



Mathematical analysis/Differential topology

## A refined estimate for the topological degree

*Une estimée raffinée du degré topologique*

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## ABSTRACT

We sharpen an estimate of [4] for the topological degree of continuous maps from a sphere  $\mathbb{S}^d$  into itself in the case  $d \geq 2$ . This provides the answer for  $d \geq 2$  to a question raised by Brezis. The problem is still open for  $d = 1$ .

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## R É S U M É

Nous affinons une estimée du degré topologique pour des applications continues d'une sphère  $\mathbb{S}^d$  dans elle-même dans le cas  $d \geq 2$ . Cela fournit la réponse pour  $d \geq 2$  à une question posée par Brezis. Le problème est encore ouvert pour  $d = 1$ .

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## 1. Introduction

Motivated by the theory of Ginzburg–Landau equations (see, e.g., [1]), Bourgain, Brezis and the author established in [4] the following theorem.

**Theorem 1.** *Let  $d \geq 1$ . For every  $0 < \delta < \sqrt{2}$ , there exists a positive constant  $C(\delta)$  such that, for all  $g \in C(\mathbb{S}^d, \mathbb{S}^d)$ ,*

$$|\deg g| \leq C(\delta) \int \int_{\substack{\mathbb{S}^d \times \mathbb{S}^d \\ |g(x) - g(y)| > \delta}} \frac{1}{|x - y|^{2d}} dx dy. \quad (1)$$

Here and in what follows, for  $x \in \mathbb{R}^{d+1}$ ,  $|x|$  denotes its Euclidean norm in  $\mathbb{R}^{d+1}$ .

The constant  $C(\delta)$  depends also on  $d$ , but for simplicity of notation, we omit  $d$ . Estimate (1) was initially suggested by Bourgain, Brezis, and Mironescu in [2]. It was proved in [3] in the case where  $d = 1$  and  $\delta$  is sufficiently small. In [9], the

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author improved (1) by establishing that (1) holds for  $0 < \delta < \ell_d = \sqrt{2 + \frac{2}{d+1}}$  with a constant  $C(\delta)$  independent of  $\delta$ . It was also shown there that (1) does not hold for  $\delta \geq \ell_d$ .

This note is concerned with the behavior of  $C(\delta)$  as  $\delta \rightarrow 0$ . Brezis [7] (see also [6, Open problem 3]) conjectured that (1) holds with

$$C(\delta) = C\delta^d, \tag{2}$$

for some positive constant  $C$  depending only on  $d$ . This conjecture is somehow motivated by the fact that (1)–(2) holds “in the limit” as  $\delta \rightarrow 0$ . More precisely, it is known that (see [8, Theorem 2])

$$\lim_{\delta \rightarrow 0} \int_{\mathbb{S}^d} \int_{\mathbb{S}^d} \frac{\delta^d}{|x - y|^{2d}} dx dy = K_d \int_{\mathbb{S}^d} |\nabla g(x)|^d dx \text{ for } g \in C^1(\mathbb{S}^d)$$

$$|g(x) - g(y)| > \delta$$

for some positive constant  $K_d$  depending only on  $d$ , and that

$$\deg g = \frac{1}{|\mathbb{S}^d|} \int_{\mathbb{S}^d} \text{Jac}(g) \text{ for } g \in C^1(\mathbb{S}^d, \mathbb{S}^d),$$

by Kronecker’s formula.

In this note, we confirm Brezis’ conjecture for  $d \geq 2$ . The conjecture is still open for  $d = 1$ . Here is the result of the note.

**Theorem 2.** *Let  $d \geq 2$ . There exists a positive constant  $C = C(d)$ , depending only on  $d$ , such that, for all  $g \in C(\mathbb{S}^d, \mathbb{S}^d)$ ,*

$$|\deg g| \leq C \int_{\mathbb{S}^d} \int_{\mathbb{S}^d} \frac{\delta^d}{|x - y|^{2d}} dx dy \text{ for } 0 < \delta < 1. \tag{3}$$

$$|g(x) - g(y)| > \delta$$

**2. Proof of Theorem 2**

The proof of Theorem 2 is in the spirit of the approach in [4,9]. One of the new ingredients of the proof is the following result [10, Theorem 1], which has its roots in [5]:

