LIFTING ENDO-p-PERMUTATION MODULES

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Abstract. We prove that all endo-p-permutation modules for a finite group are liftable from characteristic $p > 0$ to characteristic 0.

1. Introduction

Throughout we let p be a prime number and G be a finite group of order divisible by p. We let $\mathcal{O}$ denote a complete discrete valuation ring of characteristic 0 with a residue field $k := \mathcal{O}/\mathfrak{p}$ of positive characteristic p, were $\mathfrak{p} = J(\mathcal{O})$ is the unique maximal ideal of $\mathcal{O}$. Moreover, for $R \in \{\mathcal{O}, k\}$ we consider only finitely generated RG-lattices.

Amongst finitely generated kG-modules very few classes of modules are known to be liftable to $\mathcal{O}G$-lattices. Projective kG-modules are known to lift uniquely, and more generally, so do p-permutation kG-modules (see e.g. [Ben84, §2.6]). In the special case where the group G is a p-group, Alperin [Alp01] proved that endo-trivial kG-modules are liftable, and Bouc [Bou06, Corollary 8.5] observed that so are endo-permutation kG-modules as a consequence of their classification.

Passing to arbitrary groups, it is proved in [LMS16] that Alperin's result extends to endo-trivial modules over arbitrary groups. It is therefore legitimate to ask whether Bouc's result may be extended to arbitrary groups. A natural candidate for such a generalisation is the class of so-called *endo-p-permutation* kG-modules introduced by Urfer [Urf07], which are kG-modules whose k-endomorphism algebra is a p-permutation kG-module. We extend this definition to $\mathcal{O}G$-lattices and prove that any indecomposable endo-p-permutation kG-module lifts to an endo-p-permutation $\mathcal{O}G$-lattice with the same vertices.

We emphasise that our proof relies on a nontrivial result, namely the lifting of endo-permutation modules, which is a consequence of their classification. Moreover, there are two crucial points to our argument: the first one is the fact that reduction modulo $\mathfrak{p}$ applied to the class of endo-p-permutation $\mathcal{O}G$-lattices preserves both indecomposability and vertices, while the second one relies on properties of the G-algebra structure of the endomorphism ring of endo-permutation RG-lattices.

Date: August 25, 2017.

2010 *Mathematics Subject Classification.* Primary 20C20.

Key words and phrases. endo-permutation, p-permutation, source.

2. Endo-p-permutation lattices

Recall that an $\mathcal{O}G$-lattice is an $\mathcal{O}G$-module which is free as an $\mathcal{O}$-module. For $R \in \{\mathcal{O}, k\}$ an RG-lattice L is called a *p-permutation lattice* if $\mathrm{Res}^G_P(L)$ is a permutation RP-lattice for every p-subgroup P of G, or equivalently, if L is isomorphic to a direct summand of a permutation RG-lattice.

Following Urfer [Urf07], we call an RG-lattice L an *endo-p-permutation* RG-lattice if its endomorphism algebra $\mathrm{End}_R(L)$ is a p-permutation RG-lattice, where $\mathrm{End}_R(L)$ is endowed with its natural RG-module structure via the action of G by conjugation:

$$ {}^g\phi(m) = g \cdot \phi(g^{-1} \cdot m) \quad \forall\, g \in G, \forall\, \phi \in \mathrm{End}_R(L) \text{ and } \forall\, m \in L\,. $$

Equivalently, L is an endo-p-permutation RG-lattice if and only if $\mathrm{Res}^G_P(L)$ is an endo-permutation RP-lattice for a Sylow p-subgroup $P \in \mathrm{Syl}_p(G)$, or also if $\mathrm{Res}^G_Q(L)$ is an endo-permutation RQ-lattice for every p-subgroup Q of G.

This generalises the notion of an *endo-permutation* RP-lattice over a p-group P, introduced by Dade in [Dad78a, Dad78b]. In fact an RP-lattice is an endo-p-permutation RP-lattice if and only if it is an endo-permutation lattice. An endo-permutation RP-lattice M is said to be *capped* if it has at least one indecomposable direct summand with vertex P, and in this case there is in fact a unique isomorphism class of indecomposable direct summands of M with vertex P. Moreover, considering an equivalence relation called *compatibility* on the class of capped endo-permutation RP-lattices gives rise to a finitely generated abelian group $D_R(P)$, called the *Dade group* of P, whose multiplication is induced by the tensor product $\otimes_R$. For details, we refer the reader to [Dad78a] or [The95, §27-29].

