Supplementary Material: Analysis of the roots of $\beta\bar{K}(s) = -1$

Express $\bar{K}(s)$ from Eq. (16) as:

$$\bar{K}(s) = \sum \frac{\nu_i p_i}{Q(s;\nu_i)}, \qquad Q(s;\nu) = (s+1)(s+\alpha) + s\nu. \tag{A1}$$

The behavior of $K(\tau) = L^{-1}[\bar{K}(s)]$, where L^{-1} is the inverse Laplace transform operator, is determined largely by the roots of the I quadratics, $Q(s;\nu_i)$. The singularities of $\bar{K}(s)$ are given by the roots, r_i and R_i, of $Q(s;\nu_i)$:

$$\begin{bmatrix} r_i \\ R_i \end{bmatrix} = -\frac{\nu_i + \alpha + 1}{2}\left[1 \mp \sqrt{1 - \frac{4\alpha}{(\nu_i + \alpha + 1)^2}}\right], \tag{A2}$$

which shows that r_i and R_i are always real and negative since $\alpha, \nu_i > 0$.

Our main results are collected in Theorem 1, which builds upon the following Lemmas.

Lemma 1. Let $\alpha > 0$ and $\nu > 0$, then $Q(s;\nu)$ has two distinct real negative roots $R(\nu) \in (-\infty, \min(-1, -\alpha))$ and $r(\nu) \in (\max(-1, -\alpha), 0)$. Moreover, $r(\nu)$ is a strictly increasing function, and $R(\nu)$ a strictly decreasing function of ν.

Proof. Note that the notation used in Eq. (A2) is $R_i \equiv R(\nu_i)$ and similarly for r_i. For $\alpha, \nu > 0$, the roots r_i and R_i in Eq. (A2) are distinct. Furthermore, since $0 < 4\alpha/(\nu + \alpha + 1)^2 < 1$, r_i and R_i are real and $R_i < r_i$. Observe that $Q(s;\nu) \to \infty$ as $s \to \pm\infty$.

Let ν_i, ν_j be two values of $\nu > 0$, with $\nu_j > \nu_i$, with roots given by R_i, R_j, r_i, r_j. Since:

$$Q(R_i;\nu_j) = (R_i+1)(R_i+\alpha) + R_i\nu_i + R_i(\nu_j - \nu_i) = R_1(\nu_j - \nu_i) < 0, \tag{A3}$$

$Q(s;\nu_j)$ has a root $R_j < R_i$. An identical argument shows there is a root $r_j > r_i$. Thus, $R(\nu)$ and $r(\nu)$ are, respectively, decreasing and increasing functions of ν.

Since $Q(-\alpha;\ \nu) = -\alpha\nu < 0$, and $Q(-1;\ \nu) = -\nu < 0$ there is a root $R(\nu) < \min(-1, -\alpha)$ and a root $\max(-1, -\alpha) < r(\nu)$. Similarly, $Q(0,\ \nu) = \alpha > 0$, so there is a root $r(\nu) < 0$.

Remark 1.1. Observe that as $\nu \to \infty, R(\nu) \to -\infty$ and $r(\nu) \uparrow 0$.

Remark 1.2. It is also straightforward to show that $Q(s;\ \nu) < 0$ for $R(\nu) < s < r(\nu)$.

Remark 1.3. Tighter bounds on R_i and r_i can be obtained from Eq. (A2). For example, $-1 - \nu_i - \alpha < R_i < -\nu_i + \min(-1, -\alpha)$ and $\max(-1, -\alpha/(1 + \nu_i)) < r_i < -\alpha/(1 + \nu_i + \alpha)$.

Lemma 2. The function $\bar{K}(s)$ is smooth except at $s = R_i \equiv R(\nu_i)$ and $s = r_i \equiv r(\nu_i)$. At these singularities,

$$\lim_{s \to R_i^{\pm}} \bar{K}(s) = \mp\infty; \quad \lim_{s \to r_i^{\pm}} \bar{K}(s) = \mp\infty. \tag{A4}$$

Proof. By inspection.

Remark 2.1. Lemma 1 shows that the singularities are all distinct. For convenience, we index the roots R and r differently. Starting from the most negative R root, the numbering is ordered, $I, I - 1, \ldots, 1$. Starting from the most negative r root, the numbering is $1, 2, \ldots, I$. With this indexing, we have, from Lemma 1:

$$R_I < \ldots < R_2 < R_1 < \min(-1, -\alpha) < \max(-1, -\alpha) < r_1 < r_2 < \ldots < r_I < 0. \tag{A5}$$

Then, R_I and r_I correspond to the largest ν, R_{I-1} and r_{I-1} to the second largest value of ν, etc. Combining this with Remark 1.2, we see that each term in $\bar{K}(s)$ is negative for $s \in (R_1, r_1)$ and so $\bar{K}(s) < 0$ in this range. Since $\bar{K}(s)$ is continuous and bounded above on this interval, it attains a maximum value somewhere. Let this maximum value be $-1/\beta^*$, with $\beta^* > 0$, attained for some value $s = s^* \in (R_1, r_1)$. This s^* is unique, as shown below.

