
 225 shown in [2] that by taking  $U = (A^+)^T$  we can recover the Hamiltonian structure in  
 226 (17). Therefore, the reduced system to (15) becomes

$$227 \quad (19) \quad \begin{cases} \dot{y}(t) = \mathbb{J}_{2k} A^T L A y + \mathbb{J}_{2k} (A^+)^T (\mathcal{P}^T (A^+)^T)^{-1} \mathcal{P}^T \nabla_z f(Ay), \\ y(0) = A^+ z_0. \end{cases}$$

228 Note that the Hamiltonian formulation of (19) allows us to integrate it using a symplectic  
 229 integrator. This conserves the symmetry expressed in [Theorem 2.1](#) at the level  
 230 of the reduced system. It is also shown in [2, 37] that the stability of the critical points  
 231 of (15) is preserved in the reduced system and the difference of the Hamiltonians of  
 232 the two system (15) and (19) is constant. Therefore, the overall behavior (19) is close  
 233 to the full order Hamiltonian system (15). In the next subsection we discuss methods  
 234 for generating a  $\mathbb{J}_{2n}$ -symplectic basis  $A$ .

235 **3.3. Greedy generation of a  $\mathbb{J}_{2n}$ -symplectic basis.** Suppose that  $S \in \mathbb{R}^{2n \times N}$   
 236 is the snapshot matrix containing the time instances  $\{z(t_i)\}_{i=1}^N$  of the solution to (15).

237 We seek the  $\mathbb{J}_{2n}$ -symplectic basis  $A$  such that the collective symplectic projection error  
 238 of samples in  $S$  onto  $\text{colspan}(A)$  is minimized.

$$239 \quad (20) \quad \begin{aligned} & \underset{A \in \mathbb{R}^{2n \times 2k}}{\text{minimize}} && \|S - P_{I,A}^{\text{symp}}(S)\|_F^2, \\ & \text{subject to} && A^T \mathbb{J}_{2n} A = \mathbb{J}_{2k}. \end{aligned}$$

240 Here  $P_{I,A}^{\text{symp}} = AA^+$  is the symplectic projection operator with respect to the standard  
 241 Euclidean inner product onto  $\text{colspan}(A)$ . Note that  $P_{I,A}^{\text{symp}} \circ P_{I,A}^{\text{symp}} = P_{I,A}^{\text{symp}}$  [37, 2].

242 Direct approaches to solve (20) are often inefficient. Some SVD-type solutions to  
 243 (20) are proposed by [37]. However, the form of the suggested basis, e.g. the block  
 244 diagonal form suggested in [37], is not compatible with a general weight matrix  $X$ .

245 The greedy generation of a  $\mathbb{J}_{2n}$ -symplectic basis aims to find a near optimal so-  
 246 lution to (20) in an iterative process. This method increases the overall accuracy of  
 247 the basis by adding the best possible basis vectors at each iteration. Suppose that  
 248  $A_{2k} = [e_1 | \dots | e_k | \mathbb{J}_{2n}^T e_1 | \dots | \mathbb{J}_{2n}^T e_k]$  is a  $\mathbb{J}_{2n}$ -symplectic and orthonormal basis [2]. The  
 249 first step of the greedy method is to find the snapshot  $z_{k+1}$ , that is worst approximated  
 250 by the basis  $A_{2k}$ :

$$251 \quad (21) \quad z_{k+1} := \underset{z \in \{z(t_i)\}_{i=1}^N}{\text{argmax}} \|z - P_{I,A_{2k}}^{\text{symp}}(z)\|_2.$$

252 Note that if  $z_{k+1} \neq 0$  then  $z_{k+1}$  is not in  $\text{colspan}(A_{2k})$ . Then we obtain a non-trivial  
 253 vector  $e_{k+1}$  by  $\mathbb{J}_{2n}$ -orthogonalizing  $z_{k+1}$  with respect to  $A_{2k}$ :

$$254 \quad (22) \quad \tilde{z} = z_{k+1} - A_{2k} \alpha, \quad e_{k+1} = \frac{\tilde{z}}{\|\tilde{z}\|_2}.$$

255 Here,  $\alpha \in \mathbb{R}^{2k}$  are the expansion coefficients of the projection of  $z$  onto the column  
 256 span of  $A_{2k}$  where  $\alpha_i = -\Omega(z_{k+1}, \mathbb{J}_{2n}^T e_i)$  for  $i \leq k$  and  $\alpha_i = \Omega(z_{k+1}, e_i)$  for  $i > k$ .  
 257 Since  $\Omega(e_{k+1}, \mathbb{J}_{2n}^T e_{k+1}) = \|e_{k+1}\|_2^2 \neq 0$  the enriched basis  $A_{2k+2}$  reads

$$258 \quad (23) \quad A_{2k+2} = [e_1 | \dots | e_k | e_{k+1} | \mathbb{J}_{2n}^T e_1 | \dots | \mathbb{J}_{2n}^T e_{k+1}].$$

259 It is easily verified that  $A_{2k+2}$  is  $\mathbb{J}_{2n}$ -symplectic and orthonormal. This enrichment  
 260 continues until the given tolerance is satisfied. We note that the choice of the or-  
 261 thogonalization routine generally depends on the application. In this paper we use  
 262 the symplectic Gram-Schmidt (GS) process as the orthogonalization routine. How-  
 263 ever the isotropic Arnoldi method or the isotropic Lanczos method [35] are backward  
 264 stable alternatives.

265 MOR is specially useful in reducing parametric models that depend on a closed  
 266 and bounded parameter set  $\mathcal{S} \subset \mathbb{R}^d$  characterizing physical properties of the under-  
 267 lying system. The evaluation of the projection error is impractical for such problems.  
 268 The loss in the Hamiltonian function can be used as a cheap surrogate to the projec-  
 269 tion error. Suppose that a  $\mathbb{J}_{2n}$ -symplectic basis  $A_{2k}$  is given, then one selects a new  
 270 parameter  $\omega_{k+1} \in \mathcal{S}$  by greedy approach:

$$271 \quad (24) \quad \omega_{k+1} = \underset{\omega \in \mathcal{S}}{\text{argmax}} |H(z(\omega)) - H(P_{I,A}^{\text{symp}}(z(\omega)))|,$$

272 and then enriches the basis  $A_{2k}$  as discussed above. It is shown in [2] that the loss in  
 273 the Hamiltonian is constant in time. Therefore,  $\omega_{k+1}$  can be identified in the *offline*



274 *phase* before simulating the reduced order model. Note that the relation between the  
 275 projection error (21) and the error in the Hamiltonian (24) is still unknown.

276 We summarize the greedy algorithm for generating a  $\mathbb{J}_{2n}$ -symplectic basis in **Algo-**  
 277 **rithm 1**. The first loop constructs a  $\mathbb{J}_{2n}$ -symplectic basis for the Hamiltonian system  
 278 (15), and the second loop adds the nonlinear snapshots to the symplectic inverse of  
 279 the basis. We refer the reader to [2] for more details. In **section 4** we will show how  
 280 this algorithm can be generalized to support any weighted inner product.

---

**Algorithm 1** The greedy algorithm for generation of a  $\mathbb{J}_{2n}$ -symplectic basis

---

**Input:** Tolerated projection error  $\delta$ , initial condition  $z_0$ , snapshots  $\mathcal{Z} = \{z(t_i)\}_{i=1}^N$   
 and  $\mathcal{G} = \{\nabla f(z(t_i))\}_{i=1}^N$

1.  $e_1 \leftarrow \frac{z_0}{\|z_0\|_2}$
2.  $A \leftarrow [e_1 | \mathbb{J}_{2n}^T e_1]$
3.  $k \leftarrow 1$
4. **while**  $\|z - P_{I,A}^{\text{symp}}(z)\|_2 > \delta$  for any  $z \in \mathcal{Z}$
5.      $z_{k+1} := \operatorname{argmax}_{z \in \mathcal{Z}} \|z - P_{I,A}^{\text{symp}}(z)\|_2$
6.      $\mathbb{J}_{2n}$ -orthogonalize  $z_{k+1}$  to obtain  $e_{k+1}$
7.      $A \leftarrow [e_1 | \dots | e_{k+1} | \mathbb{J}_{2n}^T e_1 | \dots | \mathbb{J}_{2n}^T e_{k+1}]$
8.      $k \leftarrow k + 1$
9. **end while**
10. compute  $(A^+)^T = [e'_1 | \dots | e'_k | \mathbb{J}_{2n}^T e'_1 | \dots | \mathbb{J}_{2n}^T e'_k]$
11. **while**  $\|g - P_{I,(A^+)^T}^{\text{symp}}(g)\|_2 > \delta$  for all  $g \in \mathcal{G}$
12.      $g_{k+1} := \operatorname{argmax}_{g \in \mathcal{G}} \|g - P_{I,(A^+)^T}^{\text{symp}}(g)\|_2$
13.      $\mathbb{J}_{2n}$ -orthogonalize  $g_{k+1}$  to obtain  $e'_{k+1}$
14.      $(A^+)^T \leftarrow [e'_1 | \dots | e'_{k+1} | \mathbb{J}_{2n}^T e'_1 | \dots | \mathbb{J}_{2n}^T e'_{k+1}]$
15.      $k \leftarrow k + 1$
16. **end while**
17.  $A \leftarrow \left( \left( (A^+)^T \right)^+ \right)^T$

**Output:**  $\mathbb{J}_{2n}$ -symplectic basis  $A$ .

---

281 **4. Symplectic MOR with weighted inner product.** In this section we com-  
 282 bine the concept of model reduction with a weighted inner product, discussed in **sub-**  
 283 **section 3.1**, with the symplectic model reduction discussed in **subsection 3.2**. We  
 284 will argue that the new method can be viewed as a natural extension of the original  
 285 symplectic method. Finally, we generalize the greedy method for the symplectic basis  
 286 generation, and the symplectic model reduction of nonlinear terms to be compatible  
 287 with any non-degenerate weighted inner product.

