A refined estimate for the topological degree

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Abstract

We sharpen an estimate of [4] for the topological degree of continuous maps from a sphere $S^d$ into itself in the case $d \geq 2$. This provides the answer for $d \geq 2$ to a question raised by Brezis. The problem is still open for $d = 1$.

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1 Introduction

Motivated by the theory of Ginzburg Landau equations (see, e.g., [1]), Bourgain, Brezis, and the author established in [4]:

Theorem 1. Let $d \geq 1$. For every $0 < \delta < \sqrt{2}$, there exists a positive constant $C(\delta)$ such that, for all $g \in C(S^d, S^d)$,

$$|\text{deg } g| \leq C(\delta) \int_{S^d} \int_{S^d} \frac{1}{|x-y|^2} \, dx \, dy.$$  \hspace{1cm} (1.1)

Here and in what follows, for $x \in \mathbb{R}^{d+1}$, $|x|$ denotes its Euclidean norm in $\mathbb{R}^{d+1}$.

The constant $C(\delta)$ depends also on $d$ but for simplicity of notation we omit $d$. Estimate (1.1) was initially suggested by Bourgain, Brezis, and Mironescu in [2]. It was proved in [3] in the case where $d = 1$ and $\delta$ is sufficiently small. In [9], the author improved (1.1) by establishing that (1.1) holds for $0 < \delta < \ell_d = \sqrt{2 + \frac{2}{d+1}}$ with a constant $C(\delta)$ independent of $\delta$. It was also shown there that (1.1) does not hold for $\delta \geq \ell_d$.

This note is concerned with the behavior of $C(\delta)$ as $\delta \to 0$. Brezis [7] (see also [6, Open problem 3]) conjectured that (1.1) holds with

$$C(\delta) = C\delta^d,$$  \hspace{1cm} (1.2)

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for some positive constant $C$ depending only on $d$. This conjecture is somehow motivated by the fact that (1.1)-(1.2) holds “in the limit” as $\delta \to 0$. More precisely, it is known that (see [8, Theorem 2])

$$\lim_{\delta \to 0} \int_{|g(x) - g(y)| > \delta} \frac{\delta^d}{|x - y|^{2d}} \, dx \, dy = K_d \int_{\mathbb{S}^d} |\nabla g(x)|^d \, dx \quad \text{for } g \in C^1(\mathbb{S}^d)$$

for some positive constant $K_d$ depending only on $d$ and that

$$\deg g = \frac{1}{|\mathbb{S}^d|} \int_{\mathbb{S}^d} \text{Jac}(g) \quad \text{for } g \in C^1(\mathbb{S}^d, \mathbb{S}^d),$$

by Kronecker’s formula.

In this note, we confirm Brezis’ conjecture for $d \geq 2$. The conjecture is still open for $d = 1$. Here is the result of the note.

**Theorem 2.** Let $d \geq 2$. There exists a positive constant $C = C(d)$, depending only on $d$, such that, for all $g \in C(\mathbb{S}^d, \mathbb{S}^d)$,

$$|\deg g| \leq C \int_{|g(x) - g(y)| > \delta} \frac{\delta^d}{|x - y|^{2d}} \, dx \, dy \quad \text{for } 0 < \delta < 1. \quad (1.3)$$

## 2 Proof of Theorem 2

The proof of Theorem 2 is in the spirit of the approach in [4, 9]. One of the new ingredients of the proof is the following result [10, Theorem 1], which has its roots in [5]:

**Lemma 1.** Let $d \geq 1$, $p \geq 1$, let $B$ be an open ball in $\mathbb{R}^d$, and let $f$ be a real bounded measurable function defined in $B$. We have, for all $\delta > 0$,

$$\frac{1}{|B|^2} \int_B \int_B |f(x) - f(y)|^p \, dx \, dy \leq C_{p,d} \left( |B|^{\frac{p}{2} - 1} \int_B \int_B \frac{\delta^p}{|x - y|^{d+p}} \, dx \, dy \right)^{\frac{1}{p}}, \quad (2.1)$$

for some positive constant $C_{p,d}$ depending only on $p$ and $d$.

In Lemma 1, $|B|$ denotes the Lebesgue measure of $B$.

We are ready to present

**Proof of Theorem 2.** We follow the strategy in [4, 9]. We first assume in addition that $g \in C^1(\mathbb{S}^d, \mathbb{S}^d)$. Let $B$ be the open unit ball in $\mathbb{R}^{d+1}$ and let $u : B \to B$ be the average extension of $g$, i.e.,

$$u(X) = \int_{B(x,r)} g(s) \, ds \quad \text{for } X \in B, \quad (2.2)$$

where $x = X/|X|$, $r = 2(1 - |X|)$, and $B(x, r) := \{ y \in \mathbb{S}^d; |y - x| \leq r \}$. In this proof, $\int_D g(s) \, ds$ denotes the equantity $\frac{1}{|D|} \int_D g(s) \, ds$ for a measurable subset $D$ of $\mathbb{S}^d$ with positive ($d$-dimensional
where $g$ depends only on $\rho(x)$.