**Lemma 1.** *Let  $d \geq 1$ ,  $p \geq 1$ , let  $B$  be an open ball in  $\mathbb{R}^d$ , and let  $f$  be a real bounded measurable function defined in  $B$ . We have, for all  $\delta > 0$ ,*

$$\frac{1}{|B|^2} \int_B \int_B |f(x) - f(y)|^p dx dy \leq C_{p,d} \left( |B|^{\frac{p}{d}-1} \int_B \int_B \frac{\delta^p}{|x - y|^{d+p}} dx dy + \delta^p \right), \tag{4}$$

$$|f(x) - f(y)| > \delta$$

for some positive constant  $C_{p,d}$  depending only on  $p$  and  $d$ .

In Lemma 1,  $|B|$  denotes the Lebesgue measure of  $B$ . We are ready to present

**Proof of Theorem 2.** We follow the strategy in [4,9]. We first assume in addition that  $g \in C^1(\mathbb{S}^d, \mathbb{S}^d)$ . Let  $B$  be the open unit ball in  $\mathbb{R}^{d+1}$  and let  $u : B \rightarrow B$  be the average extension of  $g$ , i.e.

$$u(X) = \int_{B(x,r)} g(s) ds \text{ for } X \in B, \tag{5}$$

where  $x = X/|X|$ ,  $r = 2(1 - |X|)$ , and  $B(x, r) := \{y \in \mathbb{S}^d; |y - x| \leq r\}$ . In this proof,  $\int_D g(s) ds$  denotes the quantity  $\frac{1}{|D|} \int_D g(s) ds$  for a measurable subset  $D$  of  $\mathbb{S}^d$  with positive ( $d$ -dimensional Hausdorff) measure. Fix  $\alpha = 1/2$  and for every  $x \in \mathbb{S}^d$ , let  $\rho(x)$  be the length of the largest radial interval coming from  $x$  on which  $|u| > \alpha$  (possibly  $\rho(x) = 1$ ). In particular, if  $\rho(x) < 1$ , then

$$\left| \int_{B(x, 2\rho(x))} g(s) ds \right| = 1/2. \tag{6}$$

By [4, (7)], we have

$$|\deg g| \leq C \int_{\substack{\mathbb{S}^d \\ \rho(x) < 1}} \frac{1}{\rho(x)^d} dx. \tag{7}$$

Here and in what follows,  $C$  denotes a positive constant, which is independent of  $x, \xi, \eta, g$ , and  $\delta$ , and can change from one place to another.

We now implement ideas involving Lemma 1 applied with  $p = 1$ . We have, by (6),

$$\int_{B(x, 2\rho(x))} \int_{B(x, 2\rho(x))} |g(\xi) - g(\eta)| d\xi d\eta \geq \int_{B(x, 2\rho(x))} \left| g(\xi) - \int_{B(x, 2\rho(x))} g(\eta) d\eta \right| d\xi \geq C.$$

This yields, for some  $1 \leq j_0 \leq d + 1$ ,

$$\int_{B(x, 2\rho(x))} \int_{B(x, 2\rho(x))} |g_{j_0}(\xi) - g_{j_0}(\eta)| d\xi d\eta \geq C,$$

where  $g_j$  denotes the  $j$ -th component of  $g$ . It follows from (4) that, for some  $\delta_0 > 0$  ( $\delta_0$  depends only on  $d$ ) and for  $0 < \delta < \delta_0$ ,

$$\rho(x)^{1-d} \int_{B(x, 2\rho(x))} \int_{\substack{B(x, 2\rho(x)) \\ |g_{j_0}(\xi) - g_{j_0}(\eta)| > \delta}} \frac{\delta}{|\xi - \eta|^{d+1}} d\xi d\eta \geq C,$$

which implies

$$\sum_{j=1}^{d+1} \rho(x)^{1-d} \int_{B(x, 2\rho(x))} \int_{\substack{B(x, 2\rho(x)) \\ |g_j(\xi) - g_j(\eta)| > \delta}} \frac{\delta}{|\xi - \eta|^{d+1}} d\xi d\eta \geq C. \tag{8}$$