288 **4.1. Generalization of the symplectic projection.** As discussed in **subsec-**  
 289 **tion 3.1**, the error analysis of methods for solving partial differential equations often  
 290 requires the use of a weighted inner product. This is particularly important when  
 291 dealing with Hamiltonian systems, where the system energy can induce a norm that  
 292 is fundamental to the dynamics of the system.

293 Consider a Hamiltonian system of the form (15) together with the weighted inner  
 294 product defined in (10) with  $m = 2n$ . Also suppose that the solution  $z$  to (15) is  
 295 well approximated by a  $2k$  dimensional symplectic subspace with the basis matrix



296 A. We seek to construct a projection operator that minimizes the projection error  
 297 with respect to the  $X$ -norm while preserving the symplectic dynamics of (15) in the  
 298 projected space. Consider the operator  $P : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$  be defined as

$$299 \quad (25) \quad P = A\mathbb{J}_{2k}^T A^T X \mathbb{J}_{2n} X.$$

300 It is easy to show that  $P$  is idempotent if and only if

$$301 \quad (26) \quad \mathbb{J}_{2k}^T A^T X \mathbb{J}_{2n} X A = I_{2k},$$

302 in which case  $P$  is a projection operator onto  $\text{colspan}(A)$ . Suppose that  $S$  is the  
 303 snapshot matrix containing the time samples  $\{z(t_i)\}_{i=1}^N$  of the solution to (15). We  
 304 seek to find the basis  $A$  that minimizes the collective projection error of snapshots  
 305 with respect to the  $X$ -norm,

$$306 \quad (27) \quad \begin{aligned} & \underset{A \in \mathbb{R}^{2n \times 2k}}{\text{minimize}} && \sum_{i=1}^N \|z(t_i) - P(z(t_i))\|_X^2, \\ & \text{subject to} && \mathbb{J}_{2k}^T A^T X \mathbb{J}_{2n} X A = I_{2k}. \end{aligned}$$

307 By (25) we have

$$308 \quad (28) \quad \begin{aligned} \sum_{i=1}^N \|z(t_i) - P(z(t_i))\|_X^2 &= \sum_{i=1}^N \|z(t_i) - A\mathbb{J}_{2k}^T A^T X \mathbb{J}_{2n} X z(t_i)\|_X^2 \\ &= \sum_{i=1}^N \|X^{1/2} z(t_i) - X^{1/2} A\mathbb{J}_{2k}^T A^T X \mathbb{J}_{2n} X z(t_i)\|_2^2 \\ &= \|X^{1/2} S - X^{1/2} A\mathbb{J}_{2k}^T A^T X \mathbb{J}_{2n} X S\|_F^2 \\ &= \|\tilde{S} - \tilde{A}\tilde{A}^+\tilde{S}\|_F^2. \end{aligned}$$

309 Here  $\tilde{S} = X^{1/2} S$ ,  $\tilde{A} = X^{1/2} A$ , and  $\tilde{A}^+ = \mathbb{J}_{2k}^T \tilde{A}^T J_{2n}$  is the symplectic inverse of  
 310  $\tilde{A}$  with respect to the skew-symmetric matrix  $J_{2n} = X^{1/2} \mathbb{J}_{2n} X^{1/2}$ . Note that the  
 311 symplectic inverse in (28) is a generalization of the symplectic inverse introduced in  
 312 subsection 3.2. Therefore, we may use the same notation (the superscript  $+$ ) for  
 313 both. We summarized the properties of this generalization in Theorem 4.1. With this  
 314 notation, the condition (26) turns into  $\tilde{A}^+ \tilde{A} = I_{2k}$  which is equivalent to  $\tilde{A}^T J_{2n} \tilde{A} =$   
 315  $\mathbb{J}_{2k}$ . In other words, this condition implies that  $\tilde{A}$  has to be a  $J_{2n}$ -symplectic matrix.  
 316 Finally we can rewrite the minimization (27) as

$$317 \quad (29) \quad \begin{aligned} & \underset{\tilde{A} \in \mathbb{R}^{2n \times 2k}}{\text{minimize}} && \|\tilde{S} - P_{X, \tilde{A}}^{\text{symp}}(\tilde{S})\|_F, \\ & \text{subject to} && \tilde{A}^T J_{2n} \tilde{A} = \mathbb{J}_{2k}. \end{aligned}$$

318 where  $P_{X, \tilde{A}}^{\text{symp}} = \tilde{A}\tilde{A}^+$  is the symplectic projection with respect to the  $X$ -norm onto  
 319 the  $\text{colspan}(\tilde{A})$ . At first glance, the minimization (29) might look similar to (20).  
 320 However, since  $\tilde{A}$  is  $J_{2n}$ -symplectic, and the projection operator depends on  $X$ , we  
 321 need to seek an alternative approach to find a near optimal solution to (29).

322 As (20), direct approaches to solving (29) are impractical. Furthermore, there are  
 323 no SVD-type methods known to the authors, that solve (29). However, the greedy  
 324 generation of the symplectic basis can be generalized to generate a near optimal basis  
 325  $\tilde{A}$ . The generalized greedy method is discussed in subsection 4.3.

326 Now suppose that a basis  $A = X^{-1/2}\tilde{A}$ , with  $\tilde{A}$  solving (29), is available such  
 327 that  $z \approx Ay$  with  $y \in \mathbb{R}^{2k}$ , the expansion coefficients of  $z$  in the basis of  $A$ . Using  
 328 (26) we may write the reduced system to (15) as

$$329 \quad (30) \quad \dot{y} = \mathbb{J}_{2k}^T A^T X \mathbb{J}_{2n} X \mathbb{J}_{2n} L A y + \mathbb{J}_{2k}^T A^T X \mathbb{J}_{2n} X \mathbb{J}_{2n} \nabla_z f(Ay).$$

330 Since  $(\mathbb{J}_{2k}^T A^T X \mathbb{J}_{2n} X)A = I_{2k}$ , we may use the chain rule to write

$$331 \quad (31) \quad \nabla_z H(z) = (\mathbb{J}_{2k}^T A^T X \mathbb{J}_{2n} X)^T \nabla_y H(Ay).$$

332 Finally, as  $\nabla_z H(z) = Lz + \nabla_z f(z)$ , the reduced system (30) becomes

$$333 \quad (32) \quad \begin{cases} \dot{y}(t) = J_{2k} A^T L A y + J_{2k} \nabla_y f(Ay), \\ y(0) = \mathbb{J}_{2k}^T A^T X \mathbb{J}_{2n} X z_0, \end{cases}$$

334 where  $J_{2k} = \tilde{A}^+ J_{2n} (\tilde{A}^+)^T$  is a skew-symmetric matrix. The system (32) is a general-  
 335 ized Hamiltonian system with the Hamiltonian defined as  $\mathcal{H}(y) = \frac{1}{2} y^T A^T L A y + f(Ay)$ .  
 336 Therefore, a Poisson integrator preserves the symplectic symmetry associated with  
 337 (32).

338 We close this section by summarizing the properties of the symplectic inverse in  
 339 the following theorem.

340 **THEOREM 4.1.** *Let  $A \in \mathbb{R}^{2n \times 2k}$  be a  $J_{2n}$ -symplectic basis where  $J_{2n} \in \mathbb{R}^{2n \times 2n}$  is*  
 341 *a full rank and skew-symmetric matrix. Furthermore, suppose that  $A^+ = \mathbb{J}_{2k}^T A^T J_{2n}$*   
 342 *is the symplectic inverse. Then the following holds:*

- 343 1.  $A^+ A = I_{2k}$ .
- 344 2.  $(A^+)^T$  is  $J_{2n}^{-1}$ -symplectic.
- 345 3.  $\left( \left( (A^+)^T \right)^+ \right)^T = A$ .
- 346 4. Let  $J_{2n} = X^{1/2} \mathbb{J}_{2n} X^{1/2}$ . Then  $A$  is ortho-normal with respect to the  $X$ -norm,  
 347 if and only if  $(A^+)^T$  is ortho-normal with respect to the  $X^{-1}$ -norm.