We derive from (2.4) and (2.6) that, for $0 < \rho(x) < 1$, then

$$\left| \int_{B(x,2\rho(x))} g(s) \, ds \right| = 1/2. \quad (2.3)$$

By [4] (7), we have

$$|\deg g| \leq C \int_{\rho(x)<1} \frac{1}{\rho(x)^d} \, dx. \quad (2.4)$$

Here and in what follows, $C$ denotes a positive constant which is independent of $x$, $\xi$, $\eta$, $g$, and $\delta$, and can change from one place to another.

We now implement ideas involving Lemma 1 applied with $p = 1$. We have, by (2.3),

$$\int_{B(x,2\rho(x))} \int_{B(x,2\rho(x))} |g(\xi) - g(\eta)| \, d\xi \, d\eta \geq \int_{B(x,2\rho(x))} \left| \int_{B(x,2\rho(x))} g(\xi) - \int_{B(x,2\rho(x))} g(\eta) \, d\eta \right| \, d\xi \geq C.$$ 

This yields, for some $1 \leq j_0 \leq d + 1$,

$$\int_{B(x,2\rho(x))} \int_{B(x,2\rho(x))} |g_{j_0}(\xi) - g_{j_0}(\eta)| \, d\xi \, d\eta \geq C,$$

where $g_j$ denotes the $j$-th component of $g$. It follows from (2.1) that, for some $\delta_0 > 0$ ($\delta_0$ depends only on $d$) and for $0 < \delta < \delta_0$,

$$\rho(x)^{-d} \int_{B(x,2\rho(x))} \int_{B(x,2\rho(x))} \frac{\delta}{|\xi - \eta|^{d+1}} \, d\xi \, d\eta \geq C,$$

which implies

$$\sum_{j=1}^{d+1} \rho(x)^{-d} \int_{B(x,2\rho(x))} \int_{B(x,2\rho(x))} \frac{\delta}{|g_j(\xi) - g_j(\eta)|^{d+1}} \, d\xi \, d\eta \geq C. \quad (2.5)$$

Since

$$\rho(x)^{-d} \int_{B(x,2\rho(x))} \int_{B(x,2\rho(x))} \frac{\delta}{|\xi - \eta|^{d+1}} \, d\xi \, d\eta < \frac{C}{2(d+1)},$$

if $C_1 > 0$ is large enough (the largeness of $C_1$ depends only on $C$ and $d$), it follows from (2.5) that

$$\sum_{j=1}^{d+1} \rho(x)^{-d} \int_{B(x,2\rho(x))} \int_{B(x,2\rho(x))} \frac{\delta}{|g_j(\xi) - g_j(\eta)|^{d+1}} \, d\xi \, d\eta \geq C. \quad (2.6)$$

We derive from (2.4) and (2.6) that, for $0 < \delta < \delta_0$,

$$|\deg g| \leq C \int_{\rho(x)<1} \frac{1}{\rho(x)^{2d-1}} \, dx \sum_{j=1}^{d+1} \int_{B(x,2\rho(x))} \int_{B(x,2\rho(x))} \frac{\delta}{|\xi - \eta|^{d+1}} \, d\xi \, d\eta.$$
This implies, by Fubini’s theorem, that, for $0 < \delta < \delta_0$,

\[
|\deg g| \leq C \sum_{j=1}^{d+1} \frac{\delta}{|\xi_2 - \eta_2|^{d+1}} \int \int \frac{1}{\rho(x)^{2d-1}} dx.
\]  

(2.7)

We have

\[
\int_{\rho(x) > |x - \xi|} \frac{1}{\rho(x)^{2d-1}} dx \leq \int_{\rho(x) > |x - \xi|} \frac{1}{\rho(x)^{2d-1}} dx + \int_{\rho(x) > C|\xi - \eta|/\delta} \frac{1}{\rho(x)^{2d-1}} dx
\]

\[
\leq \int_{|x - \xi| > C|\xi - \eta|/\delta} \frac{C}{|x - \xi|^{2d-1}} dx + \int_{|x - \xi| \leq C|\xi - \eta|/\delta} \frac{C\delta^{d-1}}{|\xi - \eta|^{2d-1}} dx.
\]

Finally, we use the assumption that $d \geq 2$. Since $d > 1$, it follows that

\[
\int_{\rho(x) > |x - \xi|} \frac{1}{\rho(x)^{2d-1}} dx \leq \frac{C\delta^{d-1}}{|\xi - \eta|^{d-1}}.
\]

(2.8)

Combining (2.7) and (2.8) yields, for $0 < \delta < \delta_0$,

\[
|\deg g| \leq C \sum_{j=1}^{d+1} \frac{\delta^d}{|\xi_2 - \eta_2|^{2d}} \int \int \frac{1}{\rho(x)^{2d-1}} dx.
\]

(2.9)

Assertion (1.3) is now a direct consequence of (2.9) for $\delta < \delta_0$ and (1.1) for $\delta_0 \leq \delta < 1$.

The proof in the case $g \in C(S^d, S^d)$ can be derived from the case $g \in C^1(S^d, S^d)$ via a standard approximation argument. The details are omitted.

\[\square\]

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References


