Irreducible representations of simple algebraic groups in which a unipotent element is represented by a matrix with single non-trivial Jordan block

Donna Testerman and A.E. Zalesski*

1 Introduction

The representation theory of algebraic groups is based on the study of weight spaces, which are nothing other than the homogeneous components with respect to a maximal torus. As every semisimple element belongs to a maximal torus, the knowledge of weights and their multiplicities, in a given representation, yields rich information on eigenspaces of the element under consideration.

It is much harder to obtain information on properties of unipotent elements, for example, their fixed point space, their minimal polynomial, or in the best case, their Jordan block structure in a given representation. The situation is better for certain classes of elements, such as root elements, but in general, problems of this kind can be difficult.

One such question was raised several years ago by the second author, specifically:

Determine the irreducible representations ϕ of a simple algebraic group G such that, for some unipotent element u, the Jordan normal form of $\phi(u)$ has exactly one block of size greater than 1?

The main motivation for considering this question is to supply an additional tool for the recognition of linear groups via properties of a single element. However, one can also view this question as a test of how well the general theory is adapted for solving computational problems on unipotent elements.

The first contribution was made by I. Suprunenko in [17, Theorem 1.9], who solved the problem in the case where $\phi(u)$ has exactly one Jordan block. Later she obtained a solution to the above problem for classical groups [19, Theorem 3] (see [20] for the proof). The current manuscript grew out of our work on overgroups of regular elements in simple algebraic groups (see [23, 24]). At that time, Suprunenko had announced a result which can be used for solving

*Acknowledgement. A part of this work was carried out with the generous support of the Bernoulli Center, at the Swiss Federal Institute of Technology Lausanne, when the second author participated in the workshop "Local representation theory and simple groups" (2016).

the above question for elements of order p in the exceptional groups other than G_2; see Remark after Theorem 4.2 for details. We have recently learned that David Craven is working on similar questions for finite simple groups and their automorphism groups.

Our main result answers the above question by considering all unipotent elements in all simple algebraic groups of exceptional type.

Theorem 1.1. *Let G be a simply connected simple linear algebraic group of exceptional Lie type over an algebraically closed field F of characteristic $p \geq 0$, and let $u \in G$ be a nonidentity unipotent element. Let ϕ be a non-trivial irreducible representation of G. Then the Jordan normal form of $\phi(u)$ contains at most one non-trivial block if and only if G is of type G_2, u is a regular unipotent element and* $\dim \phi \leq 7$.

Theorem 1.1 remains true when replacing G by a finite quasi-simple group of Lie type, as every irreducible F-representation of such a group lifts to a representation of an appropriate simple algebraic group.

Our method is different from those used in [17, 20] and in a sense is indirect. We first consider the case where G is of type A_1, and for a representation ρ, not necessarily irreducible, we prove that the condition that $\rho(G)$ contains a unipotent element with only one non-trivial Jordan block implies that all non-zero weights of the representation are of multiplicity 1. Then we consider a special case where $p = 0$ or $|u| = p$, and use a result of [22, 13] saying that, with the exception of one class of elements in G_2, when $p = 3$, u is contained in a simple algebraic subgroup of G of type A_1. This implies that all non-zero weights of ϕ are of multiplicity 1. The irreducible representations with this property are determined in [24]; the list is very short for G of exceptional type. The Jordan normal form of all classes of unipotent elements in these representations was computed by Lawther [8]. This yields the result for $p = 0$ or $|u| = p$. If $|u| > p > 0$ then there is a suitable parabolic subgroup P such that $u \notin R_u(P)$, the unipotent radical of P. So the projection u' into a Levi subgroup L of P is non-trivial. Then one can observe that $\mathrm{Jor}(\tau(u'))$ has single non-trivial block for every composition factor τ of the restriction of ϕ to L. This allows us to use induction on the rank of G.

Notation Throughout the paper p denotes a prime or 0, and F an algebraically closed field of characteristic p. Unless otherwise stated, G is a simple simply connected algebraic group over F. All representations of G and FG-modules are rational. To say that a representation ρ of G or an FG-module M is irreducible, we write $\rho \in \mathrm{Irr}\, G$ or $M \in \mathrm{Irr}\, G$. We let $\{\alpha_1, \dots, \alpha_n\}$ be a base of the root system of G. Our labelling of Dynkin diagrams is as in [3].

For an integer $n > 0$ we denote by J_n the Jordan block of size n, that is, the $(n \times n)$-matrix with 1 at the positions (i, i) and $(i, i + 1)$ for $i = 1, \dots, n$, and 0 elsewhere. The Jordan block J_1 is called *trivial*. For a matrix x we denote by Jor(x) a Jordan normal form of x. If x is a linear transformation of a vector space V we write $\mathrm{Jor}_V(x)$ for a Jordan normal form of x, especially when we need to specify V. A diagonal matrix with diagonal entries $x_1, \dots, x_n$ is denoted by $\mathrm{diag}(x_1, \dots, x_n)$. A similar notation is used for a block-diagonal matrix.

2 Preliminaries

In Lemma 2.1 below ρ_S^{reg} denotes the FS-module afforded by the regular representation of a finite group S.

Lemma 2.1. *Let F be an algebraically closed field of characteristic $p > 0$, let G be a finite group with Sylow p-subgroup S of order p, and let M be an indecomposable FG-module. Suppose that $N_G(S)/S$ is abelian. Then there is an indecomposable FS-module L such that $M|_S = \frac{\dim M - \dim L}{|S|} \cdot \rho_S^{reg} \oplus L$ and $\dim L < p$.*

Proof. If the restriction $M|_S$ is a projective FS-module then the statement is obvious with $L = 0$. Suppose otherwise. Set $N = N_G(S)$. By [1, §19, Theorem 1], $M|_N = L \oplus P$ where P is projective, and L is indecomposable. As S is cyclic, every projective FS-module is free, so $P|_S = \frac{\dim M - \dim L}{|S|} \cdot \rho_S^{reg}$. Recall that L is uniserial ([5, Theorem VII.2.4]), that is, the submodule lattice of L is a chain. Let $S = \langle y \rangle$, and set $x = 1 - y$ in the group algebra FN, $L_0 = L$ and $L_i = x^i L$ for $i = 1, \dots, d$ assuming $L_d = 0$, $L_{d-1} \neq 0$. So $d \leq p$. Observe that L_1 is an FN-module. (Indeed, for $n \in N$ we have $nL_1 = (1 - nyn^{-1})L = (1 - y^j)L$ for some integer $j > 0$ and $1 - y^j = (1 - y) + (1 - y)y + \cdots + (1 - y)y^{j-1}$.) Therefore, L_i is an FN-module for every i. As S acts trivially on every L_i/L_{i+1}, it is completely reducible as FN-module. Since L is uniserial, every L_i/L_{i+1} is irreducible. Since N/S is abelian, $\dim(L_i/L_{i+1}) = 1$. So $d = \dim L$. Here $d < |S|$ as otherwise $L|_S$ is free and hence so is $M|_S$. This completes the proof. □