348 *Proof.* It is straightforward to show all statements using the definition of a sym-  
 349 plectic basis.  $\square$

350 **4.2. Stability Conservation.** It is shown in [37, 2] that a Hamiltonian reduced  
 351 system constructed by the projection  $P_{I,A}^{\text{symp}}$  preserves the stability of stable equilib-  
 352 rium points of (19), and therefore, preserves the overall dynamics. In this section, we  
 353 discuss that the stability of equilibrium points is also conserved using the projection  
 354 operator  $P_{X,\tilde{A}}^{\text{symp}}$ .

355 **PROPOSITION 4.2.** [10] *An equilibrium point  $z_e \in \mathbb{R}^{2n}$  is Lyapunov stable if there*  
 356 *exists a scalar function  $W : \mathbb{R}^{2n} \rightarrow \mathbb{R}$  such that  $\nabla W(z_e) = 0$ ,  $\nabla^2 W(z_e)$  is positive*  
 357 *definite, and that for any trajectory  $z(t)$  defined in the neighborhood of  $z_e$ , we have*  
 358  *$\frac{d}{dt} W(z(t)) \leq 0$ . Here  $\nabla^2 W$  is the Hessian matrix of  $W$ , and  $W$  is commonly referred*  
 359 *to as a Lyapunov function.*

360 It is shown in [2] that the stable points of the Hamiltonian reduced system con-  
 361 structed using the projection  $P_{X,\tilde{A}}^{\text{symp}}$  is Lyapunov stable. However, since the proof  
 362 only requires the conservation of the Hamiltonian and the positive definiteness of  $\mathcal{H}$ ,  
 363 the proof also holds for generalized Hamiltonian reduced systems.

364 **THEOREM 4.3.** [2] *Consider a Hamiltonian system of the form (15) together with*  
 365 *the reduced system (32). Suppose that  $z_e$  is an equilibrium point for (15) and that*  
 366  *$y_e = \tilde{A}^+ X^{1/2} z_e$ . If  $H$  (or  $-H$ ) is a Lyapunov function satisfying Proposition 4.2,*  
 367 *then  $z_e$  and  $y_e$  are Lyapunov stable equilibrium points for (15) and (32), respectively.*

368 **4.3. Greedy generation of a  $J_{2n}$ -symplectic basis.** In this section we modify  
 369 the greedy algorithm introduced in subsection 3.3 to construct a  $J_{2n}$ -symplectic basis.  
 370 Ortho-normalization is an essential step in greedy approaches to basis generation  
 371 [26, 39]. Here, we summarize a variation of the GS orthogonalization process, known  
 372 as the *symplectic GS* process.

373 Suppose that  $\Omega_{J_{2n}}$  is a symplectic form defined on  $\mathbb{R}^{2n}$  such that  $\Omega_{J_{2n}}(x, y) =$   
 374  $x^T J_{2n} y$ , for all  $x, y \in \mathbb{R}^{2n}$  and some full rank and skew-symmetric matrix  $J_{2n} =$   
 375  $X^{1/2} \mathbb{J}_{2n} X^{1/2}$ . We would like to build a basis of size  $2k + 2$  in an iterative manner and  
 376 start with some initial vector, e.g.  $e_1 = z_0 / \|z_0\|_X$ . It is known that a symplectic basis  
 377 has an even number of basis vectors [34]. We may take  $Te_1$ , where  $T = X^{-1/2} \mathbb{J}_{2n}^T X^{1/2}$ ,  
 378 as a candidate for the second basis vector. It is easily verified that  $\tilde{A}_2 = [e_1 | Te_1]$  is  $J_{2n}$ -  
 379 symplectic and consequently,  $\tilde{A}_2$  is the first basis generated by the greedy approach.  
 380 Next, suppose that  $\tilde{A}_{2k} = [e_1 | \dots | e_k | Te_1 | \dots | Te_k]$  is generated in the  $k$ th step of the  
 381 greedy method and  $z \notin \text{colspan}(\tilde{A}_{2k})$  is provided. We aim to  $J_{2n}$ -orthogonalize  $z$   
 382 with respect to the basis  $\tilde{A}_{2k}$ . This means we seek a coefficient vector  $\alpha \in \mathbb{R}^{2k}$  such  
 383 that

$$384 \quad (33) \quad \Omega_{J_{2n}}(z + \tilde{A}_{2k}\alpha, y) = 0,$$

385 for all possible  $y \in \text{colspan}(\tilde{A}_{2k})$ . It is easily checked that (33) has the unique solution  
 386  $\alpha_i = -\Omega_{J_{2n}}(z, Te_i)$  for  $i \leq k$  and  $\alpha_i = \Omega_{J_{2n}}(z, e_i)$  for  $i > k$ , i.e.,  $z$  has a unique  
 387 symplectic projection. If we take  $\tilde{z} = z + \tilde{A}_{2k}\alpha$ , then the next candidate pair of basis  
 388 vectors are  $e_{k+1} = \tilde{z} / \|\tilde{z}\|_X$  and  $Te_{k+1}$ . Finally, the basis generated at the  $(k + 1)$ -th  
 389 step of the greedy method is given by

$$390 \quad (34) \quad \tilde{A}_{2k+2} = [e_1 | \dots | e_{k+1} | Te_1 | \dots | Te_{k+1}].$$

391 **Theorem 4.4** guarantees that the column vectors of  $\tilde{A}_{2k+2}$  are linearly independent.  
 392 Furthermore, it is checked easily that  $\tilde{A}_{2k+2}$  is  $J_{2n}$ -symplectic. We note that the  
 393 symplectic GS orthogonalization process is chosen due to its simplicity. However, in  
 394 problems where there is a need for a large basis, this process might be impractical. In  
 395 such cases, one may use a backward stable routine, e.g. the isotropic Arnoldi method  
 396 or the isotropic Lanczos method [35].

397 It is well known that a symplectic basis, in general, is not norm bounded [31]. The  
 398 following theorem guarantees that the greedy method for generating a  $J_{2n}$ -symplectic  
 399 basis yields a bounded basis.

400 **THEOREM 4.4.** *The basis generated by the greedy method for constructing a  $J_{2n}$ -*  
 401 *symplectic basis is orthonormal with respect to the  $X$ -norm.*

402 *Proof.* Let  $\tilde{A}_{2k} = [e_1 | \dots | e_k | Te_1 | \dots | Te_k]$  be the  $J_{2n}$ -symplectic basis generated  
 403 at the  $k$ th step of the greedy method. Using the fact that  $A_{2k}$  is  $J_{2n}$ -symplectic, one  
 404 can check that

$$405 \quad (35) \quad \langle e_i, e_j \rangle_X = \langle Te_i, Te_j \rangle_X = \Omega_{J_{2n}}(e_i, Te_j) = \delta_{i,j}, \quad i, j = 1, \dots, k,$$

406 and

$$407 \quad (36) \quad \langle e_i, Te_j \rangle_X = \Omega_{J_{2n}}(e_i, e_j) = 0 \quad i, j = 1, \dots, k,$$

408 where  $\delta_{i,j}$  is the Kronecker delta function. This ensures that  $\tilde{A}_{2k}^T X \tilde{A}_{2k} = I_{2k}$ , i.e.,  
 409  $\tilde{A}_{2k}$  is an ortho-normal basis with respect to the  $X$ -norm.  $\square$

410 We note that if we take  $X = I_{2n}$ , then the greedy process generates a  $\mathbb{J}_{2n}$ -symplectic  
 411 basis. With this choice, the greedy method discussed above becomes identical to the  
 412 greedy process discussed in [subsection 3.3](#). Therefore, the symplectic model reduc-  
 413 tion with a weight matrix  $X$  is indeed a generalization of the method discussed in  
 414 [subsection 3.2](#).

