Stochastic partial differential equations driven by Lévy white noises: Generalized random processes, random field solutions and regularity

THÈSE N^{O} 8223 (2017)

PRÉSENTÉE LE 8 DÉCEMBRE 2017
À LA FACULTÉ DES SCIENCES DE BASE
CHAIRE DE PROBABILITÉS
PROGRAMME DOCTORAL EN MATHÉMATIQUES

ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

POUR L'OBTENTION DU GRADE DE DOCTEUR ÈS SCIENCES

PAR

Thomas Marie Jean-Baptiste HUMEAU

acceptée sur proposition du jury:

Prof. S. Morgenthaler, président du jury
Prof. R. Dalang, directeur de thèse
Prof. M. Sanz-Solé, rapporteuse
Prof. A. Basse-O'Connor, rapporteur
Prof. M. Unser, rapporteur

ÉCOLE POLYTECHNIQUE
FÉDÉRALE DE LAUSANNE

Suisse
2017

À Clotilde et Marcellin.

Remerciements

I would like to thank deeply my doctoral advisor, Professor Robert C. Dalang, for his guidance during the past four years. You have managed to mentor and motivate me through sometimes difficult moments. Your talents as a mathematician and a scrupulous proofreader have impressed me in particular during this process.

I would also like to thank the members of the jury, Professor S. Morgenthaler, president of this jury; Professor M. Unser, internal examiner; Professor M. Sanz-Solé, external examiner and Professor A. Basse-O'Connor external examiner. I have had the opportunity to learn about generalized and sparse stochastic processes from Prof. Unser during a course at EPFL, and his work led to the question that we answer in first chapter of this thesis. You have followed my results during the past four years, and have always shown enthusiasm. I also especially thank Prof. Sanz-Solé for her course on SPDEs at EPFL last winter, I enjoyed it very much and learned a lot. You always showed great interest in my work, and I will remember your encouragements.

I also want to thank my co-authors, Julien Fageot and Carsten Chong, for making me discover mathematical collaboration. I have learned a lot from you and I really enjoyed the long hours we spent on Lévy noises and SPDEs. This thesis would not be as it is without you.

I am grateful for my fellow PhD students of the probability chair, especially my officemate Fei for his kindness, his supply of cheap books and delicacies from China, and Carlo for our discussions, coffee breaks and Kaggle adventure. I especially wish all the best to the newest phD students of the probability chair, David, and more recently Hamza.

Merci aussi à Ariane pour son aide dans l'organisation de mes voyages, et à Anna pour son grand support dans les questions liées à l'École Doctorale.

Je tiens ensuite à remercier tous ceux qui m'ont accompagné durant ma formation de mathématicien, tout d'abord mon papa qui m'apprenait à programmer ma calculatrice au collège pour déterminer les heures auxquelles les aiguilles d'une montre se superposaient (je n'avais rien compris à l'époque), puis les enseignants qui m'ont marqué, comme M. Hubier en 3ème, Mme Renard en terminale, M. Colin en sup et M. Abou-Jaoudé en spé, et enfin M. Touzi et M. Tankov à l'X, qui sont à l'origine de mon intérêt pour les probabilités. Merci aussi à Pierre, avec qui j'ai passé des soirées à simuler des khôlles au tableau (avant de passer de l'autre côté quelques années plus tard), bien que tu étais beaucoup plus fort que moi, ou encore Cécilia avec qui je pouvais avoir des conversations "normales" au milieu de gens bizarres comme des physiciens et des chimistes.

Je remercie aussi les amis qui m'ont entouré, et qui ont malgré eux participé à la santé de mon esprit en me permettant de sortir des mathématiques. Je pense en particulier aux grimpeurs du cube et d'ailleurs, et bien sûr également tous les membres de Plastik.

Merci aussi à ma belle-famille, qui a suivi de loin mes études en se demandant quand est-ce que j'allais enfin chercher un "vrai" travail, mais tout en gardant un certain intérêt pour mes recherches et toujours avec beaucoup d'encouragements. Je remercie en particulier Blandine, parce que je sais que vous serez très heureuse d'être mentionnée dans une thèse de maths.

Merci à mes parents et mes frères et soeurs pour m'avoir soutenu durant mes études, et qui ont toujours montré une grande fierté. Merci à mon papa pour m'avoir orienté vers les mathématiques, et à ma maman pour m'avoir montré qu'il n'y avait pas que ça.

Merci enfin à mon épouse, Clotilde, pour m'avoir soutenu et encouragé pendant ces quatre années. Tu as su me remotiver dans les moments de découragement et tu as toujours cru en moi.

Lausanne, 13 Novembre 2017 T. H.

Abstract

In this thesis, we study various aspects of stochastic partial differential equations driven by Lévy white noise. This driving noise, which is a generalization of Gaussian white noise, can be viewed either as a generalized random process or as an independently scattered random measure. After unifying these approaches and establishing appropriate stochastic integral representations, we show that a necessary and sufficient condition for a Lévy white noise to have values in $\mathscr{S}'(\mathbb{R}^d)$, the space of tempered Schwartz distributions, is that the underlying Lévy measure have a positive absolute moment.

In the case of a linear stochastic partial differential equation with a general differential operator and driven by a symmetric pure jump Lévy white noise, we show that when the mild solution is locally Lebesgue integrable, then it is equal to the generalized solution, and that a random field representation exists for the generalized solution if and only if the fundamental solution of the operator has certain integrability properties. In that case, we show that the random field representation is equal to the mild solution. For this purpose, a new stochastic Fubini theorem is proved. These results are applied to the linear stochastic heat and wave equations driven by a symmetric α-stable noise.

We then study the non-linear stochastic heat equation driven by a general type of Lévy white noise, possibly with heavy tails and non-summable small jumps. Our framework includes in particular the α-stable noise. In the case of the equation on the whole space $\mathbb{R}^d$, we show that the law of the solution that we construct does not depend on the space variable. Then we show in various domains $D \subset \mathbb{R}^d$ that the solution u to the stochastic heat equation is such that $t \mapsto u(t,\cdot)$ has a *càdlàg* version in a fractional Sobolev space of order $r < -\frac{d}{2}$. Finally, we show that $x \mapsto u(t,x)$ (respectively $t \mapsto u(t,x)$) at a fixed time (respectively fixed space-point) has a continuous version under some optimal moment conditions. In the α-stable case, we show that for the choices of α for which this moment condition is not satisfied, the sample paths of $x \mapsto u(t,x)$ (respectively $t \mapsto u(t,x)$) are unbounded on any non-empty open subset.

Key words: Stochastic partial differential equation, stochastic heat equation, Lévy white noise, tempered distribution, generalized random process, α-stable noise, fractional Sobolev space, regularity.

Résumé

Dans cette thèse, nous étudions différents aspects des équations aux dérivées partielles stochastiques avec bruit blanc de Lévy. Ces bruits blancs de Lévy, qui sont des généralisations du bruit blanc gaussien, peuvent être vus soit comme un processus généralisé, soit comme une mesure aléatoire indépendamment répartie. Après avoir unifié ces deux points de vue et établi des représentations sous forme d'intégrales stochastiques appropriées, nous montrons qu'une condition nécessaire et suffisante pour qu'un bruit blanc de Lévy soit à valeur dans $\mathscr{S}'(\mathbb{R}^d)$, l'espace des distributions tempérées de Schwartz, est que la mesure de Lévy sous-jacente ait un moment absolu d'ordre strictement positif.

Dans le cas d'une équation aux dérivées partielles stochastique linéaire avec un opérateur différentiel général et un bruit blanc de Lévy symétrique et de saut pur, nous démontrons que lorsque la solution *mild* est localement Lebesgue intégrable, alors elle est égale à la solution généralisée, et que la solution généralisée a une représentation sous forme de champ aléatoire si et seulement si la solution fondamentale de l'opérateur différentiel vérifie certaines conditions d'intégrabilité. Dans ce cas, nous démontrons que cette représentation sous forme de champ aléatoire est égale à la solution *mild*. Nous aurons besoin dans ces démonstrations d'un nouveau théorème de Fubini stochastique que nous démontrons. Ces résultats sont ensuite appliqués au cas de l'équation de la chaleur et des ondes stochastique avec un bruit α-stable symétrique.

Ensuite, nous étudions l'équation de la chaleur non linéaire avec un bruit blanc de Lévy général qui peut avoir des queues de distribution épaisses, et des petits sauts non sommables. Notre cadre d'étude inclus en particulier le bruit α-stable. Dans le cas de l'équation sur l'espace $\mathbb{R}^d$ tout entier, nous démontrons que la loi de la solution construite ne dépend pas de la variable d'espace. Puis nous démontrons dans différents domaines $D \subset \mathbb{R}^d$ que la solution u de l'équation de la chaleur stochastique est telle que $t \mapsto u(t,\cdot)$ a une version *càdlàg* dans un espace de Sobolev fractionnaire d'ordre $r < -\frac{d}{2}$. Finalement, nous démontrons que $x \mapsto u(t,x)$ (respectivement $t \mapsto u(t,x)$) à un instant donné (respectivement à une position d'espace donnée) a une version continue sous une certaine condition de moment optimale. Dans le cas α-stable, nous démontrons que pour les choix de α pour lesquels cette condition de moment n'est pas vérifiée, les trajectoires de $x \mapsto u(t,x)$ (respectivement $t \mapsto u(t,x)$) sont non bornées sur tout ouvert non vide.

Mots clefs : Équation aux dérivées partielles stochastiques, équation de la chaleur stochastique, bruit blanc de Lévy, distribution tempérée, processus stochastique généralisé, bruit α-stable, espace de Sobolev fractionnaire, régularité.

Contents

Introduction

Stochastic differential equations and stochastic partial differential equations (SPDEs) have been an object of interest for mathematicians for several decades (see for example the seminal work of [69]), and have found applications in many different fields, such as finance (see [54, 46]), population dynamics or turbulence (see [2, 8, 9]). A stochastic partial differential equation is a mathematical model that represents the evolution of a system depending on several variables (often time and space in the case of a physical system) and perturbed by a random noise. For example, the stochastic heat equation models the evolution of the temperature in a body subject to random perturbations:

$$\frac{\partial u}{\partial t}(t,x) = \Delta u(t,x) + \sigma\left(u(t,x)\right)\dot{L}(t,x)\,, \tag{0.0.1}$$

where σ is a function that models the non-linear reactions of the system to the perturbations induced by the noise $\dot{L}$. This equation is also known as a diffusion equation, an can be used for example to represent the diffusion of a chemical agent in a solvent. It can also be used in mathematical finance to model the formation of prices via the evolution of the limit order book ([46]). Another example is the stochastic wave equation:

$$\frac{\partial^2 u}{\partial t^2}(t,x) = \Delta u(t,x) + \sigma\left(u(t,x)\right)\dot{L}(t,x)\,. \tag{0.0.2}$$

This equation describes the motion of a vibrating body, driven by the random noise $\dot{L}$. Again, the function σ models the non-linear response of the body to the random perturbations. A famous example in the SPDEs world was provided in [69]: consider a guitar left outdoors in a sandstorm. The vibrations of the guitar strings are mathematically modeled by the wave equation in dimension one (the guitar string is ideally represented by a segment of fixed length without any thickness). It would be too complex to mathematically keep track of every sand particle in the sandstorm, and they are instead modeled by a random noise $\dot{L}$. Another more realistic example is the case of the motion of a strand of DNA in a fluid, perturbed by collisions with molecules (see [21, p. 39]).

There exist several approaches to solve and study these equations, and arguably the two most frequent are the Da Prato and Zabczyk approach (see [20]) and the random field approach, initiated by [69]. The first one considers the solutions to those SPDEs as stochastic processes with values in an infinite dimensional Hilbert space. The second one considers solutions as real-valued random fields. Those two approaches are in fact closely related in many cases (see [25]), and each point of view emphasizes different characteristics of the solution. In this thesis, we will mostly consider the random field approach.

The equations (0.0.1) and (0.0.2) have already been extensively studied in the context of Gaussian noise, usually white in time (i.e. with independent values at different times), and possibly colored in space (i.e. with possible space correlations). For specific examples, the interested reader can consult introductions to the subject in [69, 21, 48]. The random field approach allows to consider sample paths regularity of the solutions, see for example [11, 22, 19, 21] for the stochastic heat and wave equations in various settings.

Lévy noises are a natural generalization of Gaussian noises that allow also for random impulses, but SPDEs in the context of Lévy noises have only recently been studied, especially with a random field approach (see [71] for the infinite dimensional approach). In [4], R. Balan has studied the existence of a solution to various types of SPDEs driven by an α-stable white noise on a bounded domain, and in [13, 14], C. Chong proved existence results for the stochastic heat equation with a more general type of noise. The case of a Lévy colored noise has been investigated in [5].

The main goal of this thesis is to study regularity properties of SPDEs driven by a Lévy white noise. As one expects, the presence of impulses in the noise creates very different behaviors for the solution to an SPDE compared to the Gaussian case. For instance, in the case of the stochastic heat or wave equation with a Gaussian noise, the solution is known to be jointly Hölder continuous in the space and time variable. With a Lévy noise with impulses, the solutions have jumps and cannot be jointly continuous. Also, the Gaussian noise allows L^2-integrability conditions, compared to the Lévy case which imposes in general L^p-integrability conditions, where possibly $0 < p < 2$.

In Chapter 1, we introduce the main notations and definitions that will be used throughout this thesis.

Then, in Chapter 2, we study two definitions of Lévy white noises. The first one uses the notion of generalized random processes developed in [36], that is, a probability measure on a space of distributions. For example, the derivative of a well chosen order of a Lévy field is shown to be a Lévy white noise in that sense. The second definition is as an independently scattered random measure, and is defined as a random process indexed by a family of sets (see [58]). It uses a decomposition of the noise related to the Lévy-Itô decomposition for Lévy fields. In Section 2.3, we show that we can extend a Lévy noise defined as a generalized random process to an independently scattered random measure (see Theorem 2.3.5). This extension allows us to use [58] to determine the largest class of deterministic functions that are integrable with respect to the noise (see Proposition 2.3.7), and in the case of the derivative of a Lévy field, we establish a stochastic integral representation valid for this class of function. It is interesting to note that this representation does not exactly correspond to the usual Lévy-Itô decomposition, but uses a truncation level for the jumps that depends on the integrand.

In Chapter 3, we study a question that was brought to our attention by M. Unser and J. Fageot in the Biomedical Imaging Group at EPFL. A Lévy white noise can usually be defined as a random element in the space of distributions $\mathscr{D}'(\mathbb{R}^d)$, and in the particular case of Gaussian white noise, it is easy to show that this random element takes values in $\mathscr{S}'(\mathbb{R}^d)$, the space of tempered Schwartz distributions. The question of interest is whether or not the same is true for any Lévy white noise. We show that this is not the case. Indeed, the positive absolute moment condition (**PAM**) was shown to be a sufficient condition in [34], and we prove in Theorems 3.1.5 and 3.2.7 that this condition is also necessary.

In Chapter 4, we restrict to the case of a linear SPDE, but with a general differential operator and driven by a Lévy white noise. The equation we study can be written as

$$Lu = \dot{X}\,,$$

where L is some partial differential operator and $\dot{X}$ is a Lévy white noise. We study two different notions of solution. The first notion is defined as a generalized stochastic process, that is, a random distribution. The second notion is the mild solution, which is a random field defined as a stochastic integral with respect to the noise. We show in Theorems 4.2.1 and 4.2.5 that when the mild solution is locally Lebesgue integrable, then it is equal to the generalized solution. Also, in Theorems 4.3.1 and 4.3.4, we show that if the generalized solution has a random field representation, then this representation must be equal to the mild solution, and in particular the fundamental solution for the operator L must have some integrability properties. An important ingredient in the proofs of these results is a new stochastic Fubini theorem in Theorem 4.2.3. This theorem is only valid for deterministic integrands, but it only needs a pathwise standard L^1 condition (for ω fixed), compared to Theorem A.0.2, where the conditions also requires to take second moments of the integrand. We then apply these results to the stochastic heat and wave equations driven by a symmetric α-stable noise in various dimensions. In particular, we show that the linear stochastic heat equation has a random field solution if and only if $\alpha < 1 + \frac{2}{d}$, where d is the dimension (see Theorem 4.4.5), and that the linear stochastic wave equation has a random field solution if and only if the dimension d is no greater than 2.

In Chapter 5, we consider the non-linear stochastic heat equation on bounded domains $D \subset \mathbb{R}^d$ or on the whole space $\mathbb{R}^d$, driven by Lévy white noises with possibly poor moments properties. We suppose in particular that the Lévy measure ν of the noise satisfies

$$\int_{|z|\leqslant 1} |z|^p \nu(\mathrm{d}z) < +\infty \qquad \text{and} \qquad \int_{|z|>1} |z|^q \nu(\mathrm{d}z) < +\infty\,, \tag{0.0.3}$$

for some p, q such that $0 < p < 1 + \frac{2}{d}$ and $\frac{p}{1+\left(1+\frac{2}{d}-p\right)} < q \leqslant p$. The condition for $|z| \leqslant 1$ concerns the summability of the small jumps, and is related to the variation of the underlying Lévy process, and the condition for $|z| > 1$ concerns the moments of the large jumps. In particular, this includes any α-stable noise with $\alpha \in (0,2)$. In the presence of jumps, contrary to the Gaussian case (see [21, 11]), the sample paths of the solution cannot be Hölder continuous in both variables, due to the discontinuity of the heat kernel at the origin. Instead, we study the regularity of $t \mapsto u(t,\cdot)$ in a fractional Sobolev space of order $r < -\frac{d}{2}$. In Theorems 5.2.7, 5.3.12 and 5.4.6, we prove that $t \mapsto u(t,\cdot)$ has a *càdlàg* version in a fractional Sobolev space of order $r < -\frac{d}{2}$. The threshold $r < -\frac{d}{2}$ is optimal, since the solution will not even take values in a fractional Sobolev space of order $r \geqslant -\frac{d}{2}$. In the case of Lévy white noise, it is interesting to note that since the law of the location of the jumps is absolutely continuous with respect to the Lebesgue measure, under some moment conditions, the solution $u(t,\cdot) \in L^q_{\mathrm{loc}}(\mathbb{R}^d)$ at any fixed time t. We also study the regularity of $t \mapsto u(t,x)$ at fixed $x \in D$, and $x \mapsto u(t,x)$ at fixed $t \in [0,T]$. Depending on the dimension and the moment properties of the noise, we show that the behavior of these partial functions can differ widely. In particular, for a fixed time $t \in [0,T]$, if $p \leqslant \frac{2}{d}$ and $p < 2$ in (0.0.3), then the mapping $x \mapsto u(t,x)$ has a continuous modification (see Propositions 5.2.10, 5.3.13 and 5.4.7). The case of a symmetric α-stable noise shows that this bound is optimal: if $\alpha \geqslant \frac{2}{d}$, then $x \mapsto u(t,x)$ is unbounded on any non-empty open subset of D (see Propositions 5.3.15 and 5.4.8). For a fixed space-point $x \in D$, if $p < 1$ in (0.0.3), then the mapping $t \mapsto u(t,x)$ has a continuous modification (see Propositions 5.2.12, 5.3.17 and 5.4.9). Again, the bound $p < 1$ is optimal since in the case of an α-stable symmetric noise, if $\alpha \geqslant 1$, then the mapping $t \mapsto u(t,x)$ is unbounded on any non-empty open interval (see Propositions 5.2.14, 5.3.19 and 5.4.11). In the case of the equation on the whole space, we also prove in Theorem 5.3.6 that the law of the solution does not depend on the space variable.

1 Notations and main definitions

In this chapter, we introduce the main notations and definitions that will be needed throughout this manuscript. In a series of inequalities, the letter C or C' will denote a generic constant whose value may vary from one line to the other. For any integer $d \geqslant 1$, and any Borel measurable subset $A \subset \mathbb{R}^d$, we denote by $\mathrm{Leb}_d\,(A)$ its Lebesgue measure. We denote $\mathbb{N}$ the set of positive integers including 0. Also, we will use the notation $\mathbb{R}_+^d := \left\{x \in \mathbb{R}^d : x_i \geqslant 0\,,\ 1 \leqslant i \leqslant d\right\}$, and $\mathbb{R}_+^* := \{x \in \mathbb{R} : x > 0\}$. For any metric space (E, m), any $r > 0$, and any $x \in E$, we denote by $B_x(r)$ the open ball of radius r in E centered at x, relative to the metric m. In the following, S will always denote a measurable subset of $\mathbb{R}^d$. In some particular cases, we will restrict to the case where S is a disjoint union of orthants of $\mathbb{R}^d$. For any real-valued function g, we denote $g_+ = \max(g, 0)$. We say a function $f : \mathbb{R} \to E$ is *càdlàg* if for any $t_0 \in \mathbb{R}$, $\lim_{t\to t_0^-} f(t)$ exists, and if $\lim_{t\to t_0^+} f(t) = f(t_0)$ (*càdlàg* stands for the French *continue à droite, avec limite à gauche*). For any *càdlàg* function f, and any $t_0 \in \mathbb{R}$, we write $f(t_0-) := \lim_{t\to t_0^-} f(t)$, and $\Delta_{t_0} f := f(t_0) - f(t_0-)$. A generalization of the *càdlàg* property for multiparameter functions will be introduced later in Definition 1.0.7. For any function $f : \mathbb{R}^d \to \mathbb{R}$, and any set $A \subset \mathbb{R}^d$, we denote by $f|_A$ the restriction of f to the set A.

For any function $\varphi : \mathbb{R}^d \to \mathbb{R}$ sufficiently smooth, and any multi-index $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d$, we write $|\alpha| := \sum_{i=1}^d \alpha_i$, and $\varphi^{(\alpha)}(x) = \frac{\partial^{|\alpha|}\varphi}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}}(x)$. Also, for any $x \in \mathbb{R}^d$, $x^\alpha := x_1^{\alpha_1} \cdots x_d^{\alpha_d}$.

We denote by $(\Omega, \mathscr{F}, \mathbb{P})$ a complete probability space. The space of random variables with finite p-th moment is denoted $L^p(\Omega)$. For $p \geqslant 1$, these spaces are Banach spaces equipped with the norm $\|\cdot\|_{L^p(\Omega)} := (\mathbb{E}\,[|\cdot|^p])^{\frac{1}{p}}$. For $0 < p < 1$, we equip these spaces with the metric $d_p(X, Y) := \mathbb{E}\,[|X - Y|^p]$, and the resulting metric space is complete. In this case, $\|\cdot\|_{L^p(\Omega)} := \mathbb{E}\,[|\cdot|^p]$. When $p = 0$, this space is the space of all random variables, equipped with the metric of convergence in probability, and we write $\|\cdot\|_{L^0(\Omega)} := \mathbb{E}\,[|\cdot| \wedge 1]$. For a random variable X and a probability measure μ, we write $X \sim \mu$ to express the fact that under $\mathbb{P}$, the random variable X has law μ. A stochastic process is a family of random variables $(X_t)_{t\in\mathbb{R}_+}$ indexed by time. A random field is the multiparameter generalization of a stochastic process: it is a family of random variables $(X_t)_{t\in T}$ indexed by $T \subset \mathbb{R}^d$, where $d \geqslant 1$. A generalized stochastic process (or generalized random field) X is a linear map from a space of test functions $\mathscr{E}$ ($\mathscr{D}(\mathbb{R}^d)$ or $\mathscr{S}(\mathbb{R}^d)$ in this thesis, see Section 2.1.1 and Section 2.1.2) to $L^0(\Omega)$. If, in addition, this map is continuous, then by [69, Corollary 4.2], it has a version $\tilde{X}$ (i.e. for any $\varphi \in \mathscr{E}$, $\langle X, \varphi \rangle = \langle \tilde{X}, \varphi \rangle$ a.s.) such that for almost all $\omega \in \Omega$, for any sequence $\varphi_n \to \varphi$ in $\mathscr{E}$ as $n \to +\infty$, $\langle \tilde{X}, \varphi_n \rangle(\omega) \to \langle \tilde{X}, \varphi \rangle(\omega)$. Then, $\tilde{X}$ defines a random element in $\mathscr{D}'(\mathbb{R}^d)$ or $\mathscr{S}'(\mathbb{R}^d)$. In this case, $\tilde{X}$ is called a continuous generalized stochastic process, or a random distribution.

Remark 1.0.1. *There exists several modes of convergence for a sequence of random variables. Let (X_n) be a sequence of random variables.*

(i) We say X_n converges almost surely (a.s.) to X as $n \to +\infty$ if there is a set $A \in \mathscr{F}$ such that $\mathbb{P}(A) = 1$ and for all $\omega \in A$, $X_n(\omega) \to X(\omega)$ as $n \to +\infty$.

(ii) We say X_n converges in probability to X as $n \to +\infty$ if for any $\varepsilon > 0$, $\mathbb{P}(|X_n - X| \geqslant \varepsilon) \to 0$ as $n \to +\infty$.

(iii) We say X_n converges in law to X as $n \to +\infty$ if for any bounded and continuous function $g : \mathbb{R} \to \mathbb{R}$, $\mathbb{E}\left[g(X_n)\right] \to \mathbb{E}\left[g(X)\right]$ as $n \to +\infty$.

It is well known that the almost sure convergence implies the convergence in probability, and the convergence in probability implies convergence in law. If the limit X is a deterministic constant, then the convergence in law is equivalent to the convergence in probability (see [31]).

Remark 1.0.2. *We say $(X_n)_{n \geqslant 1}$ is a Cauchy sequence in law if $X_n - X_m$ converges in law to 0 as $n, m \to +\infty$. More precisely, $(X_n)_{n \geqslant 1}$ is a Cauchy sequence in law if for any bounded and continuous function g, for any $\varepsilon > 0$, there is an integer $N \in \mathbb{N}$ such that for any $n, m \geqslant N$, $|\mathbb{E}\left[g(X_n - X_m)\right] - g(0)| \leqslant \varepsilon$. The term Cauchy sequence here is an extension of the usual sense, since the convergence in law is not metrizable. Then, for any $\varepsilon > 0$, there is a function g_ε bounded and continuous such that $\mathbb{1}_{|x| \geqslant \varepsilon} \leqslant g_\varepsilon(x)$ (see Figure 1.1), and $g_\varepsilon(0) = 0$. Therefore,*

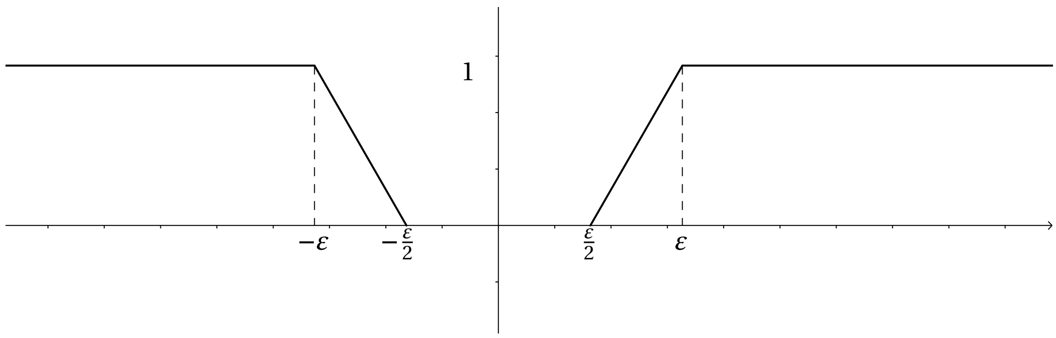

Figure 1.1 – The function g_ε.

$$\mathbb{P}(|X_n - X_m| \geqslant \varepsilon) \leqslant \mathbb{E}\left[g_\varepsilon(X_n - X_m)\right].$$

We deduce that $\mathbb{P}(|X_n - X_m| \geqslant \varepsilon) \to 0$ *as* $n, m \to +\infty$*, and* X_n *is also a Cauchy sequence in probability (which is here the usual notion of Cauchy sequence, since the convergence in probability is metrizable).*

The common theme of all the following chapters concerns Lévy processes and Lévy fields. For the convenience of the reader, we recall their definition, and introduce some of their basic properties. First, we will need the notion of Poisson random measures.

Definition 1.0.3. *Let* $(G, \mathscr{G}, \mu)$ *be a measure space, where* μ *is* σ*-finite. A Poisson random measure* M *on* G *with intensity measure* μ *is a family of random variables* $(M(A))_{A \in \mathscr{G}}$ *such that:*

(i) For any $A \in \mathscr{G}$, $M(A)$ *is a Poisson random variable with parameter* $\mu(A)$ *(with the convention that if* $\mu(A) = +\infty$*, then* $M(A) = +\infty$*).*

(ii) For any $A_1, \dots, A_n$ *disjoint subsets of* G*, the random variables* $M(A_1), \dots, M(A_n)$ *are independent.*

(iii) For any $\omega \in \Omega$*, the set function* $M(\cdot, \omega) : A \in \mathscr{G} \mapsto M(A)(\omega)$ *is a measure.*

For a Poisson random measure M *with intensity measure* μ*, we denote by* $\tilde{M} := M - \mu$ *its compensated Poisson random measure.*

It is then possible to construct an integral with respect to a Poisson random measure and with respect to a compensated Poisson random measure. For the integral of a deterministic function, we refer to the remarkably clear and detailed constructions in the lecture notes of T. Kurtz (see [49, Chapter 9]).

We can now introduce Lévy processes. They are a widely studied class of processes, and are used in many models. A Lévy measure ν is a measure on $\mathbb{R}$ such that $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}} (1 \wedge |x|^2) \, \nu(\mathrm{d}x) < +\infty$.

Definition 1.0.4. *A Lévy process* $(X_t)_{t \in \mathbb{R}_+}$ *is a real valued stochastic process such that* $X_0 = 0$ *almost surely,* X *has stationary and independent increments and* X *is stochastically continuous (that is, for any* $s \geqslant 0$, $|X_t - X_s| \to 0$ *in probability as* $t \to s$*).*

It turns out that any Lévy process has a *càdlàg* modification.

Proposition 1.0.5. *Let* X *be a Lévy process. Then* X *has a* càdlàg *version.*

Proof. See [63, Theorem 11.5]. □

In the following, whenever we introduce a Lévy process, we will always implicitly use this *càdlàg* version. For any *càdlàg* process X, we introduce the jump measure J_X by

$$J_X(A) := \sharp \{ t \geqslant 0 : \Delta X_t \neq 0, \text{and } (t, \Delta X_t) \in A \} .$$

Arguably the most important property of Lévy processes is the Lévy-Itô decomposition, and the Lévy-Khintchine that results therefrom. It gives a general result about the structure of a Lévy process, and is often the key to studying sample paths properties. In particular, it allows us to characterize the law of any Lévy process by only three parameters: a drift, a volatility and a Lévy measure. More precisely we have the following result.

Theorem 1.0.6. *Let X be a Lévy process. Then there exists a triplet (γ,σ,ν) (that we call the characteristic triplet), where $\gamma\in\mathbb{R}$, $\sigma\in\mathbb{R}_+$ and ν is a measure on $\mathbb{R}$ that does not charge $\{0\}$, such that:*

(i) For any measurable $B\subset\mathbb{R}$, $\nu(B)=\mathbb{E}\left[J_X([0,1]\times B)\right]$, and $\int_{\mathbb{R}}\left(1\wedge|z|^2\right)\nu(\mathrm{d}z)<+\infty$ (in particular, ν is a Lévy measure).

(ii) J_X is a Poisson random measure on $\mathbb{R}_+\times\mathbb{R}$ with intensity measure $Leb_1\times\nu$.

(iii) For any $t\geqslant 0$, the following decomposition holds almost surely:

$$X_t=\gamma t+\sigma W_t+\int_0^t\int_{|z|\leqslant 1} z\tilde{J}_X(\mathrm{d}s,\mathrm{d}z)+\int_0^t\int_{|z|>1} zJ_X(\mathrm{d}s,\mathrm{d}z)\,,$$

where W is a standard Brownian motion, $\tilde{J}_X:=J_X-Leb_d\times\nu$ is the compensated jump measure, and the integrals with respect to J_X and $\tilde{J}_X$ are Poisson and compensated Poisson integrals as defined in [44, Lemma 12.13]. In addition, the terms of the decomposition are independent stochastic processes.

We will also need a generalization of Lévy processes to a multiparameter framework. Essentially, we will use Lévy processes as basic mathematical objects from which to define a noise, formally defined as the distributional derivative of such a process (see Definition 2.2.7). Formally, the derivative of a sufficiently regular process defines a measure (in the case of a Lévy process, the associated noise is not quite a measure, but behaves like one in many ways). When the parameter space is one dimensional ($\mathbb{R}_+$ in the case of Lévy processes), this noise acts on a one dimensional space. Since our study includes stochastic partial differential equations (and therefore a multiparameter setting), we need to define a multiparameter Lévy field. A general presentation of this theory of *multiparameter Lévy fields* can be found in [1]; see also [26].

In the following, for any $k\in\mathbb{N}$, $\mathbf{1}_k$ (respectively $\mathbf{0}_k$, $\mathbf{2}_k$) denotes the k-dimensional vector with coordinates all equal to 1 (respectively to 0, 2). We recall that $(\Omega,\mathscr{F},\mathbb{P})$ is a complete probability space. Let $(X_t)_{t\in\mathbb{R}_+^d}$ be a d-parameter random field. For $s,t\in\mathbb{R}_+^d$ with $s=(s_1,\dots,s_d)$, $t=(t_1,\dots,t_d)$, we say that $s\leqslant t$ if $s_i\leqslant t_i$ for all $1\leqslant i\leqslant d$, and $s<t$ if $s_i<t_i$ for all $1\leqslant i\leqslant d$. For $a\leqslant b\in\mathbb{R}_+^d$, we define the box $]a,b]=\left\{t\in\mathbb{R}_+^d : a<t\leqslant b\right\}$, and the increment $\Delta_a^b X$ of X over the box $]a,b]$ by

$$\Delta_a^b X=\sum_{\varepsilon\in\{0,1\}^d}(-1)^{|\varepsilon|}X_{c_\varepsilon(a,b)}\,, \tag{1.0.1}$$

where for any $\varepsilon \in \{0,1\}^d$, we write $|\varepsilon| = \sum_{i=1}^d \varepsilon_i$ and $c_\varepsilon(a,b) \in \mathbb{R}_+^d$ is defined by $c_\varepsilon(a,b)_i = a_i \mathbb{1}_{\{\varepsilon_i=1\}} + b_i \mathbb{1}_{\{\varepsilon_i=0\}}$, for all $1 \leqslant i \leqslant d$. We can check that when $d=1$, then $\Delta_a^b X = X_b - X_a$. In fact, for all $d \geqslant 1$, $\int_{[a,b]} \varphi^{(\mathbf{1}_d)}(t)\,\mathrm{d}t = \Delta_a^b \varphi$. The next definition is a generalization of the *càdlàg* property to processes indexed by $\mathbb{R}_+^d$. We define the relations $\mathscr{R} = (\mathscr{R}_1, \dots, \mathscr{R}_d)$, where $\mathscr{R}_i$ is either $\leqslant$ or $>$, and $a \mathscr{R} b$ if and only if $a_i \mathscr{R}_i b_i$ for all $1 \leqslant i \leqslant d$.

Definition 1.0.7. *Using the terminology in [1] and [66], we say that X is* lamp *(for limit along monotone paths) if we have the following:*

(i) For all 2^d relations $\mathscr{R}$, $\lim\limits_{u \to t,\, t \mathscr{R} u} X_u$ exists.

(ii) If $\mathscr{R} = (\leqslant, \dots, \leqslant)$ then $X_t = \lim\limits_{u \to t,\, t \mathscr{R} u} X_u$.

(iii) $X_t = 0$ if $t_i = 0$ for some $1 \leqslant i \leqslant d$.

We are now ready to give the definition of a Lévy field in $\mathbb{R}_+^d$.

Definition 1.0.8. *$X = (X_t)_{t \in \mathbb{R}_+^d}$ is a d-*parameter Lévy field *if it has the following properties:*

(i) X is continuous in probability.

(ii) X is lamp almost surely.

(iii) For any sequence of disjoint boxes $]a_k, b_k]$, $1 \leqslant k \leqslant n$, the random variables $\Delta_{a_k}^{b_k} X$ are independent.

(iv) Given two boxes $]a,b]$ and $]c,d]$ in $\mathbb{R}_+^d$ such that $]a,b] + t =]c,d]$ for some $t \in \mathbb{R}^d$, the increments $\Delta_a^b X$ and $\Delta_c^d X$ are identically distributed.

The jump $\Delta_t X$ of X at time t is defined by $\Delta_t X = \lim\limits_{u \to t,\, u < t} \Delta_u^t X$.

This definition coincides with the notion of Lévy process when $d=1$. In addition, for all $t = (t_1, \dots, t_d) \in \mathbb{R}_+^d$, and for all $1 \leqslant i \leqslant d$, the process $X_\cdot^{i,t} = X_{(t_1, \dots, t_{i-1}, \cdot, t_{i+1}, \dots, t_d)}$ is a Lévy process (the notation here means that it is the process in one parameter obtained by fixing all the coordinates of t except the i-th).

The Brownian sheet is an example of such a d-parameter Lévy field. It is the analog in this framework of Brownian motion and further properties of this field are detailed in [23], [27], [47] or [69].

For all $t \in \mathbb{R}_+^d$, X_t is an infinitely divisible random variable, and by the Lévy-Khintchine formula [63, Chapter 2, Theorem 8.1], there exists real numbers γ_t, σ_t and a Lévy measure ν_t such that

$$\mathbb{E}\left(e^{iuX_t}\right) = \exp\left[iu\gamma_t - \frac{1}{2}\sigma_t^2 u^2 + \int_{\mathbb{R}} \left(e^{iuz} - 1 - iuz\mathbb{1}_{|z|\leqslant 1}\right) \nu_t(\mathrm{d}z)\right]. \tag{1.0.2}$$

The triplet $(\gamma_t, \sigma_t, \nu_t)$ is called the characteristic triplet of X_t. Since for all $1 \leqslant i \leqslant d$ and $t \in \mathbb{R}_+^d$, the process $X^{i,t}$ defined above is a Lévy process, we deduce that there exists a triplet (γ, σ, ν) where $\gamma, \sigma \in \mathbb{R}$ and ν is a Lévy measure such that $(\gamma_t, \sigma_t, \nu_t) = (\gamma, \sigma, \nu)\mathrm{Leb}_d([0,t])$. We call (γ, σ, ν) the characteristic triplet of the Lévy field X. We can now state the multidimensional analog of the Lévy-Itô decomposition, taken from [1, Theorem 4.6] particularized to the case of stationary increments (see also [26]).

Theorem 1.0.9. *Let X be a d-parameter Lévy field with characteristic triplet (γ, σ, ν). The following holds:*

(i) The jump measure J_X defined by $J_X(B) = \#\{(t, \Delta_t X) \in B\}$, for B in the Borel σ-algebra of $\mathbb{R}_+^d \times (\mathbb{R}\backslash\{0\})$, is a Poisson random measure with intensity $Leb_d \times \nu$.

(ii) For all $t \in \mathbb{R}_+^d$, we have the decomposition

$$X_t = \gamma Leb_d([0,t]) + \sigma W_t + \int_{[0,t]} \int_{|z|>1} z J_X(\mathrm{d}s, \mathrm{d}z) + \int_{[0,t]} \int_{|z|\leqslant 1} z \tilde{J}_X(\mathrm{d}s, \mathrm{d}z),$$

where W is a Brownian sheet, $\tilde{J}_X = J_X - Leb_d \times \nu$ is the compensated jump measure, and the equality holds almost surely. In addition, the terms of the decomposition are independent random fields.

2 Definitions and extensions of Lévy white noises

A noise is a mathematical object that is used in an ordinary differential equation or a partial differential equation as a source term to model some perturbations of a system. We are interested here in random perturbations of a system, that are white (in space and time when we consider an equation with a temporal component). Informally, this means that a white noise is a stochastic process with independent values at every space-time point. If it is defined, the integral process of such a noise then has to have independent increments and some regularity properties. Therefore, it makes sense to study Lévy white noises, and as we will see in Lemma 2.2.9, they can be defined as the derivatives of Lévy processes and multiparameter Lévy fields introduced respectively in Definition 1.0.4 and 1.0.8. In addition to the practical interest of such random perturbations in a context of mathematical modeling, white noises are rich mathematical objects worthy of study in their own right.

In this chapter, we introduce several pre-existing definitions of Lévy white noises, first as a probability measure on a space of distributions, and then as an independently scattered random measure, and we show that they are essentially equivalent. To that end, we present for convenience a few definitions and useful results about distribution spaces in Section 2.1, and we introduce Lévy white noises in Section 2.2. In Section 2.3, we prove that the two apparently different definitions of Lévy white noise are in fact closely related (see Theorem 2.3.5), and we prove a stochastic integral representation formula in Theorem 2.3.10.

2.1 Spaces of distributions

We will need some classical results and definitions from Laurent Schwartz's theory of distributions. For a complete presentation of this theory, we refer the reader to [64]. Distributions in the context of probability theory are also sometimes called generalized functions (see for example [36]).

2.1.1 The spaces $\mathscr{D}(\mathbb{R}^d)$ and $\mathscr{D}'(\mathbb{R}^d)$

In the following, we denote by $\mathscr{D}(\mathbb{R}^d)$ the space of C^∞ and compactly supported functions. A sequence $(\varphi_n)_{n\geqslant 0}$ of functions in $\mathscr{D}(\mathbb{R}^d)$ converges to 0 in $\mathscr{D}(\mathbb{R}^d)$ if there is a compact set $K\subset\mathbb{R}^d$ such that for all $n\in\mathbb{N}$, $\operatorname{supp}\varphi_n\subset K$, and for any multi-index $\alpha\in\mathbb{N}^d$,

$$\sup_{x\in K}|\varphi_n^{(\alpha)}(x)|\to 0, \qquad \text{as } n\to+\infty.$$

We can then define the topological dual of $\mathscr{D}(\mathbb{R}^d)$, denoted $\mathscr{D}'(\mathbb{R}^d)$, that is the space of continuous linear functionals on $\mathscr{D}(\mathbb{R}^d)$. We call it the space of distributions. In particular, a linear functional T is a distribution if and only if for each compact set $K\subset\mathbb{R}^d$, there exists a constant C and an integer p such that for any $\varphi\in\mathscr{D}(\mathbb{R}^d)$ with $\operatorname{supp}\varphi\subset K$,

$$\left|\langle T,\varphi\rangle\right|\leqslant C\max_{|\alpha|\leqslant p}\sup_{x\in K}\left|\varphi^{(\alpha)}(x)\right|.$$

In particular, it is easy to check that any locally integrable function $f:\mathbb{R}^d\to\mathbb{R}$ defines a distribution via the classical $L^2(\mathbb{R}^d)$ inner product:

$$\langle f,\varphi\rangle := \int_{\mathbb{R}^d} f(x)\varphi(x)\,\mathrm{d}x, \qquad \text{for all } \varphi\in\mathscr{D}(\mathbb{R}^d).$$

We can then define $\mathscr{C}$, the cylinder σ-algebra on this space, that is the σ-algebra generated by sets of the form

$$C=\left\{T\in\mathscr{D}'(\mathbb{R}^d):\langle T,\varphi_i\rangle\in B_i, 1\leqslant i\leqslant n\right\}, \tag{2.1.1}$$

where n is a positive integer, $(\varphi_i)_{1\leqslant i\leqslant n}$ is a family of test functions in $\mathscr{D}(\mathbb{R}^d)$, and $(B_i)_{1\leqslant i\leqslant n}$ is a collection of Borel subsets of $\mathbb{R}$. We say a sequence of distributions $(T_k)_{k\geqslant 0}$ converges to T in $\mathscr{D}'(\mathbb{R}^d)$ if for any $\varphi\in\mathscr{D}(\mathbb{R}^d)$, $\langle T_k,\varphi\rangle\to\langle T,\varphi\rangle$ as $k\to+\infty$. In other terms, we equip $\mathscr{D}'(\mathbb{R}^d)$ with the weak-$*$ topology. It is then possible to define successive derivatives of a distribution by duality, and the derivative of any order of a distribution is also a distribution.

Definition 2.1.1. *Let $T\in\mathscr{D}'(\mathbb{R}^d)$, and $k\in\mathbb{N}^d$. Then the functional $T^{(k)}$ defined by*

$$\left\langle T^{(k)},\varphi\right\rangle := (-1)^{|k|}\left\langle T,\varphi^{(k)}\right\rangle, \qquad \text{for all } \varphi\in\mathscr{D}(\mathbb{R}^d),$$

is a distribution called the derivative of order k of T.

2.1.2 The spaces $\mathscr{S}(\mathbb{R}^d)$ and $\mathscr{S}'(\mathbb{R}^d)$

The space of Schwartz's functions, also called rapidly decaying smooth functions, is denoted by $\mathscr{S}(\mathbb{R}^d)$. It is the space of smooth functions such that for any multi-indices $\alpha,\beta\in\mathbb{N}^d$,

$$\sup_{x\in\mathbb{R}^d}\left|x^\alpha\varphi^{(\beta)}(x)\right|<+\infty.$$

This explains the term "rapidly decaying", since functions in $\mathscr{S}(\mathbb{R}^d)$ and all their derivatives go to zero at infinity faster than any polynomial. We can define a family of semi-norms $\mathcal{N}_p$ on $\mathscr{S}(\mathbb{R}^d)$, where for $p \in \mathbb{N}$:

$$\mathcal{N}_p(\varphi) := \sum_{|\alpha|,|\beta| \leqslant p} \sup_{x \in \mathbb{R}^d} \left| x^\alpha \varphi^{(\beta)}(x) \right|, \qquad \text{for any } \varphi \in \mathscr{S}(\mathbb{R}^d). \tag{2.1.2}$$

This family of semi-norms defines a topology on $\mathscr{S}(\mathbb{R}^d)$, and a basis of neighborhoods of the origin for this topology is given by the family

$$\left(\left\{ \varphi \in \mathscr{S}(\mathbb{R}^d) : \mathcal{N}_p(\varphi) < \varepsilon \right\} \right)_{p \in \mathbb{N}, \varepsilon > 0},$$

since such a basis is usually given by finite intersections of sets of this form, and for all $p \in \mathbb{N}$, $\varphi \in \mathscr{S}(\mathbb{R}^d)$, $\mathcal{N}_p(\varphi) \leqslant \mathcal{N}_{p+1}(\varphi)$. Then, a sequence $(\varphi_n)_n$ converges to zero in $\mathscr{S}(\mathbb{R}^d)$ if for all $p \in \mathbb{N}$, $\mathcal{N}_p(\varphi_n) \to 0$ as $n \to +\infty$. Similar to the definition of distributions in Section 2.1.1, we can define the topological dual of $\mathscr{S}(\mathbb{R}^d)$, denoted by $\mathscr{S}'(\mathbb{R}^d)$ and called the space of tempered distributions. In other terms, a tempered distribution is a continuous linear functional on $\mathscr{S}(\mathbb{R}^d)$. Equivalently, a linear functional T on $\mathscr{S}(\mathbb{R}^d)$ is a tempered distribution of and only if there exists $C \in \mathbb{R}_+$ and $p \in \mathbb{N}$ such that for any $\varphi \in \mathscr{S}(\mathbb{R}^d)$,

$$\left| \langle T, \varphi \rangle \right| \leqslant C \mathcal{N}_p(\varphi).$$

The topology on the space of tempered distribution is the weak-$*$ topology (that is the smallest topology for which the evaluation maps $E_\varphi : T \mapsto \langle T, \varphi \rangle$ are continuous for any $\varphi \in \mathscr{S}(\mathbb{R}^d)$), and a basis for this topology is given by cylinder sets of the form

$$O = \bigcap_{i=1}^{n} \left\{ u \in \mathscr{S}'(\mathbb{R}^d) : \langle u, \varphi_i \rangle \in A_i \right\},$$

where, for all $i \leqslant n$, φ_i is an element of $\mathscr{S}(\mathbb{R}^d)$, n is an integer and A_i is an open set in $\mathbb{R}$. As in the case of classical distributions, we can take derivatives of any order of a tempered distribution (see Definition 2.1.1), and it is easily shown that $\mathscr{S}'(\mathbb{R}^d)$ is stable under differentiation.

Remark 2.1.2. *We say that a Borel function $f : \mathbb{R}^d \to \mathbb{R}$ is slowly growing if* $\sup_{t \in \mathbb{R}^d} |f(t)|(1 + |t|)^{-\alpha} < \infty$ *for some $\alpha \geqslant 0$. In this case, f defines a tempered distribution by the formula* $\langle f, \varphi \rangle = \int_{\mathbb{R}^d} f(t)\varphi(t)\,\mathrm{d}t$*, for all $\varphi \in \mathscr{S}(\mathbb{R}^d)$.*

The previous remark states that a sufficient condition for a function to define a tempered distribution is that it is slowly growing. However, as tempting as it may sound, this condition is not necessary. For example, the function $x \in \mathbb{R} \mapsto e^x \cos(e^x)$ is not slowly growing, and since it is the derivative of the bounded function $x \mapsto \sin(e^x)$, it defines a tempered distribution.

Furthermore, we have the continuous embedding $\mathscr{D}(\mathbb{R}^d) \subset \mathscr{S}(\mathbb{R}^d)$, and therefore also the continuous embedding $\mathscr{S}'(\mathbb{R}^d) \subset \mathscr{D}'(\mathbb{R}^d)$.

The space $\mathscr{S}(\mathbb{R}^d)$ is particularly convenient for Fourier analysis. We recall the definition of the

Fourier transform of a function:

Definition 2.1.3. *Let $\varphi \in \mathscr{S}(\mathbb{R}^d)$. The Fourier transform $\mathscr{F}(\varphi)$ of φ is defined by:*

$$\mathscr{F}(\varphi)(\xi) := \int_{\mathbb{R}^d} e^{-i\xi\cdot x}\varphi(x)\,\mathrm{d}x, \qquad \textit{for all } \xi \in \mathbb{R}^d.$$

It turns out that the Fourier transform is a homeomorphism of the space $\mathscr{S}(\mathbb{R}^d)$. By duality, it is then possible to define the Fourier transform of a tempered distribution.

Definition 2.1.4. *Let $T \in \mathscr{S}'(\mathbb{R}^d)$. The Fourier transform $\mathscr{F}(T)$ of T is defined by:*

$$\langle \mathscr{F}(T), \varphi\rangle := \langle T, \mathscr{F}(\varphi)\rangle, \qquad \textit{for all } \varphi \in \mathscr{S}(\mathbb{R}^d).$$

2.2 Definitions of Lévy white woise

In the theory of stochastic partial differential equations, the Gaussian white noise on $\mathbb{R}^d$, usually denoted $\dot{W}$, plays a fundamental role. It has the advantage of being very well understood, and relatively easy to work with (in general in probability theory, the Gaussian world is much nicer than its non-Gaussian counterpart). It has several equivalent definitions, but arguably the most common one is that of a centered Gaussian process indexed by $L^2(\mathbb{R}^d)$ (see for example [59, Chapter I, Definition 1.4]), with covariance given by

$$\operatorname{cov}\big(\dot{W}(f), \dot{W}(g)\big) = \langle f, g\rangle_{L^2(\mathbb{R}^d)} \qquad \text{for all } f, g \in L^2(\mathbb{R}^d).$$

It is easy to check that this is indeed a covariance function. This process is a special case of the more general class of isonormal processes on a Hilbert space (see [44, p. 251]). Also, it satisfy the following identity:

$$\mathbb{E}\Big[\big(\dot{W}(f)\big)^2\Big] = \|f\|_{L^2(\mathbb{R}^d)} \qquad \text{for all } f \in L^2(\mathbb{R}^d). \tag{2.2.1}$$

Since $\mathscr{D}(\mathbb{R}^d)$ is a subset of $L^2(\mathbb{R}^d)$, we can consider the stochastic process $\big(\dot{W}(\varphi);\ \varphi \in \mathscr{D}(\mathbb{R}^d)\big)$. This process is almost surely linear: let $\lambda \in \mathbb{R}$, and $\varphi, \psi \in \mathscr{D}(\mathbb{R}^d)$,

$$\dot{W}(\varphi + \lambda\psi) = \dot{W}(\varphi) + \lambda\dot{W}(\psi) \qquad \text{a.s.}$$

Then, let (φ_n) be a sequence of functions in $\mathscr{D}(\mathbb{R}^d)$ such that $\varphi_n \to 0$ as $n \to +\infty$ in $\mathscr{D}(\mathbb{R}^d)$. Then, by (2.2.1), and since $\mathscr{D}(\mathbb{R}^d)$ is continuously embedded in $L^2(\mathbb{R}^d)$, we deduce that $\dot{W}(\varphi_n) \to 0$ in $L^2(\Omega)$, and therefore in probability. We deduce that $\dot{W}$ is a random linear functional on $\mathscr{D}(\mathbb{R}^d)$ (see [69, p. 332]), so $\dot{W}$ defines a generalized stochastic process. This generalized stochastic process is continuous in probability, and the space $\mathscr{D}(\mathbb{R}^d)$ is nuclear (see [42, §1.5]). Therefore, we can use [69, Corollary 4.2] to deduce that there is a measurable map also denoted $\dot{W}$: $(\Omega, \mathscr{F}) \to \big(\mathscr{D}'(\mathbb{R}^d), \mathscr{C}\big)$ and for any $\varphi \in \mathscr{D}(\mathbb{R}^d)$, and almost every $\omega \in \Omega$, $\langle \dot{W}(\omega), \varphi\rangle = \dot{W}(\varphi)(\omega)$. This map is then a random variable with values in $\mathscr{D}'(\mathbb{R}^d)$, and the law of this random variable (which is then a probability measure on $\mathcal{D}'(\mathbb{R}^d)$) will be called a Gaussian white noise. With this simple example in mind, we can proceed to define more general Lévy white noises.

2.2.1 As a probability measure on $\mathcal{D}'(\mathbb{R}^d)$

A first definition of Lévy white noise comes from the theory of generalized stochastic processes. These processes are in fact random Schwartz distributions, i.e. continuous generalized stochastic processes. In [36], the authors define a Lévy white noise as a probability measure on $\left(\mathcal{D}'(\mathbb{R}^d), \mathscr{C}\right)$, where $d \geqslant 1$, $\mathcal{D}'(\mathbb{R}^d)$ is the space of distributions defined in Section 2.1.1, and $\mathscr{C}$ is the cylinder σ-algebra on this space (see (2.1.1)). The construction of such a measure is based on an infinite dimensional version of the famous Bochner's Theorem. We remind the reader that this theorem states that if a function $f : \mathbb{R} \to \mathbb{C}$ is continuous, positive definite and such that $f(0) = 1$, then it is the Fourier transform of a probability measure on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$. To introduce the generalization of this theorem, we will need the notion of nuclear spaces. We do not develop this notion here, but we refer to [67, Definition 50.1] for the interested reader. In this thesis, we will only need the following remark:

Remark 2.2.1. *The test function spaces $\mathcal{D}(\mathbb{R}^d)$ and $\mathcal{S}(\mathbb{R}^d)$ are nuclear spaces (see [67, p. 510]).*

Definition 2.2.2. *Let E be a topological vector space. A continuous function $f : E \to \mathbb{C}$ is said to be* positive definite *if for any $n \geqslant 1$, and any choice of $e_1, \dots, e_n \in E$ and $\xi_1, \dots, \xi_n \in \mathbb{C}$,*

$$\sum_{1 \leqslant i,j \leqslant n} f(e_i - e_j)\xi_i \bar{\xi_j} \geqslant 0\,.$$

Using this definition, we introduce the infinite dimensional version of Bochner's Theorem due to Minlos (see [36, Theorem 2 p.350]).

Theorem 2.2.3 (Minlos-Bochner)**.** *Let $\mathcal{N}$ be a nuclear space. If $\hat{\mu} : \mathcal{N} \to \mathbb{C}$ is continuous, positive-definite, and $\hat{\mu}(0) = 1$, then there is a unique probability measure μ on $\left(\mathcal{N}', \mathscr{C}(\mathcal{N}')\right)$, the topological dual of $\mathcal{N}$ equipped with its cylinder σ-algebra, such that*

$$\hat{\mu}(\varphi) = \int_{\mathcal{N}'} e^{i\langle T, \varphi\rangle} \mu(\mathrm{d}T)\,, \qquad \textit{for all } \varphi \in \mathcal{N}\,.$$

Using this powerful theorem, we can then proceed to build a general Lévy white noise, simply by specifying its characteristic function. In the Gaussian case, one easily shows that for any $\varphi \in \mathcal{D}(\mathbb{R}^d)$, the Fourier transform of the law of $\dot{W}$ is given by

$$\mathbb{E}\left[e^{i\langle \dot{W}, \varphi\rangle}\right] = e^{-\frac{1}{2}\|\varphi\|^2_{L^2(\mathbb{R}^d)}}\,, \qquad \text{for all } \varphi \in \mathcal{D}(\mathbb{R}^d),$$

and this Fourier transform is easily shown to satisfy the hypothesis of the Minlos-Bochner theorem. For a reader familiar with the Lévy-Khinchine representation of the law of Lévy processes (see (1.0.2)), it is therefore natural to define a Lévy white noise as a random variable

$\dot{X}$ with values in $\mathscr{D}'(\mathbb{R}^d)$ and characteristic function

$$\mathbb{E}\left[e^{i\langle \dot{X},\varphi\rangle}\right] = \exp\left[\int_{\mathbb{R}^d} \Psi(\varphi(t))\,\mathrm{d}t\right], \qquad \text{for all } \varphi \in \mathscr{D}(\mathbb{R}^d),$$

where for some σ, $\gamma \in \mathbb{R}$ and some Lévy measure ν,

$$\Psi(u) = i\gamma u - \frac{1}{2}\sigma^2 u^2 + \int_{\mathbb{R}} \left(e^{izu} - 1 - izu\mathbb{1}_{|z|\leqslant 1}\right)\nu(\mathrm{d}z). \tag{2.2.2}$$

We only need to prove that this characteristic function satisfies the hypothesis of Theorem 2.2.3. This leads to the following definition:

Definition 2.2.4. *Let* $(\Omega,\mathscr{F},\mathbb{P})$ *be a probability space. A* Lévy white noise *on* S *with characteristic triplet* (γ,σ,ν) *is a measurable mapping* $\dot{X} : (\Omega,\mathscr{F}) \to (\mathscr{D}'(\mathbb{R}^d),\mathscr{C})$ *such that for any* $\varphi \in \mathscr{D}(\mathbb{R}^d)$,

$$\mathbb{E}\left[e^{i\langle \dot{X},\varphi\rangle}\right] = \exp\left[\int_S \Psi(\varphi(t))\,\mathrm{d}t\right],$$

where Ψ *is the Lévy exponent associated to the characteristic triplet* (γ,σ,ν) *as in* (2.2.2).

To prove the existence of Lévy white noise, we need the following result,

Theorem 2.2.5 (Chapter III, Theorem 5 in [36])**.** *For any characteristic triplet* (γ,σ,ν)*, where* γ, $\sigma \in \mathbb{R}$ *and* ν *is a Lévy measure, there exists a unique probability measure* $\mu_{\dot{X}}$ *on* $(\mathscr{D}'(\mathbb{R}^d),\mathscr{C})$ *such that for any* $\varphi \in \mathscr{D}(\mathbb{R}^d)$,

$$\int_{\mathscr{D}'(\mathbb{R}^d)} e^{i\langle T,\varphi\rangle}\mu_{\dot{X}}(\mathrm{d}T) = \exp\left[\int_S \Psi(\varphi(t))\,\mathrm{d}t\right],$$

where Ψ *was defined in* (2.2.2)*. Then,* $(\Omega,\mathscr{F},\mathbb{P}) := (\mathscr{D}'(\mathbb{R}^d),\mathscr{C},\mu_{\dot{X}})$ *is a probability space, and we can define the Lévy white noise* $\dot{X}$ *on* S *with characteristic triplet* (γ,σ,ν) *via the formula*

$$\langle \dot{X}(\omega),\varphi\rangle = \langle \omega,\varphi\rangle, \qquad \textit{for all } \omega \in \Omega \textit{ and } \varphi \in \mathscr{D}(\mathbb{R}^d).$$

This construction of Lévy white noise is rather abstract. We only use the fact that a well chosen probability measure exists on a space of distributions. In the next section, we will construct a Lévy white noise directly from objects more familiar to probabilists that were introduced in Definitions 1.0.4 and 1.0.8, namely Lévy processes and Lévy fields.

Derivative of a Lévy field

We recall that a Lévy process $(X_t)_{t\in\mathbb{R}_+}$ is a real valued stochastic process such that $X_0 = 0$ almost surely, X has stationary and independent increments and X is stochastically continuous. Every Lévy process has a *càdlàg* (right continuous with left limits) modification by Proposition 1.0.5, and we will always consider such a modification in the following. An important feature of Lévy processes is the Lévy-Itô decomposition (Theorem 1.0.6): for a Lévy process X there exists a unique triplet (γ,σ,ν), where $\sigma \geqslant 0$, $\gamma \in \mathbb{R}$, and ν is a Lévy measure (in particular, ν is nonnegative and $\int_{\mathbb{R}\setminus\{0\}} \left(1 \wedge |z|^2\right) \nu(\mathrm{d}z) < +\infty$), such that the jump measure of X (denoted by J_X) is a Poisson random measure on $\mathbb{R}_+ \times \mathbb{R}\setminus\{0\}$ with intensity $\mathrm{d}t\,\nu(\mathrm{d}z)$ and X has the decomposition $X_t = \gamma t + \sigma W_t + X_t^P + X_t^M$. In this decomposition, W is a standard Brownian motion,

$$X_t^P = \int_{s\in[0,t],\,|z|>1} z J_X(\mathrm{d}s,\,\mathrm{d}z)$$

is a compound Poisson process (the term containing the large jumps of X), and

$$X_t^M = \int_{s\in[0,t],\,|z|\leqslant 1} z\,(J_X(\mathrm{d}s,\,\mathrm{d}z) - \mathrm{d}s\,\nu(\mathrm{d}z))$$

is a square integrable martingale (the term containing the small jumps of X). Since a Lévy process is *càdlàg*, it is locally Lebesgue integrable, and defines almost surely an element of $\mathscr{D}'(\mathbb{R})$ via the L^2-inner product

$$\langle X,\varphi\rangle = \int_{\mathbb{R}_+} X_t\,\varphi(t)\,\mathrm{d}t, \qquad \text{for all } \varphi \in \mathscr{D}(\mathbb{R}).$$

In particular, its derivative in the sense of distributions makes sense,

Definition 2.2.6. *Let X be a Lévy process with characteristic triplet (γ,σ,ν). The derivative of X is denoted X' and is defined by*

$$\langle X'(\omega),\varphi\rangle := -\langle X(\omega),\varphi'\rangle := -\int_{\mathbb{R}_+} X_t(\omega)\varphi'(t)\,\mathrm{d}t,$$

for all $\omega \in \Omega$ and $\varphi \in \mathscr{D}(\mathbb{R})$.

The reader might wonder what the derivative in the sense of distribution of a Lévy process might have to do with Lévy noise. The answer to this question will be provided in Lemma 2.2.9 for any Lévy process, but we can first study the example of Brownian motion. Let W be a standard Brownian motion. Then, by definition, for any $\varphi \in \mathscr{D}(\mathbb{R})$,

$$\langle W',\varphi\rangle = -\langle W,\varphi'\rangle = -\int_0^{+\infty} W_t\varphi'(t)\,\mathrm{d}t.$$

However, by Itô's formula,

$$W_t\,\varphi(t) = \int_0^t W_s\varphi'(s)\,\mathrm{d}s + \int_0^t \varphi(s)\,\mathrm{d}W_s.$$

Passing to the limit as $t \to +\infty$ (the stochastic integral is a martingale bounded in $L^2(\Omega)$), since φ has compact support, we get:

$$\int_{\mathbb{R}_+} \varphi(s)\,\mathrm{d}W_s = -\int_{\mathbb{R}_+} W_s\varphi'(s)\,\mathrm{d}s.$$

By well known results on stochastic integrals with respect to Brownian motion, the left-hand side of the previous equation is a Gaussian random variable with zero mean and variance $\|\varphi\|^2_{L^2(\mathbb{R}_+)}$. Therefore, the derivative of Brownian motion is in fact a Gaussian white noise on $\mathbb{R}_+$. More generally, we will see below that X' defines a Lévy white noise on $\mathbb{R}_+$ (see Lemma 2.2.9).

To extend this definition to any finite dimension, we use the generalization of the notion of Lévy process introduced in Definition 1.0.8, where the "time" parameter is in $\mathbb{R}^d_+$, with $d \geqslant 1$. If X is a d-parameter Lévy field, by the *lamp* property of its sample paths, it is locally integrable and defines almost surely an element of $\mathscr{D}'(\mathbb{R}^d)$ via the L^2-inner product. Similar to the one-dimensional case (see Definition 2.2.6), we can define the derivatives in the sense of distributions of this random field.

Definition 2.2.7. *Let X be a d-parameter Lévy field with characteristic triplet (γ, σ, ν). The d^{th} cross-derivative of X is denoted $X^{(\mathbf{1}_d)}$ and is defined by*

$$\left\langle X^{(\mathbf{1}_d)}, \varphi \right\rangle(\omega) := (-1)^d \left\langle X, \varphi^{(\mathbf{1}_d)} \right\rangle(\omega) := (-1)^d \int_{\mathbb{R}^d_+} X_t(\omega)\varphi^{(\mathbf{1}_d)}(t)\,\mathrm{d}t,$$

for $\omega \in \Omega$ and $\varphi \in \mathscr{D}(\mathbb{R}^d)$, where we recall that $\varphi^{(\mathbf{1}_d)} = \frac{\partial^d}{\partial t_1 \cdots \partial t_d}\varphi$.

Remark 2.2.8. *Given a d-parameter Lévy field X with characteristic triplet (γ, σ, ν) and jump measure J_X, for a suitable class of functions $\varphi : \mathbb{R}^d_+ \to \mathbb{R}$, we can define the stochastic integral*

$$\begin{aligned}\int_{\mathbb{R}^d_+} \varphi(s)\,\mathrm{d}X_s &:= \gamma \int_{\mathbb{R}^d_+} \varphi(s)\,\mathrm{d}s + \sigma \int_{\mathbb{R}^d_+} \varphi(s)\,\mathrm{d}W_s \\ &\quad + \int_{\mathbb{R}^d_+}\int_{|z|>1} z\varphi(s) J_X(\mathrm{d}s,\,\mathrm{d}z) \\ &\quad + \int_{\mathbb{R}^d_+}\int_{|z|\leqslant 1} z\varphi(s) \tilde{J}_X(\mathrm{d}s,\,\mathrm{d}z) \\ &= \gamma A_1(\varphi) + \sigma A_2(\varphi) + A_3(\varphi) + A_4(\varphi),\end{aligned} \tag{2.2.3}$$

where the first integral is a Lebesgue integral, the second integral is a Wiener integral (see [48, Chapter 2]) and the last two integrals are Poisson integrals (see [44, Lemma 12.13]) with the space $E = \mathbb{R}^d_+ \times (\mathbb{R}\backslash\{0\})$.

The next lemma relates the definition of the derivative of a d-paramteter Lévy field with the mapping $\varphi \to \int_{\mathbb{R}^d_+} \varphi(s)\,\mathrm{d}X_s$. It proves in particular that $X^{(\mathbf{1}_d)}$ is in fact a Lévy white noise on $\mathbb{R}^d_+$, as defined in Definition 2.2.4

Lemma 2.2.9. *Let X be a d-parameter Lévy field with characteristic triplet (γ, σ, ν) and jump measure J_X. Then, for all $\varphi \in \mathscr{D}(\mathbb{R}^d)$,*

$$\left\langle X^{(\mathbf{1}_d)}, \varphi \right\rangle = \int_{\mathbb{R}^d_+} \varphi(s)\,\mathrm{d}X_s.$$

Also,

$$\mathbb{E}\left[e^{i\langle X^{(\mathbf{1}_d)},\varphi\rangle}\right] = \exp\left[\int_{\mathbb{R}_+^d} \Psi(\varphi(t))\,\mathrm{d}t\right],$$

where Ψ *is the Lévy exponent associated to the characteristic triplet* (γ,σ,ν) *as in* (2.2.2)*, and* $X^{(\mathbf{1}_d)}$ *is a Lévy white noise on* $\mathbb{R}_+^d$*.*

Proof. Generically, if μ is a measure on $\mathbb{R}_+^d$ and if $x(t) := \mu([0,t])$, then $\frac{\partial^d}{\partial t_1 \cdots \partial t_d} x = \mu$ in $\mathscr{D}'(\mathbb{R}^d)$. Indeed, by (1.0.1), for any $\varphi \in \mathscr{D}(\mathbb{R}^d)$,

$$\begin{aligned}\int_{\mathbb{R}_+^d} \varphi(s)\mu(\mathrm{d}s) &= (-1)^d \int_{\mathbb{R}_+^d} \mu(\mathrm{d}s) \int_{\mathbb{R}_+^d} \mathrm{d}t\, \varphi^{(\mathbf{1}_d)}(t) \mathbb{1}_{t\geqslant s} \\ &= (-1)^d \int_{\mathbb{R}_+^d} \mathrm{d}t\, \varphi^{(\mathbf{1}_d)}(t) \int_{\mathbb{R}_+^d} \mu(\mathrm{d}s) \mathbb{1}_{t\geqslant s} \\ &= (-1)^d \int_{\mathbb{R}_+^d} \varphi^{(\mathbf{1}_d)}(t) x(t)\,\mathrm{d}t,\end{aligned} \tag{2.2.4}$$

where the second equality requires a Fubini-type theorem.

Notice that for bounded Borel sets, the set function

$$B \mapsto \tilde{X}(B) := \int_{\mathbb{R}_+^d} \mathbb{1}_B(s)\,\mathrm{d}X_s$$

defines an $L^0(\Omega,\mathscr{F},\mathbb{P})$-valued measure (see e.g. [26, Theorem 2.6]), and $X_t = \tilde{X}([0,t])$ a.s. We shall apply the argument in (2.2.4) separately to the four integrals in (2.2.3). For the first integral, the standard Fubini's theorem applies. For the second integral, since $\varphi \in L^2(\mathbb{R}^d)$, it is well defined, and since it has compact support, we use the stochastic Fubini's theorem [69, Theorem 2.6]. For the third integral, let $J_{X^P}(\mathrm{d}s,\mathrm{d}z) = \mathbb{1}_{|z|>1} J_X(\mathrm{d}s,\mathrm{d}z)$ be the jump measure of the compound Poisson part X^P of the Lévy-Itô decomposition of X. Then $J_{X^P} = \sum_{i\geqslant 1} \delta_{\tau_i}\delta_{Z_i}$, where (τ_i, Z_i) are random elements of $\mathbb{R}_+^d \times (\mathbb{R}\backslash\{0\})$, and $A_3(\varphi) = \sum_{i\geqslant 1} Z_i \varphi(\tau_i)$. For a fixed φ with compact support, this is a finite sum, so Fubini's theorem applies trivially. For the term $A_4(\varphi)$, the integral is a compensated Poisson integral, and we know that it exists (see [44, Lemma 12.13]) if and only if

$$\int_{\mathbb{R}_+^d} \int_{|z|\leqslant 1} \left(|z\varphi(s)|^2 \wedge |z\varphi(s)|\right) \mathrm{d}s\,\nu(\mathrm{d}z) < +\infty.$$

Since $\varphi \in L^2(\mathbb{R}^d)$,

$$\int_{\mathbb{R}_+^d} \int_{|z|\leqslant 1} \left(|z\varphi(s)|^2 \wedge |z\varphi(s)|\right) \mathrm{d}s\,\nu(\mathrm{d}z) \leqslant \|\varphi\|_{L^2}^2 \int_{|z|\leqslant 1} z^2 \nu(\mathrm{d}z) < +\infty.$$

For $n \in \mathbb{N}$, define

$$A_{4,n}(\varphi) := \int_{\mathbb{R}_+^d} \int_{\frac{1}{2^{n+1}} < |z| \leqslant \frac{1}{2^n}} z\varphi(t)\, \tilde{J}_X(\mathrm{d}t,\mathrm{d}z)$$

$$= \int_{\mathbb{R}_+^d} \int_{\frac{1}{2^{n+1}} < |z| \leqslant \frac{1}{2^n}} z\varphi(t)\, J_X(\mathrm{d}t, \mathrm{d}z) \\ - \int_{\mathbb{R}_+^d} \int_{\frac{1}{2^{n+1}} < |z| \leqslant \frac{1}{2^n}} z\varphi(t)\, \mathrm{d}t\, \nu(\mathrm{d}z)\,.$$

Then $A_{4,n}(\varphi)$ is a sequence of centered independent random variables (the compensated Poisson integrals are over disjoint sets) in $L^2(\Omega)$, and we know that

$$\mathbb{E}\left(\left(A_{4,n}^2(\varphi)\right)\right) = \int_{\mathbb{R}_+^d} \varphi(t)^2\, \mathrm{d}t \int_{\frac{1}{2^{n+1}} \leqslant |z| < \frac{1}{2^n}} z^2 \nu(\mathrm{d}z)\,.$$

Since ν is a Lévy measure, we see that $\sum_n \mathbb{E}\left(\left(A_{4,n}^2(\varphi)\right)\right) < \infty$ and by Kolmogorov's convergence criterion (see [31, Theorem 2.5.3]) we deduce that as $n \to +\infty$,

$$\sum_{0 \leqslant k \leqslant n} A_{4,k}(\varphi) \to \int_{\mathbb{R}_+^d} \int_{|z| \leqslant 1} z\varphi(t)\, \tilde{J}_X(\mathrm{d}t, \mathrm{d}z) = A_4(\varphi) \qquad \text{a.s.} \tag{2.2.5}$$

For each $n \in \mathbb{N}$, since the Lévy measure ν is finite on compact subsets of $\mathbb{R}_+^d \times \left[\frac{1}{2^{n+1}}, \frac{1}{2^n}\right]$, Fubini's theorem applies to the set function $B \mapsto A_{4,n}(\mathbb{1}_B)$ in the same way it did for A_3 and A_1. Therefore, letting

$$X_t^{M,n} = \int_{\mathbb{R}_+^d} \int_{\frac{1}{2^{n+1}} < |z| \leqslant \frac{1}{2^n}} z \mathbb{1}_{t \geqslant s} \tilde{J}_X(\mathrm{d}s, \mathrm{d}z)\,,$$

the argument in (2.2.4) implies that

$$A_{4,n}(\varphi) = (-1)^d \int_{\mathbb{R}_+^d} \varphi^{(\mathbf{1}_d)}(t) X_t^{M,n}\, \mathrm{d}t\,.$$

By [1, Theorem 4.6] (see also [26, Theorem 2.3]), $\sum_{0 \leqslant k \leqslant n} X_t^{M,k} \to X_t^M$, where X^M is the small jumps part of X, and the convergence is a.s., uniformly on compact subsets of $\mathbb{R}_+^d$. Since φ has compact support, (2.2.5) implies that

$$A_4(\varphi) = (-1)^d \int_{\mathbb{R}_+^d} \varphi^{(\mathbf{1}_d)}(t) X_t^M\, \mathrm{d}t\,,$$

which concludes the proof. □

We note that the law of a Lévy white noise is entirely characterized by the triplet (γ, σ, ν) (given that we use the truncation function $\mathbb{1}_{|x| \leqslant 1}$ in the Lévy-Itô decomposition). Also, $X^{(\mathbf{1}_d)}$ has support in $\mathbb{R}_+^d$. It is then easy to define a Lévy white noise on S, where S is a disjoint union of orthants by gluing independent copies of $X^{(\mathbf{1}_d)}$ on each orthant. More precisely, we have the following result,

Definition 2.2.10. *Let $S = \bigcup_{i=1}^k O_i \subset \mathbb{R}^d$ be a disjoint union of orthants of $\mathbb{R}^d$. For every $1 \leqslant i \leqslant k$, there is an isomorphism $f_i : O_i \mapsto \mathbb{R}_+^d$ (that only changes the sign of some coordinates). Then let $(X^i)_{1 \leqslant i \leqslant k}$ be a collection of independent d-parameter Lévy fields on $\mathbb{R}_+^d$ with the same characteristic triplet (γ, σ, ν). Let $X : t \in S \to \sum_{i=1}^k X_{f_i(t)}^i \mathbb{1}_{t \in O_i}$. This Lévy field defines* *a distribution (by the* lamp *property), its d^{th} cross-derivative $X^{(\mathbf{1}_d)}$ in the sense of Schwartz distributions is a* Lévy white noise on S.