Lemma 2.2. *Let $J_m \in GL_m(F)$, $J_n \in GL_n(F)$, $1 < n < m$ be Jordan blocks of size m, n respectively. Then the Jordan form of $J_m \otimes J_n$ contains at least two blocks of size greater than 1 unless $m = n = 2$ and $p \neq 2$.*

Proof. Let X be a cyclic p-group if F is a field of characteristic $p > 0$, otherwise an infinite cyclic group. Let V_m, V_n be indecomposable FX-modules of dimensions m, n, respectively. Let $V_i \subset V_m$, $V_j \subset V_m$ be submodules of dimensions i, j, respectively. Then $V_i \otimes V_j$ is a submodule of $V_m \otimes V_n$. The number of indecomposable summands of an FX-module M of dimension $\geq k$ is not less than that on any submodule of M. It follows that the result follows by induction as soon as one verifies this for $(m, n) = (2, 2), (3, 2)$.

If $m = n = 2$ then $V_2 \otimes V_2 = W_1 \oplus W_2$, where the pair $(\dim W_1, \dim W_2)$ is $(2, 2)$ if $p = 2$ and $(3, 1)$ if $p \neq 2$. (This is well known. If $p \neq 2$, see [5, Ch, VII, Theorem 2.7]. If $p = 2$ then V_2 is free, and hence so is $V_2 \otimes V_2$.) By induction, the lemma is true for $p = 2$.

Let $m = 3, n = 2$. If $p = 3$ then V_3 is free, and hence so is $V_3 \otimes V_2$. If $p \neq 2, 3$ then the lemma again follows by [5, Ch, VII, Theorem 2.7]. $\square$

Corollary 2.3. *Let G be an algebraic group and $u \in G$ unipotent. Let M be an irreducible FG-module such that $\mathrm{Jor}_M(u)$ has a single non-trivial block. Then either M is tensor-indecomposable or $G = A_1$ and $\dim M = 4$.*

We will require the following generation result, due to Guralnick and Saxl.

Lemma 2.4. [6, Theorems 5.1 and 5.4] *Let G be an exceptional finite group of Lie type, of untwisted rank l, and $x \in (G \setminus Z(G))$. Then G can be generated by $l + 3$ conjugates of x, except, possibly, for the case $G = F_4$, q even, $x^2 = 1$, where G can be generated by 8 conjugates of x.*

Lemma 2.5. *Let G be an irreducible subgroup of $GL_n(F)$ and $g \in G$. For an eigenvalue λ of g set $d = \dim(\mathrm{Id} - \lambda^{-1} g)V$. Suppose that G is generated by m conjugates of g. Then $n \leq dm$.*

In addition, if G is an exceptional group of Lie type, of untwisted rank l, then $n \leq d(l + 3)$, except, possibly, for G of type F_4, q even, $x^2 = 1$, where $n \leq 8d$.

Proof. Let $G = \langle g_1, \dots, g_m \rangle$, where g_i $(1 \leq i \leq m)$ is conjugate to g in G. Set $V' = \sum_{i=1}^{m} (\mathrm{Id} - \lambda^{-1} g_i)V$. Then $\dim V' \leq md$ and $GV' = V'$, whence $V = V'$, and the first statement follows.

If G is a finite exceptional group of Lie type then the additional statement follows from Lemma 2.4.

$\square$

3 Some representations of groups $SL_2(p)$ and $SL_2(F)$

Lemma 3.1. *Let $D = SL(2, p) \subset G = SL(2, F)$, $u \in D$ a unipotent element and let K be a tensor-decomposable irreducible FG-module. Suppose that $\mathrm{Jor}_K(u)$ contains a single non-trivial block. Then $p > 2$ and $\dim K = 4$. In addition, $K|_D$ has a composition factor of dimension 3, and u has a block of size 3.*

Proof. Let $K = K_1 \otimes K_2$, where K_1 is a tensor-indecomposable FG-module and $d := \dim K_1 > 1$. By Lemma 2.2, $\mathrm{Jor}_{K_1}(u)$ and $\mathrm{Jor}_{K_2}(u)$ consist of blocks of size at most 2. As K_1 is irreducible and tensor-indecomposable, $\mathrm{Jor}_{K_1}(u)$ consists of a single Jordan block. Therefore, $\dim K_1 = 2$ and $p \neq 2$. Obviously, $\mathrm{Jor}_{K_2}(u)$ cannot have more than one block. It follows that K_2 is tensor-indecomposable, and again by Lemma 2.2, $\dim K_2 = 2$. As $K_1|_D \cong K_2|_D$, $K|_D$ contains as a direct summand the adjoint FG-module, which is of dimension 3 for $p > 2$. $\square$

The following result is well known (see e.g. Humphreys [7, 12.4]):

Lemma 3.2. *Let E be an indecomposable rational module of composition length 2 for a simple algebraic group. Let μ, μ' be the highest weights of E/L, L, resp., where L is the maximal submodule of E. Then either $\mu < \mu'$ or $\mu > \mu'$, and in the latter case E is of shape W_μ/M, where W_μ is the Weyl module of highest weight μ and M is a submodule of W_μ.*

Corollary 3.3. *Let $G = A_1$ and let V be an FG-module. Suppose that W is either a Weyl module or indecomposable of composition length at most 2. Then all weights of W are of multiplicity 1.*

Proof. If $p = 0$ then all weights of an irreducible FG-module are well known to be of multiplicity 1, and hence so are the weights of any Weyl module of G for any $p > 0$. If W is an indecomposable of composition length 2 then, by Lemma 3.2, either W or the dual of W is a quotient of a Weyl module, whence the claim. □

Lemma 3.4. [2, Corollary 3.9] *Let $G = A_1$ and $V_{a\omega_1}, V_{b\omega_1}$ be irreducible FG-modules of highest weights $a\omega_1, b\omega_1$, respectively. Let $a = \sum_{i\geq 0} a_i p^i$ and $b = \sum_{i\geq 0} b_i p^i$ be the p-adic expansions of a and b, respectively. Let $v_p(a+1)$ denote the maximum r such that $p^r | a+1$. Suppose that there exists an indecomposable FG-module of composition length 2 with factors $V_{a\omega_1}$ and $V_{b\omega_1}$. Then there exists a natural number $k \geq v_p(a+1)$ such that $a_i = b_i$ for $i \neq k, k+1$, and $a_k = p - b_k - 2$ and $a_{k+1} = b_{k+1} \pm 1$. In particular, either $a \geq p$ or $b \geq p$.*

Corollary 3.5. *Let $p > 3$ and let G, a, b be as in Lemma 3.4. Let E be an FG-module with composition factors $V_{a\omega_1}$ and $V_{b\omega_1}$. Suppose that $a = p^i + p^j$ and $b = p^r + p^t$ where $i < j$, $r < t$. Then E is completely reducible.*

Proof. Suppose the contrary. Note that $a + 1$ is coprime to p as $p > 2$. So $v_p(a + 1) = 0$. We can assume (by swapping the modules) that $i \leq r$. Suppose that $i < r$. Then $b_i = 0$ and $a_i = 1$; by Lemma 3.4, $a_i = p - 2$, which is false as $p > 3$. So $i = r$. Then $j \neq t$, and we can assume $j < t$. Then $a_j = 1, b_j = 0$, and, by Lemma 3.4, $1 = a_j = p - 2$, a contradiction. □

Remark. The assumption on a, b in Corollary 3.5 is equivalent to saying that $V_{a\omega_1}$ and $V_{b\omega_1}$ are tensor-decomposable and $\dim V_{a\omega_1} = \dim V_{b\omega_1} = 4$.