If $P \leq G$ is a p-subgroup, we write $D_R(P)^{G-st}$ for the set of *G-stable* elements of $D_R(P)$, i.e. the set of equivalence classes $[L] \in D_R(P)$ such that

$$ \mathrm{Res}^P_{{}^xP\cap P}([L]) = \mathrm{Res}^{{}^xP}_{{}^xP\cap P} \circ c_x([L]) \in D_R({}^xP \cap P)\,, \quad \forall\, x \in G\,, $$

where c_x denotes conjugation by x.

The following results can be found in Urfer [Urf07] for the case $R = k$, under the additional assumption that k is algebraically closed. However, it is straightforward to prove that they hold for an arbitrary field k of characteristic p, and also in case $R = \mathcal{O}$.

Remark 2.1. It follows easily from the definitions that the class of endo-p-permutation RG-lattices is closed under taking direct summands, R-duals, tensor products over R, (relative) Heller translates, restriction to a subgroup, and tensor induction to an overgroup. However, this class is not closed under induction, nor under direct sums.

Two endo-p-permutation RG-lattices are called *compatible* if their direct sum is an endo-p-permutation RG-lattice.

Lemma 2.2 ([Urf07, Lemma 1.3])**.** *Let $H \leq G$ and L be an endo-p-permutation RH-lattice. Then $\mathrm{Ind}^G_H(L)$ is an endo-p-permutation RG-lattice if and only if $\mathrm{Res}^H_{{}^xH\cap H}(L)$ and $\mathrm{Res}^{{}^xH}_{{}^xH\cap H}({}^xL)$ are compatible for each $x \in G$.*

Theorem 2.3 ([Urf07, Theorem 1.5]). *An indecomposable RG-lattice L with vertex P and RP-source S is an endo-p-permutation RG-lattice if and only if S is a capped endo-permutation RP-lattice such that $[S] \in D_R(P)^{G-st}$. Moreover, in this case $\mathrm{Ind}_P^G(S)$ is an endo-p-permutation RG-lattice.*

3. Preserving indecomposability and vertices by reduction modulo $\mathfrak{p}$

For an $\mathcal{O}G$-lattice L, the reduction modulo $\mathfrak{p}$ of L is

$$L/\mathfrak{p}L \cong k \otimes_{\mathcal{O}} L\,.$$

Note that $k \otimes_{\mathcal{O}} \mathrm{End}_{\mathcal{O}}(L) \cong \mathrm{End}_k(L/\mathfrak{p}L)$. A kG-module M is said to be *liftable* if there exists an $\mathcal{O}G$-lattice $\widehat{M}$ such that $M \cong \widehat{M}/\mathfrak{p}\widehat{M}$.

Lemma 3.1. *Let L be an endo-p-permutation $\mathcal{O}G$-lattice and write $A := \mathrm{End}_{\mathcal{O}}(L)$. Then the natural homomorphism $k \otimes_{\mathcal{O}} A^G \longrightarrow (k \otimes_{\mathcal{O}} A)^G$ is an isomorphism of k-algebras.*

Proof. To begin with, consider a transitive permutation $\mathcal{O}G$-lattice $U = \mathrm{Ind}_Q^G(\mathcal{O})$. Then $Q \leq G$ is the stabiliser of $x = 1_G \otimes 1_{\mathcal{O}}$, so that

$$\{gx \mid g \in [G/Q]\}$$

is a G-invariant $\mathcal{O}$-basis of U and $U^G \cong \mathcal{O}(\sum_{g\in[G/Q]} gx)$. It follows that

$$\{1_k \otimes gx \mid g \in [G/Q]\}$$

is a G-invariant k-basis of $k \otimes_{\mathcal{O}} U$ and $(k \otimes_{\mathcal{O}} U)^G = k(\sum_{g\in[G/Q]} 1 \otimes gx)$. Therefore the restriction of the canonical surjection $U \longrightarrow k \otimes_{\mathcal{O}} U$ to the submodule U^G of G-fixed points of U has image $(k \otimes_{\mathcal{O}} U)^G$ with kernel equal to $\mathfrak{p}U^G$. Hence the canonical homomorphism