Remark 2.2.11. *By Lemma 2.2.9, $X^{(\mathbf{1}_d)}$ is a Lévy white noise on S. The Lévy white noise $X^{(\mathbf{1}_d)}$ on S is almost surely a distribution on $\mathbb{R}^d$ with support in S. Each Lévy field X^i has a decomposition in the form of Theorem 1.0.9 (ii), and we can define a Brownian sheet W on S by $W : t \in S \mapsto W_t = \sum_{i=1}^k W^i_{f_i(t)} \mathbb{1}_{t\in O_i}$ and a Poisson random measure J_X on $S \times (\mathbb{R}\backslash\{0\})$ by $J_X(A \times B) = \sum_{i=1}^k J_{X^i}(f_i(A \cap O_i) \times B)$. Its intensity measure is $\mathrm{d}s\,\nu(\mathrm{d}x)$, where $\mathrm{d}s$ denotes the d-dimensional Lebesgue measure.*

2.2.2 As an independently scattered random measure

A random measure is in general a stochastic process indexed by a family of sets, with some additivity property. In the following, we introduce the notion of *independently scattered random measure*, taken from [58, p.455].

Definition 2.2.12. *Let S be a Borel-measurable subset of $\mathbb{R}^d$. An* independently scattered random measure on *S is a stochastic process $(M(A))_{A\in\mathscr{L}(S)}$ indexed by the set $\mathscr{L}(S)$ of Borel-measurable subsets of $\mathbb{R}^d$ such that $Leb_d(A \cap S) < +\infty$, such that for any sequence $(A_i)_{i\in\mathbb{N}}$ of disjoint sets in $\mathscr{L}(S)$, the random variables $M(A_i)$, $i \geqslant 0$ are mutually independent, and if in addition $\cup_{i\in\mathbb{N}} A_i \in \mathscr{L}(S)$, then*

$$M(\cup_{i\in\mathbb{N}} A_i) = \sum_{i\in\mathbb{N}} M(A_i) \qquad a.s.,$$

where the sum converges almost surely.

Specific independently scattered random measures can be built by choosing the distribution of this stochastic process. Given the title of this thesis, it is therefore natural to expect that an indefinitely divisible law (closely related to Lévy processes) is a good candidate. This was indeed done in [58, Proposition 2.1]. More precisely, we have the following result:

Proposition 2.2.13. *Let S be a Borel-measurable subset of $\mathbb{R}^d$. For any characteristic triplet (γ, σ, ν), where $\gamma, \sigma \in \mathbb{R}$ and ν is a Lévy measure, there is an independently scattered random measure M such that for any $A \in \mathscr{L}(S)$,*

$$\mathbb{E}\left[e^{iuM(A)}\right] = \exp\left[Leb_d(A \cap S)\,\Psi(u)\right], \tag{2.2.6}$$

where

$$\Psi(u) = i\gamma u - \frac{\sigma^2 u^2}{2} + \int_{\mathbb{R}} \left(e^{iuz} - 1 - iuz\mathbb{1}_{|z|\leqslant 1}\right)\nu(\mathrm{d}z).$$

Proof. See [58, Proposition 2.1, (a) and (b)] □

2.3 Unification of these definitions and stochastic integral representations

The results of this section where obtained in collaboration with Julien Fageot, and most of the material is taken from [33].

2.3.1 Extension of Lévy white noise to an independently scattered random measure

Let S be a Borel measurable subset of $\mathbb{R}^d$. In the previous sections, we defined three closely related objects: Lévy white noise, which is a probability measure on a space of distributions, the derivative of a Lévy field, and independently scattered random measures (see respectively Definition 2.2.4, Definition 2.2.7 and Definition 2.2.12). We have already seen in Lemma 2.2.9 that the derivative of a Lévy field $X^{(\mathbf{1}_d)}$ on S is in fact a particular example of a Lévy white noise on S. In the following, we show that a Lévy white noise can be extended to define an independently scattered random measure M on S such that (2.2.6) holds. To make this extension, we basically need to prove that we can give meaning to the expression $\langle \dot{X}, \mathbb{1}_A \rangle$, where $A \in \mathscr{L}(S)$. This extension will be done in several steps, using regularizations of the indicator functions. Then, we show that when the Lévy white noise $\dot{X}$ is obtained from a Lévy field (that is, $\dot{X} = X^{(\mathbf{1}_d)}$), then it has a stochastic integral representation (see Theorem 2.3.10 *(iii)*).

In the following, let $\dot{X}$ be a general Lévy white noise on S, with characteristic triplet (γ, σ, ν).

Definition 2.3.1. *Let $\theta \in \mathscr{D}(\mathbb{R}^d)$, $\theta \geqslant 0$ and $\int_{\mathbb{R}^d} \theta(t)\,\mathrm{d}t = 1$. For $n \in \mathbb{N}$, and $t \in \mathbb{R}^d$, let $\theta_n(t) = n^d \theta(nt)$. Then, for all $\varphi \in \mathscr{D}(\mathbb{R}^d)$ and all Borel measurable sets $A \subset \mathbb{R}^d$, we define*

$$\langle \dot{X}, \varphi \mathbb{1}_A \rangle := \lim_{n \to +\infty} \langle \dot{X}, \varphi \cdot (\theta_n * \mathbb{1}_A) \rangle,$$

where the limit is in probability.

Proposition 2.3.2. *Definition 2.3.1 is well posed. In particular the limit exists and does not depend on the choice of the mollifier θ. In addition, the characteristic function of the random variable $\langle \dot{X}, \varphi \mathbb{1}_A \rangle$ is given by*

$$\Phi_{\langle \dot{X}, \varphi \mathbb{1}_A \rangle}(\xi) = \mathbb{E}\left[\exp\left(i\xi \langle \dot{X}, \varphi \mathbb{1}_A \rangle\right)\right] = \exp\left(\int_{A \cap S} \Psi(\xi \varphi(t))\,\mathrm{d}t\right), \qquad \text{for all } \xi \in \mathbb{R},$$

where Ψ is the Lévy exponent of $\dot{X}$ as in (2.2.2). *Also, for any disjoint Borel measurable sets $A, B \subset \mathbb{R}^d$, $\langle \dot{X}, \varphi \mathbb{1}_{A \cup B} \rangle = \langle \dot{X}, \varphi \mathbb{1}_A \rangle + \langle \dot{X}, \varphi \mathbb{1}_B \rangle$ almost surely, and $\langle \dot{X}, \varphi \mathbb{1}_A \rangle$ and $\langle \dot{X}, \varphi \mathbb{1}_B \rangle$ are independent.*

Proof. We first remark that for all $n \in \mathbb{N}$, the function $\varphi \cdot (\theta_n * \mathbb{1}_A)$ is in $\mathscr{D}(\mathbb{R}^d)$, therefore the random variables $Y_n = \langle \dot{X}, \varphi \cdot (\theta_n * \mathbb{1}_A) \rangle$ are well defined. It suffices to show that the sequence

$(Y_n)_{n\in\mathbb{N}}$ is Cauchy in probability. Since the convergence in law to a constant is equivalent to the convergence in probability to that constant, it suffices to show that $Y_n - Y_m$ converges to zero in law as $n, m \to +\infty$. Since $\dot{X}$ is almost surely linear, for any $n, m \in \mathbb{N}$, $Y_n - Y_m = \langle \dot{X}, \varphi \cdot ((\theta_n - \theta_m) * \mathbb{1}_A)\rangle$, and by definition, we know that for any $\xi \in \mathbb{R}$,

$$\mathbb{E}\left[\exp\left(i\xi\left\langle \dot{X}, \varphi\cdot((\theta_n-\theta_m)*\mathbb{1}_A)\right\rangle\right)\right] = \exp\left(\int_S \Psi\left(\xi\varphi(t)\,((\theta_n-\theta_m)*\mathbb{1}_A)\,(t)\right)\mathrm{d}t\right),$$

where

$$\Psi(\xi) = i\gamma\xi - \frac{\sigma^2\xi^2}{2} + \int_{|z|\leqslant 1}\left(e^{i\xi z} - 1 - i\xi z\right)\nu(\mathrm{d}z) + \int_{|z|>1}\left(e^{i\xi z}-1\right)\nu(\mathrm{d}z).$$

We treat each of the four terms of the Lévy exponent separately. Since $\varphi \in \mathscr{D}(\mathbb{R}^d)$, there is a compact set K such that $\operatorname{supp}\varphi =: K$. Then,

$$\left|\int_S i\gamma\xi\varphi(t)\,((\theta_n-\theta_m)*\mathbb{1}_A)\,(t)\,\mathrm{d}t\right| \leqslant |\gamma\xi|\|\varphi\|_\infty \|(\theta_n-\theta_m)*\mathbb{1}_A\|_{L^1(K)}.$$

It is well known that for $p \geqslant 1$ and $f \in L^p(K)$, $(\theta_n - \theta_m) * f \to 0$ in $L^p(K)$ as $n, m \to +\infty$ (see [38, Théorème 1.3.14]). Therefore, since $\mathbb{1}_A \in L^1(K)$, $\|(\theta_n-\theta_m)*\mathbb{1}_A\|_{L^1(K)} \to 0$ as $n, m \to +\infty$.

Similarly,

$$\left|\int_S \frac{\sigma^2\xi^2\varphi(t)^2\,(\theta_n-\theta_m)*\mathbb{1}_A(t)^2}{2}\,\mathrm{d}t\right| \leqslant \frac{1}{2}|\sigma^2\xi^2|\|\varphi\|_\infty^2\,\|(\theta_n-\theta_m)*\mathbb{1}_A\|_{L^2(K)}^2. \tag{2.3.1}$$

Since $\mathbb{1}_A \in L^2(K)$, $\|(\theta_n-\theta_m)*\mathbb{1}_A\|_{L^2(K)}^2 \to 0$ as $n, m \to +\infty$.

For the third term, by [44, Lemma 5.14],

$$\left|e^{i\xi z} - 1 - i\xi z\right| \leqslant \frac{1}{2}|\xi z|^2,$$

and

$$\begin{aligned}\int_S\int_{|z|\leqslant 1}&\left|e^{i\xi\varphi(t)(\theta_n-\theta_m)*\mathbb{1}_A(t)z} - 1 - i\xi\varphi(t)\,((\theta_n-\theta_m)*\mathbb{1}_A)\,(t)z\right|\nu(\mathrm{d}z)\,\mathrm{d}t \\ &\leqslant \frac{1}{2}|\xi^2|\|\varphi\|_\infty^2\left(\int_{|z|\leqslant 1} z^2\nu(\mathrm{d}z)\right)\|(\theta_n-\theta_m)*\mathbb{1}_A\|_{L^2(K)}^2\,\mathrm{d}t,\end{aligned}$$

and we conclude as for (2.3.1). The last term corresponds to the compound Poisson part of the Lévy-Itô decomposition of the Lévy noise. It is the characteristic function of the random variable $M_{n,m} := \int_{\mathbb{R}^d}\int_{\mathbb{R}} z\varphi(t)\,((\theta_n-\theta_m)*\mathbb{1}_A)\,(t)N(\mathrm{d}t,\mathrm{d}z)$ where N is a Poisson random measure on $S\times\mathbb{R}$ with intensity measure $\mathrm{d}t\,\mathbb{1}_{|z|>1}\nu(\mathrm{d}z)$, and we know that

$$M_{n,m} = \sum_{i\geqslant 1} Z_i\varphi(T_i)\,((\theta_n-\theta_m)*\mathbb{1}_A)\,(T_i),$$

for some random jump points $(T_i, Z_i)_{i\geqslant 1}$, and the sum above has finitely many terms (independently of m, n) almost surely due to the compactness of the support of φ. Indeed, with

$K = \operatorname{supp} \varphi$, we have

$$\mathbb{E}[N(K \times \mathbb{R})] = \int_{K \cap S \times \mathbb{R}} \mathrm{d}t \mathbb{1}_{|z|>1} \nu(\mathrm{d}z) \leqslant \mathrm{Leb}_d(K) \int_{|z|>1} \nu(\mathrm{d}z) < +\infty,$$

since ν is a Lévy measure, and $N(K \times \mathbb{R})$ is the random variables that counts the number of points T_i that fall in the support of φ. By Lebesgue's differentiation theorem (see [70, Chapter 7, Exercise 2]), $(\theta_n - \theta_m) * \mathbb{1}_A(t) \to 0$ as $n, m \to +\infty$ for all $t \in K \backslash H$, where H is a subset of $\mathbb{R}^d$ such that $\mathrm{Leb}_d(H) = 0$. The random times T_i have an absolutely continuous law. Indeed, for any Borel set $B \subset \mathbb{R}^d$,

$$\mathbb{P}(T_i \in B) \leqslant \mathbb{P}(N(B \times \mathbb{R}) \geqslant 1) \leqslant \mathrm{Leb}_d(B \cap S) \int_{|z|>1} \nu(\mathrm{d}z). \tag{2.3.2}$$

Therefore, for all $i \geqslant 1$, $\mathbb{P}(T_i \in H) = 0$ and $M_{n,m} \to 0$ as $n, m \to +\infty$ almost surely, hence also in law. Therefore $(Y_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in law, hence in probability (see Remark 1.0.2), and there exists a random variable Y such that $Y_n \to Y$ in probability as $n \to +\infty$. By checking the convergence of each term of the decomposition of the Lévy exponent, it is easy to see using the same estimates as above that for all $\xi \in \mathbb{R}$,

$$\int_S \Psi\left(\xi\varphi(t)\theta_n * \mathbb{1}_A(t)\right) \mathrm{d}t \to \int_S \Psi\left(\xi\varphi(t)\mathbb{1}_{t \in A}\right) \mathrm{d}t = \int_{A \cap S} \Psi\left(\xi\varphi(t)\right) \mathrm{d}t, \qquad \text{as } n \to +\infty,$$

hence

$$\mathbb{E}\left[e^{i\xi Y}\right] = \exp\left(\int_{A \cap S} \Psi\left(\xi\varphi(t)\right) \mathrm{d}t\right).$$

If $\tilde{\theta}$ is an other mollifier, and $(\tilde{Y}_n)_{n \in \mathbb{N}}$ and $\tilde{Y}$ are the associated sequence and limit, it is easy to see by linearity of $\dot{X}$ that $Y_n - \tilde{Y}_n \to 0$ in probability as $n \to +\infty$. If A and B are disjoint Borel measurable sets of $\mathbb{R}^d$, we observe that $\theta_n * \mathbb{1}_{A \cup B} = \theta_n * \mathbb{1}_A + \theta_n * \mathbb{1}_B$, which proves the decomposition $\langle \dot{X}, \varphi\mathbb{1}_{A \cup B} \rangle = \langle \dot{X}, \varphi\mathbb{1}_A \rangle + \langle \dot{X}, \varphi\mathbb{1}_B \rangle$ at the limit when $n \to +\infty$. Independence comes from the factorisation of the characteristic function of these random variables. □

From the previous definition, it is straightforward to define the random variables $\langle \dot{X}, \mathbb{1}_A \rangle$ for any bounded Borel set A. Indeed, it suffices to choose any $\varphi \in \mathscr{D}(\mathbb{R}^d)$ such that $\varphi|_A = 1$ and set $\langle \dot{X}, \mathbb{1}_A \rangle := \langle \dot{X}, \varphi\mathbb{1}_A \rangle$. This definition does not depend on the choice of φ. Indeed, if φ and ψ are two such functions, by linearity of the noise and from the expression of the characteristic function we get that $\langle \dot{X}, \varphi\mathbb{1}_A \rangle - \langle \dot{X}, \psi\mathbb{1}_A \rangle = \langle \dot{X}, (\varphi - \psi)\mathbb{1}_A \rangle = 0$, almost surely. In this particular case, we observe that the characteristic function of the noise takes the particular form

$$\mathbb{E}\left[\exp\left(i\xi\langle \dot{X}, \mathbb{1}_A \rangle\right)\right] = \exp\left(\mathrm{Leb}_d(A \cap S)\Psi(\xi)\right). \tag{2.3.3}$$

We now extend the definition to a Borel set with finite measure (but not necessarily bounded).

Definition 2.3.3. *Let $A \subset \mathbb{R}^d$ be a Borel measurable set such that $\mathrm{Leb}_d(A \cap S) < +\infty$. For $n \in \mathbb{N}^*$,* *let $A_n = A \cap [-n,n]^d$. Then we define*

$$\langle \dot{X}, \mathbb{1}_A \rangle := \lim_{n \to +\infty} \langle \dot{X}, \mathbb{1}_{A_n} \rangle ,$$

where the limit is in probability.

Proposition 2.3.4. *Definition 2.3.3 is well posed, in particular the limit exists in probability. In addition, the characteristic function of the random variable $\langle \dot{X}, \mathbb{1}_A \rangle$ is given by*

$$\mathbb{E}\left[\exp\left(i\xi \langle \dot{X}, \mathbb{1}_A \rangle\right)\right] = \exp\left(Leb_d(A \cap S)\Psi(\xi)\right), \qquad \textit{for all } \xi \in \mathbb{R}, \tag{2.3.4}$$

where ψ is the Levy exponent of $\dot{X}$. Also, for any disjoint Borel measurable sets $A, B \subset \mathbb{R}^d$, such that $Leb_d(A), Leb_d(B) < +\infty$, $\langle \dot{X}, \mathbb{1}_{A\cup B} \rangle = \langle \dot{X}, \mathbb{1}_A \rangle + \langle \dot{X}, \mathbb{1}_B \rangle$ almost surely and $\langle \dot{X}, \mathbb{1}_A \rangle$ and $\langle \dot{X}, \mathbb{1}_B \rangle$ are independent.

Proof. From the expression of the characteristic function in (2.3.3) and the fact that A_n is a bounded Borel set, it is easy to see that the sequence $\left(\langle \dot{X}, \mathbb{1}_{A_n} \rangle\right)_{n\in\mathbb{N}}$ is Cauchy in law (see Remark 1.0.2), and therefore converges in probability. The expression of the characteristic function follows from the fact that $\mathrm{Leb}_d(A \cap S) < +\infty$ and an application of the dominated convergence theorem. The last statement comes directly from an application of Proposition 2.3.2 and the expression of the characteristic functions. □

Theorem 2.3.5. *The extension of the Lévy white noise $\dot{X}$ is an independently scattered random measure in the sense of Definition 2.2.12.*

Proof. Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of disjoint sets in $\mathscr{L}(S)$, the set of Borel sets with finite Lebesgue measure. Let $k \in \mathbb{N}$ and $i_1 < \cdots < i_k \in \mathbb{N}$. We show that the random variables $\left\langle \dot{X}, \mathbb{1}_{A_{i_j}} \right\rangle$, $1 \leqslant j \leqslant k$ are independent. By linearity of the noise in Proposition 2.3.4, this fact is an immediate consequence of (2.3.4) and the additivity of Lebesgue measure. This proves that $\langle \dot{X}, \mathbb{1}_{A_n} \rangle$, $n \in \mathbb{N}$ is a sequence of independent random variables. If in addition $\sum_{n\in\mathbb{N}} \mathrm{Leb}_d(A_n \cap S) < +\infty$, $\bigcup_{n\in\mathbb{N}} A_n \in \mathscr{L}(S)$ and we need to show that

$$\langle \dot{X}, \mathbb{1}_{\bigcup_{n\in\mathbb{N}} A_n} \rangle = \sum_{n\in\mathbb{N}} \langle \dot{X}, \mathbb{1}_{A_n} \rangle ,$$

where the series converges almost surely. By the second statement of Proposition 2.3.4, it is easy to see that for any $k \in \mathbb{N}$,

$$\left\langle \dot{X}, \mathbb{1}_{\bigcup_{n=1}^k A_n} \right\rangle = \sum_{n=1}^{k} \langle \dot{X}, \mathbb{1}_{A_n} \rangle . \tag{2.3.5}$$

By the expression of the characteristic function of the left-hand side of (2.3.5), we see that $\left\langle \dot{X}, \mathbb{1}_{\bigcup_{n=1}^k A_n} \right\rangle \to \langle \dot{X}, \mathbb{1}_{\bigcup_{n\in\mathbb{N}} A_n} \rangle$ in law as $k \to +\infty$. By linearity of the noise in Proposition 2.3.4, it also converges in probability as $k \to +\infty$. Therefore the right-hand side of (2.3.5) is a sum of independent random variables that converges in probability. By [17, Theorem 5.3.4], the sum converges almost surely, which concludes the proof. □

2.3.2 Stochastic integral representations

Integrals with respect to independently scattered random measures

Let $\dot{X}$ be a Lévy white noise on S extended to an independently scattered random measure. The notion of independently scattered random measure calls for the definition of an integral. The natural way to define such an integral is to define it first for a class of simple functions (which are linear combinations of indicator functions of sets in $\mathscr{L}(S)$), and then to extend it to a class of function as large as possible using a limiting argument. In [58], B. S. Rajput and J. Rosinski have done this work and entirely characterized the set of deterministic functions f that are integrable with respect to an independently scattered random measure. In the following, we recall the definition of this integral and the mentioned characterization. Let $f = \sum_{i\leqslant n} f_i \mathbb{1}_{A_i}$, where for $i \leqslant n$, $A_i \in \mathscr{L}(S)$, and let $A \in \mathscr{B}(\mathbb{R}^d)$. Then we define

$$\langle \dot{X}, f\mathbb{1}_A \rangle := \sum_{i\leqslant n} f_i \langle \dot{X}, \mathbb{1}_{A_i\cap A} \rangle .$$

The following definition was introduced in [58, p. 460].

Definition 2.3.6. *A Borel-measurable function $f : \mathbb{R}^d \to \mathbb{R}$ is said to be $\dot{X}$-integrable if there is a sequence $(f_n)_{n\geqslant 1}$ of simple functions such that $f_n \to f$ almost everywhere as $n \to +\infty$, and such that for any $A \in \mathscr{B}(\mathbb{R}^d)$, the sequence $\left(\langle \dot{X}, f_n\mathbb{1}_A \rangle\right)_{n\geqslant 1}$ converges in probability as $n \to +\infty$. Then, for any $A \in \mathscr{B}(\mathbb{R}^d)$,*

$$\langle \dot{X}, f\mathbb{1}_A \rangle := \lim_{n\to+\infty} \langle \dot{X}, f_n\mathbb{1}_A \rangle .$$

The set of $\dot{X}$-integrable functions is denoted $L(\dot{X}, S)$.

It turns out that Definition 2.3.6 is well-posed (and in particular, the limit does not depend on the choice of the approximating sequence, see [58, p. 460]). There is in fact a simple characterization of $\dot{X}$-integrable functions that we recall in the next proposition.

Proposition 2.3.7 (Theorem 2.7 in [58])**.** *Let $f : S \to \mathbb{R}$ be a Borel measurable function. Then f is $\dot{X}$-integrable if and only if the following conditions hold:*

(i) $\int_S \left| \gamma f(t) + \int_{\mathbb{R}} z f(t) \left(\mathbb{1}_{|zf(t)|\leqslant 1} - \mathbb{1}_{|z|\leqslant 1} \right) \nu(\mathrm{d}z) \right| \mathrm{d}t < +\infty$,

(ii) $\int_S |\sigma f(t)|^2 \, \mathrm{d}t < +\infty$,

(iii) $\int_{S\times\mathbb{R}} \left(|zf(t)|^2 \wedge 1 \right) \mathrm{d}t\, \nu(\mathrm{d}z) < +\infty$.

In addition, the law of $\langle \dot{X}, f \rangle$ is characterized by

$$\mathbb{E}\left[e^{i\langle \dot{X}, f\rangle} \right] = \exp\left(\int_S \Psi(f(t))\, \mathrm{d}t \right),$$

where Ψ *is the Lévy exponent of* $\dot{X}$ *as defined in* (2.2.2).

By Lemma 2.2.9, we know that a Lévy white noise derived from a Lévy field has a stochastic integral representation. In the next section, we show that the extension of such a noise to an independently scattered random measure also has such a stochastic integral representation.

Stochastic integral representations of the Lévy white noise

For the remainder of this section we suppose that S is a union of orthants. Let X be a d-parameter Lévy field on S, with characteristic triplet (γ, σ, ν), jump measure J_X, and Lévy-Itô decomposition as in Theorem 1.0.9. We know by Lemma 2.2.9 that the d-th cross-derivative of X in the sense of distributions defines a Lévy white noise on S. This Lévy white noise is denoted by $\dot{X}$, and it can be in turn extended to an independently scattered random measure that we still denote $\dot{X}$ (see Theorem 2.3.5). This extension allows us to use the general theory of independently scattered random measures developed in [58], and define the set of $\dot{X}$-integrable functions in Definition 2.3.6. For the reader familiar with stochastic integrals with respect to Poisson random measures, condition (iii) of Proposition 2.3.7 should look familiar (see for example [44, Lemma 12.13]). It turns out that the class of $\dot{X}$ integrable functions f is exactly the class of functions for which a specific stochastic integral decomposition exists for $\langle \dot{X}, f \rangle$. We also study in the following a stochastic integral representation of $\langle \dot{X}, f \rangle$ related to the Lévy-Itô decomposition of the underlying Lévy field X. In the next definition, all the integrals with respect to W are Wiener integrals, the integrals with respect to J_X and $\tilde{J}_X$ are Poisson integrals and compensated Poisson integrals as defined in [44, Lemma 12.13].

Definition 2.3.8. *(i) A Borel measurable function* $f : S \to \mathbb{R}$ *is said to be Itô* $\dot{X}$*-integrable if the following exists:*

$$I(f) := \int_S \gamma f(t)\,\mathrm{d}t + \int_S \sigma f(t)\,\mathrm{d}W_t + \int_S \int_{|z|\leqslant 1} z f(t) \tilde{J}_X(\mathrm{d}t, \mathrm{d}z) + \int_S \int_{|z|>1} z f(t) J_X(\mathrm{d}t, \mathrm{d}z).$$

The set of Itô $\dot{X}$*-integrable functions is denoted by* $L(I, S)$.

(ii) A Borel measurable function $f : S \to \mathbb{R}$ *is said to be Poisson* $\dot{X}$*-integrable if the following exists:*

$$\begin{aligned}\tilde{I}(f) := \int_S \left(\gamma f(t) + \int_{\mathbb{R}} z f(t) \left(\mathbb{1}_{|z f(t)|\leqslant 1} - \mathbb{1}_{|z|\leqslant 1} \right) \nu(\mathrm{d}z) \right) \mathrm{d}t + \int_S \sigma f(t)\,\mathrm{d}W_t \\ + \int_S \int_{|z f(t)|\leqslant 1} z f(t) \tilde{J}_X(\mathrm{d}t, \mathrm{d}z) + \int_S \int_{|z f(t)|>1} z f(t) J_X(\mathrm{d}t, \mathrm{d}z).\end{aligned} \tag{2.3.6}$$

The set of Poisson $\dot{X}$*-integrable functions is denoted by* $L\left(\tilde{I}, S\right)$.

Remark 2.3.9. *The set* $L(I, S)$ *is exactly the class of suitable functions mentioned in Remark 2.2.8, and the decomposition given by the operator* I *is coincides with the stochastic integral introduced in* (2.2.3).

We can in fact characterize the domains of the operators I and $\tilde{I}$. The following result can be found in [33, Proposition 4.7].

Theorem 2.3.10. *The following holds:*

(i) $f \in L(I,S)$ *if and only if* $\gamma f \in L^1(S)$, $\sigma f \in L^2(S)$, *and*

$$\int_S \int_{|z|>1} \left(|zf(t)| \wedge 1\right) \mathrm{d}t\, \nu(\mathrm{d}z) + \int_S \int_{|z|\leqslant 1} \left(|zf(t)|^2 \wedge |zf(t)|\right) \mathrm{d}t\, \nu(\mathrm{d}z) < +\infty. \tag{2.3.7}$$

(ii) $L(I,S) \subset L(\tilde{I},S)$, *and for all* $f \in L(I,S)$, $I(f) = \tilde{I}(f)$ *almost surely.*

(iii) $L(\tilde{I},S) = L(\dot{X},S)$ *and for all* $f \in L(\tilde{I},S)$, $\tilde{I}(f) = \langle \dot{X}, f \rangle$ *almost surely.*

Remark 2.3.11. *If* $\gamma = 0$, *then the condition* $\gamma f \in L^1(S)$ *is vacuous. Similarly, if* $\sigma = 0$, *then the condition* $\sigma f \in L^2(S)$ *is vacuous.*

Proof of Theorem 2.3.10. We first prove *(i)*. The deterministic and Wiener integral exist under the well known conditions $\gamma f \in L^1(S)$, $\sigma f \in L^2(S)$. By [44, Lemma 12.13], the compensated Poisson and Poisson integrals exist if and only if (2.3.7) is satisfied. To verify the equality $L(\dot{X},S) = L(\tilde{I},S)$ in *(iii)*, we can use the existence criteria of the different terms in (2.3.6) (see [44, Lemma 12.13] for the Poisson and compensated Poisson integrals) to see that a function f is $\dot{X}$ integrable if and only if it is Poisson $\dot{X}$-integrable. Then, let $f \in L(I,S)$. Condition *(ii)* of Proposition 2.3.7 is satisfied. Moreover, we can see that

$$\begin{aligned}\int_S \int_{|z|>1} \left(|zf(t)| \wedge 1\right) \mathrm{d}t\, \nu(\mathrm{d}z) + \int_S \int_{|z|\leqslant 1} \left(|zf(t)|^2 \wedge |zf(t)|\right) \mathrm{d}t\, \nu(\mathrm{d}z)& \\ \geqslant \int_{S\times\mathbb{R}} \left(|zf(t)|^2 \wedge 1\right) \mathrm{d}s\, \nu(\mathrm{d}z)&,\end{aligned}$$

therefore condition *(iii)* of Proposition 2.3.7 is satisfied. Then,

$$\begin{aligned}&\int_S \int_{\mathbb{R}} \left|zf(t)\right| \left|\mathbb{1}_{|zf(t)|\leqslant 1} - \mathbb{1}_{|z|\leqslant 1}\right| \mathrm{d}t\, \nu(\mathrm{d}z) \\ &= \int_S \int_{|z|>1} \left|zf(t)\right| \mathbb{1}_{|zf(t)|\leqslant 1} \mathrm{d}t\, \nu(\mathrm{d}z) + \int_S \int_{|z|\leqslant 1} \left|zf(t)\right| \mathbb{1}_{|zf(t)|>1} \mathrm{d}t\, \nu(\mathrm{d}z) \\ &\leqslant \int_S \int_{|z|>1} \left(|zf(t)| \wedge 1\right) \mathrm{d}t\, \nu(\mathrm{d}z) + \int_S \int_{|z|\leqslant 1} \left(|zf(t)|^2 \wedge |zf(t)|\right) \mathrm{d}t\, \nu(\mathrm{d}z) < +\infty,\end{aligned} \tag{2.3.8}$$

therefore *(i)* of Proposition 2.3.7 is satisfied, and the inclusion $L(I,S) \subset L(\dot{X},S) = L(\tilde{I},S)$ is satisfied. We then show that $I(f) = \tilde{I}(f)$. We can assume without loss of generality that

$\gamma = \sigma = 0$. Then,

$$\begin{aligned} I(f) &= \int_S \int_{|z|\leqslant 1} z f(t) \tilde{J}_X(\mathrm{d}t, \mathrm{d}z) + \int_S \int_{|z|>1} z f(t) J_X(\mathrm{d}t, \mathrm{d}z) \\ &= \int_S \int_{|z|\leqslant 1} z f(t) \mathbb{1}_{|zf(t)|\leqslant 1} \tilde{J}_X(\mathrm{d}t, \mathrm{d}z) + \int_S \int_{|z|\leqslant 1} z f(t) \mathbb{1}_{|zf(t)|>1} \tilde{J}_X(\mathrm{d}t, \mathrm{d}z) \\ &\quad + \int_S \int_{|z|>1} z f(t) \mathbb{1}_{|zf(t)|\leqslant 1} J_X(\mathrm{d}t, \mathrm{d}z) + \int_S \int_{|z|>1} z f(t) \mathbb{1}_{|zf(t)|>1} J_X(\mathrm{d}t, \mathrm{d}z) . \end{aligned} \tag{2.3.9}$$

By (2.3.8), we can write

$$\begin{aligned} \int_S \int_{|z|\leqslant 1} z f(t) \mathbb{1}_{|zf(t)|>1} \tilde{J}_X(\mathrm{d}t, \mathrm{d}z) = \int_S \int_{|z|\leqslant 1} & z f(t) \mathbb{1}_{|zf(t)|>1} J_X(\mathrm{d}t, \mathrm{d}z) \\ & - \int_S \int_{|z|\leqslant 1} z f(t) \mathbb{1}_{|zf(t)|>1} \, \mathrm{d}t \, \nu(\mathrm{d}z) , \end{aligned} \tag{2.3.10}$$

and

$$\begin{aligned} \int_S \int_{|z|>1} z f(t) \mathbb{1}_{|zf(t)|\leqslant 1} J_X(\mathrm{d}t, \mathrm{d}z) = \int_S \int_{|z|>1} & z f(t) \mathbb{1}_{|zf(t)|\leqslant 1} \tilde{J}_X(\mathrm{d}t, \mathrm{d}z) \\ & + \int_S \int_{|z|>1} z f(t) \mathbb{1}_{|zf(t)|\leqslant 1} \, \mathrm{d}t \, \nu(\mathrm{d}z) . \end{aligned} \tag{2.3.11}$$

Recombining (2.3.10) and (2.3.11) in (2.3.9), we get

$$\begin{aligned} I(f) = \int_S \int_{\mathbb{R}} & z f(t) \mathbb{1}_{|zf(t)|\leqslant 1} \tilde{J}_X(\mathrm{d}t, \mathrm{d}z) + \int_S \int_{\mathbb{R}} z f(t) \mathbb{1}_{|zf(t)|>1} J_X(\mathrm{d}t, \mathrm{d}z) \\ & + \int_S \int_{\mathbb{R}} z f(t) \left(\mathbb{1}_{|zf(t)|\leqslant 1} - \mathbb{1}_{|z|\leqslant 1} \right) \mathrm{d}t \, \nu(\mathrm{d}z) = \tilde{I}(f) . \end{aligned}$$

We finally show that for $f \in L(\tilde{I}, S)$, $\langle \dot{X}, f \rangle = \tilde{I}(f)$. For $f \in \mathscr{D}(\mathbb{R}^d)$, we can use Lemma 2.2.9 to deduce that $\langle \dot{X}, f \rangle = I(f)$, and since $\mathscr{D}(\mathbb{R}^d) \subset L(I)$, we get $\langle \dot{X}, f \rangle = \tilde{I}(f)$. Then let $A \subset \mathbb{R}^d$ be a Borel set such that $\mathrm{Leb}_d(A \cap S) < +\infty$. Let $(\theta_n)_{n\geqslant 1}$ be a sequence of mollifiers as in Definition 2.3.1. Since for any $n \in \mathbb{N}$, $\langle \dot{X}, f \cdot (\theta_n * \mathbb{1}_A) \rangle = \tilde{I}(f \cdot (\theta_n * \mathbb{1}_A))$, and since $\langle \dot{X}, f \cdot (\theta_n * \mathbb{1}_A) \rangle \to \langle \dot{X}, f \mathbb{1}_A \rangle$ in probability as $n \to +\infty$, it suffices to show that $\tilde{I}(f \cdot (\theta_n * \mathbb{1}_A)) \to \tilde{I}(f \mathbb{1}_A)$ in probability as $n \to +\infty$. In fact, one easily checks that $f \cdot (\theta_n * \mathbb{1}_A)$ and $f \mathbb{1}_A \in L(I, S)$. Therefore it is enough to show that $I(f \cdot (\theta_n * \mathbb{1}_A)) \to I(f \mathbb{1}_A)$ in probability as $n \to +\infty$, and this is obtained using the linearity of I and the convergence in law of each part of the decomposition of the Lévy exponent as in the proof of Proposition 2.3.2. The same reasoning works to show $\langle \dot{X}, \mathbb{1}_A \rangle = \tilde{I}(\mathbb{1}_A)$. To extend the result to simple functions, the problem we have is that each term of the decomposition of $\tilde{I}$ is not linear (although we will see that $\tilde{I}$ is linear). Let $\alpha > 0$.

Then,

$$\begin{aligned}
&\int_{S\times\mathbb{R}} z\mathbb{1}_{t\in A}\mathbb{1}_{|z|\leqslant 1}\tilde{J}_X(\mathrm{d}t,\mathrm{d}z)+\int_{S\times\mathbb{R}} z\mathbb{1}_{t\in A}\mathbb{1}_{|z|>1}J_X(\mathrm{d}t,\mathrm{d}z)\\
&\qquad=\int_{S\times\mathbb{R}} z\mathbb{1}_{t\in A}\mathbb{1}_{|z|\leqslant \alpha}\tilde{J}_X(\mathrm{d}t,\mathrm{d}z)+\int_{S\times\mathbb{R}} z\mathbb{1}_{t\in A}\mathbb{1}_{|z|>\alpha}J_X(\mathrm{d}t,\mathrm{d}z)\\
&\qquad\quad+\int_{S\times\mathbb{R}} z\mathbb{1}_{t\in A}\left(\mathbb{1}_{|z|\leqslant\alpha}-\mathbb{1}_{|z|\leqslant 1}\right)\mathrm{d}t\,\nu(\mathrm{d}z)\\
&\qquad:=I_M^{\alpha}(A)+I_P^{\alpha}(A)+D^{\alpha}(A)\,.
\end{aligned}$$

Let $f=\sum_{i=1}^n y_i\mathbb{1}_{A_i}$ be a simple function, where for all $1\leqslant i\leqslant n$, $\mathrm{Leb}_d(A_i)<+\infty$ and $|y_i|>0$. Then,

$$\begin{aligned}
\langle\dot{X},f\rangle&=\sum_{i=1}^n y_i\langle\dot{X},\mathbb{1}_{A_i}\rangle\\
&=\sum_{i=1}^n y_i\left(\int_{A_i\cap S}\gamma\,\mathrm{d}t+\int_S\sigma\mathbb{1}_{A_i}(t)\,\mathrm{d}W_t+I_M^1(A_i)+I_P^1(A_i)\right)\\
&=\int_S\gamma f(t)\,\mathrm{d}t+\int_S\sigma f(t)\,\mathrm{d}W_t+\sum_{i=1}^n y_i\left(I_M^{|y_i|^{-1}}(A_i)+I_P^{|y_i|^{-1}}(A_i)+D^{|y_i|^{-1}}(A_i)\right).
\end{aligned}$$

Then,

$$\begin{aligned}
\tilde{I}(f)&=\int_S\left(\gamma f(t)+\int_{\mathbb{R}} zf(t)\left(\mathbb{1}_{|zf(t)|\leqslant 1}-\mathbb{1}_{|z|\leqslant 1}\right)\nu(\mathrm{d}z)\right)\mathrm{d}t\\
&\quad+\int_S\sigma f(t)\,\mathrm{d}W_t+\sum_{i=1}^n y_i\left(\int_{(A_i\cap S)\times\mathbb{R}} z\mathbb{1}_{|zy_i|\leqslant 1}\tilde{J}_X(\mathrm{d}t,\mathrm{d}z)+\int_{(A_i\cap S)\times\mathbb{R}} z\mathbb{1}_{|zy_i|>1}J_X(\mathrm{d}t,\mathrm{d}z)\right)\\
&=\int_S\gamma f(t)\,\mathrm{d}t+\int_S\sigma f(t)\,\mathrm{d}W_t+\sum_{i=1}^n y_i\left(I_M^{|y_i|^{-1}}(A_i)+I_P^{|y_i|^{-1}}(A_i)+D^{|y_i|^{-1}}(A_i)\right)\\
&=\langle\dot{X},f\rangle\,.
\end{aligned}$$

Let $f\in L(\dot{X},S)$. By definition, there is a sequence of simple functions f_n such that $f_n\to f$ almost everywhere as $n\to+\infty$, and $\langle\dot{X},f_n\rangle\to\langle\dot{X},f\rangle$ in probability as $n\to+\infty$. The proof of [58, Theorem 2.7] shows that the sequence $(f_n)_{n\geqslant 1}$ can be chosen such that for any $n\in\mathbb{N}$, $|f_n|\leqslant|f|$. We only need to show that $\tilde{I}(f_n)\to\tilde{I}(f)$ in probability as $n\to+\infty$. We show the convergence in probability of each part of the decomposition of the stochastic integral. First we deal with the Gaussian part. By classical properties of Wiener stochastic integration, we get

$$\mathbb{E}\left[\left(\int_S\sigma f_n(t)\,\mathrm{d}W_t-\int_S\sigma f(t)\,\mathrm{d}W_t\right)^2\right]=\int_S\sigma^2\left(f_n(t)-f(t)\right)^2\mathrm{d}t\to 0\qquad\text{as } n\to+\infty,$$

by the dominated convergence theorem. We deduce that $\int_S\sigma f_n(t)\,\mathrm{d}W_t\to\int_S\sigma f(t)\,\mathrm{d}W_t$ in $L^2(\Omega)$ as $n\to+\infty$, which implies the convergence in probability. Then, we show the convergence of the compensated Poisson term.

$$I_1:=\int_{S\times\mathbb{R}} zf_n(t)\mathbb{1}_{|zf_n(t)|\leqslant 1}\tilde{J}_X(\mathrm{d}t,\mathrm{d}z)-\int_{S\times\mathbb{R}} zf(t)\mathbb{1}_{|zf(t)|\leqslant 1}\tilde{J}_X(\mathrm{d}t,\mathrm{d}z)$$

$$= \int_{S\times\mathbb{R}} z\big(f_n(t)-f(t)\big)\,\mathbb{1}_{|zf(t)|\leqslant 1}\tilde{J}_X(\mathrm{d}t,\mathrm{d}z) + \int_{S\times\mathbb{R}} zf_n(t)\mathbb{1}_{|zf_n(t)|\leqslant 1,|zf(t)|>1}\tilde{J}_X(\mathrm{d}t,\mathrm{d}z)\,.$$

Each of these two integrals exist since

$$\int_{S\times\mathbb{R}} \left|z\big(f_n(t)-f(t)\big)\,\mathbb{1}_{|zf(t)|\leqslant 1}\right|^2 \wedge \left|z\big(f_n(t)-f(t)\big)\,\mathbb{1}_{|zf(t)|\leqslant 1}\right| \mathrm{d}t\,\nu(\mathrm{d}z) \\ \leqslant 4\int_{S\times\mathbb{R}} \left|zf(t)\mathbb{1}_{|zf(t)|\leqslant 1}\right|^2 \mathrm{d}t\,\nu(\mathrm{d}z)\,,$$

and

$$\int_{S\times\mathbb{R}} \left|zf_n(t)\mathbb{1}_{|zf_n(t)|\leqslant 1,|zf(t)|>1}\right|^2 \wedge \left|zf_n(t)\mathbb{1}_{|zf_n(t)|\leqslant 1,|zf(t)|>1}\right| \mathrm{d}t\,\nu(\mathrm{d}z) \\ \leqslant \int_{S\times\mathbb{R}} \left|zf_n(t)\mathbb{1}_{|zf_n(t)|\leqslant 1}\right|^2 \mathrm{d}t\,\nu(\mathrm{d}z)\,,$$

and

$$\int_{S\times\mathbb{R}} \left|zf_n(t)\mathbb{1}_{|zf_n(t)|\leqslant 1}\right|^2 \mathrm{d}t\,\nu(\mathrm{d}z) < +\infty\,,$$

by *(iii)* in Proposition 2.3.7. Furthermore, since these two integrals are compensated Poisson integrals over disjoint subsets of $S\times\mathbb{R}$, they are independent and their mean is zero. Then,

$$\mathbb{E}\big(I_1^2\big) = \int_{S\times\mathbb{R}} |z(f_n(t)-f(t)|^2\mathbb{1}_{|zf(t)|\leqslant 1}\,\mathrm{d}t\,\nu(\mathrm{d}z) \\ + \int_{S\times\mathbb{R}} |zf_n(t)|^2\mathbb{1}_{|zf_n(t)|\leqslant 1,|zf(t)|>1}\,\mathrm{d}t\,\nu(\mathrm{d}z)\,. \tag{2.3.12}$$

Observe that $|z(f_n(t)-f(t)|^2\mathbb{1}_{|zf(t)|\leqslant 1} \leqslant 4|zf(t)|^2\mathbb{1}_{|zf(t)\leqslant 1}$, and by Proposition 2.3.7 *(iii)*,

$$\int_{S\times\mathbb{R}} |zf(t)|^2\mathbb{1}_{|zf(t)\leqslant 1}\,\mathrm{d}t\,\nu(\mathrm{d}z) < +\infty\,.$$

Therefore, by the dominated convergence theorem, we get the convergence to zero of the first integral on the right-hand side of (2.3.12). Similarly,

$$|zf_n(t)|^2\mathbb{1}_{|zf_n(t)|\leqslant 1,|zf(t)|>1} \leqslant \mathbb{1}_{|zf(t)|>1}\,,$$

so again by dominated convergence, the second integral on the right-hand side of (2.3.12) converges to zero. We deduce that

$$\int_{S\times\mathbb{R}} zf_n(t)\mathbb{1}_{|zf_n(t)|\leqslant 1}\tilde{J}_X(\mathrm{d}t,\mathrm{d}z) \to \int_{S\times\mathbb{R}} zf(t)\mathbb{1}_{|zf(t)|\leqslant 1}\tilde{J}_X(\mathrm{d}t,\mathrm{d}z)\,,$$

in $L^2(\Omega)$ as $n\to+\infty$, which implies the convergence in probability. The treatment of the compound Poisson term goes as follows: let $(T_i,Z_i)\in S\times\mathbb{R}$ be the random jump points of the random measure J_X. Then,

$$\int_{S\times\mathbb{R}} zf_n(t)\mathbb{1}_{|zf_n(t)|>1}J_X(\mathrm{d}t,\mathrm{d}z) = \sum_{i\geqslant 1} Z_i f_n(T_i)\mathbb{1}_{|Z_i f_n(T_i)|>1}\,,$$

where the sum is finite almost surely. Indeed, $1_{|zf_n(t)|>1} \leqslant 1_{|zf(t)|>1}$, therefore $J_X(\{|zf_n(t)| > 1\}) \leqslant J_X(\{|zf(t)|>1\}) =: N$, and N is an almost surely finite random variable, and does not depends on n: indeed, $\mathbb{E}[N] = \int_{S\times\mathbb{R}} 1_{|zf(s)|>1}\,\mathrm{d}s\,\nu(\mathrm{d}z) < +\infty$ since $f \in L\big(\dot{X}, S\big)$ and using Proposition 2.3.7 *(iii)*. Therefore, since $f_n \to f$ almost everywhere, and since the law of T_i is absolutely continuous with respect to Lebesgue measure (see (2.3.2)), we deduce that

$$\sum_{i\geqslant 1} Z_i f_n(T_i)\mathbb{1}_{|Z_i f_n(T_i)|>1} \to \sum_{i\geqslant 1} Z_i f(T_i)\mathbb{1}_{|Z_i f(T_i)|>1} = \int_{S\times\mathbb{R}} zf(t)\mathbb{1}_{|zf(t)|>1} J_X(\mathrm{d}t,\mathrm{d}z),$$

almost surely as $n \to +\infty$, which implies the convergence in probability. We then have the following:

$$\big\langle \dot{X}, f_n \big\rangle = \tilde{I}(f_n) =: U(f_n) + \tilde{I}_W(f_n) + \tilde{I}_M(f_n) + \tilde{I}_P(f_n) \qquad \text{a.s.}$$

Also, we proved that $\tilde{I}_W(f_n)+\tilde{I}_M(f_n)+\tilde{I}_P(f_n) \to \tilde{I}_W(f)+\tilde{I}_M(f)+\tilde{I}_P(f)$ in probability as $n \to +\infty$. Also, $\big\langle \dot{X}, f_n \big\rangle$ converges in probability to $\big\langle \dot{X}, f \big\rangle$ as $n \to +\infty$, hence also in law. From these facts, we deduce that the deterministic part of the decomposition $U(f_n)$ converges as $n \to +\infty$, and from the expression of the characteristic function,

$$U(f_n) \to U(f) := \int_S \left(\gamma f(t) + \left(\int_{\mathbb{R}} zf(t)\left(\mathbb{1}_{|zf(t)|\leqslant 1} - \mathbb{1}_{|z|\leqslant 1}\right)\nu(\mathrm{d}z)\right)\right)\mathrm{d}t \qquad \text{as } n \to +\infty.$$

This concludes the proof. □

In general the inclusion $L(I,S) \subset L(\dot{X},S)$ is strict. For example, we can consider the case of an α-stable white noise $\dot{W}^\alpha$ on $\mathbb{R}^d$, $\alpha \in (0,2)$, that is a Lévy white noise on $\mathbb{R}^d$ with characteristic triplet $(0,0,\nu_\alpha(\mathrm{d}x))$, where $\nu_\alpha(\mathrm{d}z) = \frac{\mathrm{d}z}{|z|^{\alpha+1}}$. By Theorem 2.3.10, a function $f : \mathbb{R}^d \to \mathbb{R}$ is Itô $\dot{X}$-integrable if and only if (2.3.7) holds. Then, if $\alpha \neq 1$,

$$\begin{aligned}
\int_{|z|>1} \big(|zf(t)| \wedge 1\big)\nu(\mathrm{d}z) &= 2\int_0^{+\infty} |zf(t)|\mathbb{1}_{1<z\leqslant |f(t)|^{-1}}\mathbb{1}_{|f(t)|<1}\frac{\mathrm{d}z}{z^{\alpha+1}} \\
&\quad + 2\int_0^{+\infty} \mathbb{1}_{z>|f(t)|^{-1}}\mathbb{1}_{|f(t)|<1}\frac{\mathrm{d}z}{z^{\alpha+1}} \\
&\quad + 2\int_0^{+\infty} \mathbb{1}_{1<z}\mathbb{1}_{|f(t)|\geqslant 1}\frac{\mathrm{d}z}{z^{\alpha+1}} \\
&= \mathbb{1}_{|f(t)|<1}\frac{2\big(|f(t)|-|f(t)|^\alpha\big)}{\alpha-1} + \frac{2}{\alpha}\left(\mathbb{1}_{|f(t)|<1}|f(t)|^\alpha + \mathbb{1}_{|f(t)|\geqslant 1}\right).
\end{aligned}$$

Similarly,

$$\begin{aligned}
\int_{|z|\leqslant 1} \big(|zf(t)|^2 \wedge |zf(t)|\big)\nu(\mathrm{d}z) &= 2\int_0^{+\infty} |zf(t)|^2\mathbb{1}_{z\leqslant 1}\mathbb{1}_{|f(t)|<1}\frac{\mathrm{d}z}{z^{\alpha+1}} \\
&\quad + 2\int_0^{+\infty} |zf(t)|\mathbb{1}_{|f(t)|^{-1}<z\leqslant 1}\mathbb{1}_{|f(t)|\geqslant 1}\frac{\mathrm{d}z}{z^{\alpha+1}} \\
&\quad + 2\int_0^{+\infty} |zf(t)|^2\mathbb{1}_{z\leqslant |f(t)|^{-1}}\mathbb{1}_{|f(t)|\geqslant 1}\frac{\mathrm{d}z}{z^{\alpha+1}} \\
&= \frac{2}{2-\alpha}|f(t)|^2\mathbb{1}_{|f(t)|<1} + \frac{2}{\alpha-1}\big(|f(t)|^\alpha - |f(t)|\big)\mathbb{1}_{|f(t)|\geqslant 1}
\end{aligned}$$

$$+\frac{2}{2-\alpha}|f(t)|^\alpha \mathbb{1}_{|f(t)|\geqslant 1}$$

Summing those two terms which are always positive, and since $\alpha \in (0,2)$, we get that $f \in L(I,\mathbb{R}^d)$ if and only if

$$\int_{\mathbb{R}^d} \left||f(t)|^\alpha - |f(t)|\right| \mathrm{d}t + \int_{\mathbb{R}^d} |f(t)|^\alpha \,\mathrm{d}t + \mathrm{Leb}_d\left(\{t : |f(t)| \geqslant 1\}\right) < +\infty.$$

Also, by using Markov's inequality on the last term, we get that $\mathrm{Leb}_d\left(\{t : |f(t)| \geqslant 1\}\right) \leqslant \|f\|^\alpha_{L^\alpha(\mathbb{R}^d)}$. When $\alpha = 1$, we obtain

$$\int_{|z|>1} \left(|zf(t)| \wedge 1\right) \nu(\mathrm{d}z) = -\mathbb{1}_{|f(t)|<1}|f(t)|\ln\left(|f(t)|\right) + 2\left(\mathbb{1}_{|f(t)|<1}|f(t)| + \mathbb{1}_{|f(t)|\geqslant 1}\right).$$

and

$$\begin{aligned}\int_{|z|\leqslant 1} \left(|zf(t)|^2 \wedge |zf(t)|\right) \nu(\mathrm{d}z) = \frac{2}{2-\alpha}|f(t)|^2\mathbb{1}_{|f(t)|<1} + |f(t)|\ln\left(|f(t)|\right)\mathbb{1}_{|f(t)|\geqslant 1} \\ + 2|f(t)|\mathbb{1}_{|f(t)|\geqslant 1}.\end{aligned}$$

Finally, when we obtain that $f \in L(I,\mathbb{R}^d)$ if and only if

$$\int_{\mathbb{R}^d} \left||f(t)|^\alpha - |f(t)|\right| \mathrm{d}t + \int_{\mathbb{R}^d} |f(t)|^\alpha \,\mathrm{d}t < +\infty \qquad \text{when } \alpha \neq 1,$$

and

$$\int_{\mathbb{R}^d} \left|f(t)\ln\left(|f(t)|\right)\right| \mathrm{d}t + \int_{\mathbb{R}^d} |f(t)|^\alpha \,\mathrm{d}t < +\infty \qquad \text{when } \alpha = 1,$$

By Theorem 2.3.10 and Proposition 2.3.7, a function $f : \mathbb{R}^d \to \mathbb{R}$ is Poisson $\dot{X}$-integrable if and only if

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}} \left(|zf(t)|^2 \wedge 1\right) \mathrm{d}t\, \nu(\mathrm{d}z) < +\infty.$$

Then,

$$\begin{aligned}\int_{\mathbb{R}} \left(|zf(t)|^2 \wedge 1\right) \nu(\mathrm{d}z) &= 2\int_0^{+\infty} |zf(t)|^2\mathbb{1}_{z\leqslant |f(t)|^{-1}} \frac{\mathrm{d}z}{z^{\alpha+1}} + 2\int_0^{+\infty} \mathbb{1}_{|f(t)|^{-1}<z} \frac{\mathrm{d}z}{z^{\alpha+1}} \\ &= \frac{2}{2-\alpha}|f(t)|^\alpha + \frac{2}{\alpha}|f(t)|^\alpha.\end{aligned}$$

Therefore, we have that $L(\dot{W}^\alpha, \mathbb{R}^d) = L^\alpha(\mathbb{R}^d)$. Then, if we consider for example $\alpha \in (0,1)$ and $f : t \mapsto \frac{1}{t}\mathbb{1}_{t\in(0,1)}$, then f is Poisson $\dot{X}$-integrable but f is not Itô $\dot{X}$-integrable.

3 Lévy white noise as a tempered distribution

We have studied in the previous chapter several definitions of Lévy white noise. One of the points of view that we developed is that of a probability measure on a space of distributions, or equivalently a random element in $\mathscr{D}'(\mathbb{R}^d)$. In the context of SPDEs, one would like to apply various operators to this noise, for example a fractional laplacian (whose definition uses Fourier transforms). However, this space of generalized functions is too large, and some of its elements behave too wildly for Fourier analysis. In the case of Gaussian white noise, it is already known that it belongs to the nicer space of tempered distributions $\mathscr{S}'(\mathbb{R}^d)$ (see [36, 69]). The question that is of interest then is whether the same is true for all Lévy white noises. This question was brought to our attention by J. Fageot and M. Unser from the Biomedical Imaging Group at EPFL, and was partially answered in [34]. In this article, the authors gave a sufficient condition for a Lévy white noise to belong to $\mathscr{S}'(\mathbb{R}^d)$, and we show in Theorems 3.1.5 and 3.2.7 that this condition is also necessary. The results exposed in this chapter have been published in [24] and this chapter is based on this article.

This chapter is organized as follows: In Section 3.1, we treat the one dimensional case by dealing with each term of the Lévy-Itô decomposition of the noise separately. In particular, we show that only the compound Poisson part is susceptible not to belong to $\mathscr{S}'(\mathbb{R}^d)$. The generalization to higher dimensions $d \geqslant 1$ is treated in Section 3.2, and the main result is stated in Theorem 3.2.7.

3.1 Lévy processes and Lévy white noise in $\mathscr{S}'(\mathbb{R})$

In this section, we restrict to the one dimensional case, because it is conceptually simpler, and the notations are lighter. The multidimensional case treated in the next section is not fundamentally different, but some new ideas have to be introduced. Lévy processes have been introduced in Definition 1.0.4. We want to study if a Lévy process (and the noise derived from it) define a tempered distribution. Since every Lévy process has a *càdlàg* (right continuous with left limits) modification by Proposition 1.0.5, and we will always consider such a modification in the following, these processes define a class of locally Lebesgue integrable processes, and therefore define a classical distribution, that is, for any Lévy process X, the linear functional defined by the L^2-inner product

$$\varphi \mapsto \langle X, \varphi \rangle := \int_{\mathbb{R}_+} X_t \varphi(t)\,\mathrm{d}t, \qquad \varphi \in \mathscr{D}(\mathbb{R}),$$

is an element of $\mathscr{D}'(\mathbb{R})$ (see the discussion before Definition 2.2.6). A tempered distribution is a distribution with somewhat nicer integrability properties. They are sometimes viewed as distributions with sub-exponential growth, but this view is not quite accurate. Indeed, we have already seen that the function $x \mapsto e^x \cos(e^x)$ defines a tempered distribution, and has exponential growth at infinity, as it was explained after Remark 2.1.2. However, it is the very fast oscillatory behavior of this function at infinity that compensates for the growth at infinity.

For instance, the function $x \mapsto e^x$ also exhibits exponential growth at infinity, but does not define a tempered distribution. The proof of this fact is quite simple, but the idea behind it will be key to the study of Lévy processes as tempered distributions. Suppose that the function $x \mapsto e^x$ defines a tempered distribution. By definition, there exists an integer p and a constant C such that for any $\varphi \in \mathscr{S}(\mathbb{R})$, $\left|\langle \exp(\cdot), \varphi \rangle\right| \leqslant C\mathscr{N}_p(\varphi)$ (the definition of the family of semi-norms $\mathscr{N}_p$ was given in (2.1.2)). Then, let $\theta \in \mathscr{D}(\mathbb{R})$, $\theta \geqslant 0$, and for any $n \in \mathbb{N}$, and $x \in \mathbb{R}$, we define $\theta_n(x) := \theta(x-n)$. We can easily estimate that for any $p \in \mathbb{N}$, $\mathscr{N}_p(\theta_n) \leqslant C' n^p$ for some $C' > 0$. Also, we have

$$\left|\langle \exp(\cdot), \theta_n \rangle\right| = \left|\int_{\mathbb{R}} e^x \theta_n(x)\,\mathrm{d}x\right| = \left|\int_{\mathbb{R}} e^x \theta(x-n)\,\mathrm{d}x\right| = e^n \int_{\mathbb{R}} e^y \theta(y)\,\mathrm{d}y.$$

Therefore, if $x \mapsto e^x$ is a tempered distribution, we must have that $e^n \leqslant Cn^p$, for some $C \geqslant 0$ and $p \in \mathbb{N}$, which is absurd, and therefore the exponential function is not a tempered distribution.

An important feature of Lévy processes is the Lévy-Itô decomposition of Theorem 1.0.6. We will use this decomposition to treat separately each term, and we will see that the only potential obstacle for X to define a tempered distribution is in its compound Poisson part X^P. We will also see that the growth at infinity of a Lévy process is closely related to the existence of absolute moments.

For any *càdlàg* process L, we define the following subset of Ω:

$$\Omega_L = \left\{\omega \in \Omega : L(\omega) \in \mathscr{S}'(\mathbb{R})\right\}, \tag{3.1.1}$$

with the understanding that when $L(\omega) \in \mathscr{S}'(\mathbb{R})$, the continuous linear functional associated with $L(\omega)$ is given by $\langle L(\omega), \varphi \rangle = \int_{\mathbb{R}_+} L_t(\omega)\varphi(t)\,\mathrm{d}t$, for all $\varphi \in \mathscr{S}(\mathbb{R})$. Similarly, we can introduce the set

$$\Omega_{L'} = \left\{\omega \in \Omega : L'(\omega) \in \mathscr{S}'(\mathbb{R})\right\}, \tag{3.1.2}$$

where L' is the derivative in the sense of distributions of L. Since the derivative of a tempered distribution is also a tempered distribution, we obviously have $\Omega_L \subset \Omega_{L'}$. The converse inclusion is also true as we will see in Theorem 3.1.5. We can therefore use the properties of Lévy processes to study whether or not Lévy white noises are tempered distributions.

3.1.1 The case of a Lévy process with integer moments

Three terms in the Lévy-Itô decomposition have absolute moments of any order, and this leads to the following proposition. We recall the notion of slowly growing function introduced in Remark 2.1.2.

Proposition 3.1.1. *Let X be a Lévy process with characteristic triplet (γ, σ, ν) and Lévy-Itô decomposition $X_t = \gamma t + \sigma W_t + X_t^P + X_t^M$. Let $Y_t = \gamma t + \sigma W_t + X_t^M$. Then Y is slowly growing a.s., and the set Ω_Y defined as in* (3.1.1) *(with L there replaced by Y) has probability one.*

Proof. The process Y is a sum of a linear deterministic function, and of two independent square integrable Lévy processes. In particular, $\mathbb{E}(|Y_1|) < +\infty$. By the strong law of large numbers for Lévy processes in [63, Theorem 36.5], $t^{-1} Y_t \to \mathbb{E}(Y_1) = \gamma$ almost surely as $t \to +\infty$. It follows that Y is sublinear and locally bounded (by the *càdlàg* property) almost surely, so it is slowly growing. We deduce that Y is a tempered distribution almost surely by Remark 2.1.2. □

Proposition 3.1.1 tells us that the only obstacle to X defining a tempered distribution is in the compound Poisson part of the Lévy-Itô decomposition. In the next section, we study the growth at infinity of a compound Poisson process.

3.1.2 Growth of a compound Poisson process

In view of Corollary 3.1.1, it remains to determine when a compound Poisson process belongs to $\mathcal{S}'(\mathbb{R})$. We begin with two key results on the growth of a compound Poisson process. Let $X_t = \sum_{i=1}^{N_t} Z_i$ be a compound Poisson process, where N is a Poisson process with parameter λ that is independent of the sequence $(Z_i)_{i\geqslant 1}$ of i.i.d. random variables. Let $S_0 = 0$ and $(S_n)_{n\geqslant 1}$ be the sequence of jump times of X and let $T_n = S_n - S_{n-1}$. Also, let $Y_n = X_{S_n} = \sum_{i=1}^{n} Z_i$. We first show that on the set Ω_X, the compound Poisson process is slowly growing.

Proposition 3.1.2. *Let X be the compound Poisson process defined above and Ω_X the set defined in* (3.1.1). *There is a set A of probability one such that for all $\omega \in \Omega_X \cap A$, the function $t \mapsto X_t(\omega)$ is slowly growing.*

Remark 3.1.3. *We point out that this result relies on more than the piecewise constancy of a compound Poisson process. Indeed, there exist* càdlàg *piecewise constant functions in $\mathcal{S}'(\mathbb{R})$ which are not slowly growing. For example consider the function f that is equal to zero except on intervals of the form $[n, n+2^{-n}[$ where it is constant equal to $2^{\frac{n}{2}}$ for all $n \in \mathbb{N}$. Then $f \in L^1(\mathbb{R}) \subset \mathcal{S}'(\mathbb{R})$, but f is clearly not slowly growing.*

Proof of Proposition 3.1.2. The main idea is the following. Since X is constant on the interval $[S_n, S_{n+1}[$ and the jump times are rarely close together, we can build a sequence of random test functions φ_n supported just to the right of S_n, with a shrinking support, and such that $\langle X, \varphi_n \rangle = X_{S_n}$ for large enough n. The control of $|\langle X, \varphi_n \rangle|$ by a norm $\mathcal{N}_p(\varphi_n)$ leads to a bound on X_{S_n}, and then on X_t since X is piecewise constant.

For $n \geqslant 1$, the jump time S_n has Gamma distribution with parameters n and λ. For $k \geqslant 1$ to be chosen later, and $\varphi \in \mathscr{D}(\mathbb{R})$ with support in $[0,1]$, $\varphi \geqslant 0$ and $\int_{\mathbb{R}} \varphi = 1$, we consider the sequence φ_n defined by

$$\varphi_n(t) = S_n^k \varphi\left((t - S_n)\, S_n^k\right), \tag{3.1.3}$$

(see the illustration in Figure 3.1). Then

$$\operatorname{supp}(\varphi_n) \subset \left[S_n, S_n + \frac{1}{S_n^k}\right], \tag{3.1.4}$$

and $\int_{\mathbb{R}} \varphi_n = 1$. Furthermore, for any nonnegative integers p and $\alpha, \beta \leqslant p$,

$$\begin{aligned}
\sup_{t\in\mathbb{R}} \left| t^\alpha \varphi_n^{(\beta)}(t) \right| &= \sup_{t \in \left[S_n, S_n + \frac{1}{S_n^k}\right]} \left| t^\alpha \varphi_n^{(\beta)}(t) \right| \\
&\leqslant \left(S_n + \frac{1}{S_n^k}\right)^\alpha S_n^{k(\beta+1)} \sup_{t\in\mathbb{R}} \left| \varphi^{(\beta)}(t) \right|,
\end{aligned}$$

hence,

$$\mathcal{N}_p(\varphi_n)\, \mathbb{1}_{S_n \geqslant 1} \leqslant C \mathcal{N}_p(\varphi) S_n^{(p+1)k+p} \mathbb{1}_{S_n \geqslant 1}, \tag{3.1.5}$$

where $C \in \mathbb{R}$ is deterministic, nonnegative, and depends only on p. We define the events

$$A_{n,k} = \left\{ X \text{ does not jump in the interval } \left]S_n, S_n + \frac{1}{S_n^k}\right[\right\}. \tag{3.1.6}$$

Using the fact that T_{n+1} has exponential distribution with parameter λ and that T_{n+1} and S_n are independent, we have

$$\begin{aligned}
\mathbb{P}(A_{n,k}^c) &= \mathbb{P}\left\{ N_{S_n + \frac{1}{S_n^k}} - N_{S_n} \geqslant 1 \right\} \\
&= \mathbb{P}\left\{ T_{n+1} < \frac{1}{S_n^k} \right\} = \mathbb{E}\left(1 - e^{-\frac{\lambda}{S_n^k}}\right) \leqslant \mathbb{E}\left(\frac{\lambda}{S_n^k}\right).
\end{aligned}$$

The Laplace transform of S_n is $\mathbb{E}\left(e^{-tS_n}\right) = \lambda^n (t+\lambda)^{-n}$, for $t \geqslant 0$. For $n \geqslant 3$, integrating twice from t to $+\infty$, we obtain

$$\mathbb{E}\left(\frac{1}{S_n^2}\right) = \frac{\lambda^2}{(n-1)(n-2)}, \qquad n \geqslant 3. \tag{3.1.7}$$

We deduce that $\sum_n \mathbb{E}\left(S_n^{-2}\right) < +\infty$. Therefore, $\sum_n \mathbb{P}\left(A_{n,2}^c\right) < +\infty$ and by the Borel-Cantelli

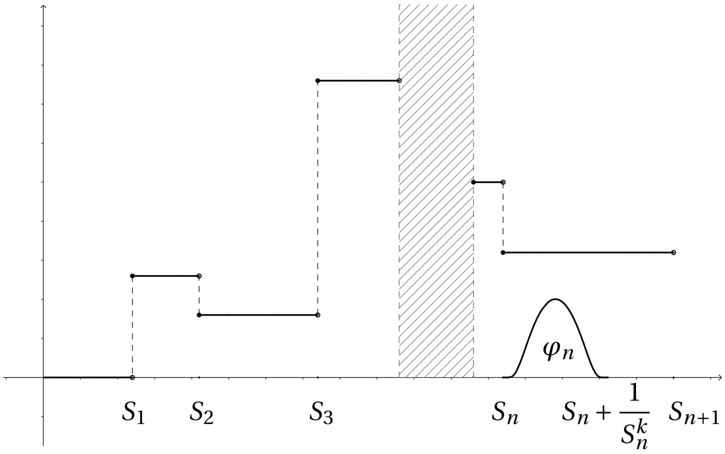

Figure 3.1 – Example of the function φ_n

Lemma,

$$\mathbb{P}\left(\limsup_{n\to+\infty} A_{n,2}^c\right) = 0\,,$$

and the set $A = \liminf_{n\to+\infty} A_{n,2}$ has probability one. Let $\omega \in A \cap \Omega_X$, and $N(\omega)$ be such that for all $n \geqslant N(\omega)$, $\omega \in A_{n,2}$. Then for $n \geqslant N(\omega)$, because of (3.1.4) and (3.1.6),

$$\langle X, \varphi_n\rangle(\omega) = X_{S_n}(\omega)\mathbb{1}_{A_{n,2}}(\omega) + \left\langle X, \varphi_n\right\rangle(\omega)\mathbb{1}_{A_{n,2}^c}(\omega) = X_{S_n}(\omega)\,. \tag{3.1.8}$$

Since $X(\omega)$ is a tempered distribution by definition of Ω_X, there is $p(\omega) \in \mathbb{N}$ and $C(\omega) \in \mathbb{R}$ such that

$$\begin{aligned}\left|\langle X, \varphi_n\rangle(\omega)\right|\mathbb{1}_{S_n(\omega)\geqslant 1} &\leqslant C(\omega)\mathcal{N}_{p(\omega)}\left(\varphi_n\right)\mathbb{1}_{S_n(\omega)\geqslant 1}\\ &\leqslant C'(\omega)S_n^{3p(\omega)+2}(\omega)\mathbb{1}_{S_n(\omega)\geqslant 1}\,,\end{aligned} \tag{3.1.9}$$

by (3.1.5) with $k = 2$. Because $S_n \to +\infty$ a.s., we can choose $N(\omega)$ such that $S_n(\omega) \geqslant 1$ for all integers $n \geqslant N(\omega)$ (replacing A by another almost sure set). From (3.1.8) and (3.1.9), we deduce that for all $\omega \in A \cap \Omega_X$,

$$\frac{\left|X_{S_n}(\omega)\right|}{S_n^{3p(\omega)+2}(\omega)} \leqslant C'(\omega) < +\infty, \qquad \text{for all } n \geqslant N(\omega)\,.$$

Let $n \geqslant N(\omega)$ and let $t \geqslant S_n(\omega)$. There is an integer $j \geqslant n$ such that $t \in \left[S_j(\omega), S_{j+1}(\omega)\right[$. Then

$$|X_t(\omega)| = |X_{S_j}(\omega)| \leqslant C'(\omega)S_j^{3p(\omega)+2}(\omega) \leqslant C'(\omega)\,t^{3p(\omega)+2}\,.$$

We deduce that

$$\limsup_{t\to+\infty} \frac{|X_t(\omega)|}{1 + t^{3p(\omega)+2}} \leqslant C'(\omega) < +\infty$$

on the set $A \cap \Omega_X$. This completes the proof. □

The next proposition recalls properties of the long term behavior of a compound Poisson process.

Proposition 3.1.4. *Let X be the compound Poisson process with jump heights* $(Z_i)_{i\geqslant 1}$ *defined at the beginning of this section.*

(i) Suppose that there is a real number $p>0$ *such that* $\mathbb{E}(|Z_1|^p)<+\infty$. *Then there is* $\alpha>0$ *such that*

$$\limsup_{t\to+\infty}\frac{|X_t|}{1+t^\alpha}<+\infty \qquad \textit{a.s.}$$

(ii) Suppose that $\mathbb{E}(|Z_1|^p)=+\infty$ *for every* $p>0$. *Then for any* $\alpha>0$,

$$\limsup_{t\to+\infty}\frac{|X_t|}{1+t^\alpha}=+\infty \qquad \textit{a.s.}$$

Proof. We use the notations introduced at the beginning of Section 3.1.2. To prove *(i)*, let $p>0$ be such that $\mathbb{E}(|Z_1|^p)<+\infty$. If $p<1$, then by the law of large numbers of Kolmogorov, Marcinkiewicz and Zygmund (see [44, Theorem 4.23]), we have $n^{-\alpha}Y_n\to 0$ a.s., with $\alpha=p^{-1}$, so $\sup_{n\geqslant 1} n^{-\alpha}|Y_n|<+\infty$ a.s. If $p\geqslant 1$, then by the strong law of large numbers, $\sup_{n\geqslant 1} n^{-1}Y_n<+\infty$. Finally, for $p>0$, we combine both cases by setting $\alpha=\max\big(p^{-1},1\big)$, so that

$$\sup_{n\geqslant 1}\frac{|Y_n|}{1+n^\alpha}<+\infty \qquad \text{a.s.} \tag{3.1.10}$$

Let $t\in\mathbb{R}_+$. There is an integer k such that $t\in[S_k,S_{k+1}[$, so that $X_t=X_{S_k}=Y_k$ and

$$\frac{|X_t|}{1+t^\alpha}\leqslant\frac{|Y_k|}{1+S_k^\alpha}=\frac{|Y_k|}{1+k^\alpha}\,\frac{1+k^\alpha}{1+S_k^\alpha}\,. \tag{3.1.11}$$

Since S_k is the sum of k i.i.d. exponential random variables with parameter $\lambda>0$, the law of large numbers tells us that $k^{-1}S_k\to\frac{1}{\lambda}$ a.s. We deduce from (3.1.11) and (3.1.10) that

$$\limsup_{t\to+\infty}\frac{|X_t|}{1+t^\alpha}<+\infty \qquad \text{a.s.,}$$

and *(i)* is proved.