We now localize the roots:

Lemma 3. There is at least one root of $\beta\bar{K}(s) = -1$ in each of the I – 1 intervals (R_{i+1}, R_i), and in each of I – 1 intervals (r_i, r_{i+1}).

Proof. Use Lemma 2 and apply the intermediate value theorem on each of the stated intervals. The function $\bar{K}(s)$ takes on every real value on each of the intervals; in particular, it takes on the value $-1/\beta$ at some point(s) in each interval.

Remark 3.1. $\beta\bar{K}(s) = -1$ has 2I roots. Lemma 3 shows that at least I – 1 'fast' roots (i.e., higher magnitude, denoted by R_i) are found in $s \in (-\infty, \min(-1, -\alpha))$ and at least I – 1 'slow' roots (i.e., lower magnitude, denoted by r_i) are in $s \in (\max(-1, -\alpha), 0)$. We isolate the other two roots below.

Lemma 4. The value $s^* \in (R_1, r_1)$ where $\bar{K}(s)$ attains its maximum value ($-1/\beta^*$) is unique. If $\beta < \beta^*$ then there is a root of $\beta\bar{K}(s) = -1$ in each of the intervals (R_1, s^*) and (s^*, r_1).

Proof. The value s^* is a stationary point of $\bar{K}(s)$. If $\beta = \beta^*$ then s^* is a real root of $\beta^*\bar{K}(s)$ = –1 with multiplicity of at least two. Along with the (at least) 2I – 2 roots of Lemma 3, this makes at least 2I roots. Hence, if there was another s^* there would be more than 2I roots, which is impossible.

Remark 4.1. Applying the intermediate value theorem on (R_1, s^*), we see that $\bar{K}(s)$ attains every value in $\left(-\infty, -1/\beta^*\right)$ somewhere on this interval. In particular, it attains the value $-1/\beta$ if $\beta < \beta^*$. The same argument works on (s^*, r_1). Thus, if $\beta < \beta^*$, we have found 2I disjoint intervals each containing at least one root. But, there are exactly $2I$ roots of the characteristic equation. Hence, for $\beta < \beta^*$ there is exactly one root in each of the stated intervals.

Remark 4.2. At $\beta = \beta^*$, the roots coalesce into a double real root, while for $\beta > \beta^*$, there are two complex roots. To complete the analysis of the location of the roots of $\beta\overline{K}(s) = -1$, we need to specify the magnitude of β^* relative to α and β. For this, observe that $s = -\alpha$ is in the interval (R_1, r_1) (Lemma 1), and that $\overline{K}(-\alpha) = -1/\alpha$. But, since $-1/\beta^*$ is the maximum value of $\overline{K}$ on (R_1, r_1), this means that $-1/\beta^* \geq -1/\alpha$, or $\beta^* \geq \alpha$. We also have the physical condition that the eroded soil is always more easily eroded than the original soil, i.e., $\beta < \alpha$. Thus, $\beta < \alpha \leq \beta^*$ or, in words, the value of β never exceeds β^*, meaning that double (or complex) roots cannot occur.

Remark 4.3. From Lemmas 3 and 4, we conclude that there is exactly one root in each of $I - 1$ intervals (R_{i+1}, R_i), and in each of $I - 1$ intervals (r_i, r_{i+1}). There are two distinct roots in the interval (R_1, r_1).

We now show how all the roots vary as a function of detachability β.

Lemma 5. The leftmost (rightmost) I roots strictly increase (decrease) with β for $\beta \in (0, \beta^*)$.

Proof. Since $\overline{K}(s)$ has one root for $s \in (R_i, R_{i+1})$, from Lemma 2 $\overline{K}(s)$ is strictly increasing on this interval. Since $-1/\beta$ increases with increasing β, so must the root of $\overline{K}(s) = -1/\beta$. A corresponding argument applies to the case $s \in (r_i, r_{i+1})$.

We now consider the pair of roots in $s \in (R_1, r_1)$.

Lemma 6. Given that $\alpha > \beta > 0$, the two roots of $\overline{K}(s) = -1/\beta$ are located in (R_1, r_1) as follows:

I $\alpha > \beta > 1$; one in $(R_1, -\alpha)$ and one in $(-\alpha, -1)$.

II $\alpha > 1 > \beta > 0$; one in $(R_1, -\alpha)$ and one in $(-1, r_1)$.

III $1 > \alpha > \beta > 0$; one in $(R_1, -1)$ and one in $(-\alpha, r_1)$.

Proof. For I: From Lemma 2, $\lim_{s \downarrow R_1} \bar{K}(s) = -\infty$ and, from Lemma 1, $R_1 < -\alpha$. Since $\bar{K}(-\alpha) = -1/\alpha > -1/\beta$, the intermediate value theorem shows there exists $s \in (R_1, -\alpha)$ satisfying $\bar{K}(s) = -1/\beta$. Also, $\bar{K}(-1) = -1 < -1/\beta$ by hypothesis, and again the intermediate value theorem shows existence of a root in $(-\alpha, -1)$.