$$k \otimes_{\mathcal{O}} U^G \longrightarrow (k \otimes_{\mathcal{O}} U)^G$$

is an isomorphism. Because taking fixed points commutes with direct sums, the latter isomorphism holds as well for every p-permutation $\mathcal{O}G$-lattice U. Therefore, writing $A = \bigoplus_{i=1}^m U_i$ as a direct sum of indecomposable p-permutation $\mathcal{O}G$-lattices, we obtain that the canonical homomorphism

$$k \otimes_{\mathcal{O}} A^G \cong \bigoplus_{i=1}^{m} k \otimes_{\mathcal{O}} U_i^G \quad \longrightarrow \quad \bigoplus_{i=1}^{m} (k \otimes_{\mathcal{O}} U_i)^G \cong (k \otimes_{\mathcal{O}} A)^G$$

is an isomorphism. □

The following characterization of vertices is well-known, but we include a proof for completeness.

Lemma 3.2. *Let $R \in \{\mathcal{O}, k\}$ and let L be an indecomposable RG-lattice. Let $L^\vee = \mathrm{Hom}_R(L, R)$ denote the R-dual of L and let*

$$\mathrm{End}_R(L) \cong L \otimes_R L^\vee \cong U_1 \oplus \cdots \oplus U_n$$

be a decomposition of $L \otimes_R L^\vee$ into indecomposable summands. Then a p-subgroup P of G is a vertex of L if and only if every U_i has a vertex contained in P and one of them has vertex P.

Proof. Suppose L has vertex P. Then L is projective relative to P and, by tensoring with $L^\vee$, we see that $L \otimes_R L^\vee$ is projective relative to P, and therefore so are $U_1, \ldots, U_n$. In other words, P contains a vertex of U_i for each $1 \leq i \leq n$. Now L is isomorphic to a direct summand of $L \otimes_R L^\vee \otimes_R L$ because the evaluation map

$$L \otimes_R L^\vee \otimes_R L \longrightarrow L\,, \qquad x \otimes \psi \otimes y \mapsto \psi(x)y$$

splits via $y \mapsto \sum_{i=1}^n y \otimes v_i^\vee \otimes v_i$, where $\{v_1, \ldots, v_n\}$ is an R-basis of L and $\{v_1^\vee, \ldots, v_n^\vee\}$ is the dual basis. Therefore L is isomorphic to a direct summand of some $U_i \otimes_R L$ (by the Krull-Schmidt theorem). If, for each $1 \leq i \leq n$, a vertex of U_i was strictly contained in P, then $U_i \otimes_R L$ would be projective relative to a proper subgroup of P, hence the direct summand L would also be projective relative to a proper subgroup of P, a contradiction. This proves that, for some i, a vertex of U_i is equal to P.

Suppose conversely that every U_i has a vertex contained in P and one of them has vertex P. Let Q be a vertex of L. By the first part of the proof, every U_i has a vertex contained in Q and one of them has vertex Q. This forces Q to be equal to P up to conjugation. □

Proposition 3.3. *If L is an indecomposable endo-p-permutation $\mathcal{O}G$-lattice with vertex $P \leq G$, then $L/\mathfrak{p}L$ is an indecomposable endo-p-permutation kG-module with vertex P.*

Proof. Set $A := \mathrm{End}_{\mathcal{O}}(L)$, so that $A^G = \mathrm{End}_{\mathcal{O}G}(L)$. First we prove that $\mathrm{End}_{kG}(L/\mathfrak{p}L) = (k \otimes_{\mathcal{O}} A)^G$ is a local algebra. Write $\psi : A^G \longrightarrow A^G/\mathfrak{p}A^G$ for the canonical homomorphism. By Nakayama's Lemma $\mathfrak{p}A^G \subseteq J(A^G)$, so that any maximal left ideal of A^G contains $\mathfrak{p}A^G$. Therefore