To prove *(ii)*, suppose that for any $p>0$, we have $\mathbb{E}(|Z_1|^p)=+\infty$. Then according to the theorem in [44] mentioned above, for any $p\in]0,1[$, $n^{-1/p}Y_n$ does not converge on a set of positive probability. Since $(Y_n)_{n\geqslant 1}$ is a sum of i.i.d. random variables, the existence of a limit at infinity is a tail event. From Kolmogorov's zero-one law, we deduce that for any $p\in]0,1[$, $n^{-1/p}Y_n$ does not converge almost surely, and, in particular,

$$\limsup_{n\to+\infty}\frac{|Y_n|}{n^{1/p}}>0 \qquad \text{a.s..} \tag{3.1.12}$$

Fix $\alpha > 0$ and let $p_1 = \frac{1}{\alpha+1} \in]0,1[$. By (3.1.12),

$$\limsup_{n\to+\infty} \frac{|Y_n|}{n^{1/p_1}} > 0 \qquad \text{a.s.} \tag{3.1.13}$$

For $t \in \mathbb{R}_+$, there is an integer k such that $t \in [S_k, S_{k+1}[$, so $X_t = X_{S_k} = Y_k$ and

$$\frac{|X_t|}{1+t^\alpha} \geqslant \frac{|Y_k|}{1+S_{k+1}^\alpha} = \frac{|Y_k|}{1+k^{1/p_1}} \frac{1+k^{1/p_1}}{1+S_{k+1}^\alpha}. \tag{3.1.14}$$

By the strong law of large numbers and the fact that ${p_1}^{-1} = \alpha+1 > \alpha$, we have that

$$\lim_{k\to\infty} \frac{1+k^{1/p_1}}{1+S_{k+1}^\alpha} = +\infty.$$

Taking the lim sup on both sides of (3.1.14) (in fact taking the limit along some subsequence), we deduce from (3.1.13) that

$$\limsup_{t\to+\infty} \frac{|X_t|}{1+t^\alpha} = +\infty \qquad \text{a.s.}$$

□

3.1.3 Lévy white noise: the general case

Let X be a Lévy process. We have already defined its distributional derivative in Definition 2.2.6, and proved that it is indeed a Lévy white noise in the sense of Definition 2.2.4. Notice that the law of the Lévy white noise $\dot{X}$ is entirely characterized by the triplet (γ,σ,ν) (given that we use the truncation function $\mathbb{1}_{|z|\leqslant 1}$ in the Lévy-Itô decomposition).

We now turn to the question of whether or not a Lévy white noise is a tempered distribution. Similar to (3.1.2), for any Lévy noise $\dot{X}$, we define the set

$$\Omega_{\dot{X}} = \left\{\omega \in \Omega : \dot{X}(\omega) \in \mathscr{S}'(\mathbb{R})\right\}, \tag{3.1.15}$$

and we have the following characterization.

Theorem 3.1.5. *Let X be a Lévy process with characteristic triplet (γ,σ,ν), and $\dot{X}$ the associated Lévy white noise. Then $\Omega_X = \Omega_{\dot{X}}$ (defined respectively in* (3.1.1) *and* (3.1.15)*), and the following holds:*

(i) If there exists $\eta > 0$ such that $\mathbb{E}(|X_1|^\eta) < +\infty$, then $\mathbb{P}(\Omega_X) = \mathbb{P}\left(\Omega_{\dot{X}}\right) = 1$.

(ii) If $\mathbb{E}(|X_1|^\eta) = +\infty$ for all $\eta > 0$, then $\mathbb{P}(\Omega_X) = \mathbb{P}\left(\Omega_{\dot{X}}\right) = 0$.

Remark 3.1.6. *If $\mathbb{E}(|X_1|^\eta) < +\infty$ for some $\eta > 0$, then we say that X has a positive absolute moment (**PAM**). Recall that for $\eta > 0$, $\mathbb{E}(|X_1|^\eta) < +\infty$ if and only if $\int_{|x|>1} |z|^\eta \nu(\mathrm{d}z) < +\infty$ (see [63,* *Theorem 25.3]). Hence the condition **PAM** can be equivalently expressed in terms of the Lévy measure ν.*

Proof. Differentiation maps $\mathscr{S}'(\mathbb{R})$ to itself, hence on Ω_X, the Lévy noise $\dot{X}$ is a tempered distribution: $\Omega_X \subset \Omega_{\dot{X}}$. We now show that $\Omega_{\dot{X}} \subset \Omega_X$. Let $\omega \in \Omega_{\dot{X}}$. Two solutions in $\mathscr{D}'(\mathbb{R})$ of the equation $u' = \dot{X}(\omega)$ differ by a constant (see [64, Théorème I, chapter II, §4 p.51]) and $X(\omega)$ is obviously one of them. Therefore, if there is a solution to this equation in $\mathscr{S}'(\mathbb{R})$, then $\omega \in \Omega_X$. To show that such a solution u exists, recall that a distribution is an element of $\mathscr{S}'(\mathbb{R})$ if and only if it is the derivative of some order of a slowly growing continuous function (see [64, Théorème VI, chapter VII, §4 p.239]): $\dot{X}(\omega) = g^{(n)}$ for some continuous slowly growing function g and some integer n. If $n \geqslant 1$, then $u = g^{(n-1)}$ is a solution in $\mathscr{S}'(\mathbb{R})$ of $u' = \dot{X}(\omega)$. If $n = 0$, then $u(t) = \int_0^t g(s)\,\mathrm{d}s$ is a slowly growing solution, therefore $u \in \mathscr{S}'(\mathbb{R})$.

To prove *(i)*, it suffices to show that $\mathbb{P}(\Omega_X) = 1$. Let $X_t = \gamma t + \sigma W_t + X_t^M + X_t^P$ be the Lévy-Itô decomposition of X. Since $\mathbb{E}(|X_1|^\eta) < +\infty$, we have $\int_{|z|>1} |z|^\eta \nu(\mathrm{d}z) < +\infty$ (see Remark 3.1.6). The jump heights $(Z_i)_{i\geqslant 1}$ of the compound Poisson part X^P are i.i.d., with law $\lambda^{-1}\mathbb{1}_{|x|>1}\nu(\mathrm{d}x)$ (where λ is a normalizing constant), therefore $\mathbb{E}(|Z_1|^\eta) < +\infty$. Then we can use Proposition 3.1.1 for the continuous and small jumps terms of the Lévy-Itô decomposition of X, and Proposition 3.1.4*(i)* for the large jumps term, to deduce that X is slowly growing. By the *càdlàg* property of X and Remark 2.1.2 we conclude that $\mathbb{P}(\Omega_X) = 1$.

To prove *(ii)*, it suffices to show that $\mathbb{P}(\Omega_X) = 0$. By Proposition 3.1.1, $\Omega_X = \Omega_{X^P}$. Also, since

$$\{\omega : t \mapsto X_t^P(\omega) \text{ is slowly growing}\} \\ \cap \{\omega : \forall \alpha > 0,\ \limsup_{t\to+\infty}(1+t^\alpha)^{-1}|X_t^P| = +\infty\} = \emptyset,$$

and under *(ii)* the second set has probability one by Proposition 3.1.4*(ii)*, we deduce from Proposition 3.1.2 that $\mathbb{P}\left(\Omega_{X^P} \cap A\right) = 0$, where A is the almost-sure set defined in Proposition 3.1.2. Therefore, $\mathbb{P}\left(\Omega_{X^P}\right) = \mathbb{P}(\Omega_X) = 0$. □

Corollary 3.1.7. *Let X be a Lévy process with characteristic triplet (γ, σ, ν), let $\dot{X}$ be the associated Lévy noise and suppose it has a **PAM**. Then there is a random tempered distribution S, that is, a measurable map from $(\Omega, \mathscr{F})$ to $\left(\mathscr{S}'(\mathbb{R}), \mathscr{B}\right)$, where $\mathscr{B}$ is the Borel σ-field for the weak-$*$ topology, such that almost surely, for all $\varphi \in \mathscr{S}(\mathbb{R})$,*

$$\langle S, \varphi \rangle = \langle \dot{X}, \varphi \rangle = -\int_{\mathbb{R}_+} X_t \varphi'(t)\,\mathrm{d}t.$$

In addition, the maps $C : \omega \mapsto C(\omega)$ and $p : \omega \mapsto p(\omega)$ such that for all $\varphi \in \mathscr{S}(\mathbb{R})$,

$$\left|\langle S, \varphi \rangle\right| \leqslant C \mathscr{N}_p(\varphi) \qquad a.s.$$

can be chosen to be $\mathscr{F}$-measurable.

Proof. See Corollary 3.2.9 □

Remark 3.1.8. *An alternate proof of the fact that $\Omega_X \supset \Omega_{\dot{X}}$ is as follows. We can restrict to the case where X is a compound Poisson process. We construct here a solution to the equation $u'(\omega) = \dot{X}(\omega)$ such that $u(\omega) \in \mathscr{S}'(\mathbb{R})$. Let $\theta \in \mathscr{D}(\mathbb{R})$ be such that $\theta \geqslant 0$, $\int_{\mathbb{R}} \theta = 1$ and $\operatorname{supp}\theta \subset [0,1]$. Then let $\varphi \in \mathscr{S}(\mathbb{R})$. There exists a function $\Phi \in \mathscr{S}(\mathbb{R})$ such that $\varphi = \Phi'$ if and only if $\int_{\mathbb{R}} \varphi = 0$ (consider $\Phi(x) = \int_{-\infty}^x \varphi(t)\,dt$ for the if part, the other direction is obvious). Then consider the linear functional A on $\mathscr{S}(\mathbb{R})$ defined by*

$$A\varphi(t) = \int_{-\infty}^{t} \left(\varphi(s) - \theta(s)\int_{\mathbb{R}} \varphi\right) \mathrm{d}s\,.$$

This functional defines an antiderivative on $\mathscr{S}(\mathbb{R})$: for any $\varphi \in \mathscr{S}(\mathbb{R})$, $A\left(\varphi'\right) = \varphi$. Also, the reader can easily check that for all $p \in \mathbb{N}$,

$$\sup_{t\in\mathbb{R}} |t|^p |A\varphi(t)| \leqslant C_p \mathscr{N}_{p+2}(\varphi)\,,$$

for some constant C_p depending only on p, and therefore, I is a continuous linear functional with values in $\mathscr{S}(\mathbb{R})$. This implies that for $\omega \in \Omega_{\dot{X}}$, we can define a tempered distribution $u(\omega)$ by

$$\left\langle u(\omega), \varphi\right\rangle = -\left\langle \dot{X}(\omega), A\varphi\right\rangle\,, \qquad \textit{for all } \varphi \in \mathscr{S}(\mathbb{R})\,.$$

This tempered distribution clearly satisfies $u'(\omega) = \dot{X}(\omega)$. By definition of $\dot{X}$, u and X only differ by a (random) constant, and so $X(\omega) \in \mathscr{S}'(\mathbb{R})$. Therefore $\Omega_{\dot{X}} \subset \Omega_X$.

3.2 Lévy fields and Lévy noise in $\mathscr{S}'(\mathbb{R}^d)$

In this section, we consider the same questions as in Section 3.1, but for a generalization of the notion of Lévy process which was defined in Definition 1.0.8, where the “time” parameter is in $\mathbb{R}_+^d$, with $d \geqslant 1$. A general presentation of this theory of *multiparameter Lévy fields* can be found in [1]; see also [26].

Let X be a d parameter Lévy field with characteristic triplet (γ, σ, ν), and let $\dot{X}$ be the d-th cross derivative of X in the sense of distributions, as defined in Definition 2.2.7. As in Section 3.1.3, note that the law of the multidimensional Lévy white noise $\dot{X}$ is entirely characterized by the triplet (γ, σ, ν) (given that we use the truncation function $\mathbb{1}_{|x|\leqslant 1}$ in the Lévy-Itô decomposition). We have seen in Lemma 2.2.9 that this definition is equivalent to the definition of Lévy white noise in Definition 2.2.4.

3.2.1 The case of a p-integrable martingale ($p > 1$)

We say that a random field M is a multiparameter martingale with respect to a filtration $\mathbb{F} = (\mathscr{F}_t)_{t\in\mathbb{R}_+^d}$ (see [47, Chapter 7, Section 2 p.233]) if M is $\mathbb{F}$-adapted, integrable, and for all $s \leqslant t \in \mathbb{R}_+^d$, then $\mathbb{E}(M_t|\mathscr{F}_s) = M_s$. We will also need the notion of commuting filtration (see

[47, Chapter 7, Section 2, Definition p.233]). A Filtration $\mathbb{F}$ is said to be commuting if for any $s,t \in \mathbb{R}_+^d$, and any bounded $\mathscr{F}_t$ measurable random variable Y,

$$\mathbb{E}\left[Y|\mathscr{F}_s\right] = \mathbb{E}\left[Y|\mathscr{F}_{s\wedge t}\right],$$

where $(s \wedge t)_i = s_i \wedge t_i$. By [47, Theorem 2.1.1 in chapter 7], to show that $\mathbb{F}$ is commuting, it suffices to show that for any $s,t \in \mathbb{R}_+^d$, $\mathscr{F}_s$ and $\mathscr{F}_t$ are conditionally independent given $\mathscr{F}_{s\wedge t}$. In particular, if X is a d-parameter Lévy field and $\mathscr{F}_t$ is the σ-algebra generated by the family $(X_s)_{s\leqslant t}$, then $\mathbb{F}$ is commuting by the independence of the increments of X (see more details in the proof of Proposition 3.2.1).

For any *lamp* random field L, we consider, similarly to (3.1.1), the event

$$\Omega_L = \left\{\omega \in \Omega : L(\omega) \in \mathscr{S}'(\mathbb{R}^d)\right\}, \tag{3.2.1}$$

with the understanding that when $L(\omega) \in \mathscr{S}'(\mathbb{R}^d)$, the continuous linear functional associated with $L(\omega)$ is $\langle L(\omega), \varphi\rangle = \int_{\mathbb{R}_+^d} L_t(\omega)\varphi(t)\,\mathrm{d}t$, for all $\varphi \in \mathscr{S}(\mathbb{R}^d)$.

Proposition 3.2.1. *Fix $p > 1$ and let $(M_t)_{t\in\mathbb{R}_+^d}$ be a multiparameter martingale with respect to a commuting filtration $(\mathscr{F}_t)_{t\in\mathbb{R}_+^d}$, such that for all $t \in \mathbb{R}_+^d$,*

$$\mathbb{E}\left(|M_t|^p\right) \leqslant \left(c\,Leb_d\left([0,t]\right)\right)^{\frac{p}{2}}$$

for some constant c. Then the set Ω_M defined as in (3.2.1) *has probability one.*

Proof. Similar to the one dimensional case, we control the supremum of $|t|^{-\alpha}|M_t|$ as $|t| \to +\infty$, or, equivalently, the supremum of $|s|^{-\alpha}|M_s|$ for $s \in \mathbb{R}_+^d \setminus [0,t]$ as $\min_{i=1,\dots,d} t_i \to +\infty$, and prove that the limit in probability of this supremum, as all the coordinates of t go to $+\infty$, is zero. The proof uses the multidimensional analog of Doob's L^p inequality: Cairoli's Strong (p,p) inequality (see [47, Chapter 7, Theorem 2.3.2]). The conditions for this Theorem are stronger than those required for Doob's L^p inequality, in particular we need the natural filtration of M (denoted by $\left(\mathscr{F}_t^M\right)_{t\geqslant 0}$) to be commuting (see [47, Chapter 7, Definition p. 233]). By [47, Chapter 7, Theorem 2.1.1], it suffices to show that for any $s,t \in \mathbb{R}_+^d$, $\mathscr{F}_s^M$ and $\mathscr{F}_t^M$ are conditionally independent given $\mathscr{F}_{s\wedge t}^M$ where $(s\wedge t)_i = s_i \wedge t_i$. We will use the independence of the increments of M to prove this result. Indeed, we can write $[0,t]\bigcup[0,s] = [0,s\wedge t]\bigcup E_1 \bigcup E_2$ where this union is disjoint and E_1 (resp. E_2) is a finite union of disjoint d-dimensional boxes in $[0,t]$ (resp. $[0,s]$). We then define $\mathscr{F}^1 = \sigma\left(\Delta_a^b M : [a,b] \subset E_1\right)$ that is the σ-algebra generated by the random variables of the form $\Delta_a^b M$ where $[a,b] \subset E_1$. Similarly we define $\mathscr{F}_2 = \sigma\left(\Delta_a^b M : [a,b] \subset E_2\right)$. Because of the independence of the increments, $\mathscr{F}^1, \mathscr{F}^2$ and $\mathscr{F}_{s\wedge t}^M$ are independent (see illustration in Figure 3.2) and we have:

$$\mathscr{F}_t^M = \mathscr{F}_{s\wedge t}^M \vee \mathscr{F}^1 \quad \text{and} \quad \mathscr{F}_s^M = \mathscr{F}_{s\wedge t}^M \vee \mathscr{F}^2.$$

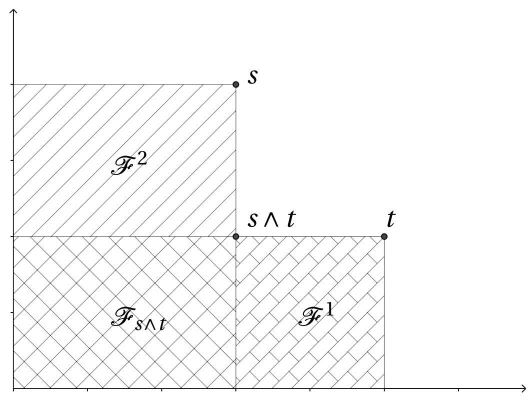

Figure 3.2 – Illustration of the commuting property when $d = 2$.

Then we deduce that for $A \in \mathscr{F}_t^M$, and $B \in \mathscr{F}_s^M$:

$$\begin{aligned}\mathbb{P}\left(A \cap B | \mathscr{F}_{s\wedge t}^M\right) &= \mathbb{E}\left(\mathbb{1}_A \mathbb{1}_B | \mathscr{F}_{s\wedge t}^M\right)\\ &= \mathbb{E}\left(\mathbb{1}_A \mathbb{E}\left(\mathbb{1}_B | \mathscr{F}_t^M\right) | \mathscr{F}_{s\wedge t}^M\right)\\ &= \mathbb{E}\left(\mathbb{1}_A \mathbb{E}\left(\mathbb{1}_B | \mathscr{F}_{s\wedge t}^M\right) | \mathscr{F}_{s\wedge t}^M\right) = \mathbb{P}\left(A | \mathscr{F}_{s\wedge t}^M\right) \mathbb{P}\left(B | \mathscr{F}_{s\wedge t}^M\right).\end{aligned}$$

So $\mathscr{F}_s^M$ and $\mathscr{F}_t^M$ are conditionally independent given $\mathscr{F}_{s\wedge t}^M$ and the filtration is commuting. For all $i \in \mathbb{N}\backslash\{0\}$, let $x_i = 2^{i-1}$ and $x_0 = 0$. For $k = (k_1, ..., k_d) \in \mathbb{N}^d$, let $a_k = (x_{k_1}, ..., x_{k_d})$, and let $b_k = (2^{k_1}, ..., 2^{k_d})$. We fix $k \in \mathbb{N}^d$, $k \neq (0, \ldots, 0)$. By using successively Jensen's inequality and Cairoli's inequality, for any $\alpha > 0$, we have

$$\begin{aligned}\mathbb{E}\left(\sup_{s\in[a_k,b_k]} \frac{|M_s|}{|s|^\alpha}\right) &\leqslant \frac{1}{|a_k|^\alpha} \mathbb{E}\left(\sup_{s\leqslant b_k} |M_s|^p\right)^{\frac{1}{p}}\\ &\leqslant \frac{c_p}{|a_k|^\alpha} \mathbb{E}\left(|M_{b_k}|^p\right)^{\frac{1}{p}} \leqslant \frac{c_p\sqrt{c\,\mathrm{Leb}_d([0,b_k])}}{|a_k|^\alpha},\end{aligned}$$

for some constant c_p depending only on p and the dimension d, where $|a_k|$ and $|s|$ denote here the Euclidian norm. Since $k_1 \vee \cdots \vee k_d \geqslant 1$, we have $|a_k| \geqslant 2^{k_1\vee\cdots\vee k_d - 1}$, hence

$$\mathbb{E}\left(\sup_{s\in[a_k,b_k]} \frac{|M_s|}{|s|^\alpha}\right) \leqslant c_p\sqrt{c}\, 2^{\frac{1}{2}\sum_{i=1}^d k_i} 2^{-\alpha(k_1\vee\cdots\vee k_d - 1)} \leqslant c_p\sqrt{c}\, 2^\alpha 2^{-\left(\frac{\alpha}{d} - \frac{1}{2}\right)\sum_{i=1}^d k_i}.$$

We choose $\alpha = \lfloor \frac{d}{2} \rfloor + 1$. Let $t \in \mathbb{R}_+^d$ be far enough from the origin (we will consider the limit as all the coordinates of t go to $+\infty$), and for all $1 \leqslant i \leqslant d$, let n_i be the largest integer such that $2^{n_i} \leqslant t_i$ and let $n = (n_1, ..., n_d)$. We can suppose that $n_i \geqslant 2$ for all $1 \leqslant i \leqslant n$. We write Ξ for the set of all relations $\mathscr{R}$ of the form $(r_1, ..., r_d)$, where for all $i \in \{1, ..., d\}$, $r_i \in \{\leqslant, \geqslant\}$ and $\mathscr{R} \neq (\leqslant, ..., \leqslant)$. Then $[0, t_n] \subset [0, t]$, where $t_n = (2^{n_1}, ..., 2^{n_d})$. The complement of the box

$[0, t_n]$ in $\mathbb{R}_+^d$ is covered by boxes of the form $[a_k, b_k]$, where $k \in \mathbb{N}^d$ and $k\mathscr{R}n$ for some $\mathscr{R} \in \Xi$. Therefore,

$$\begin{aligned}\mathbb{P}\left(\sup_{s\notin[0,t]} \frac{|M_s|}{|s|^\alpha} > \varepsilon\right) &\leqslant \mathbb{P}\left(\sup_{s\notin[0,t_n]} \frac{|M_s|}{|s|^\alpha} > \varepsilon\right)\\ &\leqslant \sum_{\mathscr{R}\in\Xi} \sum_{\substack{k\in\mathbb{N}^d\\ k\mathscr{R}n}} \mathbb{P}\left(\sup_{s\in[a_k,b_k]} \frac{|M_s|}{|s|^\alpha} > \varepsilon\right)\\ &\leqslant \frac{c_p\sqrt{c}2^\alpha}{\varepsilon} \sum_{\mathscr{R}\in\Xi} \sum_{\substack{k\in\mathbb{N}^d\\ k\mathscr{R}n}} 2^{-\left(\frac{\alpha}{d}-\frac{1}{2}\right)\sum_{i=1}^d k_i} \underset{t\twoheadrightarrow+\infty}{\longrightarrow} 0,\end{aligned}$$

where $t \twoheadrightarrow +\infty$ means that $t_1 \wedge ... \wedge t_d \to +\infty$. To check that the limit is indeed zero, one has that for any fixed $\mathscr{R} \in \Xi$, at least one of the inequalities in $\mathscr{R}$ is $\geq$. By symmetry, we can suppose that it is the first inequality. Then

$$\sum_{\substack{k\in\mathbb{N}^d\\ k\mathscr{R}n}} 2^{-\left(\frac{\alpha}{d}-\frac{1}{2}\right)\sum_{i=1}^d k_i} \leq C_{\alpha,d} \sum_{k_1\geqslant n_1} 2^{-\left(\frac{\alpha}{d}-\frac{1}{2}\right)k_1} \underset{n_1\to+\infty}{\to} 0.$$

The result follows since Ξ is a finite set. Then $\sup_{s\notin[0,t]} |s|^{-\alpha}|M_s| \to 0$ in probability as $t \twoheadrightarrow +\infty$. Then,

$$\begin{aligned}\mathbb{P}\left(\frac{|M_t|}{|t|^\alpha} \underset{|t|\to+\infty}{\longrightarrow} 0\right) = 1 &\Leftrightarrow \mathbb{P}\left(\bigcap_{n\geqslant1} \bigcup_{k\in\mathbb{N}} \bigcap_{\substack{t\in\mathbb{Q}^d\\ t_1\vee...\vee t_d\geqslant k}} \left\{\frac{|M_t|}{|t|^\alpha} \leqslant \frac{1}{n}\right\}\right) = 1\\ &\Leftrightarrow \forall n \geqslant 1,\ \mathbb{P}\left(\bigcup_{k\in\mathbb{N}} \bigcap_{\substack{t\in\mathbb{Q}^d\\ t_1\vee...\vee t_d\geqslant k}} \left\{\frac{|M_t|}{|t|^\alpha} \leqslant \frac{1}{n}\right\}\right) = 1\\ &\Leftrightarrow \forall n \geqslant 1,\ \lim_{k\to+\infty} \uparrow \mathbb{P}\left(\bigcap_{\substack{t\in\mathbb{Q}^d\\ t_1\vee...\vee t_d\geqslant k}} \left\{\frac{|M_t|}{|t|^\alpha} \leqslant \frac{1}{n}\right\}\right) = 1\\ &\Leftrightarrow \forall n \geqslant 1,\ \lim_{k\to+\infty} \downarrow \mathbb{P}\left(\sup_{t\notin[0,(k,...,k)]} \frac{|M_t|}{|t|^\alpha} > \frac{1}{n}\right) = 0,\end{aligned}$$

therefore $|t|^{-\alpha}|M_t| \to 0$ a.s as $|t| \to +\infty$. By the *lamp* property of M we deduce that M is slowly growing, and by Remark 2.1.2 we deduce that $\mathbb{P}(\Omega_M) = 1$. □

Corollary 3.2.2. *Let X be a d-parameter Lévy field with characteristic triplet (γ, σ, ν) and Lévy-Itô decomposition $X_t = \gamma Leb_d([0,t]) + \sigma W_t + X_t^P + X_t^M$ where X^P is the large jump part of the decomposition and X^M is the compensated small jumps part. Let $Y_t = \gamma Leb_d([0,t]) + \sigma W_t + X_t^M$. Then the set Ω_Y defined in* (3.2.1) *has probability one.*

Proof. The random field $\tilde{Y} = \sigma W + X^M$ is a sum of two independent square integrable martingales and by a classical result on compensated Poisson integrals and Brownian sheets (see [63, Propostion 19.5] and [21]),

$$\mathbb{E}\left(\tilde{Y}_t^2\right) = \left(\sigma^2 + \int_{|z|\leqslant 1} z^2 \nu(\mathrm{d}z)\right) \mathrm{Leb}_d\left([0,t]\right),$$

where the multiplicative constant is finite since ν is a Lévy measure. Hence $\tilde{Y}$ verifies the hypothesis of the Proposition 3.2.1 with $p = 2$, therefore it defines a tempered distribution a.s. Since $\tilde{Y}$ and Y differ by a slowly growing function, we deduce that Y is a tempered distribution almost surely. □

3.2.2 The compound Poisson sheet

Let X be a d-parameter Lévy field, and define $Y = (Y_t)_{t\in\mathbb{R}_+^d}$ as in Corollary 3.2.2. By Corollary 3.2.2, for any d-parameter Lévy field X, we have $\Omega_X \cap \Omega_Y = \Omega_{X^P} \cap \Omega_Y$. We shall prove that Ω_{X^P} has probability 0 or 1. In the one dimensional setting, we used the fact that a compound Poisson process with a **PAM** is slowly growing a.s (see Proposition 3.1.4 *(i)*). As mentioned in the Introduction, the same results in a d-dimensional setting are to the best of our knowledge unavailable, which leads us to find another approach. In the multiparameter case, we will use properties of stochastic integrals with respect to a Poisson random measure to show that under a moment condition, a compound Poisson sheet and its associated white noise define tempered distributions. While this is in principle a special case of [34, Theorem 3], in view of Corollary 3.2.2, the two statements are in fact equivalent.

Lemma 3.2.3. *Let ν be a Lévy measure and M be a Poisson random measure on $(\mathbb{R}\backslash\{0\}) \times \mathbb{R}_+^d$ with intensity measure $\mathbb{1}_{|z|>\eta}\nu(\mathrm{d}z)\,\mathrm{d}t$, where $\eta > 0$. Suppose that $\int_{|z|>\eta} |x|^\alpha \nu(\mathrm{d}z) < +\infty$ for some $\alpha > 0$ (**PAM**) and consider the compound Poisson sheet $P_t = \int_{[0,t]}\int_{|z|>\eta} xM(\mathrm{d}s,\mathrm{d}z)$. Then*

(i) M almost surely defines a tempered distribution via the formula

$$\langle M, \varphi\rangle = \int_{\mathbb{R}_+^d}\int_{|z|>\eta} M(\mathrm{d}s,\mathrm{d}z)\varphi(s)z, \qquad \varphi \in \mathscr{S}(\mathbb{R}^d). \tag{3.2.2}$$

(ii) $\mathbb{P}(\Omega_P) = 1$ and for all $\varphi \in \mathscr{S}(\mathbb{R}^d)$,

$$\langle P, \varphi\rangle := \int_{\mathbb{R}_+^d} P_s \varphi(s)\,\mathrm{d}s = \int_{\mathbb{R}_+^d}\int_{|z|>\eta} M(\mathrm{d}t,\mathrm{d}z)\int_{[t,+\infty[} \mathrm{d}s\,\varphi(s)z, \tag{3.2.3}$$

(iii) $M = P^{(\mathbf{1}_d)}$ in $\mathscr{S}'(\mathbb{R}^d)$, where we recall that $P^{(\mathbf{1}_d)} = \frac{\partial^d}{\partial t_1\cdots\partial t_d}P$.

Proof. We can suppose without loss of generality that $\eta = 1$. Since M is a Poisson random measure on $\mathbb{R}_+^d \times (\mathbb{R}\backslash\{0\})$ with jumps of size larger than 1, there are (random) points $(\tau_i, Z_i)_{i\geqslant 1} \in$

$\mathbb{R}_+^d \times (\mathbb{R}\backslash[-1,1])$ such that $M = \sum_{i\geqslant 1}\delta_{\tau_i}\delta_{Z_i}$. To prove *(i)*, we first need to check that the integral in (3.2.2) is well defined. Let $\varphi \in \mathscr{S}(\mathbb{R}^d)$. The stochastic integral is a Poisson integral, and it is well defined (as the limit in probability of Poisson integrals of elementary functions) if and only if (see [44, Lemma 12.13])

$$\int_{|z|>1}\int_{\mathbb{R}_+^d}\left(\left|z\varphi(t)\right|\wedge 1\right)\mathrm{d}t\,\nu(\mathrm{d}z) < +\infty. \tag{3.2.4}$$

Let $r \in \mathbb{N}$. There is a constant $C > 1$ such that $\sup_{t\in\mathbb{R}_+^d}(1+|t|^r)|\varphi(t)| \leqslant C < +\infty$. Then $\left|z\varphi(t)\right| \wedge 1 \leqslant \frac{C|z|}{1+|t|^r}\wedge 1$. We write V_d for the volume of the d-dimensional unit sphere. Then, for $|z|>1$,

$$\begin{aligned}\int_{\mathbb{R}_+^d}\left(\left|z\varphi(t)\right|\wedge 1\right)\mathrm{d}t &\leqslant \int_{\mathbb{R}_+^d}\left(\frac{C|z|}{1+|t|^r}\wedge 1\right)\mathrm{d}t\\ &\leqslant dV_d\int_{\mathbb{R}_+}\left(\frac{C|z|}{1+u^r}\wedge 1\right)u^{d-1}\,\mathrm{d}u\\ &\leqslant dV_d\left(\int_0^{(C|z|-1)^{\frac{1}{r}}}u^{d-1}\,\mathrm{d}u + C|z|\int_{(C|z|-1)^{\frac{1}{r}}}^{+\infty}\frac{u^{d-1}}{1+u^r}\,\mathrm{d}u\right)\\ &\leqslant V_d\,(C|z|-1)^{\frac{d}{r}} + dV_dC|z|\int_{(C|z|-1)^{\frac{1}{r}}}^{+\infty}\frac{u^{d-1}}{1+u^r}\,\mathrm{d}u.\end{aligned}$$

The last integral has to be well defined so we take $r > d$, and then

$$\int_{(C|z|-1)^{\frac{1}{r}}}^{+\infty}\frac{u^{d-1}}{1+u^r}\,\mathrm{d}u \leqslant \int_{(C|z|-1)^{\frac{1}{r}}}^{+\infty}u^{d-1-r}\,\mathrm{d}u = \frac{1}{r-d}\,(C|z|-1)^{\frac{d-r}{r}},$$

so

$$\int_{\mathbb{R}_+^d}\left(\left|z\varphi(t)\right|\wedge 1\right)\mathrm{d}t \leqslant V_d\,(C|z|-1)^{\frac{d}{r}} + \frac{dV_dC|z|}{r-d}\,(C|z|-1)^{\frac{d-r}{r}}.$$

We deduce that there exists a constant C' such that for $|z|>1$,

$$\int_{\mathbb{R}_+^d}\left(\left|z\varphi(t)\right|\wedge 1\right)\mathrm{d}t \leqslant C'|z|^{\frac{d}{r}}. \tag{3.2.5}$$

We then choose r large enough so that $\frac{d}{r} \leqslant \alpha \wedge \frac{1}{2}$, in which case the moment condition on ν gives us (3.2.4), and therefore the Poisson integral is well defined and a.s. finite. Set $g_r(t) = \frac{1}{1+|t|^r}$, $t \in \mathbb{R}_+^d$. Then for r sufficiently large,

$$\int_{\mathbb{R}_+^d}\int_{|z|>1}M(\mathrm{d}t,\mathrm{d}z)g_r(t)|z|$$

is well-defined, since by (3.2.5) and **PAM** ,

$$\int_{|z|>1}\int_{\mathbb{R}_+^d}\left(\left|zg_r(t)\right|\wedge 1\right)\mathrm{d}t\nu(\mathrm{d}z) < +\infty.$$

Since $M = \sum_i \delta_{\tau_i}\delta_{Z_i}$,

$$\langle M,\varphi\rangle = \sum_i Z_i\varphi(\tau_i)\,.$$

Now suppose $\varphi_n \to 0$ in $\mathscr{S}(\mathbb{R}^d)$. Then for large n, $|\varphi_n| \leqslant g_r$, and

$$\begin{aligned} |\langle M,\varphi_n\rangle| &= |\sum_i \varphi_n(\tau_i)Z_i| \\ &\leqslant \sum_i |Z_i| g_r(\tau_i) = \int_{\mathbb{R}^d_+}\int_{|z|>1} M(\mathrm{d}t,\mathrm{d}z)g_r(t)|z| < +\infty \qquad \text{a.s.} \end{aligned}$$

For almost all fixed $\omega \in \Omega$,

$$\varphi_n(\tau_i(\omega)) \to 0 \text{ as } n \to +\infty, \qquad |\varphi_n(\tau_i(\omega))| \leqslant g_r(\tau_i(\omega)) \qquad \text{and } \sum_i g_r(\tau_i(\omega))|Z_i(\omega)| < +\infty.$$

By the dominated convergence theorem,

$$\langle M(\omega),\varphi_n\rangle = \sum_i \varphi_n(\tau_i(\omega))Z_i(\omega) \to 0 \qquad \text{as } n \to +\infty.$$

Therefore, the linear functional $\varphi_n \mapsto \langle M(\omega),\varphi_n\rangle$ is continuous on $\mathscr{S}(\mathbb{R}^d)$, and so $M(\omega) \in \mathscr{S}'(\mathbb{R}^d)$ for a.a. $\omega \in \Omega$.

To prove *(ii)*, we first prove that the Poisson integral on the right hand side of (3.2.3) is well defined, and we will need the **PAM** condition. Let $\varphi \in \mathscr{S}(\mathbb{R}^d)$ and let $\Phi(t) = \int_{[t,+\infty[}\varphi(s)\,\mathrm{d}s$. Then (3.2.3) is well defined if

$$\int_{|z|>1}\int_{\mathbb{R}^d_+} (|z\Phi(t)| \wedge 1)\,\mathrm{d}t\,\nu(\mathrm{d}z) < +\infty. \tag{3.2.6}$$

Using (3.2.8) in Lemma 3.2.4 below, property (3.2.6) is established in the same way as (3.2.4) and, as above, the right-hand side of (3.2.3) defines almost surely a tempered distribution. Let $\varphi \in \mathscr{S}(\mathbb{R}^d)$. Then

$$\int_{\mathbb{R}^d_+}\int_{|z|>1} M(\mathrm{d}t,\mathrm{d}z)\int_{[t,+\infty[}\mathrm{d}s\,\varphi(s)z = \sum_{i\geqslant 1}\int_{\mathbb{R}^d_+} Z_i \mathbb{1}_{\tau_i\in[0,s]}\varphi(s)\,\mathrm{d}s. \tag{3.2.7}$$

Following the argument in (2.2.4), we want to be able to use Fubini's theorem to exchange the sum and the integral in the last expression. For any $\alpha \in \mathbb{N}$, by the same argument as in the proof of Lemma 3.2.4 below with $\beta = 0$,

$$\sup_{t\in\mathbb{R}^d_+}(1+|t|^\alpha)\int_{[t,+\infty[}|\varphi(s)|\,\mathrm{d}s \leqslant C\mathscr{N}_{|\alpha|+2d}(\varphi)\,.$$

As in the proof of (3.2.4), we deduce that

$$\int_{|z|>1}\int_{\mathbb{R}^d_+}\left(\left|z\int_{[t,+\infty[}|\varphi(s)|\,\mathrm{d}s\right| \wedge 1\right)\mathrm{d}t\,\nu(\mathrm{d}z) < +\infty.$$

Then $\sum_{i\geqslant 1}\int_{\mathbb{R}_+^d}|Z_i|\mathbb{1}_{\tau_i\in[0,s]}|\varphi(s)|\,\mathrm{d}s<+\infty$, and by (3.2.7) and Fubini's Theorem,

$$\int_{\mathbb{R}_+^d}\int_{|z|>1}M(\mathrm{d}t,\mathrm{d}z)\int_{[t,+\infty[}\mathrm{d}s\varphi(s)z=\int_{\mathbb{R}_+^d}\sum_{i\geqslant 1}Z_i\mathbb{1}_{\tau_i\in[0,s]}\varphi(s)\,\mathrm{d}s$$
$$=\int_{\mathbb{R}_+^d}P_s\varphi(s)\,\mathrm{d}s\,.$$

This establishes (3.2.3). Property *(iii)* now follows by replacing φ by $\varphi^{(\mathbf{1}_d)}$ in (3.2.3). $\square$

Lemma 3.2.4. *For $\varphi\in\mathscr{S}(\mathbb{R}^d)$, let Φ be the function defined by $\Phi(t)=\int_{[t,+\infty)}\varphi(s)\,\mathrm{d}s$. Let $p\in\mathbb{N}$, $\alpha,\beta\in\mathbb{N}^d$, such that $|\alpha|,|\beta|\leqslant p$. Then for all $a\in\mathbb{R}^d$, there is $C=C(p,d,a)<+\infty$, such that, for all $\varphi\in\mathscr{S}(\mathbb{R}^d)$,*

$$\sup_{t\geqslant a}\left|(1+|t^\alpha|)\Phi^{(\beta)}(t)\right|\leqslant C'\mathcal{N}_{p+2d}(\varphi)\,. \tag{3.2.8}$$

Proof. Let $t\in\mathbb{R}^d$. Then $\Phi(t)=\int_{\mathbb{R}_+^d}\varphi(s+t)\,\mathrm{d}s$, so

$$\Phi^{(\beta)}(t)=\int_{\mathbb{R}_+^d}\varphi^{(\beta)}(s+t)\,\mathrm{d}s\,.$$

Therefore,

$$\begin{aligned}\left|(1+|t^\alpha|)\Phi^{(\beta)}(t)\right|&\leqslant(1+|t^\alpha|)\int_{\mathbb{R}_+^d}\left|\varphi^{(\beta)}(s+t)\right|\mathrm{d}s\\&=(1+|t^\alpha|)\int_{\mathbb{R}_+^d}\frac{\left|\varphi^{(\beta)}(s+t)\right|\left(1+\left|(t+s)^{\alpha+\mathbf{2}_d}\right|\right)}{1+\left|(t+s)^{\alpha+\mathbf{2}_d}\right|}\,\mathrm{d}s\\&\leqslant\mathcal{N}_{p+2d}(\varphi)(1+|t^\alpha|)\int_{\mathbb{R}_+^d}\frac{1}{1+\left|(t+s)^{\alpha+\mathbf{2}_d}\right|}\,\mathrm{d}s\\&\leqslant C\mathcal{N}_{p+2d}(\varphi)\end{aligned}$$

for $t\geqslant a$, where C is a constant depending only on p, d and a. $\square$

3.2.3 Multidimensional Lévy white noise: the general case

The following lemma extends to d-parameter Lévy fields the property recalled in Remark 3.1.6.

Lemma 3.2.5. *Let X be a d-parameter Lévy field with characteristic triplet (γ,σ,ν) and let $\alpha>0$. The following are equivalent:*
(i) $\forall\,t\in\mathbb{R}_+^d$, $\mathbb{E}(|X_t|^\alpha)<+\infty$; *(ii)* $\exists\,t\in(\mathbb{R}_+\backslash\{0\})^d:\mathbb{E}(|X_t|^\alpha)<+\infty$; *(iii)* $\displaystyle\int_{|z|>1}|z|^\alpha\nu(\mathrm{d}z)<+\infty$.

Proof. Clearly, *(i)* implies *(ii)*. Suppose that *(ii)* is true for some t in $(\mathbb{R}_+\backslash\{0\})^d$. As discussed just after Definition 1.0.8, the process $X^{i,t}$ obtained by fixing all coordinates of the parameter t except the i-th is again a Lévy process with characteristic triplet $(\gamma,\sigma,\nu)\prod_{j\neq i}t_j$. By an application of [63, Theorem 25.3] we deduce that $(\prod_{j\neq i}t_i)\int_{|z|>1}|z|^\alpha\nu(\mathrm{d}z)<+\infty$ and then *(iii)* is verified. Suppose now that *(iii)* is true. Let $t \in \mathbb{R}_+^d$, and $1 \leqslant i \leqslant d$. Since $(\prod_{j\neq i} t_i) \int_{|z|>1} |z|^\alpha \nu(\mathrm{d}z) < +\infty$, another application [63, Theorem 25.3] gives us $\mathbb{E}(|X_s^{i,t}|^\alpha) < +\infty$ for all $s \in \mathbb{R}_+$. Since i and t are taken arbitrarily, we deduce *(i)*. □

We need a technical lemma that essentially states that for a compound Poisson sheet X^P, there is a well-chosen sequence $(\varphi_n)_{n\geqslant 1}$ of test-functions with suitably decreasing compact support such that X^P is constant on $\operatorname{supp}(\varphi_n)$ for n large enough (this was established in dimension one during the proof of Proposition 3.1.2).

Lemma 3.2.6. *Let X^P be a d-parameter Lévy field with jump measure J_X and characteristic triplet $(0,0,\mathbb{1}_{|z|\geqslant 1}\nu)$, where $\lambda := \int_{|z|\geqslant 1} \nu(\mathrm{d}z) < +\infty$. Let L be the compound Poisson process defined by $L_t = X^P_{(\mathbf{1}_{d-1},t)}$, and let $(S_n)_{n\geqslant 1}$ denote its sequence of jump times. Then for all $p \in \mathbb{N}$, there exists a finite non random constant C_p with the following property: for all $\omega \in \Omega$, there exists a sequence $(\varphi_n)_{n\geqslant 1}$ of functions (depending on ω) in $\mathscr{D}(\mathbb{R}^d)$ such that*

$$\mathcal{N}_p(\varphi_n)\mathbb{1}_{S_n\geqslant 1} \leqslant C_p S_n^{3d+4p}\mathbb{1}_{S_n\geqslant 1}\,, \tag{3.2.9}$$

and there exists an event Ω' such that $\mathbb{P}(\Omega') = 1$ and for all $\omega \in \Omega'$, there exists an integer $N(\omega)$ such that, for all $n \geqslant N(\omega)$, X^P is constant on the support of φ_n and

$$\left\langle X^P, \varphi_n \right\rangle(\omega) = L_{S_n}(\omega)\,. \tag{3.2.10}$$

Proof. As in the proof of Proposition 3.1.2, we will construct a sequence $(\varphi_n)_{n\geqslant 1}$ of functions with suitably decreasing compact support, and then use a Borel-Cantelli argument to show that X^P is constant on this support. Let $\varphi \in \mathscr{D}(\mathbb{R}^d)$ with $\operatorname{supp}\varphi \subset [0, \mathbf{1}_d]$ and $\int_{\mathbb{R}^d} \varphi = 1$. Similar to (3.1.3), the sequence $(\varphi_n)_{n\geqslant 1}$ is defined by

$$\varphi_n(t) = S_n^{3d}\varphi\left((t_1-1)S_n^3, \ldots, (t_{d-1}-1)S_n^3, (t_d - S_n)S_n^3\right), \quad t \in \mathbb{R}^d,$$

so that $\operatorname{supp}\varphi_n \subset \left[(\mathbf{1}_{d-1}, S_n), \left(1+\frac{1}{S_n^3}, \ldots, 1+\frac{1}{S_n^3}, S_n + \frac{1}{S_n^3}\right)\right]$ and $\int_{\mathbb{R}^d} \varphi_n = 1$. Let $p \in \mathbb{N}$. Then

$$\begin{aligned}
\mathcal{N}_p(\varphi_n)\mathbb{1}_{S_n\geqslant 1} &= \sum_{|\alpha|,|\beta|\leqslant p} \sup_{t\in\mathbb{R}^d} \left| t^\alpha \varphi_n^{(\beta)}(t)\right| \mathbb{1}_{S_n\geqslant 1} \\
&= \sum_{|\alpha|,|\beta|\leqslant p} \sup_{t\in[0,(2,\ldots,2,S_n+1)]} t^\alpha \left|\varphi_n^{(\beta)}(t)\right| \mathbb{1}_{S_n\geqslant 1} \\
&\leqslant \sum_{|\alpha|,|\beta|\leqslant p} 2^{\sum_{i=1}^{d-1}\alpha_i}(S_n+1)^{\alpha_d} \sup_{t\in\mathbb{R}^d}\left|\varphi_n^{(\beta)}(t)\right| \mathbb{1}_{S_n\geqslant 1} \\
&\leqslant \sum_{|\alpha|,|\beta|\leqslant p} 2^{\sum_{i=1}^{d-1}\alpha_i}(S_n+1)^{\alpha_d} S_n^{3\left(d+\sum_{i=1}^d \beta_i\right)} \mathcal{N}_p(\varphi)\mathbb{1}_{S_n\geqslant 1} \\
&\leqslant C_p'\mathcal{N}_p(\varphi) S_n^{3d+4p}\mathbb{1}_{S_n\geqslant 1}\,,
\end{aligned}$$

for some finite non random constant C_p'. Therefore (3.2.9) holds and $C_p := C_p'\mathcal{N}_p(\varphi)$ depends only on φ and p. Let

$$I_{n,k} = \left]\left(\mathbf{1}_{d-1}, S_n\right), \left(1+\frac{1}{S_n^k}, \ldots, 1+\frac{1}{S_n^k}, S_n+\frac{1}{S_n^k}\right)\right[,$$

and let $A_{n,k}$ be the event "X^P is constant in the box $I_{n,k}$". Clearly, (3.2.10) holds on $A_{n,k}$.

Observe that

$$\begin{aligned}\mathbb{P}(A_{n,k}^c) &= \mathbb{P}\left\{X^P \text{ has at least one jump time in the set } J_{n,k}\right\}\\ &= \mathbb{P}\left\{J_{X^P}\left((\mathbb{R}\backslash[-1,1]) \times J_{n,k}\right) \geqslant 1\right\},\end{aligned}$$

where $J_{n,k}$ is defined as the following set:

$$J_{n,k} = \left[\mathbf{0}_d, \left(1+\frac{1}{S_n^k}, \ldots, 1+\frac{1}{S_n^k}, S_n+\frac{1}{S_n^k}\right)\right[\setminus [\mathbf{0}_d, (\mathbf{1}_{d-1}, S_n)] = J_{n,k}^1 \cup J_{n,k}^2,$$

where $J_{n,k}^1$ and $J_{n,k}^2$ are disjoint sets defined by (see illustration in Figure 3.3)

$$\begin{aligned}J_{n,k}^1 &= \Big\{x \in \mathbb{R}_+^d : \forall\, 1 \leqslant i \leqslant d-1,\ x_i < 1+\frac{1}{S_n^k},\ x_d \leqslant S_n,\\ &\qquad\qquad \text{and } \exists i_0 \in \{1,\ldots,d-1\} \text{ s.t. } x_{i_0} > 1\Big\},\\ J_{n,k}^2 &= \left]\left(\mathbf{0}_{d-1}, S_n\right), \left(1+\frac{1}{S_n^k}, \ldots, 1+\frac{1}{S_n^k}, S_n+\frac{1}{S_n^k}\right)\right[.\end{aligned}$$

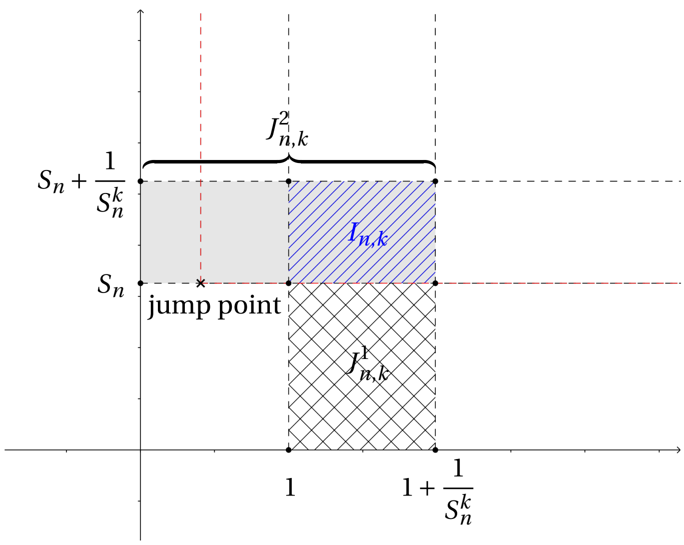

Figure 3.3 – Representation of $I_{n,k}$, $J_{n,k}^1$ and $J_{n,k}^2$ when $d = 2$.

Therefore we can write

$$\begin{aligned}\mathbb{P}(A^c_{n,k}) &= \mathbb{P}\left\{J_{X^P}\left((\mathbb{R}\setminus[-1,1])\times J^1_{n,k}\right)\right.\\ &\qquad\left.+\,J_{X^P}\left((\mathbb{R}\setminus[-1,1])\times J^2_{n,k}\right)\geqslant 1\right\}\\ &\leqslant \mathbb{P}\left\{J_{X^P}\left((\mathbb{R}\setminus[-1,1])\times J^1_{n,k}\right)\geqslant 1\right\}\\ &\qquad+\mathbb{P}\left\{J_{X^P}\left((\mathbb{R}\setminus[-1,1])\times J^2_{n,k}\right)\geqslant 1\right\}.\end{aligned}\tag{3.2.11}$$

Let $\mathscr{F}_{(\mathbf{1}_{d-1},t)} = \sigma(X_s,\ s\in[\mathbf{0}_d,(\mathbf{1}_{d-1},t)])$ and $\mathscr{F}_{(\mathbf{1}_{d-1},\infty)} = \bigvee_{t\in\mathbb{R}_+}\mathscr{F}_{(\mathbf{1}_{d-1},t)}$. We also write $H_1 = \left\{x\in\mathbb{R}^d_+ : x_1\leqslant 1,\dots,x_{d-1}\leqslant 1\right\}$. Then, due to the independence of the increments of X^P, the collection of random variables $\left(J_{X^P}((\mathbb{R}\setminus[-1,1])\times A)\right)_{A\subset\mathbb{R}^d_+\setminus H_1}$ is independent of $\mathscr{F}_{(\mathbf{1}_{d-1},\infty)}$. Since S_n is $\mathscr{F}_{(\mathbf{1}_{d-1},\infty)}$-measurable, we deduce that conditionally on S_n, the random variable

$$J_{X^P}\left((\mathbb{R}\setminus[-1,1])\times J^1_{n,k}\right)$$

has a Poisson law with parameter $\lambda\mathrm{Leb}_d(J^1_{n,k})$, where $\lambda := \int_{|z|>1}\nu(dz)$. Further, on the event $\{S_n\geqslant 1\}$,

$$\mathrm{Leb}_d\left(J^1_{n,k}\right) = \sum_{j=1}^{d-1}\binom{d-1}{j}S_n\left(\frac{1}{S_n^k}\right)^j\left(1+\frac{1}{S_n^k}\right)^{d-1-j}\leqslant 3^{d-1}S_n^{-(k-1)}.$$

Indeed, the Lebesgue measure of a subset of $J^1_{n,k}$ of vectors with exactly j components strictly greater than one is $S_n\left(\frac{1}{S_n^k}\right)^j\left(1+\frac{1}{S_n^k}\right)^{d-1-j}$, and there are $\binom{d-1}{j}$ such subsets. We deduce that

$$\begin{aligned}&\mathbb{P}\left\{J_{X^P}\left((\mathbb{R}\setminus[-1,1])\times J^1_{n,k}\right)\geqslant 1\right\}\\ &\qquad\leqslant \mathbb{P}\{S_n\leqslant 1\}+\mathbb{E}\left(\mathbb{1}_{S_n>1}\left(1-e^{-\lambda\mathrm{Leb}_d\left(J^1_{n,k}\right)}\right)\right)\\ &\qquad\leqslant \mathbb{P}\{S_n\leqslant 1\}+\lambda 3^{d-1}\mathbb{E}\left(S_n^{-(k-1)}\right).\end{aligned}\tag{3.2.12}$$

We also define a process $\tilde{L}_t = X^P_{(\mathbf{2}_{d-1},t)}$. It is a Lévy process with Lévy measure $\mu(\mathrm{d}x) = 2^{d-1}\mathbb{1}_{|x|>1}\nu(\mathrm{d}x)$. Since X^P is piecewise constant, $\tilde{L}$ is a piecewise constant Lévy process, therefore a compound Poisson process (see [63, Theorem 21.2]). On the event $\{S_n>1\}$, we have

$$J^2_{n,k}\subset[(\mathbf{0}_{d-1},S_n),(\mathbf{2}_{d-1},S_n+S_n^{-k})].$$

Therefore, if X^P has a jump point in $J^2_{n,k}$, then $\tilde{L}$ has a jump in $\left]S_n,S_n+S_n^{-k}\right[$. Let $\mathscr{G}_t = \sigma(X_u : u\in[0,(\mathbf{2}_{d-1},t])$. Then, S_n is a $\mathscr{G}$-stopping time and $\tilde{L}$ is a Lévy process adapted to the filtration $\mathscr{G}$, so by the strong Markov property, the number of jumps of the process $\hat{L}_\cdot = \tilde{L}_{\cdot+S_n}-\tilde{L}_{S_n}$ is independent of S_n and has Poisson distribution of parameter $2^{d-1}\lambda t$. Therefore we can write

$$\begin{aligned}
&\mathbb{P}\left\{J_{X^P}\left((\mathbb{R}\setminus[-1,1])\times J^2_{n,k}\right)\geqslant 1\right\}\\
&\quad\leqslant \mathbb{P}\{S_n\leqslant 1\}+\mathbb{P}\left(\left\{J_{X^P}\left((\mathbb{R}\setminus[-1,1])\times J^2_{n,k}\right)\geqslant 1\right\}\cap\{S_n>1\}\right)\\
&\quad\leqslant \mathbb{P}\{S_n\leqslant 1\}+\mathbb{P}\left\{\tilde{L}\text{ has a jump in }\left(S_n,S_n+\frac{1}{S_n^k}\right)\right\}\\
&\quad= \mathbb{P}\{S_n\leqslant 1\}+\mathbb{P}\left\{\hat{L}\text{ has a jump in }\left(0,\frac{1}{S_n^k}\right)\right\}\\
&\quad= \mathbb{P}\{S_n\leqslant 1\}+\mathbb{E}\left(1-\exp\left[-\frac{2^{d-1}\lambda}{S_n^k}\right]\right)\\
&\quad\leqslant \mathbb{P}\{S_n\leqslant 1\}+\mathbb{E}\left(\frac{2^{d-1}\lambda}{S_n^k}\right).
\end{aligned} \tag{3.2.13}$$

Using the density of the Gamma distribution, we see that

$$\mathbb{P}\{S_n\leqslant 1\}=\int_0^1\frac{\lambda^n}{(n-1)!}e^{-\lambda x}x^{n-1}\,\mathrm{d}x\leqslant\frac{\lambda^n}{(n-1)!}. \tag{3.2.14}$$

Integrating the Laplace transform of S_n as in (3.1.7), for $n\geqslant 4$, we see that

$$\mathbb{E}\left(S_n^{-3}\right)=\frac{\lambda^3}{(n-1)(n-2)(n-3)}\qquad\text{and}\qquad\mathbb{E}\left(S_n^{-2}\right)=\frac{\lambda^2}{(n-1)(n-2)}. \tag{3.2.15}$$

Then we get from (3.2.11),(3.2.12), (3.2.13) with $k=3$, (3.2.14) and (3.2.15), that for $n\geqslant 4$,

$$\begin{aligned}
\mathbb{P}\left(A^c_{n,3}\right)&\leqslant\frac{2\lambda^n}{(n-1)!}+\lambda 3^{d-1}\mathbb{E}\left(\frac{1}{S_n^2}\right)+\lambda 2^{d-1}\mathbb{E}\left(\frac{1}{S_n^3}\right)\\
&=\frac{2\lambda^n}{(n-1)!}+\frac{\lambda^2 3^{d-1}}{(n-1)(n-2)}+\frac{\lambda^3 2^{d-1}}{(n-1)(n-2)(n-3)},
\end{aligned}$$

and we deduce that $\sum_{n\geqslant 1}\mathbb{P}\left(A^c_{n,3}\right)<\infty$. By the Borel-Cantelli Lemma,

$$\mathbb{P}\left(\limsup_{n\to+\infty}A^c_{n,3}\right)=0,$$

and the set $\Omega'=\liminf_{n\to+\infty}A_{n,3}$ has probability one. This completes the proof. □

We now return to the question of whether or not a Lévy white noise is a tempered distribution. Similar to (3.1.15), for any d-dimensional Lévy noise $\dot{X}$, we define the set $\Omega_{\dot{X}}$ by

$$\Omega_{\dot{X}}=\left\{\omega\in\Omega:\dot{X}(\omega)\in\mathscr{S}'(\mathbb{R}^d)\right\}, \tag{3.2.16}$$

and we have the following characterization.

Theorem 3.2.7. *Let X be a d-parameter Lévy field with jump measure J_X and characteristic* *triplet* (γ,σ,ν) *and* $\dot{X}$ *the associated Lévy white noise. Then the following holds for the set* $\Omega_{\dot{X}}$ *defined in* (3.2.16) *and the set* Ω_X *defined as in* (3.2.1)*:*

(i) If there exists $\eta > 0$ *such that* $\mathbb{E}\left(|X_{\mathbf{1}_d}|^\eta\right) < +\infty$*, then* $\mathbb{P}(\Omega_X) = \mathbb{P}\left(\Omega_{\dot{X}}\right) = 1$.

(ii) If for all $\eta > 0$, $\mathbb{E}\left(|X_{\mathbf{1}_d}|^\eta\right) = +\infty$*, then* $\mathbb{P}(\Omega_X) = \mathbb{P}\left(\Omega_{\dot{X}}\right) = 0$.

Remark 3.2.8. *By Lemma 3.2.5, the equivalent condition mentioned in Remark 3.1.6 remains valid in the* d*-parameter case.*

As mentioned in the introduction to this section, the first assertion of Theorem 3.2.7 was established in [34, Theorem 3] using a different definition of Lévy white noise. In Lemma 2.2.9 earlier, we have shown that the two definitions are equivalent.

Proof of Theorem 3.2.7. To prove *(i)*, by the Lévy-Itô decomposition (Theorem 1.0.9), Corollary 3.2.2 and Lemma 3.2.3*(ii)*, we have $\mathbb{P}(\Omega_X) = 1$. Since derivation maps $\mathscr{S}'(\mathbb{R}^d)$ to itself, we deduce that $\mathbb{P}\left(\Omega_{\dot{X}}\right) = 1$.

To prove *(ii)*, suppose that $\dot{X}$ does not have a **PAM**. We can use Theorem 1.0.9 to decompose X into the sum of a continuous part C, a small jumps part X^M and a compound Poisson part X^P. By Corollary 3.2.2, $\mathbb{P}\left(\Omega_{C+X^M}\right) = 1$. Then we deduce that for all $\omega \in \Omega_{\dot{X}} \cap \Omega_{C+X^M}$,

$$\dot{X}^P(\omega) = \dot{X}(\omega) - \dot{C}(\omega) - \dot{X}^M(\omega) = \dot{X}(\omega) - \left(C(\omega) + X^M(\omega)\right)^{(\mathbf{1}_d)}$$

belongs to $\mathscr{S}'(\mathbb{R}^d)$. The general strategy of the proof is to construct, from the compound Poisson sheet X^P, a compound Poisson process that has the same moment properties, and show that when $\dot{X}^P \in \mathscr{S}'(\mathbb{R}^d)$, this process has polynomial growth at infinity, and this occurs with probability zero by Proposition 3.1.4*(ii)*.

We first examine the noise $\dot{X}^P$ associated with the compound Poisson part. The jump measure $J_{X^P}(\mathrm{d}s,\mathrm{d}z) = \mathbb{1}_{|z|>1} J_X(\mathrm{d}s,\mathrm{d}z)$ of X^P is a Poisson random measure on $\mathbb{R}_+^d \times (\mathbb{R}\backslash\{0\})$ and $J_{X^P} = \sum_{i\geqslant 1} \delta_{\tau_i}\delta_{Z_i}$, where $\tau_i \in \mathbb{R}_+^d$ and $|Z_i| \geqslant 1$. By Lemma 2.2.9, for all $\varphi \in \mathscr{D}(\mathbb{R}^d)$,

$$\left\langle \dot{X}^P, \varphi \right\rangle = \int_{\mathbb{R}_+^d} \int_{|z|>1} z\varphi(t) J_X(\mathrm{d}t,\mathrm{d}z) = \sum_{i\geqslant 1} Z_i \varphi(\tau_i)\ .$$

By Lemma 3.2.6, for all $\omega \in \Omega$, there exists a sequence $(\varphi_n)_{n\geqslant 1}(\omega)$ of smooth compactly supported functions such that (3.2.9) holds. Furthermore, there is an event $\Omega' \subset \Omega$ with probability one such that there is an integer $N(\omega)$ with the property that for all $n \geqslant N(\omega)$, X^P is constant on the support of $\varphi_n(\omega)$, and (3.2.10) holds. Let L be the compound Poisson process defined in Lemma 3.2.6 by $L_t = X^P_{(\mathbf{1}_{d-1},t)}$. We restrict ourselves to $\omega \in \Omega_{\dot{X}} \cap \Omega_{C+X^M} \cap \Omega'$, but we drop the dependence on ω in the following for simplicity of notation. We write $\Phi_n(t) = \int_{[t,+\infty)} \varphi_n(s)\,\mathrm{d}s$. Let $\theta \in C^\infty(\mathbb{R}^d)$ be such that $\theta = 0$ on the set $\left\{t \in \mathbb{R}^d : t_1 \wedge \ldots \wedge t_d \leqslant -1\right\}$ and $\theta = 1$ on the set $\left\{t \in \mathbb{R}^d : t_1 \wedge \ldots \wedge t_d \geqslant -\frac{1}{2}\right\}$ and such that all its derivatives are bounded. We give here an example of a construction of such a function (taken from [38]): Let $E:\mathbb{R}\longrightarrow\mathbb{R}$ defined by $E(x)=e^{-\frac{1}{x}}$ if $x>0$ and $E(x)=0$ otherwise. Then the function $F(x)=E(1+x)E\left(-x-\frac{1}{2}\right)$ belongs to $\mathscr{D}(\mathbb{R})$ with support in $]-1;-\frac{1}{2}[$. We can then set

$$I(x)=\frac{\int_{-\infty}^{x}F(t)\,\mathrm{d}t}{\int_{\mathbb{R}}F(t)\,\mathrm{d}t}\,.$$

I is a smooth function such that $I(x)=0$ if $x\leqslant -1$, and $I(x)=1$ if $x\geqslant -\frac{1}{2}$. Finally we can choose

$$\theta=\underbrace{I\otimes\ldots\otimes I}_{d\text{ times}}.$$

Then for all $n\geqslant 1, \theta\Phi_n\in\mathscr{D}(\mathbb{R}^d)\subset\mathscr{S}(\mathbb{R}^d)$. So, in particular, for all $n\geqslant 1$, since θ is constant on $\mathbb{R}_+^d$,

$$\begin{aligned}\left\langle \dot{X}^P,\theta\Phi_n\right\rangle &= (-1)^d\left\langle X^P,(\theta\Phi_n)^{(\mathbf{1}_d)}\right\rangle\\ &=(-1)^d\left\langle X^P,(\Phi_n)^{(\mathbf{1}_d)}\right\rangle=\left\langle X^P,\varphi_n\right\rangle=L_{S_n}\,,\end{aligned}$$

by (3.2.10), and since $\Omega_{\dot{X}}\cap\Omega_{C+X^M}\subset\Omega_{\dot{X}^P}$, we deduce that

$$\left|L_{S_n}\right|\leqslant C\mathscr{N}_p(\theta\Phi_n)\,,\tag{3.2.17}$$

for some real number C and integer p (both depending on ω). For $\alpha,\beta\in\mathbb{N}^d$, with $|\alpha|,|\beta|\leqslant p$, we estimate $\sup_{t\in\mathbb{R}^d}|t^\alpha(\theta\Phi_n)^{(\beta)}|$. Since all the derivatives of θ are bounded,

$$\begin{aligned}\sup_{t\in\mathbb{R}^d}\left|t^\alpha(\theta\Phi_n)^{(\beta)}(t)\right| &= \sup_{t\geqslant-\mathbf{1}_d}\left|t^\alpha(\theta\Phi_n)^{(\beta)}(t)\right|\\ &=\sup_{t\geqslant-\mathbf{1}_d}\left|t^\alpha\sum_{\gamma\leqslant\beta}\binom{\beta}{\gamma}\Phi_n^{(\gamma)}(t)\theta^{(\beta-\gamma)}(t)\right|\\ &\leqslant C_1\sum_{\gamma\leqslant\beta}\sup_{t\geqslant-\mathbf{1}_d}\left|t^\alpha\Phi_n^{(\gamma)}(t)\right|\,,\end{aligned}$$

for some constant C_1 depending only on p and θ. By (3.2.8), for some constant C_2,

$$\sup_{t\geqslant-\mathbf{1}_d}\left|t^\alpha\Phi_n^{(\gamma)}(t)\right|\mathbb{1}_{S_n\geqslant1}\leqslant C_2\mathscr{N}_{p+2d}(\varphi_n)\mathbb{1}_{S_n\geqslant1}\leqslant C_3S_n^{\tilde{p}}\mathbb{1}_{S_n\geqslant1}\,,$$

by (3.2.9), for some constant C_3 and $\tilde{p}$ independent of n. Therefore, for any integer p, there is an integer $\tilde{p}$ and a constant C depending only p and d, such that

$$\mathscr{N}_p(\theta\Phi_n)\mathbb{1}_{S_n\geqslant1}\leqslant CS_n^{\tilde{p}}\mathbb{1}_{S_n\geqslant1}\,.\tag{3.2.18}$$

We deduce from (3.2.17) and (3.2.18) that

$$\left|\frac{L_{S_n}}{S_n^{\tilde{p}}}\right|\mathbb{1}_{S_n\geqslant1}\leqslant C\mathbb{1}_{S_n\geqslant1}<+\infty\,.$$

As in the proof of Proposition 3.1.2, we deduce that for all $\omega \in \Omega_{\dot{X}} \cap \Omega_{C+X^M} \cap \Omega'$, there exists $p(\omega) \in \mathbb{N}$ and $C(\omega) \in \mathbb{R}_+$ such that

$$\limsup_{t\to+\infty} \frac{|L_t|(\omega)}{1+t^{\tilde{p}(\omega)}} \leqslant C(\omega) < +\infty .$$

Since L is a compound Poisson process with no absolute moment of any positive order (it has the same Lévy measure as X^P) we can now conclude by Proposition 3.1.4*(ii)* that $\Omega_{\dot{X}} \cap \Omega_{C+X^M} \cap \Omega'$ is contained in a set of probability zero. Since $\mathbb{P}\left(\Omega_{C+X^M} \cap \Omega'\right) = 1$, we deduce that $\mathbb{P}\left(\Omega_{\dot{X}}\right) = 0$.

By the fact that the derivative of a tempered distribution is a tempered distribution, $\Omega_X \subset \Omega_{\dot{X}}$. Therefore, $\mathbb{P}(\Omega_X) = 0$. □

Corollary 3.2.9. *Let X be a d-parameter Lévy field with characteristic triplet (γ,σ,ν), let $\dot{X}$ be the associated Lévy noise and suppose it has a **PAM**. Then there is a random tempered distribution S, that is, a measurable map from $(\Omega,\mathscr{F})$ to $\left(\mathscr{S}'(\mathbb{R}^d),\mathscr{B}\right)$, where $\mathscr{B}$ is the Borel σ-field for the weak-$*$ topology, such that almost surely, for all $\varphi \in \mathscr{S}(\mathbb{R}^d)$,*

$$\langle S,\varphi\rangle = \langle \dot{X},\varphi\rangle = (-1)^d \int_{\mathbb{R}^d_+} X_t \varphi^{(\mathbf{1}_d)}(t)\,\mathrm{d}t .$$

In addition, the maps $C:\omega \mapsto C(\omega)$ and $p:\omega\mapsto p(\omega)$ such that for all $\varphi \in \mathscr{S}(\mathbb{R}^d)$,

$$\left|\langle S,\varphi\rangle\right| \leqslant C\mathscr{N}_p(\varphi) \qquad a.s.$$

can be chosen to be $\mathscr{F}$-measurable.

Proof. We already know from Theorem 3.2.7 that $\mathbb{P}\left(\Omega_{\dot{X}}\right) = 1$. We define S to be equal to $\dot{X}$ (in $\mathscr{S}'(\mathbb{R}^d)$) on $\Omega_{\dot{X}}$ and zero elsewhere. We want to be able to consider S as a measurable map with values in $\mathscr{S}'(\mathbb{R}^d)$. We recall that a basis for the weak-$*$ topology on $\mathscr{S}'(\mathbb{R}^d)$ is given by cylinder sets of the form

$$O = \bigcap_{i=1}^{n} \left\{u \in \mathscr{S}'(\mathbb{R}) : \langle u,\varphi_i\rangle \in A_i\right\}, \tag{3.2.19}$$

where, for all $i \leqslant n$, φ_i is an element of $\mathscr{S}(\mathbb{R}^d)$, n is an integer and A_i is an open set in $\mathbb{R}$. The σ-field generated by all cylinder sets is called the cylinder σ-algebra and is denoted by $\mathscr{C}$. We first show that $S:(\Omega,\mathscr{F}) \longrightarrow \left(\mathscr{S}'(\mathbb{R}^d),\mathscr{C}\right)$ is measurable. For this, clearly, it suffices to show that for all cylinder sets O as above, the set $S^{-1}(O) = \{\omega\in\Omega : S(\omega)\in O\}$ belongs to $\mathscr{F}$. Clearly,

$$S^{-1}(O) = \bigcap_{i=1}^{n} \left\{\omega \in \Omega : \langle S(\omega),\varphi_i\rangle \in A_i\right\} .$$

The map $(t,\omega) \to X_t(\omega)$ is jointly measurable so, by Fubini's Theorem, the map $\langle S,\varphi_i\rangle : \Omega \longrightarrow \mathbb{R}$ is $\mathscr{F}$-measurable and therefore $S^{-1}(O) \in \mathscr{F}$. The Borel σ-field $\mathscr{B}$ contains $\mathscr{C}$ since every cylinder set is an open set. The converse inclusion is not immediate: see [29, Proposition

2.1] for a proof of the equality $\mathscr{B} = \mathscr{C}$. This fact is also mentioned in [35, p. 41]. For the convenience of the reader, we also include a more detailed proof that follows the argument in [29, Proposition 2.1]. We will first prove what seems to be a well-known fact (but is surprisingly difficult to find in the literature if one wants to avoid advanced notions of topology on locally convex vector spaces) that $\mathscr{S}'(\mathbb{R}^d)$ is a separable space. We know that $\mathscr{S}(\mathbb{R}^d)$ is separable for the topology described in Section 2.1.2 (see [56, 10.3.4 p. 176]), that is, there is a countable dense subset $A \subset \mathscr{S}(\mathbb{R}^d)$. It is also well known that $\mathscr{S}(\mathbb{R}^d) \subset \mathscr{S}'(\mathbb{R}^d)$ is dense for the weak-$*$ topology (that is, for any $u \in \mathscr{S}'(\mathbb{R}^d)$, there is a sequence $u_n \in \mathscr{S}(\mathbb{R}^d)$ such that for all $\varphi \in \mathscr{S}(\mathbb{R}^d)$, $\langle u_n, \varphi\rangle \to \langle u, \varphi\rangle$ as $n \to \infty$, see for example [41, Theorem 11.23]). Let $u \in \mathscr{S}'(\mathbb{R}^d)$. There is a sequence $(\varphi_n)_{n\geqslant 0}$ of functions of $\mathscr{S}(\mathbb{R}^d)$ such that $\varphi_n \to u$ in $\mathscr{S}'(\mathbb{R}^d)$ as $n \to +\infty$. Then, for each $n \geqslant 0$, there is a sequence $(\psi_{n,m})_{m\geqslant 0}$ of functions of A such that for any $p \in \mathbb{N}$, $\mathscr{N}_p(\varphi_n - \psi_{n,m}) \to 0$ as $n \to +\infty$. Let $\varepsilon > 0$ and $\theta \in \mathscr{S}(\mathbb{R}^d)$.

$$\left|\langle u,\theta\rangle - \langle \psi_{n,m},\theta\rangle\right| \leqslant \left|\langle u-\varphi_n,\theta\rangle\right| + \left|\langle \varphi_n - \psi_{n,m},\theta\rangle\right|$$

We can choose n large enough so that $\left|\langle u-\varphi_n,\theta\rangle\right| \leqslant \varepsilon$. Then, since $\psi_{n,m} \to \varphi_n$ in $\mathscr{S}(\mathbb{R}^d)$, the convergence is also in $L^2(\mathbb{R}^d)$ and for m large enough, $\left|\langle \varphi_n - \psi_{n,m},\theta\rangle\right| \leqslant \varepsilon$. We deduce that there is a sequence of functions $\tilde{\psi}_n$ in A such that $\langle u - \tilde{\psi}_n, \theta\rangle \to 0$ as $n \to +\infty$ for any $\theta \in \mathscr{S}(\mathbb{R}^d)$. Therefore, A is dense in $\mathscr{S}'(\mathbb{R}^d)$ for the weak-$*$ topology, and $\mathscr{S}'(\mathbb{R}^d)$ is separable.

Following the lines of the proof of [29, Proposition 2.1], we define

$$M_j = \left\{ u \in \mathscr{S}'(\mathbb{R}^d) : \sup_{\varphi\in\mathscr{S}(\mathbb{R}^d)} \frac{|\langle u,\varphi\rangle|}{\mathscr{N}_j(\varphi)} \leqslant j \right\}.$$

Then $\mathscr{S}'(\mathbb{R}^d) = \bigcup_{j=0}^{+\infty} M_j$. Indeed, the first inclusion $\mathscr{S}'(\mathbb{R}^d) \supset \bigcup_{j=0}^{+\infty} M_j$ is obvious. For the other inclusion, let $u \in \mathscr{S}'(\mathbb{R}^d)$. By definition, there is a $p \in \mathbb{N}$ and $C \in \mathbb{R}$, such that for any $\varphi \in \mathscr{S}(\mathbb{R}^d)$,

$$\left|\langle u,\varphi\rangle\right| \leqslant C\mathscr{N}_p(\varphi).$$

Since the family of semi norms $\left(\mathscr{N}_p\right)_{p\geqslant 0}$ is increasing, we deduce that for all $\varphi \in \mathscr{S}(\mathbb{R}^d)$,

$$\sup_{\varphi\in\mathscr{S}(\mathbb{R}^d)} \frac{|\langle u,\varphi\rangle|}{\mathscr{N}_{p\vee\lceil C\rceil}(\varphi)} \leqslant p \vee \lceil C\rceil,$$

that is $u \in M_{p\vee\lceil C\rceil}$.