Lemma 3.6. *Let $D = SL(2, p)$ and let S be an indecomposable FD-module. Let $u \in D$, $o(u) = p$ and suppose that $\mathrm{Jor}_S(u)$ contains a single non-trivial block. Then $\dim S \leq p + 1$, and one of the following holds (where l is the composition length of S):*

(1) *$l = 1$ and $\dim S \leq p$;*

(2) *$l = 2$, $p > 2$ and $\dim S \in \{p - 1, p + 1\}$ or $p = 2$ and $\dim S = 2$;*

(3) *$l = 3$, $p > 3$, $\dim S = p+1$ and the dimensions of the composition factors of S are $2, p - 3, 2$;*

(4) *S has a composition factor of dimension $p-2$ and all other factors are trivial.*

In addition, if $\dim S \geq p$ *then* $\mathrm{Jor}_S(u)$ *contains a block of size* p.

Proof. The first claim is well known if S is irreducible. Suppose S is reducible, and set $U = \langle u \rangle$. By Lemma 2.1, the Jordan form of u is $(m \cdot J_p, J_d)$ for some $d < p$. By assumption, $0 \leq m \leq 1$, and $m = 1$ implies $d \leq 1$. Therefore, $\dim S \leq p+1$. The additional claim (after item (4)) is obvious.

Consider the options for l. If $l = 1$ then the dimension of S is well known to be at most p. Suppose that $l > 1$ and $p = 2$. Then $D \cong SL_2(2)$, and the non-trivial composition factors of S are projective D-modules. So either S is irreducible or trivial on the subgroup of D of order 3. Then $\dim S = 2$. Let $p > 2$. If $l = 2$ then $\dim S = p-1$ or $p+1$, see [2, p. 49] or [7, p.111]. If $l \geq 3$ then (3) and (4) follow by applying (2) to the factors of S of composition length 2. Indeed, let T be an indecomposable submodule of S of composition length 2. By (2), $\dim T = p \pm 1$, and $\dim S \leq p+1$ by the above. So $\dim T = p-1$, and $m := \dim S/T \leq 2$. Let d, e be the dimensions of the composition factors of T, where $d + e = p - 1$. As S is indecomposable, there is an indecomposable quotient of S of dimension $d+m$ or $e+m$. By (2), $d+m$ or $e+m$ equals $p-1$. We may assume that $d+m = p-1$ (by reordering d, e). If $m = 2$ then $d = p-3$ and $e = 2$ so (3) holds. Here $p \neq 3$ as $d \neq 0$. If $m = 1$ then $d = p-2$ and $e = 1$, that is (4) holds. □

The following fact is trivial but it is convenient to state it explicitly as this is frequently used.

Lemma 3.7. *Let M be an FG-module, and $u \in G$ unipotent. Suppose that* $\mathrm{Jor}_M(u)$ *contains a single non-trivial block. Then the Jordan form of u on any submodule or quotient module of M contains at most one non-trivial block. The same is true for every quotient M_2/M_1, where $M_1 \subset M_2$ are FG-submodules of M.*

Proof. Indeed, u has a single block of size $k > 1$ on M if and only if the module $(u-1)M$ is uniserial as an $F\langle u \rangle$-module. This property is inherited by submodules. Applying this to the dual of M, we get the result for quotient modules. These also imply the result for M_2/M_1. □

Lemma 3.8. *Let G be of type A_1, and let $u \in G$ be a unipotent element. Let M be an FG-module and M_0 the maximal trivial submodule of M. Suppose that* $\mathrm{Jor}_M(u)$ *contains a single non-trivial block. Then the composition series of M contains at most two non-trivial terms. More precisely, one of the following holds:*

(A) *the composition series of M contains at most one non-trivial term; or*

(B) *$p > 2$, the composition length of M/M_0 is 2 and $p+1 \le \dim M/M_0 \le p+2$.*

Moreover, in case (B), we have:

(C) *If $a\omega_1, b\omega_1$ are the highest weights of the composition factors of M/M_0, with $a \ge b$, then $a \ge p$.*

Proof. For $p = 0$ the lemma is trivial. So we assume $p > 0$. Obviously, we may assume that M is indecomposable. Let D denote the subgroup of G isomorphic to $SL_2(p)$. Then $M|_D = S \oplus T$, where T is a trivial D-module and S is an indecomposable one. We first prove (A) and (B), in a sequence of steps (1) to (11).

(1) Every submodule of M/M_0 is indecomposable. In particular, the socle of M/M_0 is irreducible.

Indeed, if L is a submodule of M/M_0 and $L = L_1 \oplus L_2$, where L_1, L_2 are non-zero FG-modules, then one of them is trivial by Lemma 3.7, which contradicts Lemma 3.4 and the definition of M_0.

(2) Let $M_1 \subset M_2$ be FG-submodules of M. Suppose that $\mathrm{Jor}_{M_2/M_1}(u)$ has a block of size p. Then $M_1 \subseteq M_0$ and M/M_2 is a trivial FG-module.

Indeed, M_2/M_1 has an indecomposable $F\langle u\rangle$-submodule X of dimension p. Hence X is projective and injective, so $M/M_1|_{\langle u\rangle} = X \oplus Y$, where Y is an $F\langle u\rangle$-module. By Lemma 3.7, Y is a trivial $F\langle u\rangle$-module. As $X \subset M_2/M_1$, it follows that u is trivial on M/M_2, and hence M/M_2 is a trivial FG-module. Applying this to the dual of M, we observe that u acts trivially on M_1. So the claim follows.

(3) Let K be a composition factor of M. If $\mathrm{Jor}_K(u)$ has a block of size p then statement (A) holds.

This follows from (2).

(4) If $p = 2$ then the statement (A) holds.

Indeed, in this case $|u| = 2$ and M has a non-trivial composition factor K, say. Then u must have a block of size 2 on K, so the result is true by (3).

From now on we assume $p > 2$.

(5) If M has a composition factor K of dimension p then (A) holds.

