Representing groups against all odds

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C'était le nombre ce serait pire non
davantage ni moins indifféremment
mais autant le hasard

S. Mallarmé, *Un coup de dés*

Abstract

We investigate how probability tools can be useful to study representations of non-amenable groups. A suitable notion of "probabilistic subgroup" is proposed for locally compact groups, and is valuable to induction of representations. Nonamenable groups admit nonabelian free subgroups in that measure-theoretical sense. Consequences for affine actions and for unitarizability are then drawn. In particular, we obtain a new characterization of amenability via some affine actions on Hilbert spaces.

Along the way, various fixed-point properties for groups are studied. We also give a survey of several useful facts about group representations on Banach spaces, continuity of group actions, compactness of convex hulls in locally convex spaces, and measurability pathologies in Banach spaces.

Keywords: amenability, group representation, fixed-point property, von Neumann problem, Dixmier problem, induced representation, tychomorphism, Krein space.

Résumé

Improbables représentations de groupes

Nous étudions des représentations de groupes non moyennables à l'aide d'outils de la théorie des probabilités. Une notion de « sous-groupe probabiliste » est proposée pour les groupes localement compacts et une certaine induction de représentations reste possible dans ce cadre. Les groupes non moyennables contiennent un sous-groupe libre dans ce sens probabiliste, ce qui a des conséquences pour les actions affines et pour l'unitarizabilité. En particulier, nous obtenons une nouvelle caractérisation de la moyennabilité via les actions affines sur les espaces hilbertiens.

Chemin faisant, nous étudions diverses propriétés de point fixe pour les groupes et passons en revue divers faits fort utiles concernant les représentations de groupes sur les espaces de Banach, la continuité des actions de groupes, la compacité des enveloppes convexes dans les espaces localement convexes et l'inclémence de la mesurabilité dans les espaces de Banach.

Mots-clefs : moyennabilité, représentation de groupes, propriété de point fixe, problème de von Neumann, problème de Dixmier, représentation induite, tychomorphisme, espace de Krein.

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CONTENTS

INTRODUCTION

LINUS: Lucy, how much is six from four?
LUCY: Six from four?! You can't subtract six from four... You can't subtract a bigger number from a smaller number.
LINUS: You can if you're stupid!

C. SCHULZ, *Peanuts*, August 27, 1957

THERE IS NO MEANINGFUL WAY to define the *area* of an *arbitrary* subset of the plane. Let alone the *volume* of a subset of the space. These defeats of measure theory were definitively identified in 1929 as being caused by the so-called *non-amenability* of some groups of transformations. This thesis furbishes weapons for a revenge: measure theory can nonetheless be useful to understand nonamenable groups.

Freeness in nonamenable groups

Paradoxes in measure theory flourished in the early twentieth century. The most famous is due to Banach and Tarski: you can cut a ball into five pieces, translate and rotate them, and then get two brand new balls of the same radius than the initial one. Quite annoying for any notion of "volume", which should, intuitively, be invariant under rotations and translations.

In his seminal 1929 paper *Zur allgemeinen Theorie des Maßes* [**vN29**], von Neumann shed light on measure theory by a breakthrough idea: instead of considering a measure on a space which is required to be invariant under some group of transformations, he directly considered a measure on the group *itself*. This change of point of view—looking at a mathematical theory through the prism of group theory—is characteristic of many landmarks in modern mathematics: von Neumann did for measure theory what was done earlier by Galois for algebraic equations, by Klein for geometry, by Poincaré for topology.

This new point of view naturally leads to the notion of an *amenable* group[1], that is, a group that carries a suitable notion of "measure", called a *mean*. Von Neumann thus subsumed many paradoxes in measure theory (the original Banach–Tarski paradox, its generalisation to higher-dimensional spaces, an earlier related paradox of Hausdorff, as well as its own planar version of the Banach–Tarski paradox) into a lack of *amenability* of some groups of transformations of the space. Tarski then quickly showed that paradoxical decompositions can only be due to such a lack of amenability [**Tar38**, Satz 3.8].

The easiest example of a nonamenable group is the free group[2]. Actually, the groups related to the above-mentioned paradoxes are nonamenable precisely because they contain a free subgroup. This raised a natural question: do all nonamenable groups have a free subgroup? This problem, that an apocryphal tradition named "von Neumann problem", remained defiantly open for some fifty years before negative answers were at last found (see [**Mon13**] and the references therein for various examples of nonamenable groups without free subgroups).

The good news with these negative answers is that all that fuss around non-amenability was not a pedantic way to hide prosaic considerations about free subgroups. The bad news is that our knowledge of these friendly free groups seems unlikely to help to understand all nonamenable groups.

Yet we should not be discouraged. After all, the definition of amenability definitely belong to measure theory, not to algebra, so it should not be a surprise that we cannot characterize non-amenability via such an algebraic object as a subgroup. Actually, many results in this area can be rephrased as an answer to following question: to which extent can non-amenability be characterized as "a form of freeness", that is, via some (necessarily nonalgebraic) feature of the free groups? In other words, is the *trace* left by free groups in some mathematical theory actually characteristic of non-amenability?

Combinatorics provide first answers. If we build a free group "step by step", starting from some finite subset and recursively adding new elements, we get *many* new elements at each step. This idea can be formally stated in two directions. We can fix some "building" subset S and remark that translating any finite subset F by elements of S produce a good *proportion* of elements that were not already in F. This phenomenon indeed characterizes

[1]We anachronistically use the standard modern terminology—von Neumann used the term "measurable group", whose meaning shifted in between. The term *amenable* has been coined in the late forties by Day. The tradition of dubious puns is nowadays well established in infinite group theory.

[2]For this introduction, technical terminology is simplified: "group" means "countable discrete group" and "free group" means "countable nonabelian free group".

non-amenability [**Føl55**]. More elementarily, we can also simply *count* the elements built at each step and consider the asymptotical behaviour of this number—called the *growth* of a group. And indeed, like a free group, a nonamenable group must be of *exponential* growth. But here the link is disappointing, since it is not a complete characterization: some amenable groups are of exponential growth too[3].

Other answers came from geometry. From a geometer's point of view, a free group is a tree. And indeed, trees can be found inside Cayley graphs of nonamenable groups thanks to Tarski's above-mentioned result, and this characterizes nonamenability of finitely generated groups. Another point of view is provided by fixed-point properties. By the very universal property of free groups, a fixed-point property for groups is usually either enjoyed by all groups or not enjoyed by free groups. And indeed, some fixed-point properties can characterize amenability [**Day61**, **Ric67**].

Another answer came, ironically, from measure theory. The question is, loosely speaking: do measured spaces make a difference between nonamenable groups and groups with free subgroups? The above-mentioned early contribution of Tarski already indicates a negative answer from the point of view of paradoxes. But a more striking answer was given in 2009 by Gaboriau and Lyons: the orbit equivalence relation of some Bernoulli shift of any nonamenable group *contains* the orbit equivalence relation of a free action of a free group. We think of that result as "nonamenable groups contain a probabilistic free subgroup", a phrase that will be given a rigorous meaning in this thesis.

Fine, some freeness can be found in nonamenable groups. But via combinatorics, geometry or measure: the algebraic structure of free groups must be completely diluted in these points of view. For sure we cannot use that freeness to bootstrap some very concrete knowledge about the free groups, like its representations. Yet we can.

The idea that these smells of freeness in nonamenable groups can actually be used to understand nonamenable groups seemed to have ceased from being fanciful after Gaboriau and Lyons's breakthrough [**Mon06**, Section 5]. In her thesis [**Eps08**], I. Epstein used the latter to provide a continuum of orbit inequivalent actions for any nonamenable group. Epstein and Monod [**EM09**] provided the first examples of nonunitarizable groups not containing a free subgroup thanks to similar measure-theoretical ideas. Monod and Ozawa [**MO10**] used the Gaboriau–Lyons theorem to show nonunitarizability of some Burnside groups. Monod and the author [**GM17**] investigated affine actions of nonamenable groups on Hilbert spaces via this probabilistic free subgroup.

The goal of this thesis is to provide some framework for this measure-theoretical point of view on nonamenability. The idea of "containing a probabilistic subgroup" is given a

[3] Actually, the exact link between amenability and growth is still far from being understood. See e.g. [**Kel14**, **Mon17**].

rigorous meaning through the notion of *tychomorphism*[4]. Loosely speaking, a tychomorphism from a group H to another group G is a measured space that believes that H is a subgroup of G. The Gaboriau—Lyons theorem then yields a tychomorphism from free groups to any nonamenable group. Nonalgebraic though it is, this notion can still be used to induce linear and affine representations of groups. These induction processes were at the core of [**MO10**] and [**GM17**].

Organization and survey of the contents

Our main results, in contrast to geometry, can be reached through a royal road: read Definitions 2.1.1 and 2.2.1 (tychomorphisms), admit Theorem 3.2.1 (Gaboriau–Lyons), focus on Theorem 4.3.12 (induction techniques), and then jump to the proofs of Theorem 5.2.21 and 6.1.5 (applications to affine actions and unitarizability). But highways are the most boring routes, and we hope that the reader will find entertaining the countryside detours that we made in order to behold (or, sometimes, build) the landscapes. Here is a hitchhiker map.

The first chapter recalls some well-known vocabulary and tools that will be used throughout our work. The focus is on topological groups and amenability. We recall in particular some structure theory results thanks to which we can often, for applications, restrict without loss of generality our attention to some tractable groups (Polish locally compact groups, Lie groups, etc.).

Chapter 2 lays the basic features of *tychomorphisms*, the probabilistic variant of a subgroup. After the definitions, we explain how two elementary facts about subgroups (the transitivity of the relation “being a closed subgroup” and the possibility to lift a discrete free subgroup from a quotient) still hold for tychomorphisms. A first link with amenability is then proved.

Chapter 3 produces the main examples of nontrivial tychomorphisms, namely tychomorphisms from nonabelian free groups to any nonamenable group (Theorem 3.2.1). This is basically the above-mentioned Gaboriau–Lyons theorem, in a locally compact setting.