Since

$$\rho(x)^{1-d} \int_{B(x, 2\rho(x))} \int_{\substack{B(x, 2\rho(x)) \\ |\xi - \eta| > C_1 \rho(x) \delta}} \frac{\delta}{|\xi - \eta|^{d+1}} d\xi d\eta < \frac{C}{2(d+1)},$$

if  $C_1 > 0$  is large enough (the largeness of  $C_1$  depends only on  $C$  and  $d$ ), it follows from (8) that

$$\sum_{j=1}^{d+1} \rho(x)^{1-d} \int_{B(x, 2\rho(x))} \int_{\substack{B(x, 2\rho(x)) \\ |g_j(\xi) - g_j(\eta)| > \delta \\ |\xi - \eta| \leq C \rho(x) \delta}} \frac{\delta}{|\xi - \eta|^{d+1}} d\xi d\eta \geq C. \tag{9}$$

We derive from (7) and (9) that, for  $0 < \delta < \delta_0$ ,

$$|\deg g| \leq C \int_{\substack{\mathbb{S}^d \\ \rho(x) < 1}} \frac{1}{\rho(x)^{2d-1}} dx \sum_{j=1}^{d+1} \int_{B(x, 2\rho(x))} \int_{\substack{B(x, 2\rho(x)) \\ |g_j(\xi) - g_j(\eta)| > \delta \\ |\xi - \eta| \leq C \rho(x) \delta}} \frac{\delta}{|\xi - \eta|^{d+1}} d\xi d\eta.$$

This implies, by Fubini’s theorem, that, for  $0 < \delta < \delta_0$ ,

$$|\deg g| \leq C \sum_{j=1}^{d+1} \int_{\mathbb{S}^d} \int_{\substack{\mathbb{S}^d \\ |g_j(\xi) - g_j(\eta)| > \delta}} \frac{\delta}{|\xi - \eta|^{d+1}} d\xi d\eta \int_{\substack{\rho(x) \geq C|\xi - \eta|/\delta \\ 2\rho(x) > |x - \xi|}} \frac{1}{\rho(x)^{2d-1}} dx. \tag{10}$$

We have

$$\int_{\substack{2\rho(x) > |x-\xi| \\ \rho(x) \geq C|\xi-\eta|/\delta}} \frac{1}{\rho(x)^{2d-1}} dx \leq \int_{\substack{2\rho(x) > |x-\xi| \\ |x-\xi| > C|\xi-\eta|/\delta}} \frac{1}{\rho(x)^{2d-1}} dx + \int_{\substack{\rho(x) \geq C|\xi-\eta|/\delta \\ |x-\xi| \leq C|\xi-\eta|/\delta}} \frac{1}{\rho(x)^{2d-1}} dx$$

$$\leq \int_{|x-\xi| > C|\xi-\eta|/\delta} \frac{C}{|x-\xi|^{2d-1}} dx + \int_{|x-\xi| \leq C|\xi-\eta|/\delta} \frac{C\delta^{2d-1}}{|\xi-\eta|^{2d-1}} dx.$$

Finally, we use the assumption that  $d \geq 2$ . Since  $d > 1$ , it follows that

$$\int_{\rho(x) > |x-\xi|} \frac{1}{\rho(x)^{2d-1}} dx \leq \frac{C\delta^{d-1}}{|\xi-\eta|^{d-1}}. \tag{11}$$

Combining (10) and (11) yields, for  $0 < \delta < \delta_0$ ,

$$|\deg g| \leq C \sum_{j=1}^{d+1} \int_{\mathbb{S}^d} \int_{\mathbb{S}^d} \frac{\delta^d}{|\xi-\eta|^{2d}} d\xi d\eta. \tag{12}$$

$|g_j(\xi) - g_j(\eta)| > \delta$

Assertion (3) is now a direct consequence of (12) for  $\delta < \delta_0$  and (1) for  $\delta_0 \leq \delta < 1$ .

The proof in the case  $g \in C(\mathbb{S}^d, \mathbb{S}^d)$  can be derived from the case  $g \in C^1(\mathbb{S}^d, \mathbb{S}^d)$  via a standard approximation argument. The details are omitted.  $\square$

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