Since $\dim K = p$, K is tensor-indecomposable, and hence $K|_D$ is irreducible. Then it is a projective D-module and $\mathrm{Jor}_K(u)$ consists of a single block of size p. So the claim follows from (3).

(6) Let $M_1 \subset M_2 \subset M_3$ be FG-submodules of M such that M_2/M_1 is irreducible and M_3/M_1 is indecomposable. If M_3/M_2 is trivial then (A) holds.

Set $K = M_2/M_1$ and $L = M_3/M_1$. Suppose the contrary. Then it suffices to handle the case where $\dim(L/K) = 1$. If $\dim K \leq p-1$ then $\dim L \leq p$. By [11, Theorem 2], L is completely reducible, contrary to the assumption. So $\dim K > p$ by (5). Therefore, K is tensor-decomposable. By Lemma 3.1, $\dim K = 4$ and $p > 3$ by Lemma 3.1 and (3). Then $\dim L = 5 \leq p$, so L is decomposable by [11, Theorem 2]. This is a contradiction.

(7) Either statement (A) holds or all composition factors of M/M_0 are non-trivial and M/M_0 is uniserial.

If M/M_0 has a trivial composition factor then the claim follows from (6). Otherwise, this follows from Lemma 3.7 and (1).

(8) Suppose that M has a tensor-decomposable composition factor K, say. Then either (A) holds and $\dim K = 4$ or $p > 3$ and (B) holds.

By Lemmas 3.7 and 3.1, $\dim K = 4$, the composition factors of $K|_D$ are of dimensions $1, 3$ and $\mathrm{Jor}_K(u)$ has a block of size 3. Suppose that (A) is false. Then $p > 3$ by (3). Furthermore, K can be included in a subquotient L, say, of composition length 2. Let K' be the second factor of L. By (7), K' is non-trivial and L is indecomposable.

Suppose first that K' is tensor-decomposable. Then $\dim K' = 4$ and, by Lemma 3.5, L is completely reducible, which is false.

So K' is tensor indecomposable, and hence $K'|_D$ is irreducible. Set $m = \dim K'$, so $\dim L = 4 + m$. Then $1 < m \leq p$, and $m < p$ by (5). Then $L|_D$ has composition factors of dimensions $3, 1, m$, and hence is decomposable (otherwise contradicts Lemma 3.6(3)). As $\mathrm{Jor}_L(u)$ has a single non-trivial block, it follows that $L|_D$ contains an indecomposable submodule X, say, with composition factors of dimensions $3, m$. By Lemma 3.6, we have $3 + m = p - 1$ or $p + 1$. In the former case $\dim L = p$, which is false in view of [11, Theorem 2]. So $3+m = p+1$, and hence $\dim L = p+2$. Furthermore, by Lemma 3.6, $\mathrm{Jor}_X(u)$ contains a block of size p. Let $L = M_2/M_1$ for some FG-modules $M_1 \subset M_2$. Then, by (2), M/M_2 and M_1 are trivial FG-modules. So we deduce that $M_1 = M_0$ and $M = M_2$, i.e. $L = M/M_0$, so (B) follows.

(9) If the restriction $(M/M_0)|_D$ has a trivial composition factor then (A) or (B) holds.

Suppose the contrary. Then by (7), all composition factors of M/M_0 are non-trivial, and tensor-indecomposable factors remain irreducible upon restriction to D. So one of the composition factors of M is tensor-decomposable, which contradicts (8).

(10) Either (A) or (B) holds, or the restriction $(M/M_0)|_D$ is indecomposable and has no trivial composition factor.

Suppose that neither (A) nor (B) holds. Then, by (9), $(M/M_0)|_D$ has no trivial composition factor. By Lemma 3.1, every composition factor of M/M_0 is tensor-indecomposable, and hence irreducible for D. Then $(M/M_0)|_D$ is indecomposable in view of Lemma 3.7.

(11) Statement (A) or (B) holds.

Suppose the contrary. Then, by (8), the composition factors of M are tensor-indecomposable and hence are irreducible for D. By (10), $(M/M_0)|_D$ is indecomposable with no trivial composition factor. Then, by Lemma 3.6, $p > 2$ and $\dim(M/M_0) \leq p+1$. If $\dim(M/M_0) \leq p$ then M/M_0 is completely reducible by [11], which contradicts (1).

So we have $p > 2$ and $\dim(M/M_0) = p+1$, and all composition factors are irreducible for D. If case (3) of Lemma 3.6 holds, then M/M_0 has composition length 3 with tensor indecomposable factors of dimension 2, $p-3$, 2, contradicting Lemma 3.4. So Lemma 3.6(2) must hold and M/M_0 has composition length 2 as in (B).

Finally, statement (C) follows from (B) and Lemma 3.4. □

Lemma 3.9. *Let $G \cong A_1$ and let M be an FG-module. Let $u \in G$ be a unipotent element. Suppose that $\mathrm{Jor}_M(u)$ contains a single non-trivial block. Then all non-zero weights of M, with respect to a fixed maximal torus of G, are of multiplicity 1.*

Proof. For $p = 0$ the lemma is trivial, for $p > 0$ this follows from Lemma 3.8 and Corollary 3.3. □

Lemma 3.10. *Let G be a simple algebraic group and $X \cong A_1$ a subgroup of G. Let $u \in X$ be a unipotent element and M an FG-module. Suppose that $\mathrm{Jor}_M(u)$ contains a single non-trivial block. Then all non-zero weights of M, with respect to a fixed maximal torus of G, are of multiplicity 1. Moreover M is tensor-indecomposable, unless $p \neq 2$, $G = A_1$ and $\dim M = 4$.*

Proof. Suppose the contrary, and fix a maximal torus T of G, and a maximal torus T_1 of X with $T_1 \subset T$. Let M_λ be a T-weight space of weight $\lambda \neq 0$ such that $\dim M_\lambda > 1$. Then $\dim M_{w(\lambda)} > 1$ for every $w \in W$, where W is the Weyl group of G. By Lemma 3.9, T_1 acts trivially on $M_{w(\lambda)}$ for every $w \in W$. Recall that the weights of G are elements of $\mathrm{Hom}(T, GL_1(F))$, which is a $\mathbb{Z}$-lattice of rank r equal to the rank of G. The Weyl group acts on T and hence on $\mathrm{Hom}(T, GL_1(F))$, so W is realized as a subgroup of $GL_r(\mathbb{Z})$. Let R be the vector space over the rational number field $\mathbb{Q}$ spanned by the weights, and this yields an embedding of W into $GL_r(\mathbb{Q})$. It is well known that W is an irreducible subgroup of $GL_r(\mathbb{Q})$. The subspace of R spanned by $\{w(\lambda) : w \in W\}$ is W-stable, and hence coincides with R. Therefore, every weight μ can be written as $\sum_{w\in W} a_w w(\lambda)$ with $a_w \in \mathbb{Q}$. Let m be an integer such that $ma_w \in \mathbb{Z}$ for every $w \in W$. Then $m\mu = \sum_{w\in W}(ma_w)w(\lambda)$, where the coefficients ma_w are integers. This implies that $(m\mu)(T_1) = 1$, whence $\mu(T_1^m) = 1$. Note that for every $t_1 \in T_1$ there is an element $t \in T_1$ such that $t^m = t_1$, in other words $T_1 = T_1^m$. Therefore, $\mu(T_1) = 1$. This is true for every weight μ of T. This implies that T_1 acts trivially on M, which is a contradiction.