415 We notice that  $X^{1/2}$  does not explicitly appear in (32). Therefore, it is desirable  
 416 to compute  $A_{2k} = X^{-1/2}\tilde{A}_{2k}$  without requiring the computation of the matrix square  
 417 root of  $X$ . It is easily checked that the matrix  $B_{2k} := X^{1/2}\tilde{A}_{2k} = XA_{2k}$  is  $\mathbb{J}_{2n}$ -  
 418 symplectic and orthonormal. Reformulation of condition (33) yields

$$419 \quad (37) \quad \Omega_{\mathbb{J}_{2n}}(w + B_{2k}\alpha, \bar{y}) = 0, \quad \forall \bar{y} \in \text{colspan}(B_{2k}),$$

420 where  $w = X^{1/2}z$ . From (22) we know that (37) has the unique solution  $\alpha_i =$   
 421  $-\Omega_{\mathbb{J}_{2n}}(z, \mathbb{J}_{2n}^T \hat{e}_i)$  for  $i \leq k$  and  $\alpha_i = \Omega_{\mathbb{J}_{2n}}(z, \hat{e}_i)$  for  $i > k$ , where  $\hat{e}_i$  is the  $i$ th column  
 422 vector of  $B_{2k}$ . Furthermore, we take

$$423 \quad (38) \quad \hat{e}_{k+1} = \hat{z}/\|\hat{z}\|_2, \quad \hat{z} = w + B_{2k}\alpha,$$

424 as the next enrichment vector to construct

$$425 \quad (39) \quad B_{2(k+1)} = [\hat{e}_1 | \dots | \hat{e}_{k+1} | \mathbb{J}_{2n}^T \hat{e}_1 | \dots | \mathbb{J}_{2n}^T \hat{e}_{k+1}].$$

426 One can recover  $e_{k+1}$  from the relation  $e_{k+1} = X^{-1/2}\hat{e}_{k+1}$ . However, since we are  
 427 interested in the matrix  $A_{2(k+1)}$  and not  $\tilde{A}_{2(k+1)}$ , we can solve the system  $XA_{2(k+1)} =$   
 428  $B_{2(k+1)}$  for  $A_{2(k+1)}$ . This procedure eliminates the computation of  $X^{1/2}$ .

429 For identifying the best vectors to be added to a set of basis vectors, we may use  
 430 similar error functions to those introduced in [subsection 3.3](#). The projection error can  
 431 be used to identify the snapshot that is worst approximated by a given basis  $\tilde{A}_{2k}$ :

$$432 \quad (40) \quad z_{k+1} := \operatorname{argmax}_{z \in \{z(t_i)\}_{i=1}^N} \|z - P(z)\|_X.$$

433 Where  $P$  is defined in (25). Alternatively we can use the loss in the Hamiltonian  
 434 function in (24) for parameter dependent problems. We summarize the greedy method  
 435 for generating a  $J_{2n}$ -symplectic matrix in [Algorithm 2](#).

436 It is shown in [2] that under natural assumptions on the solution manifold of (15),  
 437 the original greedy method for symplectic basis generation converges exponentially  
 438 fast. We expect the generalized greedy method, equipped with the error function  
 439 (40), to converge as fast, since the  $X$ -norm is topologically equivalent to the standard  
 440 Euclidean norm [20], for a full rank matrix  $X$ .

441 **4.4. Efficient evaluation of nonlinear terms.** The evaluation of the nonlin-  
 442 ear term in (32) still retains a computational complexity proportional to the size of  
 443 the full order system (15). To overcome this, we take an approach similar to [subsec-](#)  
 444 [tion 3.2](#). The DEIM approximation of the nonlinear term in (32) yields

$$445 \quad (41) \quad \dot{y} = J_{2k}A^T L A y + \tilde{A}^+ X^{1/2} \mathbb{J}_{2n} U (\mathcal{P}^T U)^{-1} \mathcal{P}^T \nabla_z f(Ay).$$

446 Here  $U$  is a basis constructed from the nonlinear snapshots  $\{\nabla_z f(z(t_i))\}_{i=1}^N$ , and  $\mathcal{P}$   
 447 is the interpolating index matrix [15]. As discussed in [subsection 3.2](#), for a general  
 448 choice of  $U$ , the reduced system (32) does not retain a Hamiltonian form. Since  
 449  $(\tilde{A}^+ X^{1/2})A = I_{2k}$  applying the chain rule on (41) yields

$$450 \quad (42) \quad \dot{y} = J_{2k}A^T L A y + \tilde{A}^+ X^{1/2} \mathbb{J}_{2n} U (\mathcal{P}^T U)^{-1} \mathcal{P}^T (\tilde{A}^+ X^{1/2})^T \nabla_y f(Ay).$$

---

**Algorithm 2** The greedy algorithm for generation of a  $J_{2n}$ -symplectic basis

---

**Input:** Tolerated projection error  $\delta$ , initial condition  $z_0$ , the snapshots  $\mathcal{Z} = \{Xz(t_i)\}_{i=1}^N$ , full rank matrix  $X = X^T > 0$

1.  $z_1 = Xz(0)$
2.  $P = A\mathbb{J}_{2k}^T A^T X\mathbb{J}_{2n}$
3.  $\hat{e}_1 \leftarrow z_1 / \|z_1\|_2$
4.  $B \leftarrow [\hat{e}_1 | \mathbb{J}_{2n}^T \hat{e}_1]$
5.  $k \leftarrow 1$
6. **while**  $\|z - Pz\|_X > \delta$  for any  $z \in \mathcal{Z}$
7.      $z_{k+1} := \operatorname{argmax}_{z \in \mathcal{Z}} \|z - Pz\|_X$
8.      $\mathbb{J}_{2n}$ -orthogonalize  $z_{k+1}$  to obtain  $\hat{e}_{k+1}$
9.      $B \leftarrow [\hat{e}_1 | \dots | \hat{e}_{k+1} | \mathbb{J}_{2n}^T \hat{e}_1 | \dots | \mathbb{J}_{2n}^T \hat{e}_{k+1}]$
10.     $k \leftarrow k + 1$
11. **end while**
12. solve  $XA = B$  for  $A$

**Output:** The reduced basis  $A$

---

451 Freedom in the choice of the basis  $U$  allows us to require  $U = X^{1/2}(\tilde{A}^+)^T$ . This  
 452 reduces the complex expression in (42) to

453 (43) 
$$\dot{y} = J_{2k} A^T L A y + J_{2k} \nabla_y f(Ay),$$

454 and hence we recover the Hamiltonian structure. The reduced system then yields

455 (44) 
$$\begin{cases} \dot{y}(t) = J_{2k} A^T L A y + J_{2k} (\mathcal{P}^T X \mathbb{J}_{2n} X A \mathbb{J}_{2k})^{-1} \mathcal{P}^T \nabla_z f(z), \\ y(0) = \mathbb{J}_{2k}^T A^T X J X z_0. \end{cases}$$

456 We now discuss how to ensure that  $X^{1/2}(\tilde{A}^+)^T$  is a basis for the nonlinear snapshots.  
 457 Note that if  $z \in \operatorname{colspan}(X^{1/2}(\tilde{A}^+)^T)$  then  $X^{-1/2}z \in \operatorname{colspan}((\tilde{A}^+)^T)$ . Therefore,  
 458 it is sufficient to require  $(\tilde{A}^+)^T$  to be a basis for  $\{X^{-1/2}\nabla_z f(z(t_i))\}_{i=1}^N$ . **Theorem 4.1**  
 459 suggests that  $(\tilde{A}^+)^T$  is a  $J_{2n}^{-1}$ -symplectic basis and that the transformation between  
 460  $\tilde{A}$  and  $(\tilde{A}^+)^T$  does not affect the symplectic feature of the bases. Consequently, from  
 461  $A$  we may compute  $(\tilde{A}^+)^T$  and enrich it with snapshots  $\{X^{-1/2}\nabla_z f(z(t_i))\}_{i=1}^N$ . Once  
 462  $(\tilde{A}^+)^T$  represents the nonlinear term with the desired accuracy, we may compute  $\tilde{A} =$   
 463  $\left(\left((\tilde{A}^+)^T\right)^+\right)^T$  to obtain the reduced basis for (44). **Theorem 4.1** implies that  $(\tilde{A}^+)^T$   
 464 is ortho-normal with respect to the  $X^{-1}$ -norm. This affects the ortho-normalization  
 465 process. We note that greedy approaches to basis generation do not generally result  
 466 in a minimal basis.

467 As discussed in [subsection 4.3](#) it is desirable to eliminate the computation of  
 468  $X^{\pm 1/2}$ . Having  $z \in \operatorname{colspan}(X^{1/2}(\tilde{A}^+)^T)$  implies that  $X^{-1}z \in \operatorname{colspan}(\mathbb{J}_{2n}^T X A \mathbb{J}_{2n})$ .  
 469 Note that [Algorithm 2](#) constructs a  $\mathbb{J}_{2n}$ -symplectic matrix  $XA$  and  $\mathbb{J}_{2n}^T X A \mathbb{J}_{2n}$  is the  
 470 symplectic inverse of  $XA$  with respect to the standard symplectic matrix  $\mathbb{J}_{2n}$ . Given  
 471  $e$  as a candidate for enriching  $X^{1/2}(\tilde{A}^+)^T$  we may instead enrich  $\mathbb{J}_{2n}^T X A \mathbb{J}_{2n}$  with  $\hat{e}$ ,  
 472 that solves  $X\hat{e} = e$ .