Let $k \in \mathbb{N}$. We claim that for any $u_0 \in M_k$, the countable family of cylinder sets of the form

$$E^k_\varepsilon(u_0,\varphi_1,\ldots,\varphi_n) = \bigcap_{i=1}^{n} \left\{u \in M_k : \left|\langle u,\varphi_i\rangle - \langle u_0,\varphi_i\rangle\right| < \varepsilon\right\},$$

where $\varepsilon \in \mathbb{Q}_+$, $n \in \mathbb{N}$ and $\varphi_i \in A$, is a basis of neighborhoods of u_0 for the restriction of the weak-$*$ topology to M_k, that is, for any neighborhood V of u_0 in M_k, there is a set $E^k_\varepsilon(u_0,\varphi_1,\ldots,\varphi_n) \subset V$ for some $\varepsilon \in \mathbb{Q}_+$ and some $\varphi_1,\ldots,\varphi_n \in A$. Let V be a neighborhood of u_0. By definition there is an open set $U \subset V$ such that $u_0 \in U$. The set U can be written as a union (not necessarily countable) of sets of the form (3.2.19). Therefore, there is an integer n, functions $\varphi_1, \dots, \varphi_n \in \mathscr{S}(\mathbb{R}^d)$ and open sets $A_1, \dots, A_n \subset \mathbb{R}$ such that

$$u_0 \in O = \bigcap_{i=1}^{n} \left\{ u \in \mathscr{S}'(\mathbb{R}) : \langle u, \varphi_i \rangle \in A_i \right\} \subset V .$$

Since A_i is an open set, there exists $\alpha_i > 0$ such that $B(\langle u_0, \varphi_i \rangle, \alpha_i) \subset A_i$, where $B(x,r)$ denotes the open ball centered at x of radius r. Let $\alpha = \min_{i \leqslant n} \alpha_i$. Let $\varepsilon \in \mathbb{Q}_+^*$ and $\tilde{\varphi}_1, \dots, \tilde{\varphi}_n \in A$. Let $u \in E_\varepsilon^k(u_0, \tilde{\varphi}_1, \dots, \tilde{\varphi}_n)$.

$$\begin{aligned} \left| \langle u, \varphi_i \rangle - \langle u_0, \varphi_i \rangle \right| &\leqslant \left| \langle u, \varphi_i \rangle - \langle u, \tilde{\varphi}_i \rangle \right| + \left| \langle u, \tilde{\varphi}_i \rangle - \langle u_0, \tilde{\varphi}_i \rangle \right| + \left| \langle u_0, \tilde{\varphi}_i \rangle - \langle u_0, \varphi_i \rangle \right| \\ &\leqslant k\mathscr{N}_k(\varphi_i - \tilde{\varphi}_i) + \varepsilon + k\mathscr{N}_k(\varphi_i - \tilde{\varphi}_i) . \end{aligned}$$

We choose $\tilde{\varphi}_i \in A$ such that $\max_{i \leqslant n} \mathscr{N}_k(\varphi_i - \tilde{\varphi}_i) \leqslant \frac{\alpha}{8k}$ and choose $\varepsilon \leqslant \frac{\alpha}{4}$. Then,

$$\left| \langle u, \varphi_i \rangle - \langle u_0, \varphi_i \rangle \right| \leqslant \frac{\alpha}{2} < \alpha .$$

Therefore, for all $i \leqslant n$, $\langle u, \varphi_i \rangle \in A_i$, that is, $u \in O$ and we deduce that $E_\varepsilon^k(u_0, \tilde{\varphi}_1, \dots, \tilde{\varphi}_n) \subset O$, which proves the claim.

Now, let U be an open set for the weak-$*$ topology on $\mathscr{S}'(\mathbb{R}^d)$. Let

$$\tilde{U} := \bigcup_{u \in \cap A \cap U} \bigcup_{p=1}^{+\infty} \bigcup_{\varepsilon \in \mathbb{Q}_+^*} \bigcup_{n \in \mathbb{N}} \bigcup_{\Phi \in A^n} \bigcup_{E_\varepsilon^p(u, \Phi_1, \dots, \Phi_n) \subset U} E_\varepsilon^k(u, \Phi_1, \dots, \Phi_n) . \tag{3.2.20}$$

We claim that $\tilde{U} = U$. Only the inclusion $U \subset \tilde{U}$ is not obvious. Let $u_0 \in U$. There is a $k \in \mathbb{N}$ such that $u_0 \in M_k$. Since U is open, it is a neighborhood of u_0, and as such, there is an integer n, functions $\varphi_1, \dots, \varphi_n \in \mathscr{S}(\mathbb{R}^d)$ and open sets $A_1, \dots, A_n \subset \mathbb{R}$ such that

$$u_0 \in O = \bigcap_{i=1}^{n} \left\{ u \in \mathscr{S}'(\mathbb{R}) : \langle u, \varphi_i \rangle \in A_i \right\} \subset U .$$

Then there is $\varepsilon \in \mathbb{Q}_+^*$ such that for all $i \leqslant n$, $B(\langle u_0, \varphi_i \rangle, \varepsilon) \subset A_i$. The set A is dense in $\mathscr{S}'(\mathbb{R}^d)$, therefore there is a function $\tilde{u} \in A \cap U$ such that for all $i \leqslant n$, $|\langle u_0 - \tilde{u}, \varphi_i \rangle| \leqslant \frac{\varepsilon}{18}$. Then $\tilde{u} \in M_{\tilde{k}}$ for some $\tilde{k} \in \mathbb{N}$. Since the semi-norms $\mathscr{N}_p$ are increasing, we have that $M_p \subset M_{p+1}$ for all $p \in \mathbb{N}$. Let $p = k \vee \tilde{k}$. Then $\tilde{u}, u_0 \in M_p$. Since A is dense in $\mathscr{S}(\mathbb{R}^d)$, we can find $\Phi \in A^n$ such that $p\mathscr{N}_p(\varphi_i - \Phi_i) \leqslant \frac{\varepsilon}{18}$, for all $i \leqslant n$. Then $u_0 \in E_{\frac{\varepsilon}{3}}^p(\tilde{u}, \Phi_1, \dots, \Phi_n) \subset O \subset U$ and then $u_0 \in \tilde{U}$ and $U = \tilde{U}$.

Since all the unions in (3.2.20) are countable, we deduce that any open set can be written as a countable union of cylinder sets, therefore $\mathscr{B} \subset \mathscr{C}$. Since we have already pointed out that the converse inclusion holds, it follows that, $\mathscr{C} = \mathscr{B}$.

Therefore, the map $S : (\Omega, \mathscr{F}) \longrightarrow \left(\mathscr{S}'(\mathbb{R}^d), \mathscr{B}\right)$ is measurable, and S defines a random tempered distribution.

Furthermore, since the space $\mathscr{S}(\mathbb{R}^d)$ is separable (see [56, 10.3.4 p.176]), we let A be a countable dense subset. Then the measurability of the maps C and p comes from the fact that we can choose

$$p(\omega) = \min\left\{ p \in \mathbb{N} : \sup_{\varphi \in A} \frac{\left|\langle S, \varphi\rangle\right|}{\mathscr{N}_p(\varphi)}(\omega) < +\infty \right\},$$

and

$$C(\omega) = \sup_{\varphi \in A} \frac{\left|\langle S, \varphi\rangle\right|}{\mathscr{N}_{p(\omega)}(\varphi)}(\omega).$$

□

Remark 3.2.10. *In dimension one, we used the map A in Remark 3.1.8 to give an alternate proof of the inclusion $\Omega_{\dot{X}} \subset \Omega_X$. The analog of this map A in higher dimensions also exists. Let $\theta \in \mathscr{D}(\mathbb{R})$ such that $\theta \geqslant 0$, $\operatorname{supp}\theta \subset [0,1]$ and $\int_{\mathbb{R}} \theta = 1$. We write $\tilde{\theta} = \theta \otimes \cdots \otimes \theta$ the d^{th}-order tensor product of θ with itself: $\tilde{\theta}(s_1, \ldots, s_d) = \theta(s_1)\cdots\theta(s_d)$. Let $\varphi \in \mathscr{S}(\mathbb{R}^d)$. Define*

$$A_d\varphi(t) = \int_{(-\infty, t]} \mathrm{d}s \int_{\mathbb{R}^d} \mathrm{d}r\, \tilde{\Delta}_r^s\left(\varphi, \tilde{\theta}\right),$$

where

$$\tilde{\Delta}_r^s\left(\varphi, \theta\right) = \sum_{\varepsilon \in \{0,1\}^d} (-1)^{|\varepsilon|} \varphi(c_\varepsilon(r,s))\tilde{\theta}(c_{1-\varepsilon}(r,s)),$$

and $c_\varepsilon(r,s)$ was defined just after (1.0.1). It is easy to see that if $\varphi = \varphi_1 \otimes \cdots \otimes \varphi_d$, where $\varphi_1, \ldots, \varphi_d \in \mathscr{S}(\mathbb{R})$, then $A_d\varphi = \left(A_1\varphi_1\right) \otimes \cdots \otimes \left(A_1\varphi_d\right)$, where A_1 coincides with the map A of Remark 3.1.8. Then, since A was built as an antiderivative, for such φ,

$$A_d\left(\frac{\partial^d \varphi}{\partial t_1 \cdots \partial t_d}\right) = \varphi. \tag{3.2.21}$$

We have already shown that A_1 maps continuously $\mathscr{S}(\mathbb{R})$ to itself. We equip $\mathscr{S}(\mathbb{R}) \otimes \cdots \otimes \mathscr{S}(\mathbb{R})$ with the topology π generated by the family of semi-norms $\mathscr{N}_{p_1,\ldots,p_d}(\varphi_1 \otimes \cdots \otimes \varphi_d) = \prod_{i=1}^d \mathscr{N}_{p_i}(\varphi_i)$. Then $A_d : \mathscr{S}(\mathbb{R}) \otimes \cdots \otimes \mathscr{S}(\mathbb{R}) \to \mathscr{S}(\mathbb{R}) \otimes \cdots \otimes \mathscr{S}(\mathbb{R})$ is continuous (and then uniformly continuous by linearity). We denote $\mathscr{S}(\mathbb{R})\hat{\otimes}_\pi \cdots \hat{\otimes}_\pi \mathscr{S}(\mathbb{R})$ the completion of $\mathscr{S}(\mathbb{R}) \otimes \cdots \otimes \mathscr{S}(\mathbb{R})$. By [67, Theorem 51.6], $\mathscr{S}(\mathbb{R})\hat{\otimes}_\pi \cdots \hat{\otimes}_\pi \mathscr{S}(\mathbb{R}) \simeq \mathscr{S}(\mathbb{R}^d)$, therefore A_d extends (by uniform continuity) to a continuous linear map from $\mathscr{S}(\mathbb{R}^d)$ to itself. Formula (3.2.21) is true by linearity for $\varphi \in \mathscr{S}(\mathbb{R}) \otimes \cdots \otimes \mathscr{S}(\mathbb{R})$. Let $\varphi \in \mathscr{S}(\mathbb{R}^d)$. There is a sequence $(\varphi_n)_{n\geqslant 1}$ of elements of $\mathscr{S}(\mathbb{R}) \otimes \cdots \otimes \mathscr{S}(\mathbb{R})$ such that $\varphi_n \to \varphi$ in $\mathscr{S}(\mathbb{R}^d)$. Since derivation is a continuous map from $\mathscr{S}(\mathbb{R}^d)$ to itself, we deduce that (3.2.21) holds for any $\varphi \in \mathscr{S}(\mathbb{R}^d)$.

4 Random field solutions to a linear SPDE driven by Lévy white noise

In this chapter, we study two different notions of solutions to a linear stochastic partial differential equation driven by Lévy white noise. On the one hand, from the random field approach to SPDEs, we have the concept of mild solution, which is a random field solution defined as the convolution of a Green's function of the differential operator with the noise. The mild solution is therefore defined as a stochastic integral, and some conditions are needed for its existence. For example, the simple case of Gaussian white noise requires the Green's function to be square integrable. The theory of stochastic integration of deterministic functions with respect to Lévy white noise has been detailed in Chapter 2. In particular, we recalled that in [58], Rajput and Rosinski determined the space of integrable deterministic functions with respect to an independently scattered random measure, and our unification of these random measures with Lévy white noises (see Section 2.3.1) allows us to use the integrability criterions of Proposition 2.3.7. On the other hand, from the general theory on (deterministic) partial differential equations, we have the notion of weak solutions, or solutions in the sense of distributions. The terms "weak" and "distribution" in the context of probability theory and stochastic analysis can be confusing, since they are usually used for other notions. We will instead use the terms generalized solutions and generalized stochastic processes or random fields, in the spirit of the book [36].

Existence conditions for those two types of solutions are easy to derive, and in this chapter we are interested in the link between a mild solution and a generalized solution to a linear stochastic partial differential equation. The questions we study are the following:

(1) When it can be defined, is a mild solution also a generalized solution?

(2) When a generalized solution exists, can it be a mild solution, and under what conditions?

(3) What can be said in the case of the stochastic heat equation, or the stochastic wave equation, driven by an α-stable noise?

To answer these questions, we first introduce two different notions of solution to a linear SPDE in Section 4.1. Then, in Section 4.2, we provide an answer to question (1), first in the α-stable case in Theorem 4.2.1, and then in a more general case in Theorem 4.2.5. To prove these results, we also establish a new stochastic Fubini theorem in Theorem 4.2.3 that is interesting in its own right. Then, Section 4.3 deals with question (2), and a necessary condition for the generalized solution to have a random field representation is given in Theorem 4.3.1 for the α-stable case, and in Theorem 4.3.4 for a more general case. Finally, Section 4.4 deals with question (3), where we study applications of these results to the case of the stochastic heat equation and the stochastic wave equation in various dimensions. The main results can be found in Theorem 4.4.5 and Theorem 4.4.10.

Let $\dot{X}$ be a symmetric pure jump Lévy white noise on S, where S is a Borel measurable subset of $\mathbb{R}^d$, with characteristic triplet $(0,0,\nu)$. We consider the extension of $\dot{X}$ (still denoted by $\dot{X}$) to an independently scattered random measure (see Section 2.3.1), and we use the integration theory of deterministic functions with respect to $\dot{X}$ (see Definition 2.3.6). The characteristic triplet is always relative to a truncation function, and as we do throughout all this thesis, we will use the truncation function $z \to \mathbb{1}_{|z|\leqslant 1}$. More precisely, ν is a symmetric Lévy measure, and for any measurable set $A \subset S$ with finite Lebesgue measure,

$$\mathbb{E}\left(e^{iu\dot{X}(A)}\right) = \exp\left[\mathrm{Leb}_d(A)\int_{\mathbb{R}}\left(e^{iuz} - 1 - iuz\mathbb{1}_{|z|\leqslant 1}\right)\nu(\mathrm{d}z)\right], \qquad u \in \mathbb{R}.$$

We recall that from Definition 2.3.6, the set of functions that are integrable with respect to $\dot{X}$ is denoted $L(\dot{X}, S)$. We consider the stochastic partial differential equation

$$Lu = \dot{X}, \tag{4.0.1}$$

where L is a partial differential operator with adjoint L^* (think of L as the heat or wave operator typically). Let us consider a fundamental solution $\rho \in \mathscr{D}'(\mathbb{R}^d)$ to this equation, that is a solution to $L\rho = \delta_0$ in $\mathscr{D}'(\mathbb{R}^d)$. We recall the definition of the convolution between a distribution ρ and a smooth function with compact support φ.

$$\varphi * \rho(t) := \langle \rho, \varphi(t - \cdot)\rangle .$$

Note that in general, this convolution is a C^∞ function. Also, for $\varphi \in \mathscr{D}(\mathbb{R}^d)$, we define $\langle \check{\rho}, \varphi\rangle := \langle \rho, \check{\varphi}\rangle$, where for all $t \in \mathbb{R}^d$, $\check{\varphi}(t) := \varphi(-t)$.

4.1 Notions of solution to a linear SPDE

We introduce two different notions of solutions to the linear SPDE (4.0.1) with associated fundamental solution ρ. Notice that in this framework, we are only considering the case where the Green's function of the operator L is given by a shift of a fundamental solution.

4.1.1 Generalized solution

In the following we will need a hypothesis on the fundamental solution ρ of the differential operator L.

(H1) ρ is such that for any $\varphi \in \mathscr{D}(\mathbb{R}^d)$, the convolution $\varphi * \check{\rho}$ is in $L(\dot{X}, S)$.

The case where the noise is a symmetric α-stable noise for some $\alpha \in (0,2)$ is already quite rich, and provides some insights into the general theory. In this framework, by the example at the end of Chapter 2, **(H1)** becomes

(H1') ρ is such that for any $\varphi \in \mathscr{D}(\mathbb{R}^d)$, the convolution $\varphi * \check{\rho}$ is in $L^\alpha(S)$.

We can then define a generalized solution to (4.0.1).

Definition 4.1.1. *Assume **(H1)**. The* generalized solution *to the stochastic partial differential equation* (4.0.1) *is the linear functional u_{gen} on $\mathscr{D}(\mathbb{R}^d)$ such that for all $\varphi \in \mathscr{D}(\mathbb{R}^d)$,*

$$\langle u_{gen}, \varphi \rangle := \langle \dot{X}, \varphi * \check{\rho} \rangle . \tag{4.1.1}$$

Remark 4.1.2. *The generalized solution is in general not a distribution, since it may not define a continuous linear functional on $\mathscr{D}(\mathbb{R}^d)$. We may require additional properties on ρ to have this property.*

Remark 4.1.3. *The functional u_{gen} is a solution to* (4.0.1) *in the weak sense: for $\varphi \in \mathscr{D}(\mathbb{R}^d)$,*

$$\langle Lu_{gen}, \varphi \rangle = \langle u_{gen}, L^* \varphi \rangle = \langle \dot{X}, \left(L^* \varphi\right) * \check{\rho} \rangle .$$

Also,

$$\left(L^* \varphi\right) * \check{\rho}(t) = \langle \check{\rho}, L^* \varphi(t - \cdot) \rangle = \langle \rho, L^* \varphi(t + \cdot) \rangle = \langle \delta_0, \varphi(t + \cdot) \rangle = \varphi(t) ,$$

Therefore, for all $\varphi \in \mathscr{D}(\mathbb{R}^d)$,

$$\langle Lu_{gen}, \varphi \rangle = \langle \dot{X}, \varphi \rangle .$$

A generalized solution cannot in general be evaluated pointwise. However, a generalized function (i.e. a distribution in the sense of Schwartz) can sometimes be represented as a true function. This is the motivation for the following definition.

Definition 4.1.4. *We say a generalized stochastic process u has a random field representation if there exists a jointly measurable random field $(Y_t)_{t \in \mathbb{R}^d}$ such that Y has almost surely locally integrable sample paths, and for any $\varphi \in \mathscr{D}(\mathbb{R}^d)$,*

$$\langle u, \varphi \rangle = \int_{\mathbb{R}^d} Y_t \varphi(t) \, \mathrm{d}t \qquad a.s. \tag{4.1.2}$$

The generalized stochastic processes that have a random field representation are exactly those which can be evaluated pointwise. For example, the Dirac distribution δ_0 does not have a random field representation.

4.1.2 Mild solution

Generalized solutions are a useful generalization of classical solutions to a partial differential equation. However, non-linear operations on generalized functions are in general hard to define, and we are often interested in finding solutions that can be evaluated pointwise. One particularly popular type of solution from the SPDE literature is the notion of mild solution. Essentially, it consists in writing the equation in an integral form, making use of the fundamental solution. A mild solution is then defined as a fixed point of this integral formulation of our SPDE. Consequently, this fixed point formulation also gives us a way to prove the existence of a solution via a Picard iteration scheme. In order to be able to define a mild solution to (4.0.1), we will need another hypothesis on the fundamental solution ρ.

(H2) For any $t \in \mathbb{R}^d$, $\rho(t-\cdot) \in L(\dot{X}, S)$.

Again, in the case where the noise is a symmetric α-stable noise for some $\alpha \in (0,2)$ **(H2)** becomes

(H2') For any $t \in \mathbb{R}^d$, $\rho(t-\cdot) \in L^\alpha(S)$.

Definition 4.1.5. *Under hypothesis **(H2)**, we define the* mild solution *of* (4.0.1) *via the formula*

$$u_{mild}(t) := \left\langle \dot{X}, \rho(t-\cdot) \right\rangle . \tag{4.1.3}$$

Remark 4.1.6. *When it exists, the mild solution is always a random field, while the generalized solution is defined as a distribution. It might turn out that the generalized solution has a random field representation, and we can then wonder if this representation is the mild solution. This question is investigated in the sequel.*

The random field u_{mild} defined in (4.1.3) has a jointly measurable version. This is a consequence of the following proposition.

Proposition 4.1.7. *Let $f : \mathbb{R}^n \times \mathbb{R}^d \to \mathbb{R}$ be a Borel measurable function such that for any $t \in \mathbb{R}^n$, $f(t,\cdot) \in L\left(\dot{X}, S\right)$. For any $t \in \mathbb{R}^n$, let*

$$u(t) = \left\langle \dot{X}, f(t,\cdot) \right\rangle .$$

Then the random field u has a jointly measurable version.

Proof. Here, we prove this result for a general Lévy white noise, in particular we do not assume symmetry nor the fact that $\sigma = \gamma = 0$. The proof of this result can be found in [7, p. 926], but we reproduce it for the convenience of the reader. By [18, Theorem 3] and by the remark that follows in [18], the existence of a jointly measurable version of s is equivalent to the measurability of the map $t \in \mathbb{R}^n \mapsto u(t) \in L^0(\Omega)$, where the topology on $L^0(\Omega)$ is generated by the metric of convergence in probability. This map can be factorized as the composition of $\psi_1 : t \in \mathbb{R}^n \mapsto f(t,\cdot) \in L\big(\dot{X}, S\big)$ with $\psi_2 : \varphi \in L\big(\dot{X}, S\big) \mapsto \langle \dot{X}, \varphi \rangle \in L^0(\Omega)$. By [58, Theorem 3.3], the mapping ψ_2 is continuous. Since $L\big(\dot{X}, S\big)$ is a separable vector space (see [45, Theorem 2.12]), it suffices to show that for any $g \in L\big(\dot{X}, S\big)$ and $r \in \mathbb{R}_+^*$, the set $\big\{t \in \mathbb{R}^n : \|f(t,\cdot) - g(\cdot)\|_{\dot{X}} < r\big\}$ is a Borel set, where by definition, $\|f(t,\cdot) - g(\cdot)\|_{\dot{X}} = \inf\Big\{c > 0 : \int_{\mathbb{R}^d} \Phi\Big(\frac{|f(t,s)-g(s)|}{c}\Big)\, \mathrm{d}s \leqslant c\Big\} \in \mathbb{R}$, and $\Phi(r) = \big|\gamma r + \int_{\mathbb{R}} zr\,\big(\mathbb{1}_{|zr|\leqslant 1} - \mathbb{1}_{|z|\leqslant 1}\big)\,\nu(\mathrm{d}z)\big| + \sigma^2 r^2 + \int_{\mathbb{R}} (|rz|^2 \wedge 1)\,\nu(\mathrm{d}z)$. The function Φ is continuous (see [58, Lemma 3.1]), therefore by the joint measurability of f, we can deduce that $\alpha(t) = \|f(t,\cdot) - g(\cdot)\|_{\dot{X}}$ is the hitting time of a closed set by a continuous (adapted to the trivial filtration) process (here t plays the role of ω and c the role of time), therefore it is measurable. □

In particular, we deduce from Proposition 4.1.7 that the mild solution has a jointly measurable version.

4.2 When a mild solution is also a generalized solution

The two notions of solutions can be related, but are in general not equivalent. The notion of generalized solution seems more general, but in order to compare the two, we at least need that the mild solution has locally integrable sample paths since we want to integrate it against any test function $\varphi \in \mathscr{D}(\mathbb{R}^d)$. We first point out that the generalized and mild solutions depend on the choice of the fundamental solution ρ. Therefore, once the choice of the fundamental solution has been made, it makes sense to study *the* mild solution and *the* generalized solution. For the remainder of this section, we fix the choice of a fundamental solution to the operator L. We recall that the generalized solutions u_{gen} and the mild solution u_{mild} (under **(H1)** and **(H2)**) are defined by:

$$\langle u_{\text{gen}}, \varphi \rangle := \langle \dot{X}, \varphi * \check{\rho} \rangle\,, \qquad \text{for all } \varphi \in \mathscr{D}(\mathbb{R}^d)\,.$$

and

$$u_{\text{mild}}(t) := \langle \dot{X}, \rho(t-\cdot) \rangle\,, \qquad \text{for all } t \in \mathbb{R}^d\,.$$

Therefore, in general, if u_{mild} has locally integrable sample paths, then for any $\varphi \in \mathscr{D}(\mathbb{R}^d)$,

$$\langle u_{\text{mild}}, \varphi \rangle := \int_{\mathbb{R}^d} u_{\text{mild}}(t)\varphi(t)\,\mathrm{d}t = \int_{\mathbb{R}^d} \langle \dot{X}, \rho(t-\cdot) \rangle \varphi(t)\,\mathrm{d}t\,.$$

We see in particular, that if we can exchange the stochastic integral and the Lebesgue integral, then we get

$$\langle u_{\text{mild}}, \varphi \rangle = \Big\langle \dot{X}, \int_{\mathbb{R}^d} \rho(t-\cdot)\varphi(t)\,\mathrm{d}t \Big\rangle = \langle \dot{X}, \varphi * \check{\rho} \rangle = \langle u_{\text{gen}}, \varphi \rangle\,.$$

Therefore, conditional on the validity of the exchange of the order of integration, $u_{\text{mild}} = u_{\text{gen}}$ in the sense of generalized stochastic processes, and in order to answer the question of when the mild solution is also the generalized solution, we need a stochastic Fubini theorem.

4.2.1 When a mild solution is also a generalized solution: the α-stable case

We first deal with this question in the case of an α-stable symmetric noise, where many results are known. Let $\dot{W}^\alpha$ be an α-stable symmetric Lévy white noise on S, where S is a Borel measurable subset of $\mathbb{R}^d$, with characteristic triplet $(0,0,\nu_\alpha)$, where $\nu_\alpha(\mathrm{d}x) = \frac{1}{2|x|^{\alpha+1}}\,\mathrm{d}x$. The characteristic function of $\dot{W}^\alpha$ is given by

$$\mathbb{E}\left(e^{iu\dot{W}^\alpha(A)}\right) = \exp\left[-\mathrm{Leb}_d(A)\,|u|^\alpha\right], \qquad u \in \mathbb{R},$$

for any measurable set $A \subset S$ with finite Lebesgue measure. This notion coincides with that of a symmetric α-stable random measure developed in [62, §3.3]. Since the skewness parameter β vanishes, it is well known that a function $f : \mathbb{R}^d \to \mathbb{R}$ is $\dot{W}^\alpha$-integrable if and only if $f \in L^\alpha(S)$ (see [62, §3.2] and the example after the proof of Theorem 2.3.10 in Chapter 2).

Theorem 4.2.1. *Assume **(H2')**. Let u_{mild} be a jointly measurable version of the mild solution to* (4.0.1) *defined in* (4.1.3). *For any $\varphi \in \mathscr{D}(\mathbb{R}^d)$, let $\mu_\varphi(\mathrm{d}t) = |\varphi(t)|\,\mathrm{d}t$.*

(i) If $\alpha > 1$, and for any $\varphi \in \mathscr{D}(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d}\left(\int_S |\rho(t-s)|^\alpha\,\mathrm{d}s\right)^{\frac{1}{\alpha}}\mu_\varphi(\mathrm{d}t) < +\infty, \tag{4.2.1}$$

then u_{mild} is the generalized solution to (4.0.1).

(ii) If $\alpha = 1$, and ρ is such that for any $\varphi \in \mathscr{D}(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d}\mu_\varphi(\mathrm{d}t)\int_S \mathrm{d}s\,|\rho(t-s)|\left[1+\log_+\left(\frac{|\rho(t-s)|\int_{\mathbb{R}^d}\mu_\varphi(\mathrm{d}r)\int_S \mathrm{d}v\,|\rho(r-v)|}{\left(\int_S|\rho(t-v)|\,\mathrm{d}v\right)\left(\int_{\mathbb{R}^d}|\rho(r-s)|\mu_\varphi(\mathrm{d}r)\right)}\right)\right] < +\infty. \tag{4.2.2}$$

then u_{mild} is the generalized solution to (4.0.1).

(iii) If $\alpha < 1$, and ρ is such that for any $\varphi \in \mathscr{D}(\mathbb{R}^d)$,

$$\int_S\left(\int_{\mathbb{R}^d}|\rho(t-s)|\mu_\varphi(\mathrm{d}t)\right)^\alpha \mathrm{d}s < +\infty, \tag{4.2.3}$$

then u_{mild} is the generalized solution to (4.0.1).

Remark 4.2.2. *We also have the following:*

(1) Condition (4.2.1) *is equivalent to:*

$$t \mapsto \|\rho(t-\cdot)\|_{L^\alpha(S)} \in L^1_{loc}(\mathbb{R}^d)\,.$$

(a) *If* $S = \mathbb{R}^d_+$*, and* $\rho(t) = 0$ *for all* $t \in \mathbb{R}^d \setminus \mathbb{R}^d_+$*, then* (4.2.1) *is equivalent to* $\rho \in L^\alpha_{loc}(\mathbb{R}^d_+)$*.*

(b) *If* $S = \mathbb{R}_+ \times \mathbb{R}^{d-1}$*, and* $\rho(t,x) = 0$ *if* $t < 0$*, then* (4.2.1) *is equivalent to*

$$\forall t > 0,\ \int_0^t \int_{\mathbb{R}^{d-1}} |\rho(s,y)|^\alpha \,\mathrm{d}s\,\mathrm{d}y < +\infty\,.$$

(2) *Similarly, condition* (4.2.3) *is equvivalent to:*

$$\textit{for any compact } K \subset \mathbb{R}^d,\ \int_S \left(\int_K |\rho(t-s)|\,\mathrm{d}t\right)^\alpha \mathrm{d}s < +\infty\,.$$

(a) *If* $S = \mathbb{R}^d_+$*, and* $\rho(t) = 0$ *for all* $t \in \mathbb{R}^d \setminus \mathbb{R}^d_+$*, then* (4.2.3) *is equivalent to* $\rho \in L^1_{loc}(\mathbb{R}^d_+)$*.*

Proof of Theorem 4.2.1. We begin with *(i)*. As mentioned above, we need a stochastic Fubini theorem to exchange the Lebesgue integral and the stochastic integral. In the context of α-stable random measures, much more is known than in the case of a general Lévy noise. For instance, the book [62] studies α-stable process in general, and everything that we need about α-stable random measures is detailed in this book with much clarity. More precisely, [62, Theorem 11.3.2] gives necessary and sufficient conditions for u_{mild} to have almost surely locally integrable sample paths, and [62, Theorem 11.4.1] provides a stochastic Fubini theorem when those conditions are met. Indeed, since $\varphi \in \mathscr{D}(\mathbb{R}^d)$, the measure μ_φ is finite. By (4.2.1) and [62, Theorem 11.3.2], $\int_{\mathbb{R}^d} |u_{\text{mild}}(t)|\mu_\varphi(\mathrm{d}t) < +\infty$ a.s (that is, the sample paths of u_{mild} are almost surely locally integrable, and u_{mild} defines a generalized random process). By the stochastic Fubini Theorem in [62, Theorem 11.4.1],

$$\int_{\mathbb{R}^d} u_{\text{mild}}(t)\varphi(t)\,\mathrm{d}t = \int_S \left(\int_{\mathbb{R}^d} \rho(t-s)\varphi(t)\,\mathrm{d}t\right) \dot{W}^\alpha(\mathrm{d}s) = \left\langle \dot{W}_\alpha, \varphi * \check{\rho} \right\rangle\,.$$

Therefore, for any $\varphi \in \mathscr{D}(\mathbb{R}^d)$,

$$\left\langle u_{\text{mild}}, \varphi \right\rangle = \left\langle \dot{W}_\alpha, \varphi * \check{\rho} \right\rangle =: \left\langle u_{\text{gen}}, \varphi \right\rangle\,,$$

and therefore $u_{\text{mild}} = u_{\text{gen}}$.

The proof of *(ii)* and *(iii)* follows the same steps, with the difference that the conditions (4.2.2) and (4.2.3) are necessary to apply [62, Theorem 11.3.2] when $\alpha = 1$ or $\alpha > 1$. □

The careful reader may wonder if **(H1')** is satisfied in these cases, since it is a necessary condition for the existence of the generalized solution. In fact, (4.2.2) and (4.2.3) immediately imply Hyposthesis **(H1')** when $\alpha \leqslant 1$, and by [65, A.1], (4.2.1) also implies **(H1')** when $\alpha > 1$.

4.2.2 A Stochastic Fubini Theorem

In this section, we suppose that the driving noise $\dot{X}$ is a pure jump symmetric Lévy white noise, that is, a Lévy white noise with characteristic triplet $(0,0,\nu)$, where ν is a symmetric Lévy measure. We can no longer rely on the pre-existing work on α-stable random measures exposed in [62], and we need another version of a stochastic Fubini theorem. For convenience, we provide here a Fubini's theorem for integrals with respect to this Lévy noise. Such stochastic Fubini theorems for L^0-valued random measures already exist in the literature. For instance, [50, Corollary 1] is more general (it deals with stochastic integrands), but its assumptions are hard to check (it relies on a localizing sequence). Furthermore, integration of non-deterministic processes with respect to Lévy white noises usually relies on a space-time framework, where the time component is critical for the definition of predictable processes.

Theorem 4.2.3. *Let $\dot{X}$ be a symmetric pure jump Lévy white noise on $S \subset \mathbb{R}^d$, with characteristic triplet $(0,0,\nu)$ and jump measure J. Let $f : S \times \mathbb{R}^n \mapsto \mathbb{R}$ such that for any $t \in \mathbb{R}^n$, $f(\cdot,t) \in L(\dot{X},S)$, and let μ be a finite measure on $\mathbb{R}^n$. Suppose that*

$$\int_{\mathbb{R}^n} \left|\left\langle \dot{X}, f(\cdot,t)\right\rangle\right| \mu(\mathrm{d}t) < +\infty, \qquad a.s. \tag{4.2.4}$$

Then, for almost all $s \in S$, $f(s,\cdot) \in L^1(\mu)$, and the function $\mu \circledast f : s \mapsto \int_{\mathbb{R}^n} f(s,t)\mu(\mathrm{d}t)$ is in $L(\dot{X},S)$, and

$$\int_{\mathbb{R}^n} \left\langle \dot{X}, f(\cdot,t)\right\rangle \mu(\mathrm{d}t) = \left\langle \dot{X}, \mu \circledast f\right\rangle \qquad a.s. \tag{4.2.5}$$

Remark 4.2.4. *We emphasize that the $\circledast$ operation is not commutative. In particular, it involves a measure and a measurable function whose roles are not interchangeable.*

Proof of Theorem 4.2.3. The main probability space is $(\Omega, \mathscr{F}, \mathbb{P})$. Since μ is a finite measure, we can suppose without loss of generality that it is a probability measure on $\mathbb{R}^n$. Let $\left(\Omega', \mathscr{F}', \mathbb{P}'\right)$ be a probability space, and $(T_i)_{i\geqslant 1}$ be a sequence of i.i.d. random variables on this space with law μ. We write $\mathbb{E}'$ for the expectation with respect to the probability measure $\mathbb{P}'$. In this framework, (4.2.4) is equivalent to

$$\mathbb{E}'\left(\left|\left\langle \dot{X}, f(\cdot,T_1)\right\rangle\right|\right) < +\infty \qquad \mathbb{P}-\text{a.s.}$$

More precisely, there is a set $\Omega_1 \subset \Omega$ such that $\mathbb{P}(\Omega_1) = 1$, and for any $\omega \in \Omega_1$,

$$\mathbb{E}'\left(\left|\left\langle \dot{X}, f(\cdot,T_1)\right\rangle(\omega)\right|\right) < +\infty.$$

By the strong law of large numbers, for any $\omega \in \Omega_1$, there is a set $\Omega_1'(\omega) \subset \Omega'$ such that $\mathbb{P}'\left(\Omega_1'(\omega)\right) = 1$ and for any $\omega' \in \Omega_1'(\omega)$,

$$\frac{1}{n}\sum_{i=1}^{n} \left\langle \dot{X}, f(\cdot, T_i(\omega'))\right\rangle(\omega) \to \mathbb{E}'\left(\left\langle \dot{X}, f(\cdot,T_1)\right\rangle(\omega)\right) \qquad \text{as } n \to +\infty. \tag{4.2.6}$$

We define

$$A=\left\{(\omega,\omega')\in\Omega\times\Omega' : (4.2.6)\text{ happens}\right\}.$$

Then, for $\omega\in\Omega$, let

$$A_\omega=\left\{\omega'\in\Omega' : (\omega,\omega')\in A\right\}.$$

By the previous argument, for any $\omega\in\Omega_1$, $\mathbb{P}'(A_\omega)=1$, and we deduce that $\mathbb{P}\times\mathbb{P}'(A)=1$.

For any $n\in\mathbb{N}$, $s\in S$ and $\omega'\in\Omega'$, we set $f_n(s,\omega')=\frac{1}{n}\sum_{i=1}^n f(s,T_i(\omega'))$. Then, $f_n(\cdot,\omega')\in L(\dot{X},S)$ by linearity of this space. For any $\omega'\in\Omega'$, there is a set $\Omega_n(\omega')\subset\Omega$ such that $\mathbb{P}\left(\Omega_n(\omega')\right)=1$ and for any $\omega\in\Omega_n(\omega')$,

$$\frac{1}{n}\sum_{i=1}^n\left\langle\dot{X},f(\cdot,T_i(\omega'))\right\rangle(\omega)=\left\langle\dot{X},f_n(\cdot,\omega')\right\rangle(\omega).\tag{4.2.7}$$

For any $\omega'\in\Omega'$, the set $\Omega_\infty(\omega')=\bigcap_{n=1}^{+\infty}\Omega_n(\omega')$ is such that $\mathbb{P}\left(\Omega_\infty(\omega')\right)=1$ and for any $\omega\in\Omega_\infty(\omega')$, (4.2.7) holds for all $n\in\mathbb{N}$. We define

$$B=\left\{(\omega,\omega')\in\Omega\times\Omega' : (4.2.7)\text{ happens for all } n\in\mathbb{N}\right\}.$$

Then, for $\omega'\in\Omega'$, let

$$B^{\omega'}=\left\{\omega\in\Omega : (\omega,\omega')\in B\right\}.$$

By the previous argument, for any $\omega'\in\Omega'$, $\mathbb{P}\left(B^{\omega'}\right)=1$, and we deduce that

$$\int_{\Omega'}\mathbb{P}\left(B^{\omega'}\right)\mathbb{P}'(\mathrm{d}\omega')=1.$$

By Fubini's theorem, we deduce that

$$\int_{\Omega'}\left(\int_\Omega\mathbb{1}_{(\omega,\omega')\in A\cap B}\,\mathbb{P}(\mathrm{d}\omega)\right)\mathbb{P}'(\mathrm{d}\omega')=\int_\Omega\left(\int_{\Omega'}\mathbb{1}_{(\omega,\omega')\in A\cap B}\,\mathbb{P}'(\mathrm{d}\omega')\right)\mathbb{P}(\mathrm{d}\omega)=1.$$

Let $\omega'\in\Omega$. We define

$$(A\cap B)^{\omega'}=\left\{\omega\in\Omega : (\omega,\omega')\in A\cap B\right\}.$$

Then, by Fubini's theorem,

$$\begin{aligned}0&=\int_{\Omega'}\mathbb{P}'(\mathrm{d}\omega')\int_\Omega\mathbb{P}(\mathrm{d}\omega)(1-\mathbb{1}_{(\omega,\omega')\in A\cap B})\\&=\int_{\Omega'}\mathbb{P}'(\mathrm{d}\omega')\int_\Omega\mathbb{P}(\mathrm{d}\omega)(1-\mathbb{1}_{\omega\in(A\cap B)^{\omega'}})\\&=\int_{\Omega'}\mathbb{P}'(\mathrm{d}\omega')\left(1-\mathbb{P}\left((A\cap B)^{\omega'}\right)\right),\end{aligned}$$

and for $\mathbb{P}'$-almost all $\omega'\in\Omega'$, $\mathbb{P}\left((A\cap B)^{\omega'}\right)=1$. In other words, for $\mathbb{P}'$-almost all $\omega'\in\Omega'$,

$$\frac{1}{n}\sum_{i=1}^n\left\langle\dot{X},f(\cdot,T_i(\omega'))\right\rangle(\omega)=\left\langle\dot{X},f_n(\cdot,\omega')\right\rangle(\omega)\to\mathbb{E}'\left(\left\langle\dot{X},f(\cdot,T_1)\right\rangle(\omega)\right)\qquad\text{as } n\to+\infty,$$

for $\mathbb{P}$-almost all $\omega \in \Omega$. In particular, for $\mathbb{P}'$-almost all $\omega' \in \Omega$, the sequence of random variables $\left(\langle \dot{X}, f_n(\cdot,\omega')\rangle\right)_{n\geqslant 1}$ on $(\Omega, \mathscr{F}, \mathbb{P})$ is a Cauchy sequence in probability. By $\mathbb{P}$-a.s. linearity of $\dot{X}$ and the isomorphism property in [58, Theorem 3.4], we deduce that $\left(f_n(\cdot,\omega')\right)_{n\geqslant 1}$ is a Cauchy sequence in $L(\dot{X}, S)$. By completeness, for $\mathbb{P}'$-almost all $\omega' \in \Omega'$, there is a function $\tilde{f}(\cdot,\omega') \in L(\dot{X}, S)$ such that $f_n(\cdot,\omega') \to \tilde{f}(\cdot,\omega')$ as $n \to +\infty$ in $L(\dot{X}, S)$ (see [58] for the definition of that convergence, in particular, it implies the convergence in measure on compact subsets of S). By (4.2.4) and [60, Theorem 6], for almost every $s \in S$, $\int_{\mathbb{R}^d} |f(s,t)|\mu(\mathrm{d}t) < +\infty$, that is $\mathbb{E}'\left(|f(s,T_1)|\right) < +\infty$. By the strong law of large numbers, we deduce that for almost all $s \in S$, there is a set Ω'_s such that $\mathbb{P}'(\Omega'_s) = 1$ and for any $\omega' \in \Omega'_s$,

$$f_n(s,\omega') \to \mathbb{E}'\left(f(s,T_1)\right) \text{ as } n \to +\infty. \tag{4.2.8}$$

Let $C = \left\{(s,\omega') \in S \times \Omega' : \text{(4.2.8) holds}\right\}$, for $s \in S$, $C_s = \left\{\omega' \in \Omega' : (s,\omega') \in C\right\}$, and for $\omega' \in \Omega'$, $C^{\omega'} = \left\{s \in S : (s,\omega') \in C\right\}$. Since for almost all $s \in S$, $\mathbb{P}'(C_s) = 1$, by Fubini's theorem, we have

$$0 = \int_S \mathrm{d}s \int_{\Omega'} \mathbb{P}(\mathrm{d}\omega')\left(1 - \mathbb{1}_{(s,\omega')\in C}\right) = \int_{\Omega'} \mathbb{P}'(\mathrm{d}\omega')\mathrm{Leb}_d\left(\left(C^{\omega'}\right)^c\right).$$

We deduce that for almost all $\omega' \in \Omega'$, (4.2.8) holds for almost every $s \in S$. We can then drop the dependence in ω', so that there is a sequence $(t_i)_{i\geqslant 1}$ of deterministic times (for $\mathbb{P}$) in $\mathbb{R}^n$ such that

$$\frac{1}{n}\sum_{i=1}^n f(s,t_i) \to \mu \circledast f(s) \qquad \text{a.e. in } s \text{ as } n \to +\infty, \tag{4.2.9}$$

$$\frac{1}{n}\sum_{i=1}^n \langle \dot{X}, f(\cdot,t_i)\rangle = \left\langle \dot{X}, \frac{1}{n}\sum_{i=1}^n f(\cdot,t_i)\right\rangle \to \int_{\mathbb{R}^d} \langle \dot{X}, f(\cdot,t)\rangle \mu(\mathrm{d}t) \qquad \mathbb{P}-\text{a.s.}, \tag{4.2.10}$$

as $n \to +\infty$, and

$$\frac{1}{n}\sum_{i=1}^n f(\cdot,t_i) \to \tilde{f}(\cdot) \qquad \text{in } L(\dot{X}, S) \text{ as } n \to +\infty. \tag{4.2.11}$$

Since convergence in $L\left(\dot{X}, S\right)$ implies convergence almost everywhere along a subsequence (see [58, p. 466]), by uniqueness of the limit we get from (4.2.9) and (4.2.11) that $\mu \circledast f = \tilde{f}$ almost everywhere (and hence $\tilde{f}$ does not depend on ω'), and $\frac{1}{n}\sum_{i=1}^n f(\cdot,t_i) \to \mu \circledast f$ in $L(\dot{X}, S)$. Therefore,

$$\left\langle \dot{X}, \frac{1}{n}\sum_{i=1}^n f(\cdot,t_i)\right\rangle \to \langle \dot{X}, \mu \circledast f\rangle \qquad \text{as } n \to +\infty, \tag{4.2.12}$$

in $\mathbb{P}$-probability. By uniqueness of the limit, gathering (4.2.10) and (4.2.12), we deduce that $\mathbb{P}$-almost surely, (4.2.5) holds. □

4.2.3 When a mild solution is also a generalized solution: the general case

In this section, we suppose again that the driving noise $\dot{X}$ is a pure jump symmetric Lévy white noise, that is, a Lévy white noise with characteristic triplet $(0, 0, \nu)$, where ν is a symmetric Lévy measure. We can now apply Theorem 4.2.3 to our problem.

Theorem 4.2.5. *Assume **(H2)**. Let u_{mild} be a measurable version of the mild solution to* (4.0.1) *defined in* (4.1.3)*. Suppose that the sample paths of u_{mild} are almost surely locally integrable with respect to Lebesgue measure. Then $u_{mild} = u_{gen}$ in the sense of generalized stochastic processes.*

Proof. By definition, we have the formula

$$u_{\text{mild}}(t) = \left\langle \dot{X}, \rho(t-\cdot) \right\rangle .$$

Let $\varphi \in \mathscr{D}(\mathbb{R}^d)$, and let $\mu_\varphi^+(\mathrm{d}t) := \varphi_+(t)\,\mathrm{d}t$ and $\mu_\varphi^-(\mathrm{d}t) := \varphi_-(t)\,\mathrm{d}t$, where $\varphi_+ = \max(\varphi, 0)$ and $\varphi_- = max(-\varphi, 0)$ are, respectively, the positive and negative parts of φ. These two measures are finite, and are the positive and negative part of the signed measure $\mu_\varphi(\mathrm{d}t) := \varphi(t)\,\mathrm{d}t$. Since u_{mild} has almost surely locally integrable sample paths,

$$\int_{\mathbb{R}^d} |u_{\text{mild}}(t)|\, \mu_\varphi^\pm(\mathrm{d}t) < +\infty .$$

Therefore, we can apply Theorem 4.2.3 separately with the positive and negative part of μ_φ, and recombining them together yields:

$$\left\langle u_{\text{mild}}, \varphi \right\rangle := \int_{\mathbb{R}^d} u_{\text{mild}}(t)\varphi(t)\,\mathrm{d}t = \int_{\mathbb{R}^d} \left\langle \dot{X}, \rho(t-\cdot) \right\rangle \mu_\varphi(\mathrm{d}t) = \left\langle \dot{X}, \varphi * \check{\rho} \right\rangle = \left\langle u_{\text{gen}}, \varphi \right\rangle ,$$

which proves the claim. □

Remark 4.2.6. *In the α-stable case, we had a necessary and sufficient condition for the sample paths of the mild solution to be locally integrable. In the general case, we do not have such precise statement, we only have the necessary condition of [60, Theorem 6].*

Again, one might wonder if Hyposthesis **(H1)** is satisfied, and it turns out that $\varphi * \check{\rho} = \mu_\varphi \circledast f$, where $f(s,t) := \rho(t-s)$, and by Theorem 4.2.3, $\mu_\varphi \circledast f$ is $\dot{X}$-integrable, so **(H1)** is satisfied and the generalized solution is well defined.

Having almost surely locally integrable sample paths is the minimum requirement for a stochastic process to be considered as a generalized stochastic process, since we need to be able to integrate it against any test function. Essentially, Theorem 4.2.5 states that if the mild solution can be considered as a generalized stochastic process, then it must be equal to the generalized solution.

4.3 Necessary condition for the existence of a random field solution

We have seen in the previous section, that under the minimum requirement that the mild solution has locally integrable sample paths (and therefore can be considered as a generalized stochastic process), it is equal (in the sense of generalized stochastic processes) to the generalized solution. To further investigate the link between these two notions of solutions, one can try to answer the following question: when it exists, is the generalized solution "equal" to the mild solution. Of course, stated that way, this question is ill-posed, since we are trying to compare two mathematical objects living in different spaces: on the one hand, the generalized solution is a random linear functional on the space of compactly supported smooth functions, and on the other hand, the mild solution is a random field. Furthermore, we need hypothesis **(H2)** to hold in order to be able to define the mild solution. At the beginning of this chapter , we introduced in Definition 4.1.4 the notion of a random field representation of a generalized stochastic process. Then, stated more precisely, the question we answer in this section is the following:

"Suppose that **(H1)** is satisfied. Then, the generalized solution can be defined as in Definition 4.1.1. Suppose also that the generalized solution has a random field representation Y. Then, is **(H2)** satisfied, and if so, is Y the mild solution?"

In the case of a Gaussian noise, that can be spatially correlated, this question has already been investigated under slightly different assumptions in [22, Theorem 11]. Transposed to our framework, this theorem in particular implies that in the case of an SPDE driven by Gaussian white noise (in space and time), if the generalized solution has a random field representation, then the fundamental solution of this SPDE is necessarily square integrable. We extend this kind of statement to the more general setting of symmetric Lévy white noises.

4.3.1 Necessary condition for the existence of a random field solution: α-stable case

Again, we first restrict to the case of a symmetric α-stable noise, for some $\alpha \in (0,2)$, where the existing theory is more developed.

Theorem 4.3.1. *Assume **(H1')**. Let u_{gen} be the generalized solution to* (4.0.1) *defined by* (4.1.1). *Suppose that u_{gen} has a random field representation X in the sense of Definition 4.1.2, that is there exists a jointly measurable random field $(X_t)_{t\in\mathbb{R}^d}$ such that X has almost surely locally integrable sample paths, and for any $\varphi \in \mathscr{D}(\mathbb{R}^d)$,*

$$\langle u_{gen}, \varphi \rangle = \int_{\mathbb{R}^d} X_t \varphi(t)\,\mathrm{d}t \qquad a.s. \tag{4.3.1}$$

*Then, for almost all $t \in \mathbb{R}^d$, $\rho(t-\cdot) \in L^\alpha(S)$ (i.e. **(H2')** is satisfied almost everywhere), and*

$$X_t = \langle \dot{W}^\alpha, \rho(t-\cdot) \rangle = u_{mild}(t) \qquad a.s. \quad a.e. \tag{4.3.2}$$

Furthermore, for any $\psi \in \mathscr{D}(\mathbb{R}^d)$,

(i) If $\alpha > 1$,

$$\int_{\mathbb{R}^d} \left(\int_S \left|\rho(t-s)\right|^\alpha \mathrm{d}s \right)^{\frac{1}{\alpha}} |\psi(t)|\,\mathrm{d}t < +\infty. \tag{4.3.3}$$

(ii) If $\alpha = 1$,

$$\int_{\mathbb{R}^d} \mathrm{d}t \int_S \mathrm{d}s\, |\rho(t-s)\psi(t)| \left[1 + \log_+\left(\frac{|\rho(t-s)| \int_{\mathbb{R}^d} \int_S |\rho(r-v)|\,\mathrm{d}v |\psi(r)|\,\mathrm{d}r}{\left(\int_S |\rho(t-v)|\,\mathrm{d}v\right)\left(\int_{\mathbb{R}^d} |\rho(r-s)\psi(r)|\,\mathrm{d}r\right)}\right)\right] < +\infty. \quad (4.3.4)$$

(iii) If $\alpha < 1$,

$$\int_S \left(\int_{\mathbb{R}^d} \left|\rho(t-s)\psi(t)\right| \mathrm{d}t\right)^\alpha \mathrm{d}s < +\infty. \quad (4.3.5)$$

Remark 4.3.2. *Equivalently, under the hypothesis of Theorem 4.3.1, we get instead of* (4.3.3) *and* (4.3.5) *respectively,*

$$t \mapsto \|\rho(t-\cdot)\|_{L^\alpha(S)} \in L^1_{loc}(\mathbb{R}^d)\,, \quad (4.3.6)$$

and

$$\text{for any compact } K \subset \mathbb{R}^d,\ \int_S \left(\int_K |\rho(t-s)|\,\mathrm{d}t\right)^\alpha \mathrm{d}s < +\infty. \quad (4.3.7)$$

In particular, if $S = \mathbb{R}^d$, $\|\rho(t-\cdot)\|_{L^\alpha(S)}$ *does not depend on* t, *and* (4.3.6) *is verified for any* $\rho \in L^\alpha(S)$. *From* (4.3.7), *we get that* $\rho \in L^1_{loc}(\mathbb{R}^d)$. *The case of* $S = \mathbb{R}_+ \times \mathbb{R}^{d-1}$ *and the heat equation is studied in Section 4.4.*

Proof of Theorem 4.3.1. There exists a set $\tilde{\Omega} \subset \Omega$ of probability one such that for all $\omega \in \tilde{\Omega}$, the function $t \mapsto X_t(\omega)$ is locally integrable. Without loss of generality, we can suppose that $\Omega = \tilde{\Omega}$. Let $\varphi \in \mathscr{D}(\mathbb{R}^d)$ be such that $\varphi \geqslant 0$, $\operatorname{supp}\varphi \subset B(0,1)$ and $\int_{\mathbb{R}^d} \varphi = 1$. For each $t \in \mathbb{R}^d$ and $n \in \mathbb{N}$, we define $\varphi_n^t(\cdot) = n^d \varphi(n(\cdot - t))$. Let $Y^n : (t,\omega) \mapsto \langle X(\omega), \varphi_n^t \rangle$. Then,

$$Y_t^n(\omega) = \int_{\mathbb{R}^d} X_s(\omega) n^d \varphi(n(s-t))\,\mathrm{d}s = \int_{\mathbb{R}^d} X_{r+t}(\omega) n^d \varphi(nr)\,\mathrm{d}r\,. \quad (4.3.8)$$

Consider $f : (t,s,\omega) \mapsto (t+s,\omega)$. The function f is measurable as a map from $(\mathbb{R}^d \times \mathbb{R}^d \times \Omega, B(\mathbb{R}^d) \otimes B(\mathbb{R}^d) \otimes \mathscr{F})$ to $(\mathbb{R}^d \times \Omega, B(\mathbb{R}^d) \otimes \mathscr{F})$, and $X_{r+t}(\omega) = X \circ f(r,t,\omega)$. Since X is a jointly measurable process, and by Fubini's theorem, we deduce from the second equality in (4.3.8) that Y^n is a jointly measurable process. We define the set

$$A = \left\{(t,\omega) : \langle X(\omega), \varphi_n^t \rangle \to X_t(\omega) \text{ as } n \to +\infty\right\}.$$

We can write

$$A = \bigcap_{n \in \mathbb{N}*} \bigcup_{N \in \mathbb{N}} \bigcap_{k \geqslant N} \left\{(t,\omega) : \left|Y_t^k(\omega) - X_t(\omega)\right| \leqslant \frac{1}{n}\right\},$$

and since Y^n and X are both jointly measurable processes, $A \in \mathscr{B}(\mathbb{R}^d) \otimes \mathscr{F}$. By Lebesgue's differentiation theorem (see [70, Chapter 7, Exercise 2]), for any $\omega \in \Omega$, $\int_{\mathbb{R}^d} \mathbb{1}_{(t,\omega)\in A^c}\,\mathrm{d}t = 0$. Then, by Fubini's theorem,

$$0 = \mathbb{E}\left(\int_{\mathbb{R}^d} \mathbb{1}_{(t,\omega)\in A^c}\,\mathrm{d}t\right) = \int_{\mathbb{R}^d} \mathbb{P}\left(\{\omega : (t,\omega) \in A^c\}\right) \mathrm{d}t\,.$$

Therefore, there is a non random set $\tilde{A} \subset \mathbb{R}^d$ such that $\mathrm{Leb}_d\left(\tilde{A}\right) = 0$ and for all $t \notin \tilde{A}$,

$$\mathbb{P}\left\{\omega : (t,\omega) \in A^c\right\} = 0,$$

that is, $\mathbb{P}\left\{\langle X, \varphi_n^t\rangle \to X_t \text{ as } n \to +\infty\right\} = 1$.

By well known properties of a symmetric α-stable noise, for any $f \in L^\alpha(S)$ (see for instance [62, Proposition 3.4.1]),

$$\mathbb{E}\left(e^{i\langle \dot{W}^\alpha, f\rangle}\right) = e^{-\|f\|^\alpha_{L^\alpha(S)}}, \tag{4.3.9}$$

where $\|f\|^\alpha_{L^\alpha(S)} = \int_S |f(x)|^\alpha \,\mathrm{d}x$. Therefore,

$$\mathbb{E}\left(e^{i\langle u_{\mathrm{gen}}, \varphi\rangle}\right) = e^{-\|\varphi * \check{\rho}\|^\alpha_{L^\alpha}} = \mathbb{E}\left(\exp\left(i\int_{\mathbb{R}^d} X_s \varphi(s)\,\mathrm{d}s\right)\right). \tag{4.3.10}$$

Let $t_0 \in \tilde{A}^c$. Then $\langle X, \varphi_n^{t_0}\rangle \to X_{t_0}$ almost surely as $n \to +\infty$. We define $\rho_n^{t_0} = \varphi_n^{t_0} * \check{\rho} \in L^\alpha(S)$ by **(H1)**. By (4.3.10), for $n, m \in \mathbb{N}$,

$$e^{-\|\rho_n^{t_0} - \rho_m^{t_0}\|^\alpha_{L^\alpha}} = \mathbb{E}\left(\exp\left(i\int_{\mathbb{R}^d} X_s\left(\varphi_n^{t_0}(s) - \varphi_m^{t_0}(s)\right)\mathrm{d}s\right)\right) \to 1 \qquad \text{as } n, m \to +\infty. \tag{4.3.11}$$

We deduce that $(\rho_n^{t_0})_{n\geqslant 1}$ is a Cauchy sequence in $L^\alpha(S)$. By completeness of this space, there is a function $g^{t_0} \in L^\alpha(S)$ such that

$$\rho_n^{t_0} \to g^{t_0}, \qquad \text{in } L^\alpha(S) \text{ as } n \to +\infty. \tag{4.3.12}$$

Furthermore, we know from the theory of generalized functions that $\varphi_n^{t_0} \to \delta_{t_0}$ in $\mathscr{D}'(\mathbb{R}^d)$ as $n \to +\infty$. Therefore,

$$\rho_n^{t_0} \to \delta_{t_0} * \check{\rho}, \qquad \text{in } \mathscr{D}'(\mathbb{R}^d) \text{ as } n \to +\infty. \tag{4.3.13}$$

From (4.3.12) and (4.3.13), we would like to deduce that $\delta_{t_0} * \check{\rho} = g^{t_0}$ in some sense. The left-hand side of this equality is a generalized function, and the right-hand side is defined as an element of $L^\alpha(S) \subset D'(S)$. Therefore, the right space to show this equality is $\mathscr{D}'(S)$. If this equality is true, it means that $s \mapsto \rho(t_0 - s)$ can be considered as a function in $L^\alpha(S)$. To prove this equality, we need to show that for any $\theta \in \mathscr{D}(S)$, $\langle \delta_{t_0} * \check{\rho}, \theta\rangle = \langle g^{t_0}, \theta\rangle$. In the case $\alpha \geqslant 1$, by Hölder's inequality,

$$|\langle g^{t_0} - \rho_n^{t_0}, \theta\rangle| \leqslant \int_S \left|g^{t_0}(s) - \rho_n^{t_0}(s)\right| |\theta(s)|\,\mathrm{d}s \leqslant \|g^{t_0} - \rho_n^{t_0}\|_{L^\alpha(S)} \|\theta\|_{L^{\frac{\alpha}{\alpha-1}}(S)}.$$

Passing to the limit as $n \to +\infty$, we get that for all $t_0 \in \tilde{A}^c$, $\delta_{t_0} * \check{\rho} = g^{t_0} \in L^\alpha(S)$ in $\mathscr{D}'(S)$. Then, in the sense of distributions, $\check{\rho} = \delta_{-t_0} * \delta_{t_0} * \check{\rho} = \delta_{-t_0} * g^{t_0}$. Therefore, in the sense of distributions, ρ is equal to the function $t \in \mathbb{R}^d \mapsto g^{t_0}(t_0 - t)$, which therefore does not depend on t_0, and is such that for almost all $t \in \mathbb{R}^d$, $\delta_t * \check{\rho} = \rho(t - \cdot) \in L^\alpha(S)$. Also, for any $t \in \tilde{A}^c$,

$$\langle X, \varphi_n^t\rangle = \langle u_{\mathrm{gen}}, \varphi_n^t\rangle = \langle \dot{W}^\alpha, \varphi_n^t * \check{\rho}\rangle = \langle \dot{W}^\alpha, \rho_n^t\rangle,$$

and $\langle X, \varphi_n^t \rangle \to X_t$ almost surely as $n \to +\infty$, and

$$\rho_n^t \to g^t = \delta_t * \check{\rho}, \qquad \text{in } L^\alpha(S) \text{ as } n \to +\infty. \tag{4.3.14}$$

Then, $\langle \dot{W}^\alpha, \rho_n^t \rangle \to \langle \dot{W}^\alpha, \rho(t-\cdot) \rangle$ in probability as $n \to +\infty$. Therefore (4.3.2) holds. Since we used Hölder's inequality, this method does not work in the case $\alpha < 1$, and does not imply (4.3.3), (4.3.4) or (4.3.5). We therefore develop a different proof that works for any $\alpha \in (0,2)$.

If $\alpha \in (0,2)$ is arbitrary, we deduce from (4.3.9) and (4.3.11) that $\langle \dot{W}^\alpha, \rho_n^{t_0} - g^{t_0} \rangle \to 0$ in law as $n \to +\infty$, and by [44, Lemma 4.7], the convergence is also in probability. By almost sure linearity, we deduce that $\langle \dot{W}^\alpha, \rho_n^{t_0} \rangle \to \langle \dot{W}^\alpha, g^{t_0} \rangle$ in probability as $n \to +\infty$. By uniqueness of the limit, and since $\langle \dot{W}^\alpha, \rho_n^{t_0} \rangle = \langle u_{\text{gen}}, \varphi_n^{t_0} \rangle = \langle X, \varphi_n^{t_0} \rangle$,

$$X_{t_0} = \langle \dot{W}^\alpha, g^{t_0} \rangle, \qquad \text{a.s. for any } t_0 \in \tilde{A}^c.$$

For any $(t,s) \in \mathbb{R}^d \times S$, let

$$g(t,s) = \limsup_{n \to +\infty} \rho_n^t(s). \tag{4.3.15}$$

Then $(t,s) \mapsto g(t,s)$ is measurable, and for $t \in \tilde{A}^c$, $g(t,\cdot) = g^t(\cdot)$ almost everywhere. Therefore

$$X_{t_0} = \langle \dot{W}^\alpha, g(t_0,\cdot) \rangle, \qquad \text{a.s. for any } t_0 \in \tilde{A}^c.$$

Let $\psi \in \mathscr{D}(\mathbb{R}^d)$. Then, $\mu_\psi(\mathrm{d}t) := \psi(t)\,\mathrm{d}t$ is a finite signed measure, that we can decompose into positive and negative parts μ_ψ^+ and μ_ψ^-. Since X is almost surely locally integrable,

$$\int_{\mathbb{R}^d} |X_t| \mu_\psi^+(\mathrm{d}t) < +\infty, \qquad \text{and} \qquad \int_{\mathbb{R}^d} |X_t| \mu_\psi^-(\mathrm{d}t) < +\infty \qquad \text{a.s.}$$

By [62, Theorem 11.3.2], if $\alpha > 1$, we get

$$\int_{\mathbb{R}^d} \left(\int_S |g(t,s)|^\alpha \,\mathrm{d}s \right)^{\frac{1}{\alpha}} |\psi(t)|\,\mathrm{d}t < +\infty, \tag{4.3.16}$$

if $\alpha = 1$, we get

$$\int_{\mathbb{R}^d} \mathrm{d}t \int_S \mathrm{d}s |g(t,s)\psi(t)| \left[1 + \log_+ \left(\frac{|g(t,s)| \int_{\mathbb{R}^d} \int_S |g(r,v)|\,\mathrm{d}v |\psi(r)|\,\mathrm{d}r}{\left(\int_S |g(t,v)|\,\mathrm{d}v \right) \left(\int_{\mathbb{R}^d} |g(r,s)\psi(r)|\,\mathrm{d}r \right)} \right) \right] < +\infty, \tag{4.3.17}$$

and if $\alpha < 1$, we get

$$\int_S \left(\int_{\mathbb{R}^d} \left| g(t,s)\psi(t) \right| \mathrm{d}t \right)^\alpha \mathrm{d}s < +\infty. \tag{4.3.18}$$

By the generalized Minkowsky inequality (see [65, A.1]) and by (4.3.16), when $\alpha > 1$,

$$\left(\int_S \left| \int_{\mathbb{R}^d} \left| g(t,s)\psi(t) \right| \mathrm{d}t \right|^\alpha \mathrm{d}s \right)^{\frac{1}{\alpha}} \leqslant \int_{\mathbb{R}^d} \left(\int_S |g(t,s)|^\alpha \,\mathrm{d}s \right)^{\frac{1}{\alpha}} |\psi(t)|\,\mathrm{d}t < +\infty.$$

In particular, we see that for almost all $s \in S$, $t \mapsto g(t,s)$ is locally integrable (and therefore defines a distribution). By [62, Theorem 11.4.1], we can exchange the stochastic integral with the Lebesgue integral in (4.3.1):

$$\int_{\mathbb{R}^d} X_t \mu_\psi(\mathrm{d}t) = \int_{\mathbb{R}^d} \langle \dot{W}^\alpha, g(t,\cdot)\rangle \psi(t)\,\mathrm{d}t = \left\langle \dot{W}^\alpha, \int_{\mathbb{R}^d} \psi(t) g(t,\cdot)\,\mathrm{d}t \right\rangle \qquad \text{a.s.} \tag{4.3.19}$$

We define $\int_{\mathbb{R}^d} \psi(t) g(t,s)\,\mathrm{d}t =: \psi \circledast g(s)$ (this operation on ψ and g is not commutative). From (4.3.1) and (4.3.19), we get

$$\langle \dot{W}^\alpha, \psi \circledast g\rangle - \langle \dot{W}^\alpha, \psi * \check{\rho}\rangle = \langle \dot{W}^\alpha, \psi \circledast g - \psi * \check{\rho}\rangle = \langle X, \psi\rangle - \langle u_{\mathrm{gen}}, \psi\rangle = 0 \text{ a.s.}, \tag{4.3.20}$$

and by (4.3.9), we deduce that $\|\psi \circledast g - \psi * \check{\rho}\|_{L^\alpha}^\alpha = 0$. Then, for any $\psi \in \mathscr{D}(\mathbb{R}^d)$, there is a set B_ψ such that $\mathrm{Leb}_d(B_\psi) = 0$ and for any $s \in S \setminus B_\psi$, $\psi \circledast g(s) = \psi * \check{\rho}(s)$. Since $\mathscr{D}(\mathbb{R}^d)$ is separable, there is a countable dense subset $D \subset \mathscr{D}(\mathbb{R}^d)$. Let

$$B = \bigcup_{\psi \in D} B_\psi, \qquad \mathrm{Leb}_d(B) = 0.$$

Then, for all $s \in S \setminus B$, for all $\psi \in D$,

$$\langle g(\cdot,s), \psi\rangle = \psi \circledast g(s) = \psi * \check{\rho}(s) = \langle \rho, \psi(s+\cdot)\rangle = \langle \delta_s * \rho, \psi\rangle.$$

Since two distributions equal on a dense set are equal everywhere by continuity, we get that for all $s \in S \setminus B$, $g(\cdot,s) = \delta_s * \rho$ in $\mathscr{D}'(\mathbb{R}^d)$. Then, $\rho = \delta_{-s} * g(\cdot,s)$ in $\mathscr{D}'(S)$, and ρ is a function depending only on the $t \in \mathbb{R}^d$ variable, more precisely for almost all $t \in \mathbb{R}^d$, $\rho(t) = g(t+s,s)$ which does not depend on s. Then, for almost all $(t,s) \in \mathbb{R}^d \times S$, $g(t,s) = \rho(t-s)$. By definition of g in (4.3.15) and by (4.3.14), we deduce that ρ is a function such that for almost all $t \in \mathbb{R}^d$, $\rho(t-\cdot) \in L^\alpha(S)$. Also, from (4.3.16), (4.3.17) and (4.3.18), we get (4.3.3), (4.3.4) and (4.3.5). □

Remark 4.3.3. *The proof of the result in the case $\alpha \geqslant 1$ proves that the result is still valid in the case of Gaussian white noise: it is essentially equivalent to taking $\alpha = 2$.*

4.3.2 Necessary condition for the existence of a random field solution: symmetric pure jump Lévy noise.

We now consider the more general case of a symmetric pure jump Lévy noise $\dot{X}$. Similarly to the α-stable case, we can obtain a necessary condition for the existence of a random field solution.

Theorem 4.3.4. *Assume **(H1)**. Let u_{gen} be the generalized solution to* (4.0.1) *defined by* (4.1.1)*. Suppose that u_{gen} has a random field representation Y in the sense of Definition 4.1.2, that is there exists a jointly measurable random field $(Y_t)_{t\in\mathbb{R}^d}$ such that Y has almost surely locally integrable sample paths, and for any $\varphi \in \mathscr{D}(\mathbb{R}^d)$,*

$$\langle u_{gen}, \varphi\rangle = \int_{\mathbb{R}^d} Y_t \varphi(t)\,\mathrm{d}t \qquad a.s.$$

Then, for almost all $t \in \mathbb{R}^d$, $\rho(t-\cdot) \in L(\dot{X}, S)$ (i.e. ***(H2)*** *is satisfied almost everywhere), and*

$$Y_t = \langle \dot{X}, \rho(t-\cdot)\rangle = u_{mild}(t) \qquad a.s. \quad a.e.$$

Proof. We use the same notations as in the proof of Theorem 4.3.1. By the same reasoning as in the proof of Theorem 4.3.1, there is a non random set $\tilde{A} \subset \mathbb{R}^d$ such that $\mathrm{Leb}_d(\tilde{A}) = 0$ and for all $t \notin \tilde{A}$, $\mathbb{P}\{\langle Y, \varphi_n^t\rangle \to Y_t \text{ as } n \to +\infty\} = 1$. Then, as before we define $\rho_n^{t_0} = \varphi_n^{t_0} * \check{\rho} \in L(\dot{X}, S)$. For $n, m \in \mathbb{N}$,

$$\mathbb{E}\left(e^{i\left\langle \dot{X}, \rho_n^{t_0} - \rho_m^{t_0}\right\rangle}\right) = \mathbb{E}\left(e^{i\int_{\mathbb{R}^d} Y_s\left(\varphi_n^{t_0}(s) - \varphi_m^{t_0}(s)\right)\mathrm{d}s}\right) \to 1 \qquad \text{as } n, m \to +\infty.$$

We deduce that $\langle \dot{X}, \rho_n^{t_0} - \rho_m^{t_0}\rangle$ converges to zero in probability. Since $\dot{X}$ is symmetric, the linear mapping $f \in L(\dot{X}, S) \mapsto \langle \dot{X}, f\rangle \in L^0(\Omega)$ is an isomorphism (see [58, Theorem 3.4]). In particular the inverse map is continuous, therefore the sequence $(\rho_n^{t_0})_{n\in\mathbb{N}}$ is Cauchy in $L(\dot{X}, S)$. This space is complete, therefore there is a function g^{t_0} such that $\rho_n^{t_0} \to g^{t_0}$ in $L(\dot{X}, S)$. For any $(t, s) \in \mathbb{R}^d \times S$, let

$$g(t,s) = \limsup_{n\to+\infty} \rho_n^t(s).$$

Then $(t, s) \mapsto g(t, s)$ is measurable, and for $t \in \tilde{A}^c$, $g(t,\cdot) = g^t(\cdot)$ almost everywhere. Also we get that for almost all $t_0 \in \mathbb{R}^d$, $Y_{t_0} = \langle \dot{X}, g^{t_0}\rangle$ almost surely. Also, since Y has almost surely locally integrable sample paths, for any $\psi \in \mathscr{D}(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} \left|\langle \dot{X}, g^t\rangle\right| \mu_\psi(\mathrm{d}t) < +\infty \qquad \text{a.s.},$$

where $\mu_\psi(\mathrm{d}t) = |\psi(t)|\,\mathrm{d}t$. By Theorem 4.2.3,

$$\int_{\mathbb{R}^d} \langle \dot{X}, g^t\rangle \psi(t)\,\mathrm{d}t = \langle \dot{X}, \psi \circledast g\rangle \qquad \text{a.s.}$$

Therefore, for any $\psi \in \mathscr{D}(\mathbb{R}^d)$,

$$\langle \dot{X}, \psi \circledast g\rangle = \int_{\mathbb{R}^d} \langle \dot{X}, g^t\rangle \psi(t)\,\mathrm{d}t = \int_{\mathbb{R}^d} Y_t \psi(t)\,\mathrm{d}t = \langle \dot{X}, \check{\rho} * \psi\rangle \qquad \text{a.s.},$$

where the last equality is by Definition 4.1.1. Therefore, for almost every $s \in S$, $\psi \circledast g(s) = \psi * \check{\rho}(s)$. We can then conclude as in the proof of Theorem 4.3.1 after (4.3.20). $\square$

4.4 Examples

In this section, we give some examples of application of Theorems 4.2.1 and 4.3.1. We focus on two well known stochastic partial differential equations: the linear heat equation and the linear wave equation, in various dimensions. We restrict to the case of a symmetric α-stable noise, as the choice of the parameter $\alpha \in (0, 2)$ will be enough to capture the different cases.

4.4.1 The stochastic heat equation

Let $\dot{W}^\alpha$ be an α-stable symmetric noise on $\mathbb{R}_+ \times \mathbb{R}^d$. The heat operator H in dimension d is a constant coefficient partial differential operator given by

$$H = \frac{\partial}{\partial t} - \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2}.$$

A fundamental solution ρ_H for this operator is given by the formula

$$\rho_H(t,x) = \frac{1}{(4\pi t)^{\frac{d}{2}}} \exp\left(-\frac{|x|^2}{4t}\right) \mathbb{1}_{t>0}. \tag{4.4.1}$$

We consider the following Cauchy problem

$$\begin{cases} Hu = \dot{W}^\alpha \\ u(0,\cdot) = 0 \end{cases}. \tag{4.4.2}$$

Existence of the generalized solution

We wish to define the generalized solution of this equation associated with the fundamental solution ρ_H.