For the second assertion in the lemma see Corollary 2.3. □

Theorem 3.11. *Let G be a simple algebraic group, $u \in G$ a unipotent element and M an FG-module. Suppose that $\mathrm{Jor}_M(u)$ contains a single non-trivial block. If $p > 0$ and $u^p = 1$, or if $p = 0$, then all non-zero weights of M, with respect to a fixed maximal torus of G, are of multiplicity 1, unless possibly $G = G_2, p = 3$ and u lies in the class $A_1^{(3)}$ as in* [13].

Proof. By the main results of [22, 13], every element of order p in a simple algebraic group in defining characteristic p is contained in a simple algebraic subgroup of type A_1, with the exception of the class of elements labelled $A_1^{(3)}$ in $G = G_2$ when $p = 3$. If $p = 0$, every unipotent element is well known to lie in a subgroup of type A_1. So the statement follows from Lemma 3.10. □

4 Representations of groups of exceptional type

In view of Theorem 3.11, it is useful to know which irreducible representations of exceptional algebraic groups have all non-zero weights of multiplicity 1. Moreover, for our application to the question about the Jordan block structure of unipotent elements in the representation space, Corollary 2.3 shows that we can restrict our attention to tensor-indecomposable representations. We have the following result taken from [24].

Lemma 4.1. *Let G be a simple algebraic group of exceptional type and let M be a tensor-indecomposable irreducible FG-module with highest weight $\omega \neq 0$. Suppose that all non-zero weights of M are of multiplicity one. Then $(G, \omega) \in \{(E_6, p^a\omega_i), i = 1, 2, 6, (E_7, p^a\omega_j), j = 1, 7, (E_8, p^a\omega_8), (F_4, p^a\omega_k), k = 1, 4, (G_2, p^a\omega_l), l = 1, 2\}$, for some $a \geq 0$.*

Theorem 4.2. *Let G be a simple algebraic group of exceptional type and let $1 \neq u \in G$ be a unipotent element of order p. Let M be an irreducible FG-module. Then one of the following holds:*

(a) $\mathrm{Jor}_M(u)$ *contains at least two non-trivial blocks;*

(b) $G = G_2$, $\omega = p^a\omega_1$, $p \geq 7$, *u is regular and* $\mathrm{Jor}_M(u)$ *has precisely one block;*

(c) $G = G_2, p = 3$ *and u lies in the class labelled* $A_1^{(3)}$.

Proof. By Corollary 2.3, M is tensor indecomposable. As $|u| = p$, Theorem 3.11 leaves us with inspection of the unipotent block structure for the cases listed in Lemma 4.1, unless we are in the situation of (c) above. The Jordan block structure of all unipotent elements in the representations listed in Lemma 4.1 has been computed by Lawther [8]. We see that either (a) or (b) holds. (Note that the case $|u| = 2$ can be deduced from classification of irreducible linear groups generated by transvections, see for instance [12].) $\square$

Remark. Under the assumptions of Theorem 4.2 and assuming in addition that $G \neq G_2$, a result of Suprunenko [18, Theorem 1] allows one to reduce the problem under discussion to an analysis of modules M of dimension at most $4(l + 3)$ and a small list of further exceptions, most of which can be handled using the Tables of [8]. However, the proof of Suprunenko's result announced in 2005 has not been yet published, so we prefer to avoid using it. In addition, the method based on Theorem 3.11 can also lead to an alternative proof of a similar theorem for classical algebraic groups.

We wish to extend Theorem 4.2 to the case $|u| > p$. For this we use induction. From Lemma 3.7 we get the following

Lemma 4.3. *Let P be a parabolic subgroup in G with Levi factor L and unipotent radical U. Let $u \in P$ be unipotent and let u' denote the projection of u into L. Let V be an irreducible FG-module and $0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_k = V$ be a socle series of $V|_P$ (that is, U acts trivially on the factors V_i/V_{i-1}). Suppose that the Jordan form of u on V has at most one block of size greater than 1. Then the same holds for the Jordan form of u' on every factor V_i/V_{i-1}.*

Lemma 4.4. *Let G be a simple algebraic group in characteristic $p > 0$, P a parabolic subgroup and $U = R_u(P)$ the unipotent radical of P. If U is abelian then the exponent of U equals p. If U is nilpotent of class 2 then U is of exponent p or 4.*

Proof. It is well known that U is generated by root subgroups U_α for some sets of roots α, and each U_α is an abelian group of exponent p. This implies the statement if U is abelian. Otherwise, let U' be the derived subgroup of U. Then for every $x \in U$ the mapping $u \to [x, u]$ $(u \in U)$ yields a group homomorphism $U \to U'$. Now U' is of exponent p as so is U/U'. If $p = 2$ then U is of exponent 4. If $p > 2$ then $(xu)^p = x^p u^p [u, x]^{p(p-1)/2} = x^p u^p$ as $[u, x]^{p(p-1)/2} = 1$. This easily implies the lemma. $\square$

Remark. Below we will apply Lemma 4.4 to groups G of type E_6 or F_4, and to the maximal parabolic subgroups P corresponding to nodes $1, 6$ for E_6 and $1, 4$ for F_4. Then U is of nilpotency class 2. This follows from the description of U in [4, 4.4] for node 1 of E_6, and for node 6 this follows too as the graph automorphism of E_6 permutes the nodes 1,6. For $G = F_4$ this similarly follows from [4, 4.5] for node 1, and for node 4 from data at [4, p. 19]. (More precisely, U is generated by the root subgroups U_α, where α runs over positive roots whose expression in terms of simple roots contains α_4, and U contains a normal subgroup R generated by U_α for α such that the root α_4 occurs in such expressions with coefficient 2. As no positive root of F_4 has α_4-coefficient greater than 2, the claim follows from [4, 4.8(i)].)

Lemma 4.5. *Let G be a simple exceptional algebraic group of rank l in defining characteristic p, and $u \in G$ unipotent. Let M be an irreducible G-module such that $\mathrm{Jor}_M(u)$ contains a single non-trivial block. Let $k \geq 0$ be an integer such that $|u^{p^k}| = p$. Then $\dim M \leq (p-1)p^k(l+3)$, unless possibly G is of type F_4, $p = 2$, where $\dim M \leq 2^{k+3}$.*

Proof. As $\mathrm{Jor}_M(u)$ contains a single non-trivial block, it follows that $\mathrm{Jor}_M(u^{p^k})$ contains at most p^k non-trivial blocks, each of size at most p. Therefore, $\dim(\mathrm{Id} - u^{p^k})M \leq (p-1)p^k$. There is a finite group $G_1 \subset G$ of Lie type such that $u \in G_1$ and M is an irreducible FG_1-module. So the result follows by applying Lemma 2.5 to u^{p^k}. $\square$

Example. If $|u| \leq 4$ then $\dim M \leq 2(l+3)$ or 16 for F_4. If $|u| = 9$ then $\dim M \leq 6(l+3)$.