473 Since  $\mathbb{J}_{2n}^T X A \mathbb{J}_{2n}$  is  $\mathbb{J}_{2n}$ -symplectic the projection operator onto the column span  
 474 of  $\mathbb{J}_{2n}^T X A \mathbb{J}_{2n}$  can be constructed as  $Q = \mathbb{J}_{2n}^T X A \mathbb{J}_{2n} A^T X$ . Given a nonlinear snap-

475 shot  $z$ , we may need to project the vector  $X^{-1}z$  onto  $\text{colspan}(\mathbb{J}_{2n}^T X A \mathbb{J}_{2n})$ . However,  
 476  $Q(X^{-1}z) = \mathbb{J}_{2n}^T X A \mathbb{J}_{2n} A^T z$  and thus, the matrix  $X^{-1}$  does not appear explicitly.  
 477 This process eliminates the computation of  $X^{\pm 1/2}$ . We summarize the process of  
 478 generating a basis for the nonlinear terms in [Algorithm 3](#).

---

**Algorithm 3** Generation of a basis for nonlinear terms

---

**Input:** Tolerated projection error  $\delta$ ,  $\mathbb{J}_{2n}$ -symplectic basis  $B = XA$  of size  $2k$ , the snapshots  $\mathcal{G} = \{\nabla_z f(z(t_i))\}_{i=1}^N$ , full rank matrix  $X = X^T > 0$

1.  $Q \leftarrow \mathbb{J}_{2n}^T X A \mathbb{J}_{2n} A^T$
2. compute  $(B^+)^T = \mathbb{J}_{2n}^T B \mathbb{J}_{2n} = [e_1 | \dots | e_k | \mathbb{J}_{2n}^T e_1 | \dots | \mathbb{J}_{2n}^T e_k]$
3. **while**  $\|g - Qg\|_2 > \delta$  for any  $g \in \mathcal{G}$
4.      $g_{k+1} := \underset{g \in \mathcal{G}}{\text{argmax}} \|g - Qg\|_2$
5.     solve  $Xe = g_{k+1}$  for  $e$
6.      $\mathbb{J}_{2n}$ -orthogonalize  $e$  to obtain  $e_{k+1}$
7.      $(B^+)^T \leftarrow [e_1 | \dots | e_{k+1} | \mathbb{J}_{2n}^T e_1 | \dots | \mathbb{J}_{2n}^T e_{k+1}]$
8.      $k \leftarrow k + 1$
9. **end while**
10. compute  $XA = \left( (B^+)^T \right)^+$

**Output:**  $\mathbb{J}_{2n}$ -symplectic basis  $XA$

---

479 **4.5. Offline/online decomposition.** Model order reduction becomes particu-  
 480 larly useful for parameter dependent problems in multi-query settings. For the pur-  
 481 pose the of most efficient computation, it is important to delineate high dimensional  
 482 ( $\mathcal{O}(n^\alpha)$ ) offline computations from low dimensional ( $\mathcal{O}(k^\alpha)$ ) online ones, for some  
 483  $\alpha \in \mathbb{N}$ . Time intensive high dimensional quantities are computed only once for a  
 484 given problem in the offline phase and the cheaper low dimensional computations  
 485 can be performed in the online phase. This segregation or compartmentalization of  
 486 quantities, according to their computational cost, is referred to as the offline/online  
 487 decomposition.

488 More precisely, one can decompose the computations into the following stages:  
 489 *Offline stage:* Quantities in this stage are computed only once and then used in the  
 490 online stage.

- 491 1. Generate the weighted snapshots  $\{Xz(t_i)\}_{i=1}^N$  and the snapshots of the non-  
 492 linear term  $\{\nabla_z f(z(t_i))\}_{i=1}^N$
- 493 2. Generate a  $J_{2n}$ -symplectic basis for the solution snapshots and the snapshots  
 494 of the nonlinear terms, following [Algorithms 2](#) and [3](#), respectively.
- 495 3. Assemble the reduced order model [\(44\)](#).

496 *Online stage:* The reduced model [\(44\)](#) is solved for multiple parameter sets and the  
 497 output is extracted.

498 **5. Numerical results.** Let us now discuss the performance of the symplectic  
 499 model reduction with a weighted inner product. In [subsections 5.1](#) and [5.2](#) we apply  
 500 the model reduction to equations of a vibrating elastic beam without and with cavity,  
 501 respectively. And we examine the evaluation of the nonlinear terms in the model  
 502 reduction of the sine-Gordon equation, in section [subsection 5.3](#).



FIG. 1. (a) initial condition and a snapshot of the 3D beam. (b) initial condition and a snapshot of the 2D beam with cavity.

503 **5.1. The elastic beam equation.** Consider the equations governing small de-  
 504 formations of a clamped elastic body  $\Gamma \subset \mathbb{R}^3$  as

$$505 \quad (45) \quad \begin{cases} u_{tt}(t, x) = \nabla \cdot \sigma + f, & x \in \Gamma, \\ u(0, x) = \vec{0}, & x \in \Gamma, \\ \sigma \cdot n = \tau, & x \in \partial\Gamma_\tau, \\ u(t, x) = \vec{0}, & x \in \partial\Gamma \setminus \partial\Gamma_\tau, \end{cases}$$

506 and

$$507 \quad (46) \quad \sigma = \lambda(\nabla \cdot u)I + \mu(\nabla u + (\nabla u)^T).$$

508 Here  $u : \Gamma \rightarrow \mathbb{R}^3$  is the unknown displacement vector field, subscript  $t$  denotes deriva-  
 509 tive with respect to time,  $\sigma : \Gamma \rightarrow \mathbb{R}^{3 \times 3}$  is the stress tensor,  $f$  is the body force per  
 510 unit volume,  $\lambda$  and  $\mu$  are Lamé's elasticity parameters for the material in  $\Gamma$ ,  $I$  is the  
 511 identity tensor,  $n$  is the outward unit normal vector at the boundary and  $\tau : \partial\Gamma_\tau \rightarrow \mathbb{R}^3$   
 512 is the traction at a subset of the boundary  $\partial\Gamma_\tau$  [32]. We refer to Figure 1(a) for a  
 513 snapshot of the elastic beam.

514 We define a vector valued function space as  $V = \{u \in (L^2(\Gamma))^3 : \|\nabla u_i\|_2 \in L^2, i =$   
 515  $1, 2, 3, u = \vec{0} \text{ on } \partial\Gamma_\tau\}$ , equipped with the standard  $L^2$  inner product  $(\cdot, \cdot) : V \times V \rightarrow \mathbb{R}$ ,  
 516 and seek the solution to (45). To derive the weak formulation of (45), we multiply it  
 517 with the vector valued test function  $v \in V$ , integrate over  $\Gamma$ , and use integration by  
 518 parts to get

$$519 \quad (47) \quad \int_\Gamma u_{tt} \cdot v \, dx = - \int_\Gamma \sigma : \nabla v \, dx + \int_{\partial\Gamma_\tau} (\sigma \cdot n) \cdot v \, ds + \int_\Gamma f \cdot v \, dx,$$

520 where  $\sigma : \nabla v = \sum_{i,j} \sigma_{ij}(\nabla v)_{ji}$  is the tensor inner product. Note that the skew-  
 521 symmetric part of  $\nabla v$  vanishes over the product  $\sigma : \nabla v$ , since  $\sigma$  is symmetric. By  
 522 prescribing the boundary conditions to (47) we recover

$$523 \quad (48) \quad \int_\Gamma u_{tt} \cdot v \, dx = - \int_\Gamma \sigma : \text{Sym}(\nabla v) \, dx + \int_{\partial\Gamma_\tau} \tau \cdot v \, ds + \int_\Gamma f \cdot v \, dx,$$

524 with  $\text{Sym}(\nabla v) = (\nabla v + (\nabla v)^T)/2$ . The variational form associated to (45) is

$$525 \quad (49) \quad (u_{tt}, v) = -a(u, v) + b(v), \quad u, v \in V,$$



526 where

$$527 \quad (50) \quad a(u, v) = \int_{\Gamma} \sigma : \text{Sym}(\nabla v) \, dx, \quad b(v) = \int_{\partial\Gamma_{\tau}} \tau \cdot v \, ds + \int_{\Gamma} f \cdot v \, dx.$$