Proposition 4.4.1. *For any choice of $\alpha \in (0,2)$ and $d \geqslant 1$, the generalized solution to the linear stochastic heat equation driven by a symmetric α-stable noise is well defined.*

Proof. We have to check for which combination of α and d the convolution $\varphi * \check{\rho}_H$ is in $L^\alpha(\mathbb{R}_+ \times \mathbb{R}^d)$ for any $\varphi \in \mathscr{D}(\mathbb{R}^{d+1})$ (see **(H1')**). We have that for $\varphi \in \mathscr{D}(\mathbb{R}^{d+1})$ and $(t,x) \in \mathbb{R} \times \mathbb{R}^d$,

$$\varphi * \check{\rho}_H(t,x) = \int_t^{+\infty} \mathrm{d}s \int_{\mathbb{R}^d} \mathrm{d}y \frac{1}{(4\pi(s-t))^{\frac{d}{2}}} \exp\left(-\frac{|y-x|^2}{4(s-t)}\right) \varphi(s,y).$$

Since φ has compact support, we see from this formula that there is a $T \in \mathbb{R}_+$ such that for any $t \geqslant T$ and $x \in \mathbb{R}^d$, $\varphi * \check{\rho}_H(t,x) = 0$. Therefore, we need to check that $\varphi * \check{\rho}_H$ is in $L^\alpha([0,T] \times \mathbb{R}^d)$ for any $T \in \mathbb{R}_+$ and $\varphi \in \mathscr{D}(\mathbb{R}^{d+1})$. The function $\varphi * \check{\rho}_H$ is smooth, so we only need to check integrability for x in neighborhood of infinity. Then, for some compact $K \subset \mathbb{R}^d$, for x large enough,

$$\begin{aligned} |\varphi * \check{\rho}_H(t,x)| \leqslant |\varphi| * \check{\rho}_H(t,x) &= \mathbb{1}_{t \leqslant T} \int_t^T \mathrm{d}s \int_K \mathrm{d}y \frac{1}{(4\pi(s-t))^{\frac{d}{2}}} \exp\left(-\frac{|y-x|^2}{4(s-t)}\right) |\varphi(s,y)| \\ &\leqslant \mathbb{1}_{t \leqslant T} \|\varphi\|_\infty \int_t^T \mathrm{d}s \int_K \mathrm{d}y \frac{1}{(4\pi(T-t))^{\frac{d}{2}}} \exp\left(-\frac{|y-x|^2}{4(T-t)}\right), \end{aligned}$$

where the second inequality comes from the fact that for $|x|$ large enough, the function $s \in [t,T] \mapsto \frac{1}{(4\pi(s-t))^{\frac{d}{2}}} \exp\left(-\frac{|y-x|^2}{4(s-t)}\right)$ is non-decreasing and realizes its maximum at $s = T$. Then, using the inequality

$$|y-x|^2 \geqslant \frac{1}{2}|x|^2 - |y|^2\,, \tag{4.4.3}$$

we get

$$\begin{aligned}|\varphi * \check{\rho}_H(t,x)| &\leqslant \mathbb{1}_{t\leqslant T} \frac{\|\varphi\|_\infty}{(4\pi)^{\frac{d}{2}}}(T-t)^{-\frac{d}{2}+1}\int_K \mathrm{d}y \exp\left(-\frac{|y-x|^2}{4(T-t)}\right)\\ &\leqslant \mathbb{1}_{t\leqslant T} \frac{\|\varphi\|_\infty}{(4\pi)^{\frac{d}{2}}}(T-t)^{-\frac{d}{2}+1}\exp\left(-\frac{|x|^2}{8(T-t)}\right)\int_K \exp\left(-\frac{|y|^2}{4(T-t)}\right)\mathrm{d}y\,.\end{aligned}$$

We evaluate the integral and deduce that

$$|\varphi * \check{\rho}_H(t,x)| \leqslant \mathbb{1}_{t\leqslant T}\|\varphi\|_\infty T \exp\left(-\frac{|x|^2}{8T}\right). \tag{4.4.4}$$

From (4.4.4) we deduce that $\varphi * \check{\rho}_H$ has compact support in the time variable (uniformly with respect to the space variable), and has rapid decay in the space variable. Therefore $\varphi * \check{\rho}_H \in L^\alpha([0,T]\times\mathbb{R}^d)$ for any $\alpha \in \mathbb{R}_+$. We deduce that the stochastic linear heat equation driven by symmetric α-stable noise always has a generalized solution u_{gen} defined by

$$\langle u_{\text{gen}}, \varphi\rangle := \langle \dot{W}^\alpha, \varphi * \check{\rho}_H\rangle\,, \qquad \text{for all } \varphi \in \mathscr{D}(\mathbb{R}^{d+1}). \tag{4.4.5}$$

Furthermore, from (4.4.4), we get that

$$\|\varphi * \check{\rho}_H\|_{L^\alpha([0,T]\times\mathbb{R}^d)} \leqslant C\|\varphi\|_\infty\,,$$

for some constant C that depends on the support of φ. Therefore, if φ_n is a sequence of test functions in $\mathscr{D}(\mathbb{R}^{d+1})$ such that $\varphi_n \to 0$ in $\mathscr{D}(\mathbb{R}^{d+1})$, then

$$\mathbb{E}\left[e^{i\xi\langle u_{\text{gen}},\varphi_n\rangle}\right] = e^{-|\xi|^\alpha \|\varphi_n * \check{\rho}_H\|^\alpha_{L^\alpha([0,T]\times\mathbb{R}^d)}} \to 1\,, \qquad \text{as } n \to +\infty.$$

Therefore, $\langle u_{\text{gen}}, \varphi_n\rangle \to 0$ in law as $n \to +\infty$, and since convergence in law to a constant is equivalent to the convergence in probability to this constant, we deduce that $\langle u_{\text{gen}}, \varphi_n\rangle \to 0$ in probability as $n \to +\infty$. Therefore, u_{gen} defines a linear functional on $\mathscr{D}(\mathbb{R}^{d+1})$ that is continuous in probability. The space $\mathscr{D}(\mathbb{R}^{d+1})$ is nuclear (see Remark 2.2.1), so by [69, Corollary 4.2] u_{gen} has an almost surely continuous version (and therefore u_{gen} defines a continuous generalized stochastic process). □

Remark 4.4.2. *The previous proof is still valid if we formally replace α by 2, and therefore the same result is true in the Gaussian case.*

Existence of the mild solution

The criterion for the existence of the mild solution to the linear stochastic heat equation (4.4.2) is already known (see [4]).

Proposition 4.4.3. *The mild solution to the linear stochastic heat equation driven by a symmetric α-stable noise exists if and only if*

$$\alpha < 1 + \frac{2}{d}. \tag{4.4.6}$$

In this case,

$$u_{mild}(t,x) := \left\langle \dot{W}^{\alpha}, \rho_H(t-\cdot, x-\cdot)\right\rangle. \tag{4.4.7}$$

Proof. The mild solution of (4.4.2) associated with ρ_H is well defined if and only if the following integral is finite for any $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$ (see **(H2')**):

$$\begin{aligned}\int_{\mathbb{R}_+} \mathrm{d}s \int_{\mathbb{R}^d} \mathrm{d}y\, \rho_H(t-s, x-y)^{\alpha} &= \int_0^t \mathrm{d}s \frac{1}{(4\pi s)^{\alpha\frac{d}{2}}} \int_{\mathbb{R}^d} \mathrm{d}y \exp\left(-\frac{\alpha|y|^2}{4s}\right) \\ &= \int_0^t \mathrm{d}s \frac{1}{(4\pi s)^{\frac{d}{2}(\alpha-1)}\,\alpha^{\frac{d}{2}}},\end{aligned}$$

and the last integral is finite if and only if

$$\alpha < 1 + \frac{2}{d}.$$

Then, by Definition 4.1.5,

$$u_{\text{mild}}(t,x) := \left\langle \dot{W}^{\alpha}, \rho_H(t-\cdot, x-\cdot)\right\rangle.$$

□

Existence of a random field solution

We have seen in the previous section that for any α and d, it is possible to define the mild solution u_{gen}, and that the mild solution u_{mild} exists if and only if $\alpha < 1 + \frac{2}{d}$. We now apply the results of Theorem 4.2.1 and Theorem 4.3.1 to learn more about the relations between those two notions of solution.

Proposition 4.4.4. *The generalized solution u_{gen} to the linear stochastic heat equation driven by a symmetric α-stable noise has a random field representation X if and only if* (4.4.6) *is satisfied, and in that case, this random field representation X is equal to u_{mild} almost everywhere almost surely.*

Proof. If $\alpha \in \left(1, 1+\frac{d}{2}\right)$, then from Theorem 4.2.1 *(i)*, we deduce that u_{mild} is almost surely equal to u_{gen} (the condition is immediately verified since μ_φ is a finite measure with support in a compact set). Similarly, for any $\varphi \in \mathscr{D}(\mathbb{R}^{d+1})$, if $\alpha < 1$, by (4.4.4), $|\check{\rho}_H| * |\varphi| \in L^\alpha(\mathbb{R}_+ \times \mathbb{R}^d)$, hence by Theorem 4.2.1 *(iii)*, the mild solution of the stochastic heat equation u_{mild} is equal to the generalized solution u_{gen}. The case $\alpha = 1$ is slightly more involved, since we need to check condition (4.2.2). Let $\varphi \in \mathscr{D}(\mathbb{R}^{d+1})$. First, we have

$$\int_{\mathbb{R}_+ \times \mathbb{R}^d} \rho_H(t-s, x-v)\,\mathrm{d}v = t\mathbb{1}_{t>0},$$

and for any $x \in \mathbb{R}_+$, $\log_+(x) \leqslant |\log(x)|$, therefore, for $t > 0$

$$\begin{aligned}\log_+ &\left(\frac{\rho_H(t-s,x-y) \int_{\mathbb{R}^{d+1}} \int_{\mathbb{R}_+ \times \mathbb{R}^d} |\rho_H(u-v,r-w)|\,\mathrm{d}v\,\mathrm{d}w\,\mu_\varphi(\mathrm{d}u,\mathrm{d}r)}{\left(\int_{\mathbb{R}_+\times\mathbb{R}^d} \rho_H(t-v,x-w)\,\mathrm{d}v\,\mathrm{d}w\right)\left(\int_{\mathbb{R}^{d+1}} \rho_H(u-s,r-y)\mu_\varphi(\mathrm{d}u,\mathrm{d}r)\right)} \right) \\
&\leqslant \left|\log\left(\rho_H(t-s,x-y)\right)\right| + \left|\log\left(\int_{\mathbb{R}^{d+1}} u\mu_\varphi(\mathrm{d}u,\mathrm{d}r)\right)\right| + \left|\log(t)\right| \\
&\quad + \left|\log\left(\check{\rho}_H * |\varphi|(s,y)\right)\right|.\end{aligned}$$

Hence, to have (4.2.2), we need to check the finiteness of the following integrals:

$$\begin{aligned}
I_I &:= \int_{\mathbb{R}_+\times\mathbb{R}^d} \left(\check{\rho}_H * |\varphi|\right)(s,y)\,\mathrm{d}s\,\mathrm{d}y, \\
I_2 &:= \int_{\mathbb{R}_+\times\mathbb{R}^d} \left(\check{\rho}_H \left|\log\left(\check{\rho}_H\right)\right|\right) * |\varphi|(s,y)\,\mathrm{d}s\,\mathrm{d}y, \\
I_3 &:= \int_{\mathbb{R}_+\times\mathbb{R}^d} \left(\int_{\mathbb{R}^{d+1}} \rho_H(t-s,x-y)|\log(t)\varphi(t,x)|\,\mathrm{d}t\,\mathrm{d}x\right)\mathrm{d}s\,\mathrm{d}y, \\
I_4 &:= \int_{\mathbb{R}_+\times\mathbb{R}^d} \left|\log\left(\check{\rho}_H * |\varphi|(s,y)\right)\right| \left(\check{\rho}_H * |\varphi|\right)(s,y)\,\mathrm{d}s\,\mathrm{d}y.
\end{aligned}$$

The case of I_1 has already been treated after (4.4.4), and for I_3, we can simply permute the integrals and get

$$I_3 = \int_{\mathbb{R}^{d+1}} |t\mathbb{1}_{t>0} \log(t)\varphi(t,x)|\,\mathrm{d}t\,\mathrm{d}x < +\infty.$$

For I_2 and I_4, by the same considerations as for the case $\alpha \neq 1$, we need to check that for any $\varphi \in \mathscr{D}(\mathbb{R}^{d+1})$,

$$\begin{aligned}
&(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d \mapsto \left|\check{\rho}_H \log(\check{\rho}_H)\right| * |\varphi|(t,x), \\
\text{and } &(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d \mapsto \left(\check{\rho}_H * |\varphi|\right)(t,x) \left|\log(\check{\rho}_H * |\varphi|(t,x))\right|,
\end{aligned}$$

are in $L^1([0,T]\times\mathbb{R}^d)$ for any $T \in \mathbb{R}_+$. By (4.4.4), we get that $\left(\check{\rho}_H * |\varphi|\right)\left|\log(\check{\rho}_H * |\varphi|)\right| \in L^1([0,T] \times \mathbb{R}^d)$. Then,

$$\begin{aligned}\left|\check{\rho}_H \log(\check{\rho}_H)\right| * |\varphi|(t,x) = \mathbb{1}_{t \leqslant T} \int_t^T \mathrm{d}s \int_K \mathrm{d}y \frac{1}{(4\pi(s-t))^{\frac{d}{2}}} \exp\left(-\frac{|y-x|^2}{4(s-t)}\right) \\
\times \left|-\frac{d}{2}\log\left(4\pi(s-t)\right) - \frac{|y-x|^2}{4(s-t)}\right| |\varphi(s,y)|.\end{aligned}$$

We use the triangular inequality and treat each term separately. Again, by continuity (since $|\varphi|$ is continuous and has compact support), we are only concerned about integrability near a neighborhood of infinity. By Lemma A.0.5, for $|x|$ large enough,

$$\begin{aligned}
J_1 := \mathbb{1}_{t\leqslant T}\int_t^T \mathrm{d}s\int_K \mathrm{d}y \frac{\pi|y-x|^2}{(4\pi(s-t))^{\frac{d}{2}+1}}\exp\left(-\frac{|y-x|^2}{4(s-t)}\right)|\varphi(s,y)| \\
\leqslant \mathbb{1}_{t\leqslant T}\|\varphi\|_\infty \int_K \mathrm{d}y \frac{\pi|y-x|^2}{(4\pi(T-t))^{\frac{d}{2}}}\exp\left(-\frac{|y-x|^2}{4(T-t)}\right).
\end{aligned}$$

Then, using (4.4.3), and letting $\beta = \sup_{y\in K}|y|$,

$$\begin{aligned}
J_1 &\leqslant \mathbb{1}_{t\leqslant T}\|\varphi\|_\infty \frac{\pi\left(\beta+|x|\right)^2}{(4\pi(T-t))^{\frac{d}{2}}}\exp\left(-\frac{|x|^2}{8(T-t)}\right)\int_K \exp\left(\frac{|y|^2}{4(T-t)}\right)\mathrm{d}y \\
&\leqslant \mathbb{1}_{t\leqslant T}\|\varphi\|_\infty \pi\left(\beta+|x|\right)^2\exp\left(-\frac{|x|^2}{8T}\right).
\end{aligned}$$

We deduce that J_1 has rapid decay in the space variable as $|x|\to+\infty$ and compact support in time. Also, since for large x the function $s\in[t,T]\mapsto \frac{1}{(4\pi(s-t))^{\frac{d}{2}}}\exp\left(-\frac{|y-x|^2}{4(s-t)}\right)$ is non-decreasing and realizes its maximum at $s=T$, and using (4.4.3),

$$\begin{aligned}
J_2 &:= \mathbb{1}_{t\leqslant T}\int_t^T \mathrm{d}s\int_K \mathrm{d}y \frac{1}{(4\pi(s-t))^{\frac{d}{2}}}\exp\left(-\frac{|y-x|^2}{4(s-t)}\right)\frac{d}{2}\left|\log(4\pi(s-t))\right||\varphi(s,y)| \\
&\leqslant \mathbb{1}_{t\leqslant T}\frac{d}{2}\|\varphi\|_\infty (4\pi(T-t))^{-\frac{d}{2}}\left(\int_K \exp\left(-\frac{|y-x|^2}{4(T-t)}\right)\mathrm{d}y\right)\left(\int_0^{T-t}\left|\log(4\pi s)\right|\mathrm{d}s\right) \\
&\leqslant \mathbb{1}_{t\leqslant T}\frac{d}{2}\|\varphi\|_\infty \exp\left(-\frac{|x|^2}{8T}\right)\int_0^T\left|\log(4\pi s)\right|\mathrm{d}s.
\end{aligned}$$

We deduce that J_2 has rapid decay in the space variable as $|x|\to+\infty$ and compact support in time. Therefore, $\left|\check{\rho}_H\log(\check{\rho}_H)\right| * |\varphi| \in L^1([0,T]\times\mathbb{R}^d)$ for any $T\in\mathbb{R}_+$. Hence by Theorem 4.2.1 *(ii)*, the mild solution u_{mild} of the stochastic heat equation in the case $\alpha=1$ is also equal to u_{gen}.

Furthermore, if u_{gen} has a random field representation Y in the sense of Definition 4.1.2, then, by Theorem 4.3.1, necessarily $\rho_H\in L^\alpha([0,T]\times\mathbb{R}^d)$ for any $T>0$, which is equivalent to (4.4.6), and the random field representation Y is equal to the mild solution u_{mild} almost everywhere a.s. Therefore, a necessary and sufficient condition for the existence of a random field solution to the stochastic heat equation (4.4.2) is that $\alpha < 1+\frac{2}{d}$. □

We therefore have the following theorem:

Theorem 4.4.5. *The generalized solution u_{gen} to the stochastic heat equation* (4.4.2) *defined by* (4.4.5) *always exists. The mild solution u_{mild} defined by* (4.4.7) *exists if and only if*

$$\alpha < 1+\frac{2}{d}, \tag{4.4.8}$$

Furthermore, a random field representation X of the generalized solution exists if and only if (4.4.8) *is satisfied and in this case, for almost all* $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$,

$$X_{t,x} = \langle \dot{W}^\alpha, \rho_H(t-\cdot, x-\cdot)\rangle = u_{mild}(t,x) \qquad a.s. \quad a.e.$$

4.4.2 The stochastic wave equation

We now consider the stochastic wave equation. This is an equation of hyperbolic type, therefore we expect different conclusions from the case of the stochastic heat equation in the previous section. For an overview of this SPDE, see [21]. Let $\dot{W}^\alpha$ be an α-stable symmetric noise on $\mathbb{R}_+ \times \mathbb{R}^d$. The wave operator O in dimension d is a constant coefficient partial differential operator given by

$$O = \frac{\partial^2}{\partial t^2} - \sum_{i=1}^d \frac{\partial^2}{\partial x_i^2}.$$

The fundamental solution of this operator is a function only in dimension one and two. In dimension one it is given by

$$\rho_1^O(t,x) = \frac{1}{2}\mathbb{1}_{|x|\leqslant t} \qquad \text{for all } (x,t) \in \mathbb{R}^2,$$

and in dimension two by

$$\rho_2^O(t,x) = \frac{1}{2\pi}\frac{1}{\sqrt{t^2-|x|^2}}\mathbb{1}_{|x|<t} \qquad \text{for all } (t,x) \in \mathbb{R}\times\mathbb{R}^2.$$

In dimension $d \geqslant 3$, the fundamental solution is a distribution that can be characterized by its Fourier transform in the space variable x.

We consider the following Cauchy problem

$$\begin{cases} Ou = \dot{W}^\alpha, \\ u(0,\cdot) = 0, \\ \frac{\partial u}{\partial t}(0,\cdot) = 0. \end{cases} \tag{4.4.9}$$

When it exists (i.e. under **(H1')**), we recall that the generalized solution u_{gen} is given by

$$\langle u_{\text{gen}}, \varphi\rangle := \langle \dot{W}^\alpha, \varphi * \check{\rho}_d^O\rangle, \qquad \text{for all } \varphi \in \mathscr{D}(\mathbb{R}^{d+1}). \tag{4.4.10}$$

Existence of the generalized solution

We first study the existence of the generalized solution in various dimensions $d \geqslant 1$.

Proposition 4.4.6. *In any dimension $d \geqslant 1$, the generalized solution u_{gen} to the linear stochastic wave equation driven by a symmetric α-stable noise always exists.*

Proof. We need to check wether **(H1')** is satisfied.

$\underline{d=1}$: We need to check that for any $\varphi \in \mathscr{D}(\mathbb{R}^2)$, the convolution $\varphi * \check{\rho}_1^O$ is in $L^\alpha(\mathbb{R}_+ \times \mathbb{R})$. We get

$$\varphi * \check{\rho}_1^O(t,x) = \int_0^{+\infty} \mathrm{d}s \int_{-s}^{s} \mathrm{d}y\, \varphi(s+t, y+x),$$

and we can see from this expression that it is a smooth function with compact support, hence in $L^\alpha(\mathbb{R}_+ \times \mathbb{R})$.

$\underline{d=2}$: Let $\varphi \in \mathscr{D}(\mathbb{R}^3)$. We check wether or not for some $\alpha \in (0,2)$, the function $\varphi * \check{\rho}_2^O \in L^\alpha(\mathbb{R}_+ \times \mathbb{R}^d)$. By standard properties of the convolution, $\varphi * \check{\rho}_2^O$ is a smooth function. Let $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^2$. Then,

$$\varphi * \check{\rho}_2^O(t,x) = \int_{\mathbb{R}} \mathrm{d}s \int_{\mathbb{R}^2} \mathrm{d}y\, \rho_2^O(s-t, y-x)\varphi(s,y).$$

Since φ has compact support and ρ_2^O has support in the set $\{(t,x) \in \mathbb{R}_+ \times \mathbb{R}^2 : |x| \leqslant t\}$, we can write

$$\varphi * \check{\rho}_2^O(t,x) = \mathbb{1}_{t \leqslant T} \int_t^T \mathrm{d}s \int_{B_x(T-t)} \mathrm{d}y\, \rho_2^O(s-t, y-x)\varphi(s,y),$$

for some $T \in \mathbb{R}_+$, where $B_x(r)$ is the open ball of radius r centered at x. We see in this expression that the convolution has compact support in space and time, since if x is far enough from the support of φ, the integrand is zero. We deduce that for any $\alpha \in (0,2)$, $\varphi * \check{\rho}_2^O \in L^\alpha(\mathbb{R}_+ \times \mathbb{R}^d)$, and the generalized solution to the stochastic linear wave equation in dimension 2 always exists.

$\underline{d \geqslant 3}$: For any $\varphi \in \mathscr{D}(\mathbb{R} \times \mathbb{R}^d)$, the function $\varphi * \check{\rho}_d^O$ is smooth. Furthermore, for physical reasons, it is usually more sensible to consider a fundamental solution with support in the light cone, that is the set $\{(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d : |x| \leqslant t\}$. Such fundamental solutions always exist in any dimension (see [38, Proposition 11.3.1]). By the same type of considerations on the support of the convolution $\varphi * \check{\rho}_d^O$ as in dimension one and two, we see that this function has compact support, therefore $\varphi * \check{\rho}_d^O \in L^\alpha(\mathbb{R}_+ \times \mathbb{R}^d)$ for any $\alpha \in (0,2)$. We deduce that the generalized solution always exists. □

Existence of the mild solution

Proposition 4.4.7. *The mild solution to the stochastic wave equation driven by a symmetric α-stable noise exists only in dimensions one and two regardless of the parameter $\alpha \in (0,2)$.*

Proof. $\underline{d=1}$: There is a mild solution to the wave equation driven by α-stable noise if and only if for any $(t,x) \in \mathbb{R}^+ \times \mathbb{R}$, $\rho_1^O(t-\cdot, x-\cdot) \in L^\alpha(\mathbb{R}_+ \times \mathbb{R})$ (see **(H2')**). Therefore, for any $T>0$, we need to check the finiteness of the integral

$$\int_0^T \mathrm{d}t \int_{\mathbb{R}} \mathrm{d}x\, \rho_1^O(t,x)^\alpha = \int_0^T \mathrm{d}t \int_{\mathbb{R}} \mathrm{d}x\, \frac{1}{2^\alpha} \mathbb{1}_{|x| \leqslant t} = \frac{T^2}{2^\alpha}. \tag{4.4.11}$$

We deduce that the mild solution exists for any choice of $\alpha \in (0,2)$.

$\underline{d=2}$: The mild solution exists if and only if $\rho_2^O(t-\cdot,x-\cdot) \in L^\alpha(\mathbb{R}_+\times\mathbb{R})$ for any $(t,x)\in\mathbb{R}_+\times\mathbb{R}^2$ (see **(H2')**). We have

$$\begin{aligned}
\|\rho_2^O(t-\cdot,x-\cdot)\|^{\alpha\vee 1}_{L^\alpha(\mathbb{R}_+\times\mathbb{R}^2)} &= \int_0^t \mathrm{d}s \int_{\mathbb{R}^2} \mathrm{d}y \frac{1}{(2\pi)^\alpha \left((t-s)^2-|x-y|^2\right)^{\frac{\alpha}{2}}} \\
&= \frac{1}{(2\pi)^\alpha}\int_0^t \mathrm{d}s \int_{|u|\leqslant s} \mathrm{d}u \frac{1}{(s^2-|u|^2)^{\frac{\alpha}{2}}}.
\end{aligned}$$

Using a polar coordinates change of variables, we get

$$\|\rho_2^O(t-\cdot,x-\cdot)\|^{\alpha\vee 1}_{L^\alpha(\mathbb{R}_+\times\mathbb{R}^2)} = \frac{1}{(2\pi)^{\alpha-1}}\int_0^t \mathrm{d}s \int_0^s \mathrm{d}r \frac{r}{(s-r)^{\frac{\alpha}{2}}(s+r)^{\frac{\alpha}{2}}}.$$

This integral is finite if and only if $\frac{\alpha}{2}<1$, that is $\alpha<2$. We can further evaluate this integral and we get

$$\begin{aligned}
\|\rho_2^O(t-\cdot,x-\cdot)\|^{\alpha\vee 1}_{L^\alpha(\mathbb{R}_+\times\mathbb{R}^2)} &= \frac{1}{(2\pi)^{\alpha-1}}\int_0^t \mathrm{d}s \int_0^s \frac{\mathrm{d}r}{2} \frac{2r}{(s^2-r^2)^{\frac{\alpha}{2}}} \\
&= \frac{1}{(2\pi)^{\alpha-1}}\int_0^t \mathrm{d}s \frac{s^{2-\alpha}}{2-\alpha} = \frac{t^{3-\alpha}}{(2\pi)^{\alpha-1}(2-\alpha)(3-\alpha)}.
\end{aligned} \tag{4.4.12}$$

Therefore, in dimension 2, there is always a mild solution to the linear stochastic wave equation with α-stable noise.

$\underline{d\geqslant 3}$: Since fundamental solutions of the wave equation in dimension $d\geqslant 3$ are not functions, there is no mild solution. □

Remark 4.4.8. *From this proof, we can deduce the already known result in the Gaussian case (see [21, p. 46]) that a solution to the linear stochastic wave equation only exists in dimension one.*

Existence of a random field solution

Proposition 4.4.9. *The generalized solution u_{gen} to the linear stochastic wave equation driven by a symmetric α-stable noise has a random field representation if and only if $d\leqslant 2$, and in that case, this random field representation is equal to u_{mild} almost everywhere almost surely.*

Proof. $\underline{d=1}$: We check if the mild solution is equal to the generalized solution solution using Theorem 4.2.1 and the Remark 4.2.2. If $\alpha>1$, it suffices to check that $\{\|\rho_1^O(t-\cdot,x-\cdot)\|_{L^\alpha(\mathbb{R}_+\times\mathbb{R})}, (t,x)\in\mathbb{R}_+\times\mathbb{R}\} \in L^1_{\mathrm{loc}}(\mathbb{R}_+\times\mathbb{R})$, which is the case by (4.4.11). In the case $\alpha<1$, we can check that for any compact $K\subset\mathbb{R}^2$,

$$\int_{\mathbb{R}_+\times\mathbb{R}} \mathrm{d}s\,\mathrm{d}y \left(\int_K \mathrm{d}t\,\mathrm{d}x |\rho_1^O(t-s,x-y)|\right)^\alpha < +\infty.$$

It is easy to see that the function $(s,y) \mapsto \int_K \mathrm{d}t\,\mathrm{d}x |\rho_1^O(t-s,x-y)|$ has compact support, which suffices to prove the claim. In the case $\alpha = 1$ we check that for any compact set $K \subset \mathbb{R}^2$,

$$\begin{aligned} &\int_K \mathrm{d}t\,\mathrm{d}x \int_{\mathbb{R}_+\times\mathbb{R}} \mathrm{d}s\,\mathrm{d}y |\rho_1^O(t-s,x-y)| \Bigg[1+ \\ &\log_+\left(\frac{|\rho_1^O(t-s,x-y)| \int_K \mathrm{d}u\,\mathrm{d}r \int_{\mathbb{R}_+\times\mathbb{R}} \mathrm{d}v\,\mathrm{d}w |\rho_1^O(u-v,r-w)|}{\left(\int_{\mathbb{R}_+\times\mathbb{R}} \mathrm{d}v\,\mathrm{d}w |\rho_1^O(t-v,x-w)|\right)\left(\int_K |\rho_1^O(u-s,r-y)|\,\mathrm{d}u\,\mathrm{d}r\right)}\right)\Bigg] < +\infty. \end{aligned} \tag{4.4.13}$$

Indeed, $\int_{\mathbb{R}_+\times\mathbb{R}} \mathrm{d}v\,\mathrm{d}w |\rho_1^O(t-v,x-w)| = \frac{t^2}{2}$, and $\int_K \mathrm{d}u\,\mathrm{d}r \int_{\mathbb{R}_+\times\mathbb{R}} \mathrm{d}v\,\mathrm{d}w |\rho_1^O(u-v,r-w)| = C$ for some constant C depending on K. Therefore, (4.4.13) is bounded by

$$\int_K \mathrm{d}t\,\mathrm{d}x \int_{\mathbb{R}_+\times\mathbb{R}} \mathrm{d}s\,\mathrm{d}y |\rho_1^O(t-s,x-y)| \left[1+\log_+\left(\frac{|\rho_1^O(t-s,x-y)|C}{\frac{t^2}{2}\left(\int_K |\rho_1^O(u-s,r-y)|\,\mathrm{d}u\,\mathrm{d}r\right)}\right)\right].$$

We have mentionned before that for any $x>0$, $\log_+(x) \leqslant |\log(x)|$. Therefore, we need to check that the following integrals are finite:

$$\begin{aligned} J_1 &:= \int_K \mathrm{d}t\,\mathrm{d}x \int_{\mathbb{R}_+\times\mathbb{R}} \mathrm{d}s\,\mathrm{d}y |\rho_1^O(t-s,x-y)| \left|\log\left(\rho_1^O(t-s,x-y)\right)\right|, \\ J_2 &:= \int_K \mathrm{d}t\,\mathrm{d}x \int_{\mathbb{R}_+\times\mathbb{R}} \mathrm{d}s\,\mathrm{d}y |\rho_1^O(t-s,x-y)| \left|\log\left(\frac{t^2}{2}\right)\right|, \\ J_3 &:= \int_K \mathrm{d}t\,\mathrm{d}x \int_{\mathbb{R}_+\times\mathbb{R}} \mathrm{d}s\,\mathrm{d}y |\rho_1^O(t-s,x-y)| \left|\log\left(\int_K |\rho_1^O(u-s,r-y)|\,\mathrm{d}u\,\mathrm{d}r\right)\right|. \end{aligned}$$

By (4.4.11) and integrability of the logarithm near the origin, $J_2 < +\infty$. Also, ρ_1^O is bounded, and the integration domain is bounded, so $J_1 < +\infty$ and $J_3 < +\infty$. Therefore, for any $\alpha \in (0,2)$, the mild solution is equal to the generalized solution.

<u>$d=2$:</u> In the case where $\alpha > 1$, by (4.4.12), $(t,x) \in \mathbb{R}_+\times\mathbb{R}^2 \to \|\rho_2^O(t-\cdot,x-\cdot)\|_{L^\alpha(\mathbb{R}_+\times\mathbb{R}^2)}$ does not depend on x and is continuous in the t variable, therefore (4.2.1) is verified, and the mild solution is equal to the generalized solution.

In the case where $\alpha < 1$, we know from previous considerations that for any test function φ, $|\check{\rho}_2^O| * \varphi$ is smooth with compact support, therefore (4.2.3) is verified, and the mild solution is equal to the generalized solution.

The case $\alpha = 1$ is again more involved, since we need to check the unfriendly expression (4.2.2). We check that for any compact set $K \subset \mathbb{R}^3$,

$$\begin{aligned} &\int_K \mathrm{d}t\,\mathrm{d}x \int_{\mathbb{R}_+\times\mathbb{R}^2} \mathrm{d}s\,\mathrm{d}y |\rho_2^O(t-s,x-y)| \Bigg[1+ \\ &\log_+\left(\frac{|\rho_2^O(t-s,x-y)| \int_K \mathrm{d}u\,\mathrm{d}r \int_{\mathbb{R}_+\times\mathbb{R}^2} \mathrm{d}v\,\mathrm{d}w |\rho_2^O(u-v,r-w)|}{\left(\int_{\mathbb{R}_+\times\mathbb{R}^2} \mathrm{d}v\,\mathrm{d}w |\rho_2^O(t-v,x-w)|\right)\left(\int_K |\rho_2^O(u-s,r-y)|\,\mathrm{d}u\,\mathrm{d}r\right)}\right)\Bigg] < +\infty. \end{aligned} \tag{4.4.14}$$

Indeed, by (4.4.12), $\int_{\mathbb{R}_+\times\mathbb{R}^2} \mathrm{d}v\,\mathrm{d}w |\rho_2^O(t-v, x-w)| = C_1 t^2$, and $\int_K \mathrm{d}u\,\mathrm{d}r \int_{\mathbb{R}_+\times\mathbb{R}^2} \mathrm{d}v\,\mathrm{d}w |\rho_2^O(u-v, r-w)| = C_2$ for some constants C_1, C_2 depending on K. Therefore, (4.4.14) is bounded by

$$\int_K \mathrm{d}t\,\mathrm{d}x \int_{\mathbb{R}_+\times\mathbb{R}^2} \mathrm{d}s\,\mathrm{d}y |\rho_2^O(t-s, x-y)| \left[1 + \log_+\left(\frac{|\rho_2^O(t-s, x-y)| C_2}{C_1 t^2 \left(\int_K |\rho_2^O(u-s, r-y)| \,\mathrm{d}u\,\mathrm{d}r\right)}\right)\right].$$

We have mentionned before that for any $x > 0$, $\log_+(x) \leqslant |\log(x)|$. Therefore, we need to check that the following integrals are finite:

$$\begin{aligned}
\tilde{J}_1 &:= \int_K \mathrm{d}t\,\mathrm{d}x \int_{\mathbb{R}_+\times\mathbb{R}^2} \mathrm{d}s\,\mathrm{d}y |\rho_2^O(t-s, x-y)| \left|\log\left(\rho_2^O(t-s, x-y)\right)\right|, \\
\tilde{J}_2 &:= \int_K \mathrm{d}t\,\mathrm{d}x \int_{\mathbb{R}_+\times\mathbb{R}^2} \mathrm{d}s\,\mathrm{d}y |\rho_2^O(t-s, x-y)| \left|\log\left(C_1 t^2\right)\right|, \\
\tilde{J}_3 &:= \int_K \mathrm{d}t\,\mathrm{d}x \int_{\mathbb{R}_+\times\mathbb{R}^2} \mathrm{d}s\,\mathrm{d}y |\rho_2^O(t-s, x-y)| \left|\log\left(\int_K |\rho_2^O(u-s, r-y)| \,\mathrm{d}u\,\mathrm{d}r\right)\right|.
\end{aligned}$$

By, (4.4.12) and integrability of the logarithm around the origin, $\tilde{J}_2 < +\infty$. Then,

$$\tilde{J}_1 = \int_K \mathrm{d}t\,\mathrm{d}x \int_{[0,t]\times\mathbb{R}^2} \mathrm{d}s\,\mathrm{d}y \mathbb{1}_{|y|\leqslant s} | \frac{1}{8\pi\sqrt{s^2 - |y|^2}} \left|\log\left(\frac{1}{s^2 - |y|^2}\right)\right| < +\infty.$$

Changing to polar coordinates, we get

$$\tilde{J}_1 = \int_K \mathrm{d}t\,\mathrm{d}x \int_{[0,t]} \mathrm{d}s \int_0^s \mathrm{d}r | \frac{r}{4\sqrt{s^2 - r^2}} \left|\log\left(s^2 - r^2\right)\right| < +\infty,$$

since $x \to |\log(x)||x|^{-\frac{1}{2}}$ is integrable on a neighborhood of the origin.

Finally, by Fubini's theorem,

$$\tilde{J}_3 = \int_{\mathbb{R}_+\times\mathbb{R}^2} \mathrm{d}s\,\mathrm{d}y \int_K \mathrm{d}t\,\mathrm{d}x |\rho_2^O(t-s, x-y)| \left|\log\left(\int_K |\rho_2^O(u-s, r-y)| \,\mathrm{d}u\,\mathrm{d}r\right)\right|.$$

Using the inequality $|x\log(x)| \leqslant |x|^{\frac{3}{2}}$, we get

$$\begin{aligned}
\tilde{J}_3 &\leqslant \int_{\mathbb{R}_+\times\mathbb{R}^2} \mathrm{d}s\,\mathrm{d}y \left(\int_K \mathrm{d}t\,\mathrm{d}x |\rho_2^O(t-s, x-y)|\right)^{\frac{3}{2}} \\
&\leqslant (\mathrm{Leb}_3(K))^{\frac{1}{2}} \int_{\mathbb{R}_+\times\mathbb{R}^2} \mathrm{d}s\,\mathrm{d}y \int_K \mathrm{d}t\,\mathrm{d}x |\rho_2^O(t-s, x-y)|^{\frac{3}{2}} \\
&= (\mathrm{Leb}_3(K))^{\frac{1}{2}} \int_K \mathrm{d}t\,\mathrm{d}x \int_{\mathbb{R}_+\times\mathbb{R}^2} \mathrm{d}s\,\mathrm{d}y |\rho_2^O(t-s, x-y)|^{\frac{3}{2}} \\
&= (\mathrm{Leb}_3(K))^{\frac{1}{2}} \int_K \mathrm{d}t\,\mathrm{d}x \frac{4t^{\frac{3}{2}}}{5\sqrt{2\pi}} < +\infty,
\end{aligned}$$

where we have used Hölder's inequality in the second line, and (4.4.12) in the last line. Therefore, for any $\alpha \in (0,2)$, the mild solution is equal to the generalized solution.

$\underline{d \geqslant 3}$: By Theorem 4.3.1, there cannot be any random field representation of the generalized solution, since $\rho_d^O \notin L^\alpha\left([0,T] \times \mathbb{R}^d\right)$. □

We summarize these results in the following theorem:

Theorem 4.4.10. *The generalized solution u_{gen} to the stochastic wave equation* (4.4.9) *defined by* (4.4.10) *always exists. The mild solution u_{mild} defined by* (4.4.7) *exists if and only if $d \leqslant 2$. Furthermore, a random field representation X of the generalized solution exists if and only if $d \leqslant 2$, and in this case, for almost all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$,*

$$X_{t,x} = \left\langle \dot{W}^\alpha, \rho_d^O(t-\cdot, x-\cdot)\right\rangle = u_{mild}(t,x) \qquad a.s. \quad a.e.$$

5 Some properties of the solution to the stochastic heat equation driven by heavy-tailed noise

In the case of the stochastic heat equation with Gaussian noise, the Hölder joint regularity of the mild solution has been proved in many cases (see [21] for the white noise case, or [48, 11, 40] for more general cases). In the presence of jumps, already in the linear case, this type of regularity is not relevant for the study of the mild solution to the stochastic heat equation. Indeed, due to the singularity of the Gaussian kernel at the origin, each jump of the noise creates a Dirac mass for the solution. In other terms, the mild solution of this equation in the linear case is a Dirac mass at each space-time jump point of the noise. Furthermore, these space-time jump points can form a dense subset of $[0,T]\times\mathbb{R}^d$. Therefore, we study two other types of regularity of the mild solution $\big(u(t,x);(t,x)\in[0,T]\times\mathbb{R}^d\big)$. The first one is the regularity of $t\mapsto u(t,\cdot)$ viewed as a mapping to a fractional Sobolev space (see Sections 5.2.1, 5.3.2 and 5.4.1), and the second is the regularity of the partial process obtained from u when fixing either the time or space coordinate.

The results detailed in this chapter will be published in [15].

This chapter is organized as follows: we first briefly introduce in Section 5.1 the stochastic integral with respect to Lévy white noise for integrands that are no longer deterministic. Then, Section 5.2 deals with the regularity properties of the mild solution to the stochastic heat equation on a bounded space interval in dimension one (see Theorem 5.2.7 and Propositions 5.2.10 and 5.2.12). Section 5.3 extends this study to the case of the equation on the whole space (see Theorem 5.3.12 and Propositions 5.3.13 and 5.3.17), and we prove a result on the stationarity in the space variable of the mild solution in Theorem 5.3.6. Finally, Section 5.4 studies regularity properties of the solution to the equation on a smooth and bounded domain in dimensions $d\geqslant 2$ (see Theorem 5.4.6 and Propositions 5.4.7 and 5.4.9).

5.1 Stochastic integration with respect to Lévy white noises

In Chapter 2, we introduced the notion of Lévy white noise, and using the related notion of independently scattered random measures (see Theorem 2.3.5), we recalled the construction of Wiener-type integrals, that is integrals of deterministic functions with respect to these noises (see Definition 2.3.6). Then, we used those integrals to discuss various questions about stochastic partial differential equations in linear cases. Indeed, the mild solution of a linear SPDE is defined as a stochastic convolution of the fundamental solution of the equation (which is a deterministic function) with the driving Lévy white noise. When there is a temporal component, in the non-linear case with a multiplicative noise, the mild solutions is defined via the formula

$$u(t,x) = \int_0^t \int_{\mathbb{R}^d} G(t-s,x-y)\sigma\left(u(s,y)\right) L(\mathrm{d}s,\mathrm{d}y)\,,$$

where $\dot{L}$ is a Lévy white noise, σ is a measurable function that encodes the non-linearity, and G is the fundamental solution of the equation. Here the term $\sigma\left(u(s,y)\right)$ is stochastic, and we therefore cannot use the Wiener-type integrals. A theory of such stochastic integrals has been widely studied, first in the purely temporal case with the seminal work of Itô (see [57] for a modern exposition of this theory). The extension to a spatio-temporal framework is handled via the notion of L^p-valued random measure, and the integration theory with respect to such measure is developed in [10]. We introduce here some key features of this integration theory, and we heavily rely on [12, Chapter 1] for this introduction. We recall that $(\Omega,\mathscr{F},\mathbb{P})$ is a complete probability space. The predictable σ-field $\mathscr{P}$ on $\Omega\times\mathbb{R}_+$ is the σ-field generated by the continuous processes (viewed as mappings from $\Omega\times\mathbb{R}_+$). Using the notations of [43, Chapter II], we define $\tilde{\mathscr{P}} := \mathscr{P}\otimes\mathscr{B}(\mathbb{R}^d)$, where $\mathscr{B}(\mathbb{R}^d)$ is the Borel σ-field on $\mathbb{R}^d$. A random field $u:\Omega\times\mathbb{R}_+\times\mathbb{R}^d\to\mathbb{R}$ is said to be predictable if it is $\tilde{\mathscr{P}}$-measurable. Let $\dot{L}$ be a Lévy white noise, with characteristic triplet $\left(b,\rho,\nu\right)$, and underlying jump measure J. Stochastic integration of predictable random fields with respect to Gaussian white noise is well known (see for example [21]), so we can suppose that $b=\rho=0$. Then, for any set $A\in\mathscr{B}(\mathbb{R}_+\times\mathbb{R}^d)$ with finite Lebesgue measure, by Theorem 2.3.10, we can assume that

$$\begin{aligned}\dot{L}(A) &= \int_{\mathbb{R}_+\times\mathbb{R}^d}\int_{|z|\leqslant 1} z\mathbb{1}_{(s,y)\in A}\,\tilde{J}(\mathrm{d}s,\mathrm{d}y,\mathrm{d}z) + \int_{\mathbb{R}_+\times\mathbb{R}^d}\int_{|z|>1} z\mathbb{1}_{(s,y)\in A}\,J(\mathrm{d}s,\mathrm{d}y,\mathrm{d}z)\\ &=: \dot{L}^1(A)+\dot{L}^2(A)\,,\end{aligned}$$

In this representation, the set A is not random, and we would like to extend this definition to any predictable set $\tilde{A}\in\tilde{\mathscr{P}}$. We suppose additionally that $\tilde{A}\subset\Omega\times[0,k]\times[-k,k]^d$ for some $k\in\mathbb{N}$. Then, if we denote by (T_i,Y_i,Z_i) the atoms of the Poisson random measure J, since $\tilde{A}$ is almost surely bounded, we can define the following finite sum:

$$\dot{L}^2\left(\tilde{A}\right)(\omega) := \sum_{i\geqslant 1} Z_i(\omega)\mathbb{1}_{|Z_i|(\omega)>1}\mathbb{1}_{(\omega,T_i(\omega),Y_i(\omega))\in\tilde{A}}\,,\qquad \text{for all } \omega\in\Omega\,.$$

Now, we would like to define $\dot{L}^1(\tilde{A})$. More generally, in [43, Chapter II], a theory of integration of a suitable class of stochastic processes with respect to a compensated Poisson random measure is exposed. Let $(H_{t,x})_{(t,x)\in\mathbb{R}_+\times\mathbb{R}^d}$ be a predictable random field. For $\omega\in\Omega$, we define

$$\tilde{H}_t(\omega)=\sum_{i\geqslant 1} Z_i(\omega)\mathbb{1}_{|Z_i(\omega)|\leqslant 1}\mathbb{1}_{T_i(\omega)=t}H_{T_i(\omega),Y_i(\omega)}(\omega)\,. \tag{5.1.1}$$

Then, adapting from [43, Chapter II, Definition 1.27], we say that H is in $G_{\mathrm{loc}}(\dot{L}^1)$ if there exists a sequence of stopping times T_n, $T_n\to+\infty$ a.s. as $n\to+\infty$, such that for any $t\geqslant 0$,

$$\mathbb{E}\left[\left(\sum_{s\leqslant t\wedge T_n}\tilde{H}_r^2\right)^{\frac{1}{2}}\right]<+\infty\,. \tag{5.1.2}$$

Then, for any H in $G_{\mathrm{loc}}(\dot{L}^1)$, using [43, Chapter I, Theorem 4.56], there exists a purely discontinuous martingale X such that ΔX (the jump process of X) is indistinguishable from $\tilde{H}$. Then, following [43, Chapter I, Definition 1.27], we define

$$\int_0^t\int_{\mathbb{R}^d}H_{s,y}L^1(\mathrm{d}s,\mathrm{d}y)=\int_0^t\int_{\mathbb{R}^d}\int_{|z|\leqslant 1} zH_{s,y}\tilde{J}(\mathrm{d}s,\mathrm{d}y,\mathrm{d}z):=X_t\,. \tag{5.1.3}$$

For $\tilde{A}\in\tilde{\mathscr{P}}$ with $\tilde{A}\subset\Omega\times[0,k]\times[-k,k]^d$ for some $k\in\mathbb{N}$, consider the random field $W:(\omega,t,x)\mapsto W_{(t,x)}(\omega)=\mathbb{1}_{(\omega,t,x)\in\tilde{A}}$. This random field is predictable and we see that $W\in G_{\mathrm{loc}}(\dot{L}^1)$. Indeed, by (5.1.1)

$$\tilde{W}_t(\omega)=\sum_{i\geqslant 1}\mathbb{1}_{T_i(\omega)=t}\mathbb{1}_{(\omega,T_i(\omega),Y_i(\omega))\in\tilde{A}}Z_i(\omega)\mathbb{1}_{|Z_i(\omega)|\leqslant 1}\,.$$

Also,

$$\sum_{s\leqslant t}\tilde{W}_s^2(\omega)=\sum_{i:\,T_i(\omega)\leqslant t}\mathbb{1}_{(\omega,T_i(\omega),Y_i(\omega))\in A}Z_i^2(\omega)\mathbb{1}_{|Z_i|(\omega)\leqslant 1}\,.$$

Then, we only need to check (5.1.2).

$$\begin{aligned}
\mathbb{E}\left[\left(\sum_{s\leqslant t}\tilde{W}_s^2(\cdot)\right)^{\frac{1}{2}}\right] &\leqslant \left(\mathbb{E}\left[\sum_{s\leqslant t}\tilde{W}_s^2(\cdot)\right]\right)^{\frac{1}{2}}\\
&\leqslant\left(\mathbb{E}\left[\sum_{i:T_i(\cdot)\leqslant t}\mathbb{1}_{(\cdot,T_i(\cdot),Y_i(\cdot))\in\tilde{A}}Z_i^2(\cdot)\mathbb{1}_{|Z_i|(\cdot)\leqslant 1}\right]\right)^{\frac{1}{2}}\\
&\leqslant\left(\mathbb{E}\left[\sum_{i:T_i(\cdot)\leqslant t}\mathbb{1}_{(T_i(\cdot),Y_i(\cdot))\in[0,k]\times[-k,k]^d}Z_i^2(\cdot)\mathbb{1}_{|Z_i|(\cdot)\leqslant 1}\right]\right)^{\frac{1}{2}}\\
&\leqslant\left(\int_{[0,k]\times[-k,k]^d}\left(\int_{|z|\leqslant 1}z^2\,\nu(\mathrm{d}z)\right)\mathrm{d}s\,\mathrm{d}y\right)^{\frac{1}{2}}<+\infty\,.
\end{aligned}$$

Therefore, the integral of W with respect to $\dot{L}^1$ can be defined as in (5.1.3). Then,

$$\dot{L}^1\left(\tilde{A}\right):=\int_{\mathbb{R}_+\times\mathbb{R}^d}\int_{|z|\leqslant 1}zW_{(s,y)}\tilde{J}(\mathrm{d}s,\mathrm{d}y,\mathrm{d}z)\,,$$

and $\dot{L}\left(\tilde{A}\right):=\dot{L}^1\left(\tilde{A}\right)+\dot{L}^2\left(\tilde{A}\right)$.

Then, by [12, Remark 1.4.4], this extension of $\dot{L}$ is a Lévy basis, and one can proceed with the construction of a stochastic integral of predictable processes using the Daniell mean as in [12, Chapter 1, Section 2]. We briefly give the construction of this stochastic integral with respect to $\dot{L}$. The space $\mathscr{S}_L$ of simple predictable random fields is the set of random fields S that can be written

$$S(\omega,t,x)=\sum_{i=1}^{n}a_i\mathbb{1}_{A_i},$$

where $n\in\mathbb{N}$, $a_i\in\mathbb{R}$ and A_i is a predictable set such that $A_i\subset\Omega\times[0,k_i]\times[-k_i,k_i]^d$ for some $k_i\in\mathbb{N}$. For such a simple predictable random field, we define

$$\int_{\mathbb{R}_+\times\mathbb{R}^d}S(s,y)L(\mathrm{d}s,\mathrm{d}y):=\sum_{i=1}^{n}a_i\dot{L}(A_i).$$

Then, for a random field $H:\Omega\times\mathbb{R}_+\times\mathbb{R}^d\to\mathbb{R}$ and $p\geqslant 0$, the Daniell mean $\|H\|_{p,L}^{D}$ is defined by

$$\|H\|_{p,L}^{D}:=\inf_{K\in\mathscr{S}_L^{\uparrow},|H|\leqslant K}\ \sup_{S\in\mathscr{S}_L,|S|\leqslant K}\left\|\int_{\mathbb{R}_+\times\mathbb{R}^d}S(s,y)L(\mathrm{d}s,\mathrm{d}y)\right\|_{L^p(\Omega)},$$

where $\mathscr{S}_L^{\uparrow}$ is the set of positive random fields $K:\Omega\times\mathbb{R}_+\times\mathbb{R}^d\to\mathbb{R}$ which are the pointwise supremum of simple predictable random fields. Then, a random fields $H:\Omega\times\mathbb{R}_+\times\mathbb{R}^d\to\mathbb{R}$ is L^p-integrable with respect to $\dot{L}$ if there is a sequence $(S_n)_{n\geqslant 1}$ of simple predictable random fields such that $\|S_n-H\|_{p,L}^{D}\to 0$ as $n\to+\infty$. Then, the stochastic integral of H with respect to $\dot{L}$ is defined by

$$\int_{\mathbb{R}_+\times\mathbb{R}^d}H(s,y)L(\mathrm{d}s,\mathrm{d}y):=\lim_{n\to+\infty}\int_{\mathbb{R}_+\times\mathbb{R}^d}S_n(s,y)L(\mathrm{d}s,\mathrm{d}y),$$

where the limit is in $L^p(\Omega)$ and does not depend on the choice of the sequence S_n. Interestingly, in [16], the authors obtained an explicit characterization of the random fields that are integrable with respect to a Lévy basis.

Theorem 5.1.1 (Theorem 4.1 and Remark 4.4 in [16]). *Let $\dot{L}$ be a Lévy basis with characteristic triplet (b,ρ,ν). A predictable random field $H:(\omega,t,x)\mapsto H_{t,x}(\omega)$ is L^0-integrable with respect to $\dot{L}$ if and only if ω-almost surely, $H(\omega)$ satisfies the conditions (i)-(iii) of Proposition 2.3.7.*

5.2 The SHE driven by heavy-tailed noise: equation on $[0,T]\times[0,\pi]$

Fix T>0. We consider the stochastic heat equation driven by a Lévy white noise in $[0,T]\times[0,\pi]$ with Dirichlet boundary conditions:

$$\begin{cases}\frac{\partial u}{\partial t}(t,x)=\frac{\partial^2 u}{\partial x^2}(t,x)+\sigma(u(t,x))\dot{L}(t,x), & (t,x)\in(0,T)\times(0,\pi),\\ u(t,0)=u(t,\pi)=0, & \text{for all } t\in[0,T],\\ u(0,x)=0, & \text{for all } x\in[0,\pi],\end{cases}\tag{5.2.1}$$

where $\sigma:\mathbb{R}\to\mathbb{R}$ is a Lipschitz function and L is a pure jump Lévy white noise. More precisely, we suppose that

$$\begin{aligned} L(\mathrm{d}t,\mathrm{d}x) &= b\,\mathrm{d}t\,\mathrm{d}x + \int_{|z|\leqslant 1} z\tilde{J}(\mathrm{d}t,\mathrm{d}x,\mathrm{d}z) + \int_{|z|>1} zJ(\mathrm{d}t,\mathrm{d}x,\mathrm{d}z) \\ &=: L^B(\mathrm{d}t,\mathrm{d}x) + L^M(\mathrm{d}t,\mathrm{d}x) + L^P(\mathrm{d}t,\mathrm{d}x), \end{aligned} \tag{5.2.2}$$

where $b\in\mathbb{R}$, J is a Poisson random measure on $[0,\infty)\times[0,\pi]\times\mathbb{R}$ with intensity $\mathrm{d}t\,\mathrm{d}x\,\nu(\mathrm{d}z)$, and $\tilde{J}$ is the compensated version of J. The measure ν is a Lévy measure, that is, $\nu(\{0\})=0$ and $\int_{\mathbb{R}}\left(z^2\wedge 1\right)\nu(\mathrm{d}z)<+\infty$.

The Green's function of the heat operator on the bounded domain $[0,T]\times[0,\pi]$ is given by

$$G(t;x,y) := \frac{2}{\pi}\sum_{k=1}^{+\infty}\sin(kx)\sin(ky)e^{-k^2t}\mathbb{1}_{t\geqslant 0}. \tag{5.2.3}$$

By definition, a mild solution to (5.2.1) is a predictable random field

$$u = (u(t,x),\,(t,x)\in[0,T]\times[0,\pi]),$$

such that for all $(t,x)\in[0,T]\times[0,\pi]$,

$$u(t,x) = \int_0^t\int_0^\pi G(t-s;x,y)\sigma(u(s,y))L(\mathrm{d}s,\mathrm{d}y). \tag{5.2.4}$$

Similarly to [14], we then define the random times

$$\tau_N = \inf\left\{t\geqslant 0 : J\left([0,t]\times[0,\pi]\times[-N,N]^c\right)\neq 0\right\}.$$

Also,

$$\mathbb{E}\left[J\left([0,t]\times[0,\pi]\times[-N,N]^c\right)\right] = \int_0^t\int_0^\pi\int_{\mathbb{R}}\mathbb{1}_{|z|>N}\,\mathrm{d}s\,\mathrm{d}y\,\nu(\mathrm{d}z)<+\infty. \tag{5.2.5}$$

Therefore, $(\tau_N)_{N\geqslant 1}$ is an increasing sequence of stopping times such that $\tau_N>0$ and $\tau_N\to+\infty$ almost surely as $N\to+\infty$. In fact, we have that for almost all $\omega\in\Omega$, there exists an integer $R(\omega)$ such that for any $N\geqslant R(\omega)$, $\tau_N(\omega)>T$. We use these stopping times to truncate the noise, and we can define $L_N := L\mathbb{1}_{t\leqslant\tau_N}$. Then,

$$L_N(\mathrm{d}t,\mathrm{d}x) = b_N\,\mathrm{d}t\,\mathrm{d}x + \int_{|z|\leqslant N} z\tilde{J}(\mathrm{d}t,\mathrm{d}x,\mathrm{d}z), \tag{5.2.6}$$

where $b_N := b-\int_{1<|z|\leqslant N} z\nu(\mathrm{d}z)$.

Proposition 5.2.1. *Let $\sigma:\mathbb{R}\to\mathbb{R}$ be a Lipschitz function and let L be a pure jump Lévy white noise as in* (5.2.2)*. Then there exists, up to modifications, a unique predictable random field $u=(u(t,x),\,(t,x)\in[0,T]\times[0,\pi])$ such that for any $p<3$ and $N\in\mathbb{N}$,*

$$\sup_{(t,x)\in[0,T]\times[0,\pi]}\mathbb{E}\left[|u(t,x)|^p\mathbb{1}_{t\leqslant\tau_N}\right]<+\infty,$$

and for any $(t,x) \in [0,T]\times[0,\pi]$,

$$u(t,x) = \int_0^t \int_0^\pi G(t-s;x,y)\sigma(u(s,y))L(\mathrm{d}s,\mathrm{d}y) \qquad a.s. \tag{5.2.7}$$

Remark 5.2.2. *Let* $u_N(t,x) = u(t,x)\mathbb{1}_{t\leqslant\tau_N}$. *Then,* u_N *is clearly a mild solution to the truncated equation:*

$$u_N(t,x) = \int_0^t \int_0^\pi G(t-s;x,y)\sigma\left(u_N(s,y)\right)L_N(\mathrm{d}s,\mathrm{d}y) \qquad a.s. \tag{5.2.8}$$

Proof of Proposition 5.2.1. By [6, (B.5)], we know that $G(t;x,y) \leqslant C\rho_H(t,x-y)$ for any $(t,x,y) \in [0,T]\times[0,\pi]^2$, where ρ_H is the Gaussian density function $\rho_H(t,x) = \frac{1}{\sqrt{4\pi t}}e^{-\frac{x^2}{4t}}$ as in (4.4.1) with $d=1$. Also, since $[0,\pi]$ is a bounded interval, $\nu([0,T]\times[0,\pi]\times(-1,1)^c) < +\infty$ and a.s., there is only a finite number of jumps larger than 1 in $[0,T]\times[0,\pi]$. Consequently, (1) to (4) of Assumption B of [13] are satisfied, and we can apply [13, Theorem 3.5] to obtain the existence of a unique predictable random field u satisfying (5.2.7) and

$$\sup_{(t,x)\in[0,T]\times[0,\pi]} \mathbb{E}\left[|u(t,x)|^2\mathbb{1}_{t\leqslant\tau_N}\right] < +\infty, \tag{5.2.9}$$

for any $N\in\mathbb{N}$. Since $\int_0^T\int_{\mathbb{R}}\rho_H^p(t,x)\,\mathrm{d}t\,\mathrm{d}x < +\infty$ for all $p<3$ and the Lévy measure ν satisfies $\int_{|z|\leqslant N}|z|^p\nu(\mathrm{d}z) < +\infty$ for any $p\in[2,3)$, we can use Theorem A.0.1 to further improve (5.2.9) to

$$\sup_{(t,x)\in[0,T]\times[0,\pi]} \mathbb{E}\left[|u(t,x)|^p\mathbb{1}_{t\leqslant\tau_N}\right] < +\infty$$

for every $p<3$. More precisely, the only step in the proof of [13, Theorem 3.5] where $p\leqslant 2$ is assumed is the moment estimate given in [13, Lemma 6.1(2)]. We now elaborate how this estimate can be extended to exponents $2<p<3$. Because of the stopping time τ_N, it suffices to consider the noise L_N introduced in (5.2.6). Then, for any predictable processes ϕ_1 and ϕ_2,

$$\begin{aligned}
&\mathbb{E}\left[\left|\int_0^t\int_0^\pi G(t-s;x,y)\left(\sigma(\phi_1(s,y))-\sigma(\phi_2(s,y))\right)L_N(\mathrm{d}s,\mathrm{d}y)\right|^p\right] \\
&\quad\leqslant C\mathbb{E}\left[\left|\int_0^t\int_0^\pi G(t-s;x,y)\left(\sigma(\phi_1(s,y))-\sigma(\phi_2(s,y))\right)\mathrm{d}s\,\mathrm{d}y\right|^p\right] \\
&\qquad + C\mathbb{E}\left[\left|\int_0^t\int_0^\pi\int_{|z|\leqslant N} G(t-s;x,y)\left(\sigma(\phi_1(s,y))-\sigma(\phi_2(s,y))\right)z\tilde{J}(\mathrm{d}s,\mathrm{d}y,\mathrm{d}z)\right|^p\right].
\end{aligned}$$

We consider the measure $G(t-s;x,y)\,\mathrm{d}s\,\mathrm{d}y$ in the first integral, and we use Hölder's inequality and Theorem A.0.1 *(iii)* for the second to get the upper bound

$$\begin{aligned}
&C\left(\int_0^t\int_0^\pi |G(t-s;x,y)|\,\mathrm{d}s\,\mathrm{d}y\right)^{p-1}\int_0^t\int_0^\pi |G(t-s;x,y)|\mathbb{E}[|\phi_1(s,y)-\phi_2(s,y)|^p]\,\mathrm{d}s\,\mathrm{d}y \\
&\quad + C\mathbb{E}\left[\left(\int_0^t\int_0^\pi\int_{|z|\leqslant N}|G(t-s;x,y)|^2|\phi_1(s,y)-\phi_2(s,y)|^2|z|^2\,\mathrm{d}s\,\mathrm{d}y\,\nu(\mathrm{d}z)\right)^{\frac{p}{2}}\right]
\end{aligned}$$

$$+C\mathbb{E}\left[\int_0^t\int_0^\pi\int_{|z|\leqslant N}|G(t-s;x,y)|^p|\phi_1(s,y)-\phi_2(s,y)|^p|z|^p\,\mathrm{d}s\,\mathrm{d}y\,\nu(\mathrm{d}z)\right].$$

Similarly, we consider the measure $|G(t-s;x,y)|^2\,\mathrm{d}s\,\mathrm{d}y$ to apply again Hölder's inequality to the second term of the sum above, and we get the upper bound

$$\begin{aligned}
&C\left(\int_0^t\int_0^\pi|G(t-s;x,y)|\,\mathrm{d}s\,\mathrm{d}y\right)^{p-1}\int_0^t\int_0^\pi|G(t-s;x,y)|\mathbb{E}[|\phi_1(s,y)-\phi_2(s,y)|^p]\,\mathrm{d}s\,\mathrm{d}y\\
&\quad+C\left(\int_0^t\int_0^\pi|G(t-s;x,y)|^2\,\mathrm{d}s\,\mathrm{d}y\right)^{\frac{p}{2}-1}\int_0^t\int_0^\pi|G(t-s;x,y)|^2\mathbb{E}[|\phi_1(s,y)-\phi_2(s,y)|^p]\,\mathrm{d}s\,\mathrm{d}y\\
&\quad+C\int_0^t\int_0^\pi|G(t-s;x,y)|^p\mathbb{E}[|\phi_1(s,y)-\phi_2(s,y)|^p]\,\mathrm{d}s\,\mathrm{d}y,
\end{aligned}$$

where the constants above do not depend on ϕ_1 and ϕ_2, and we used the fact that

$$\int_{|z|\leqslant N}|z|^p\nu(\mathrm{d}z)<+\infty.$$

Then, since $G(t;x,y)\leqslant C\rho_H(t,x-y)$ for any $(t,x,y)\in[0,T]\times[0,\pi]^2$, and $\int_0^T\int_{\mathbb{R}}\rho_H^p(t,x)\,\mathrm{d}t\,\mathrm{d}x<+\infty$ for all $p<3$ and $T\geqslant 0$, this upper bound becomes

$$\begin{aligned}
&C\left(\int_0^t\int_0^\pi\rho_H(t-s,x-y)\,\mathrm{d}s\,\mathrm{d}y\right)^{p-1}\int_0^t\int_0^\pi|G(t-s;x,y)|\mathbb{E}[|\phi_1(s,y)-\phi_2(s,y)|^p]\,\mathrm{d}s\,\mathrm{d}y\\
&\quad+C\left(\int_0^t\int_0^\pi\rho_H^2(t-s,x-y)\,\mathrm{d}s\,\mathrm{d}y\right)^{\frac{p}{2}-1}\int_0^t\int_0^\pi|G(t-s;x,y)|^2\mathbb{E}[|\phi_1(s,y)-\phi_2(s,y)|^p]\,\mathrm{d}s\,\mathrm{d}y\\
&\quad+C\int_0^t\int_0^\pi|G(t-s;x,y)|^p\mathbb{E}[|\phi_1(s,y)-\phi_2(s,y)|^p]\,\mathrm{d}s\,\mathrm{d}y.
\end{aligned}$$

Since $|x|^2\leqslant|x|+|x|^p$ for all $p\geqslant 2$, we finally get the upper bound

$$\begin{aligned}
&\mathbb{E}\left[\left|\int_0^t\int_0^\pi G(t-s;x,y)\left(\sigma(\phi_1(s,y))-\sigma(\phi_2(s,y))\right)L_N(\mathrm{d}s,\mathrm{d}y)\right|^p\right]\\
&\qquad\leqslant C\int_0^t\int_0^\pi(|G(t-s;x,y)|+|G(t-s;x,y)|^p)\mathbb{E}[|\phi_1(s,y)-\phi_2(s,y)|^p]\,\mathrm{d}s\,\mathrm{d}y,
\end{aligned}\tag{5.2.10}$$

which is exactly the needed extension of [13, Lemma 6.1(2)] to complete the proof.

Our setting is much less general than the one of [13, Theorem 3.5], and the proof of this result simplifies considerably. We provide here a version of this simplified proof for the convenience of the reader. We will use a classical Picard iteration scheme to prove the existence of a solution to the truncated equation (5.2.8) and to prove the moment bound (5.2.9). Let $u_N^0:=0$, and for any $n\geqslant 0$,

$$u_N^{n+1}(t,x):=\int_0^t\int_0^\pi G(t-s;x,y)\sigma\left(u_N^n(s,y)\right)L_N(\mathrm{d}s,\mathrm{d}y).$$

Then, we define $Y_N^n := u_N^{n+1} - u_N^n$, and by (5.2.10), we have

$$\| Y_N^n(t,x) \|_{L^p(\Omega)}^p \leqslant C \int_0^t \int_0^\pi (|G(t-s;x,y)| + |G(t-s;x,y)|^p) \| Y_N^{n-1}(s,y) \|_{L^p(\Omega)}^p \, \mathrm{d}s \, \mathrm{d}y.$$

Therefore, we can apply [13, Lemma 6.4, (1)] to the sequence $v_n(t,x) := \| Y_N^n(t,x) \|_{L^p(\Omega)}$, and we get that

$$\sum_{n=1}^{+\infty} \sup_{(t,x)\in[0,T]\times[0,\pi]} \| Y_N^n(t,x) \|_{L^p(\Omega)} < +\infty.$$

Therefore, u_N^n converges to some limit u_N in $L^p(\Omega)$, uniformly in space and time. Applying (5.2.10) to $\phi_1 = u_N$ and $\phi_2 = u_N^{n-1}$, we get that

$$\sup_{(t,x)\in[0,T]\times[0,\pi]} \left\| u_N^n(t,x) - \int_0^t \int_0^\pi G(t-s;x,y)\sigma(u_N(s,y)) L_N(\mathrm{d}s,\mathrm{d}y) \right\|_{L^p(\Omega)} \to 0, \qquad \text{as } n \to +\infty,$$

which implies that

$$u_N(t,x) = \int_0^t \int_0^\pi G(t-s;x,y)\sigma(u_N(s,y)) L_N(\mathrm{d}s,\mathrm{d}y), \qquad \text{a.s.}$$

Therefore, u_N is a mild solution to (5.2.8), and

$$\sup_{(t,x)\in[0,T]\times[0,\pi]} \mathbb{E}\left[|u_N(t,x)|^p \right] < +\infty \qquad \text{for all } p < 3.$$

The uniqueness statement is an application of (5.2.10) and [13, Lemma 6.4, (3)]. Then, as in the proof of [13, Theorem 3.5], $u := u_1 \mathbb{1}_{[\![0,\tau_1]\!]} + \sum_{N=2}^{+\infty} u_N \mathbb{1}_{]\!]\tau_{N-1},\tau_N]\!]}$ is a mild solution to (5.2.7). □

The properties that we consider in the following concern sample path regularity properties of the mild solution of the stochastic heat equation, and, by stationary convergence of u_N (defined on $[0,T]\times[0,\pi]$ by (5.2.8)) to u, these properties are identical to those of u_N in the previous proposition for N sufficiently large. The value of the parameter N has no importance in our study, so we can suppose that $N = 1$ and drop the dependency in N. Therefore, in the following, we will always consider the solution to the integral equation

$$\begin{aligned} u(t,x) = b \int_0^t \int_0^\pi & G(t-s;x,y)\sigma(u(s,y)) \, \mathrm{d}s \, \mathrm{d}y \\ & + \int_0^t \int_0^\pi G(t-s;x,y)\sigma(u(s,y)) L^M(\mathrm{d}s,\mathrm{d}y), \end{aligned} \tag{5.2.11}$$

where L^M is the martingale part of the noise as defined in (5.2.2), and the solution to (5.2.4) will have the same sample path regularity properties as the solution of (5.2.11). Also, for any $p < 3$, the solution to (5.2.11) has uniformly bounded moments of order p.

5.2.1 The fractional Sobolev spaces $H_r([0,\pi])$

For any function $f \in L^2([0,\pi])$, we can define its Fourier sine coefficients

$$a_n(f) = \frac{2}{\pi}\int_0^\pi f(x)\sin(nx)\,\mathrm{d}x\,, \qquad n\in\mathbb{N}. \tag{5.2.12}$$

Then, by Parseval's identity,

$$\frac{2}{\pi}\|f\|_{L^2}^2 = \sum_{n\geqslant 1} a_n(f)^2\,.$$

For any $r \geqslant 0$, we define $H_r([0,\pi])$ as the subspace of $L^2([0,\pi])$ such that

$$\|f\|_{H_r}^2 := \sum_{n\geqslant 1}\left(1+n^2\right)^r a_n(f)^2 < +\infty.$$

It is a Hilbert space for the inner product given by

$$\langle f,g\rangle_{H_r} := \sum_{n\geqslant 1}\left(1+n^2\right)^r a_n(f)a_n(g)\,.$$

For $r>0$, we define $H_{-r}([0,\pi])$ as the dual space of $H_r([0,\pi])$, that is, the space of continuous linear functionals on $H_r([0,\pi])$. Then, for any $r>0$, $H_{-r}([0,\pi])$ is isomorphic to the space of sequences $b=(b_n)_{n\geqslant 1}$ such that

$$\|b\|_{H_{-r}}^2 := \sum_{n\geqslant 1}\left(1+n^2\right)^{-r} b_n^2 < +\infty.$$

More precisely, for $r>0$ and $f\in H_{-r}([0,\pi])$, the coefficients b_n are given by $b_n = f(\sin(n\cdot))$. Then, $\|f\|_{H_{-r}} = \|b\|_{H_{-r}}$ and the duality between $H_{-r}([0,\pi])$ and $H_r([0,\pi])$ is given by

$$\langle b,g\rangle = \sum_{n\geqslant 1} b_n a_n(g) \leqslant \|b\|_{H_{-r}}\|g\|_{H_r}\,.$$

For example, it is easy to check that $\delta_x \in H_r([0,\pi])$ for any $x\in(0,\pi)$ and $r<-\frac{1}{2}$. Indeed, $\delta_x(\sin(n\cdot)) = \sin(nx)$, and for any $r<-\frac{1}{2}$,

$$\|\delta_x\|_{H_r}^2 = \sum_{n\geqslant 1}\left(1+n^2\right)^r\sin^2(nx) \leqslant \sum_{n\geqslant 1}\left(1+n^2\right)^r < +\infty.$$

5.2.2 Existence of a *càdlàg* solution in $H_r([0,\pi])$

In order to motivate why we consider the fractional Sobolev space $H_r([0,\pi])$, we start with a special case. Suppose that $b=0$ and that ν is a symmetric measure with $\nu(\mathbb{R})<+\infty$. Then we can rewrite $L=\sum_{i\geqslant 1} Z_i\delta_{(T_i,X_i)}$, and

$$u(t,x) = \sum_{i\geqslant 1} G(t-T_i;x,X_i)Z_i\sigma\left(u(T_i,X_i)\right)\,,$$

where the sum is finite. In this case, it suffices to check whether for fixed $i \geqslant 1$, $t \mapsto G(t-T_i;\cdot,X_i)$ is càdlàg in $H_r([0,\pi])$. Using the series representation

$$G(t-T_i;x,X_i) = \frac{2}{\pi}\sum_{k=1}^{+\infty}\sin(kx)\sin(kX_i)e^{-k^2(t-T_i)}\mathbb{1}_{t\geqslant T_i},$$

we immediately see that the function $x \mapsto G(t-T_i;x,X_i)$ belongs to $H_r([0,\pi])$ if and only if

$$\sum_{k=1}^{+\infty}(1+k^2)^r\sin(kX_i)^2e^{-2k^2(t-T_i)}\mathbb{1}_{t\geqslant T_i} < +\infty.$$

This is the case for any $r\in\mathbb{R}$ if $t \neq T_i$. However, for $t = T_i$, we have to restrict to $r < -\frac{1}{2}$. For the càdlàg property, the only point where a problem might appear is at $t = T_i$. At this point, the existence of a left limit is obvious since $G(t-T_i;\cdot,X_i) = 0$ for any $t < T_i$. For right-continuity, we use the fact that $(1-e^{-k^2h})^2 \leqslant k^{2\varepsilon}h^{\varepsilon}$ for any $0<\varepsilon<-\frac{1}{2}-r$, so

$$\begin{aligned}\|G(h;\cdot,X_i)-G(0;\cdot,X_i)\|_{H_r}^2 &= \sum_{k=1}^{+\infty}(1+k^2)^r\sin(kX_i)^2\left(1-e^{-k^2h}\right)^2\\ &\leqslant \sum_{k=1}^{+\infty}(1+k^2)^rk^{2\varepsilon}h^{\varepsilon} \leqslant Ch^{\varepsilon}\to 0 \qquad \text{as } h\to 0.\end{aligned}$$

Therefore, $t\mapsto u(t,\cdot)$ is càdlàg in $H_r([0,\pi])$. For the general case, we first treat the drift term in the following proposition.