The next three chapters give applications of tychomorphisms. Each one starts with the relevant context and appeals to tychomorphisms only near its end, hence a good portion can be read independently of the first three chapters. Chapter 4 explains how subgroups can be replaced by tychomorphisms when inducing group representations. We

[4] “Chance”, “randomness” are among the many possible translations for ἡ τύχη.

took there the opportunity to recall some facts about linear and affine representations of groups. These are well-known but either rarely stated for non-isometric representations or scattered in the literature.

The goal of Chapter 5 is to study the interplay between some group properties and some fixed-point properties. Its first section gives some general facts about actions on compact convex sets. The second one then studies three group properties—finiteness, compactness, amenability—in terms of affine actions. The induction techniques of the preceding chapter are ultimately used to prove Theorem 5.2.21, a new characterization of amenability via affine actions on Hilbert spaces.

Chapter 6 aims at giving a locally compact counterpart to the Monod–Ozawa theorem. Some background on unitarizability of locally compact groups is given, and this chapter ends by some considerations on stability properties of unitarizability.

We finally discuss in Chapter 7 some natural questions raised by our works and summarize there the open problems encountered along the text.

Four appendices complete our results. Appendix A is a mere checking of the validity of some disintegration in the definition of a measured equivalence relation, needed in Section 3.1.

Appendix B investigates some broad class of locally convex spaces (encompassing notably all Banach spaces with their norm, weak or weak-$*$ topologies). This class naturally occurs when considering fixed-point properties of compact groups, and is also relevant for integration theory.

Appendix C studies in full generality the interplay between joint and orbital continuities for group actions. Most well-known facts on that topic express themselves naturally in the unifying framework of uniform spaces.

Lastly, Appendix D reviews a few pathological behaviours of measurability in Banach spaces. Its purpose is to justify the technical caution of our results of Section 4.3.

Non-survey of the non-contents

We tried to achieve an impossible balance between readability, self-containment, reasonable length, and exhaustiveness. Whether we succeeded or not, this has led us to the following choices for the main part of the text:

- Many remarks have been included, but all can be skipped at a first reading.

- Many natural considerations and some technical arguments, which would have unbearably held up the reading, have been postponed to appendices. In particular, the Bourbakian reader frustrated by our next two choices should find there some good bones to gnaw at.

- The topological vector spaces we consider are most often Banach spaces. In Chapter 5, more general locally convex spaces appear, but a small apparatus (definition of a convex set, dual spaces, adjoint map, weak topology) is largely enough to get the main ideas of the proofs. Moreover, Theorem 5.2.21 of that chapter only deals with Hilbert spaces.

- The use of uniform structures has been kept to a minimum. The natural uniformities associated to a topological group indeed appear but, actually, their main use is the consideration of the subspaces of bounded continuous scalar maps that are left or right-uniformly continuous. Hence the reader can as well focus on the specific definition of the latter (recalled in Section 1.2.D) and forget about the framework of uniform structures.

- Orbit equivalence and above all measure equivalence between discrete groups are a heuristic guide for tychomorphisms, but are not, strictly speaking, needed for the logical exposition of the latter. We thence did not include a full account of these equivalence theories, for which we refer to the nice surveys of Gaboriau [**Gab10a**] and Furman [**Fur11**].

- We do not introduce the background of percolation theory that is used to prove Theorem 3.2.1. Indeed, this theorem has been proved by Gaboriau and Lyons for discrete groups and, although it requires some effort to handle the locally compact case, the percolation part of our proof is a mere adaptation of Gaboriau and Lyons's original proof. For the convenience of the reader, we instead suggest a "reading path" to extract quickly from the abundant literature on percolation the few concepts and theorems we need.

- The background on equivalence relations that is given in Section 3.1.B should be enough to grasp the statements and the examples of Section 3.1, but an understanding of the details of their proofs would probably require some acquaintance with descriptive set theory.

On topological groups

We have tried, as much as possible, to give results for topological groups instead of discrete groups. The first reason is provided by standard examples. Many groups (such as groups of homeomorphisms or of isometries) arise in some topological or geometric contexts—and are therefore likely to carry a natural topology. It would be uselessly restrictive to forget about this precious topological information when trying to understand a group.

Another reason, more philosophical, is our strong belief that discrete groups should not be considered as particularly special among topological groups. Problems in infinite group theory usually find themselves a natural topological formulation, and discrete groups sit therefore in a more natural class of groups, depending on the context. For instance, countable discrete groups are particular cases of both locally compact σ-compact groups and of Polish groups. We can also meditate on the difficulty to *delineate* discrete groups among topological groups (other than by the very definition of discreteness), whereas locally compact groups are, more or less, characterized by the existence of an invariant measure.

In our work, the natural topological context is local compactness. Nonetheless, we insist on the fact that the results presented here are *not* technical glosses around elementary results for discrete groups. To our knowledge, the main theorems of this work cannot be proved elementarily for discrete groups. Actually, this also motivates to work in the locally compact world: the proofs are not that much harder there[5]. Of course, we do not say that no shortcut is available for discrete groups (the whole Section 3.1, for instance, boils down to Remark 3.1.20 for discrete groups) and our locally compact proofs rely shamelessly on the solution to the fifth Hilbert problem. But we mean that the main ideas are already illuminating and instructive in the discrete case. As a consequence, the reader unfamiliar with the topological toolkit needed to work with locally compact groups can safely, at a first reading, assume all the groups involved to be discrete, and already grasp a good idea of our work.

[5]This general leitmotiv guided our research. However, in two places of this text, a result is known only in the discrete case (see Proposition 4.3.10(4) and Problem 11).

SOME CONVENTIONS

- Our groups usually carry a topology, possibly the discrete one. If we want to emphasize that some result does not involve topology, we will speak about "abstract" groups.
- We include "Hausdorff" in the definition of compactness. In any case, topological spaces are often tacitly assumed to be Hausdorff. A few non-Hausdorff spaces occur here and there but not in an essential way.
- The base field for our vector spaces is indifferently **R** or **C**. For geometric considerations (Chapter 5 and Appendix B), we prefer **R**; for analytic ones (Chapter 6), **C**.
- "Countable" means "finite or countably infinite".
- From the standard but non-Bourbakian terminology we used, let us recall: σ-*algebra* (collection of subsets closed under countable unions and complementation), σ-*compact* (countable union of compact sets), G_δ (countable intersection of open sets), F_σ (countable union of closed sets).
- When X is a topological space, we will write $\mathscr{C}(X)$ for the space of continuous functions defined on X and $\mathscr{C}_b(X)$ for the subspace of bounded continuous functions. These functions take values in **R** or **C** depending on the context.

1 General Tools

We recall here some well-known facts and terminology about group actions, topological groups, and amenability, that will be used throughout the rest of this work. The purpose of this chapter is not to give a full account on these notions but more modestly to give a slight insight of these tools, as well as some references to the relevant literature.

1.1 Group actions

Group actions play the first violins of this work. Here is the (often standard) terminology we will use.

A group action is nothing more than a "concrete" realisation of a group, that is, a morphism α from a group G to the automorphism group of some structure X. Instead of the morphism α, it is usually more convenient to consider the following maps: the *action map*,

$$\begin{aligned} \mathrm{G} \times \mathrm{X} &\to \mathrm{X} \\ (g, x) &\mapsto \alpha(g)(x), \end{aligned}$$

and, for any $x \in \mathrm{X}$, the associated *orbit map*

$$\begin{aligned} \mathrm{G} &\to \mathrm{X} \\ g &\mapsto \alpha(g)(x). \end{aligned}$$

We will often save us tiresome parentheses by writing $g \cdot x$ or even gx instead of $\alpha(g)(x)$.

In this work, groups act on six main kinds of structures.

Actions on a set X**.** This is the general case: the morphism α lies in Sym(X) and there is nothing else to say.

Actions on a topological space X**.** The morphism α lies in Homeo(X). As our groups also carry a topology, these actions can be considered as continuous in two meaningful ways. The strongest one is to require the continuity of the action map (we call it the *joint continuity*). A less demanding but often satisfactory condition is to require all orbit maps to be continuous (the *orbital continuity* or *separate continuity*).

Actions on a topological vector space X**.** The morphism α lies in GL(X) (linear action) or in Aff(X) (affine action). The continuity conditions of the above case can now be declined with the various topologies naturally carried by X. Moreover, they have a fruitful interplay with the bornological structure of X, especially when the latter is a Banach space. More on that topic in Section 4.1.

Actions on a compact convex subset X **of a locally convex space.** The morphism α lies in the group of convex homeomorphisms of X. These actions play an important geometric role for amenability. We will investigate in Section 5.1 to which extent they are a specific case of the preceding ones.

Actions on a measurable space X**.** Here the morphism α lies in $\mathrm{Aut}_{\mathrm{m}}(\mathrm{X})$, the subgroup of Sym(X) made of bimeasurable bijections. Endowing our topological groups with their Borel structure, two measurability conditions can again be considered: the measurability of the action map, or that of all the orbit maps. The former will appear in Section 3.1, the latter in Appendix D.

Actions on a measure space (X, μ)**.** Specific case of the preceding one, the morphism α lies in $\mathrm{Aut}_{[\mu]}(\mathrm{X})$, the subgroup of $\mathrm{Aut}_{\mathrm{m}}(\mathrm{X})$ made of maps that preserve the *class* of the measure. In other words, $\mathrm{Aut}_{[\mu]}(\mathrm{X})$ is the subgroup that preserves the null sets. We will also say in that case that G acts *non-singularly* on (X, μ). We emphasize again that we do not assume by default that G preserves the measure μ.

In addition, two other kinds of group actions will appear, but with a less central role.

Actions on a uniform space. Some continuity phenomena that occur for actions on Banach spaces actually fit in the more general framework of actions on uniform spaces. For the sake of readability, we have chosen to focus, in the main part of the text, on the easier case of Banach spaces, but these phenomena are investigated in due generality in Appendix C.

Actions on a graph X**.** The automorphism group Aut(X) is the group of bijections of the vertex set that preserve the edge set. Of marginal interest for us, these actions will only appear in Section 3.2 as a tool to build actions on some measure space.