$$\psi^{-1}(J(A^G/\mathfrak{p}A^G)) = \psi^{-1}\left(\bigcap_{\mathfrak{m}\in \mathrm{Maxl}(A^G/\mathfrak{p}A^G)} \mathfrak{m}\right) = \bigcap_{\substack{\mathfrak{a}\in \mathrm{Maxl}(A^G)\\ \mathfrak{a}\supseteq \mathfrak{p}A^G}} \mathfrak{a} = J(A^G)\,,$$

where Maxl denotes the set of maximal left ideals of the considered ring. Thus ψ induces an isomorphism $A^G/J(A^G) \cong (k \otimes_{\mathcal{O}} A^G)/J(k \otimes_{\mathcal{O}} A^G)$. Now $k \otimes_{\mathcal{O}} A^G \cong (k \otimes_{\mathcal{O}} A)^G$ as k-algebras, by Lemma 3.1. Therefore it follows that

$$\mathrm{End}_{kG}(L/\mathfrak{p}L)/J(\mathrm{End}_{kG}(L/\mathfrak{p}L)) \cong (k \otimes_{\mathcal{O}} A)^G/J((k \otimes_{\mathcal{O}} A)^G) \cong A^G/J(A^G)\,.$$

This is a skew-field since we assume that L is indecomposable. Hence $L/\mathfrak{p}L$ is indecomposable.

For the second claim, let P be a vertex of L. Let $L^\vee$ denote the $\mathcal{O}$-dual of L and consider a decomposition of $\mathrm{End}_{\mathcal{O}}(L)$ into indecomposable summands

$$\mathrm{End}_{\mathcal{O}}(L) \cong L \otimes_{\mathcal{O}} L^\vee \cong U_1 \oplus \cdots \oplus U_n\,.$$

Then there is also a decomposition

$$\mathrm{End}_k(L/\mathfrak{p}L) \cong k \otimes_{\mathcal{O}} \mathrm{End}_{\mathcal{O}}(L) \cong U_1/\mathfrak{p}U_1 \oplus \cdots \oplus U_n/\mathfrak{p}U_n\,.$$

Since L is an endo-p-permutation $\mathcal{O}G$-lattice, U_i is a p-permutation module for each $1 \leq i \leq n$. Therefore the module $U_i/\mathfrak{p}U_i$ is indecomposable and the vertices of U_i and $U_i/\mathfrak{p}U_i$ are the same (see [The95, Proposition 27.11]). By Lemma 3.2, every U_i has a vertex contained in P and one of them has vertex P. Therefore every $U_i/\mathfrak{p}U_i$ has a vertex contained in P and one of them has vertex P. By Lemma 3.2 again, P is a vertex of $L/\mathfrak{p}L$. □

4. Lifting endo-p-permutation kG-modules

We are going to use the fact that the sources of endo-p-permutation kG-modules are liftable. However, a random lift of the sources will not suffice and our next lemma deals with this question.

Lemma 4.1. *Let P be a p-group. If S is an indecomposable endo-permutation kP-module with vertex P such that $[S] \in D_k(P)^{G-st}$, then there exists an endo-permutation $\mathcal{O}P$-lattice $\widehat{S}$ lifting S such that $[\widehat{S}] \in D_{\mathcal{O}}(P)^{G-st}$.*

Proof. As a consequence of the classification of endo-permutation modules, Bouc proved that every endo-permutation kP-module is liftable [Bou06, Corollary 8.5]. Therefore S is liftable to an $\mathcal{O}P$-lattice $\widehat{S}$, i.e. $\widehat{S}/\mathfrak{p}\widehat{S} \cong S$. Note that $\widehat{S}$ is not unique because $\widehat{S} \otimes_{\mathcal{O}} L$ also lifts S for any one-dimensional $\mathcal{O}P$-lattice L. This is because $L/\mathfrak{p}L \cong k$ since the trivial module k is the only one-dimensional kP-module up to isomorphism. However, the lifted P-algebra $\mathrm{End}_{\mathcal{O}}(\widehat{S})$ is unique up to isomorphism and we can choose $\widehat{S}$ to be the unique $\mathcal{O}P$-lattice with determinant 1 which lifts S (see [The95, Lemma 28.1]). This choice of an $\mathcal{O}P$-lattice with determinant 1 is made possible because the dimension of $\widehat{S}$ is prime to p (see [The95, Corollary 28.11]).