Lemma 5.2.3. *Assume that L is pure jump Lévy noise as in* (5.2.2) *and u be the unique solution to* (5.2.11)*. Then,*

$$F(t,x) = \int_0^t\int_0^\pi G(t-s;x,y)\sigma(u(s,y))\,\mathrm{d}s\,\mathrm{d}y$$

is jointly continuous in $(t,x)\in[0,T]\times[0,\pi]$. In particular, for every $r\leqslant 0$, the process

$$t\mapsto F(t,\cdot),$$

is continuous in $H_r([0,\pi])$.

Proof. By the series representation of the Green's function (5.2.3),

$$F(t,x) = \frac{2}{\pi}\sum_{k=1}^{+\infty}\sin(kx)\int_0^t\int_0^\pi e^{-k^2(t-s)}\sin(ky)\sigma(u(s,y))\,\mathrm{d}s\,\mathrm{d}y.$$

We see that each term of the sum of F is jointly continuous in (t,x). Hence, to give a direct proof of the continuity of F (which follows from abstract regularity results on the solution of the heat equation), it suffices to show the uniform convergence of the series. Using Hölder's inequality and the fact that u has uniformly bounded moments of any order $p<3$, we obtain this from

$$
\begin{aligned}
&\mathbb{E}\left[\sum_{k=1}^{+\infty}\sup_{(t,x)\in[0,T]\times[0,\pi]}\left|\sin(kx)\int_0^t\int_0^\pi e^{-k^2(t-s)}\sin(ky)\sigma(u(s,y))\,\mathrm{d}s\,\mathrm{d}y\right|\right]\\
&\quad\leqslant C\sum_{k=1}^{+\infty}\mathbb{E}\left[\sup_{(t,x)\in[0,T]\times[0,\pi]}\left(\int_0^t e^{-\frac{5}{3}k^2(t-s)}\,\mathrm{d}s\right)^{3/5}\left(\int_0^t\int_0^\pi|\sigma(u(s,y))|^{\frac{5}{2}}\,\mathrm{d}s\,\mathrm{d}y\right)^{2/5}\right]\\
&\quad\leqslant C\sum_{k=1}^{+\infty}\left(\int_0^T e^{-\frac{5}{3}k^2 s}\,\mathrm{d}s\right)^{3/5}\left(\int_0^T\int_0^\pi\mathbb{E}[|\sigma(u(s,y))|^{\frac{5}{2}}]\,\mathrm{d}s\,\mathrm{d}y\right)^{2/5}\\
&\quad\leqslant C\sum_{k=1}^{+\infty}\frac{1}{k^{\frac{6}{5}}}<+\infty.
\end{aligned}
$$

Then, to prove the continuity of $t\mapsto F(t,\cdot)$ in $H_r([0,\pi])$, it suffices to show the continuity in $L^2([0,\pi])$. Indeed, $L^2([0,\pi])\subset H_r([0,\pi])$, and the embedding is continuous. The continuity in $L^2([0,\pi])$ follows from the fact that since $[0,T]\times[0,\pi]$ is compact, F is in fact uniformly continuous on this domain. □

Case where σ is bounded

We are assuming in this section that σ is bounded. We consider the solution u to (5.2.11) with $b=0$. In order to study the fractional Sobolev regularity of $t\mapsto u(t,\cdot)$, we need to calculate the sine Fourier coefficients $a_k(u(t,\cdot))$ defined in (5.2.12). To lighten the notations, in what follows, we will denote these coefficients $a_k(t)$. Then, by definition, for $k\geqslant 1$,

$$
\begin{aligned}
a_k(t) &:= \frac{2}{\pi}\int_0^\pi u(t,x)\sin(kx)\,\mathrm{d}x\\
&= \frac{2}{\pi}\int_0^\pi\left(\int_0^t\int_0^\pi\int_{|z|\leqslant 1}\sin(kx)zG(t-s;x,y)\sigma\left(u(s,y)\right)\tilde{J}(\mathrm{d}s,\mathrm{d}y,\mathrm{d}z)\right)\mathrm{d}x.
\end{aligned}
$$

We want to apply Theorem A.0.2 to be able to exchange the stochastic integral and the Lebesgue integral. The condition we need to check is the following:

$$
\int_0^\pi\mathrm{d}x\int_0^t\mathrm{d}s\int_0^\pi\mathrm{d}y\int_{|z|\leqslant 1}\nu(\mathrm{d}z)\mathbb{E}\left[\left|zG(t-s;x,y)\sigma\left(u(s,y)\right)\sin(kx)\right|^2\right]<+\infty. \tag{5.2.13}
$$

Since σ is bounded, and since ν is a Lévy measure, this condition is equivalent to

$$
\int_0^\pi\mathrm{d}x\int_0^t\mathrm{d}s\int_0^\pi\mathrm{d}y\left|G(s;x,y)\right|^2<+\infty.
$$

Using again [6, (B.5)], we know that $G(t;x,y)\leqslant C\rho_H(t,x-y)$ for any $(t,x,y)\in[0,T]\times[0,\pi]^2$, and since the heat kernel is in $L^p([0,T]\times\mathbb{R})$ for any $p<3$ (see (4.4.6) and above), we deduce that (5.2.13) holds. Therefore,

$$
a_k(t)=\int_0^t\int_0^\pi\int_{|z|\leqslant 1}z\sigma\left(u(s,y)\right)\frac{2}{\pi}\left(\int_0^\pi\sin(kx)G(t-s;x,y)\,\mathrm{d}x\right)\tilde{J}(\mathrm{d}s,\mathrm{d}y,\mathrm{d}z).
$$

From the series representation of G in (5.2.3) and the fact that this series representation converges uniformly in the x variable, we have that

$$\frac{2}{\pi}\int_0^\pi \sin(kx)G(t-s;x,y)\,\mathrm{d}x = \sin(ky)e^{-k^2(t-s)}\,.$$

Therefore,

$$a_k(t) = \frac{2}{\pi}e^{-k^2 t}\int_0^t\int_0^\pi \sigma\left(u(s,y)\right)\sin(ky)e^{k^2 s}L^M(\mathrm{d}s,\mathrm{d}y)\,. \tag{5.2.14}$$

Remark 5.2.4. *The calculation of the Fourier sine coefficients above is in fact valid in a more general case: if σ is no longer bounded, but has at most linear growth, then we know by Proposition 5.2.1 that u has uniformly bounded moments of order 2, and the same reasoning applies.*

In the following, we define

$$\begin{aligned} I_a^b(k) :&= \int_a^b\int_0^\pi \sin(ky)e^{k^2 s}\sigma\left(u(s,y)\right)L^M(\mathrm{d}s,\mathrm{d}y) \\ &= \int_a^b\int_0^\pi\int_{|z|\leqslant 1} z\sin(ky)e^{k^2 s}\sigma\left(u(s,y)\right)\left(J(\mathrm{d}s,\mathrm{d}y,\mathrm{d}z) - \mathrm{d}s\,\mathrm{d}y\,\nu(\mathrm{d}z)\right). \end{aligned} \tag{5.2.15}$$

By the Bichteler-Jacod inequalities for compensated Poisson random measures (see Theorem A.0.1 *(iii)*) and by the boundedness of σ, we can estimate the second and fourth moments of these stochastic integrals:

$$\begin{aligned} \mathbb{E}\left[I_a^b(k)^2\right] &\leqslant C\int_a^b\int_0^\pi \sin^2(ky)e^{2k^2 s}\mathbb{E}\left[\left|\sigma\left(u(s,y)\right)\right|^2\right]\mathrm{d}s\,\mathrm{d}y \\ &\leqslant \tilde{C}\int_a^b\int_0^\pi \sin^2(ky)e^{2k^2 s}\,\mathrm{d}s\,\mathrm{d}y\,, \end{aligned} \tag{5.2.16}$$

and

$$\begin{aligned} \mathbb{E}\left[I_a^b(k)^4\right] &\leqslant C\Bigg(\mathbb{E}\left[\left(\int_a^b\int_0^\pi \sin^2(ky)e^{2k^2 s}\left|\sigma\left(u(s,y)\right)\right|^2\mathrm{d}s\,\mathrm{d}y\right)^2\right] \\ &\qquad + \int_a^b\int_0^\pi \sin^4(ky)e^{4k^2 s}\mathbb{E}\left[\left|\sigma\left(u(s,y)\right)\right|^4\right]\mathrm{d}s\,\mathrm{d}y\Bigg) \\ &\leqslant \tilde{C}\left(\left(\int_a^b\int_0^\pi \sin^2(ky)e^{2k^2 s}\,\mathrm{d}s\,\mathrm{d}y\right)^2 + \int_a^b\int_0^\pi \sin^4(ky)e^{4k^2 s}\,\mathrm{d}s\,\mathrm{d}y\right), \end{aligned} \tag{5.2.17}$$

where the constant $\tilde{C}$ also accounts for $\int_{|z|\leqslant 1} z^2\nu(\mathrm{d}z)$ and $\int_{|z|\leqslant 1} z^4\nu(\mathrm{d}z)$. Since ν is a Lévy measure, these integrals are finite.

Also, for $0 \leqslant a < b \leqslant c < d \leqslant T$, since σ is bounded,

$$\begin{aligned}
\mathbb{E}\left[\left(I_a^b(k) I_c^d(j)\right)^2\right] &= \mathbb{E}\left[\left(\int_c^d \int_0^\pi I_a^b(k) \sin(jy) e^{j^2 s} \sigma\left(u(s,y)\right) L^M(\mathrm{d}s, \mathrm{d}y)\right)^2\right] \\
&\leqslant C \int_c^d \int_0^\pi \mathbb{E}\left[I_a^b(k)^2 \sigma\left(u(s,y)\right)^2\right] \sin^2(jy) e^{2j^2 s} \mathrm{d}s\,\mathrm{d}y \\
&\leqslant C \left(\int_a^b \int_0^\pi \sin^2(ky) e^{2k^2 s} \mathrm{d}s\,\mathrm{d}y\right)\left(\int_c^d \int_0^\pi \sin^2(jy) e^{2j^2 s} \mathrm{d}s\,\mathrm{d}y\right) \\
&\leqslant C \frac{e^{2k^2 b} - e^{2k^2 a}}{2k^2} \frac{e^{2j^2 d} - e^{2j^2 c}}{2j^2},
\end{aligned} \tag{5.2.18}$$

where we used the fact that $I_a^b(k)$ is $\mathscr{F}_c$-measurable, and (5.2.16) in the second inequality. In the case where σ is constant, this inequality can be obtained more simply using the independence of $I_a^b(k)$ and $I_c^d(j)$.

Proposition 5.2.5. *Let L be a pure jump Lévy white noise, and let σ be a bounded and Lipschitz function. Let u be the mild solution to the stochastic heat equation* (5.2.1)*. Then, for any $r < -\frac{1}{2}$, the stochastic process $(u(t,\cdot))_{t\geqslant 0}$ has a càdlàg version in $H_r([0,\pi])$.*

Proof. By the argument just before Section 5.2.1, we can suppose that u is the solution to (5.2.11). Furthermore, by Lemma 5.2.3, we may further assume that $L = L_1 = L^M$. Then, by (5.2.14), we have the Fourier sine coefficients of $t \mapsto u(t,\cdot)$

$$a_k(t) := \frac{2}{\pi} e^{-k^2 t} \int_0^t \int_0^\pi \sin(ky) e^{k^2 s} \sigma\left(u(s,y)\right) L^M(\mathrm{d}s, \mathrm{d}y). \tag{5.2.19}$$

We will use Theorem A.0.3 to show the existence of a *càdlàg* version of $t \mapsto u(t,\cdot)$. By Lemma A.0.4, u has a separable version that is jointly continuous in probability, and therefore $t \mapsto u(t,\cdot)$ has a version that is continuous in probability as a process with values in $L^2([0,\pi])$ (and therefore in $H_r([0,\pi])$ since $r < -\frac{1}{2}$). Then, it suffices to show that for any $t \in [0,T]$, $u(t,\cdot) \in H_r([0,\pi])$, and that for some $\delta > 0$,

$$\mathbb{E}\left[\|u(t+h,\cdot) - u(t,\cdot)\|_{H_r}^2 \|u(t-h,\cdot) - u(t,\cdot)\|_{H_r}^2\right] \leqslant C h^{1+\delta},$$

for any $h \in (0,1)$. We first need to check that

$$\sum_{k=1}^{+\infty} (1+k^2)^r a_k^2(t) < +\infty, \qquad \text{a.s.} \tag{5.2.20}$$

By (5.2.16),

$$\mathbb{E}\left[a_k^2(t)\right] \leqslant C \int_0^t \int_0^\pi \sin^2(ky) e^{-2k^2(t-s)} \mathrm{d}s\,\mathrm{d}y \leqslant CT,$$

where C is a constant independent of k. Since $r < -\frac{1}{2}$, we deduce (5.2.20). Then,

$$\|u(t+h,\cdot)-u(t,\cdot)\|_{H_r}^2 = \sum_{k=1}^{+\infty}(1+k^2)^r\left(a_k(t+h)-a_k(t)\right)^2,$$

and

$$\|u(t-h,\cdot)-u(t,\cdot)\|_{H_r}^2 = \sum_{j=1}^{+\infty}(1+j^2)^r\left(a_j(t-h)-a_j(t)\right)^2.$$

Using (5.2.15) and (5.2.19),

$$\begin{aligned}a_k(t+h)-a_k(t) &= -\frac{2}{\pi}e^{-k^2t}\left[(1-e^{-k^2h})I_0^t(k)-e^{-k^2h}I_t^{t+h}(k)\right]\\ a_j(t-h)-a_j(t) &= -\frac{2}{\pi}e^{-j^2(t-h)}\left[(e^{-j^2h}-1)I_0^{t-h}(j)+e^{-j^2h}I_{t-h}^t(j)\right].\end{aligned}$$

Therefore, using the classical inequality $(a+b)^2 \leqslant 2(a^2+b^2)$,

$$\begin{aligned}&\|u(t+h,\cdot)-u(t,\cdot)\|_{H_r}^2\|u(t-h,\cdot)-u(t,\cdot)\|_{H_r}^2\\ &\quad\leqslant \frac{16}{\pi^4}\sum_{k,j\geqslant 1}(1+k^2)^r(1+j^2)^r\left(|A_1(j,k)|+|A_2(j,k)|+|A_3(j,k)|+|A_4(j,k)|\right)^2\\ &\quad\leqslant C\sum_{k,j\geqslant 1}(1+k^2)^r(1+j^2)^r\left(A_1(j,k)^2+A_2(j,k)^2+A_3(j,k)^2+A_4(j,k)^2\right),\end{aligned} \tag{5.2.21}$$

for some constant C, where

$$\begin{aligned}A_1(j,k) &:= e^{-k^2t}e^{-j^2(t-h)}(1-e^{-k^2h})(e^{-j^2h}-1)I_0^t(k)I_0^{t-h}(j),\\ A_2(j,k) &:= e^{-k^2t}e^{-j^2(t-h)}(1-e^{-k^2h})e^{-j^2h}I_0^t(k)I_{t-h}^t(j),\\ A_3(j,k) &:= e^{-k^2t}e^{-j^2(t-h)}e^{-k^2h}(e^{-j^2h}-1)I_t^{t+h}(k)I_0^{t-h}(j),\\ A_4(j,k) &:= e^{-k^2t}e^{-j^2(t-h)}e^{-k^2h}e^{-j^2h}I_t^{t+h}(k)I_{t-h}^t(j).\end{aligned}$$

We treat each of the four terms separately. Intuitively, the last two terms will be the easiest to deal with. Indeed, we can use (5.2.18), which gives an order h from $\mathbb{E}\left[\left(I_t^{t+h}(k)I_0^{t-h}(j)\right)^2\right]$ and an order h from $(e^{-j^2h}-1)$ for the term A_3. Also, (5.2.18) gives an order h^2 from the term A_4. This suggests an order h^2 for these two terms. We will see in the following that this is indeed the case.

$\underline{A_1(j,k):}$

$$\begin{aligned}\mathbb{E}\left[A_1(j,k)^2\right] &= e^{-2k^2t}e^{-2j^2(t-h)}(1-e^{-k^2h})^2(e^{-j^2h}-1)^2\mathbb{E}\left[\left(I_0^t(k)I_0^{t-h}(j)\right)^2\right]\\ &\leqslant Ce^{-2k^2t}e^{-2j^2(t-h)}(1-e^{-k^2h})^2(e^{-j^2h}-1)^2\\ &\qquad\times\mathbb{E}\left[\left(I_0^{t-h}(k)I_0^{t-h}(j)\right)^2+\left(I_{t-h}^t(k)I_0^{t-h}(j)\right)^2\right]\\ &=: \tilde{A}_1(j,k)+\tilde{A}_2(j,k).\end{aligned}$$

By (5.2.18), we can write

$$\begin{aligned}\mathbb{E}\left[\left(I_{t-h}^{t}(k)I_0^{t-h}(j)\right)^2\right] &\leqslant C\,\frac{e^{2j^2(t-h)}-1}{2j^2}\,\frac{e^{2k^2t}-e^{2k^2(t-h)}}{2k^2}\\ &\leqslant e^{2j^2(t-h)}\,\frac{1-e^{-2j^2(t-h)}}{2j^2}\,Ce^{2k^2t}\,\frac{1-e^{-2k^2h}}{2k^2}\\ &\leqslant C\,\frac{e^{2k^2t}e^{2j^2(t-h)}}{2j^2}\,h,\end{aligned}$$

where we used $1-e^{-2k^2h}\leqslant 2k^2h$ and $1-e^{-2j^2(t-h)}\leqslant 1$ in the last inequality. Finally, since $(1-e^{-k^2h})^2\leqslant 1$ and $(1-e^{-j^2h})^2\leqslant j^2h$, we deduce that

$$\tilde{A}_2(j,k)\leqslant Ch^2. \tag{5.2.22}$$

Also, by the Cauchy-Schwarz inequality,

$$\mathbb{E}\left[\left(I_0^{t-h}(k)I_0^{t-h}(j)\right)^2\right]\leqslant \mathbb{E}\left[I_0^{t-h}(k)^4\right]^{\frac12}\mathbb{E}\left[I_0^{t-h}(j)^4\right]^{\frac12}. \tag{5.2.23}$$

By (5.2.17) and subadditivity of the square root,

$$\begin{aligned}\mathbb{E}\left[I_0^{t-h}(k)^4\right]^{\frac12} &\leqslant C\left(\left(\int_0^{t-h}\int_0^{\pi}\sin^2(ky)e^{2k^2s}\,\mathrm{d}s\,\mathrm{d}y\right)^2+\int_0^{t-h}\int_0^{\pi}\sin^4(ky)e^{4k^2s}\,\mathrm{d}s\,\mathrm{d}y\right)^{\frac12}\\ &\leqslant C\left(\frac{e^{2k^2(t-h)}-1}{2k^2}+\left(\frac{e^{4k^2(t-h)}-1}{4k^2}\right)^{\frac12}\right)\\ &\leqslant Ce^{2k^2t}\left(\frac{e^{-2k^2h}-e^{-2k^2t}}{2k^2}+\left(\frac{e^{-4k^2h}-e^{-4k^2t}}{4k^2}\right)^{\frac12}\right)\\ &\leqslant Ce^{2k^2t}\left(\frac{1}{2k^2}+\frac{1}{2k}\right).\end{aligned} \tag{5.2.24}$$

Let $0<\delta<\frac32$, to be chosen later. Then, multiplying each term by $(1-e^{-k^2h})^2$ and using $(1-e^{-k^2h})^2\leqslant k^2h$ for the first term of the sum, and

$$(1-e^{-k^2h})^2=(1-e^{-k^2h})^{\frac12+\delta}(1-e^{-k^2h})^{\frac32-\delta}\leqslant k^{1+2\delta}h^{\frac12+\delta}$$

for the second term of the sum, we get

$$(1-e^{-k^2h})^2\mathbb{E}\left[I_0^{t-h}(k)^4\right]^{\frac12}\leqslant Ce^{2k^2t}\left(h+k^{2\delta}h^{\frac12+\delta}\right). \tag{5.2.25}$$

A similar calculation yields

$$(1-e^{-j^2h})^2\mathbb{E}\left[I_0^{t-h}(j)^4\right]^{\frac12}\leqslant Ce^{2j^2(t-h)}\left(h+j^{2\delta}h^{\frac12+\delta}\right). \tag{5.2.26}$$

Then, we combine (5.2.23), (5.2.25) and (5.2.26) to obtain

$$\tilde{A}_1(j,k) \leqslant C\left(h^2 + j^{2\delta}k^{2\delta}h^{1+2\delta}\right). \tag{5.2.27}$$

Therefore, (5.2.22) and (5.2.27) give

$$\mathbb{E}\left[A_1(j,k)^2\right] \leqslant C\left(h^2 + j^{2\delta}k^{2\delta}h^{1+2\delta}\right).$$

$\underline{A_2(j,k)}$: We treat this term in a similar way to $A_1(j,k)$:

$$\begin{aligned}
\mathbb{E}\left[A_2(j,k)^2\right] &= e^{-2k^2t}e^{-2j^2(t-h)}(1-e^{-k^2h})^2e^{-2j^2h}\mathbb{E}\left[\left(I_0^t(k)I_{t-h}^t(j)\right)^2\right] \\
&\leqslant Ce^{-2k^2t}e^{-2j^2(t-h)}(1-e^{-k^2h})^2e^{-2j^2h}\mathbb{E}\left[\left(I_0^{t-h}(k)I_{t-h}^t(j)\right)^2 + \left(I_{t-h}^t(k)I_{t-h}^t(j)\right)^2\right] \\
&=: B_1(j,k) + B_2(j,k).
\end{aligned}$$

In the same way as for the term $\tilde{A}(j,k)$, we get

$$B_1(j,k) \leqslant Ch^2. \tag{5.2.28}$$

We use the Cauchy-Schwarz inequality to deal with the term $B_2(j,k)$:

$$B_2(j,k) \leqslant Ce^{-2k^2t}e^{-2j^2(t-h)}(1-e^{-k^2h})^2e^{-2j^2h}\mathbb{E}\left[I_{t-h}^t(k)^4\right]^{\frac{1}{2}}\mathbb{E}\left[I_{t-h}^t(j)^4\right]^{\frac{1}{2}}.$$

We can deal with the second expectation as in (5.2.24), and we get

$$\begin{aligned}
\mathbb{E}\left[I_{t-h}^t(j)^4\right]^{\frac{1}{2}} &\leqslant Ce^{2j^2t}\left(\frac{1-e^{-2j^2h}}{2j^2} + \left(\frac{1-e^{-4j^2h}}{4j^2}\right)^{\frac{1}{2}}\right) \\
&\leqslant Ce^{2j^2t}\left(h+\sqrt{h}\right).
\end{aligned}$$

Similarly,

$$\mathbb{E}\left[I_{t-h}^t(k)^4\right]^{\frac{1}{2}} \leqslant Ce^{2k^2t}\left(h+\sqrt{h}\right).$$

Also, for $0<\delta<1$, since $(1-e^{-k^2h})^2 \leqslant k^{2\delta}h^{\delta}$, we get

$$B_2(j,k) \leqslant Ck^{2\delta}h^{\delta}\left(h+\sqrt{h}\right)^2 \leqslant Ck^{2\delta}h^{1+\delta}. \tag{5.2.29}$$

By (5.2.28) and (5.2.29),

$$\mathbb{E}\left[A_2(j,k)^2\right] \leqslant Ck^{2\delta}h^{1+\delta}.$$

$\underline{A_3(j,k)}$: By (5.2.18),

$$\begin{aligned}
\mathbb{E}\left[A_3(j,k)^2\right] &= e^{-2k^2 t} e^{-2j^2(t-h)} e^{-2k^2 h} (e^{-j^2 h}-1)^2 \mathbb{E}\left[\left(I_t^{t+h}(k) I_0^{t-h}(j)\right)^2\right] \\
&\leqslant C e^{-2k^2 t} e^{-2j^2(t-h)} e^{-2k^2 h} (e^{-j^2 h}-1)^2 \, \frac{e^{2j^2(t-h)}-1}{2j^2} \, \frac{e^{2k^2(t+h)}-e^{2k^2 t}}{2k^2} \\
&\leqslant C\,(e^{-j^2 h}-1)^2 \, \frac{1-e^{-2j^2(t-h)}}{2j^2} \, \frac{1-e^{-2k^2 h}}{2k^2} \\
&\leqslant C\, \frac{(e^{-j^2 h}-1)^2}{2j^2} \, \frac{1-e^{-2k^2 h}}{2k^2}\,.
\end{aligned}$$

Then, since $(1-e^{-j^2 h})^2 \leqslant j^2 h$ and $1-e^{-2k^2 h} \leqslant 2k^2 h$, we get

$$\mathbb{E}\left[A_3(j,k)^2\right] \leqslant C h^2\,.$$

$\underline{A_4(j,k)}$: Again, by (5.2.18),

$$\begin{aligned}
\mathbb{E}\left[A_4(j,k)^2\right] &= e^{-2k^2 t} e^{-2j^2(t-h)} e^{-2k^2 h} e^{-2j^2 h} \mathbb{E}\left[\left(I_t^{t+h}(k) I_{t-h}^{t}(j)\right)^2\right] \\
&\leqslant C e^{-2k^2 t} e^{-2j^2(t-h)} e^{-2k^2 h} e^{-2j^2 h} \, \frac{e^{2j^2 t}-e^{2j^2(t-h)}}{2j^2} \, \frac{e^{2k^2(t+h)}-e^{2k^2 t}}{2k^2} \\
&\leqslant C\, \frac{1-e^{-2j^2 h}}{2j^2} \, \frac{1-e^{-2k^2 h}}{2k^2}\,.
\end{aligned}$$

Therefore, as for the previous term we get

$$\mathbb{E}\left[A_4(j,k)^2\right] \leqslant C h^2\,.$$

Then, for every $r < -\frac{1}{2}$, we can pick $0 < \delta < 1$ such that $r + \delta < -\frac{1}{2}$. Then,

$$\mathbb{E}\left[\|u(t+h,\cdot)-u(t,\cdot)\|_{H_r}^2 \, \|u(t-h,\cdot)-u(t,\cdot)\|_{H_r}^2\right] \leqslant C h^{1+\delta}\,.$$

By Theorem A.0.3, we deduce that $(u(t,\cdot))_{t\geqslant 0}$ has a càdlàg version in $H_r([0,\pi])$ for any $r < -\frac{1}{2}$. □

Remark 5.2.6. *The result of Proposition 5.2.5 is in fact valid for any random field u whose sine Fourier coeficients can be written in the form*

$$a_k(u(t,\cdot)) = C e^{-k^2 t} \int_0^t \int_0^\pi \sin(ky) e^{k^2 s} Z(s,y) L(\mathrm{d}s,\mathrm{d}y)\,,$$

where Z is a predictable and bounded random field.

General case

If σ is not bounded, we show that the conclusion of Proposition 5.2.5 is still valid.

Theorem 5.2.7. *Let $\sigma : \mathbb{R} \to \mathbb{R}$ be a Lipschitz continuous function, and let L be a pure jump Lévy white noise. Let u be the mild solution to the stochastic heat equation* (5.2.1) *constructed in Proposition 5.2.1. Then, for any $r < -\frac{1}{2}$, the stochastic process $(u(t,\cdot))_{t\geqslant 0}$ has a càdlàg version in $H_r([0,\pi])$.*

Remark 5.2.8. *Let D be a bounded and smooth domain $D \subset \mathbb{R}^d$, or $D = \mathbb{R}^d$. In the case of a very simple noise with only one deterministic jump of size one at position (s,y), the solution u to the stochastic heat equation on D with this simple noise can be written as*

$$u(t,x) = G_D(t-s;x,y)\,, \qquad (t,x) \in [0,T] \times D$$

where G_D is the Green's function for the heat operator on the domain D with Dirichlet boundary conditions. Then, at $t = s$, we can formally write $u(s,\cdot) = C\delta_y$, where δ_y is the Dirac distribution at y. Then, it is easy to show that δ_y does not belong to any fractional Sobolev space of order $r \geqslant -\frac{d}{2}$. Therefore, the constraint $r < -\frac{1}{2}$ in Theorem 5.2.7 is optimal, and the constraints $r < -\frac{d}{2}$ in Theorems 5.3.12 and 5.4.6 are also optimal.

Proof of Theorem 5.2.7. By the argument just before Section 5.2.1 and by Lemma 5.2.3, we can suppose that u is the solution to (5.2.11) with $L_1 = L^M$. Let $\sigma_n(u) = \sigma(u)\mathbb{1}_{|u|\leqslant n}$. We define

$$u_n(t,x) = \int_0^t \int_0^\pi G(t-s;x,y)\sigma_n(u(s,y))L^M(\mathrm{d}s,\mathrm{d}y)\,.$$

By Remark 5.2.4, the Fourier sine coefficients of $t \mapsto u(t,\cdot) - u_n(t,\cdot)$ are given by

$$a_{k,n}(t) = \frac{2}{\pi}\int_0^t \int_0^\pi \sin(ky)e^{-k^2(t-s)}\left(\sigma(u(s,y)) - \sigma_n(u(s,y))\right)L^M(\mathrm{d}s,\mathrm{d}y)\,.$$

Therefore, for any $t \in [0,T]$,

$$\|u(t,\cdot) - u_n(t,\cdot)\|_{H_r}^2 = \sum_{k=1}^{+\infty}(1+k^2)^r a_{k,n}^2(t)\,. \tag{5.2.30}$$

For conciseness of the notation, we write $\sigma(u(s,y)) - \sigma_n(u(s,y)) = \sigma_{(n)}(s,y)$. Then, we use the following identity to rewrite the coefficient $a_{k,n}(t)$ as a semimartingale:

$$e^{-k^2(t-s)} = 1 - \int_s^t k^2 e^{-k^2(t-r)}\,\mathrm{d}r\,.$$

Using Theorem A.0.2, we get

$$a_{k,n}(t) = \frac{2}{\pi}\left(\int_0^t \int_0^\pi \sin(ky)\sigma_{(n)}(s,y)L^M(\mathrm{d}s,\mathrm{d}y)\right.$$

$$
\begin{aligned}
&\quad - \int_0^t \int_0^\pi \sin(ky) \left(\int_s^t k^2 e^{-k^2(t-r)} \, \mathrm{d}r \right) \sigma_{(n)}(s,y) L^M(\mathrm{d}s, \mathrm{d}y) \Big) \\
&= \frac{2}{\pi} \Big(\int_0^t \int_0^\pi \sin(ky) \sigma_{(n)}(s,y) L^M(\mathrm{d}s, \mathrm{d}y) \\
&\quad - \int_0^t \left(\int_0^r \int_0^\pi \sin(ky) k^2 e^{-k^2(t-r)} \sigma_{(n)}(s,y) L^M(\mathrm{d}s, \mathrm{d}y) \right) \mathrm{d}r \Big) .
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\left| a_{k,n}(t) \right| &\leqslant \frac{2}{\pi} \left| \int_0^t \int_0^\pi \sin(ky) \sigma_{(n)}(s,y) L^M(\mathrm{d}s, \mathrm{d}y) \right| \\
&\quad + \frac{2}{\pi} \int_0^t k^2 e^{-k^2(t-r)} \left| \int_0^r \int_0^\pi \sin(ky) \sigma_{(n)}(s,y) L^M(\mathrm{d}s, \mathrm{d}y) \right| \mathrm{d}r \\
&\leqslant \frac{2}{\pi} \left| \int_0^t \int_0^\pi \sin(ky) \sigma_{(n)}(s,y) L^M(\mathrm{d}s, \mathrm{d}y) \right| \\
&\quad + \frac{2}{\pi} \sup_{r \in [0,t]} \left| \int_0^r \int_0^\pi \sin(ky) \sigma_{(n)}(s,y) L^M(\mathrm{d}s, \mathrm{d}y) \right| \left(\int_0^t k^2 e^{-k^2(t-r)} \, \mathrm{d}r \right) \\
&\leqslant C \sup_{r \in [0,t]} \left| \int_0^r \int_0^\pi \sin(ky) \sigma_{(n)}(s,y) L^M(\mathrm{d}s, \mathrm{d}y) \right| ,
\end{aligned}
$$

where C does not depend on k. Then, using Theorem A.0.1 *(i)*, we deduce that

$$
\mathbb{E} \left[\sup_{t \in [0,T]} a_{k,n}^2(t) \right] \leqslant C \int_0^T \int_0^\pi \sin^2(ky) \mathbb{E} \left[\sigma_{(n)}^2(s,y) \right] \mathrm{d}s \, \mathrm{d}y , \tag{5.2.31}
$$

where the constant C includes $\int_{|z| \leqslant 1} z^2 \nu(\mathrm{d}z)$. Since ν is a Lévy measure, this integral is finite. Furthermore, by Hölder's inequality and Markov's inequality,

$$
\begin{aligned}
\mathbb{E} \left[\sigma_{(n)}^2(s,y) \right] &= \mathbb{E} \left[\sigma \left(u(s,y) \right)^2 \mathbb{1}_{|u(s,y)| > n} \right] \\
&\leqslant \mathbb{E} \left[\sigma \left(u(s,y) \right)^{\frac{8}{3}} \right]^{\frac{3}{4}} \left[\mathbb{P} \left(|u(s,y)| > n \right) \right]^{\frac{1}{4}} \\
&\leqslant C \left(1 + \mathbb{E} \left[\left| u(s,y) \right|^{\frac{8}{3}} \right] \right)^{\frac{3}{4}} \frac{\mathbb{E} \left[\left| u(s,y) \right|^{\frac{8}{3}} \right]^{\frac{1}{4}}}{n^{\frac{2}{3}}} \\
&\leqslant \frac{C}{n^{\frac{2}{3}}} \left(1 + \mathbb{E} \left[\left| u(s,y) \right|^{\frac{8}{3}} \right] \right) .
\end{aligned} \tag{5.2.32}
$$

By Proposition 5.2.1, the solution u has uniformly bounded moments in space and time of any order $p < 3$, we deduce that

$$
\mathbb{E} \left[\sigma_{(n)}^2(s,y) \right] \leqslant \frac{C}{n^{\frac{2}{3}}} , \tag{5.2.33}
$$

for some constant C. By (5.2.31) and (5.2.33), we obtain

$$
\mathbb{E} \left[\sup_{t \in [0,T]} a_{k,n}^2(t) \right] \leqslant \frac{C}{n^{\frac{2}{3}}} .
$$

By (5.2.30), we deduce that for any $r < -\frac{1}{2}$,

$$\begin{aligned}\mathbb{E}\left[\sup_{t\in[0,T]} \|u(t,\cdot)-u_n(t,\cdot)\|_{H_r}^2\right] &\leqslant \sum_{k=1}^{+\infty}(1+k^2)^r\,\mathbb{E}\left[\sup_{t\in[0,T]} a_{k,n}^2(t)\right] \\ &\leqslant \frac{C}{n^{\frac{2}{3}}} \to 0 \qquad \text{as } n\to+\infty.\end{aligned}$$

Therefore, $\sup_{t\in[0,T]}\|u(t,\cdot)-u_n(t,\cdot)\|_{H_r} \to 0$ in $L^2(\Omega)$ as $n\to+\infty$, and there is a subsequence $(n_k)_{k\geqslant 0}$ such that $\sup_{t\in[0,T]}\|u(t,\cdot)-u_{n_k}(t,\cdot)\|_{H_r}\to 0$ almost surely as $k\to+\infty$. This means that $u_{n_k}(t,\cdot)$ converges to $u(t,\cdot)$ in $H_r([0,\pi])$ uniformly in time for any $r<-\frac{1}{2}$. Since σ_{n_k} is bounded, $t\mapsto u_{n_k}(t,\cdot)$ has by Proposition 5.2.5 and Remark 5.2.6 a càdlàg version in $H_r([0,\pi])$. Therefore, $t\mapsto u(t,\cdot)$ has a càdlàg version in $H_r([0,\pi])$ for any $r<-\frac{1}{2}$. □

Remark 5.2.9. *The result of Theorem 5.2.7 is in fact valid for any random field u that can be written in the form*

$$u(t,x) = C\int_0^t\int_0^\pi G(t-s;x,y)Z(s,y)L(\mathrm{d}s,\mathrm{d}y),$$

where Z is a predictable random field such that for some $p>2$,

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}} \mathbb{E}\left[|Z(t,x)|^p\mathbb{1}_{t\leqslant\tau_N}\right] < \infty$$

for every $N\in\mathbb{N}$. Indeed, as before it is enough to show the result for

$$u_N(t,x) = C\int_0^t\int_0^\pi G(t-s;x,y)Z(s,y)L_N(\mathrm{d}s,\mathrm{d}y),$$

and then restrict to $N=1$ and $b=0$ using Lemma 5.2.3. Replacing $\sigma_{(n)}$ by $Z_{(n)} := Z\mathbb{1}_{|Z|>n}$ in the proof of Theorem 5.2.7, we can use Hölder's inequality as in (5.2.32) *to get*

$$\begin{aligned}\mathbb{E}\left[Z_{(n)}^2(s,y)\mathbb{1}_{s\leqslant\tau_1}\right] &\leqslant \left(\mathbb{E}\left[Z^p(s,y)\mathbb{1}_{s\leqslant\tau_1}\right]\right)^{\frac{2}{p}}\left[\mathbb{P}\left(Z(s,y)\mathbb{1}_{s\leqslant\tau_1}>n\right)\right]^{\frac{p-2}{p}} \\ &\leqslant \frac{\mathbb{E}\left[Z^p(s,y)\mathbb{1}_{s\leqslant\tau_1}\right]}{n^{p-2}} \\ &\leqslant \frac{\sup_{(t,x)\in[0,T]\times\mathbb{R}}\mathbb{E}\left[|Z(t,x)|^p\mathbb{1}_{t\leqslant\tau_1}\right]}{n^{p-2}}.\end{aligned}$$

This bound replaces (5.2.33) *and the remainder of the proof carries through.*

5.2.3 Continuity in space at a fixed time

We have studied the sample path regularity of the mild solution u constructed in Proposition 5.2.1, with σ globally Lipschitz, viewed as a stochastic process with values in a space of distributions. However, because of the absolute continuity of the law of the jump locations of the noise, if we fix a time t and look at the sample path regularity of the solution in space, then, with probability one, we do not meet any of these jumps. In fact, we will prove in this section that for any fixed time $t\in[0,T]$, the paths of the process $x\mapsto u(t,x)$ are almost surely continuous.

Proposition 5.2.10. *Let L be a pure jump Lévy white noise with Lévy measure ν, and suppose that for some $0<p<2$, $\int_{|z|\leqslant 1}|z|^p\nu(\mathrm{d}z)<\infty$. Let $t\in[0,T]$ be fixed. Then the process $x\mapsto u(t,x)$ has a locally γ-Hölder continuous modification for any $\gamma<\left(2-\left(\frac{3}{2}\vee p\right)\right)\left(\frac{3}{2}\vee p\right)^{-1}$.*

Proof. By the argument just before Section 5.2.1, we can suppose that u is the solution to (5.2.11). By Hölder's inequality for the drift term, and the Bichteler-Jacod inequality in Theorem A.0.1 *(ii)* for the martingale term, we have for any $q\in\left(\frac{3}{2}\vee p,2\right)$,

$$\begin{aligned}
&\mathbb{E}\left[\left|u(t,x)-u(t,y)\right|^q\right]\\
&\quad\leqslant C\int_0^t\int_0^\pi|G(t-s;x,w)-G(t-s;y,w)|^q\,\mathbb{E}\left[|\sigma(u(s,w))|^q\right]\mathrm{d}s\,\mathrm{d}w\\
&\qquad+C\int_0^t\int_0^\pi\int_{|z|\leqslant 1}|z|^q\left|G(t-s;x,w)-G(t-s;y,w)\right|^q\mathbb{E}\left[|\sigma(u(s,w))|^q\right]\mathrm{d}s\,\mathrm{d}w\,\nu(\mathrm{d}z).
\end{aligned}$$

Since $q>p$ and since $\int_{|z|\leqslant 1}|z|^p\nu(\mathrm{d}z)<\infty$, we deduce that $\int_{|z|\leqslant 1}|z|^q\nu(\mathrm{d}z)<\infty$. Also, by Proposition 5.2.1, the mild solution u has a second moment that is uniformly bounded in space and time. Therefore, since $|\sigma(x)|^q\leqslant C\left(1+|x|^q\right)$, we get

$$\mathbb{E}\left[\left|u(t,x)-u(t,y)\right|^q\right]\leqslant C\int_0^t\int_0^\pi\left|G(t-s;x,w)-G(t-s;y,w)\right|^q\mathrm{d}s\,\mathrm{d}w.$$

By [6, Lemma B.1, (B.8)], which applies since $q>\frac{3}{2}$, we get

$$\mathbb{E}\left[\left|u(t,x)-u(t,y)\right|^q\right]\leqslant C|x-y|^{3-q}.$$

By Kolmogorov's continuity criterion (see [44, Theorem 3.23]), we get the result. □

Remark 5.2.11. *In particular, any α-stable noise with $\alpha\in(0,2)$ satisfies the hypothesis of Proposition 5.2.10 with any $p\in(\alpha,2)$.*

5.2.4 Continuity in time at a fixed space point

We now look at the regularity of the mild solution in time at a fixed space point.

Proposition 5.2.12. *Let L be a pure jump Lévy white noise with Lévy measure ν. Suppose that for some $0<p<1$, $\int_{|z|\leqslant 1}|z|^p\nu(\mathrm{d}z)<\infty$. Let $x\in[0,\pi]$ be fixed. Then the process $t\mapsto u(t,x)$ has a continuous modification.*

Proof. By the argument just before Section 5.2.1, we can suppose that u is the solution to (5.2.11). By Lemma 5.2.3, $(t,x)\mapsto F(t,x)=\int_0^t\int_0^\pi G(t-s;x,y)\sigma(u(s,y))\,\mathrm{d}s\,\mathrm{d}y$ is jointly continuous in $(t,x)\in[0,T]\times[0,\pi]$. Then, regarding the stochastic integral with respect to L^M, we observe that the jumps of the noise are summable, hence, upon changing the value of b, it is sufficient to consider the uncompensated process

$$\int_0^t \int_0^\pi \int_{\mathbb{R}} zG(t-s;x,y)\sigma(u(s,y))J(\mathrm{d}s,\mathrm{d}y,\mathrm{d}z) = \sum_{i\geqslant 1} Z_i G(t-T_i;x,Y_i)\sigma(u(T_i,Y_i))\,, \qquad (5.2.34)$$

where (T_i, Y_i, Z_i) are the jump points of the underlying Poisson random measure J. For any fixed $(x,y)\in[0,\pi]^2$, $x\neq y$, we have by [6, (B.7)] that $t\mapsto G(t;x,y)$ is a continuous function on $\mathbb{R}$. We show that the sum in (5.2.34) converges uniformly in $t\in[0,T]$. To do this we can split the sum depending on the distance of the jump Y_i to x. Indeed, for $|x-y|\leqslant\sqrt{2T}$, by Lemma A.0.5, we have

$$\sup_{t\in[0,T]} G(t;x,y)\leqslant \sup_{t\in[0,T]}\frac{C}{\sqrt{t}}e^{-\frac{(x-y)^2}{4t}} = \frac{C'}{|x-y|}\,,$$

for some constant C'. Also, if $|x-y|>\sqrt{2T}$, then

$$\sup_{t\in[0,T]} G(t;x,y)\leqslant \sup_{t\in[0,T]}\frac{C}{\sqrt{t}}e^{-\frac{(x-y)^2}{4t}} = \frac{C}{\sqrt{T}}e^{-\frac{(x-y)^2}{4T}}\,.$$

Since $0<p<1$,

$$\begin{aligned}
&\mathbb{E}\left[\left(\sum_{i\geqslant 1}\sup_{t\in[0,T]}|Z_iG(t-T_i;x,Y_i)\sigma(u(T_i,Y_i))|\right)^p\right]\\
&\quad\leqslant \mathbb{E}\left[\sum_{i\geqslant 1}|Z_i|^p\,|\sigma(u(T_i,Y_i))|^p\sup_{t\in[0,T]}|G(t-T_i;x,Y_i)|^p\right]\\
&\quad\leqslant C\sup_{(s,y)\in[0,T]\times[0,\pi]}\mathbb{E}\left[\left|\sigma\left(u(s,y)\right)\right|^p\right]\left(\int_0^T\int_0^\pi\int_{|z|\leqslant 1}|z|^p\frac{1}{|x-y|^p}\mathbb{1}_{|x-y|\leqslant\sqrt{2(T-s)}}\,\mathrm{d}s\,\mathrm{d}y\,\nu(\mathrm{d}z)\right.\\
&\qquad\left.+\int_0^T\int_0^\pi\int_{|z|\leqslant 1}|z|^p\rho_H(T-s,x-y)^p\mathbb{1}_{|x-y|>\sqrt{2(T-s)}}\,\mathrm{d}s\,\mathrm{d}y\,\nu(\mathrm{d}z)\right)<+\infty,
\end{aligned}$$

where we used the fact that $\rho_H\in L^p([0,T]\times\mathbb{R})$ since $p<3$ (see (4.4.6) and above), and the fact that $\int_{|z|\leqslant 1}|z|^p\nu(\mathrm{d}z)<+\infty$. This concludes the proof. □

Remark 5.2.13. *In particular, any α-stable noise with $\alpha\in(0,1)$ satisfies the hypothesis of Proposition 5.2.12. The next section shows that for $\alpha\geqslant 1$, the situation is completely different.*

The case of an α-stable noise with $1\leqslant\alpha<2$

In this section, we consider the stochastic heat equation on $[0,T]\times[0,\pi]$ with Dirichlet boundary conditions, with additive α-stable noise L_α on $[0,T]\times[0,\pi]$. More precisely, L_α is a Lévy white noise with characteristic triplet $(0,0,\nu_\alpha)$, where $\nu_\alpha(\mathrm{d}z):=(c_+\mathbb{1}_{z>0}+c_-\mathbb{1}_{z<0})\frac{1}{|z|^{\alpha+1}}$. This noise then coincides with the notion of α-stable random measure studied in [62].

Proposition 5.2.14. *Let $\alpha\in[1,2)$, and let u be the mild solution of the stochastic heat equation* *with additive α-stable noise:*

$$u(t,x)=\int_0^t\int_0^\pi G(t-s;x,y)L_\alpha(\mathrm{d}s,\mathrm{d}y).$$

Then for any $x\in(0,\pi)$, there is a set $N_x\subset\Omega$ of probability one such that for any $\omega\in N_x$, $t\mapsto u(t,x)(\omega)$ is unbounded on any non-empty open interval.

Proof. Fix $x\in(0,\pi)$. Observe that the process $(X(t),t\in[0,T])$ defined by

$$X(t)=u(t,x)=\int_0^t\int_0^\pi G(t-s;x,y)L_\alpha(\mathrm{d}s,\mathrm{d}y)$$

is an α-stable process given in the "standard form" of [62, (10.1.1)] with the measurable space $E=[0,T]\times[0,\pi]$, and the control measure $\mathrm{d}s\,\mathrm{d}y$. Let $T^*=[t_1,t_2]$, with $0\leqslant t_1<t_2\leqslant T$. We shall check that the necessary condition [62, (10.2.14)] for sample path boundedness in [62, Theorem 10.2.3] is not satisfied, in particular, that

$$\int_0^T\int_0^\pi\left(\sup_{t\in T^*}G(t-s;x,y)\right)^\alpha\mathrm{d}s\,\mathrm{d}y=+\infty. \tag{5.2.35}$$

Indeed, observe that the integral is bounded below by

$$\begin{aligned}\int_{t_1}^{t_2}\int_0^\pi\sup_{t\in[t_1,t_2]}G(t-s;x,y)^\alpha\,\mathrm{d}s\,\mathrm{d}y&=\int_0^{t_2-t_1}\int_0^\pi\sup_{t\in[t_1,t_2]}G(t-t_1-r;x,y)^\alpha\,\mathrm{d}r\,\mathrm{d}y\\&=\int_0^{t_2-t_1}\int_0^\pi\sup_{u\in[0,t_2-t_1]}G(u-r;x,y)^\alpha\,\mathrm{d}r\,\mathrm{d}y\\&\geqslant\int_0^{t_2-t_1}\int_{x-\varepsilon}^{x+\varepsilon}\sup_{u\in[0,t_2-t_1]}G(u-r;x,y)^\alpha\,\mathrm{d}r\,\mathrm{d}y\\&=\int_0^{t_2-t_1}\int_{x-\varepsilon}^{x+\varepsilon}\sup_{v\in[0,t_2-t_1-r]}G(v;x,y)^\alpha\,\mathrm{d}r\,\mathrm{d}y\\&=\int_0^{t_2-t_1}\int_{x-\varepsilon}^{x+\varepsilon}\sup_{v\in[0,s]}G(v;x,y)^\alpha\,\mathrm{d}s\,\mathrm{d}y,\end{aligned}$$

for any fixed $\varepsilon>0$ such that $[x-\varepsilon,x+\varepsilon]\subset(0,\pi)$. We now use the representation of the Green's function

$$G(t;x,y)=\frac{1}{\sqrt{4\pi t}}\sum_{k\in\mathbb{Z}}\left(\exp\left(-\frac{(y-x-2k\pi)^2}{4t}\right)-\exp\left(-\frac{(y+x-2k\pi)^2}{4t}\right)\right)$$

given in [6, (B.2)] to see that

$$G(t;x,y)=\rho_H(t,x-y)+H(t;x,y),$$

where $(t,y)\mapsto H(t;x,y)$ is smooth and bounded on $[0,T]\times[x-\varepsilon,x+\varepsilon]$ (cf. the proof of [69, Corollary 3.4]). In particular, for $y\in[x-\varepsilon,x+\varepsilon]$, $G(v;x,y)\geqslant\rho_H(v,x-y)-C_0$ with some constant C_0. In view of the study of the maximum of $t \mapsto \rho_H(t, x-y)$ in the proof of Proposition 5.2.12, we have

$$\int_0^T \int_0^\pi \left(\sup_{t\in T^*} G(t-s;x,y)\right)^\alpha \mathrm{d}s\,\mathrm{d}y \geqslant \int_0^{t_2-t_1} \int_{|x-y|\leqslant \varepsilon\wedge \frac{C}{2C_0}\wedge\sqrt{2s}} \left(\frac{C}{|x-y|} - C_0\right)^\alpha \mathrm{d}s\,\mathrm{d}y = +\infty,$$

and (5.2.35) is proved. $\square$

Remark 5.2.15. *The sample path behaviour in the α-stable case is due to the singularity of the Lévy measure at the origin. Indeed, we can write*

$$\begin{aligned} L_\alpha(\mathrm{d}s,\mathrm{d}y) &= \int_{|z|\leqslant 1} z\tilde{J}(\mathrm{d}s,\mathrm{d}y,\mathrm{d}z) + \int_{|z|>1} zJ(\mathrm{d}s,\mathrm{d}y,\mathrm{d}z) \\ &= L_\alpha^1(\mathrm{d}s,\mathrm{d}y) + L_\alpha^2(\mathrm{d}s,\mathrm{d}y). \end{aligned}$$

Since L_α^2 satisfies the hypothesis of Proposition 5.2.12 (there are no small jumps), the solution v to the SHE driven by this noise is such that for any $x \in [0,\pi]$, the process $t \mapsto v(t,x)$ has a continuous modification. Let w be solution to the SHE driven by L_α^1. Since Proposition 5.2.14 applies to $v+w$, for any $x \in [0,\pi]$, the process $t \mapsto w(t,x)$ is unbounded on any open interval almost surely. Then, we can extend the result of Proposition 5.2.14 to any noise with a small jump density comparable to the α-stable case. More precisely, suppose that L is a pure jump Lévy noise with Lévy measure ν and jump measure J, such that there is $\delta > 0$ such that $\nu(\mathrm{d}z) = \frac{f(z)}{|z|^{\alpha+1}}\,\mathrm{d}z$ on $[-\delta,\delta]$, and such that $f(0) \neq 0$. Suppose also that for some $0 < q < 1$,

$$\int_{-\delta}^{\delta} \frac{|f(z)-f(0)|}{|z|^{\alpha+1}} |z|^q \,\mathrm{d}z < +\infty. \tag{5.2.36}$$

This condition forces f to be continuous at 0 with a certain regularity at the origin. Then, we can write

$$\nu(\mathrm{d}z) = \frac{f(z)-f(0)}{|z|^{\alpha+1}} \mathbb{1}_{z\in[-\delta,\delta]}\,\mathrm{d}z + \frac{f(0)}{|z|^{\alpha+1}} \mathbb{1}_{z\in[-\delta,\delta]}\,\mathrm{d}z + \mathbb{1}_{z\in[-\delta,\delta]^c}\nu(\mathrm{d}z).$$

Therefore, we can write

$$\begin{aligned} \underbrace{\nu(\mathrm{d}z) + \frac{\big(f(z)-f(0)\big)_-}{|z|^{\alpha+1}} \mathbb{1}_{z\in[-\delta,\delta]}\,\mathrm{d}z}_{:=\nu_1(\mathrm{d}z)} &= \underbrace{\frac{\big(f(z)-f(0)\big)_+}{|z|^{\alpha+1}} \mathbb{1}_{z\in[-\delta,\delta]}\,\mathrm{d}z}_{:=\nu_2(\mathrm{d}z)} \\ &\quad + \underbrace{\frac{f(0)}{|z|^{\alpha+1}} \mathbb{1}_{z\in[-\delta,\delta]}\,\mathrm{d}z}_{:=\nu_3(\mathrm{d}z)} + \underbrace{\mathbb{1}_{z\in[-\delta,\delta]^c}\nu(\mathrm{d}z)}_{:=\nu_4(\mathrm{d}z)}. \end{aligned}$$

For $1 \leqslant i \leqslant 4$, let J_i be a Poisson random measure with intensity measure ν_i. We assume that the J_i are independent. Let L_i be the pure jump Lévy noise associated with J_i. We consider the mild solution u_i to the linear SHE with driving noise L_i. Then $u_1 + u$ has the same law as $u_2 + u_3 + u_4$. By (5.2.36) *and Proposition 5.2.12, for any fixed $x \in [0,\pi]$, $t \mapsto u_1(t,x)$ and $t \mapsto u_2(t,x)$ have* *a continuous modification. By the same Proposition 5.2.12, since L_4 does not have any small jumps, for any fixed $x\in[0,\pi]$, $t\mapsto u_4(t,x)$ has a continuous modification. Then, by the same argument as above for the regularity of $t\mapsto w(t,x)$, we deduce that for any $x\in[0,\pi]$, $t\mapsto u_3(t,x)$ is unbounded on any open interval. Therefore, for any $x\in[0,\pi]$, $t\mapsto u(t,x)$ is unbounded on any non-empty open interval. For example, this extension includes tempered stable Lévy noise, i.e. pure jump Lévy noise with Lévy measure $\nu(\mathrm{d}z)=\frac{e^{-\lambda|z|}}{|z|^{\alpha+1}}$, $\alpha\in[1,2)$.*

5.3 The SHE driven by heavy-tailed noise: equation on $[0,T]\times\mathbb{R}^d$

In [14], C. Chong proved the existence of a solution to the following stochastic heat equation (SHE) under some fairly general assumptions on the driving noise. In particular, his results include the cases of α-stable noises.

$$\begin{cases} \frac{\partial u}{\partial t}(t,x)=\Delta u(t,x)+\sigma(u(t,x))\dot{L}(t,x), & (t,x)\in[0,T]\times\mathbb{R}^d, \\ u(0,x)=0, & \text{for all } x\in\mathbb{R}^d, \end{cases} \tag{5.3.1}$$

where $\dot{L}$ is a Lévy white noise with characteristic triplet $(b,0,\nu)$. More precisely, we suppose that

$$\begin{aligned} L(\mathrm{d}t,\mathrm{d}x)&=b\,\mathrm{d}t\,\mathrm{d}x+\int_{|z|\leqslant 1} z\tilde{J}(\mathrm{d}t,\mathrm{d}x,\mathrm{d}z)+\int_{|z|>1} zJ(\mathrm{d}t,\mathrm{d}x,\mathrm{d}z) \\ &=: L^B(\mathrm{d}t,\mathrm{d}x)+L^M(\mathrm{d}t,\mathrm{d}x)+L^P(\mathrm{d}t,\mathrm{d}x), \end{aligned} \tag{5.3.2}$$

where $b\in\mathbb{R}$, J is a Poisson random measure on $[0,\infty)\times\mathbb{R}^d\times\mathbb{R}$ with intensity $\mathrm{d}t\,\mathrm{d}x\,\nu(\mathrm{d}z)$, and $\tilde{J}$ is the compensated version of J. The measure ν is a Lévy measure, that is, $\nu(\{0\})=0$ and $\int_{\mathbb{R}}\left(z^2\wedge 1\right)\nu(\mathrm{d}z)<+\infty$. We suppose that the following hypothesis hold:

(H3) The function $\sigma:\mathbb{R}\to\mathbb{R}$ is globally Lipschitz.

(H4) There exists $0<p<1+\frac{2}{d}$ and $\frac{p}{1+\left(1+\frac{2}{d}-p\right)}<q\leqslant p$ such that

$$\int_{|z|\leqslant 1}|z|^p\nu(\mathrm{d}z)+\int_{|z|>1}|z|^q\nu(\mathrm{d}z)<+\infty.$$

If $p<1$, we assume that

$$b_0:=b-\int_{|z|\leqslant 1} z\nu(\mathrm{d}z)=0.$$

The heat kernel for the stochastic heat equation (5.3.1) on this domain is given by the usual Gaussian density function

$$\rho_H(t,x-y)=\frac{1}{(4\pi t)^{\frac{d}{2}}}e^{-\frac{|x-y|^2}{4t}}.$$

By definition, a mild solution of (5.3.1) is a predictable random field u such that for any $(t,x)\in[0,T]\times\mathbb{R}^d$,

$$u(t,x)=\int_0^t\int_{\mathbb{R}^d}\rho_H(t-s,x-y)\sigma\left(u(s,y)\right)L(\mathrm{d}s,\mathrm{d}y)\qquad \text{a.s.} \tag{5.3.3}$$

In [14], C. Chong introduced a sequence of stopping times to truncate the large jumps of the noise in a suitable manner. This particular truncation does not delete all the large jumps, provided they are far enough away from the origin in space. More precisely, let $h:\mathbb{R}^d \to \mathbb{R}$ be defined by

$$h: x \mapsto 1+|x|^{\eta} \qquad \forall x \in \mathbb{R}^d, \tag{5.3.4}$$

for some η to be chosen later. Then, for any $N \in \mathbb{N}$,

$$\tau_N = \inf\{t \geqslant 0 : J([0,t] \times \{(x,z) : |z| > Nh(x)\}) > 0\}, \tag{5.3.5}$$

where η has to be chosen. We can then define a truncated noise from (5.3.2), for $N \geqslant 1$,

$$L_N(\mathrm{d}t,\mathrm{d}x) = b\,\mathrm{d}t\,\mathrm{d}x + \int_{|z|\leqslant 1} z\tilde{J}(\mathrm{d}t,\mathrm{d}x,\mathrm{d}z) + \int_{1<|z|<Nh(x)} zJ(\mathrm{d}t,\mathrm{d}x,\mathrm{d}z) \tag{5.3.6}$$

By definition, a mild solution to the truncated equation (5.3.1) where $\dot{L}$ is replaced by $\dot{L}_N$, is a predictable random field u_N such that for any $(t,x) \in [0,T]\times\mathbb{R}^d$,

$$u_N(t,x) = \int_0^t \int_{\mathbb{R}^d} \rho_H(t-s,x-y)\sigma\left(u_N(s,y)\right) L_N(\mathrm{d}s,\mathrm{d}y) \qquad \text{a.s.} \tag{5.3.7}$$

Proposition 5.3.1. *Let L be a Lévy white noise with characteristic triplet $(b,0,\nu)$ as in* (5.3.2)*, $\sigma:\mathbb{R}\to\mathbb{R}$, and $p,q\in\mathbb{R}_+$ such that **(H3)** and **(H4)** are satisfied. Let $\eta > \frac{d}{q}$ in* (5.3.4)*. Then for any $N \geqslant 1$, $\tau_N > 0$ and $\tau_N \to +\infty$ a.s. as $N \to +\infty$. Also, for any $N \geqslant 1$, there is a solution u_N to* (5.3.7) *such that for any $N \geqslant 1$ and $R > 0$,*

$$\sup_{(t,x)\in[0,T]\times[-R,R]^d} \mathbb{E}\left[|u_N(t,x)|^p\right] < +\infty. \tag{5.3.8}$$

Then, the random field u defined by $u(t,x)\mathbb{1}_{t\leqslant\tau_N} = u_N(t,x)$, is a mild solution to (5.3.1).

Proof of Proposition 5.3.1. The existence of the mild solution u is a direct application of [14, Theorem 3.1]. The moment property (5.3.8) can be deduced from [14, Theorem 3.1] if $d \geqslant 2$, since in that case, $p < 1+\frac{2}{d} \leqslant 2$. In the case of $d = 1$ and $2 < p < 3$, we need an extension of [14, Lemma 3.3(2)]. Up to the stopping time τ_N, the noise L coincides with L_N as defined in (5.3.6). For this noise, combining the arguments given in the proof of [14, Lemma 3.3(2)] and the proof of Proposition 5.2.1, one obtains

$$\begin{aligned}\mathbb{E}&\left[\left|\int_0^t\int_{\mathbb{R}} \rho_H(t-s,x-y)\left(\sigma(\phi_1(s,y))-\sigma(\phi_2(s,y))\right)L_N(\mathrm{d}s,\mathrm{d}y)\right|^p\right]\\ &\leq C\int_0^t\int_{\mathbb{R}}(\rho_H(t-s,x-y)+\rho_H^p(t-s,x-y))\mathbb{E}[|\phi_1(s,y)-\phi_2(s,y)|^p]h(y)^{p-q}\,\mathrm{d}s\,\mathrm{d}y,\end{aligned}$$

for all predictable processes ϕ_1 and ϕ_2 and some constant $C > 0$ independent thereof. The remainder of the proof is now identical to that of [14, Theorem 3.1]. □

5.3.1 Stationarity of the solution

We recall that the function $h:\mathbb{R}^d\to\mathbb{R}$ was defined by $h(x)=1+|x|^\eta$ for $x\in\mathbb{R}^d$. We assume that $\eta>\frac{d}{q}$. For any $a\in\mathbb{R}^d$ and $N\in\mathbb{N}$, we define the family of stopping times τ_N^a by

$$\tau_N^a := \inf\{t\geqslant 0 : J([0,t]\times\{(x,z):|z|>Nh(x-a)\})>0\}\,.$$

In particular, τ_N^0 is the same as τ_N defined in (5.3.5). Since the intensity measure of the Poisson random measure J is invariant under translation in the space variable, τ_N^a has the same law as τ_N, and the conclusions of [14, Lemma 3.2] are valid for τ_N^a. In particular, for any $N\geqslant 1$, almost surely $\tau_N^a>0$, and $\tau_N^a\nearrow+\infty$ as $N\nearrow+\infty$. Furthermore, by definition, on the event $\{t\leqslant\tau_N^a\}$, $L(\mathrm{d}t,\mathrm{d}x)=L_N^a(\mathrm{d}t,\mathrm{d}x)$, where

$$L_N^a(\mathrm{d}t,\mathrm{d}x):=b\,\mathrm{d}t\,\mathrm{d}x+\rho W(\mathrm{d}t,\mathrm{d}x)+\int_{|z|\leqslant 1} z\tilde{J}(\mathrm{d}t,\mathrm{d}x,\mathrm{d}z)+\int_{1<|z|\leqslant Nh(x-a)} zJ(\mathrm{d}t,\mathrm{d}x,\mathrm{d}z)\,.$$

For $N\in\mathbb{N}$, and $a\in\mathbb{R}^d$, we now consider the truncated stochastic heat equation

$$\begin{cases}\frac{\partial u_N^a}{\partial t}(t,x)=\Delta u_N^a(t,x)+\sigma(u_N^a(t,x))\dot{L}_N^a(t,x)\,, & (t,x)\in[0,T]\times\mathbb{R}^d\,,\\ u_N^a(0,x)=0 & \text{for all } x\in\mathbb{R}^d\,.\end{cases}\tag{5.3.9}$$

More precisely, we say that u_N^a is a mild solution to (5.3.9) if for any $(t,x)\in[0,T]\times\mathbb{R}^d$,

$$u_N^a(t,x)=\int_0^t\int_{\mathbb{R}^d}\rho_H(t-s,x-y)\sigma\left(u_N^a(s,y)\right)L_N^a(\mathrm{d}s,\mathrm{d}y)\,,\qquad \text{a.s.}$$

Proposition 5.3.2. *Let L be a Lévy white noise with characteristic triplet $(b,0,\nu)$ as in (5.3.2), $\sigma:\mathbb{R}\to\mathbb{R}$, and $p,q\in\mathbb{R}_+$ such that **(H3)** and **(H4)** are satisfied. Then for any $N\in\mathbb{N}$ and $a\in\mathbb{R}^d$, there is a unique mild solution u_N^a to the truncated SHE (5.3.9) such that for any $R\in\mathbb{R}_+$,*

$$\sup_{(t,x)\in[0,T]\times[-R,R]^d}\mathbb{E}\left[\left|u_N^a(t,x)\right|^p\right]<+\infty.$$

Proof. This results comes from a simple adaptation to the proof of [14, Theorem 3.1]. Indeed, the only difference is the shift by a of the truncation function h. The result on the moments is a direct consequence of [14, Theorem 3.1] if $p\leqslant 2$, and can be extended to $p<1+\frac{2}{d}$ similarly to the proof of Proposition 5.3.1. □

Lemma 5.3.3. *Let $a,b\in\mathbb{R}^d$, and $N\in\mathbb{N}$. Then, for any $(t,x)\in[0,T]\times\mathbb{R}^d$,*

$$u_N^a(t,x)\mathbb{1}_{t\leqslant\tau_N^a\wedge\tau_N^b}=u_N^b(t,x)\mathbb{1}_{t\leqslant\tau_N^a\wedge\tau_N^b}\,,\qquad a.s.\tag{5.3.10}$$

Proof. First of all, it is clear that on the event $\{t\leqslant\tau_N^a\wedge\tau_N^b\}$, we have the equality $L_N^a=L_N^b$. Then, we use the construction of the solutions u_N^a and u_N^b via a Picard iteration scheme. We will in fact show a stronger result, namely that at each step of the scheme,

$$u_N^{a,n}(t,x)\mathbb{1}_{t\leqslant\tau_N^a\wedge\tau_N^b} = u_N^{b,n}(t,x)\mathbb{1}_{t\leqslant\tau_N^a\wedge\tau_N^b}, \qquad \text{a.s.} \tag{5.3.11}$$

We prove this result by induction. For the initialization step, we obviously have $u_N^{a,0}(t,x) = u_N^{b,0}(t,x) = 0$ almost surely. Then, suppose that for some $n\geqslant 0$, (5.3.11) holds. Then, by definition,

$$\begin{aligned}
u_N^{a,n+1}(t,x)\mathbb{1}_{t\leqslant\tau_N^a\wedge\tau_N^b} &= \mathbb{1}_{t\leqslant\tau_N^a\wedge\tau_N^b}\int_0^t\int_{\mathbb{R}^d}\rho_H(t-s,x-y)\sigma\left(u_N^{a,n}(s,y)\right)L_N^a(\mathrm{d}s,\mathrm{d}y)\\
&= \mathbb{1}_{t\leqslant\tau_N^a\wedge\tau_N^b}\int_0^t\int_{\mathbb{R}^d}\rho_H(t-s,x-y)\sigma\left(u_N^{b,n}(s,y)\right)\mathbb{1}_{s\leqslant\tau_N^a\wedge\tau_N^b}L_N^a(\mathrm{d}s,\mathrm{d}y)\\
&= \mathbb{1}_{t\leqslant\tau_N^a\wedge\tau_N^b}\int_0^t\int_{\mathbb{R}^d}\rho_H(t-s,x-y)\sigma\left(u_N^{b,n}(s,y)\right)\mathbb{1}_{s\leqslant\tau_N^a\wedge\tau_N^b}L_N^b(\mathrm{d}s,\mathrm{d}y)\\
&= \mathbb{1}_{t\leqslant\tau_N^a\wedge\tau_N^b}\int_0^t\int_{\mathbb{R}^d}\rho_H(t-s,x-y)\sigma\left(u_N^{b,n}(s,y)\right)L_N^b(\mathrm{d}s,\mathrm{d}y)\\
&= u_N^{b,n+1}(t,x)\mathbb{1}_{t\leqslant\tau_N^a\wedge\tau_N^b}.
\end{aligned}$$

Then, since $u_N^{a,n}(t,x)\to u_N^a(t,x)$ and $u_N^{b,n}(t,x)\to u_N^b(t,x)$ as $n\to+\infty$ in $L^p(\Omega)$, we deduce that for any $(t,x)\in[0,T]\times\mathbb{R}^d$, (5.3.10) holds. □

In [22, Definition 5.1], R. Dalang introduced the property (S) for a stochastic process and a martingale measure, which is a sort of stationarity property in the space variable. In our case, the noise in not necessarily a martingale measure, but we can use a similar definition:

Definition 5.3.4. *We say the family of random fields u_N^a has property (S) if the law of the process*

$$\left(\left(u_N^a(t,a+x),(t,x)\in[0,T]\times\mathbb{R}^d\right);\left(L_N^a\left([0,t]\times(a+B)\right),(t,B)\in[0,T]\times\mathscr{B}_b(\mathbb{R}^d)\right)\right),$$

does not depend on a.

Lemma 5.3.5. *The family of random fields u_N^a has property (S).*

Proof. Similarly to the proof of Lemma 5.3.3, we prove by induction using the Picard iteration scheme that for any $n\in\mathbb{N}$, $u_N^{a,n}$ has the property (S) introduced in Definition 5.3.4, where

$$\begin{cases} u_N^{a,0}(t,x) := 0, & \text{for all } (t,x)\in[0,T]\times\mathbb{R}^d\\ u_N^{a,n+1}(t,x) := \int_0^t\int_{\mathbb{R}^d}\rho_H(t-s,x-y)\sigma\left(u_N^{a,n}(s,y)\right)L_N^a(\mathrm{d}s,\mathrm{d}y), & \text{for all } (t,x)\in[0,T]\times\mathbb{R}^d\end{cases} \tag{5.3.12}$$

In the following, to lighten the notations, we drop the subscript N and assume without loss of generality that $N=1$. For the initialization step, we obviously have $u^{a,0}(t,x+a)=0=u^{0,0}(t,x)$. Then, we assume that $u^{a,n}$ has the property (S). Since $u^{a,n+1}$ is defined via a stochastic integral in (5.3.12), we can use the same argument as in [22, Lemma 18], since the proof only relies on the fact that the noise has a law that is invariant under translation in the space variable. □

Theorem 5.3.6. *For any $a\in\mathbb{R}^d$, the random field $(u(t,a+x);(t,x)\in[0,T]\times\mathbb{R}^d)$ has the same law as the random field $(u(t,x);(t,x)\in[0,T]\times\mathbb{R}^d)$.*

Proof. By (5.3.10) in Lemma 5.3.3, $u_N^a(t,a+x)\mathbb{1}_{t\leqslant\tau_N^a\wedge\tau_N^0}=u_N^0(t,a+x)\mathbb{1}_{t\leqslant\tau_N^a\wedge\tau_N^0}$ almost surely. Taking the stationary limit as $N\to+\infty$, we get that $u^a(t,x)=u^0(t,x)$ almost surely for any $(t,x)\in[0,T]\times\mathbb{R}^d$. Also, by the property (S) of the family of random fields $(u_N^a)_{a\in\mathbb{R}^d}$ (see Lemma 5.3.5), the random field $(u_N^a(t,a+x);(t,x)\in[0,T]\times\mathbb{R}^d)$ has the same law as the random field $(u_N^0(t,x);(t,x)\in[0,T]\times\mathbb{R}^d)$. Again, taking the stationary limit as $N\to+\infty$, we get that the random field $(u^a(t,a+x);(t,x)\in[0,T]\times\mathbb{R}^d)$ has the same law as the random field $(u^0(t,x);(t,x)\in[0,T]\times\mathbb{R}^d)$. Therefore, the random field $(u^0(t,a+x);(t,x)\in[0,T]\times\mathbb{R}^d)$ has the same law as the random field $(u^0(t,x);(t,x)\in[0,T]\times\mathbb{R}^d)$. □

5.3.2 Existence of a *càdlàg* solution in $H_{r,loc}(\mathbb{R}^d)$

In the following, we are interested in the sample path regularity of the solution to (5.3.1). The mild solution u_N to the truncated equation (5.3.7) converges to the mild solution u to the stochastic heat equation (5.3.3), and the convergence is stationary. In fact, $u(t,x)=u_N(t,x)$ on the event $\{t\leqslant\tau_N\}$. Therefore, the sample path properties of u and u_N are the same, and we can and we will restrict to the study of the regularity of the sample paths of u_N. Furthermore, we can without loss of generality assume that $N=1$. Therefore, we suppose that

$$u(t,x)=\int_0^t\int_{\mathbb{R}^d}\rho_H(t-s,x-y)\sigma\left(u(s,y)\right)L_1\left(\mathrm{d}s,\mathrm{d}y\right),$$

where L_1 is the truncated noise from (5.3.6) with $N=1$. Since we are in the context of a noise with jumps, the local properties of the heat kernel forces the solution to be essentially equal to a Dirac mass at each jump point. Even in the linear case, under some moment assumptions on the jumps, the mild solution is a well defined random field, but has infinite value at every jump point. The situation gets even stranger in the case of an α-stable noise, where the jumps of the noise are known to form a dense subset of $[0,T]\times\mathbb{R}^d$. Therefore, the classical pathwise regularity properties that one gets in the case of a Gaussian noise (see for example [69] and [21]) are not relevant here. Instead, we consider the mild solution $u:t\mapsto u(t,\cdot)$ as a distribution-valued process in a local fractional Sobolev space, and prove that it has a càdlàg version in this space.

Definition 5.3.7. *The fractional Sobolev space of order $r\in\mathbb{R}$ is denoted $H_r(\mathbb{R}^d)$ and defined by*

$$H_r(\mathbb{R}^d):=\left\{f\in\mathscr{S}'(\mathbb{R}^d):\xi\mapsto\left(1+|\xi|^2\right)^{\frac{r}{2}}\mathscr{F}(f)(\xi)\in L^2(\mathbb{R}^d)\right\}.$$

This space is equipped with the norm $\|f\|_{H_r(\mathbb{R}^d)}:=\left\|(1+|\cdot|^2)^{\frac{r}{2}}\mathscr{F}(f)(\cdot)\right\|_{L^2(\mathbb{R}^d)}$.

We also define the local Sobolev space $H_{r,loc}(\mathbb{R}^d)$:

$$H_{r,loc}(\mathbb{R}^d):=\left\{f\in\mathscr{S}'(\mathbb{R}^d):\left(\forall\theta\in\mathscr{D}(\mathbb{R}^d):\theta f\in H_r(\mathbb{R}^d)\right)\right\}.$$

Then, we say that a sequence $(f_n)_{n\geqslant 1}$ *of elements of* $H_{r,loc}(\mathbb{R}^d)$ *converges to* f *in this space if for any* $\theta\in\mathscr{D}(\mathbb{R}^d)$, $\theta f_n\to\theta f$ *in* $H_r(\mathbb{R}^d)$ *as* $n\to+\infty$.