We will continuously build new actions from old ones. This process is sometimes trivial and sometimes demanding, as in Chapter 4. We will make the following broad terminological distinction:

- An action is *deduced* from another one when the acting group is the same. The canonical example is, starting from an action of G on a set X, the action built by precomposition on some set Y of functions defined on X (that is, for $f \in Y$, we define $g \cdot f$ by $(g \cdot f)(x) = f(g^{-1}x)$).
- An action is *induced* from another one when different acting groups are involved. This will be the core of Chapter 4.

We recall also that the *core* of an action α is merely its kernel, that is the subgroup of all elements acting trivially. Accordingly, the *core* of a subgroup K of a group G, written Core(K), is the intersection of all the conjugates of K, since it is but the core of the action of G on G/K by left translations.

A last remark on our terminology: all our actions are what are sometimes called left actions. We will never consider the so-called right actions (that is, left actions of the opposite group). As a result, in our text, phrases like "left action" or "right action" should be understood as shortcuts for, for instance, "action *by* translation on the left" or "on the right", depending on the context.

1.2 Topological groups

1.2.A Topology

The basic objects of this work are topological groups. Most often, our groups will be locally compact and σ-compact. We recall that, for a locally compact space X, the following topological properties are equivalent:

1. X is second countable;
2. X is σ-compact and metrizable;
3. X is separable and metrizable;
4. X is Polish.

In particular, the gaps from separability or σ-compactness to second countability are filled by metrizability. For groups, the Kakutani–Kodaira theorem states that this obstruction actually dwells in arbitrarily small subgroups.

Proposition 1.2.1 (Kakutani–Kodaira). — *Let* G *be a* σ*-compact locally compact group. For any sequence* U_n *of identity neighborhoods, there is a compact normal subgroup* K *such that* $K \subseteq \bigcap_n U_n$ *and the quotient* G/K *is metrizable (hence, in particular, second countable and Polish).*

For an elementary proof, cf. [**CH16**, Theorem 2.B.6]. The original proof [**KK44**, Satz 6] used the regular representation, we will sketch it in passing in Remark 4.1.15.

We will most often use only half the power of the Kakutani–Kodaira theorem, namely that we can find *some* compact group modulo which we get metrizability. However, the flexibility allowed by the choice of any sequence of neighborhoods is sometimes useful, as in the following corollary.

Corollary 1.2.2. — *Let* G *be an infinite compact group. Then* G *admits an infinite metrizable quotient.*

Proof. — By the Kakutani–Kodaira theorem, it suffices to find suitable identity neighborhoods U_n so that any group in their intersection must have infinite index. Define the *index* of any *subset* of G as the minimal number of left translates needed to cover the whole G. This definition coincides with the usual one for subgroups and the index is obviously nonincreasing (for the inclusion). By (pre)compactness, this index is finite for any open set. Hence we only need to show that an infinite compact group contains identity neighborhoods of arbitrarily large finite index. Let $g_1, \ldots, g_n$ be n distinct elements of G. As G is Hausdorff, we can find an identity neighborhood U such that $g_i U \cap g_j U = \emptyset$ for any $i \neq j$. Let V be any identity neighborhood such that $V^{-1}V \subseteq U$; a routine argument shows that the index of V is at least n. ■

Of course, a non-compact group cannot have a compact subgroup of finite index, so much so that Corollary 1.2.2 and Proposition 1.2.1 yield together the fact that *any infinite locally compact* σ*-compact group admits an infinite metrizable quotient.* This conclusion does not hold without σ-compactness, since there exist locally compact groups that are topologically simple and non-metrizable (see [**CH16**, Example 2.B.8]).

We recall that, by Baire's theorem, any locally compact space and any complete metrizable space are *Baire* (i.e., every countable intersection of dense open subsets is dense, or, equivalently, every countable union of closed subsets with empty interior still has an empty interior). In particular, locally compact groups and Polish groups are Baire.

1.2.B Measure

A left (resp. right) Haar measure on a group is a nonzero Radon measure that is invariant under left (resp. right) translations. We recall that a *Radon measure* on a topological space is a measure defined on the Borel σ-algebra that is locally finite (every point admits a neighborhood of finite measure) and inner regular (the measure of any Borel set is the supremum of the measures of the compact sets contained in it). Locally compact groups admit a left Haar measure (which is moreover unique up to a scaling factor), and this property more or less characterizes these groups (see e.g. [**Hal50**, §62]).

Let G be a locally compact group. When a left Haar measure m_G has been chosen on G, we will write by $\check{m}_G$ the corresponding right Haar measure (defined by the image of m_G under the inversion map, so $d\check{m}_G(x) = dm_G(x^{-1})$). The *modulus* is the homomorphism $\Delta_G\colon G \to \mathbf{R}_+^*$ defined by

$$dm_G(xg) = \Delta_G(g) dm_G(x)$$

(this equation is linear in the measure, hence the definition of the modulus does not depend on the choice of m_G). Moreover, we have the formula $\Delta_G d\check{m}_G = dm_G$. A group is *unimodular* if Δ_G is trivial (equivalently, if Haar measures are simultaneously left and right invariant).

When there is no confusion to be feared, we will write $|A|$ for the left Haar measure of a Borel set A. This means in particular that the statement involved is invariant under rescaling.

There is a strong interplay between topology and measure in a locally compact group. We record for instance the following easy fact.

Lemma 1.2.3. — *Let* G *be a locally compact group. For any Borel set* $E \subset G$ *of finite measure, the function*

$$f\colon G \to \mathbf{R}\colon g \mapsto |gE \triangle E|$$

is continuous.

Proof. — Let $\varepsilon > 0$. By the regularity of the measure, there exist a compact set $C \subseteq E$ and an open set $U \supseteq C$ such that $|E \setminus C| < \varepsilon$ and $|U \setminus C| < \varepsilon$. By continuity of group operations, there exists an open symmetric identity neighborhood V such that $VC \subseteq U$.

Now for any $g, h \in \mathrm{G}$ such that $g^{-1}h \in \mathrm{U}$, we get

$$\begin{aligned}|f(g) - f(h)| &\leqslant |g\mathrm{E} \bigtriangleup h\mathrm{E}| \\ &\leqslant |g\mathrm{E} \bigtriangleup g\mathrm{C}| + |g\mathrm{C} \bigtriangleup h\mathrm{C}| + |h\mathrm{C} \bigtriangleup h\mathrm{E}| \\ &\leqslant 2\varepsilon + \left|g^{-1}h\mathrm{C} \setminus \mathrm{C}\right| + \left|h^{-1}g\mathrm{C} \setminus \mathrm{C}\right| \\ &\leqslant 2\varepsilon + 2|\mathrm{VC} \setminus \mathrm{C}| \leqslant 4\varepsilon,\end{aligned}$$

hence f is continuous (all these computations rely on the triangle inequality $|\mathrm{X} \bigtriangleup \mathrm{Y}| \leqslant |\mathrm{X} \bigtriangleup \mathrm{Z}| + |\mathrm{Z} \bigtriangleup \mathrm{Y}|$). ■

We now review the interplay between the Haar measure of a subgroup and the Haar measure of an ambient group.

The easiest case is that of an open subgroup H in a group G. Choose any left Haar measure m_{G} on G. Since H is open, subsets of H that are Borel (resp. open, compact) in H stay so in G and H has positive measure. Hence the restriction of m_{G} on the subalgebra of Borel subsets of H is a left Haar measure on H.

Conversely, any left Haar measure m_{H} on H extends to G as follows. Choose any set X of representatives for the left cosets G/H and define m on the Borel sets of G as

$$m(\mathrm{A}) = \sum_{x \in \mathrm{X}} m_{\mathrm{H}}(x^{-1}\mathrm{A} \cap \mathrm{H}).$$

Then m is a left Haar measure on G, whose definition is independent of the choice of X.

Closed non-open subgroups of G have measure zero, hence restriction is useless in this case, but some kind of disintegration of measures holds. For normal subgroups, a convenient disintegration uses a Haar measure of the quotient.

Proposition 1.2.4 (Weil's integration formula). — *Let* G *be a locally compact group,* N *a closed normal subgroup of* G *and* Q *the quotient of* G *by* N. *Let* m_{G}, m_{N} *and* m_{Q} *be respective left Haar measures. Then, up to a constant factor,* m_{Q} *is the quotient measure* $m_{\mathrm{G}}/m_{\mathrm{N}}$. *In particular, for any* $f \in \mathscr{C}(\mathrm{G})$ *with compact support,*

$$\int_{\mathrm{G}} f(g)\ \mathrm{d}m_{\mathrm{G}}(g) = \int_{\mathrm{Q}} \left(\int_{\mathrm{N}} f(xh)\ \mathrm{d}m_{\mathrm{N}}(h)\right)\ \mathrm{d}m_{\mathrm{Q}}(x\mathrm{N}). \tag{1.1}$$

See [**Bou63**, vii, §2, n° 7, prop. 10] for a proof. Observe that, since m_{N} is left invariant, the map $x \mapsto \int_{\mathrm{N}} f(xh)\ \mathrm{d}m_{\mathrm{N}}(h)$ is constant on left cosets of N, hence Formula (1.1) makes sense.

Proposition 1.2.4 and the above discussion about subgroups readily imply the following.

Corollary 1.2.5. — *Let* G *be a locally compact group with modulus* Δ_G *and* H *be a closed subgroup of* G, *with modulus* Δ_H.

1. *Assume that* H *is either open or normal. Then the modulus* Δ_H *of* H *is the restriction of* Δ_G *to* H.
2. *Assume that* H *is open. Then* H *is unimodular if and only if* G *is so.*
3. *Assume that* H *is normal. If* G *is unimodular, then so is* H.
4. *The kernel of* Δ_G *is the largest closed normal unimodular subgroup of* G.

Remark 1.2.6. — Apart from Corollary 1.2.5, unimodularity is not transmitted to closed subgroups. Indeed, any locally compact group G embeds as a closed subgroup of the unimodular group $\mathbf{R} \rtimes G$, where the action of G on $\mathbf{R}$ is given by multiplication by the modulus. For unimodularity of $\mathbf{R} \rtimes G$, cf. Proposition 14 and the following Remarque in [**Bou63**, VII, §2, n° 9].