528 To obtain the FEM discretization of (49), we triangulate the domain  $\Gamma$  and define  
 529 vector valued piece-wise linear basis functions  $\{\phi_i\}_{i=1}^{N_h}$ , referred to as the *hat functions*.  
 530 We define the FEM space  $V_h$ , an approximation of  $V$ , as the span of those basis  
 531 functions. Projecting (49) onto  $V_h$  yields the discretized weak form

$$532 \quad (51) \quad ((u_h)_{tt}, v_h) = -a(u_h, v_h) + b(v_h), \quad u_h, v_h \in V_h.$$

533 Any particular function  $u_h$  can be expressed as  $u_h = \sum_{i=1}^{N_h} q_i \phi_i$ , where  $q_i$ ,  $i =$   
 534  $1, \dots, N_h$ , are the expansion coefficients. Therefore, by choosing test functions  $v_h =$   
 535  $\phi_i$ ,  $i = 1, \dots, N_h$ , we obtain the ODE system

$$536 \quad (52) \quad M\ddot{q} = -Kq + g_q.$$

537 where  $q = (q_1, \dots, q_{N_h})^T$  are unknowns, the *mass matrix*  $M \in \mathbb{R}^{N_h \times N_h}$  is given as  
 538  $M_{i,j} = (\phi_i, \phi_j)$ , the *stiffness matrix*  $K \in \mathbb{R}^{N_h \times N_h}$  is given as  $K_{i,j} = a(\phi_j, \phi_i)$  and  
 539  $g_q = (b(v_1), \dots, b(v_{N_h}))^T$ . Now introduce the canonical coordinate  $p = M\dot{q}$  to recover  
 540 the Hamiltonian system

$$541 \quad (53) \quad \dot{z} = \mathbb{J}_{2N_h} Lz + g_{qp},$$

542 where

$$543 \quad (54) \quad z = \begin{pmatrix} q \\ p \end{pmatrix}, \quad L = \begin{pmatrix} K & 0 \\ 0 & M^{-1} \end{pmatrix}, \quad g_{qp} = \begin{pmatrix} 0 \\ g_q \end{pmatrix},$$

544 together with the Hamiltonian function  $H(z) = \frac{1}{2}z^T Lz + z^T \mathbb{J}_{2N_h}^T g_{qp}$ . An appropriate  
 545 FEM setup leads to a symmetric and positive-definite matrix  $L$ . Hence, it seems  
 546 natural to take  $X = L$ , the energy matrix associated to (53). The system parameters  
 547 are summarized in the table below. For further information regarding the problem,  
 548 we refer to [32].

Domain shape	box: $l_x = 1, l_y = 0.2, l_z = 0.2$
Time step-size	$\Delta t = 0.01$
Gravitational force	$f = (0, 0, -0.4)^T$
Traction	$\tau = \vec{0}$
Lamé parameters	$\lambda = 1.25, \mu = 1.0$
Degrees of freedom	$2N_h = 1650$

549  
 550 Projection operators  $P_{X,V}$ ,  $P_{I,A}^{\text{symp}}$  and  $P_{X,\bar{A}}^{\text{symp}}$  are constructed following [subsections 3.1](#)  
 551 [to 3.3](#), respectively, with  $\sigma = 5 \times 10^{-4}, 2 \times 10^{-4}$  and  $1 \times 10^{-4}$ . In order to apply a  
 552 symplectic time integrator, we first compute the transformation  $J_{2k} = \mathcal{T} \mathbb{J}_{2k} \mathcal{T}^T$  using  
 553 the symplectic GS method with complete pivoting. The reduced systems, obtained  
 554 from  $P_{I,A}^{\text{symp}}$  and  $P_{X,\bar{A}}^{\text{symp}}$ , are then integrated in time using the Störmer-Verlet scheme  
 555 to generate the temporal snapshots. The reduced system obtained from  $P_{X,V}$  is  
 556 integrated using a second order implicit Runge-Kutta method. Note that the Störmer-  
 557 Verlet scheme is not used since the canonical form of a Hamiltonian system is destroyed  
 558 when  $P_{X,V}$  is applied.

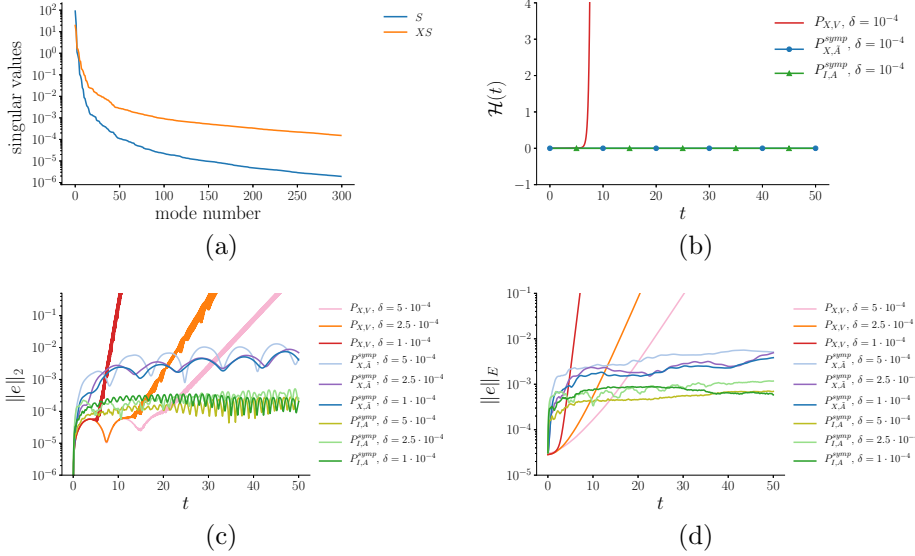


FIG. 2. Numerical results related to the beam equation. (a) the decay of the singular values. (b) conservation of the Hamiltonian. (c) error with respect to the 2-norm. (d) error with respect to the  $X$ -norm.

559 Figure 2(a) shows the decay of the singular values of the temporal snapshots  $S$   
 560 and  $XS$ , respectively. The difference in the decay indicates that the reduced sys-  
 561 tems constructed using  $P_{I,\bar{A}}^{\text{symmp}}$  and  $P_{X,\bar{A}}^{\text{symmp}}$  would have different sizes to achieve similar  
 562 accuracy.

563 Figure 2(b) shows the conservation of the Hamiltonian for the methods discussed  
 564 above. This confirms that the symplectic methods preserve the Hamiltonian and the  
 565 system energy. However, the Hamiltonian blows up for the reduced system constructed  
 566 by the projection  $P_{X,V}$ .

567 Figure 2(c) shows the  $L^2$  error between the projected systems and the full order  
 568 system, defined as

$$569 \quad (55) \quad \|e\|_{L^2} = \sqrt{(e, e)} \approx \sqrt{(q - \hat{q})^T M (q - \hat{q})},$$

570 where  $e \in V$  is the error function and  $\hat{q} \in \mathbb{R}^{2n}$  is an approximation for  $q$ . We notice  
 571 that the reduced system obtained by the non-symplectic method is unstable and the  
 572 reduced system, constructed using  $P_{X,V}$ , is more unstable as  $k$  increases. On the other  
 573 hand, the symplectic methods yield a stable reduced system. Although the system,  
 574 constructed by the projection  $P_{X,\bar{A}}^{\text{symmp}}$ , is not based on the 2-norm projection, the error  
 575 remains bounded with respect to the 2-norm.

576 We define the energy norm  $\|\cdot\|_E : V \rightarrow \mathbb{R}$  as

$$577 \quad (56) \quad \|(u, \dot{u})\|_E = \sqrt{a(u, u) + (\dot{u}, \dot{u})} \approx \|z\|_X.$$

578 Figure 2(d) shows the MOR error with respect to the energy norm. We observe that  
 579 the classical model reduction method based on the projection  $P_{X,V}$  does not yield a  
 580 stable reduced system. However, the symplectic methods provide a stable reduced  
 581 system. We observe that the original symplectic approach also provides an accurate

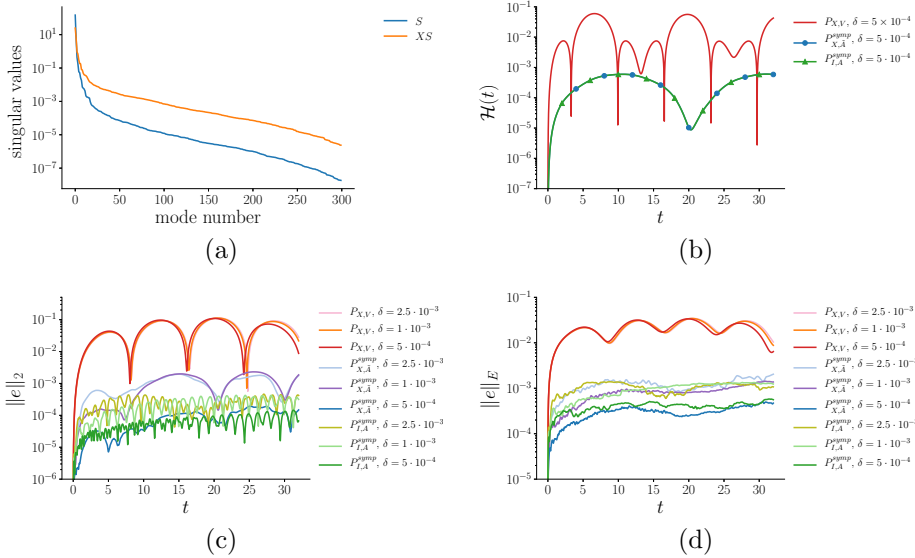


FIG. 3. Numerical results related to the beam with cavity. (a) the decay of the singular values. (b) conservation of the Hamiltonian. (c) error with respect to the 2-norm. (d) error with respect to the energy norm.