The Fourier transform of a tempered distribution was introduced in Definition 2.1.4. In other words, the fractional Sobolev space $H_r(\mathbb{R}^d)$ is the space of tempered distributions whose Fourier transform is a function with sufficient polynomial decay at infinity. Heuristically, we know that the Fourier transform exchanges regularity with decay at infinity, so the fractional Sobolev spaces can be understood as a space of function with some regularity properties.

The study of the regularity of $u: t\mapsto u(t,\cdot)$ in $H_{r,\mathrm{loc}}(\mathbb{R}^d)$ follows the same path as for the case of a bounded interval in dimension one: we first study the case where σ is a bounded function, then we extend to a more general class of functions.

The case where σ is bounded

We first state a Lemma that allows us to deal with the drift term of u.

Lemma 5.3.8. *Let* Z *be a bounded random field. Let*

$$F(t,x):=\int_0^t\int_{\mathbb{R}^d}\rho_H(t-s,x-y)Z(s,y)\,\mathrm{d}s\,\mathrm{d}y.$$

Then the process $t\mapsto F(t,\cdot)$ *is continuous in* $H_{r,loc}(\mathbb{R}^d)$ *for any* $r<-\frac{d}{2}$.

Proof. Let $u\leqslant t\in[0,T]$, and let $x,z\in\mathbb{R}^d$. By boundedness of Z, and using [61, Lemma A.2] and [61, Lemma A.3]

$$\begin{aligned}|F(t,x)-F(u,z)|&\leqslant C\Big(\int_0^u\int_{\mathbb{R}^d}\big|\rho_H(t-s,x-y)-\rho_H(u-s,z-y)\big|\,\mathrm{d}s\,\mathrm{d}y\\&\qquad+\int_u^t\int_{\mathbb{R}^d}\big|\rho_H(t-s,x-y)\big|\,\mathrm{d}s\,\mathrm{d}y\Big)\\&\leqslant C\Big(\int_0^u\int_{\mathbb{R}^d}\big|\rho_H(t-s,x-y)-\rho_H(u-s,x-y)\big|\,\mathrm{d}s\,\mathrm{d}y\\&\qquad+\int_0^u\int_{\mathbb{R}^d}\big|\rho_H(u-s,x-y)-\rho_H(u-s,z-y)\big|\,\mathrm{d}s\,\mathrm{d}y+(t-u)\Big)\\&\leqslant C\big(|t-u|\log(|t-u|)+|x-z|+|t-u|\big).\end{aligned}$$

The case $t\leqslant u$ is similar, and we deduce that F has continuous sample paths. Therefore, $t\mapsto F(t,\cdot)$ is continuous in $H_{0,\mathrm{loc}}(\mathbb{R}^d)=L^2_{\mathrm{loc}}(\mathbb{R}^d)$. We deduce in particular that the process $t\mapsto F(t,\cdot)$ is continuous in $H_{r,\mathrm{loc}}(\mathbb{R}^d)$ for any $r<-\frac{d}{2}$. □

Proposition 5.3.9. *Let* L *be a pure jump Lévy white noise with Lévy measure* ν *and jump measure* J *such that* ***(H4)*** *is satisfied for some* $p,q\in\mathbb{R}$. *Let* σ *be a bounded and Lipschitz function. Let* u *be the mild solution to the stochastic heat equation* (5.3.1) *constructed in*

Proposition 5.3.1. Then, for any $r<-\frac{d}{2}$, the stochastic process $(u(t,\cdot))_{t\geqslant 0}$ has a càdlàg version in $H_{r,loc}(\mathbb{R}^d)$.

Proof. As recalled in the beginning of Section 5.3.2, u is the stationary limit of the solutions to the truncated equation u_N. Therefore, u and u_N have the same sample path properties, and we can suppose without loss of generality that

$$u(t,x)=\int_0^t\int_{\mathbb{R}^d}\rho_H(t-s,x-y)\sigma\big(u(s,y)\big)\,L_N(\mathrm{d}s,\mathrm{d}y)\qquad\text{a.s.},$$

where

$$\begin{aligned}L_N(\mathrm{d}t,\mathrm{d}x)&=b\,\mathrm{d}t\,\mathrm{d}x+\int_{|z|\leqslant 1}z\tilde{J}(\mathrm{d}t,\mathrm{d}x,\mathrm{d}z)+\int_{1<|z|<N(1+|x|^{\eta})}zJ(\mathrm{d}t,\mathrm{d}x,\mathrm{d}z)\\&=:b\,\mathrm{d}t\,\mathrm{d}x+L^M(\mathrm{d}t,\mathrm{d}x)+L^P_N(\mathrm{d}t,\mathrm{d}x).\end{aligned}$$

To make the notations lighter, we suppose also in the following that $N=1$. The drift term has already been dealt with in Lemma 5.3.8, and we can suppose that $b=0$. In the following, we also define $Z(s,y)=\sigma\big(u(s,y)\big)$. By boundedness of σ, Z is a bounded random field. We separate the random field u into three parts, that are each treated separately. Let $A>0$.

$$\begin{aligned}u(t,x)&=\int_0^t\int_{\mathbb{R}^d}\rho_H(t-s,x-y)Z(s,y)\mathbb{1}_{y\in[-2A,2A]^d}L_1(\mathrm{d}s,\mathrm{d}y)\\&\quad+\int_0^t\int_{\mathbb{R}^d}\rho_H(t-s,x-y)Z(s,y)\mathbb{1}_{y\notin[-2A,2A]^d}L^M(\mathrm{d}s,\mathrm{d}y)\\&\quad+\int_0^t\int_{\mathbb{R}^d}\rho_H(t-s,x-y)Z(s,y)\mathbb{1}_{y\notin[-2A,2A]^d}L^P_1(\mathrm{d}s,\mathrm{d}y)\\&:=u^1(t,x)+u^2(t,x)+u^3(t,x).\end{aligned}$$

$\underline{u^1(t,x)}$: Since we are only integrating over a compact set in space, we can use again a stopping argument and assume that there is no jump larger than 1. Then we rewrite

$$u^1(t,x)=\int_0^t\int_{\mathbb{R}^d}\int_{\mathbb{R}}\rho_H(t-s,x-y)Z(s,y)z\mathbb{1}_{|z|\leqslant 1}\mathbb{1}_{y\in[-2A,2A]^d}\tilde{J}(\mathrm{d}s,\mathrm{d}y,\mathrm{d}z).$$

Let $\varphi\in\mathscr{S}(\mathbb{R}^d)$. Then,

$$\begin{aligned}\big\langle\mathscr{F}\left(u^1(t,\cdot)\right),\varphi\big\rangle&=\big\langle u^1(t,\cdot),\mathscr{F}\left(\varphi\right)\big\rangle=\int_{\mathbb{R}^d}u^1(t,x)\mathscr{F}(\varphi)(x)\,\mathrm{d}x\\&=\int_{\mathbb{R}^d}\left(\int_0^t\int_{\mathbb{R}^d}\int_{\mathbb{R}}\rho_H(t-s,x-y)Z(s,y)z\mathbb{1}_{|z|\leqslant 1}\mathbb{1}_{y\in[-2A,2A]^d}\tilde{J}(\mathrm{d}s,\mathrm{d}y,\mathrm{d}z)\right)\mathscr{F}(\varphi)(x)\,\mathrm{d}x.\end{aligned}$$

We would like to permute the stochastic integral and the Lebesgue integral. We proceed using a limiting argument. For any $0<\varepsilon<1$, let

$$
\begin{aligned}
u_\varepsilon^1(t,x) &:= \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}} \rho_H(t-s,x-y) Z(s,y) z \mathbb{1}_{\varepsilon<|z|\leqslant 1} \mathbb{1}_{y\in[-2A,2A]^d} \tilde{J}(\mathrm{d}s,\mathrm{d}y,\mathrm{d}z) \\
&= \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}} \rho_H(t-s,x-y) Z(s,y) z \mathbb{1}_{\varepsilon<|z|\leqslant 1} \mathbb{1}_{y\in[-2A,2A]^d} J(\mathrm{d}s,\mathrm{d}y,\mathrm{d}z) \\
&\quad - \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}} \rho_H(t-s,x-y) Z(s,y) z \mathbb{1}_{\varepsilon<|z|\leqslant 1} \mathbb{1}_{y\in[-2A,2A]^d} \,\mathrm{d}s\,\mathrm{d}y\,\nu(\mathrm{d}z)\,.
\end{aligned}
$$

Because the integration variable y is in a compact set, and because the jump sizes are bounded below by $\varepsilon>0$, the stochastic integral is a sum of an a.s. finite number of elements. Then,

$$
\begin{aligned}
&\int_{\mathbb{R}^d} u_\varepsilon^1(t,x) \mathscr{F}(\varphi)(x)\,\mathrm{d}x \\
&\quad = \int_{\mathbb{R}^d} \left(\int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}} \mathscr{F}(\varphi)(x) \rho_H(t-s,x-y) Z(s,y) z \mathbb{1}_{\varepsilon<|z|\leqslant 1} \mathbb{1}_{y\in[-2A,2A]^d} J(\mathrm{d}s,\mathrm{d}y,\mathrm{d}z) \right) \mathrm{d}x \\
&\qquad - \int_{\mathbb{R}^d} \left(\int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}} \mathscr{F}(\varphi)(x) \rho_H(t-s,x-y) Z(s,y) z \mathbb{1}_{\varepsilon<|z|\leqslant 1} \mathbb{1}_{y\in[-2A,2A]^d} \,\mathrm{d}s\,\mathrm{d}y\,\nu(\mathrm{d}z) \right) \mathrm{d}x \\
&\quad = \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}} \left(\int_{\mathbb{R}^d} \mathscr{F}(\varphi)(x) \rho_H(t-s,x-y)\,\mathrm{d}x \right) Z(s,y) z \mathbb{1}_{\varepsilon<|z|\leqslant 1} \mathbb{1}_{y\in[-2A,2A]^d} J(\mathrm{d}s,\mathrm{d}y,\mathrm{d}z) \\
&\qquad - \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}} \left(\int_{\mathbb{R}^d} \mathscr{F}(\varphi)(x) \rho_H(t-s,x-y)\,\mathrm{d}x \right) Z(s,y) z \mathbb{1}_{\varepsilon<|z|\leqslant 1} \mathbb{1}_{y\in[-2A,2A]^d} \,\mathrm{d}s\,\mathrm{d}y\,\nu(\mathrm{d}z)\,,
\end{aligned}
$$

where we used a Fubini theorem on the deterministic integral. Then,

$$
\int_{\mathbb{R}^d} \mathscr{F}(\varphi)(x) \rho_H(t-s,x-y)\,\mathrm{d}x = \int_{\mathbb{R}^d} e^{-i\xi\cdot y-(t-s)|\xi|^2} \varphi(\xi)\,\mathrm{d}\xi\,.
$$

Therefore,

$$
\int_{\mathbb{R}^d} u_\varepsilon^1(t,x) \mathscr{F}(\varphi)(x)\,\mathrm{d}x = \int_{\mathbb{R}^d} \hat{u}_\varepsilon^1(t,\xi) \varphi(\xi)\,\mathrm{d}\xi\,, \tag{5.3.13}
$$

where

$$
\hat{u}_\varepsilon^1(t,\xi) := \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}} e^{-i\xi\cdot y-(t-s)|\xi|^2} Z(s,y) z \mathbb{1}_{\varepsilon<|z|\leqslant 1} \mathbb{1}_{y\in[-2A,2A]^d} \tilde{J}(\mathrm{d}s,\mathrm{d}y,\mathrm{d}z)\,. \tag{5.3.14}
$$

Let $\beta \in \left[p\vee 1, \left(1+\frac{2}{d}\right)\wedge 2\right)$. Then, using Hölder's inequality,

$$
\begin{aligned}
&\mathbb{E}\left[\left| \int_{\mathbb{R}^d} \left(u_\varepsilon^1(t,x) - u^1(t,x) \right) \mathscr{F}(\varphi)(x)\,\mathrm{d}x \right|^\beta \right] \\
&\quad \leqslant \mathbb{E}\left[\left(\int_{\mathbb{R}^d} \left| u_\varepsilon^1(t,x) - u^1(t,x) \right| \left| \mathscr{F}(\varphi)(x) \right| \mathrm{d}x \right)^\beta \right] \\
&\quad \leqslant \mathbb{E}\left[\int_{\mathbb{R}^d} \left| u_\varepsilon^1(t,x) - u^1(t,x) \right|^\beta \left| \mathscr{F}(\varphi)(x) \right| \mathrm{d}x \left(\int_{\mathbb{R}^d} \left| \mathscr{F}(\varphi)(x) \right| \mathrm{d}x \right)^{\beta-1} \right] \\
&\quad \leqslant C \int_{\mathbb{R}^d} \mathbb{E}\left[\left| u_\varepsilon^1(t,x) - u^1(t,x) \right|^\beta \right] \left| \mathscr{F}(\varphi)(x) \right| \mathrm{d}x\,.
\end{aligned} \tag{5.3.15}
$$

Then, by Theorem A.0.1 *(ii)*, and by the boundedness of Z

$$
\begin{aligned}
&\sup_{x\in\mathbb{R}^d}\mathbb{E}\left[\left|u^1_\varepsilon(t,x)-u^1(t,x)\right|^\beta\right]\\
&\quad=\sup_{x\in\mathbb{R}^d}\mathbb{E}\left[\left|\int_0^t\int_{\mathbb{R}^d}\int_{\mathbb{R}}\rho_H(t-s,x-y)Z(s,y)z\mathbb{1}_{|z|\leqslant\varepsilon}\mathbb{1}_{y\in[-2A,2A]^d}\tilde{J}(\mathrm{d}s,\mathrm{d}y,\mathrm{d}z)\right|^\beta\right]\\
&\quad\leqslant C\sup_{x\in\mathbb{R}^d}\int_0^t\int_{\mathbb{R}^d}\int_{\mathbb{R}}\mathbb{E}\left[\left|\rho_H(t-s,x-y)Z(s,y)z\mathbb{1}_{|z|\leqslant\varepsilon}\mathbb{1}_{y\in[-2A,2A]^d}\right|^\beta\right]\mathrm{d}s\,\mathrm{d}y\,\nu(\mathrm{d}z)\\
&\quad\leqslant C\left(\int_0^t\int_{\mathbb{R}^d}\rho_H(s,y)^\beta\,\mathrm{d}s\,\mathrm{d}y\right)\left(\int_{|z|\leqslant\varepsilon}|z|^\beta\nu(\mathrm{d}z)\right).
\end{aligned}
\tag{5.3.16}
$$

Since $\beta<1+\frac{2}{d}$, $\int_0^t\int_{\mathbb{R}^d}\rho_H(s,y)^\beta\,\mathrm{d}s\,\mathrm{d}y<+\infty$, and since $\beta\geqslant p$, $\int_{|z|\leqslant\varepsilon}|z|^\beta\nu(\mathrm{d}z)\to0$ as $\varepsilon\to0$. We deduce that

$$\sup_{x\in\mathbb{R}^d}\mathbb{E}\left[\left|u^1_\varepsilon(t,x)-u^1(t,x)\right|^\beta\right]\to0,\qquad\text{as }\varepsilon\to0.\tag{5.3.17}$$

Using (5.3.17) in (5.3.15), we deduce that for any $t\in[0,T]$,

$$\int_{\mathbb{R}^d}u^1_\varepsilon(t,x)\mathscr{F}(\varphi)(x)\,\mathrm{d}x\to\int_{\mathbb{R}^d}u^1_\varepsilon(t,x)\mathscr{F}(\varphi)(x)\,\mathrm{d}x,\qquad\text{as }\varepsilon\to0,\text{ in }L^\beta(\Omega).\tag{5.3.18}$$

Similar to (5.3.14), we define

$$\hat{u}^1(t,\xi):=\int_0^t\int_{\mathbb{R}^d}\int_{\mathbb{R}}e^{-i\xi\cdot y-(t-s)|\xi|^2}Z(s,y)z\mathbb{1}_{|z|\leqslant1}\mathbb{1}_{y\in[-2A,2A]^d}\tilde{J}(\mathrm{d}s,\mathrm{d}y,\mathrm{d}z).$$

Replacing $\rho_H(t-s,x-y)$ by $e^{-i\xi\cdot y-(t-s)|\xi|^2}$ in (5.3.16), we get similarly

$$
\begin{aligned}
&\sup_{\xi\in\mathbb{R}^d}\mathbb{E}\left[\left|\hat{u}^1_\varepsilon(t,\xi)-\hat{u}^1(t,\xi)\right|^\beta\right]\\
&\quad\leqslant C\sup_{\xi\in\mathbb{R}^d}\int_0^t\int_{\mathbb{R}^d}\int_{\mathbb{R}}\mathbb{E}\left[\left|e^{-i\xi\cdot y-(t-s)|\xi|^2}Z(s,y)z\mathbb{1}_{|z|\leqslant\varepsilon}\mathbb{1}_{y\in[-2A,2A]^d}\right|^\beta\right]\mathrm{d}s\,\mathrm{d}y\,\nu(\mathrm{d}z)\\
&\quad\leqslant C\left(\int_0^t\int_{\mathbb{R}^d}\mathbb{1}_{y\in[-2A,2A]^d}\,\mathrm{d}s\,\mathrm{d}y\right)\left(\int_{|z|\leqslant\varepsilon}|z|^\beta\nu(\mathrm{d}z)\right).
\end{aligned}
$$

We deduce that, as for (5.3.18),

$$\int_{\mathbb{R}^d}\hat{u}^1_\varepsilon(t,\xi)\varphi(\xi)\,\mathrm{d}x\to\int_{\mathbb{R}^d}\hat{u}^1_\varepsilon(t,\xi)\varphi(\xi)\,\mathrm{d}\xi,\qquad\text{as }\varepsilon\to0,\text{ in }L^\beta(\Omega).\tag{5.3.19}$$

Using (5.3.13), (5.3.18) and (5.3.19), and by uniqueness of the limit, we deduce that for any $t\in[0,T]$, almost surely,

$$\int_{\mathbb{R}^d}u^1(t,x)\mathscr{F}(\varphi)(x)\,\mathrm{d}x=\int_{\mathbb{R}^d}\hat{u}^1(t,\xi)\varphi(\xi)\,\mathrm{d}\xi.$$

In particular,

$$\begin{aligned}\mathscr{F}\left(u^1(t,\cdot)\right)(\xi) &= e^{-|\xi|^2 t}\int_0^t\int_{\mathbb{R}^d}\int_{\mathbb{R}} e^{-i\xi\cdot y}e^{s|\xi|^2}Z(s,y)z\mathbb{1}_{|z|\leqslant 1}\mathbb{1}_{y\in[-2A,2A]^d}\tilde{J}(\mathrm{d}s,\mathrm{d}y,\mathrm{d}z)\\ &=: a_\xi(t)\,,\end{aligned} \tag{5.3.20}$$

Remark 5.3.10. *This calculation for the Fourier transform of $x\mapsto u^1(t,x)$ is still valid if Z is not necessarily bounded, but has locally uniformly bounded moments in space and time of order β.*

We can then define, for $0\leqslant a<b\leqslant T$ and $\xi\in\mathbb{R}^d$,

$$I_a^b(\xi) := \int_a^b\int_{\mathbb{R}^d}\int_{\mathbb{R}} e^{-i\xi\cdot y}e^{s|\xi|^2}Z(s,y)z\mathbb{1}_{|z|\leqslant 1}\mathbb{1}_{y\in[-2A,2A]^d}\tilde{J}(\mathrm{d}s,\mathrm{d}y,\mathrm{d}z)\,.$$

In particular,

$$\left\|u^1(t+h,\cdot)-u^1(t,\cdot)\right\|^2_{H_r(\mathbb{R}^d)} = \int_{\mathbb{R}^d}(1+|\xi|^2)^r\left|a_\xi(t+h)-a_\xi(t)\right|^2\mathrm{d}\xi\,,$$

and

$$\left\|u^1(t-h,\cdot)-u^1(t,\cdot)\right\|^2_{H_r(\mathbb{R}^d)} = \int_{\mathbb{R}^d}(1+|\xi|^2)^r\left|a_\xi(t-h)-a_\xi(t)\right|^2\mathrm{d}\xi\,.$$

The function $t\mapsto e^{-|\xi|^2 t}$ is continuous, and the stochastic integral in $a_\xi(t)$ exists in $L^2(\Omega)$. Therefore, $t\mapsto a_\xi(t)$ is continuous in $L^2(\Omega)$. Furthermore,

$$\mathbb{E}\left[\left|a_\xi(t)\right|^2\right]\leqslant C\frac{1-e^{-2|\xi|^2 t}}{2|\xi|^2}\leqslant C\,,$$

for some constant C that does not depend on ξ. Therefore, by the dominated convergence theorem,

$$\mathbb{E}\left[\left\|u^1(t+h,\cdot)-u^1(t,\cdot)\right\|^2_{H_r(\mathbb{R}^d)}\right]\to 0\,,\qquad \text{as } h\to 0\,,$$

and the process $t\mapsto u(t,\cdot)$ is continuous in $L^2(\Omega)$ (and therefore in probability) as a process with values in $H_r(\mathbb{R}^d)$.

We now realize that the situation is very similar to the case of the equation on the bounded interval $[0,\pi]$. In fact, we can do the same estimates that are carried out in the proof of Proposition 5.2.5 for the case of the equation on a bounded interval in dimension 1, using the following replacements:

$$\begin{aligned}[0,\pi] &\leftrightarrow [-2A,2A]^d\,,\\ k &\leftrightarrow \xi\,,\\ \sin(ky) &\leftrightarrow e^{-i\xi\cdot y}\,.\end{aligned}$$

By following exactly the proof of the case of a bounded interval in space as in (5.2.21), we deduce that

$$
\begin{aligned}
&\|u^1(t+h,\cdot)-u^1(t,\cdot)\|^2_{H_r(\mathbb{R}^d)}\|u^1(t-h,\cdot)-u^1(t,\cdot)\|^2_{H_r(\mathbb{R}^d)}\\
&\qquad\leqslant C\int_{\mathbb{R}^d}\mathrm{d}\xi_1\int_{\mathbb{R}^d}\mathrm{d}\xi_2\left(1+|\xi_1|^2\right)^r\left(1+|\xi_2|^2\right)^r\left(A_1(\xi_1,\xi_2)^2+A_2(\xi_1,\xi_2)^2\right.\\
&\qquad\qquad\left.+A_3(\xi_1,\xi_2)^2+A_4(\xi_1,\xi_2)^2\right),
\end{aligned}
$$

for some constant C, and where

$$
\begin{aligned}
A_1(\xi_1,\xi_2)&:=e^{-\xi_2^2 t}e^{-\xi_1^2(t-h)}(1-e^{-\xi_2^2 h})(e^{-\xi_1^2 h}-1)I_0^t(\xi_2)I_0^{t-h}(\xi_1),\\
A_2(\xi_1,\xi_2)&:=e^{-\xi_2^2 t}e^{-\xi_1^2(t-h)}(1-e^{-\xi_2^2 h})e^{-\xi_1^2 h}I_0^t(\xi_2)I_{t-h}^t(\xi_1),\\
A_3(\xi_1,\xi_2)&:=e^{-\xi_2^2 t}e^{-\xi_1^2(t-h)}e^{-\xi_2^2 h}(e^{-\xi_1^2 h}-1)I_t^{t+h}(\xi_2)I_0^{t-h}(\xi_1),\\
A_4(\xi_1,\xi_2)&:=e^{-\xi_2^2 t}e^{-\xi_1^2(t-h)}e^{-\xi_2^2 h}e^{-\xi_1^2 h}I_t^{t+h}(\xi_2)I_{t-h}^t(\xi_1).
\end{aligned}
$$

By the same arguments, we deduce that

$$
\begin{aligned}
\mathbb{E}\left[|A_1(\xi_1,\xi_2)|^2\right]&\leqslant C\left(h^2+\xi_1^{2\delta}\xi_2^{2\delta}h^{1+2\delta}\right),\\
\mathbb{E}\left[|A_2(\xi_1,\xi_2)|^2\right]&\leqslant C\xi_2^{2\delta}h^{1+\delta},\\
\mathbb{E}\left[|A_3(\xi_1,\xi_2)|^2\right]&\leqslant Ch^2,\\
\mathbb{E}\left[|A_4(\xi_1,\xi_2)|^2\right]&\leqslant Ch^2,
\end{aligned}
$$

where $\delta>0$ can be chosen in the range $\left(0,\frac{3}{2}\right)$. Then, choosing $\delta>0$ such that $\delta+r<-\frac{d}{2}$ (which is possible since $r<-\frac{d}{2}$), we deduce that

$$
\mathbb{E}\left[\|u^1(t+h,\cdot)-u^1(t,\cdot)\|^2_{H_r(\mathbb{R}^d)}\|u^1(t-h,\cdot)-u^1(t,\cdot)\|^2_{H_r(\mathbb{R}^d)}\right]\leqslant Ch^{1+\delta}.
$$

By Theorem A.0.3, we deduce that $(u^1(t,\cdot))_{t\geqslant 0}$ has a càdlàg version in $H_r(\mathbb{R}^d)$ for any $r<-\frac{d}{2}$.

$\underline{u^2(t,x):}$ Let $x\in[-A,A]$. Since $y\notin[-2A,2A]^d$, the heat kernel is smooth. Let $f:\mathbb{R}^d\to\mathbb{R}$ be a smooth function. Then, for any $a,b\in\mathbb{R}^d$ such that $a_i\leqslant b_i$ for all $1\leqslant i\leqslant d$,

$$
\begin{aligned}
f(b_1,\dots,b_d)&=f(a_1,\dots,a_d)\\
&\qquad+\sum_{i=1}^d\sum_{1\leqslant k_1<\cdots<k_i\leqslant d}\int_{a_{k_1}}^{b_{k_1}}\mathrm{d}r_{k_1}\cdots\int_{a_{k_i}}^{b_{k_i}}\mathrm{d}r_{k_i}\partial_{x_{k_1}}\dots\partial_{x_{k_i}}f(c_{\mathbf{k}}(a,r)),
\end{aligned}
\tag{5.3.21}
$$

where for any $\mathbf{k}=(k_1,\dots,k_i)$, with $k_1<\cdots<k_i$, we define $(c_{\mathbf{k}}(a,r))_j:=a_i\mathbb{1}_{j\notin\mathbf{k}}+r_j\mathbb{1}_{j\in\mathbf{k}}$, where $1\leqslant j\leqslant d$. This formula is easily proved by induction on the dimension. Then, using this

formula (5.3.21) with $a = (s, -A, \dots, -A)$ and $b = (t, x)$, we get

$$\begin{aligned}
&\rho_H(t-s, x-y) = \\
&\quad 0 + \sum_{i=1}^{d+1} \sum_{1 \leqslant k_1 < \cdots < k_i \leqslant d+1} \int_{a_{k_1}}^{b_{k_1}} \mathrm{d}r_{k_1} \cdots \int_{a_{k_i}}^{b_{k_i}} \mathrm{d}r_{k_i} \partial_{x_{k_1}} \dots \partial_{x_{k_i}} \rho_H \left(c_{\mathbf{k}}(a, r) - (s, y)\right) \\
&\quad = \sum_{i=1}^{d} \sum_{1 \leqslant k_1 < \cdots < k_i \leqslant d} \int_s^t \mathrm{d}u \int_{-A}^{x_{k_1}} \mathrm{d}r_{k_1} \cdots \int_{-A}^{x_{k_i}} \mathrm{d}r_{k_i} \partial_{x_{k_1}} \dots \partial_{x_{k_i}} \partial_t \rho_H \left(u - s, c_{\mathbf{k}}(-\mathbf{A}, r) - y\right),
\end{aligned} \tag{5.3.22}$$

where $\mathbf{A} := (A, \dots, A)$. Using (5.3.22), we have

$$\begin{aligned}
u^2(t, x) &= \int_0^t \int_{\mathbb{R}^d} \rho_H(t-s, x-y) Z(s, y) \mathbb{1}_{y \notin [-2A, 2A]^d} L^M(\mathrm{d}s, \mathrm{d}y) \\
&= \sum_{i=1}^{d} \sum_{1 \leqslant k_1 < \cdots < k_i \leqslant d} \int_0^t \mathrm{d}u \int_{-A}^{x_{k_1}} \mathrm{d}r_{k_1} \cdots \int_{-A}^{x_{k_i}} \mathrm{d}r_{k_i} \\
&\qquad \left(\int_0^u \int_{\mathbb{R}^d} \partial_{x_{k_1}} \dots \partial_{x_{k_i}} \partial_t \rho_H \left(u - s, c_{\mathbf{k}}(-\mathbf{A}, r) - y\right) Z(s, y) \mathbb{1}_{y \notin [-2A, 2A]^d} L^M(\mathrm{d}s, \mathrm{d}y) \right),
\end{aligned}$$

where we have used Theorem A.0.2, since Z is bounded, $\int_{|z| \leqslant 1} |z|^2 \nu(\mathrm{d}z) < +\infty$, and since the heat kernel is smooth for $|x-y| \geqslant A$. We see from this expression that u^2 is jointly continuous in (t, x). By the argument at the end of the proof of Lemma 5.3.8, we deduce that $t \mapsto u^2(t, \cdot) \mathbb{1}_{[-A, A]^d}$ is continuous in $H_r(\mathbb{R}^d)$ for every $r \leqslant 0$.

$\underline{u^3(t, x):}$ This process takes into account only the jumps that are far away from x, but that can be arbitrarily large. We can write u^3 as a sum:

$$\begin{aligned}
u^3(t, x) &= \int_0^t \int_{\mathbb{R}^d} \rho_H(t-s, x-y) Z(s, y) \mathbb{1}_{y \notin [-2A, 2A]^d} L_1^P(\mathrm{d}s, \mathrm{d}y) \\
&= \sum_{i \geqslant 1} \rho_H(t - T_i, x - X_i) Z(T_i, X_i) Z_i \mathbb{1}_{X_i \notin [-2A, 2A]^d, 1 < |Z_i| < 1 + |X_i|^\eta, T_i \leqslant t}.
\end{aligned}$$

We first realize that each term of this sum is jointly continuous in $(t, x) \in [0, T] \times [-A, A]^d$. We show that this sum converges uniformly in $(t, x) \in [0, T] \times [-A, A]^d$. We choose A such that $T < \frac{A^2}{2d}$. Then, by Lemma A.0.5, the maximum of the function $t \mapsto \rho_H(t, x - X_i)$ is attained at $t = T$, since $|x - X_i| > A$.

$$\begin{aligned}
\sup_{t \leqslant T, x \in [-A, A]^d} \rho_H(t - T_i, x - X_i) &\leqslant \sup_{x \in [-A, A]^d} \frac{C}{T^{\frac{d}{2}}} e^{-\frac{|x - X_i|^2}{4T}} \\
&\leqslant \frac{C}{T^{\frac{d}{2}}} e^{-\frac{|p_A(X_i) - X_i|^2}{4T}},
\end{aligned}$$

where p_A is the projection on the convex set $[-A,A]^d$. Then, let $\beta = 1\wedge q$. We have

$$\begin{aligned}
&\mathbb{E}\left[\left(\sum_{i\geqslant 1}\sup_{t\leqslant T, x\in[-A,A]^d}\left|\rho_H(t-T_i,x-X_i)Z(T_i,X_i)Z_i\mathbb{1}_{X_i\notin[-2A,2A]^d,\,1<|Z_i|<1+|X_i|^\eta,\,T_i\leqslant t}\right|\right)^\beta\right]\\
&\qquad\leqslant \frac{C}{T^{\frac{\beta d}{2}}}\mathbb{E}\left[\left(\sum_{i\geqslant 1}\left|e^{-\frac{|p_A(X_i)-X_i|^2}{4T}}Z_i\mathbb{1}_{X_i\notin[-2A,2A]^d,\,1<|Z_i|,\,T_i\leqslant T}\right|\right)^\beta\right]\\
&\qquad\leqslant \frac{C}{T^{\frac{\beta d}{2}}}\mathbb{E}\left[\sum_{i\geqslant 1}\left|e^{-\frac{|p_A(X_i)-X_i|^2}{4T}}Z_i\mathbb{1}_{X_i\notin[-2A,2A]^d,\,1<|Z_i|,\,T_i\leqslant T}\right|^\beta\right]\\
&\qquad\leqslant C\int_0^T\int_{y\notin[-2A,2A]^d}\int_{|z|>1}|z|^\beta e^{-\beta\frac{|p_A(y)-y|^2}{4T}}\,\mathrm{d}s\,\mathrm{d}y\,\nu(\mathrm{d}z)<+\infty.
\end{aligned}$$

Therefore, the sum defining u^3 converges uniformly in $(t,x)\in[0,T]\times[-A,A]^d$, and u^3 is jointly continuous. By the argument at the end of the proof of Lemma 5.3.8, we deduce that $t\mapsto u^3(t,\cdot)$ is continuous in $H_{r,\mathrm{loc}}(\mathbb{R}^d)$ for every $r\leqslant 0$.

Finally since the choice of A was arbitrary, we conclude that $t\mapsto u(t,\cdot)=u^1(t,\cdot)+u^2(t,\cdot)+u^3(t,\cdot)$ has a càdlàg version in $H_{r,\mathrm{loc}}(\mathbb{R}^d)$ for every $r<-\frac{d}{2}$. □

The case where σ is unbounded

We can extend Proposition 5.3.9 to the case where σ is unbounded but satisfies some additional assumptions. We start with a result similar to Lemma 5.3.8 to deal with the drift part of the solution u.

Lemma 5.3.11. *Let Z be a random field such that for some $\varepsilon>0$,*

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d}\mathbb{E}\left[|Z(t,x)|^{2+\varepsilon}\right]<+\infty.$$

Let

$$F(t,x):=\int_0^t\int_{\mathbb{R}^d}\rho_H(t-s,x-y)Z(s,y)\,\mathrm{d}s\,\mathrm{d}y.$$

Then the process $t\mapsto F(t,\cdot)$ is continuous in $H_{r,loc}(\mathbb{R}^d)$ for any $r<-\frac{d}{2}$.

Proof. Let $A>0$. We can write

$$\begin{aligned}
F(t,x)&=\int_0^t\int_{\mathbb{R}^d}\rho_H(t-s,x-y)Z(s,y)\mathbb{1}_{y\in[-2A,2A]^d}\,\mathrm{d}s\,\mathrm{d}y\\
&\qquad+\int_0^t\int_{\mathbb{R}^d}\rho_H(t-s,x-y)Z(s,y)\mathbb{1}_{y\notin[-2A,2A]^d}\,\mathrm{d}s\,\mathrm{d}y\\
&=F_1(t,x)+F_2(t,x).
\end{aligned}$$

We deal with each term separately.

$\underline{F_1(t,x)}$: For $\xi \in \mathbb{R}^d$,

$$\begin{aligned}\mathscr{F}(F_1(t,\cdot))(\xi) &= \int_{\mathbb{R}^d} e^{-i\xi\cdot x}\left(\int_0^t\int_{\mathbb{R}^d}\rho_H(t-s,x-y)Z(s,y)\mathbb{1}_{y\in[-2A,2A]^d}\,\mathrm{d}s\,\mathrm{d}y\right)\mathrm{d}x\\ &= \int_0^t\int_{\mathbb{R}^d} e^{-i\xi\cdot y-|\xi|^2(t-s)}Z(s,y)\mathbb{1}_{y\in[-2A,2A]^d}\,\mathrm{d}s\,\mathrm{d}y.\end{aligned}$$

Then, we define

$$F_{1,n}(t,x) = \int_0^t\int_{y\in[-2A,2A]^d}\rho_H(t-s,x-y)\mathbb{1}_{y\in[-2A,2A]^d}Z_n(s,y)\,\mathrm{d}s\,\mathrm{d}y,$$

where $Z_n(s,y) = Z(s,y)\mathbb{1}_{|Z(s,y)|\leqslant n}$. Then, for $(t,x)\in[0,T]\times\mathbb{R}^d$,

$$F_1(t,x)-F_{1,n}(t,x) = \int_0^t\int_{y\in[-2A,2A]^d}\rho_H(t-s,x-y)\left(Z(s,y)-Z_n(s,y)\right)\mathrm{d}s\,\mathrm{d}y,$$

and

$$\|F_1(t,\cdot)-F_{1,n}(t,\cdot)\|_{H_r}^2 = \int_{\mathbb{R}^d}(1+|\xi|^2)^r\left|\mathscr{F}\left(F_1(t,\cdot)-F_{1,n}(t,\cdot)\right)(\xi)\right|^2\mathrm{d}\xi.$$

Then,

$$\mathscr{F}\left(F_1(t,\cdot)-F_{1,n}(t,\cdot)\right)(\xi) = \int_0^t\int_{y\in[-2A,2A]^d}e^{-i\xi\cdot y}e^{-(t-s)|\xi|^2}Z_{(n)}(s,y)\,\mathrm{d}s\,\mathrm{d}y,$$

where $Z_{(n)} := Z\mathbb{1}_{|Z|>n}$. Then,

$$e^{-|\xi|^2(t-s)} = 1-\int_s^t|\xi|^2e^{-|\xi|^2(t-r)}\,\mathrm{d}r, \tag{5.3.23}$$

and by Theorem A.0.2, we get

$$\begin{aligned}\mathscr{F}\left(F_1(t,\cdot)-F_{1,n}(t,\cdot)\right)(\xi) &= \int_0^t\int_{y\in[-2A,2A]^d}e^{-i\xi\cdot y}Z_{(n)}(s,y)\,\mathrm{d}s\,\mathrm{d}y\\ &\quad-\int_0^t\int_{y\in[-2A,2A]^d}e^{-i\xi\cdot y}\left(\int_s^t|\xi|^2e^{-|\xi|^2(t-r)}\,\mathrm{d}r\right)Z_{(n)}(s,y)\,\mathrm{d}s\,\mathrm{d}y\\ &= \int_0^t\int_{y\in[-2A,2A]^d}e^{-i\xi\cdot y}Z_{(n)}(s,y)\,\mathrm{d}s\,\mathrm{d}y\\ &\quad-\int_0^t\left(\int_0^r\int_{y\in[-2A,2A]^d}e^{-i\xi\cdot y}|\xi|^2e^{-|\xi|^2(t-r)}Z_{(n)}(s,y)\,\mathrm{d}s\,\mathrm{d}y\right)\mathrm{d}r.\end{aligned}$$

Therefore,

$$\begin{aligned}\mathbb{E}\left[\left|\mathscr{F}\left(F_1(t,\cdot)-F_{1,n}(t,\cdot)\right)(\xi)\right|\right] &\leqslant \mathbb{E}\left[\left|\int_0^t\int_{y\in[-2A,2A]^d}e^{-i\xi\cdot y}Z_{(n)}(s,y)\,\mathrm{d}s\,\mathrm{d}y\right|\right]\\ &\quad+\mathbb{E}\left[\int_0^t|\xi|^2e^{-|\xi|^2(t-r)}\left|\int_0^r\int_{y\in[-2A,2A]^d}e^{-i\xi\cdot y}Z_{(n)}(s,y)\,\mathrm{d}s\,\mathrm{d}y\right|\mathrm{d}r\right]\end{aligned}$$

$$
\begin{aligned}
&\leqslant \mathbb{E}\left[\left|\int_0^t \int_{y\in[-2A,2A]^d} e^{-i\xi\cdot y} Z_{(n)}(s,y)\,\mathrm{d}s\,\mathrm{d}y\right|\right] \\
&\qquad + \mathbb{E}\left[\sup_{r\in[0,t]}\left|\int_0^r \int_{y\in[-2A,2A]^d} e^{-i\xi\cdot y} Z_{(n)}(s,y)\,\mathrm{d}s\,\mathrm{d}y\right|\right] \\
&\qquad\qquad \times\left(\int_0^t |\xi|^2 e^{-|\xi|^2(t-r)}\,\mathrm{d}r\right) \\
&\leqslant C \sup_{(s,y)\in[0,T]\times[-2A,2A]^d} \mathbb{E}\left[Z_{(n)}^2(s,y)\right].
\end{aligned}
$$

Furthermore, by Hölder's inequality and Markov's inequality,

$$
\begin{aligned}
\mathbb{E}\left[Z_{(n)}^2(s,y)\right] &= \mathbb{E}\left[Z(s,y)^2 \mathbb{1}_{|Z(s,y)|>n}\right] \\
&\leqslant \mathbb{E}\left[\left|Z(s,y)\right|^{2+\varepsilon}\right]^{\frac{2}{2+\varepsilon}} \left[\mathbb{P}\left(|Z(s,y)|>n\right)\right]^{\frac{\varepsilon}{2+\varepsilon}} \\
&\leqslant \mathbb{E}\left[\left|Z(s,y)\right|^{2+\varepsilon}\right]^{\frac{2}{2+\varepsilon}} \frac{\mathbb{E}\left[\left|Z(s,y)\right|\right]^{\frac{\varepsilon}{2+\varepsilon}}}{n^{\frac{\varepsilon}{2+\varepsilon}}}.
\end{aligned}
$$

We deduce that

$$
\sup_{(s,y)\in[0,T]\times[-2A,2A]^d} \mathbb{E}\left[Z_{(n)}^2(s,y)\right] \leqslant \frac{C}{n^{\frac{\varepsilon}{2+\varepsilon}}}, \tag{5.3.24}
$$

for some constant C. By (5.3.24), we deduce that

$$
\mathbb{E}\left[\sup_{t\in[0,T]}\left|\mathscr{F}\left(F_1(t,\cdot)-F_{1,n}(t,\cdot)\right)(\xi)\right|^2\right] \leqslant \frac{C}{n^{\frac{\varepsilon}{2+\varepsilon}}}.
$$

We deduce that for any $r<-\frac{d}{2}$,

$$
\mathbb{E}\left[\sup_{t\in[0,T]}\|F_1(t,\cdot)-F_{1,n}(t,\cdot)\|_{H_r}^2\right] \leqslant \frac{C}{n^{\frac{q\varepsilon}{2+\varepsilon}}} \to 0 \qquad \text{as } n\to+\infty.
$$

Since Z_n is bounded, we can apply Lemma 5.3.8 to $F_{1,n}$, and we deduce that $t\mapsto F_{1,n}(t,\cdot)$ is continuous in $H_{r,\mathrm{loc}}(\mathbb{R}^d)$ for any $r<-\frac{d}{2}$. Then, $\sup_{t\in[0,T]}\|F_1(t,\cdot)-F_{1,n}(t,\cdot)\|_{H_r(\mathbb{R}^d)}\to 0$ in $L^2(\Omega)$ as $n\to+\infty$, and there is a subsequence $(n_k)_{k\geqslant 0}$ such that $\sup_{t\in[0,T]}\|F_1(t,\cdot)-F_{1,n_k}(t,\cdot)\|_{H_r(\mathbb{R}^d)}\to 0$ almost surely as $k\to+\infty$. This means that $F_{1,n_k}(t,\cdot)$ converges to $F_1(t,\cdot)$ in $H_r(\mathbb{R}^d)$ uniformly in time for any $r<-\frac{d}{2}$. Therefore, $t\mapsto F_1(t,\cdot)$ is continuous in $H_{r,\mathrm{loc}}(\mathbb{R}^d)$ for any $r<-\frac{d}{2}$.

$\underline{F_2(t,x)}$: We prove that the function $(t,x)\mapsto F_2(t,x)$ is jointly continuous on $[0,T]\times[-A,A]^d$. Indeed, since $x\in[-A,A]^d$ and $y\notin[-2A,2A]^d$, the heat kernel is smooth on the domain of integration. By (5.3.22),

$$
\begin{aligned}
&\rho_H(t-s,x-y) = \\
&\quad = \sum_{i=1}^d \sum_{1\leqslant k_1<\cdots<k_i\leqslant d} \int_s^t \mathrm{d}u \int_{-A}^{x_{k_1}} \mathrm{d}r_{k_1}\cdots\int_{-A}^{x_{k_i}} \mathrm{d}r_{k_i}\partial_{x_{k_1}}\ldots\partial_{x_{k_i}}\partial_t\rho_H\left(u-s,c_{\mathbf{k}}(-\mathbf{A},r)-y\right),
\end{aligned}
$$

where $\mathbf{A} := (A, \dots, A)$. Therefore, we have

$$
\begin{aligned}
F_2(t,x) &= \int_0^t \int_{\mathbb{R}^d} \rho_H(t-s,x-y) Z(s,y) \mathbb{1}_{y\notin[-2A,2A]^d} \,\mathrm{d}s\,\mathrm{d}y \\
&= \sum_{i=1}^d \sum_{1\leqslant k_1<\dots<k_i\leqslant d} \int_0^t \mathrm{d}u \int_{-A}^{x_{k_1}} \mathrm{d}r_{k_1} \cdots \int_{-A}^{x_{k_i}} \mathrm{d}r_{k_i} \\
&\qquad \left(\int_0^u \int_{\mathbb{R}^d} \partial_{x_{k_1}} \dots \partial_{x_{k_i}} \partial_t \rho_H \left(u-s, c_{\mathbf{k}}(-\mathbf{A}, r) - y\right) Z(s,y) \mathbb{1}_{y\notin[-2A,2A]^d} \,\mathrm{d}s\,\mathrm{d}y \right),
\end{aligned}
$$

where we have used a Fubini theorem, since Z has uniformly bounded moments of order 2, and ρ_H and all its derivatives are smooth and have exponential decay at infinity. Therefore, $(t,x) \mapsto F_2(t,x)$ is jointly continuous on $[0,T] \times [-A,A]^d$. By the argument at the end of Lemma 5.3.8, we conclude that $t \mapsto F_2(t,\cdot)$ is continuous in $H_{r,\mathrm{loc}}(\mathbb{R}^d)$.

□

Theorem 5.3.12. *Let L be a pure jump Lévy white noise with Lévy measure ν and jump measure J such that **(H4)** is satisfied for some $p, q \in \mathbb{R}$. Let σ be a Lipschitz continuous function. Assume also that there is $\gamma > 0$ with $2\gamma < q$ such that $|\sigma(x)| \leqslant C(1+|x|^\gamma)$, for all $x \in \mathbb{R}$. Let u be the mild solution to the stochastic heat equation* (5.3.1) *in $\mathbb{R}^d$ constructed in Proposition 5.3.1. Then, for any $r < -\frac{d}{2}$, the stochastic process $(u(t,\cdot))_{t\geqslant 0}$ has a càdlàg version in $H_{r,loc}(\mathbb{R}^d)$.*

Proof. Again, by the stopping time argument just after the proof Proposition 5.3.1, we can suppose that u is the solution to (5.3.7) with $N = 1$. By [14, Theorem 3.8], and since $|\sigma(x)| \leqslant C(1+|x|^\gamma)$, for some $\varepsilon > 0$ with $(2+\varepsilon)\gamma \leqslant q$,

$$
\sup_{t\in[0,T],x\in\mathbb{R}^d} \mathbb{E}\left[|\sigma(u(t,x))|^{2+\varepsilon}\right] \leqslant C\left(1+\mathbb{E}\left[|u(t,x)|^{(2+\varepsilon)\gamma}\right]\right) \leqslant C\left(1+\mathbb{E}\left[|u(t,x)|^q\right]\right) < +\infty. \quad (5.3.25)
$$

The drift part has already been taken care of in Lemma 5.3.11, so we can suppose that $b = 0$. Then, looking at the proof of the joint continuity of u^2 and u^3 in the proof of Proposition 5.3.9, we realize that we only need that $Z(t,x) = \sigma(u(t,x))$ have uniformly bounded moments of order 2. Indeed, this condition is needed to apply the stochastic Fubini Theorem A.0.2. Therefore, we can restrict to studying the regularity of the processes u^1, where we recall that

$$
u^1(t,x) = \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}} \rho_H(t-s,x-y) \sigma\left(u(s,y)\right) z \mathbb{1}_{|z|\leqslant 1} \mathbb{1}_{y\in[-2A,2A]^d} \tilde{J}(\mathrm{d}s,\mathrm{d}y,\mathrm{d}z).
$$

Let $\sigma_n(u) = \sigma(u)\mathbb{1}_{|u|\leqslant n}$. We define

$$
u_n^1(t,x) = \int_0^t \int_{y\in[-2A,2A]^d} \rho_H(t-s,x-y) \sigma_n(u(s,y)) L^M(\mathrm{d}s,\mathrm{d}y).
$$

Then, for $(t,x) \in [0,T] \times \mathbb{R}^d$,

$$
u^1(t,x) - u_n^1(t,x) = \int_0^t \int_{y\in[-2A,2A]^d} \rho_H(t-s,x-y) \left(\sigma(u(s,y)) - \sigma_n(u(s,y))\right) L^M(\mathrm{d}s,\mathrm{d}y),
$$

and

$$\|u^1(t,\cdot)-u^1_n(t,\cdot)\|^2_{H_r(\mathbb{R}^d)}=\int_{\mathbb{R}^d}(1+|\xi|^2)^r\left|\mathscr{F}\left(u^1(t,\cdot)-u^1_n(t,\cdot)\right)(\xi)\right|^2\mathrm{d}\xi\,. \tag{5.3.26}$$

For conciseness of the notation, we write $\sigma(u(s,y))-\sigma_n(u(s,y))=\sigma_{(n)}(s,y)$. By Remark 5.3.10, similarly to (5.3.20),

$$\mathscr{F}\left(u^1(t,\cdot)-u^1_n(t,\cdot)\right)(\xi)=\int_0^t\int_{y\in[-2A,2A]^d}\int_{\mathbb{R}}e^{-i\xi\cdot y}e^{-(t-s)|\xi|^2}\sigma_{(n)}(s,y)z\mathbb{1}_{|z|\leqslant 1}\tilde{J}(\mathrm{d}s,\mathrm{d}y,\mathrm{d}z)$$

Then, using (5.3.23), and Theorem A.0.2, we get

$$\begin{aligned}\mathscr{F}\left(u^1(t,\cdot)-u^1_n(t,\cdot)\right)(\xi)&=\int_0^t\int_{y\in[-2A,2A]^d}e^{-i\xi\cdot y}\sigma_{(n)}(s,y)L^M(\mathrm{d}s,\mathrm{d}y)\\&\quad-\int_0^t\int_{y\in[-2A,2A]^d}e^{-i\xi\cdot y}\left(\int_s^t|\xi|^2e^{-|\xi|^2(t-r)}\,\mathrm{d}r\right)\sigma_{(n)}(s,y)L^M(\mathrm{d}s,\mathrm{d}y)\\&=\int_0^t\int_{y\in[-2A,2A]^d}e^{i\xi\cdot y}\sigma_{(n)}(s,y)L^M(\mathrm{d}s,\mathrm{d}y)\\&\quad-\int_0^t\left(\int_0^r\int_{y\in[-2A,2A]^d}e^{-i\xi\cdot y}|\xi|^2e^{-|\xi|^2(t-r)}\sigma_{(n)}(s,y)L^M(\mathrm{d}s,\mathrm{d}y)\right)\mathrm{d}r\,.\end{aligned}$$

Therefore,

$$\begin{aligned}\left|\mathscr{F}\left(u^1(t,\cdot)-u^1_n(t,\cdot)\right)(\xi)\right|&\leqslant\left|\int_0^t\int_{y\in[-2A,2A]^d}e^{-i\xi\cdot y}\sigma_{(n)}(s,y)L^M(\mathrm{d}s,\mathrm{d}y)\right|\\&\quad+\int_0^t|\xi|^2e^{-|\xi|^2(t-r)}\left|\int_0^r\int_{y\in[-2A,2A]^d}e^{-i\xi\cdot y}\sigma_{(n)}(s,y)L^M(\mathrm{d}s,\mathrm{d}y)\right|\mathrm{d}r\\&\leqslant\left|\int_0^t\int_{y\in[-2A,2A]^d}e^{-i\xi\cdot y}\sigma_{(n)}(s,y)L^M(\mathrm{d}s,\mathrm{d}y)\right|\\&\quad+\sup_{r\in[0,t]}\left|\int_0^r\int_{y\in[-2A,2A]^d}e^{-i\xi\cdot y}\sigma_{(n)}(s,y)L^M(\mathrm{d}s,\mathrm{d}y)\right|\\&\qquad\times\left(\int_0^t|\xi|^2e^{-|\xi|^2(t-r)}\,\mathrm{d}r\right)\\&\leqslant C\sup_{r\in[0,t]}\left|\int_0^r\int_{y\in[-2A,2A]^d}e^{-i\xi\cdot y}\sigma_{(n)}(s,y)L^M(\mathrm{d}s,\mathrm{d}y)\right|,\end{aligned}$$

where C does not depend on ξ. Then, using Theorem A.0.1 *(ii)*, we deduce that

$$\mathbb{E}\left[\sup_{t\in[0,T]}\left|\mathscr{F}\left(u^1(t,\cdot)-u^1_n(t,\cdot)\right)(\xi)\right|^2\right]\leqslant C\int_0^T\int_{y\in[-2A,2A]^d}\mathbb{E}\left[\sigma^2_{(n)}(s,y)\right]\mathrm{d}s\,\mathrm{d}y\,, \tag{5.3.27}$$

where the constant C includes $\int_{|z|\leqslant 1}z^2\nu(\mathrm{d}z)$. Since ν is a Lévy measure, this integral is finite.

Furthermore, by Hölder's inequality and Markov's inequality, for $\varepsilon > 0$,

$$\begin{aligned}\mathbb{E}\left[\sigma_{(n)}^2(s,y)\right] &= \mathbb{E}\left[\sigma\left(u(s,y)\right)^2 \mathbb{1}_{|u(s,y)|>n}\right] \\ &\leqslant \mathbb{E}\left[\left|\sigma\left(u(s,y)\right)\right|^{2+\varepsilon}\right]^{\frac{2}{2+\varepsilon}} \left[\mathbb{P}\left(|u(s,y)|>n\right)\right]^{\frac{\varepsilon}{2+\varepsilon}} \\ &\leqslant \mathbb{E}\left[\left|\sigma\left(u(s,y)\right)\right|^{2+\varepsilon}\right]^{\frac{2}{2+\varepsilon}} \frac{\mathbb{E}\left[\left|u(s,y)\right|^q\right]^{\frac{\varepsilon}{2+\varepsilon}}}{n^{\frac{q\varepsilon}{2+\varepsilon}}}.\end{aligned}$$

As in (5.3.25), the solution u (which is the solution to the truncated equation due to a stopping time argument) has uniformly bounded moments in space and time order q, and by (5.3.25), we deduce that

$$\sup_{(s,y)\in[0,T]\times[-2A,2A]^d} \mathbb{E}\left[\sigma_{(n)}^2(s,y)\right] \leqslant \frac{C}{n^{\frac{q\varepsilon}{2+\varepsilon}}}, \tag{5.3.28}$$

for some constant C. By (5.3.27) and (5.3.28), we obtain

$$\mathbb{E}\left[\sup_{t\in[0,T]} \left|\mathscr{F}\left(u^1(t,\cdot)-u_n^1(t,\cdot)\right)(\xi)\right|^2\right] \leqslant \frac{C}{n^{\frac{q\varepsilon}{2+\varepsilon}}}.$$

By (5.3.26), we deduce that for any $r < -\frac{d}{2}$,

$$\mathbb{E}\left[\sup_{t\in[0,T]} \|u^1(t,\cdot)-u_n^1(t,\cdot)\|^2_{H_r(\mathbb{R}^d)}\right] \leqslant \frac{C}{n^{\frac{q\varepsilon}{2+\varepsilon}}} \to 0 \qquad \text{as } n\to+\infty.$$

Therefore, $\sup_{t\in[0,T]} \|u^1(t,\cdot)-u_n^1(t,\cdot)\|_{H_r(\mathbb{R}^d)} \to 0$ in $L^2(\Omega)$ as $n\to+\infty$, and there is a subsequence $(n_k)_{k\geqslant 0}$ such that $\sup_{t\in[0,T]} \|u^1(t,\cdot)-u_{n_k}^1(t,\cdot)\|_{H_r(\mathbb{R}^d)} \to 0$ almost surely as $k\to+\infty$. This means that $u_{n_k}^1(t,\cdot)$ converges to $u^1(t,\cdot)$ in $H_r(\mathbb{R}^d)$ uniformly in time for any $r<-\frac{d}{2}$. Since σ_{n_k} is bounded, $t\mapsto u_{n_k}^1(t,\cdot)$ has by Proposition 5.3.9 a càdlàg version in $H_{r,\mathrm{loc}}(\mathbb{R}^d)$, and $t\mapsto u^1(t,\cdot)$ has a càdlàg version in $H_{r,\mathrm{loc}}(\mathbb{R}^d)$. Therefore, $t\mapsto u(t,\cdot)$ has a càdlàg version in $H_{r,\mathrm{loc}}(\mathbb{R}^d)$ for any $r<-\frac{d}{2}$. □

5.3.3 Continuity in space at fixed time

Proposition 5.3.13. *Let L be a pure jump Lévy white noise with Lévy measure ν and jump measure J. Suppose there exists $p,q\in\mathbb{R}_+$ such that **(H4)** is satisfied. Suppose also that $p<\frac{2}{d}$ and $p<2$. Furthermore, let σ be a Lipschitz continuous function satisfying*

$$|\sigma(x)| \le C(1+|x|^\gamma), \qquad x\in\mathbb{R}, \tag{5.3.29}$$

for some $C>0$ and $\gamma\in[0,q/p]$, and let u be the mild solution of (5.3.1) *constructed in Proposition 5.3.1. Then, for any $t\in[0,T]$, the process $x\mapsto u(t,x)$ has a continuous modification.*

Proof. By Proposition 5.3.1, there is stationary convergence of the mild solution u_N to the truncated equation defined in (5.3.7). We recall that

$$u_N(t,x)=\int_0^t\int_{\mathbb{R}^d}\rho_H(t-s,x-y)\sigma\left(u_N(s,y)\right)L_N(\mathrm{d}s,\mathrm{d}y)\,,$$

and

$$L_N(\mathrm{d}s,\mathrm{d}y)=b\,\mathrm{d}s\,\mathrm{d}y+\int_{|z|\leqslant 1}z\tilde{J}(\mathrm{d}s,\mathrm{d}y,\mathrm{d}z)+\int_{|z|>1}z\mathbb{1}_{|z|<N(1+|y|^\eta)}J(\mathrm{d}s,\mathrm{d}y,\mathrm{d}z)\,.$$

Therefore, we can suppose that $u=u_N$ for some $N\geqslant 1$, and for conciseness of the notation, we can suppose without loss of generality that $N=1$. As shown in [14, Theorem 3.8],

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d}\mathbb{E}\left[|\sigma(u(t,x))|^p\right]\leqslant C\left(1+\sup_{(t,x)\in[0,T]\times\mathbb{R}}\mathbb{E}\left[|u(t,x)|^q\right]\right)<\infty\,. \tag{5.3.30}$$

We prove the claim using different approaches that depend on the value of p.

$\underline{1<p<2}$: By the $p\leqslant\frac{2}{d}$ hypothesis, this can only happen when $d=1$, and we write

$$u(t,x)=A+B\,,$$

where

$$\begin{aligned}A&:=b\int_0^t\int_{\mathbb{R}}\rho_H(t-s,x-y)\sigma\left(u(s,y)\right)\mathrm{d}s\,\mathrm{d}y\\&\quad+\int_0^t\int_{\mathbb{R}}\int_{|z|\leqslant 1}z\rho_H(t-s,x-y)\sigma\left(u(s,y)\right)\tilde{J}(\mathrm{d}s,\mathrm{d}y,\mathrm{d}z)\,,\\B&:=\int_0^t\int_{\mathbb{R}}\int_{|z|>1}z\rho_H(t-s,x-y)\mathbb{1}_{|z|<1+|y|^\eta}\sigma\left(u(s,y)\right)J(\mathrm{d}s,\mathrm{d}y,\mathrm{d}z)\,.\end{aligned} \tag{5.3.31}$$

Also, by (5.3.30), we can use a Kolmogorov continuity-type argument similar to [61, Théorème 2.4.1] to deduce the existence of a continuous modification of A in the space variable x. More precisely, for $x,z\in\mathbb{R}^d$,

$$\begin{aligned}&\mathbb{E}\Bigg[\Bigg|b\int_0^t\int_{\mathbb{R}}\left(\rho_H(t-s,x-y)-\rho_H(t-s,z-y)\right)\sigma\left(u(s,y)\right)\mathrm{d}s\,\mathrm{d}y\\&\qquad+\int_0^t\int_{\mathbb{R}}\int_{|z|\leqslant 1}z\left(\rho_H(t-s,x-y)-\rho_H(t-s,z-y)\right)\sigma\left(u(s,y)\right)\tilde{J}(\mathrm{d}s,\mathrm{d}y,\mathrm{d}z)\Bigg|^p\Bigg]\\&\leqslant C\Bigg(\mathbb{E}\Bigg[\Bigg|b\int_0^t\int_{\mathbb{R}}\left(\rho_H(t-s,x-y)-\rho_H(t-s,z-y)\right)\sigma\left(u(s,y)\right)\mathrm{d}s\,\mathrm{d}y\Bigg|^p\Bigg]\\&\qquad+\mathbb{E}\Bigg[\Bigg|\int_0^t\int_{\mathbb{R}}\int_{|z|\leqslant 1}z\left(\rho_H(t-s,x-y)-\rho_H(t-s,z-y)\right)\sigma\left(u(s,y)\right)\tilde{J}(\mathrm{d}s,\mathrm{d}y,\mathrm{d}z)\Bigg|^p\Bigg]\Bigg)\,.\end{aligned}$$

Then, using Hölder's inequality, and (5.3.30),

$$\begin{aligned}
&\mathbb{E}\left[\left|b\int_0^t\int_{\mathbb{R}}\left(\rho_H(t-s,x-y)-\rho_H(t-s,z-y)\right)\sigma\left(u(s,y)\right)\mathrm{d}s\,\mathrm{d}y\right|^p\right]\\
&\quad\leqslant\left(\int_0^t\int_{\mathbb{R}}\left|\rho_H(t-s,x-y)-\rho_H(t-s,z-y)\right|\mathbb{E}\left[\left|\sigma\left(u(s,y)\right)\right|^p\right]\mathrm{d}s\,\mathrm{d}y\right)\\
&\qquad\times\left(\int_0^t\int_{\mathbb{R}}\left|\rho_H(t-s,x-y)-\rho_H(t-s,z-y)\right|\mathrm{d}s\,\mathrm{d}y\right)^{p-1}\\
&\quad\leqslant C\left(\int_0^t\int_{\mathbb{R}}\left|\rho_H(t-s,x-y)-\rho_H(t-s,z-y)\right|\mathrm{d}s\,\mathrm{d}y\right)^p.
\end{aligned}$$

By [61, Lemme A2], we deduce that

$$\mathbb{E}\left[\left|b\int_0^t\int_{\mathbb{R}}\left(\rho_H(t-s,x-y)-\rho_H(t-s,z-y)\right)\sigma\left(u(s,y)\right)\mathrm{d}s\,\mathrm{d}y\right|^p\right]\leqslant C|x-z|^p. \tag{5.3.32}$$

Then, using Theorem A.0.1 *(ii)* and (5.3.30),

$$\begin{aligned}
&\mathbb{E}\left[\left|\int_0^t\int_{\mathbb{R}}\int_{|z|\leqslant1}z\left(\rho_H(t-s,x-y)-\rho_H(t-s,z-y)\right)\sigma\left(u(s,y)\right)\tilde{J}(\mathrm{d}s,\mathrm{d}y,\mathrm{d}z)\right|^p\right]\\
&\quad\leqslant C\int_0^t\int_{\mathbb{R}}\int_{|z|\leqslant1}|z|^p\left|\rho_H(t-s,x-y)-\rho_H(t-s,z-y)\right|^p\mathbb{E}\left[\left|\sigma\left(u(s,y)\right)\right|^p\right]\mathrm{d}s\,\mathrm{d}y\,\nu(\mathrm{d}z)\\
&\quad\leqslant C\int_0^t\int_{\mathbb{R}}\left|\rho_H(t-s,x-y)-\rho_H(t-s,z-y)\right|^p\mathrm{d}s\,\mathrm{d}y.
\end{aligned}$$

By [61, Lemme A2], we deduce that

$$\begin{aligned}
&\mathbb{E}\left[\left|\int_0^t\int_{\mathbb{R}}\int_{|z|\leqslant1}z\left(\rho_H(t-s,x-y)-\rho_H(t-s,z-y)\right)\sigma\left(u(s,y)\right)\tilde{J}(\mathrm{d}s,\mathrm{d}y,\mathrm{d}z)\right|^p\right]\\
&\qquad\leqslant\begin{cases}C|x-z|^p, & \text{if } p<\frac{3}{2},\\ C|x-z|^{\frac{3}{2}}\log(|x-z|) & \text{if } p=\frac{3}{2},\\ C|x-z|^{3-p} & \text{if } p>\frac{3}{2}.\end{cases}
\end{aligned}\tag{5.3.33}$$

By (5.3.32) and (5.3.33), and the Kolmogorov continuity criterion, we deduce the existence of a continuous modification of A in the space variable x.

The term B in (5.3.31) is a sum of a possibly infinite number of terms. Each term is continuous in x. Let $\beta\leqslant q\wedge1$. Then, by (5.3.30), and using **(H4)**, we get for any $x_0\in\mathbb{R}$,

$$\begin{aligned}
&\mathbb{E}\left[\sup_{x:|x-x_0|\leq1}\left|\int_0^t\int_{\mathbb{R}}\int_{|z|>1}z\rho_H(t-s,x-y)\mathbb{1}_{|z|<1+|y|^\eta}\sigma\left(u(s,y)\right)J(\mathrm{d}s,\mathrm{d}y,\mathrm{d}z)\right|^\beta\right]\\
&\quad\leqslant\int_0^t\int_{\mathbb{R}}\int_{|z|>1}|z|^\beta\sup_{x:|x-x_0|\leq1}\rho_H(t-s,x-y)^\beta\mathbb{1}_{|z|<1+|y|^\eta}\mathbb{E}\left[\left|\sigma\left(u(s,y)\right)\right|^\beta\right]\mathrm{d}s\,\mathrm{d}y\,\nu(\mathrm{d}z)
\end{aligned}$$

$$\leqslant C\int_0^t\int_{\mathbb{R}}\sup_{x:|x-x_0|\le 1}\rho_H(t-s,x-y)^\beta\,\mathrm{d}s\,\mathrm{d}y$$
$$\leqslant C\left(\int_0^t\int_{y:|y-x_0|\le 1}\rho_H(t-s,0)^\beta\,\mathrm{d}s\,\mathrm{d}y+2\int_0^t\int_{x_0+1}^{\infty}\rho_H(t-s,x_0+1-y)^\beta\,\mathrm{d}s\,\mathrm{d}y\right)<+\infty.$$

We deduce that the convergence of the sum defining $B(t,x)$ is uniform in x in a ball around x_0, which proves the claim for $1<p<2$.

$\underline{p\leqslant 1}$: Then, we write (recall that u is set to u_N with $N=1$)

$$u(t,x)=A(t,x)+B(t,x)+C(t,x),$$

where

$$A(t,x):=\int_0^t\int_{\mathbb{R}^d}\int_{|z|\leqslant 1}z\rho_H(t-s,x-y)\sigma\big(u(s,y)\big)\,J(\mathrm{d}s,\mathrm{d}y,\mathrm{d}z),\qquad\text{and}$$
$$B(t,x):=\int_0^t\int_{\mathbb{R}^d}\int_{|z|>1}z\rho_H(t-s,x-y)\mathbb{1}_{|z|<1+|y|^\eta}\sigma\big(u(s,y)\big)\,J(\mathrm{d}s,\mathrm{d}y,\mathrm{d}z),\qquad\text{and}$$
$$C(t,x):=b_0\mathbb{1}_{p=1}\int_0^t\int_{\mathbb{R}^d}\rho_H(t-s,x-y)\sigma\big(u(s,y)\big)\,\mathrm{d}s\,\mathrm{d}y,$$

where b_0 was defined in **(H4)**, and $b_0=0$ if $p<1$, since $p<\frac{2}{d}$. The process C is non-zero only in the case where $d=1$ and $p=1$. Then, we can apply [61, Théorème 2.3.2] to conclude that for any fixed time t, the process $x\mapsto C(t,x)$ has a continuous modification. The term A is a sum of a possibly infinite number of terms. Each term is continuous in x because a.s., no jump time occurs at time t. Then, again by (5.3.30), and using **(H4)**, we get for any $x_0\in\mathbb{R}$,

$$\mathbb{E}\left[\left(\int_0^t\int_{\mathbb{R}^d}\int_{|z|\leqslant 1}\sup_{x:|x-x_0|\le 1}\left|z\rho_H(t-s,x-y)\sigma\big(u(s,y)\big)\right|J(\mathrm{d}s,\mathrm{d}y,\mathrm{d}z)\right)^p\right]$$
$$\leqslant\int_0^t\int_{\mathbb{R}^d}\int_{|z|\leqslant 1}|z|^p\sup_{x:|x-x_0|\le 1}\rho_H(t-s,x-y)^p\mathbb{E}\left[\left|\sigma\big(u(s,y)\big)\right|^p\right]\,\mathrm{d}s\,\mathrm{d}y\,\nu(\mathrm{d}z)$$
$$\leqslant C\int_0^t\int_{\mathbb{R}^d}\sup_{x:|x-x_0|\le 1}\rho_H(t-s,x-y)^p\,\mathrm{d}s\,\mathrm{d}y$$
$$\leqslant C\left(\int_0^t\int_{y:|y-x_0|\le 1}\rho_H(t-s,0)^p\,\mathrm{d}s\,\mathrm{d}y+\int_0^t\int_{y:|y-x_0|>1}(4\pi(t-s))^{-\frac{pd}{2}}e^{-\frac{pd(y,B_{x_0}(1))^2}{4(t-s)}}\,\mathrm{d}s\,\mathrm{d}y\right)$$
$$\leqslant C\left(\int_0^t\int_{y:|y-x_0|\le 1}\rho_H(t-s,0)^p\,\mathrm{d}s\,\mathrm{d}y+\int_0^t\int_{|y|>1}(4\pi(t-s))^{-\frac{pd}{2}}e^{-\frac{p(|y|-1)^2}{4(t-s)}}\,\mathrm{d}s\,\mathrm{d}y\right)<+\infty,$$

since $p<\frac{2}{d}$, where $d(y,B_{x_0}(1))$ is the distance from y to the ball of radius 1 centered at x_0. We deduce that the convergence of the sum defining $A(t,x)$ is uniform in x in a ball around x_0, which proves the continuity of the process $x\mapsto A(t,x)$. The continuity of $x\mapsto B(t,x)$ for fixed t was proved after (5.3.33) above. □

Remark 5.3.14. *For example, in dimension $d=1$, if L is an α-stable noise for some $\alpha\in(0,2)$ (with no drift when $\alpha<1$), we can choose any $p\in\big(\alpha,\frac{4\alpha}{1+\alpha}\wedge 2\big)$ and $q\in\left(\frac{p}{4-p},\alpha\right)$, so the previous* *proposition asserts that the sample paths of the mild solution for a fixed time t are always almost surely continuous in x (when σ satisfies* (5.3.29)*). Indeed, since $p > \alpha$ and $q < \alpha$, $\int_{|z|\leqslant 1} |z|^p \nu(\mathrm{d}z) < +\infty$, and $\int_{|z|>1} |z|^q \nu(\mathrm{d}z) < +\infty$. Furthermore, the interval $\left(\frac{p}{4-p}, \alpha\right)$ is non-empty if and only if $p < \frac{4\alpha}{1+\alpha}$. Note that the condition $\frac{p}{1+(1+\frac{2}{d}-p)} < q$ in **(H4)** becomes $\frac{p}{4-p} < q$ since $d = 1$ here.*

The case of an α-stable noise, $\frac{2}{d} \leqslant \alpha < 2$

In this section, we suppose that the noise is an α-stable noise L_α on $[0,T] \times \mathbb{R}^d$, for some $\alpha \in [\frac{2}{d}, 2)$.

Proposition 5.3.15. *Let u be the mild solution of the stochastic heat equation with additive α-stable noise for some $\alpha \in [\frac{2}{d}, 2)$:*

$$u(t,x) = \int_0^t \int_{\mathbb{R}^d} \rho_H(t-s, x-y) L_\alpha(\mathrm{d}s, \mathrm{d}y).$$

For any $t \in [0,T]$, there is a set $N_t \subset \Omega$ of probability one such that for any $\omega \in N_t$, $x \mapsto u(t,x)(\omega)$ is unbounded on any non-empty open interval.