For closed subgroups, we need to restrict ourselves to second-countable groups, so that our groups become standard from the measure-theoretical point of view. We first recall the following result on quotients of Polish groups.

Proposition 1.2.7. — *Let* G *be a Polish group and* $H \leqslant G$ *a closed subgroup.*

1. *The quotient space* $H\backslash G$ *is a Polish space (in particular, a Borel standard space).*
2. *The projection* $\pi\colon G \to H\backslash G$ *admits a Borel section (that is, a Borel map* $\sigma\colon H\backslash G \to G$ *such that* $\pi \circ \sigma = \mathrm{id}$*).*

Proof. — See [**Bou74**, IX, §3, n° 1, prop. 4] and [**Bou74**, IX, §6, n° 9, cor. 2]. ∎

We can now state the disintegration result involving Haar measures of general closed subgroups, due to Ripley [**Rip76**, Theorem 1]. For more background on disintegration of measures and pseudo-images, see Appendix A.

Proposition 1.2.8 (Ripley). — *Let* G *be a second-countable locally compact group and* $H \leqslant G$ *a closed subgroup. Let* m_G *and* m_H *be respective left Haar measures. Then there exists a Radon* σ*-finite measure* ν *on the quotient space* $H\backslash G$, *which is a pseudo-image of*

m_{G} *via the projection and such that, for each Borel subset* B *of* G, *we have*

$$m_{\mathrm{G}}(\mathrm{B}) = \int_{\mathrm{H}\backslash\mathrm{G}} m_{\mathrm{H}}(\mathrm{B}x^{-1} \cap \mathrm{H})\ \mathrm{d}\nu(\mathrm{H}x).$$

In particular, for each Borel section $\sigma\colon \mathrm{H}\backslash\mathrm{G} \to \mathrm{G}$ *of the projection map, the map*

$$(\mathrm{H}, m_{\mathrm{H}}) \times (\mathrm{H}\backslash\mathrm{G}, \nu) \to (\mathrm{G}, m_{\mathrm{G}})\colon (h, x\mathrm{H}) \mapsto h\sigma(x\mathrm{H})$$

is an H*-equivariant isomorphism of measured spaces (where* G *is endowed with the* H*-action by left translations and* $\mathrm{H} \times \mathrm{H}\backslash\mathrm{G}$ *is endowed with the product action of the* H*-action by left translations and the trivial action).*

Of course, there is nothing particular with right cosets and the above two results can be stated (and will be used) for the quotient G/H.

1.2.C Structure

In order to understand a topological group, it is useful to be able to decompose it into more tractable groups. The basic decomposition for a topological group G is the following short exact sequence

$$1 \to \mathrm{G}^{\mathrm{o}} \to \mathrm{G} \to \mathrm{G}/\mathrm{G}^{\mathrm{o}} \to 1$$

where G^{o} is the connected component of the identity. In particular, a topological group is always an extension of a totally disconnected group by a connected group.

For locally compact groups, structure results for both extreme terms of the above exact sequence are available. The connected component can be approached by Lie methods, thanks to the solution to Hilbert's fifth problem. See [**MZ55**, Section 4.6] for a proof of the following result.

Proposition 1.2.9. — *Let* G *be a connected locally compact group. Then for any identity neighborhood* U*, there exists a compact normal subgroup* K *contained in* U *such that* G/K *is a Lie group.*

The totally disconnected quotient can be studied through its family of compact open *subgroups* thanks to van Dantzig's theorem (see [**Bou71**, III, §4, n° 6, cor. 1] for a proof).

Proposition 1.2.10 (van Dantzig). — *Let* G *be a totally disconnected locally compact group. Then there exists a base of identity neighborhoods made of compact open sub-*

groups.

We will recall below (Proposition 1.3.1) another useful structure result for locally compact groups.

1.2.D Uniformity

We recall that a topological group is naturally endowed with several uniform structures. We follow Bourbaki's terminology by calling the *left* uniform structure the one whose vicinities are defined by $x^{-1}y \in \mathrm{V}$, where V runs among an identity neighborhood base, and the *right* uniform structure the one whose vicinities are defined by $xy^{-1} \in \mathrm{V}$. Consequently, a right-uniformly continuous function is a uniformly continuous function for the right uniform structure; for scalar functions, that reads: for any $\varepsilon > 0$, there is an identity neighborhood U in G such that

$$\sup_{x \in \mathrm{G}} |f(x) - f(ux)| \leqslant \varepsilon$$

for any $u \in \mathrm{U}$.

We will write $\mathscr{C}_{\mathrm{rucb}}(\mathrm{G})$ (resp. $\mathscr{C}_{\mathrm{lucb}}(\mathrm{G})$) for the closed subspaces of $\mathscr{C}_{\mathrm{b}}(\mathrm{G})$ made of right-uniformly continuous (resp. left-uniformly continuous) functions. Observe that the G-actions on $\mathscr{C}_{\mathrm{b}}(\mathrm{G})$ deduced from the left or right translations in the argument *each* preserve *both* spaces $\mathscr{C}_{\mathrm{rucb}}(\mathrm{G})$ and $\mathscr{C}_{\mathrm{lucb}}(\mathrm{G})$. It is however important to consider the "switched" situation (that is, left translations for $\mathscr{C}_{\mathrm{rucb}}(\mathrm{G})$ or right translations for $\mathscr{C}_{\mathrm{lucb}}(\mathrm{G})$) when we need some continuity (see Corollary C.2.3).

The inversion map $g \mapsto g^{-1}$ is an isomorphism of uniform structure between the right and the left uniformities. Consequently, we can unambiguously call a topological group *complete* (resp. *precompact*) if it is complete (resp. precompact) for any of these uniformities. Equivalently, a precompact group is isomorphic (as a topological group) to a dense subgroup of a compact group. Indeed, the completion of a group with respect to the *upper bound* of the left and the right uniformities is again a topological group [**RD81**, Proposition 10.12(c)]; moreover, this upper bound is again precompact if both one-sided uniformities are precompact [**RD81**, Lemma 9.12].

1.3 Amenability

Amenability is a group property introduced by von Neumann in order to understand the Banach–Tarski paradox. It enjoys dozens of equivalent definitions—as a matter of fact, this thesis will add yet another one, and extend to locally compact groups two others that were previously known for discrete groups. We recall here briefly the main definitions and properties. Proofs can be found e.g. in [**Gre69**] and [**GH17**].

Let G be a topological group and E be a subspace of $\ell^\infty(\mathrm{G})$ (or of $\mathrm{L}^\infty(\mathrm{G})$ if G is locally compact), containing the constant functions. A *mean* on E is a positive normalized linear form, that is, a linear map $\mathfrak{m}\colon \mathrm{E} \to \mathbf{R}$ such that $\mathfrak{m}(\mathbb{1}_\mathrm{G}) = 1$ and that $\mathfrak{m}(f) \geqslant 0$ whenever $f \geqslant 0$. A mean is left (resp. right) invariant if for any $f \in \mathrm{E}$ and $g \in \mathrm{G}$, $\mathfrak{m}(g \cdot f) = \mathfrak{m}(f)$, where $(g \cdot f)(x) = f(g^{-1}x)$ (resp. $(g \cdot f)(x) = f(xg)$), provided of course that E is invariant for this action.

Amenability of a general topological group is defined by the existence of a left invariant mean on the space of right-uniformly continuous bounded functions $\mathscr{C}_{\mathrm{rucb}}(\mathrm{G})$. Equivalently, using the map $f \mapsto \check{f}$ defined by $\check{f}(x) = f(x^{-1})$, we can ask for a right invariant mean on the space of left-uniformly continuous bounded functions. This definition is equivalent to a geometric fixed-point property (see Proposition 5.2.20), as well as to the existence of invariant Radon measures for actions on compact spaces, see [**GH17**, Proposition 3.6].

Fortunately, for locally compact groups, amenability is equivalent to the existence of a "better" mean, namely, a left and right invariant mean on the whole space $\mathrm{L}^\infty(\mathrm{G})$, see [**Gre69**, Theorem 2.2.1]. We will repeatedly and without notice use this flexibility allowed by locally compact groups to work in $\mathrm{L}^\infty(\mathrm{G})$ instead of $\mathscr{C}_{\mathrm{rucb}}(\mathrm{G})$.

Examples of amenable locally compact groups include compact groups, virtually solvable groups (in particular, abelian groups), groups of subexponential growth, ... The canonical example of a nonamenable group is a nonabelian free group.

The class of all amenable groups is stable by quotient, by extension, and by direct limit. In particular, a locally compact group is amenable if and only if all its compactly generated subgroups are amenable.

A closed subgroup of an amenable group need *not* be amenable [**Har73**, Proposition 1.(iii)]. However, closed subgroups of amenable *locally compact* groups are amenable; in particular, subgroups of discrete amenable groups are amenable.

Therefore, any locally compact group containing a discrete nonabelian free subgroup will fail to be amenable. But there also exist discrete nonamenable groups without free subgroups.

We will write $M_1(G)$ for the set[1] of means on $\mathscr{C}_{\mathrm{rucb}}(G)$; it is a nonempty compact convex subset of the dual of $\mathscr{C}_{\mathrm{rucb}}(G)$ (endowed with the weak-$*$ topology). The set of means is an invariant subspace of $\mathscr{C}_{\mathrm{rucb}}(G)'$, where the latter is endowed with the usual contragredient representation of the representation on $\mathscr{C}_{\mathrm{rucb}}(G)$ by left translation. A left invariant mean is then nothing but a fixed point in $M_1(G)$.

Loosely speaking, amenability witnesses a kind of "tamability" of the group: technically, the possibility to take average allows many combinatorial proofs for finite groups to be adapted to amenable ones. However, we point out that this intuition is misleading outside the locally compact world. Non-locally compact groups (even Polish ones) have a tendency to be amenable not because of their nice structure, but because they are too huge to let their pathology expresses itself inside compact spaces such as the set of means. For interesting examples of amenable non-locally compact groups, see [**GH17**].