582 solution with respect to the energy norm. Nevertheless, the relation between the two  
 583 norms depends on the problem set up and the choice of discretization [17].

584 **5.2. Elastic beam with cavity.** In this section we investigate the performance  
 585 of the proposed method on a two dimensional elastic beam that contains a cavity. In  
 586 this case a nonuniform triangulated mesh is desirable to balance the computational  
 587 cost of a FEM discretization with the numerical error around the cavity. Figure 1(a)  
 588 shows the nonuniform mesh used in this section. System parameters are taken to  
 589 be identical to those in subsection 5.1. Numerical parameters are summarized in the  
 590 table below.

cavity width	$l_c = 0.1$
Time step-size	$\Delta t = 4 \times 10^{-4}$
Degrees of freedom	$2N_h = 744$

592 Figure 3(a) shows the decay of the singular values for the snapshot matrix  $S$  and  
 593  $XS$ . The divergence of the two curves indicates that to obtain the same accuracy  
 594 in the reduced system, the basis constructed from  $S$  and  $XS$  would have different  
 595 sizes. Projection operators  $P_{X,A}$ ,  $P_{I,A}^{\text{sym}}$  and  $P_{X,A}^{\text{sym}}$  are constructed according to the  
 596 subsections 3.1 to 3.3. The truncation error is set to  $\delta = 2.5 \times 10^{-3}$ ,  $\delta = 1 \times 10^{-3}$   
 597 and  $\delta = 5 \times 10^{-4}$  in Algorithms 1 and 2

598 The 2-norm error and the error in the energy norm are presented in Figure 3(c)  
 599 and Figure 3(d), respectively. We notice that although the non-symplectic method  
 600 is bounded, it contains larger error compared to the symplectic methods. Moreover,  
 601 we notice that the error generated by the symplectic methods is consistently reduced  
 602 under basis enrichment. It is observed that in the energy norm, the projection  $P_{X,A}^{\text{sym}}$

603 provides a more accurate solution (compare to [Figure 2](#)). This is because on a nonuni-  
 604 form mesh, the weight matrix  $X$  associates higher weights to the elements that are  
 605 subject to larger error. Therefore, we expect the reduced system constructed with the  
 606 projection  $P_{X,\bar{A}}^{\text{symp}}$  to outperform the one constructed with  $P_{I,A}^{\text{symp}}$  on a highly nonuni-  
 607 form mesh.

608 [Figure 3\(b\)](#) shows the error in the Hamiltonian. Comparing to [Figure 2](#), we notice  
 609 that the energy norm helps with the boundedness of the non-symplectic method.  
 610 However, the symplectic methods preserves the Hamiltonian at a higher accuracy

611 **5.3. The sine-Gordon equation.** The sine-Gordon equation arises in differ-  
 612 ential geometry and quantum physics [36], as a nonlinear generalization of the linear  
 613 wave equation of the form

$$614 \quad (57) \quad \begin{cases} u_t(t, x) = v, & x \in \Gamma, \\ v_t(t, x) = u_{xx} - \sin(u), \\ u(t, 0) = 0, \\ u(t, l) = 2\pi. \end{cases}$$

615 Here  $\Gamma = [0, l]$  is a line segment and  $u, v : \Gamma \rightarrow \mathbb{R}$  are scalar functions. The Hamilto-  
 616 nian associated with (57) is

$$617 \quad (58) \quad H(q, p) = \int_{\Gamma} \frac{1}{2}v^2 + \frac{1}{2}u_x^2 + 1 - \cos(u) \, dx.$$

618 One can verify that  $u_t = \delta_v H$  and  $v_t = -\delta_u H$ , where  $\delta_v, \delta_u$  are standard variational  
 619 derivatives. The sine-Gordon equation admits the soliton solution

$$620 \quad (59) \quad u(t, x) = 4\arctan \left( \exp \left( \pm \frac{x - x_0 - ct}{\sqrt{1 - c^2}} \right) \right),$$

621 where  $x_0 \in \Gamma$  and the plus and minus signs correspond to the *kink* and the *anti-kink*  
 622 solutions, respectively. Here  $c, |c| < 1$ , is the arbitrary wave speed. We discretize the  
 623 segment into  $n$  equi-distant grid point  $x_i = i\Delta x, i = 1, \dots, n$ . Furthermore, we use  
 624 standard finite-differences schemes to discretize (57) and obtain

$$625 \quad (60) \quad \dot{z} = \mathbb{J}_{2n} L z + \mathbb{J}_{2n} g(z) + \mathbb{J}_{2n} c_b.$$

626 Here  $z = (q^T, p^T)^T, q(t) = (u(t, x_1), \dots, u(t, x_N))^T, p(t) = (v(t, x_1), \dots, v(t, x_N))^T,$   
 627  $c_b$  is the term corresponding to the boundary conditions and

$$628 \quad (61) \quad L = \begin{pmatrix} D_x^T D_x & 0_N \\ 0_N & I_n \end{pmatrix}, \quad g(z) = \begin{pmatrix} \sin(q) \\ \vec{0} \end{pmatrix},$$

629 where  $D_x$  is the standard matrix differentiation operator. We may take  $X = L$  as the  
 630 weight matrix associated to (60). The discrete Hamiltonian, takes the form

$$631 \quad (62) \quad H_{\Delta x} = \Delta x \cdot \frac{1}{2} \|p\|_2^2 + \Delta x \cdot \|D_x q\|_2^2 + \sum_{i=1}^n \Delta x \cdot (1 - \cos(q_i)).$$

632 The system parameters are given as

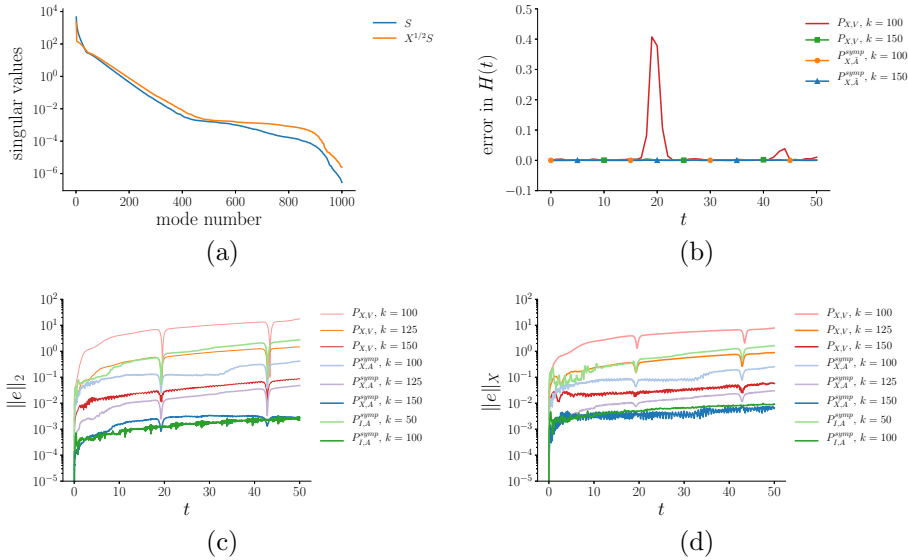


FIG. 4. Numerical results related to the sine-Gordon equation. (a) the decay of the singular values. (b) error in the Hamiltonian. (c) error with respect to the 2-norm. (d) error with respect to the energy norm.