Proof. Fix $t \in [0,T]$. Observe that the process $(Y(x), x \in \mathbb{R}^d)$ defined by

$$Y(x) = u(t,x) = \int_0^t \int_{\mathbb{R}^d} \rho_H(t-s, x-y) L_\alpha(\mathrm{d}s, \mathrm{d}y)$$

is an α-stable process given in the "standard form" of [62, (10.1.1)] with the measurable space $E = [0,T] \times \mathbb{R}^d$, and the control measure $\mathrm{d}s\,\mathrm{d}y$. We shall check that the necessary condition [62, (10.2.14)] for sample path boundedness in [62, Theorem 10.2.3] is not satisfied, in particular that for any $x_1 < x_2$, and $X^* = [x_1, x_2]^d$,

$$\int_0^t \int_{\mathbb{R}^d} \left(\sup_{x \in X^*} \rho_H(t-s, x-y)\right)^\alpha \mathrm{d}s\,\mathrm{d}y = +\infty. \tag{5.3.34}$$

This integral is bounded below by

$$\int_0^t \int_{X^*} \sup_{x \in X^*} \rho_H(t-s, x-y)^\alpha \,\mathrm{d}s\,\mathrm{d}y \geqslant \int_0^t \int_{X^*} \frac{1}{(4\pi(t-s))^{\frac{\alpha d}{2}}} \mathrm{d}s\,\mathrm{d}y = +\infty,$$

and (5.3.34) is proved. □

Remark 5.3.16. *Let u be as in Proposition 5.3.15 with $d = 1$. Then the interval $\left[\frac{2}{d}, 2\right)$ is empty and by Remark 5.3.14, $x \mapsto u(t,x)$ is continuous (as for the SHE on a bounded interval (see Remark 5.2.11)).*

5.3.4 Continuity in time at a fixed space point

Proposition 5.3.17. *Let L be a pure jump Lévy white noise with Lévy measure ν and jump measure J. Suppose there exists $p,q\in\mathbb{R}_+$ such that **(H4)** is satisfied, and such that $p<1$. Furthermore, let σ be a Lipschitz continuous function satisfying* (5.3.29)*, and let u be the mild solution to* (5.3.1) *constructed in Proposition 5.3.1. Then, for any $x\in\mathbb{R}^d$, the process $t\mapsto u(t,x)$ has a continuous modification.*

Proof. Again, by a stopping time argument, it suffices to show the regularity of u_N for any $N\geqslant 1$. Without loss of generality, we can therefore suppose that u solves

$$u(t,x)=\int_0^t\int_{\mathbb{R}^d}\rho_H(t-s,x-y)\sigma\left(u(s,y)\right)L_1(\mathrm{d}s,\mathrm{d}y),$$

where

$$L_1(\mathrm{d}s,\mathrm{d}y)=b\,\mathrm{d}t\,\mathrm{d}x+\int_{|z|\leqslant 1}z\tilde{J}(\mathrm{d}s,\mathrm{d}y,\mathrm{d}z)+\int_{|z|>1}z\mathbb{1}_{|z|<1+|y|^\eta}J(\mathrm{d}s,\mathrm{d}y,\mathrm{d}z).$$

By **(H4)**, and since $p<1$, $b_0=b-\int_{|z|\leqslant 1}z\nu(\mathrm{d}z)=0$. Then,

$$\begin{aligned}
u(t,x)&=\int_0^t\int_{\mathbb{R}^d}\rho_H(t-s,x-y)\sigma\left(u(s,y)\right)L_1(\mathrm{d}s,\mathrm{d}y)\\
&=\int_0^t\int_{\mathbb{R}^d}\int_{|z|\leqslant 1}z\rho_H(t-s,x-y)\sigma\left(u(s,y)\right)J(\mathrm{d}s,\mathrm{d}y,\mathrm{d}z)\\
&\quad+\int_0^t\int_{\mathbb{R}^d}\int_{|z|>1}z\rho_H(t-s,x-y)\mathbb{1}_{|z|<1+|y|^\eta}\sigma\left(u(s,y)\right)J(\mathrm{d}s,\mathrm{d}y,\mathrm{d}z)\\
&=:A(t,x)+B(t,x).
\end{aligned}$$

For the continuity of the term A, we can use that $\mathbb{E}[|\sigma(u(t,x))|^p]$ is uniformly bounded for $(t,x)\in[0,T]\times\mathbb{R}^d$ by (5.3.30), together with [61, Théorème 2.2.2]: hypotheses (H_2) and (H_3) there are satisfied, and

$$\int_{|z|\leqslant 1}|z|\nu(\mathrm{d}z)<+\infty\quad\text{and}\quad\int_{|z|\leqslant 1}|z|\left|\log(|z|)\right|\nu(\mathrm{d}z)<+\infty,$$

which proves $(H_1)(1)$ and $(H_5)(1)$ there. The term B is a sum of a possibly infinite number of terms, each of which is a continuous function of t. Let $\beta\leqslant q\wedge 1$. The maximum of the function $t\mapsto\rho_H(t,x)$ is attained at $t=\frac{|x|^2}{2d}$ and is equal to $\frac{C}{|x|^d}$ for some constant C. Then, again using (5.3.30), we obtain

$$\begin{aligned}
&\mathbb{E}\left[\left(\int_0^T\int_{\mathbb{R}^d}\int_{|z|>1}\sup_{r\in[0,T]}\left|\mathbb{1}_{s\leqslant r}\,z\rho_H(r-s,x-y)\mathbb{1}_{|z|<1+|y|^\eta}\sigma\left(u(s,y)\right)\right|J(\mathrm{d}s,\mathrm{d}y,\mathrm{d}z)\right)^\beta\right]\\
&\quad\leqslant\int_0^T\int_{\mathbb{R}^d}\int_{|z|>1}|z|^\beta\sup_{r\in[0,T]}\mathbb{1}_{s\leqslant r}\rho_H(r-s,x-y)^\beta\mathbb{1}_{|z|<1+|y|^\eta}\mathbb{E}\left[\left|\sigma\left(u(s,y)\right)\right|^\beta\right]\,\mathrm{d}s\,\mathrm{d}y\,\nu(\mathrm{d}z)
\end{aligned}$$

$$\leqslant C\int_0^T\int_{\mathbb{R}^d}\sup_{r\in[0,T]}\mathbb{1}_{s\leqslant r}\rho_H(r-s,x-y)^\beta\,\mathrm{d}s\,\mathrm{d}y$$
$$\leqslant C\bigg(\int_0^T\int_{|x-y|\leqslant\sqrt{2d(T-s)}}\frac{C}{|x-y|^{d\beta}}\,\mathrm{d}s\,\mathrm{d}y$$
$$+\int_0^T\int_{|x-y|>\sqrt{2d(T-s)}}\rho_H(T-s,x-y)^\beta\,\mathrm{d}s\,\mathrm{d}y\bigg)<+\infty.$$

We deduce that the convergence of the sum defining $B(t,x)$ is uniform in $t\in[0,T]$. This proves the continuity statement. □

Remark 5.3.18. *In particular, any α-stable noise with $\alpha\in(0,1)$ satisfies the hypothesis of Proposition 5.3.17. Indeed, since we must have $p>\alpha$, the condition $p<1$ imposes $\alpha<1$. Conversely, it is immediate to see that for $\alpha<1$, one can choose p,q such that **(H4)** is satisfied. The next section shows that for $\alpha\geqslant 1$, the situation is completely different.*

The case of an α-stable noise, $1\leqslant\alpha<2$

In this section, we suppose that the noise is an α-stable noise L_α on $[0,T]\times\mathbb{R}^d$, for some $\alpha\in[1,2)$.

Proposition 5.3.19. *Let u be the mild solution constructed in Proposition 5.3.1 of the stochastic heat equation with additive α-stable noise for some $\alpha\in[1,2)$:*

$$u(t,x)=\int_0^t\int_{\mathbb{R}^d}\rho_H(t-s;x-y)L_\alpha(\mathrm{d}s,\mathrm{d}y).$$

For any $x\in\mathbb{R}^d$, there is a set $N_x\subset\Omega$ of probability one such that for any $\omega\in N_x$, $t\mapsto u(t,x)(\omega)$ is unbounded on any non-empty open interval.

Proof. Fix $x\in\mathbb{R}^d$. Observe that the process $(Y(t),t\in[0,T])$ defined by

$$Y(t)=u(t,x)=\int_0^t\int_{\mathbb{R}^d}\rho_H(t-s,x-y)L_\alpha(\mathrm{d}s,\mathrm{d}y)$$

is an α-stable process given in the "standard form" of [62, (10.1.1)] with the measurable space $E=[0,T]\times\mathbb{R}^d$, and the control measure $\mathrm{d}s\,\mathrm{d}y$. We shall check that the necessary condition [62, (10.2.14)] for sample path boundedness in [62, Theorem 10.2.3] is not satisfied, in particular that for any $0\leqslant t_1<t_2\leqslant T$, and $T^*=[t_1,t_2]$,

$$\int_{t_1}^{t_2}\int_{\mathbb{R}^d}\left(\sup_{t\in T^*}\rho_H(t-s,x-y)\right)^\alpha\mathrm{d}s\,\mathrm{d}y=+\infty. \tag{5.3.35}$$

Indeed, observe that the integral is bounded below by

$$\int_{t_1}^{t_2}\int_{\mathbb{R}^d}\sup_{t\in[t_1,t_2]}\rho_H(t-s,x-y)^\alpha\,\mathrm{d}s\,\mathrm{d}y\geqslant\int_0^{t_2-t_1}\int_{\mathbb{R}^d}\sup_{v\in[0,s]}\rho_H(v,x-y)^\alpha\,\mathrm{d}s\,\mathrm{d}y.$$

In view of the study of the maximum of $t \mapsto \rho_H(t, x-y)$ in the proof of Proposition 5.3.17, we have

$$\int_{t_1}^{t_2} \int_{\mathbb{R}^d} \left(\sup_{t \in T^*} \rho_H(t-s, x-y) \right)^{\alpha} \mathrm{d}s\, \mathrm{d}y \geqslant \int_0^{t_2-t_1} \int_{|x-y| \leqslant \sqrt{2ds}} \frac{C}{|x-y|^{d\alpha}} \mathrm{d}s\, \mathrm{d}y = +\infty,$$

and (5.3.35) is proved. □

5.4 The SHE driven by heavy-tailed noise: equation on a smooth and bounded domain D in dimension $d \geqslant 2$

Let D be a smooth and bounded domain in $\mathbb{R}^d$, where $d \geqslant 2$. We consider the stochastic heat equation driven by a Lévy white noise in $[0,T] \times D$ with Dirichlet boundary conditions:

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) = \Delta u(t,x) + \sigma(u(t,x))\dot{L}(t,x), & (t,x) \in [0,T] \times D, \\ u(t,x) = 0, & \text{for all } (t,x) \in [0,T] \times \partial D, \\ u(0,x) = 0, & \text{for all } x \in D, \end{cases} \tag{5.4.1}$$

where σ is a Lipschitz function and L is a pure jump Lévy white noise. More precisely, we suppose that

$$\begin{aligned} L(\mathrm{d}t, \mathrm{d}x) &= b\, \mathrm{d}t\, \mathrm{d}x + \int_{|z| \leqslant 1} z \tilde{J}(\mathrm{d}t, \mathrm{d}x, \mathrm{d}z) + \int_{|z|>1} z J(\mathrm{d}t, \mathrm{d}x, \mathrm{d}z) \\ &=: L^B(\mathrm{d}t, \mathrm{d}x) + L^M(\mathrm{d}t, \mathrm{d}x) + L^P(\mathrm{d}t, \mathrm{d}x), \end{aligned} \tag{5.4.2}$$

where $b \in \mathbb{R}$, J is a Poisson random measure on $[0,\infty) \times D \times \mathbb{R}$ with intensity $\mathrm{d}t\, \mathrm{d}x\, \nu(\mathrm{d}z)$, and $\tilde{J}$ is the compensated version of J. The measure ν is a Lévy measure, that is, $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}} \left(z^2 \wedge 1\right) \nu(\mathrm{d}z) < +\infty$.

(H5) There exists $0 < p < 1 + \frac{2}{d}$ such that

$$\int_{|z| \leqslant 1} |z|^p \nu(\mathrm{d}z) < +\infty.$$

If $p < 1$, we assume that

$$b_0 := b - \int_{|z| \leqslant 1} z \nu(\mathrm{d}z) = 0. \tag{5.4.3}$$

The Green's function of the heat operator on the bounded domain $[0,T] \times D$ is denoted by $G_D(t;x,y)$, for all $(t,x,y) \in [0,T] \times D \times D$. By definition, a mild solution to (5.4.1) is a predictable random field $u = (u(t,x), (t,x) \in [0,T] \times D)$ such that for all $(t,x) \in [0,T] \times D$,

$$u(t,x) = \int_0^t \int_D G_D(t-s;x,y) \sigma(u(s,y)) L(\mathrm{d}s, \mathrm{d}y). \tag{5.4.4}$$

Similar to [14], we define the stopping times $\tau_N = \inf\left\{t \geqslant 0 : J([0,t] \times D \times [-N,N]^c) \neq 0\right\}$. By the same calculation as in (5.2.5), $(\tau_N)_{N\geqslant 1}$ is an increasing sequence of stopping times such that $\tau_N > 0$ and $\tau_N \to +\infty$ almost surely as $N \to +\infty$. In fact, we have that for almost all $\omega \in \Omega$, there exists an integer $R(\omega)$ such that for any $N \geqslant R(\omega)$, $\tau_N(\omega) > T$. We use these stopping times to truncate the noise, and we can define $L_N := L\mathbb{1}_{t\leqslant\tau_N}$. Then,

$$L_N(\mathrm{d}t, \mathrm{d}x) = b_N \,\mathrm{d}t\,\mathrm{d}x + \int_{|z|\leqslant N} z\tilde{J}(\mathrm{d}t, \mathrm{d}x, \mathrm{d}z), \tag{5.4.5}$$

where $b_N := b - \int_{1<|z|\leqslant N} z\nu(\mathrm{d}z)$.

Proposition 5.4.1. *Let $\sigma : \mathbb{R} \to \mathbb{R}$ be a Lipschitz function and let L be a pure jump Lévy white noise as in* (5.4.2) *such that **(H5)** is satisfied for some $p \in \mathbb{R}$. Then there exists, up to modifications, a unique predictable random field u such that*

$$\sup_{(t,x)\in[0,T]\times D} \mathbb{E}\left[|u(t,x)|^p \mathbb{1}_{t\leqslant\tau_N}\right] < +\infty,$$

and for any $(t, x) \in [0,T] \times D$,

$$u(t,x) = \int_0^t \int_D G_D(t-s; x, y)\sigma(u(s,y))L(\mathrm{d}s, \mathrm{d}y) \qquad a.s. \tag{5.4.6}$$

Proof. By [28, Corollary 3.2.8],

$$G_D(t; x, y) \leqslant \frac{C}{t^{\frac{d}{2}}} e^{-\frac{|x-y|^2}{6t}}. \tag{5.4.7}$$

Also, since D is a bounded domain, $\nu\left([0,T] \times D \times (-1,1)^c\right) < +\infty$ and a.s., there is only a finite number of jumps larger than 1 in $[0,T] \times D$. Consequently, (1) to (4) of Assumption B of [13] are satisfied, and we can apply [13, Theorem 3.5] to obtain the existence of a unique predictable random field u satisfying (5.4.6) and

$$\sup_{(t,x)\in[0,T]\times D} \mathbb{E}\left[|u(t,x)|^p \mathbb{1}_{t\leqslant\tau_N}\right] < +\infty.$$

□

Remark 5.4.2. *Let $u_N(t,x) = u(t,x)\mathbb{1}_{t\leqslant\tau_N}$. Then u_N is clearly a mild solution to the truncated equation*

$$u_N(t,x) = \int_0^t \int_D G_D(t-s; x, y)\sigma\left(u_N(s,y)\right) L_N(\mathrm{d}s, \mathrm{d}y) \qquad a.s.$$

Furthermore, $u_N \to u$ as $N \to +\infty$ almost surely, and the convergence is stationary.

The problems we consider in the following are about sample path regularity properties of the mild solution of the stochastic heat equation, and, by stationary convergence of u_N to u, these properties are identical to those of u_N defined in Remark 5.4.2 for N sufficiently large. The value of the parameter N has no importance in our study, so we can suppose that $N = 1$ and drop the dependency in N. Therefore, in the following, we will always consider the solution to the integral equation

$$\begin{aligned} u(t,x) = b \int_0^t \int_D G_D(t-s;x,y)\sigma(u(s,y))\,\mathrm{d}s\,\mathrm{d}y \\ + \int_0^t \int_D G_D(t-s;x,y)\sigma(u(s,y))L^M(\mathrm{d}s,\mathrm{d}y)\,, \end{aligned} \tag{5.4.8}$$

and the solution to (5.4.4) will have the same sample path regularity properties.

Remark 5.4.3. *Since there exists $p < 1+\frac{2}{d}$ such that $\int_{|z|\leqslant 1} |z|^p \nu(\mathrm{d}z) < +\infty$, for any $\beta \in [p, 1+\frac{2}{d})$, $\int_{|z|\leqslant 1} |z|^\beta \nu(\mathrm{d}z) < +\infty$. Therefore, we can apply Proposition 5.4.1 with $p = \beta$ and we obtain that the solution u to* (5.4.8) *has uniformly bounded moments of order β for any $\beta < 1+\frac{2}{d}$:*

$$\sup_{(t,x)\in[0,T]\times D} \mathbb{E}\left[|u(t,x)|^\beta \mathbb{1}_{t\leqslant\tau_N}\right] < +\infty, \qquad \text{for any } \beta < 1+\frac{2}{d}. \tag{5.4.9}$$

5.4.1 The fractional Sobolev spaces $H_r(D)$

Let $D \subset \mathbb{R}^d$ be a bounded domain with a smooth boundary in the sense of [51, (7.10) p. 38]. The operator $(-\Delta)$ on D with vanishing Dirichlet boundary conditions admits a complete orthonormal system in $L^2(D)$ of smooth eigenfunctions $(\Phi_j)_{j\geqslant 1}$, with eigenvalues $(\lambda_j)_{j\geqslant 1}$. Then we have the following properties (see for example [69, Chapter V, p. 343]:

$$\sum_{j\geqslant 1} (1+\lambda_j)^r < +\infty, \qquad \text{for any } r < -\frac{d}{2}, \tag{5.4.10}$$

and

$$\|\Phi_j\|_\infty \leqslant C(1+\lambda_j)^{\frac{\alpha}{2}}, \qquad \text{for any } \alpha > \frac{d}{2}. \tag{5.4.11}$$

The Green's function of the heat operator on D has the representation:

$$G_D(t;x,y) = \sum_{j\geqslant 1} \Phi_j(x)\Phi_j(y)e^{-\lambda_j t}, \qquad \text{for all } x,y \in D.$$

For any function $f \in L^2(D)$,

$$f(x) = \sum_{j\geqslant 1} a_j(f)\Phi_j(x), \qquad \text{for all } x \in D, \tag{5.4.12}$$

where $a_j(f) = \langle f, \Phi_j\rangle_{L^2(D)}$. For any $r \geqslant 0$, we define $H_r(D)$ as the set of functions $f \in L^2(D)$ such that

$$\|f\|_{H_r}^2 := \sum_{j\geqslant 1} (1+\lambda_j)^r a_j(f)^2 < +\infty.$$

This is a Hilbert space for the inner product given by

$$\langle f, g\rangle_{H_r} := \sum_{j\geqslant 1} \left(1+\lambda_j\right)^r a_j(f) a_j(g).$$

For $r > 0$, we define $H_{-r}(D)$ as the dual space of $H_r(D)$, that is, the space of continuous linear functionals on $H_r(D)$. Then, for any $r > 0$, $H_{-r}(D)$ is isomorphic to the space of sequences $b = (b_n)_{n\geqslant 1}$ such that

$$\|b\|^2_{H_{-r}} := \sum_{j\geqslant 1} \left(1+\lambda_j\right)^{-r} b_j^2 < +\infty.$$

More precisely, for $r > 0$ and $f \in H_{-r}(D)$, the coefficients b_j are given by $b_j = f(\Phi_j)$. Then $\|f\|_{H_{-r}} = \|b\|_{H_{-r}}$ and the duality between $H_{-r}(D)$ and $H_r(D)$ is given by

$$\langle b, g\rangle = \sum_{j\geqslant 1} b_j a_j(g) \leqslant \|b\|_{H_{-r}} \|g\|_{H_r}.$$

5.4.2 Existence of a *càdlàg* solution in $H_r(D)$

We start with a key proposition, which tells us that the Green's function of the heat operator on a bounded domain D essentially has the same singularities as the heat kernel. This result is taken from [32, Theorem 1].

Proposition 5.4.4. *Let ρ_H be the heat kernel in $\mathbb{R}^d$. Then*

$$G_D(t;x,y) = \rho_H(t,x-y) + H(t;x,y), \tag{5.4.13}$$

where H is a function such that for any $\varepsilon > 0$, $(t,x,y) \mapsto H(t;x,y)$ is smooth on $[0,T]\times D \times B^c_\varepsilon(\partial D)$, where $B_\varepsilon(\partial D)$ is the ε-neighborhood of the boundary of D:

$$B_\varepsilon(\partial D) = \bigcup_{y\in\partial D} \left(B(y,\varepsilon)\cap D\right),$$

and the complement is taken in D.

Proof. We will use [32, Theorem 1]. This theorem states that G_D can be decomposed into the sum of the fundamental solution of the heat operator (that is, the heat kernel ρ_H here) and a function H. Furthermore, the function H satisfies the estimates (6.1) of [32, Theorem 1] with $|x-\xi|$ replaced by $|x-\xi| + d(\xi,\partial D)$. Translated to our setting, replacing ξ by y, since y is at distance at least ε from the boundary of D, we deduce from (6.1) the smoothness of $(t,x,y) \mapsto H(t;x,y)$ on $[0,T]\times D\times B^c_\varepsilon(\partial D)$. □

Remark 5.4.5. *Our definition of fractional Sobolev spaces is based on the spectral powers of the Dirichlet Laplacian. More precisely, for any $r \geqslant 0$, $H_r(D)$ is the domain of $(-\Delta)^{\frac{r}{2}}$. We define several other Sobolev spaces: for $m \in \mathbb{N}$, as in [51, (1.3) p.3],*

$$H^m(D) = \left\{u\,:\, u^{(\alpha)} \in L^2(D),\ \textit{for all}\ |\alpha| \leqslant m\right\}.$$

Then, by [51, Définition 9.1 p. 45], for $s \geqslant 0$ and letting m be the smallest even integer such that $m \geqslant r$,

$$H^r(D) := \left[H^m(D), L^2(D)\right]_{1-\frac{r}{m}},$$

where the right-hand side is the notation of [51, Definition 2.1 p.12] for interpolation spaces (and notice that s is an upper (not lower) index). Then as in [51, (11.1) p.60], for $r \geqslant 0$, $H_0^r(D)$ is defined as the closure of $\mathscr{D}(D)$ (the set of C^∞ functions with compact support included in D) in $H^r(D)$. Following [39, Definition 8.1], for $r > \frac{1}{2}$, we define $H_B^r(D)$ to be the closed subspace of $H^r(D)$ such that its elements are equal to zero on the boundary ∂D. Finally, we point out that $H_r(\mathbb{R}^d)$ defined in Definition 5.3.7 coincides with the definition of $H^r(\mathbb{R}^d)$ in [51, (7.1) p. 35]. Then, for $m \in \mathbb{N}$, by [51, Definition 2.1 p.12], for $\theta \in [0,1]$,

$$\left[H_B^m(D), L^2(D)\right]_\theta = dom\left(\Lambda^{1-\theta}\right),$$

for some self-adjoint positive operator Λ in $L^2(D)$ with domain $H_B^m(D)$ (see also [51, Remarque 2.3 p.13]). The power in this case is to be understood as the spectral power of the operator. In particular, we can choose $\Lambda = (-\Delta)^{\frac{m}{2}}$, where Δ is the Dirichlet Laplacian, and the power $\frac{m}{2}$ is to be understood as the composition of partial differential operators (note that the power $\frac{m}{2}$ is an integer since m was chosen even). Then, by [39, Théorème 8.1],

$$\left[H_B^m(D), L^2(D)\right]_\theta = H_B^{m(1-\theta)}(D).$$

Choosing $\theta = 1 - \frac{r}{m}$, we deduce that

$$H_B^r(D) = dom\left(\Lambda^{\frac{r}{m}}\right).$$

Let $f \in L^2(D)$ as in (5.4.12). Then,

$$\Lambda^{\frac{r}{m}} f = \sum_{j\geqslant 1} \mu_j^{\frac{r}{m}} a_j(f)\Phi_j,$$

where $\mu_j = \lambda_j^{\frac{m}{2}}$ is the j^{th} eigenvalue of Λ. The previous sum has a meaning in $L^2(D)$ if and only if

$$\sum_{j\geqslant 1} \lambda_j^r \left|a_j(f)\right|^2 < +\infty.$$

We deduce that

$$H_B^r(D) = dom\left(\Lambda^{\frac{r}{m}}\right) = H_r(D).$$

Therefore, by [39, Théorème 8.1] and the discussion that follows, $H_r(D) \subset H^r(D)$, and the embedding is continuous. In the case where $r \leqslant \frac{1}{2}$, we have by [55, p. 740]

$$H_r(D) = \begin{cases} H^r(D) & \text{if } r < \frac{1}{2}, \\ H_{00}^{\frac{1}{2}}(D) & \text{if } r = \frac{1}{2}, \end{cases}$$

where $H_{00}^{\frac{1}{2}}(D)$ is the Lions-Magenes space (see [51, Théorème 11.7]). Also, by [51, Chapitre I, Théorème 11.7],

$$H_{00}^{\frac{1}{2}}(D) \subset H_{0}^{\frac{1}{2}}(D),$$

where the inclusion is continuous. In addition, for any $r \geqslant 0$, $H_0^r(D) \subset H^r(D)$, where the inclusion is continuous. Therefore, for any $r \geqslant 0$, $H_r(D) \subset H^r(D)$, where the inclusion is continuous. Then, by [51, Théorème 9.1], any function $u \in H^r(D)$ is the restriction to D of a function $\tilde{u} \in H^r(\mathbb{R}^d) = H_r(\mathbb{R}^d)$ such that by the proof of [51, Théorème 9.2], there is a constant C that does not depend on u such that $\|\tilde{u}\|_{H^r(\mathbb{R}^n)} \leqslant C\|u\|_{H^r(D)}$. Therefore the embedding is continuous. Finally, for any $r \geqslant 0$, $H_r(D) \subset H^r(D) \subset H_r(\mathbb{R}^d)$, where the embeddings are continuous. By duality, for any $r \leqslant 0$, $H_r(\mathbb{R}^d) \subset H^r(D) \subset H_r(D)$, where the embeddings are continuous.

In conclusion, for any $r \leqslant 0$, if $t \mapsto u(t,\cdot)$ is càdlàg in $H_{r,loc}(\mathbb{R}^d)$, then for any $\theta \in \mathscr{D}(\mathbb{R}^d)$, $t \mapsto \theta(\cdot)u(t,\cdot)$ is càdlàg in $H_r(\mathbb{R}^d)$, hence $t \mapsto u(t,\cdot)|_D$ is càdlàg in $H_r(D)$.

Theorem 5.4.6. *Let $\sigma : \mathbb{R} \to \mathbb{R}$ be a Lipschitz continuous function, and let L be a Lévy white noise as in* (5.4.2) *such that **(H5)** is satisfied for some $p > 0$. Suppose also that $|\sigma(x)| \leqslant C(1+|x|^\gamma)$ for some $\gamma < \frac{1}{2} + \frac{1}{d}$. Then the solution to the (SHE) defined in Proposition 5.4.1 has a càdlàg solution in $H_r(D)$ for any $r < -\frac{d}{2}$.*

Proof. By the stopping time argument exposed after the proof of Proposition 5.4.1, we can suppose that u satisfies (5.4.8). In the following, we therefore suppose that $L = L_1$ as in (5.4.5). Also, since $\gamma < \frac{1}{2} + \frac{1}{d}$, we deduce that $2\gamma < 1 + \frac{2}{d}$, and for δ small enough, $2\gamma + 2\gamma\delta < 1 + \frac{2}{d}$. Therefore, by (5.4.9),

$$\sup_{(t,x)\in[0,T]\times D} \mathbb{E}\left[\sigma(u(t,x))^{2+\delta}\right] \leqslant \sup_{(t,x)\in[0,T]\times D} C\left(1 + \mathbb{E}\left[|u(t,x)|^{2\gamma+2\gamma\delta}\right]\right) < +\infty. \tag{5.4.14}$$

Step 1: Let $\varepsilon > 0$, and

$$u_\varepsilon(t,x) = \int_0^t \int_D G_D(t-s;x,y)\sigma\left(u(s,y)\right)\mathbb{1}_{y\in B_\varepsilon^c(\partial D)} L(\mathrm{d}s,\mathrm{d}y).$$

Then, by (5.4.13), we can write

$$\begin{aligned} u_\varepsilon(t,x) = &\int_0^t \int_D \rho_H(t-s;x-y)\sigma\left(u(s,y)\right)\mathbb{1}_{y\in B_\varepsilon^c(\partial D)} L(\mathrm{d}s,\mathrm{d}y) \\ &+ \int_0^t \int_D H(t-s;x,y)\sigma\left(u(s,y)\right)\mathbb{1}_{y\in B_\varepsilon^c(\partial D)} L(\mathrm{d}s,\mathrm{d}y) \\ =: &\, u_\varepsilon^1(t,x) + u_\varepsilon^2(t,x). \end{aligned}$$

For u_ε^1, by (5.4.14), we can follow the proof of the càdlàg property of u^1 in the proofs of Propositions 5.3.9 and 5.3.12 to get that $t \mapsto u_\varepsilon^1(t,\cdot)$ has a càdlàg version in $H_{r,\mathrm{loc}}(\mathbb{R}^d)$ for any $r < -\frac{d}{2}$, and by Remark 5.4.5, it has a càdlàg version in $H_r(D)$ for any $r < -\frac{d}{2}$. Then, since

$(t,x,y) \mapsto H(t;x,y)$ is smooth on $[0,T] \times D \times B_\varepsilon^c(\partial D)$ by Proposition 5.4.4, we can mimic the proof of the joint continuity of u^2 in the proofs of Propositions 5.3.9 and 5.3.12 to get that $(t,x) \mapsto u_\varepsilon^2(t,x)$ is jointly continuous. Since D is bounded, we deduce that u_ε^2 is uniformly continuous, and then that $t \mapsto u_\varepsilon^2(t,\cdot)$ is continuous in $H_r(D)$ for any $r \leqslant 0$. Therefore, $t \mapsto u_\varepsilon(t,\cdot)$ has a càdlàg version in $H_r(D)$ for any $r < -\frac{d}{2}$.

Step 2: By definition,

$$u_\varepsilon^3(t,x) := u(t,x) - u_\varepsilon(t,x) = \int_0^t \int_D G_D(t-s;x,y)\sigma\left(u(s,y)\right) \mathbb{1}_{y \in B_\varepsilon(\partial D)} L(\mathrm{d}s,\mathrm{d}y).$$

Then,

$$\|u_\varepsilon^3(t,\cdot)\|_{H_r(D)}^2 = \sum_{k \geqslant 1} (1+\lambda_k)^r \left(a_k^\varepsilon(t)\right)^2,$$

where

$$\begin{aligned}
a_k^\varepsilon(t) :&= \int_D \Phi_k(x)\left(u_\varepsilon^3(t,x)\right)\mathrm{d}x \\
&= \int_D \Phi_k(x)\left(\int_0^t \int_D G_D(t-s;x,y)\sigma\left(u(s,y)\right)\mathbb{1}_{y \in B_\varepsilon(\partial D)} L(\mathrm{d}s,\mathrm{d}y)\right)\mathrm{d}x \\
&= \int_D \Phi_k(x)\left(b\int_0^t \int_D G_D(t-s;x,y)\sigma\left(u(s,y)\right)\mathbb{1}_{y \in B_\varepsilon(\partial D)}\,\mathrm{d}s\,\mathrm{d}y\right)\mathrm{d}x \\
&\quad + \int_D \Phi_k(x)\left(\int_0^t \int_D \int_{|z|\leqslant 1} G_D(t-s;x,y)\sigma\left(u(s,y)\right)\mathbb{1}_{y \in B_\varepsilon(\partial D)} \tilde{J}(\mathrm{d}s,\mathrm{d}y,\mathrm{d}z)\right)\mathrm{d}x.
\end{aligned}$$

A simple Fubini theorem on the Lebesgue integral allows us to change the order of integration. For the stochastic integral, we can use a limiting argument similar to the one exposed in the proof of (5.3.20), with $p \vee 1 \leqslant \beta < 1 + \frac{2}{d}$. Therefore,

$$a_k^\varepsilon(t) = \int_0^t \int_D e^{-\lambda_k(t-s)}\Phi_k(y)\sigma\left(u(s,y)\right)\mathbb{1}_{y \in B_\varepsilon(\partial D)} L(\mathrm{d}s,\mathrm{d}y).$$

We use the identity $e^{-\lambda_k(t-s)} = 1 - \int_s^t \lambda_k e^{-\lambda_k(t-r)}\,\mathrm{d}r$, and Theorem A.0.2:

$$\begin{aligned}
a_k^\varepsilon(t) &= \int_0^t \int_D \Phi_k(y)\sigma\left(u(s,y)\right)\mathbb{1}_{y \in B_\varepsilon(\partial D)} L(\mathrm{d}s,\mathrm{d}y) \\
&\qquad - \int_0^t \int_D \Phi_k(y)\left(\int_s^t \lambda_k e^{-\lambda_k(t-r)}\,\mathrm{d}r\right)\sigma\left(u(s,y)\right)\mathbb{1}_{y \in B_\varepsilon(\partial D)} L(\mathrm{d}s,\mathrm{d}y) \\
&= \int_0^t \int_D \Phi_k(y)\sigma\left(u(s,y)\right)\mathbb{1}_{y \in B_\varepsilon(\partial D)} L(\mathrm{d}s,\mathrm{d}y) \\
&\qquad - \int_0^t \left(\int_0^r \int_D \Phi_k(y)\lambda_k e^{-\lambda_k(t-r)}\sigma\left(u(s,y)\right)\mathbb{1}_{y \in B_\varepsilon(\partial D)} L(\mathrm{d}s,\mathrm{d}y)\right)\mathrm{d}r.
\end{aligned}$$

Therefore,

$$\begin{aligned}\left|a_k^\varepsilon(t)\right| &\leqslant \sup_{t\in[0,T]}\left|\int_0^t\int_D \Phi_k(y)\sigma\left(u(s,y)\right)\mathbb{1}_{y\in B_\varepsilon(\partial D)}L(\mathrm{d}s,\mathrm{d}y)\right|\left(1+\int_0^t \lambda_k e^{-\lambda_k(t-r)}\,\mathrm{d}r\right)\\ &\leqslant C\sup_{t\in[0,T]}\left|\int_0^t\int_D \Phi_k(y)\sigma\left(u(s,y)\right)\mathbb{1}_{y\in B_\varepsilon(\partial D)}L(\mathrm{d}s,\mathrm{d}y)\right|.\end{aligned}$$

Also, using the fact that $L(\mathrm{d}s,\mathrm{d}y)=b\,\mathrm{d}s\,\mathrm{d}y+L^M(\mathrm{d}s,\mathrm{d}y)$, we have

$$\begin{aligned}&\mathbb{E}\left[\sup_{t\in[0,T]}\left|\int_0^t\int_D \Phi_k(y)\sigma\left(u(s,y)\right)\mathbb{1}_{y\in B_\varepsilon(\partial D)}L(\mathrm{d}s,\mathrm{d}y)\right|^2\right]\\ &\qquad\leqslant C\int_0^T\int_D \Phi_k(y)^2\mathbb{E}\left[\left|\sigma\left(u(s,y)\right)\right|^2\right]\mathbb{1}_{y\in B_\varepsilon(\partial D)}\,\mathrm{d}s\,\mathrm{d}y.\end{aligned}$$

Using (5.4.14) and Hölder's inequality, we deduce that for some $\delta>0$ small enough,

$$\begin{aligned}\mathbb{E}\left[\sup_{t\in[0,T]}\left|a_k^\varepsilon(t)\right|^2\right] &\leqslant C\int_0^T\int_D \Phi_k(y)^2\mathbb{E}\left[\left|\sigma\left(u(s,y)\right)\right|^2\right]\mathbb{1}_{y\in B_\varepsilon(\partial D)}\,\mathrm{d}s\,\mathrm{d}y\\ &\leqslant C\left(\int_D \Phi_k(y)^{2+\delta}\,\mathrm{d}y\right)^{\frac{2}{2+\delta}}\left(\mathrm{Leb}_d\left(B_\varepsilon(\partial D)\right)\right)^{\frac{\delta}{2+\delta}}\\ &\leqslant C\left\|\Phi_k\right\|_{L^2(D)}^{\frac{4}{2+\delta}}\left\|\Phi_k\right\|_\infty^{\frac{2\delta}{2+\delta}}\left(\mathrm{Leb}_d\left(B_\varepsilon(\partial D)\right)\right)^{\frac{\delta}{2+\delta}}.\end{aligned}$$

We can now use (5.4.11) with $\alpha=d$ and the fact that $\|\Phi_k\|_{L^2(D)}=1$ to get

$$\mathbb{E}\left[\sup_{t\in[0,T]}\left|a_k^\varepsilon(t)\right|^2\right]\leqslant C(1+\lambda_k)^{\frac{d\delta}{2+\delta}}\left(\mathrm{Leb}_d\left(B_\varepsilon(\partial D)\right)\right)^{\frac{\delta}{2+\delta}}.$$

By (5.4.10), and since $r<-\frac{d}{2}$, we can choose δ small enough such that $\sum_{k\geqslant 1}(1+\lambda_k)^{r+\frac{d\delta}{2+\delta}}<+\infty$. Then,

$$\mathbb{E}\left[\sup_{t\in[0,T]}\left\|u_\varepsilon^3(t,\cdot)\right\|_{H_r(D)}^2\right]\leqslant C\left(\mathrm{Leb}_d\left(B_\varepsilon(\partial D)\right)\right)^{\frac{\delta}{2+\delta}}\to 0\qquad\text{as }\varepsilon\to 0.$$

We deduce that $u_\varepsilon^3(t,\cdot)\to 0$ in $H_r(D)$ uniformly in $t\in[0,T]$.

Step 3: We have

$$u(t,x)=u_\varepsilon^1(t,x)+u_\varepsilon^2(t,x)+u_\varepsilon^3(t,x),$$

where by Step 1, $t\mapsto u_\varepsilon^1(t,\cdot)$ has a càdlàg version in $H_{r,\mathrm{loc}}(\mathbb{R}^d)$ for any $r<-\frac{d}{2}$, and by Remark 5.4.5, it has a càdlàg version in $H_r(D)$ for any $r<-\frac{d}{2}$. Also, $t\mapsto u_\varepsilon^2(t,\cdot)$ is continuous in $H_r(D)$ for any $r\leqslant 0$. By Step 2, $u_\varepsilon^3(t,\cdot)\to 0$ in $H_r(D)$ uniformly in $t\in[0,T]$, therefore, $t\mapsto u(t,\cdot)$ has a càdlàg version in $H_r(D)$. □

5.4.3 Continuity in space at fixed time

Proposition 5.4.7. *Let L be a pure jump Lévy white noise as in* (5.4.2) *with Lévy measure ν. Suppose that for some $0 < p \leqslant \frac{2}{d}$, $\int_{|z|\leqslant 1} |z|^p \nu(\mathrm{d}z) < \infty$. Assume as in* (5.4.3) *that $b_0 = b - \int_{|z|\leqslant 1} z\nu(\mathrm{d}z) = 0$. Let $t \in [0,T]$ be fixed. Then the process $x \mapsto u(t,x)$ defined in Proposition 5.4.1 is continuous.*

Proof. By the stopping time argument developed after the proof of Proposition 5.4.1, we can suppose that u is solution to (5.4.8). Since $d \geqslant 2$ and $p \leqslant \frac{2}{d}$, the jumps are summable and we can write:

$$u(t,x) = \sum_{i\geqslant 1} Z_i G_D(t-T_i;x,X_i)\sigma\left(u(T_i,X_i)\right).$$

Since a.s., no jumps occurs at time t, each term of this sum is continuous in x, and we now prove that the convergence is uniform in x on compact sets. Also, by (5.4.7), we can use the same estimates as in the proof of Proposition 5.3.13 (case $p \leqslant 1$) for the uniform convergence of the sum defining the term A, so we deduce that $x \mapsto u(t,x)$ is continuous. □

Case of an α-stable noise, $\frac{2}{d} \leqslant \alpha < 2$.

In this section, we suppose that the noise is an α-stable noise L_α on $[0,T]\times D$, for some $\alpha \in [\frac{2}{d}, 2)$.

Proposition 5.4.8. *Let u be the mild solution of the stochastic heat equation with additive α-stable noise, as defined in Proposition 5.4.1:*

$$u(t,x) = \int_0^t \int_D G_D(t-s;x,y) L_\alpha(\mathrm{d}s,\mathrm{d}y).$$

For any $t \in [0,T]$, there is a set $N_t \subset \Omega$ of probability one such that for any $\omega \in N_t$, $x \mapsto u(t,x)(\omega)$ is unbounded on any non-empty open subset of D.

Proof. Fix $t \in [0,T]$. Observe that the process $(Y(x), x \in D)$ defined by

$$Y(x) = u(t,x) = \int_0^t \int_D G_D(t-s;x,y) L_\alpha(\mathrm{d}s,\mathrm{d}y)$$

is an α-stable process given in the "standard form" of [62, (10.1.1)] with the measurable space $E = [0,T]\times D$, and the control measure $\mathrm{d}s\,\mathrm{d}y$. We shall check that the necessary condition [62, (10.2.14)] for sample path boundedness in [62, Theorem 10.2.3] is not satisfied, in particular that for any $x_0 \in \mathring{D}$, and δ such that $X^* := B_{x_0}(\delta) \subset D$,

$$\int_0^t \int_D \left(\sup_{x\in X^*} G_D(t-s,x,y)\right)^\alpha \mathrm{d}s\,\mathrm{d}y = +\infty. \tag{5.4.15}$$

By [68, Theorem 2 and Lemma 9], for any $x, y \in X^*$,

$$G_D(t-s, x, y) \geqslant C \frac{e^{-\frac{|x-y|^2}{4t}}}{(4\pi t)^{\frac{d}{2}}}, \tag{5.4.16}$$

(instead of this sophisticated estimate, we could use (5.4.13)). Therefore,

$$\begin{aligned}\int_0^t \int_D \sup_{x\in X^*} G_D(t-s;x,y)^\alpha \, \mathrm{d}s \, \mathrm{d}y &\geqslant \int_0^t \int_{X^*} \sup_{x\in X^*} G_D(t-s;x,y)^\alpha \, \mathrm{d}s \, \mathrm{d}y \\ &\geqslant C \int_0^t \int_{X^*} \frac{1}{(4\pi(t-s))^{\frac{\alpha d}{2}}} \, \mathrm{d}s \, \mathrm{d}y = +\infty,\end{aligned}$$

and (5.4.15) is proved. □

5.4.4 Continuity in time at a fixed space point

The next result is similar to Propositions 5.2.12 and 5.3.17.

Proposition 5.4.9. *Let L be a pure jump Lévy white noise with Lévy measure ν. Suppose that for some $0 < p < 1$, $\int_{|z|\leqslant 1} |z|^p \nu(\mathrm{d}z) < \infty$. Assume as in (5.4.3) that $b_0 = b - \int_{|z|\leqslant 1} z\nu(\mathrm{d}z) = 0$. Let $x \in D$ be fixed. Then the process $t \mapsto u(t,x)$ has a continuous modification.*

Proof. Regarding the stochastic integral with respect to L^M, we observe that the jumps of the noise are summable, hence, since $b_0 = 0$, it is sufficient to consider the uncompensated process

$$\int_0^t \int_D \int_{\mathbb{R}} z G_D(t-s;x,y)\sigma(u(s,y)) J(\mathrm{d}s,\mathrm{d}y,\mathrm{d}z) = \sum_{i\geqslant 1} Z_i G_D(t-T_i;x,Y_i)\sigma(u(T_i,Y_i)), \tag{5.4.17}$$

where (T_i, Y_i, Z_i) are the jump points of the underlying Poisson random measure J. For any fixed $(x,y) \in D^2$, $x \neq y$, we have by (5.4.13) that $t \mapsto G_D(t;x,y)$ is a continuous function on $\mathbb{R}$. We show that the sum in (5.4.17) converges uniformly in $t \in [0,T]$. To do this we can split the sum depending on the distance of the jump Y_i to x. Indeed, by (5.4.7), for $|x-y| \leqslant \sqrt{3dT}$, we have

$$\sup_{t\in[0,T]} G_D(t;x,y) \leqslant \sup_{t\in[0,T]} \frac{C}{t^{\frac{d}{2}}} e^{-\frac{(x-y)^2}{6t}} = \frac{C'}{|x-y|^d},$$

for some constant C'. Also, if $|x-y| > \sqrt{3dT}$, then

$$\sup_{t\in[0,T]} G_D(t;x,y) \leqslant \sup_{t\in[0,T]} \frac{C}{t^{\frac{d}{2}}} e^{-\frac{(x-y)^2}{6t}} = \frac{C}{T^{\frac{d}{2}}} e^{-\frac{(x-y)^2}{6T}}.$$

Since $0 < p < 1$,

$$\mathbb{E}\left[\left(\sum_{i\geqslant 1} \sup_{t\in[0,T]} |Z_i G_D(t-T_i;x,Y_i)\sigma(u(T_i,Y_i))|\right)^p\right]$$

$$\leqslant \mathbb{E}\left[\sum_{i\geqslant 1} |Z_i|^p |\sigma(u(T_i, Y_i))|^q \sup_{t\in[0,T]} |G(t-T_i; x, Y_i)|^p\right]$$
$$\leqslant C \sup_{(s,y)\in[0,T]\times D} \mathbb{E}\left[\left|\sigma\left(u(s,y)\right)\right|^p\right]\left(\int_0^T\int_D\int_{|z|\leqslant 1} |z|^p \frac{1}{|x-y|^{pd}} \mathbb{1}_{|x-y|\leqslant\sqrt{3d(T-s)}}\, \mathrm{d}s\,\mathrm{d}y\,\nu(\mathrm{d}z)\right.$$
$$\left. + \int_0^T\int_D\int_{|z|\leqslant 1} |z|^p \frac{1}{(T-s)^{d\frac{p}{2}}} e^{-\frac{p(x-y)^2}{6(T-s)}} \mathbb{1}_{|x-y|>\sqrt{3d(T-s)}}\, \mathrm{d}s\,\mathrm{d}y\,\nu(\mathrm{d}z)\right) < +\infty,$$

which concludes the proof. □

Remark 5.4.10. *In particular, any α-stable noise with $\alpha \in (0,1)$ satisfies the hypothesis of Proposition 5.4.9, as occurred in Remarks 5.2.13 and 5.3.18. The next section shows that for $\alpha \geqslant 1$, the situation is completely different.*

The case of an α-stable noise, $1 \leqslant \alpha < 2$

In this section, we consider the stochastic heat equation on $[0,T]\times D$ with Dirichlet boundary conditions, with additive α-stable noise L_α on $[0,T]\times D$. We establish the analog of Propositions 5.2.14 and 5.3.19

Proposition 5.4.11. *Let u be the mild solution of the stochastic heat equation with additive α-stable noise, $1\leqslant\alpha<2$:*

$$u(t,x) = \int_0^t\int_D G_D(t-s;x,y) L_\alpha(\mathrm{d}s,\mathrm{d}y).$$

Then for any $x\in D$, there is a set $N_x \subset \Omega$ of probability one such that for any $\omega\in N_x$, $t\mapsto u(t,x)(\omega)$ is unbounded on any non-empty open interval.

Proof. Fix $x\in D$. Observe that the process $(X(t), t\in[0,T])$ defined by

$$X(t) = u(t,x) = \int_0^t\int_D G_D(t-s;x,y) L_\alpha(\mathrm{d}s,\mathrm{d}y)$$

is an α-stable process given in the "standard form" of [62, (10.1.1)] with the measurable space $E=[0,T]\times D$, and the control measure $\mathrm{d}s\,\mathrm{d}y$. Let $T^* = [t_1,t_2]$, with $0\leqslant t_1 < t_2 \leqslant T$. We shall check that the necessary condition [62, (10.2.14)] for sample path boundedness in [62, Theorem 10.2.3] is not satisfied, in particular, that

$$\int_0^T\int_D\left(\sup_{t\in T^*} G_D(t-s;x,y)\right)^\alpha \mathrm{d}s\,\mathrm{d}y = +\infty. \tag{5.4.18}$$

Indeed, observe that the integral is bounded below by

$$\int_{t_1}^{t_2}\int_D \sup_{t\in[t_1,t_2]} G_D(t-s;x,y)^\alpha\,\mathrm{d}s\,\mathrm{d}y = \int_0^{t_2-t_1}\int_D \sup_{t\in[t_1,t_2]} G_D(t-t_1-r;x,y)^\alpha\,\mathrm{d}r\,\mathrm{d}y$$

$$
\begin{aligned}
&= \int_0^{t_2-t_1} \int_D \sup_{u \in [0,t_2-t_1]} G_D(u-r;x,y)^\alpha \, \mathrm{d}r \, \mathrm{d}y \\
&\geqslant \int_0^{t_2-t_1} \int_{B_x(\varepsilon)} \sup_{u \in [0,t_2-t_1]} G_D(u-r;x,y)^\alpha \, \mathrm{d}r \, \mathrm{d}y \\
&= \int_0^{t_2-t_1} \int_{B_x(\varepsilon)} \sup_{v \in [0,t_2-t_1-r]} G_D(v;x,y)^\alpha \, \mathrm{d}r \, \mathrm{d}y \\
&= \int_0^{t_2-t_1} \int_{B_x(\varepsilon)} \sup_{v \in [0,s]} G_D(v;x,y)^\alpha \, \mathrm{d}s \, \mathrm{d}y,
\end{aligned}
$$

for any fixed $\varepsilon > 0$ such that $B_x(\varepsilon) \subset D$. We now use (5.4.16), and the study of the maximum of $t \mapsto G_D(t, x-y)$ in the proof of Proposition 5.4.9, to get

$$
\int_0^T \int_D \left(\sup_{t \in T^*} G_D(t-s;x,y) \right)^\alpha \mathrm{d}s \, \mathrm{d}y \geqslant \int_0^{t_2-t_1} \int_{|x-y| \leqslant \varepsilon \wedge \sqrt{2ds}} \frac{C}{|x-y|^{\alpha d}} \, \mathrm{d}s \, \mathrm{d}y = +\infty,
$$

and (5.4.18) is proved. $\square$

A Appendix

In this appendix, we state some useful theorems using the notations of this thesis. Most of the time, these theorems will not be stated in their full generality, but will be instead adapted to our framework.

The first theorem is a result from [52, Theorem 1], and is a series of inequalities for moments of stochastic integrals with respect to compensated Poisson random measures. These inequalities are sometimes called Bichteler-Jacod inequalities.

Theorem A.0.1. *Let J be a Poisson random measure on $\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}$, $d \geqslant 1$, with intensity measure $\mathrm{d}t\,\mathrm{d}x\,\nu(\mathrm{d}z)$. Let $\tilde{J}$ be the associated compensated Poisson random measure. Let $H : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$ be predictable random field. Assume that*

$$\int_{[0,t]\times\mathbb{R}^d\times\mathbb{R}} \left(\left|H(s,y)z\right|^2 \wedge \left|H(s,y)z\right|\right) \mathrm{d}s\,\mathrm{d}y\,\nu(\mathrm{d}z) < +\infty.$$

Then, $I_t(H) := \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}} zH(s,y)\tilde{J}(\mathrm{d}s,\mathrm{d}y,\mathrm{d}z)$ is well defined and we have the following estimates:

(i) If $p \in (0,2]$, then

$$\mathbb{E}\left[\sup_{s\in[0,t]} |I_s(H)|^p\right] \leqslant C_p \mathbb{E}\left[\left(\int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}} |zH(s,y)|^2 \,\mathrm{d}s\,\mathrm{d}y\,\nu(\mathrm{d}z)\right)^{\frac{p}{2}}\right].$$

(ii) If $p \in [1,2]$, then

$$\mathbb{E}\left[\sup_{s\in[0,t]} |I_s(H)|^p\right] \leqslant C_p \mathbb{E}\left[\left(\int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}} |zH(s,y)|^p \,\mathrm{d}s\,\mathrm{d}y\,\nu(\mathrm{d}z)\right)\right].$$

(iii) If $p \geqslant 2$*, then*

$$\mathbb{E}\left[\sup_{s\in[0,t]} |I_s(H)|^p\right] \leqslant C_p \mathbb{E}\left[\left(\int_0^t \int_{\mathbb{R}^d}\int_{\mathbb{R}} |zH(s,y)|^2 \,\mathrm{d}s\,\mathrm{d}y\,\nu(\mathrm{d}z)\right)^{\frac{p}{2}}\right] + C_p \mathbb{E}\left[\left(\int_0^t \int_{\mathbb{R}^d}\int_{\mathbb{R}} |zH(s,y)|^p \,\mathrm{d}s\,\mathrm{d}y\,\nu(\mathrm{d}z)\right)\right].$$

The next result is a stochastic Fubini theorem taken from [3, Theorem 5] for stochastic integrals with respect to compensated Poisson random measures.

Theorem A.0.2. *Let* J *be a Poisson random measure on* $\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}$*,* $d \geqslant 1$*, with intensity measure* $\mathrm{d}t\,\mathrm{d}x\,\nu(\mathrm{d}z)$*. Let* $\tilde{J}$ *be the associated compensated Poisson random measure. Let* $H : [0,T]\times\mathbb{R}^d\times\mathbb{R}^d \to \mathbb{R}$ *be predictable random field. Let* μ *be a finite measure on* $\mathbb{R}^d$*, and assume that*

$$\int_{\mathbb{R}^d}\mu(\mathrm{d}x)\int_0^T \mathrm{d}s\int_{\mathbb{R}^d}\mathrm{d}y\int_{\mathbb{R}}\nu(\mathrm{d}z)\mathbb{E}\left[\left|zH(s,x,y)\right|^2\right] < +\infty.$$

Then, for any $0 \leqslant t \leqslant T$*,*

$$\int_{\mathbb{R}^d}\left(\int_0^t\int_{\mathbb{R}^d\times\mathbb{R}} zH(s,x,y)\tilde{J}(\mathrm{d}s,\mathrm{d}y,\mathrm{d}z)\right)\mu(\mathrm{d}x) = \int_0^t\int_{\mathbb{R}^d\times\mathbb{R}}\left(\int_{\mathbb{R}^d} zH(s,x,y)\mu(\mathrm{d}x)\right)\tilde{J}(\mathrm{d}s,\mathrm{d}y,\mathrm{d}z),$$

almost surely.

In Chapter 5, we use extensively a sufficient condition for the existence of a *càdlàg* version of a stochastic process with values in a Hilbert space. This result can be found in [37, §4, Theorem 1, p.179], which gives a sufficient condition for the absence of discontinuities of the second kind for a stochastic process with values in a complete metric space. In the case of a Hilbert space, their result particularizes as follows.

Theorem A.0.3. *Let* H *be an Hilbert space equipped with the norm* $\|\cdot\|_H$*. Let* $(X_t)_{t\geqslant 0}$ *be a* H*-valued separable stochastic process that is continuous in probability. Suppose that for any* $t \geqslant 0$ *and any* $0 \leqslant h \leqslant t$*,*

$$\mathbb{E}\left[\|X_{t+h}-X_t\|_H^2\,\|X_t - X_{t-h}\|_H^2\right] \leqslant Ch^{1+r},$$

for some $r > 0$*. Then the process* X *has a* càdlàg *version.*

Theorem A.0.3 will be useful to obtain the existence of a *càdlàg* version of $t \mapsto u(t,\cdot)$ in a fractional Sobolev space, where u is the mild solution to the stochastic heat equation under some additional assumptions. We provide the following technical lemma to prove that a certain type of stochastic integral with respect to a Poisson random measure is continuous in probability.

Lemma A.0.4. *Let* D *be either* $[0,\pi]$*,* $\mathbb{R}^d$ *or a bounded and smooth domain in* $\mathbb{R}^d$*. Let* J *be a Poisson random measure on* $\mathbb{R}_+\times D\times\mathbb{R}$*, with intensity measure* $\mathrm{d}t\,\mathrm{d}x\,\nu(\mathrm{d}z)$*, where* ν *is a Lévy* *measure and* $\int_{|z|\leqslant 1}|z|^\beta \nu(\mathrm{d}z) < +\infty$ *for some* $1 \leqslant \beta < 1 + \frac{2}{d}$. *Let* Z *be a random field such that*

$$\sup_{(t,x)\in[0,T]\times D} \mathbb{E}\left[\left|Z(s,y)\right|^\beta\right] < +\infty .$$

Then, the random field $(t,x) \mapsto u(t,x)$ *defined by*

$$u(t,x) = \int_0^t \int_D \int_{|z|\leqslant 1} z G_D(t-s;x,y) Z(s,y) \tilde{J}(\mathrm{d}s, \mathrm{d}y, \mathrm{d}z) ,$$

has a separable version that is continuous in probability, where G_D *is the Green's function of the heat operator on the domain* D.

Proof. Using [37, Theorem 1], we deduce that u has a separable version, that we will still denote by u. Let $(t,x),(r,z) \in [0,T]\times D$. We suppose that $t \geqslant r$. Then,

$$\begin{aligned}
u(t,x) - u(r,z) &= u(t,x) - u(r,x) + u(r,x) - u(r,z) \\
&= \int_0^r \int_D \int_{|z|\leqslant 1} z\left(G_D(t-s;x,y) - G_D(r-s;x,y)\right) Z(s,y)\tilde{J}(\mathrm{d}s, \mathrm{d}y, \mathrm{d}z) \\
&\quad + \int_r^t \int_D \int_{|z|\leqslant 1} z G_D(t-s;x,y) Z(s,y)\tilde{J}(\mathrm{d}s, \mathrm{d}y, \mathrm{d}z) \\
&\quad + \int_0^r \int_D \int_{|z|\leqslant 1} z\left(G_D(r-s;x,y) - G_D(r-s;z,y)\right) Z(s,y)\tilde{J}(\mathrm{d}s, \mathrm{d}y, \mathrm{d}z) .
\end{aligned}$$

Therefore,

$$\mathbb{E}\left[|u(t,x) - u(r,z)|^\beta\right] \leqslant C\left(I_1 + I_2 + I_3\right) ,$$

where

$$\begin{aligned}
I_1 &= \mathbb{E}\left[\left|\int_0^r \int_D \int_{|z|\leqslant 1} z\left(G_D(t-s;x,y) - G_D(r-s;x,y)\right) Z(s,y)\tilde{J}(\mathrm{d}s, \mathrm{d}y, \mathrm{d}z)\right|^\beta\right] , \\
I_2 &= \mathbb{E}\left[\left|\int_r^t \int_D \int_{|z|\leqslant 1} z G_D(t-s;x,y) Z(s,y)\tilde{J}(\mathrm{d}s, \mathrm{d}y, \mathrm{d}z)\right|^\beta\right] , \\
I_3 &= \mathbb{E}\left[\left|\int_0^r \int_D \int_{|z|\leqslant 1} z\left(G_D(r-s;x,y) - G_D(r-s;z,y)\right) Z(s,y)\tilde{J}(\mathrm{d}s, \mathrm{d}y, \mathrm{d}z)\right|^\beta\right] .
\end{aligned}$$

Using Theorem A.0.1 *(ii)*, we get

$$\begin{aligned}
I_1 &\leqslant C \int_0^r \int_D \left|G_D(t-s;x,y) - G_D(r-s;x,y)\right|^\beta \mathrm{d}s\,\mathrm{d}y , \\
I_2 &\leqslant C \int_r^t \int_D \left|G_D(t-s;x,y)\right|^\beta \mathrm{d}s\,\mathrm{d}y , \\
I_3 &\leqslant C \int_0^r \int_D \left|G_D(r-s;x,y) - G_D(r-s;z,y)\right|^\beta \mathrm{d}s\,\mathrm{d}y .
\end{aligned}$$

Since $\beta < 1 + \frac{2}{d}$, $(s, y) \mapsto G_D(t-s; x, y) \in L^\beta([0,T] \times D)$. Therefore,

$$I_2 \to 0, \qquad \text{as } r \to t.$$

We can rewrite I_1 as

$$I_1 \leqslant C \int_0^T \|G_D(t-s; x, \cdot) - G_D(r-s; x, \cdot)\|_{L^\beta(D)}^\beta \, \mathrm{d}s.$$

Since $\beta \geqslant 1$, we can use the fact that simple functions are dense in $L^\beta\big(\mathbb{R}, L^\beta(D)\big)$ (see [30, Chapter III, Corollary 3.8 p. 125]), the proof of [53, Chapter XIII, Corollary 1.2] applies (see also [19, Proposition 4.1]), and we deduce that

$$I_1 \to 0 \qquad \text{as } r \to t.$$

Also, from Proposition 5.4.4 we can decompose $G_D(r-s; x, y) = \rho_H(r-s; x-y) + H(r-s; x, y)$, where since $x \in D$ is fixed, $(s, y) \mapsto H(r-s; x, y)$ is smooth on $[0,T] \times D$. Therefore,

$$\begin{aligned} I_3 \leqslant C \bigg(&\int_0^T \int_{\mathbb{R}^d} \big|\rho_H(r-s; x-y) - \rho_H(r-s; z-y)\big|^\beta \, \mathrm{d}s \, \mathrm{d}y \\ &+ \int_0^T \int_D \big|H(r-s; x, y) - H(r-s; z, y)\big|^\beta \, \mathrm{d}s \, \mathrm{d}y \bigg). \end{aligned}$$

Then,

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}^d} \big|\rho_H(r-s; x-y) - \rho_H(r-s; z-y)\big|^\beta \, \mathrm{d}s \, \mathrm{d}y \\ &\qquad = \int_{\mathbb{R}^d} \big\|\rho_H(r-\cdot; x-y) - \rho_H(r-\cdot; z-y)\big\|_{L^\beta([0,T])}^\beta \, \mathrm{d}y, \end{aligned}$$

and we can conclude as for I_1 that

$$\int_0^r \int_{\mathbb{R}^d} \big|\rho_H(r-s; x-y) - \rho_H(r-s; z-y)\big|^\beta \, \mathrm{d}s \, \mathrm{d}y \to 0, \qquad \text{as } z \to x.$$

The function H is non-zero if and only if D is a bounded domain, and by smoothness of H and the dominated convergence theorem,

$$\int_0^r \int_D \big|H(r-s; x, y) - H(r-s; z, y)\big|^\beta \, \mathrm{d}s \, \mathrm{d}y \to 0, \qquad \text{as } z \to x.$$

Therefore,

$$I_3 \to 0 \qquad \text{as } z \to x.$$

We conclude that $u(r,z) \to u(t,x)$ as $(r,z) \to (t,x)$ in $L^\beta(\Omega)$, and therefore u is continuous in probability. □

The next result is a technical lemma that we use many times in this thesis: it concerns the maximum in time of the heat kernel.

Lemma A.0.5. *For $d \geqslant 1$, recall that*

$$\rho_H(t,x) = \frac{C}{t^{\frac{d}{2}}} e^{-\frac{|x|^2}{4t}},$$

for some constant C that we do not need to specify here. Then,

(i) If $T < \frac{|x|^2}{2d}$, then

$$\sup_{t\in[0,T]} \rho_H(t,x) = \frac{C}{T^{\frac{d}{2}}} e^{-\frac{|x|^2}{4T}}.$$

(ii) If $T \geqslant \frac{|x|^2}{2d}$, then

$$\sup_{t\in[0,T]} \rho_H(t,x) = \frac{C'}{|x|^d},$$

for some constant C'.

Proof. We study the maximum of the function $t \mapsto \rho_H(t,x)$. The derivative of this function is given by

$$\frac{\partial \rho_H}{\partial t}(t,x) = C t^{-\frac{d}{2}-2}\left(\frac{|x|^2}{4} - \frac{d}{2}t\right) e^{-\frac{|x|^2}{4t}}.$$

This derivatives cancels if and only if $t = \frac{|x|^2}{2d}$, and is positive for values of t smaller than this threshold, and negative otherwise. This point is therefore a maximum, and plugging this value of t in the expression of ρ_H yields the result. □

Bibliography

[1] Adler, R. J., Monrad, D., Scissors, R. H., and Wilson, R. Representations, decompositions and sample function continuity of random fields with independent increments. *Stochastic Process. Appl.*, 15(1):3–30, 1983.

[2] Allen, E. J. Derivation of stochastic partial differential equations for size- and age-structured populations. *Journal of Biological Dynamics*, 3(1):73–86, 2009. PMID: 22880751.

[3] Applebaum, D. Martingale-valued measures, Ornstein-Uhlenbeck processes with jumps and operator self-decomposability in Hilbert space. In *In memoriam Paul-André Meyer: Séminaire de Probabilités XXXIX*, volume 1874 of *Lecture Notes in Math.*, pp. 171–196. Springer, Berlin, 2006.

[4] Balan, R. M. SPDEs with α-stable Lévy noise: a random field approach. *Int. J. Stoch. Anal.*, pp. Art. ID 793275, 22, 2014.

[5] Balan, R. M. Integration with respect to Lévy colored noise, with applications to SPDEs. *Stochastics*, 87(3):363–381, 2015.

[6] Bally, V., Millet, A., and Sanz-Solé, M. Approximation and support theorem in Hölder norm for parabolic stochastic partial differential equations. *Ann. Probab.*, 23(1):178–222, 1995.

[7] Barndorff-Nielsen, O. E. and Basse-O'Connor, A. Quasi Ornstein-Uhlenbeck processes. *Bernoulli*, 17(3):916–941, 2011.

[8] Barndorff-Nielsen, O. E., Benth, F. E., and Veraart, A. E. D. Ambit processes and stochastic partial differential equations. In *Advanced mathematical methods for finance*, pp. 35–74. Springer, Heidelberg, 2011.

[9] Barndorff-Nielsen, O. E. and Schmiegel, J. Ambit processes: with applications to turbulence and tumour growth. In *Stochastic analysis and applications*, volume 2 of *Abel Symp.*, pp. 93–124. Springer, Berlin, 2007.

[10] Bichteler, K. and Jacod, J. Random measures and stochastic integration. In *Theory and application of random fields (Bangalore, 1982)*, volume 49 of *Lecture Notes in Control and Inform. Sci.*, pp. 1–18. Springer, Berlin, 1983.

[11] Chen, L. and Dalang, R. C. Hölder-continuity for the nonlinear stochastic heat equation with rough initial conditions. *Stoch. Partial Differ. Equ. Anal. Comput.*, 2(3):316–352, 2014.

[12] Chong, C. *Tempo-Spatial Stochastic Integral Processes: Theory and Applications.* Dissertation, Technische Universität München, München, 2015.

[13] Chong, C. Lévy-driven volterra equations in space and time. *Journal of Theoretical Probability*, pp. 1–45, 2016.

[14] Chong, C. Stochastic PDEs with heavy-tailed noise. *Stochastic Process. Appl.*, 127(7):2262–2280, 2017.

[15] Chong, C., Dalang, R. C., and Humeau, T. Paths properties of solutions to the stochastic heat equation with heavy-tailed noise. *In preparation*, 2017.

[16] Chong, C. and Klüppelberg, C. Integrability conditions for space-time stochastic integrals: theory and applications. *Bernoulli*, 21(4):2190–2216, 2015.

[17] Chung, K. L. *A course in probability theory.* Academic Press [A subsidiary of Harcourt Brace Jovanovich, Publishers], New York-London, second edition, 1974. Probability and Mathematical Statistics, Vol. 21.

[18] Cohn, D. L. Measurable choice of limit points and the existence of separable and measurable processes. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, 22:161–165, 1972.

[19] Conus, D. *The Non-linear Stochastic Wave Equation in High Dimensions: Existence, Hölder-continuity and Itô-Taylor Expansion.* PhD thesis, EPFL, 2008.

[20] Da Prato, G. and Zabczyk, J. *Stochastic Equations in Infinite Dimensions.* Cambridge University Press, 1992.

[21] Dalang, R., Khoshnevisan, D., Mueller, C., Nualart, D., and Xiao, Y. *A minicourse on stochastic partial differential equations*, volume 1962 of *Lecture Notes in Mathematics.* Springer-Verlag, Berlin, 2009. Held at the University of Utah, Salt Lake City, UT, May 8–19, 2006, Edited by Khoshnevisan and Firas Rassoul-Agha.

[22] Dalang, R. C. Extending the martingale measure stochastic integral with applications to spatially homogeneous s.p.d.e.'s. *Electron. J. Probab.*, 4:no. 6, 29 pp. (electronic), 1999.

[23] Dalang, R. C. Level sets and excursions of the Brownian sheet. In *Topics in spatial stochastic processes (Martina Franca, 2001)*, volume 1802 of *Lecture Notes in Math.*, pp. 167–208. Springer, Berlin, 2003.

[24] Dalang, R. C. and Humeau, T. Lévy processes and Lévy white noise as tempered distributions. *Ann. Probab.*, To appear.

[25] Dalang, R. C. and Quer-Sardanyons, L. Stochastic integrals for spde's: a comparison. *Expositiones Mathematicae*, 2011.

[26] Dalang, R. C. and Walsh, J. B. The sharp Markov property of Lévy sheets. *Ann. Probab.*, 20(2):591–626, 1992.

[27] Dalang, R. C. and Walsh, J. B. The sharp Markov property of the Brownian sheet and related processes. *Acta Math.*, 168(3-4):153–218, 1992.

[28] Davies, E. B. *Heat kernels and spectral theory*, volume 92 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1990.

[29] Dobrushin, R. L. and Minlos, R. A. A study of the properties of generalized Gaussian random fields. In *Problems in mechanics and mathematical physics (Russian)*, pp. 117–165. Izdat. "Nauka", Moscow, 1976.

[30] Dunford, N. and Schwartz, J. T. *Linear operators. Part I.* Wiley Classics Library. John Wiley & Sons, Inc., New York, 1988. General theory, With the assistance of William G. Bade and Robert G. Bartle, Reprint of the 1958 original, A Wiley-Interscience Publication.

[31] Durrett, R. *Probability: theory and examples.* Cambridge University Press, Cambridge, 4th edition, 2010.

[32] Èĭdel'man, S. D. and Ivasišen, S. D. Investigation of the Green's matrix of a homogeneous parabolic boundary value problem. *Trudy Moskov. Mat. Obšč.*, 23:179–234, 1970.

[33] Fageot, J. and Humeau, T. Unified View on Lévy White Noises: General Integrability Conditions and Applications to Linear SPDE. *ArXiv e-prints*, (1708.02500), August 2017.

[34] Fageot, J., Amini, A., and Unser, M. On the continuity of characteristic functionals and sparse stochastic modeling. *J. Fourier Anal. Appl.*, 20(6):1179–1211, 2014.

[35] Fernique, X. Processus linéaires, processus généralisés. *Ann. Inst. Fourier (Grenoble)*, 17(fasc. 1):1–92, 1967.

[36] Gel'fand, I. M. and Vilenkin, N. Y. *Generalized functions. Vol. 4.* Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London, 1964 [1977].

[37] Gīhman, Ĭ. Ī. and Skorohod, A. V. *The theory of stochastic processes. I.* Springer-Verlag, New York-Heidelberg, 1974. Translated from the Russian by S. Kotz, Die Grundlehren der mathematischen Wissenschaften, Band 210.

[38] Golse, F. *Distributions, Analyse de Fourier, équations aux dérivées partielles.* Ecole Polytechnique, 2010.

[39] Grisvard, P. Caractérisation de quelques espaces d'interpolation. *Arch. Rational Mech. Anal.*, 25:40–63, 1967.

[40] Hu, Y., Huang, J., Nualart, D., and Tindel, S. Stochastic heat equations with general multiplicative Gaussian noises: Hölder continuity and intermittency. *Electron. J. Probab.*, 20:no. 55, 50, 2015.

[41] Hunter, J. K. and Nachtergaele, B. *Applied analysis.* World Scientific Publishing Co., Inc., River Edge, NJ, 2001.

[42] Itô, K. *Foundations of stochastic differential equations in infinite-dimensional spaces*, volume 47 of *CBMS-NSF Regional Conference Series in Applied Mathematics.* Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1984.

[43] Jacod, J. and Shiryaev, A. N. *Limit theorems for stochastic processes*, volume 288 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences].* Springer-Verlag, Berlin, second edition, 2003.

[44] Kallenberg, O. *Foundations of modern probability.* Springer-Verlag, New York, 2nd edition, 2002.

[45] Kasperski, A. Approximation in Musielak-Orlicz sequence vector spaces of multifunctions. In *Monograph on the occasion of 100th birthday anniversary of Zygmunt Zahorski*, pp. 203–218. Wydaw. Politech. Śl., Gliwice, 2015.

[46] Keller-Ressel, M. and Müller, M. S. A stefan-type stochastic moving boundary problem. *Stochastics and Partial Differential Equations: Analysis and Computations*, 4(4):746–790, Dec 2016.

[47] Khoshnevisan, D. *Multiparameter processes - An introduction to random fields.* Springer-Verlag, New York, 2002.

[48] Khoshnevisan, D. *Analysis of stochastic partial differential equations*, volume 119 of *CBMS Regional Conference Series in Mathematics.* American Mathematical Society, Providence, RI, 2014.

[49] Kurtz, T. Lecture notes on stochastic analysis, 2007.

[50] Lebedev, V. A. Fubini's theorem for parameter-dependent stochastic integrals with respect to L^0-valued random measures. *Teor. Veroyatnost. i Primenen.*, 40(2):313–323, 1995.

[51] Lions, J.-L. and Magenes, E. *Problèmes aux limites non homogènes et applications. Vol. 1.* Travaux et Recherches Mathématiques, No. 17. Dunod, Paris, 1968.

[52] Marinelli, C. and Röckner, M. On maximal inequalities for purely discontinuous martingales in infinite dimensions. In *Séminaire de Probabilités XLVI*, volume 2123 of *Lecture Notes in Math.*, pp. 293–315. Springer, Cham, 2014.

[53] Mikusiński, J. *The Bochner integral.* Birkhäuser Verlag, Basel-Stuttgart, 1978. Lehrbücher und Monographien aus dem Gebiete der exakten Wissenschaften, Mathematische Reihe, Band 55.

[54] Musiela, M. and Zariphopoulou, T. Stochastic partial differential equations and portfolio choice. In *Contemporary quantitative finance*, pp. 195–216. Springer, Berlin, 2010.

[55] Nochetto, R. H., Otárola, E., and Salgado, A. J. A PDE approach to fractional diffusion in general domains: a priori error analysis. *Found. Comput. Math.*, 15(3):733–791, 2015.

[56] Pietsch, A. *Nuclear locally convex spaces.* Springer-Verlag, New York-Heidelberg, 1972.

[57] Protter, P. E. *Stochastic integration and differential equations*, volume 21 of *Stochastic Modelling and Applied Probability.* Springer-Verlag, Berlin, 2005. Second edition. Version 2.1, Corrected third printing.

[58] Rajput, B. S. and Rosiński, J. Spectral representations of infinitely divisible processes. *Probab. Theory Related Fields*, 82(3):451–487, 1989.

[59] Revuz, D. and Yor, M. *Continuous Martingales and Brownian Motion*, volume 293 of *A series of comprehensive studies in mathematics.* Springer, 1999.

[60] Rosiński, J. On path properties of certain infinitely divisible processes. *Stochastic Process. Appl.*, 33(1):73–87, 1989.

[61] Saint Loubert Bié, E. Étude d'une EDPS conduite par un bruit poissonnien. *Probab. Theory Related Fields*, 111(2):287–321, 1998.

[62] Samorodnitsky, G. and Taqqu, M. S. *Stable non-gaussian random processes: Stochastic models with infinite variance.* Stochastic Modeling. Chapman & Hall, 1994.

[63] Sato, K.-I. *Lévy processes and infinitely divisible distributions*, volume 68 of *Cambridge Studies in Advanced Mathematics.* Cambridge University Press, Cambridge, 2013.

[64] Schwartz, L. *Théorie des distributions.* Publications de l'Institut de Mathématique de l'Université de Strasbourg, No. IX-X. Hermann, Paris, 1966.

[65] Stein, E. M. *Singular integrals and differentiability properties of functions.* Princeton Mathematical Series, No. 30. Princeton University Press, Princeton, N.J., 1970.

[66] Straf, M. L. Weak convergence of stochastic processes with several parameters. In *Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability (Univ. California, Berkeley, Calif., 1970/1971), Vol. II: Probability theory*, pp. 187–221. Univ. California Press, Berkeley, Calif., 1972.

[67] Trèves, F. *Topological vector spaces, distributions and kernels.* Academic Press, New York-London, 1967.

[68] Van den Berg, M. A Gaussian lower bound for the Dirichlet heat kernel. *Bull. London Math. Soc.*, 24(5):475–477, 1992.

[69] Walsh, J. B. An introduction to stochastic partial differential equations. In *École d'été de probabilités de Saint-Flour, XIV—1984*, volume 1180 of *Lecture Notes in Math.*, pp. 265–439. Springer, Berlin, 1986.

[70] Wheeden, R. L. and Zygmund, A. *Measure and integral.* Marcel Dekker, Inc., New York-Basel, 1977. An introduction to real analysis, Pure and Applied Mathematics, Vol. 43.

[71] Zabczyk, J. and Peszat, S. *Stochastic Partial Differential Equations with Lévy Noise: An Evolution Equation Approach.* Cambridge University Press, 2007.

CV

Thomas Humeau was born on January 27^{th}, 1990, in Colmar, France. He attended high school in the Lycée Jean-Jaurès in Reims, France, and received his *baccalauréat* in June 2007. From September 2007 to July 2009, he went to the Lycée Sainte Geneviève in Versailles, France, for the *classe préparatoires* which prepare for the entrance exam to the French *Grandes écoles*. He entered École polytechnique in Palaiseau, France, in September 2009 to study applied mathematics. He did a research internship under the supervision of P. Tankov in the University Paris-Diderot on the simulation of killed Lévy processes, for which he was awarded the *prix du stage de recherche*. He received the *diplôme d'ingénieur* from the École polytechnique in 2012, and the *diplôme de l'École polytechnique* in 2013. For the academic year 2012-2013, he was awarded a studentship from Trinity College to attend the University of Cambridge, UK, and obtained a Master of advanced study in mathematics with distinction, for which he was awarded a Senior scholarship. He moved to Switzerland in September 2013 to pursue a PhD in the chair of probability at EPFL under the supervision of Professor R.C. Dalang.