In the study of infinite groups, the amenable groups are, if not an easy case, at least one for which a powerful tool, the mean, is available. It is therefore useful to have structure results in order to simplify the study of nonamenable groups. This is done by removing as much amenability as possible from our group. More precisely, the stability properties allow to define the *amenable radical*, written $\mathrm{Ramen}(G)$, as the largest normal amenable subgroup of G. It is indeed a radical: the quotient $G/\mathrm{Ramen}(G)$ has no nontrivial normal amenable subgroup. In particular, if G itself is not amenable, then $G/\mathrm{Ramen}(G)$ is not amenable either. The following structure result shows the relevance of the amenable radical.

Proposition 1.3.1 (Burger–Monod). — *Let* G *be a locally compact group. The quotient* $G/\mathrm{Ramen}(G)$ *has an open characteristic subgroup of finite index that splits as* $S \times D$, *where* S *is a connected semisimple Lie group with no compact factor and trivial center, and* D *is a totally disconnected group.*

The precise definition of the above-mentioned characteristic subgroup is irrelevant for our purposes (since it is open and of finite index, hence topologically, measurably and geometrically similar to $G/\mathrm{Ramen}(G)$) but can be found in [**BM02**, Section 3.3]. Observe that the amenable radical contains in particular all normal compact subgroups, hence the subgroup $S \times D$ is metrizable if G is σ-compact by the Kakutani–Kodaira theorem.

[1] This notation should not be confused with $\mathscr{M}(G)$, the set of Borel measures on G.

2 TYCHOMORPHISMS

> Die Frage, ob man zur Lösung der mathematischen Probleme die Anschauung brauche, muß dahin beantwortet werden, daß eben die Sprache hier die nötige Anschauung liefert[1].
>
> L. WITTGENSTEIN, *Tractatus logico-philosophicus*, 6.233

TYCHOMORPHISMS PROVIDE A COMMON FRAMEWORK for closed subgroups and measure equivalence. They enjoy three fundamental features: tychomorphisms can be composed; tychomorphisms from a free group to a quotient group can be lifted; amenability can be pulled back through tychomorphisms. The most interesting examples of tychomorphisms will be provided in the next chapter and applications will be given in Chapters 4 to 6.

2.1 Amplifications

We recall that a *measured* G*-space* is a measured space endowed with a non-singular action of a group G. *Isomorphisms* of measured spaces are understood as measure-preserving Borel isomorphisms of conull measurable subsets.

Definition 2.1.1. — *Let* (Σ, m) *and* (Σ', m') *be two measured* G*-spaces. We say that*

[1] *To the question whether we need intuition to solve a mathematical problem, we must answer that the intuition is precisely provided by the language itself.*

(Σ', m') *is an* amplification *of* (Σ, m) *if there exists a* G*-equivariant isomorphism*

$$(\Sigma', m') \simeq (\Sigma, m) \times (X, \nu),$$

where (X, ν) *is a standard measured space endowed with the trivial* G*-action. The amplification is said* finite *if the measure* ν *is finite. The space* (X, ν) *is called the* inert factor *(or* G*-inert factor if we need to be more precise).*

Observe that G preserves the measure m if and only if it preserves the measure m'. Being an amplification is obviously a reflexive transitive relation between measured G-space, which is moreover invariant upon dilating measures by a constant factor. In particular, if an amplification is finite, we can always assume, up to rescaling either m or m', that ν is a probability measure.

We will mostly be interested in amplifications of the group G itself, endowed with a left or right Haar measure (and acting on itself by left translation). When the group is unimodular, we will say unambiguously that Σ is an amplification of G.

Example 2.1.2 (Closed subgroup). — Let H be a closed subgroup of a locally compact second-countable group G. Then (G, m_G) is an amplification of (H, m_H) for the H-actions by left translations. Indeed, by Ripley's disintegration theorem (Proposition 1.2.8), there exists a measure ν on the Borel standard space $H\backslash G$ such that (G, m_G) is H-equivariantly isomorphic to $(H, m_H) \times (H\backslash G, \nu)$.

Example 2.1.3 (Discrete groups). — Let Γ be a discrete countable group. A measured Γ-space (Σ, m) is an amplification of Γ if and only if the Γ-action on Σ admits a measurable fundamental domain of positive measure (by a *fundamental domain*, we mean: a subset X such that, up to null sets, translates are pairwise disjoint and cover Σ). The condition is obviously necessary and, if $X \subset \Sigma$ is a fundamental domain of positive measure, then (Σ, m) is Γ-equivariantly isomorphic to $(\Gamma, \#) \times (X, m|_X)$.

Let (Σ', m') be an amplification of (Σ, m) and choose some G-equivariant isomorphism

$$(\Sigma', m') \simeq (\Sigma, m) \times (X, \nu).$$

By composing this isomorphism with the projections over each factor of the second term, we get two measurable maps:

- the *retraction* $\chi\colon \Sigma' \to \Sigma$, which is G-equivariant;
- the *contraction* $\pi\colon \Sigma' \to X$, which is G-invariant.

The measure m (resp. ν) is a pseudo-image of m' through the retraction (resp. the contraction). If the amplification is finite, we even have the equality $\chi_*(m') = \nu(X)m$.

Example 2.1.4. — Let us look again at the example of a group G considered as an amplification of one of its closed subgroups H (Example 2.1.2). Let $\sigma\colon H\backslash G \to G$ be a measurable section of the projection $\pi\colon G \to H\backslash G$ (Proposition 1.2.7). Then a retraction is given by $g \mapsto g\sigma(\pi(g))^{-1}$ and the contraction is nothing but the projection π.

2.2 Tychomorphisms

Here comes the main object of this chapter, the probabilistic generalisation of a closed subgroup. We recall that, unless otherwise stated, groups act on themselves by left translations.

Definition 2.2.1. — *Let* G *and* H *be two locally compact second-countable groups. A* tychomorphism *from* H *to* G *is a measured* $(G \times H)$*-space* (Σ, m) *that is, as an* H*-space, an amplification of* (H, m_H) *and, as a* G*-space, a finite amplification of* $(G, \check{m}_G)$.

Observe that, by definition, the H-action on the tychomorphism (Σ, m) preserves the measure, whereas the G-action does so if and only if G is unimodular. To avoid ponderous notations we will often drop the measure of the inert factors of the tychomorphisms. We will be more careful for the Haar measures, since our groups are not unimodular in general. But we can nonetheless rescale their Haar measures and hence always assume that the G-inert factor of the tychomorphism has total mass one.

Intuitively, a tychomorphism from H to G can be thought as a measured space that "contains" H and is "not too different" from G. It is a generalisation of both measure equivalence (which corresponds to the case where G and H are unimodular and Σ is also a finite amplification of H) and closed subgroups, as shown by the next example.

Example 2.2.2. — Let H be a closed subgroup of a locally compact second-countable group G. Then there is a tychomorphism from H to G. Indeed, consider the measured space $\Sigma = (G, m_G)$ endowed with the action of H by left translations and with the action of G by right translations. These actions commute and we already know by Example 2.1.2 that (G, m_G) is an amplification of (H, m_H). Moreover, the inversion map $g \mapsto g^{-1}$ gives a G-equivariant isomorphism between (G, m_G) and $(G, \check{m}_G)$, the latter being endowed with the G-action by left translations.

The next example identifies the trivial objects associated to that notion.

Example 2.2.3. — Let H be a second-countable *compact* group. Then there is a tychomorphism from H to any locally compact second-countable group G: simply consider $\Sigma = G \times H$. On the other hand, if there is a tychomorphism Σ from a group H to a second-countable compact group G, then H must be compact too, since Σ is a finite measure space that is an amplification of H.

2.3 Cocycles and actions

Tychomorphisms being in particular amplifications, we can also associate to them retractions and contractions. But the commutativity of the actions gives rise to a fundamental feature of tychomorphisms: their link with measurable cocycles.

Let (Σ, m) be a tychomorphism from H to G and consider the G-equivariant isomorphism

$$(\Sigma, m) = (G, \check{m}_G) \times (X, \theta)$$

given by the definition. The H-action on Σ commutes with the G-action, so if we transport the former through the above isomorphism, we can write the H-action as

$$h \cdot (g, x) = (g\alpha(h, x)^{-1}, h \cdot x), \tag{2.1}$$

where the H-action on X is non-singular and the measurable map $\alpha \colon H \times X \to G$ is a *measurable cocycle*, that is satisfies, for all $h, h' \in H$ and almost all $x \in X$:

$$\alpha(hh', x) = \alpha(h, h' \cdot x)\alpha(h', x)$$
$$\alpha(e_H, x) = e_G$$

Moreover, as H preserves the measure m and acts on the right on G (which is endowed with a right Haar measure), H must also preserve the measure of the inert part (X, θ).

If we apply this same argument for the G-action through the H-equivariant isomorphism

$$(\Sigma, m) = (H, m_H) \times (Y, \eta),$$

we also get a non-singular G-action on (Y, η) and a measurable cocycle $\beta \colon G \times Y \to H$. However, the measure η is not necessarily preserved by G, since m is not G-invariant and m_H is not necessarily right-invariant. Actually, the Radon–Nikodym derivative of translates of η is given by

$$\frac{\mathrm{d}g\eta}{\mathrm{d}\eta}\colon \mathrm{Y} \to \mathbf{R}\colon y \mapsto \Delta_{\mathrm{G}}(g)\Delta_{\mathrm{H}}(\beta(g^{-1}, y)),$$

as it can be easily checked thanks to Fubini's theorem.

We point out however that a tychomorphism carries more information than a pair of cocycles as above, because of the amplification requirement. More precisely, Formula (2.1) alone does not render the fact that the measure $(\mathrm{G} \times \mathrm{H})$-space is also an H-amplification (compare e.g. the proof of Lemma 2.4.3).