Lemma 4.6. *Let G be of type E_6, E_7, E_8 or F_4, and let $1 \neq u \in G$ be a unipotent element. Let V be an irreducible G-module such that $\dim V > 1$. Then $\mathrm{Jor}_V(u)$ has at least two non-trivial blocks.*

Proof. By Lemma 3.1, we can assume that $V = V_\omega$ is tensor-indecomposable, and hence without loss of generality the highest weight ω of V is p-restricted. In view of Theorem 4.2 we can assume that $|u| > p$. If $|u| = 4$ then $\dim M \leq 22$ (see Example following Lemma 4.5); however G is well known to have no non-trivial irreducible representation of degree less than 25. So we can assume that $|u| > 2p$. Suppose the contrary, that $\mathrm{Jor}_V(u)$ has a single non-trivial block.

Suppose first that G is of type E_6. Let P_i, $i = 1, 6$, be a maximal parabolic of G corresponding to nodes 1, respectively 6, of the Dynkin diagram of G. Let L_i be a Levi subgroup of P_i and L_i' the derived subgroup of L_i. Then L_i' is a simple group of type D_5. By Lemma 4.4, $u \notin R_u(P)$. Let u' be the projection of u into L_i. Then $1 \neq u' \in L_i'$. By Lemma 4.3, the Jordan form of u' has at most one non-trivial block on every composition factor of the restriction of V to L_i'. Let λ be the highest weight of a non-trivial composition factor.

Recall that L_1' is generated by the root subgroups $U_{\pm\alpha_i}$ with $i \in \{2,3,4,5,6\}$, and L_2' is generated by the root subgroups $U_{\pm\alpha_i}$ with $i \in \{1,2,3,4,5\}$. Let $\omega = \sum a_i\omega_i$, where $\omega_1, \dots, \omega_6$ are the fundamental weights of E_6 and $0 \leq a_i < p$ for $i = 1, \dots, 6$. Let $\lambda_1, \dots, \lambda_5$ be the fundamental weights of D_5. By Smith's theorem [15], the restriction of V to L_1' contains a composition factor of highest weight $\sum_{i=1}^5 a_i\lambda_i$, and the restriction of V to L_2' has a composition factor of highest weight $a_6\lambda_1 + a_5\lambda_2 + a_4\lambda_3 + a_3\lambda_4 + a_2\lambda_5$. (The root ordering for $L_2' \cong D_5$ is inverse of that in L_1'.) By a result of Suprunenko [20, Theorem 3] for $p > 2$

and [19, Theorem 3] for $p = 2$, $\lambda = p^m\omega_1$ for some integer $m > 0$. Applying this to L'_1, we obtain $a_1 \le 1$, $a_2 = \cdots = a_5 = 0$, applying to L'_2, we get $a_6 \le 1$. So we are left to examine the cases where $\omega \in \{\omega_1, \omega_6, \omega_1 + \omega_6\}$. By Lawther [8, p. 4136], $\mathrm{Jor}_V(u)$ has at least two non-trivial blocks for $\omega = \omega_1$ and ω_6.

Let $\omega = \omega_1 + \omega_6$. Then $\omega - \alpha_1$, respectively $\omega - \alpha_6$, affords the highest weight of an FL_i-composition factor for $i = 1$, resp. $i = 2$, with highest weight $\lambda_1 + \lambda_4$. This again contradicts [20, Theorem 3] for $p > 2$ and [19, Theorem 3] for $p = 2$.

If G is of type E_7 then G has a parabolic subgroup P whose Levi factor L' is of type E_6, and $R_u(P)$ is abelian ([4, 4.4]), and hence of exponent p. As above we deduce that $u \notin R_u(P)$. If $\omega = \omega_1$, ω affords an L'-composition factor of highest weight ω_1 for $L' = E_6$, and if $\omega = \omega_7$, the weight $\omega - \alpha_7$ affords an L'-composition factor which is again one of the 27-dimensional irreducible E_6-modules. But this contradicts the conclusion of the previous paragraph. In an entirely similar way, and using [4, 4.4], the case $G = E_8$ follows from that for E_7.

Let $G = F_4$, and let $\omega = \sum a_i\omega_i$, where $\omega_1, \dots, \omega_4$ are the fundamental weights of F_4. Let P_i, $i = 1, 4$, be a maximal parabolic of G corresponding to nodes 1 or 4 of the Dynkin diagram of G. Let L_i be a Levi subgroup of P_i and L'_i the derived subgroup of L_i. Then L'_4 is simple of type B_3, and L'_1 is simple of type C_3. Let $\lambda_1, \lambda_2, \lambda_3$ be the fundamental weights of L'_4 and μ_1, μ_2, μ_3 the fundamental weights of L'_1. As above, by Smith's theorem, the restriction of V to L'_4 contains a composition factor with highest weight $a_1\lambda_1 + a_2\lambda_2 + a_3\lambda_3$, and the restriction of V to L'_1 has a composition factor with highest weight $a_4\mu_1 + a_3\mu_2 + a_2\mu_3$. Applying [20, Theorem 3] to L'_4, we get $a_1 = 1, a_2 = a_3 = 0$; applying to L'_2 [20, Theorem 3] for $p > 2$ and [19, Theorem 3] for $p = 2$, we get $a_4 \le 1$. So we have to examine the cases $\omega \in \{\omega_1, \omega_4, \omega_1 + \omega_4\}$. By Lawther [8, p. 4134, 4135], $\mathrm{Jor}_V(u)$ has at least two non-trivial blocks for $\omega = \omega_1$ and ω_4. If $p = 2$ and $\omega = \omega_1 + \omega_4$ then V is tensor-decomposable (see [16, Corollary of Theorem 41]), so $\mathrm{Jor}_V(u)$ has at most two non-trivial blocks by Lemma 2.3.