In order to prove that $[\widehat{S}]$ is G-stable in the Dade group, we note that the determinant 1 is preserved by conjugation and by restriction. Therefore, the equality

$$\mathrm{Res}^{P}_{{}^xP\cap P}([S]) = \mathrm{Res}^{{}^xP}_{{}^xP\cap P} \circ c_x([S]) \in D_k({}^xP \cap P)\,, \quad \forall\, x \in G$$

implies an equality for the unique lifts with determinant 1

$$\mathrm{Res}^{P}_{{}^xP\cap P}([\widehat{S}]) = \mathrm{Res}^{{}^xP}_{{}^xP\cap P} \circ c_x([\widehat{S}]) \in D_{\mathcal{O}}({}^xP \cap P)\,, \quad \forall\, x \in G\,.$$

This proves that $[\widehat{S}] \in D_{\mathcal{O}}(P)^{G-st}$, completing the proof. □

Theorem 4.2. *Let M be an indecomposable endo-p-permutation kG-module, and let $P \leq G$ be a vertex of M. Then there exists an indecomposable endo-p-permutation $\mathcal{O}G$-lattice $\widehat{M}$ with vertex P such that $\widehat{M}/\mathfrak{p}\widehat{M} \cong M$.*

Proof. Let P be a vertex of M and S be a kP-source of M. By Theorem 2.3, S is a capped endo-permutation kP-module such that $[S] \in D_k(P)^{G-st}$. By Lemma 4.1, S lifts to an endo-permutation $\mathcal{O}P$-lattice $\widehat{S}$ such that $[\widehat{S}] \in D_{\mathcal{O}}(P)^{G-st}$. Moreover $\mathrm{Ind}_P^G(\widehat{S})$ is an endo-p-permutation $\mathcal{O}G$-lattice, by Lemma 2.2 and the fact that $[\widehat{S}]$ is G-stable. Now consider a decomposition of $\mathrm{Ind}_P^G(\widehat{S})$ into indecomposable summands

$$\mathrm{Ind}_P^G(\widehat{S}) = L_1 \oplus \cdots \oplus L_s \quad (s \in \mathbb{N})\,.$$

By Remark 2.1, each of the lattices L_i $(1 \leq i \leq s)$ is an endo-p-permutation $\mathcal{O}G$-lattice. Then, by Proposition 3.3,

$$\mathrm{Ind}_P^G(S) \cong \mathrm{Ind}_P^G(\widehat{S})/\mathfrak{p}\,\mathrm{Ind}_P^G(\widehat{S}) \cong L_1/\mathfrak{p}L_1 \oplus \cdots \oplus L_s/\mathfrak{p}L_s$$

is a decomposition of $\mathrm{Ind}_P^G(S)$ into indecomposable summands which preserves the vertices of the indecomposable summands. Because S is a source of M, there exists an index $1 \leq i \leq s$ such that $M \cong L_i/\mathfrak{p}L_i$. Then $\widehat{M} := L_i$ lifts M. □

Remark 4.3. In [BK06], Boltje and Külshammer consider the class of *modules with an endo-permutation source*, which also play a role in the study of Morita equivalences, as observed by Puig [Pui99]. In recent work of Kessar and Linckelmann [KL17], it is proved that in odd characteristic any Morita equivalence with an endo-permutation source is liftable from k to $\mathcal{O}$, under the assumption that k is algebraically closed.

As a typical example, we remark that simple modules for p-soluble groups are known to be instances of modules with an endo-permutation source (see [The95, Theorem 30.5]) and they are also known to be liftable to characteristic zero (Fong-Swan Theorem). Urfer proved in his Ph.D. thesis [Urf06] that such simple modules are endo-p-permutation modules in case they are not induced from proper subgroups, but in general they need not be endo-p-permutation.

One may ask whether our result extends to kG-modules with an endo-permutation source, i.e. whose class in the Dade group is not necessarily G-stable. We do not have an answer to this question. Our proof that endo-p-permutation modules are liftable to characteristic zero does not seem to extend to this larger class of modules, because it relies on the fact that the endomorphism algebra is a p-permutation module.

Acknowledgments. The authors are grateful to Nadia Mazza for useful discussions.

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