2.4 Operations on tychomorphisms

Lemma 2.4.1 (Composition). — *Let* G_1, G_2 *and* H *be three second-countable locally compact groups. If there are tychomorphisms from* G_1 *to* H *and from* H *to* G_2*, then there is one from* G_1 *to* G_2*.*

Proof. — Write the two given tychomorphisms as

$$(\mathrm{G}_1, m_{\mathrm{G}_1}) \times \mathrm{X}_1 \simeq \Sigma_1 \simeq (\mathrm{H}, \check{m}_{\mathrm{H}}) \times \mathrm{Y}_1$$
$$(\mathrm{H}, m_{\mathrm{H}}) \times \mathrm{Y}_2 \simeq \Sigma_2 \simeq (\mathrm{G}_2, \check{m}_{\mathrm{G}_2}) \times \mathrm{X}_2,$$

where Y_1 and X_2 have finite measure. Let $\alpha_i\colon \mathrm{G}_i \times \mathrm{Y}_i \to \mathrm{H}$ be the associated cocycles and consider the non-singular actions of G_i on Y_i. Consider the measure space $\Sigma = \mathrm{Y}_1 \times \mathrm{Y}_2 \times (\mathrm{H}, m_{\mathrm{H}})$ endowed with the following non-singular actions of $\mathrm{G}_1 \times \mathrm{G}_2$:

$$(g_1, g_2)(z_1, z_2, h) = (g_1 z_1, g_2 z_2, \alpha_1(g_1, z_1) h \alpha_2(g_2, z_2)^{-1}).$$

Thanks to the isomorphisms given by the tychomorphisms, we see that there is a G_2-equivariant isomorphism between Σ and $\mathrm{Y}_1 \times \mathrm{X}_2 \times (\mathrm{G}_2, \check{m}_{\mathrm{G}_2})$. As Y_1 and X_2 have finite measure, Σ is a finite amplification of $(\mathrm{G}, \check{m}_{\mathrm{G}_2})$. Similarly, after intertwining Σ by the isomorphism induced by the inverse map $\mathrm{H} \to \mathrm{H}\colon h \mapsto h^{-1}$, we get a G_1-equivariant isomorphism between Σ and $\mathrm{Y}_2 \times \mathrm{X}_1 \times (\mathrm{G}_1, m_{\mathrm{G}_1})$. Hence Σ is indeed a tychomorphism from G_1 to G_2. ■

A first consequence of composition is the fact that, for unimodular groups, the *existence* of a tychomorphism is invariant for measure equivalence in both source and target; that is, if there is a tychomorphism from H to G, then there is one from H′ to G′ for any groups H′ and G′ that are measure-equivalent to H and G, respectively.

Another consequence of composition is the fact that tychomorphisms can be built by composing successively measure equivalences and inclusions of subgroups. Whether all tychomorphisms arise that way is a natural question (although, as we will argue, not a crucial one), that we will discuss in Section 2.7.

Remark 2.4.2. — The composition of tychomorphisms is compatible with the composition of cocycles. Let us use again the notations of Lemma 2.4.1. The action of G_1 on the inert part $Y_1 \times X_2$ is given by

$$g \cdot (y, x) = (g \cdot y, \alpha_1(g, y) \cdot x)$$

and the associated cocycle is

$$(\beta_2 \circ \alpha_1)(g, y, x) := \beta_2(\alpha_1(g, y), x),$$

where $\beta\colon H \times X_2 \to G_2$ is the associated cocycle for the action of H on X_2. Similar formulæ hold for the action of G_2 on $Y_2 \times X_1$.

Our main concern in this thesis will be about tychomorphisms from a free group. The universal property of free groups implies that discrete free groups can be lifted (preserving discreteness) from a quotient group. A similar statement holds for tychomorphisms.

Lemma 2.4.3 (Lifting of free groups). — *Let* G *be a locally compact second-countable group,* N *a closed normal subgroup, and* $Q = G/N$ *the corresponding quotient. If there is a tychomorphism from a free group* F_r *of rank* r $(0 \leqslant r \leqslant \aleph_0)$ *to* Q*, then there is also one from* F_r *to* G*.*

Proof. — Let $(\Sigma, m) \simeq (Q, \check{m}_Q) \times (X, \nu)$ be a tychomorphism from F_r to Q with $\alpha\colon F_r \times X \to Q$ the associated cocycle and let $\mathcal{F}$ be a measurable fundamental domain of positive measure for the action of F_r. We will show that $(\widetilde{\Sigma}, \widetilde{m}) = (G, \check{m}_G) \times (X, \nu)$, which is obviously a finite amplification of $(G, \check{m}_G)$, can be endowed with a commuting measure-preserving action of F_r that admits a measurable fundamental domain, hence is a tychomorphism from F_r to G.

Let S be a basis of the free group F_r and choose any Borel section $\sigma\colon Q \to G$ of the projection $\pi\colon G \to Q$ (Proposition 1.2.7). As any element of F_r can be written in a unique way as a reduced S-word, there is an essentially unique cocycle $\widetilde{\alpha}\colon F_r \times X \to G$ such that $\widetilde{\alpha}(s, x) = \sigma \circ \alpha(s, x)$ for all $s \in S$ and almost all $x \in X$. This cocycle defines an action of

F_r on $\widetilde{\Sigma}$ by

$$w \cdot (g, x) = (g\widetilde{\alpha}(w, x)^{-1}, w \cdot x) \qquad (w \in F_r, g \in G, x \in X).$$

It is immediate that this action commutes with the G-action and preserves the measure, as the action of F_r on X preserves the measure ν (Section 2.3). We then only need to show that this action admits a fundamental domain. By Weil's integration formula (Proposition 1.2.4), the measured space $\widetilde{\Sigma}$ is isomorphic to $(N, m_N) \times (\Sigma, m)$ for some choice of Haar measure on N. By transporting the F_r-action via this isomorphism, we can check that

$$w \cdot (n, s) = (n', ws) \qquad (w \in F_r, n \in N, s \in \Sigma)$$

for some $n' \in N$. The precise value of the latter[2] is irrelevant for our purpose, the point is that the F_r-action on $\widetilde{\Sigma}$ is a twisted product of the given action on Σ by some F_r-cocycle on N, hence admits $N \times \mathcal{F}$ as a fundamental domain of positive measure. ■

2.5 Link with amenability

Tychomorphisms are intimately related to amenability, as we will constantly see. The very motivation for their definition was the following striking measurable solution to von Neumann problem.

Theorem 2.5.1. — *Let* G *be a locally compact second-countable group. Then* G *is nonamenable if and only if there is a tychomorphism from a nonabelian free group to* G.

Proof plan. — The "if" part is an easy generalisation to tychomorphisms of the fact that amenability passes to closed subgroups (of locally compact groups), see Proposition 2.5.2 below. For discrete groups, the "only if" part is a landmark of measured group theory, known as the Gaboriau–Lyons theorem. We will explain in Chapter 3 the technical modifications needed for the locally compact case, see Theorem 3.2.1. □

So let us start with the easy step of this theorem.

[2]Namely, $n' = \sigma\left(q\alpha(w,x)^{-1}\right)^{-1}\sigma(q)n\widetilde{\alpha}(w,x)^{-1}$, where $s = (q, x)$ is the value given by the decomposition $\Sigma \simeq Q \times X$.

Proposition 2.5.2. — *Let* G *be an amenable locally compact second-countable group. If there is a tychomorphism from another locally compact second-countable group* H *to* G*, then* H *is amenable.*

We give here two elementary proofs, one via the mean aspect of amenability and one via its nice fixed-point characterization. Incidentally, we observe that the induction techniques of Chapter 4, namely Proposition 4.3.5 and Theorem 4.3.12, yield two other (uselessly far-fetched) proofs, via respectively the bounded cohomology characterization of amenability or another fixed-point property (Theorem 5.2.21).

Proof (Invariant mean). — Let $\mathfrak{m}\colon \mathrm{L}^\infty(\mathrm{G}\times\mathrm{X}) \to \mathrm{L}^\infty(\mathrm{X})$ be an invariant "vector-valued" mean, that is, an invariant positive linear map such that $\mathfrak{m}(\mathbb{1}_{\mathrm{G}\times\mathrm{X}}) = \mathbb{1}_{\mathrm{X}}$, where the invariance is understood with respect to the G-action by *right* translations. The existence of such a mean for amenable groups can be proved via the Day–Rickert fixed-point property, along the lines of the first part of the proof of Proposition 5.2.20 (see also [**Mon01**, Section 5.3] for a detailed study of this tool).

Consider now the following composition of maps

$$\mathrm{L}^\infty(\mathrm{H}) \to \mathrm{L}^\infty(\Sigma) \simeq \mathrm{L}^\infty(\mathrm{G}\times\mathrm{X}) \xrightarrow{\mathfrak{m}} \mathrm{L}^\infty(\mathrm{X}) \to \mathbf{R},$$

where

- the first arrow is the precomposition by a retraction of Σ onto H;
- the isomorphism is given by the identification of Σ as a finite amplification of G;
- the last arrow is given by averaging over the finite measure space X.

One checks immediately that this composition is a left invariant mean on $\mathrm{L}^\infty(\mathrm{H})$. ■

Proof (Fixed point). — Along the lines of [**Zim78**, Theorem 2.1], we sketch a second proof, less straightforward but more geometric. This proof purposefully anticipates material from subsequent chapters (see in particular Proposition 5.2.20 for the fixed-point characterization of amenability). Let K be a compact convex set on which H acts jointly continuously by affine homeomorphisms. Consider the set

$$\mathrm{L} = \{f\colon \Sigma \to \mathrm{K} \mid f \text{ is measurable and for all } h \in \mathrm{H},\ f(hs) = hf(s) \text{ for almost all } s \in \Sigma\}$$

There is a natural action of G on L given by precomposition on Σ. Using the fixed-point property of the amenable group G, we get a G-fixed point $f \in \mathrm{L}$. We can view the latter as an H-equivariant map from X to K. Pushing the normalised measure of X (which is H-invariant) onto K via this map, we get an invariant probability measure on K. The barycenter of the latter is thus the sought fixed point.