Domain length	$l = 50$
No. grid points	$n = 500$
Time step-size	$\Delta t = 0.01$
Wave speed	$c = 0.2$

633

634 The midpoint scheme (7) is used to integrate (57) in time and generate the snapshot  
 635 matrix  $S$ . Similar to the previous subsection, projection operators  $P_{X,V}$ ,  $P_{I,A}^{\text{symplectic}}$  and  
 636  $P_{X,A}^{\text{symplectic}}$  are used to construct a reduced system. To accelerate the evaluation of the  
 637 nonlinear term, the symplectic methods discussed in subsections 3.1 and 3.2 are cou-  
 638 pled with the projection operators  $P_{I,A}^{\text{symplectic}}$  and  $P_{X,A}^{\text{symplectic}}$ , respectively. Furthermore, the  
 639 DEIM approximation is used for the efficient evaluation of the reduced system, ob-  
 640 tained by the projection  $P_{X,V}$ . The midpoint rule is also used to integrate the reduced  
 641 systems in time. Figure 4 shows the numerical results corresponding to the reduced  
 642 models without approximating the nonlinearity, while the results corresponding to  
 643 the accelerated evaluation of the nonlinear term are presented in Figure 5.

644 Figure 4(a) shows the decay of the singular values of matrices  $S$  and  $XS$ . As in  
 645 the previous section, we observe a saturation in the decay of the singular values of  $XS$   
 646 compared to the singular values of  $S$ . This indicates that the reduced basis, based  
 647 on a weighted inner product, should be chosen to be larger to provide an accuracy  
 648 similar to based on the Euclidean inner product. Put differently, unweighted reduced  
 649 bases, when compared to the weighted ones, may be highly inaccurate in reproducing  
 650 underlying physical properties of the system.

651 Figure 4(b) displays the error in the Hamiltonian. It is observed again that the  
 652 symplectic approaches conserve the Hamiltonian. However, the classical approaches  
 653 do not necessarily conserve the Hamiltonian. We point out that using the projection  
 654 operator  $P_{X,V}$  ensures the boundedness of the Hamiltonian. The contrary is observed

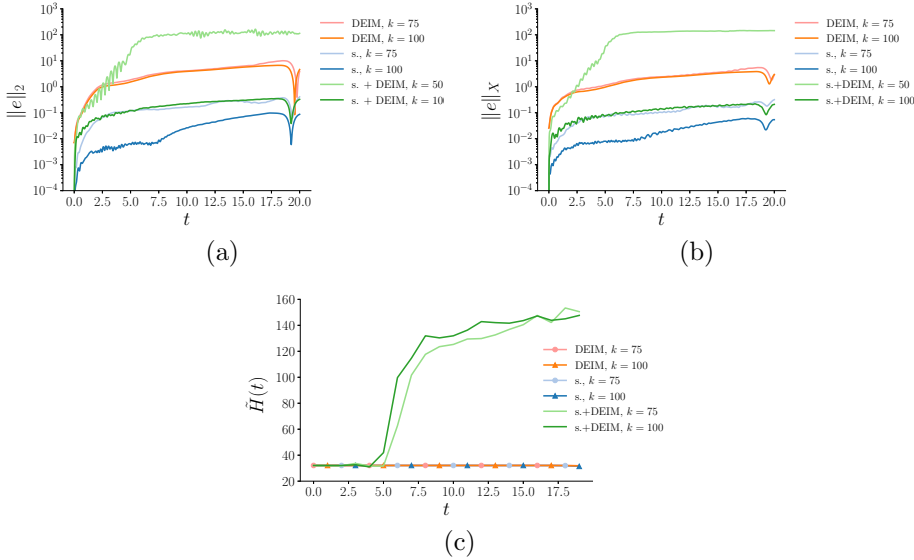


FIG. 5. Numerical results related to the sine-Gordon equation with efficient evaluation of the nonlinear terms. Here, “DEIM” indicates classical model reduction with the DEIM, “s.+DEIM” indicates symplectic model reduction with the DEIM and “s.” indicates symplectic model reduction with symplectic treatment of the nonlinear term. (a) error with respect to the Euclidean norm. (b) error with respect to the  $X$ -norm. (c) error in the Hamiltonian.

655 when we apply the POD with respect to the Euclidean inner-product, i.e. applying  
 656 the projection operator  $P_{I,V}$ . This can be seen in the results presented in [37], where  
 657 the unboundedness of the Hamiltonian is observed when  $P_{I,V}$  is applied to the sine-  
 658 Gordon equation. Nevertheless, only the symplectic model reduction consistently  
 659 preserves the Hamiltonian.

660 Figure 4(c) shows the error with respect to the Euclidean inner-product between  
 661 the solution of the projected systems and the original system. The behavior of the  
 662 solution is investigated for  $k = 100$ ,  $k = 125$  and  $k = 150$ . We observe that all  
 663 systems which are projected with respect to the  $X$ -norm are bounded. As the results  
 664 in [37] suggest, the Euclidean inner-product does not necessarily yield a bounded  
 665 reduced system. Moreover, we notice that the symplectic projection  $P_{X,A}^{\text{symP}}$  results in  
 666 a substantially more accurate reduced system compared to the reduced system yielded  
 667 from  $P_{X,V}$ . This is because the overall behavior of the original system is translated  
 668 correctly to the reduced system constructed with the symplectic projection.

669 The error with respect to the  $X$ -norm between the solution of the original system  
 670 and the projected systems is presented in Figure 4(d). We see that the behavior of  
 671 the  $X$ -norm error is similar to the Euclidean norm, however the growth of the error  
 672 is slower for methods based on a weighted inner product. Note that the connection  
 673 between the error in the Euclidean norm and the  $X$ -norm is problem and discretiza-  
 674 tion dependent. We also observed that symplectic methods are substantially more  
 675 accurate.

676 Figure 5 shows the performance of the different model reduction methods, when  
 677 an efficient method is adopted in evaluating the nonlinear term in (60). This figure  
 678 compares the symplectic approaches against non-symplectic methods. For all simu-  
 679 lations, the size of the reduced basis for (60) is chosen to be  $k = 100$ . The size of

680 the basis of the nonlinear term is then taken as  $k_n = 75$  and  $k_n = 100$ . For symplec-  
 681 tic methods, a basis for the nonlinear term is constructed according to [Algorithm 3](#),  
 682 whereas for non-symplectic methods, the DEIM is applied. Note that for symplectic  
 683 methods, the basis for the nonlinear term is added to the symplectic basis  $A$ . This  
 684 means that the size of the reduced system is larger compared to the classical approach.

685 [Figure 5\(a\)](#) and [Figure 5\(b\)](#) show the error with respect to the Euclidean norm and  
 686 the  $X$ -norm between the solution of the projected systems compared to the solution  
 687 of the original system, respectively. We observe that all solutions are bounded and the  
 688 behavior of the error in the Euclidean norm and the  $X$ -norm is similar. We observe  
 689 that enriching the DEIM basis does not increase the overall accuracy of the system  
 690 projected using  $P_{X,V}$ . Furthermore, applying the DEIM to a symplectic reduced  
 691 system also destroys the symplectic nature of the reduced system, as suggested in  
 692 [subsection 4.4](#). Therefore, it is essential to adopt a symplectic approach to reduce the  
 693 complexity of the evaluation of the nonlinear terms. We observe that the symplectic  
 694 method presented in [subsection 4.4](#) provides not only an accurate approximation of  
 695 the nonlinear term, but also preserves the symplectic structure of the reduced system.  
 696 Moreover, enriching such a basis consistently increases the accuracy of the solution,  
 697 as suggested in [Figure 5\(a\)](#) and [Figure 5\(b\)](#).

698 [Figure 5\(b\)](#) shows the conservation of the Hamiltonian for different methods. It  
 699 is again visible that applying the DEIM to a symplectic reduced system destroys the  
 700 Hamiltonian structure, therefore the Hamiltonian is not preserved.

701 **6. Conclusion.** We present a model reduction routine that combines the classic  
 702 model reduction method, defined with respect to a weighted inner product, with  
 703 symplectic model reduction. This allows the reduced system to be defined with respect  
 704 to the norms and inner-products that are natural to the problem and most suitable  
 705 for the method of discretization. Furthermore, the symplectic nature of the reduced  
 706 system preserves the Hamiltonian structure of the original system, which results in  
 707 robustness and enhanced stability in the reduced system.

708 We demonstrate that including the weighted inner-product in the symplectic  
 709 model reduction can be viewed as a natural extension of the unweighted symplec-  
 710 tic method. Therefore, the stability preserving properties of the symplectic method  
 711 generalize naturally to the new method.

712 Numerical results suggest that classic model reduction methods with respect to a  
 713 weighted inner product can help with the boundedness of the system. However, only  
 714 the symplectic treatment can consistently increase the accuracy of the reduced system.  
 715 This is consistent with the fact the symplectic methods preserve the Hamiltonian  
 716 structure.

717 We also show that to accelerate the evaluation of the nonlinear terms, adopting  
 718 a symplectic approach is essential. This allows an accurate reduced model that is  
 719 consistently enhanced when the basis for the nonlinear term is enriched.

720 Hence, the symplectic model-reduction with respect to a weighted inner product  
 721 can provide an accurate and robust reduced system that allows the use of the norms  
 722 and inner products most appropriate to the problem.

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