For II: From Lemma 2, $\lim_{s \uparrow r_1} \bar{K}(s) = -\infty$ and, from Lemma 1, $r_1 > -1$. Since $\bar{K}(-1) = -1 > -1/\beta$ for this case, the intermediate value theorem shows existence of a root in $(-1, r_1)$. Since $-\alpha < -\beta$ and $\bar{K}(-\alpha) = -1/\alpha > -1/\beta$, the intermediate value theorem shows there is a root in $(R_1, -\alpha)$.

For III: From Lemma 2, $\lim_{s \downarrow R_1} \bar{K}(s) = -\infty$ and, from Lemma 1, $R_1 < -1$. Since $\bar{K}(-1) = -1 > -1/\beta$ for this case, the intermediate value theorem shows there is a root in $(R_1, -1)$. Also, from Lemma 1, $r_1 > -\alpha$. Recalling that $\bar{K}(-\alpha) = -1/\alpha > -1/\beta$ and $\lim_{s \uparrow r_1} \bar{K}(s) = -\infty$, the intermediate value theorem shows there is root in $(-\alpha, r_1)$.

Remark 6.1. If $\beta = 1$ then $s = -1$ is a root of $\beta \bar{K}(s) = -1$. Similarly, if $\alpha = \beta$ (meaning that the deposited soil has the same cohesion as the original soil, which is not physically realistic), then $s = -\alpha$ is a root.

By this sequence of Lemmas, the following theorem is proved.

Theorem 1. Assume $p_i > 0$, $\alpha > \beta > 0$. The $2I$ roots of $\bar{K}(s) = -1/\beta$ have the properties:

(i) All the roots are real, simple and negative.

(ii) There are I roots in the interval $(-\infty, \min(-\alpha, -1))$.

(iii) There are $I - 1$ roots in the interval $(\max(-\alpha, -1), 0)$.

(iv) The location of the final root depends on the values of α and β relative to -1 as specified in Lemma 6.

Roots in (ii) are denoted as fast, those in (iii) are called slow. We refer to the root in (iv) as the intermediate root. The bounds on this root for $\alpha > \beta > 1$ can be far apart, particularly if $\alpha \gg 1$. The bounds for this case are sharpened below.

Theorem 2. Let $\alpha > \beta > 0$, then lower, s_L, and upper, s_U, bounds on the intermediate root are given by

$$s_L > \max\left(s_{\min}, s_{\max} - \frac{\beta \sum \frac{v_i p_i}{r_i - R_i}}{1 - \beta \sum \frac{v_i p_i}{(s_{\min} - R_i)(r_i - R_i)}} \right), \tag{A6}$$

and

$$s_U < \min\left(s_{\max}, r_I - \frac{\beta \sum \frac{v_i p_i}{r_i - R_i}}{1 - \beta \sum \frac{v_i p_i}{(s_{\max} - R_i)(r_i - R_i)}} \right), \tag{A7}$$

where, from Lemma 6, $(s_{\min}, s_{\max})$ are defined as:

$$(s_{\min}, s_{\max}) = \begin{cases} (-\alpha, -1), & \alpha > \beta > 1 \\ (-1, r_1), & \alpha > 1 > \beta \\ (-\alpha, r_1), & 1 > \alpha > \beta. \end{cases} \tag{A8}$$

Proof. Write $\beta \bar{K}(s) = -1$ as

$$-\frac{1}{\beta}=\sum_{i=1}^{I}\frac{v_i p_i}{r_i-R_i}\left(\frac{1}{s-r_i}-\frac{1}{s-R_i}\right). \tag{A9}$$

For the lower bound Eq. (A9) becomes

$$\begin{aligned}-\frac{1}{\beta}&>\sum_{i=1}^{I}\frac{v_i p_i}{r_i-R_i}\left(\frac{1}{s-r_i}-\frac{1}{s_{\min}-R_i}\right)\\&>\sum_{i=1}^{I}\frac{v_i p_i}{r_i-R_i}\left(\frac{1}{s-s_{\max}}-\frac{1}{s_{\min}-R_i}\right),\end{aligned} \tag{A10}$$

which on rearranging for *s* gives the bound of inequality (A6). The upper bound is found analogously as

$$\begin{aligned}-\frac{1}{\beta}&<\sum_{i=1}^{I}\frac{v_i p_i}{r_i-R_i}\left(\frac{1}{s-r_i}-\frac{1}{s_{\max}-R_i}\right)\\&<\sum_{i=1}^{I}\frac{v_i p_i}{r_i-R_i}\left(\frac{1}{s-r_I}-\frac{1}{s_{\max}-R_i}\right),\end{aligned} \tag{A11}$$

resulting in inequality (A7)

Theorem 3. $\bar{K}(s)$ has an upper bound of *B*/*s*.

Proof. Since

$$(s+1)(s+\alpha)+sv>s(\alpha+v+1), \tag{A12}$$

then,

$$\bar{K}(s)=\sum\frac{v_i p_i}{(s+1)(s+\alpha)+sv_i}<\frac{1}{s}\sum\frac{v_i p_i}{\alpha+v_i}=\frac{B}{s}. \tag{A13}$$