Let $\omega = \omega_1 + \omega_4$ and $p > 2$. By Lübeck [9], $\dim V_\omega = 1053$. As $|u| > p > 2$, we have $|u| \le 3^3, 5^2, 7^2, 11^2$ for $p = 3, 5, 7, 11$ respectively, see [8]. By Lemma 4.5, we have $\dim M \le 7 \cdot 10 \cdot 11 = 770$. This is a contradiction. $\square$

Now we consider the remaining cases for the group $G = G_2$. So from now on we assume $G = G_2$, that is, the unipotent elements which are either of order greater than p or the one class of elements of order 3 for $p = 3$ which do not lie in any A_1-type subgroup [13, Theorem 5.1]. We fix a maximal torus T of G and root subgroups with respect to T. For all roots α, let $x_\alpha : \mathbf{G}_a \to G$ be an isomorphism whose image is the T-root subgroup U_α corresponding to α. By [8], for example, we are left to consider the following:

a) u is regular and hence conjugate to $x_{-\alpha_1}(1)x_{-\alpha_2}(1)$, $p \le 5$, and u has order p^2 or 8.

b) u is in the class $G_2(a_1)$, $p = 2$, u is conjugate to $x_{\alpha_2}(1)x_{3\alpha_1+\alpha_2}(1)$ and has order 4.

c) u has order 3 and is conjugate to $x_{2\alpha_1+\alpha_2}(1)x_{3\alpha_1+2\alpha_2}(1)$.

We note that the Jordan block structure of all unipotent elements acting on the irreducible modules with highest weight ω_1, or ω_2 for $p \neq 3$, is given in Lawther [8]. We use this to show:

Lemma 4.7. *Let $1 \neq u \in G = G_2$ be unipotent and let V be one of the two irreducible FG-modules with highest weight ω_1 or ω_2. Then $\mathrm{Jor}_V(u)$ has a single non-trivial block if and only if one of the following holds:*
(1) *u is regular and $\omega = \omega_1$.*
(2) *u is regular, $p = 3$ and $\omega = \omega_2$.*

Proof. For the weight $\omega = \omega_2, p \neq 3$ and $\omega = \omega_1$ the statement follows directly from [8, Table 1]. So it remains to consider the case of the irreducible module $V_G(\omega_2)$, when $p = 3$. We apply the exceptional graph automorphism of G and see that any element acting with only one non-trivial Jordan block on V_{ω_2} must have image an element acting with only one non-trivial Jordan block on V_{ω_1}. Then, by the above remarks, the image of the element under the graph automorphism must be regular, which means the element itself is regular. The result follows. $\square$

Proposition 4.8. *Let $G = G_2$ and let $u \in G$ be unipotent. Let V be an irreducible G-module with highest weight ω. Then $\mathrm{Jor}_V(u)$ has a single non-trivial block if and only if $\omega = p^k\omega_1$ or $p = 3$ and $\omega = p^k\omega_2$ for some integer $k \geq 0$.*

Proof. Suppose the contrary. By Corollary 2.3, V is tensor-indecomposable, so we may then assume that V is p-restricted. Note that u is conjugate to a unipotent element of $G_2(p)$, and the restriction of a p-restricted irreducible representation to $G_2(p)$ remains irreducible. So it suffices to deal with $G = G_2(p)$, which we assume in some cases below. By Lemma 4.7, we can assume that $\omega \neq \omega_1, \omega_2$.

Let $p = 2$. Then $|u| \leq 8$. Let u_2 be a power of u such that $|u_2| = 2$. By Lemma 4.5, we get $\dim V \leq 20$. However, as ω is 2-restricted and $\omega \neq \omega_1, \omega_2$, we have $\omega = \omega_1 + \omega_2$. By [9], $\dim V = 64$ in this case, which is a contradiction.

For elements of order p for $p = 3$, as in c) above, we similarly obtain the bound $\dim V \leq 10$, whereas the minimal dimension of an irreducible representation of G with highest weight $\omega \neq 0, \omega_1, \omega_2$ exceeds 26 [10]. So we are left with the case $|u| = 9$ for $p = 3$ and $|u| = 25$ for $p = 5$. In these cases u is a regular unipotent element of G, say $u = x_{-\alpha_1}(1)x_{-\alpha_2}(1)$.

Let $P_i \leq G$ be the parabolic subgroup with $P_i \supseteq B^-$, the Borel subgroup generated by the maximal torus T and the root subgroups corresponding to negative roots, and whose Levi factor L_i satisfies $L_i' = \langle U_{\pm\alpha_i}\rangle$. Set $Q_i = R_u(P_i)$

and let $\pi_i : P_i \to L_i$ be the canonical projection. Set $u_i = \pi_i(u) \in L_i'$. As $u \in P_i$, u stabilizes the commutator series $V \supset [V, Q_i] \supset [[V, Q_i], Q_i] \supset \cdots$, and acts on the quotients, via the element u_i. Then by Lemma 3.7, the matrix of u in its action on every subquotient has at most one non-trivial Jordan block and, by Lemma 3.8(C), each of these subquotients can have at most one non-trivial p-restricted irreducible constituent. Note that we will abuse notation and write ω_i for the restriction of ω_i to $T \cap L_i'$. Setting $[V, Q_i^0] = V$, let $[V, Q_i^d] = [[V, Q_i^{d-1}]Q_i]$ for $d \geq 1$; we will use the following result from [14, 2.3]:

If $p = 3$, assume $\omega = r\omega_1$, for some r. Fix an integer $d \geq 0$. Then the quotient $[V, Q_i^d]/[V, Q_i^{d+1}]$ is isomorphic to the direct sum of those weight spaces of V of the form $\omega - d\alpha_j - m\alpha_i$, for some $m \geq 0$ and where $\{i, j\} = \{1, 2\}$.

Case $p = 3$.

Note that V is tensor-indecomposable if and only if $\omega = 2\omega_1$ or $2\omega_2$ (we ignore the cases $\omega = \omega_1, \omega_2$ by the above), see [16, Corollary of Theorem 41].

The class of regular elements is invariant under the graph automorphism of G and so the Jordan block structure of a regular element on $V_{a\omega_1}$ is the same as the Jordan block structure of this element on $V_{a\omega_2}$. So it suffices to deal with $\omega = 2\omega_1$. Using the above quoted result [14, 2.3], we see that $[V, Q_1^1]/[V, Q_1^2]$ has three FL_1'-composition factors, afforded by $\omega - \alpha_1 - \alpha_2$ and $\omega - 2\alpha_1 - \alpha_2$, (the latter weight has multiplicity 2, see [10], and affords two composition factors) with highest weights $3\omega_1$, respectively ω_1. This contradicts Lemma 3.8(G).

Case $p = 5$.

Throughout, we will refer to [10] for weight multiplicities, without further reference.

Consider first the modules $V = V_{a\omega_1}$, $a = 2, 3, 4$. Here the FL_1'-module $[V, Q_1]/[V, Q_1^2]$ has FL_1-composition factors of highest weights $(a+1)\omega_1$ and $(a-1)\omega_1$, afforded by $\omega - \alpha_1 - \alpha_2$, respectively $\omega_1 - 2\alpha_1 - \alpha_2$. Then Lemma 3.8 implies that $a = 4$. But in this case the second weight has multiplicity 2 and affords a third non-trivial composition factor, contradicting Lemma 3.8(C).