The only gap in the above sketch is to make rigorous the fact that L is a compact convex set on which G acts jointly continuously (so that we can indeed apply the fixed-point property of G). The surest way to fill it with the tools appearing in this work is to observe that we only need to prove orbital continuity (Lemma C.1.9) and that we may assume that K is a weakly-$*$ compact subset of the dual E' of a Banach space E (see Proposition 5.1.7). We can therefore see L as a weakly-$*$ compact subset of the Banach space $L^\infty_{w*}(\Sigma, E')$ of bounded scalarly weakly-$*$ measurable maps (see Section 4.3.B). The action of G is then orbitally continuous, being the contragredient of the action by precomposition on $L^1(\Sigma, E)$. Continuity of the latter can be seen, for instance, with the G-equivariant identification

$$L^1(G \times X, E) \simeq L^1(G)\widehat{\otimes} L^1(X)\widehat{\otimes} E, \qquad \blacksquare$$

where the actions on $L^1(X)$ and on E are trivial. (This identification is yielded by Grothendieck's theorem on projective tensor products [**Gro55**, I §2 n° 2, Théorème 2], see also [**Ryan02**, Section 2.3].)

2.6 Discrete tychomorphisms

We show here that “interesting” tychomorphisms must carry a “genuine” measure. We call *atom* a point x of a measure space such that $\{x\}$ is measurable and has positive measure. A measure space is *atomless* if it has no atom; it is on the contrary *discrete* if, up to a null set, each point is an atom. We will call a tychomorphism *discrete* if the underlying measure space is discrete.

Proposition 2.6.1. — *Let* G *and* H *be two second-countable locally compact groups and* Σ *be a tychomorphism from* H *to* G.

If Σ *has an atom, then both* G *and* H *are discrete and* H *is virtually a subgroup of* G *(that is, there is a finite-index subgroup of* H *which is isomorphic to a subgroup of* G*).*

In particular, if a group admits a tychomorphism with an atom from a nonabelian free group, then it actually admits a nonabelian free subgroup.

Proof. — The decomposition of Σ as amplifications of G and H shows that if Σ has an atom, then so do the groups G and H. Since Haar measures are invariant and inner regular, these groups must then be discrete.

Write $\Sigma \simeq \mathrm{G} \times \mathrm{X}$, with an associated cocycle $\alpha\colon \mathrm{H} \times \mathrm{X} \to \mathrm{G}$, and choose an atom $(g_0, x_0) \in \mathrm{G} \times \mathrm{X}$; in particular, x_0 is an atom of the inert part X. As H preserves the measure on X and the latter has finite measure, the orbit of x_0 must be finite. Let N be the stabiliser of x_0, which is therefore of finite index in H. Since N fixes x_0, the map $\alpha(\cdot, x_0)$ produces a morphism from N to G. Moreover, this morphism must be injective, as N acts (essentially) freely on Σ. Hence H is indeed virtually a subgroup of G.

The statement about free groups is a straightforward consequence of the Nielsen–Schreier theorem about subgroups of free groups. ■

2.7 Subgroups, measure equivalence, and tychomorphisms

We saw in Section 2.4 that tychomorphisms can be composed, and in Section 2.2 that the basic examples of tychomorphisms are given by subgroups and measure equivalence. We can therefore ask whether all tychomorphisms arise by composing the latter. More precisely, if Σ is a tychomorphism from H to G, do there exist groups $\mathrm{H}_1, \ldots, \mathrm{H}_n$ and $\mathrm{G}_1, \ldots, \mathrm{G}_{n-1}$ such that Σ is the composition of the following chain

$$\mathrm{H} \leqslant \mathrm{H}_1 \overset{\mathrm{ME}}{\simeq} \mathrm{G}_1 \leqslant \mathrm{H}_2 \overset{\mathrm{ME}}{\simeq} \mathrm{G}_2 \leqslant \cdots \leqslant \mathrm{H}_{n-1} \overset{\mathrm{ME}}{\simeq} \mathrm{G}_{n-1} \leqslant \mathrm{H}_n \overset{\mathrm{ME}}{\simeq} \mathrm{G}?$$

Of course, we may have $\mathrm{H} = \mathrm{H}_1$ or $\mathrm{H}_n = \mathrm{G}$.

It is possible that the answer is positive, even maybe with a uniform, small bound on the length n of the above chain. However, this would not overwhelm the theory, because the yoga of tychomorphisms is precisely that, if you can prove something for closed subgroups and for measure-equivalent groups, then you should be able to prove it at once for tychomorphisms—and this unifying framework make their interest.

Let us still have a look at this question for specific small chains. If H admits a tychomorphism to G, must H be measure-equivalent to a subgroup of G? The answer is no. For instance, consider a nonamenable Tarski monster G. The subgroups of G are either finite or G itself, hence give rise to only two-measure equivalence classes, one of them containing only finite groups. We therefore only need to produce a tychomorphism to G from an infinite group H which is not measure-equivalent to G. As G is not amenable, it admits tychomorphisms from any free group of countable rank (Theorem 3.2.1); we can choose for instance $\mathrm{H} = \mathbf{Z}$, which is amenable, hence not measure-equivalent to G. (We can also reason with $\mathrm{H} = \mathrm{F}_\infty$, which is not measure-equivalent to G because the latter is finitely generated, hence has a finite first ℓ^2-Betti number, whereas $\beta_1(\mathrm{F}_\infty) = \infty$

and ℓ^2-Betti numbers of measure-equivalent discrete groups are proportional by [**Gab02**, Theorem 6.3].)

Symmetrically, we can ask whether any tychomorphism from H to a unimodular group G decomposes as an inclusion of H into a group G′ that is measure-equivalent to G. It is harder to exclude such a possibility, because H may embed as a subgroup of groups falling into infinitely many measure equivalence classes[3]. In particular, considering the tychomorphisms from the free groups, we may ask for the following strongly negative answers to the von Neumann problem (stated here for discrete groups for the sake of simplicity).

Problem 1. — Find a nonamenable group such that no group in its measure-equivalence class contains a nonabelian free subgroup. Is there even a nonamenable torsion group whose measure-equivalence class contains only torsion groups?

When beholding these problems, keep in mind that determining entirely the measure-equivalence class of a nonamenable group is always a hard problem. Moreover, known measure-equivalence invariants (ratio of ℓ^2-Betti number, Kazhdan's property (T), some non-vanishing results in bounded cohomology, Haagerup property, . . .) do not pass to subgroups or are enjoyed by free groups, hence cannot be used in combination with a tychomorphism from a nonabelian free groups to a nonamenable group.

[3]Trivial examples are given by finite groups or **Z**. Nontrivial examples can be found among rigidity results for products [**Gab02**, Corollaire 0.3], [**MS06**, Theorem 1.16].

3 Sources of tychomorphisms

HAVING LAID THE BASIC FEATURES of tychomorphisms, we need now both interesting examples and unexpected applications. This chapter is devoted to the former. The first section will develop a general machinery to produce tychomorphisms from suitable equivalence relations—which is reminiscent of the link between orbit and measure equivalence. We then use it in the second section to get tychomorphisms from nonabelian free groups to any nonamenable group.

The latter result was proved, for discrete groups and in a slightly different language, by Gaboriau and Lyons in [**GL09**]. The technical shackles inherent to the locally compact case are precisely dealt with by the machinery of the first section, but the rest of the proof follows the same idea.

This chapter is undoubtedly the most technical, appealing to tools from descriptive set theory and percolation. However, the other chapters rest *only* on Theorem 3.2.1, namely on the *existence* of interesting tychomorphisms for nonamenable groups (whatever they look like). In other words, this chapter may safely be skipped at a first reading, taking Theorem 3.2.1 for granted.

3.1 From equivalence relations to tychomorphisms

3.1.A Motivation: closedness without topology

One of the main tools of measured group theory is the orbit equivalence relation induced by a group action on a probability space. We will show in this chapter how the inclusion of orbit equivalence relations is related to tychomorphisms. This link is rather limpid for discrete groups, but some technical difficulties arise for locally compact groups: indeed, tychomorphisms of the latter generalise *closed* subgroups, and can certainly not encompass all subgroups in view of the applications to amenability (Proposition 2.5.2), since locally compact amenable groups may well have nonamenable but non-closed subgroups.

Hence some closedness condition has to be imposed on the subrelations we will consider. Unfortunately, even when the orbit equivalence relation carries some topology, it would be far too restrictive to look at the closed subrelations. Indeed, the subrelation induced by a closed subgroup has no reason to be closed in the induced topology, even for discrete groups[1].

On the other end of the "regularity" spectrum, the Borel structure of the equivalence relation is too loose. Indeed, the orbit equivalence relation induced by an action of a countable group by Borel transformations is always itself Borel, hence we cannot distinguish discrete and non-discrete countable subgroups by the Borel structure of their orbit equivalence relations (which is not a surprise, since the Borel structure induced on a countable subset by a Hausdorff topology is always trivial).

So we are faced by the following problem: "*if* G *is a topological group and* $H \leqslant G$ *an abstract subgroup, how can we check that* H *is closed in* G without *having direct access to the topology of* G*?*". We are allowed to use the Borel structure that H inherits from G, but we just saw that we will need more. In its full generality, this problem is obviously unsolvable, but, perhaps surprisingly, it does admit a solution for Polish locally compact groups.

First, recall that a subgroup H of a locally compact group G is closed if and only if it is locally compact for the induced topology[2]. Moreover, locally compact groups can be characterized via their Haar measures (cf. [**Hal50**, §62]). Lastly, Ripley's theorem (Proposition 1.2.8) relates the Haar measure of a closed subgroup and the Haar measure of the ambient group. Putting all these facts together, we get the following measure-theoretical characterization of closedness: a Borel subgroup $H \leqslant G$ of a Polish locally compact group is closed if and only if

- there exists a left-invariant Borel measure m_H on H (for the Borel structure induced on H by that of G);
- there is an isomorphism of measured spaces $\varphi\colon (H, m_H) \times (B, \nu) \to (G, m_G)$, where (B, ν) is some standard measured space and m_G is a Haar measure on G;

[1] A concrete example is given by the group $G = \mathbf{Z}$ acting on the circle by irrational rotations, and the subrelation induced by the subgroup $H = 2\mathbf{Z}$.