Now turn to the modules whose highest weight is of the form $b\omega_2$. For $V_{2\omega_2}$, the FL_2'-module $[V, Q_2^3]/[V, Q_2^4]$ has composition factors of highest weights $3\omega_2$, ω_2 and ω_2, afforded by $\omega - 3\alpha_1 - \alpha_2$, respectively $\omega - 3\alpha_1 - 2\alpha_2$, the latter having multiplicity 3 in V. This contradicts Lemma 3.8(C).

For $V_{3\omega_2}$, the FL_2'-module $[V, Q_2^3]/[V, Q_2^4]$ has composition factors of highest weights $4\omega_2$ and $2\omega_2$, afforded by $\omega - 3\alpha_1 - \alpha_2$, respectively $\omega - 3\alpha_1 - 2\alpha_2$, contradicting Lemma 3.8(C).

Finally, for the FG-module $V_{4\omega_2}$, we consider the action of the parabolic subgroup P_1. The FL_1'-module $[V, Q_1^3]/[V, Q_1^4]$ has a composition factor R of dimension 10 whose highest weight is $9\omega_1$ (afforded by the weight $\omega - 3\alpha_2$). Then R is a tensor product of modules of dimensions 2 and 5, which contradicts Lemma 2.2.

We now turn to modules $V_{a\omega_1+b\omega_2}$, where $0 < a, b < 5$. By [21, 1.35], the weight $\omega - \alpha_1 - \alpha_2$ has multiplicity 2 in V if and only if $(3b+a+3) \not\equiv 0 \mod 5$. If $\omega - \alpha_1 - \alpha_2$ has multiplicity 2, the FL_i-module $[V, Q_i]/[V, Q_i^2]$ has composition factors of highest weights $a+3$ and $a+1$, or $b+1$ and $b-1$, for $i = 1$, respectively 2, afforded by $\omega - \alpha_j$ and $\omega - \alpha_j - \alpha_i$, where $\{i, j\} = \{1, 2\}$. Now using repeatedly Lemma 3.8 and Lemma 2.2, we deduce that $\omega \in \{\omega_1 + 2\omega_2, 3\omega_1 + b\omega_2\ (b = 1, 3, 4), 4\omega_1 + \omega_2\}$. If $b > 1$, the weight $\omega - 2\alpha_2$ affords an FL_1-composition factor of $[V, Q_1^2]/[V, Q_1^3]$ which is tensor decomposable and contradicts Lemma 2.2. If $\omega = a\omega_1 + \omega_2$, for $a = 3, 4$, then $\omega - 3\alpha_1$ and $\omega - 3\alpha_1 - \alpha_2$ afford FL_2-composition factors of $[V, Q_2^3]/[V, Q_2^4]$, of highest weights $4\omega_2$, respectively $2\omega_2$, contradicting Lemma 3.8(C). □

This completes the consideration of the remaining cases for the group $G = G_2$ and together with Theorem 4.2 completes the proof of Theorem 1.1.

References

[1] J. Alperin, Local representation theory, Cambridge Univ. Press, Cambridge, 1986.

[2] H. Andersen, J. Jorgensen, and P. Landrock, The projective indecomposable modules of $SL(2, p^a)$, *Proc. London Math. Soc.* (3) 46 (1983), 38 – 52.

[3] N. Bourbaki, Groupes et algèbres de Lie, Ch. IV-VI, Hermann, Paris, 1968.

[4] W. Curtis, W. Kantor and G. Seitz, The 2-transitive permutation representations of the finite Chevalley groups, *Trans. Amer. Math. Soc.* 218 (1976), 1 – 59.

[5] W. Feit, The representation theory of finite groups, North-Holland, Amsterdam, 1982.

[6] R. Guralnick and J. Saxl, Generation of finite almost simple groups by conjugates, *J. Algebra* 268 (2003), 519 – 571.

[7] J. Humphreys, Modular representations of finite groups of Lie type, Cambridge Univ. Press, Cambridge, 2006.

[8] R. Lawther, Jordan block sizes of unipotent elements in exceptional algebraic groups, *Comm. in Algebra* 23 (1995), 4125 – 4156; correction, ibid, 26 (1998), 2709.

[9] F. Lübeck, Small degree representations of finite Chevalley groups in defining characterisitic, *LMS J. Comput. Math.* 4 (2001), 135 – 169.

[10] F. Lübeck, Tables of weight multiplicities, on-line data, see http://www.math.rwth-aachen.de/~Frank.Luebeck/chev/WMSmall/index.html

[11] G.J. McNinch, Dimensional criteria for semisimplicity of representations, *Proc. London Math. Soc.* (3) 76 (1998), 95 – 149 .

[12] H. Pollatsek, Irreducible groups generated by transvections over finite fields of characteristic two, *J. Algebra* 39 (1976), 328 – 333.

[13] R. Proud, J. Saxl and D. Testerman, Subgroups of type A_1 containing a fixed unipotent element in an algebraic group, *J. Algebra* 231 (2000), 53 – 66.

[14] G.M. Seitz, The maximal subgroups of classical algebraic groups, *Memoirs Amer. Math. Soc.* no. 365, Amer. Math. Soc., Providence, (1987).

[15] S. Smith, Irreducible modules and parabolic subgroups, *J. Algebra* 75 (1982), 286 – 289.

[16] R. Steinberg, Lectures on Chevalley groups, Mimeographed Notes, Yale Univ., 1967.

[17] I.D. Suprunenko, Irreducible representations of simple algebraic groups containing matrices with big Jordan blocks, *Proc. London Math. Soc.* 71 (1995), 281 – 332.

[18] I.D. Suprunenko, Unipotent elements of prime order in irreducible representations of exceptional algebraic groups: the second Jordan block (in Russian), *Dokl. Nat. Acad. Nauk Belarusi* 49 (2005), no. 4, 5 –9.

[19] I. D. Suprunenko, Unipotent elements of nonprime order in representations of the classical groups: two big Jordan blocks (in Russian), *Dokl. Nat. Akad. Nauk Belarusi* 55 (2011), no. 4, p. 21–26.

[20] I.D. Suprunenko, Unipotent elements of non-prime order in representations of the classical algebraic groups: two big Jordan blocks, *J. Math. Sci.* 199 (2014), 350 – 374.

[21] D. Testerman, Irreducible subgroups of exceptional algebraic groups, *Memoirs Amer. Math. Soc.* no. 390, Amer. Math. Soc., Providence, (1988).

[22] D. Testerman, A_1-type overgroups of elements of order p in semisimple algebraic groups and the associated finite groups, *J. Algebra* 177 (2008), 34 – 76.

[23] D. Testerman and A.E. Zalesski, Irreducibility in algebraic groups and regular unipotent elements, *Proc. Amer. Math. Soc.* 141 (2013), 13 – 28,

[24] D. Testerman and A.E. Zalesski, Subgroups of simple algebraic groups containing maximal tori and representations with multiplicity 1 non-zero weights, *Transformation Groups* 20 (2015), 831 – 861.