[2] The "only if" direction is true for locally compact spaces, the "if" one is specific to topological groups, cf. [**HR79**, Theorem 5.11]

- the above isomorphism is H-equivariant.

Well, this seems to be a rather complicated characterization of closedness. But the point is that it is purely measure-theoretical, allowing to a suitable translation into the realm of equivalence relations. The latter, when they are relevant for group theory, are indeed usually not given on a topological space but rather on a measured space[3]. This translation, that requires some stamina, will give rise to the notion of *measured tilings* in Section 3.1.E, after some preliminary remarks on measured equivalence relations in Section 3.1.D. For the sake of a better understanding, we develop first in Section 3.1.C the easier notion of a *tiling* of a relation by a subrelation, which corresponds, for groups, to the fact that a closed subgroup of a Polish group produces a "nice" partition of the ambient group, in the sense that equivalence classes (the cosets) are isomorphic to each other and the quotient space is standard. In other words, tilings correspond to the last two points of the above characterization of closedness, forgetting about the measure.

But let us first start with some reminders on equivalence relations.

3.1.B Background on equivalence relations

For any set X, we will denote by Δ the *diagonal* of $\mathrm{X}\times\mathrm{X}$, that is, $\Delta = \{(x,x) \mid x \in \mathrm{X}\}$, and call *flip* the map $\mathrm{X}\times\mathrm{X} \to \mathrm{X}\times\mathrm{X}\colon (x,y)\mapsto(y,x)$. Recall that an *equivalence relation* $\mathscr{R}$ on a set X is a subset of $\mathrm{X}\times\mathrm{X}$ that contains the diagonal, is invariant under the flip, and is transitive, that is, if (x,y) and (y,z) are in $\mathscr{R}$, then so is (x,z). The *equivalence class* in $\mathscr{R}$ of a point $x\in\mathrm{X}$ is the set $\{y\in\mathrm{X} \mid (x,y)\in\mathscr{R}\}$, it will be denoted by $[x]_{\mathscr{R}}$ or $\mathscr{R}_x$. The latter notation will also be used more generally to denote the *fiber* over x for any subset A in a product $\mathrm{X}\times\mathrm{Y}$ (that is, the set $\{y\in\mathrm{Y} \mid (x,y)\in\mathrm{A}\}$). In this asymmetrical situation, the symbols and the context will make clear with which factor of the product we are working.

We will typically be interested in equivalence relations on a standard Borel space X, or even on a Polish space X. An equivalence relation will be called *Borel* (resp. *analytic*) if it is Borel (resp. analytic) as a subset of $\mathrm{X}\times\mathrm{X}$; it is called *countable* if all its equivalence classes are countable.

An important source of examples of equivalence relations is given by group actions. For a group G acting on a set X, we define the *orbit equivalence relation* $\mathscr{R}_{\mathrm{G}}$ as the set of couples (x,y) such that there exists a $g\in\mathrm{G}$ with $y=gx$. At the core of all the arguments

[3]It often happens that the measured space at hand is isomorphic to a topological one (for instance, for standard spaces), or even that the relation is isomorphic to one produced by a continuous action of a topological group (cf. [**Kan08**, Proposition 4.3.3]). But these isomorphisms are usually neither canonical nor explicit, hence not very useful when we will need to consider two relations together, for instance when one is a subrelation of the other.

in this chapter is the following easy observation: the orbit equivalence relation $\mathscr{R}_G$ is *itself* endowed with two commuting actions of G, called respectively *on the left* and *on the right*, that are deduced from the action of G on the first and on the second coordinates. More precisely, $\mathscr{R}_G$ is, by definition, an invariant subset of $X \times X$ when the latter is endowed with the diagonal action of $G \times G$.

Caveat! The orbit equivalence relation induced by a Borel[4] or even continuous action is not necessarily itself a Borel equivalence relation! It is however the case for Borel actions of Polish locally compact groups (in particular, discrete countable groups), to which we will therefore restrict our attention[5].

Remark 3.1.1. — Both assumptions on the group are needed. For general Polish groups, we can only get the analyticity of the orbit equivalence relation (trivially for continuous actions; see [**Kan08**, Proposition 4.3.2] for Borel actions), although the equivalence classes are always Borel [**Kan08**, Proposition 4.3.5]. Moreover, we cannot get more regularity for continuous actions: actually, Borel actions of Polish groups can be made continuous by endowing X with another topology that produces the same Borel sets [**BK96**, Theorem 5.2.1]. For a Polish group action inducing a non-Borel orbit equivalence relation, see [**BK96**, 3.2]. For characterizations of Polish group actions inducing Borel equivalence relations, see [**BK96**, 7].

On the other hand, local compactness alone produces even less regular orbit equivalence relations. For instance, choose some non-Borel subgroup H of the circle, with countable index[6]. Endowed with the discrete topology, H is a locally compact non-Polish group that acts continuously on the Polish space $\mathbf{S}^1$. But, by assumption, the classes of the orbit equivalence relation are not Borel. Actually, they are even not analytic, since there are countably many of them.

So our focus in this chapter (as in the preceding one) will be on Polish locally compact groups. As usual, the structure theory tools for general locally compact groups will ultimately allow us to reduce ourself to that case for the applications of the other chapters.

We will consider inclusions of equivalence relations on the same space X. We say that an inclusion $\mathscr{R} \subseteq \mathscr{S}$ has *index* κ if all equivalence classes of $\mathscr{S}$ contain κ equivalence classes of $\mathscr{R}$, where κ is a fixed cardinal. This is a natural assumption to have some "uniformity" in the way $\mathscr{R}$ embeds into $\mathscr{S}$. The prototypical example is given by the

[4]An action of a topological group G on X is called *Borel* if the action map $G \times X \to X$ is Borel. Of course, the notion of a Borel action makes sense for any group endowed with a Borel structure, but we will not need this generality as all our groups are endowed with a topological structure.

[5]The proof is fairly easy for continuous actions, since all orbits are therefore F_σ, see e.g. [**Gao09**, Theorem 8.2.2]. But the continuity of the action is not a restriction by [**Var63**, Theorem 3.2] or [**BK96**, Theorem 5.2.1].

[6]See Footnote 6 on p. 105 for a construction of such a subgroup.

orbit equivalence relations of an inclusion of groups $H \leqslant G$, where the action of H is the restriction of the action of G.

For Borel equivalence relations given on a measure space (X, ν), the above properties are understood "almost surely". For instance, "*countable*" means "for almost all x, the equivalence class of x is countable" and "*index* κ" means "for almost all x, the equivalence class $\mathscr{S}_x$ contains κ equivalence classes from $\mathscr{R}$".

3.1.C Tilings

Definition 3.1.2. — *An inclusion of Borel equivalence relations $\mathscr{R} \subseteq \mathscr{S}$ is said* tilable *if there exists a standard Borel space* B *and a Borel isomorphism $\Phi \colon \mathscr{R} \times B \to \mathscr{S}$ such that*

$$\pi_1 \circ \Phi = \pi_1',$$

where π_1 (resp. π_1') is the projection on the first coordinate of $\mathscr{S}$ (resp. on the first coordinate of the first variable of $\mathscr{R} \times B$). The pair (B, Φ) is called a tiling.

In other words, there is a Borel family of Borel isomorphisms $\Phi_x \colon [x]_{\mathscr{R}} \times B \to [x]_{\mathscr{S}}$ such that

$$\Phi((x, y), b) = (x, \Phi_x(y, b)).$$

Observe in particular that a tilable inclusion must have index $|B|$.

In this chapter, we will repeatedly illustrate our definitions with two kinds of nice Borel equivalence relations (the only ones we will use in applications): the countable ones and the orbit equivalence ones. Let us start by the easier case, the latter.

Lemma 3.1.3. — *Let* G *be a Polish locally compact group with a Borel free action on a standard Borel space* X *and* H *a subgroup of* G*. Let $\mathscr{R}$, $\mathscr{S}$ be the orbit equivalence relations induced by* H *and* G*, respectively. If* H *is closed in* G*, then $\mathscr{R} \subseteq \mathscr{S}$ is tilable.*

Proof. — As H is closed in G, there is a Borel section $\sigma \colon G/H \to G$ of the projection map, that is, $\sigma(gH) \in gH$ for any $g \in G$. Moreover, the quotient space G/H is standard Borel as a quotient of a Polish group by a closed subgroup (Proposition 1.2.7). Therefore, the Borel map

$$\Phi \colon \mathscr{R} \times G/H \to \mathscr{S} \colon (x, y, gH) \mapsto (x, \sigma(gH)y),$$

which is surjective by definition of σ and injective by freeness of the G-action, is a Borel isomorphism (see e.g. [**Sri98**, Theorem 4.5.1]) that tiles the inclusion $\mathscr{R} \subseteq \mathscr{S}$. ■

Let us now investigate tilings of countable equivalence relations; of course, some index assumption on the inclusion is necessary.

Proposition 3.1.4. — *Let* X *be a Polish space and* $\mathscr{R} \subseteq \mathscr{S} \subseteq \mathrm{X}\times\mathrm{X}$ *be two Borel countable equivalence relations. Assume moreover that* $\Delta \subseteq \mathscr{R}$ *and* $\mathscr{R} \subseteq \mathscr{S}$ *have index* κ *and* λ*, respectively. Then there is a countable family* $\{f_{k,\ell}\}_{(k,\ell)\in\kappa\times\lambda}$ *of Borel maps* $\mathrm{X} \to \mathrm{X}$ *such that:*

- *for each* $x \in \mathrm{X}$*,* $f_{(\cdot,\cdot)}(x)$ *is a bijection from* $\kappa \times \lambda$ *to* $\mathscr{S}_x$*;*
- *for each* $x \in \mathrm{X}$*, for each* $\ell \in \lambda$*,* $f_{(\cdot,\ell)}(x)$ *is a bijection from* κ *to* $\mathscr{R}_{f_{0,\ell}(x)}$*.*

In particular,

- $\mathscr{S}$ *is covered by the graphs of* $f_{k,\ell}$*;*
- *for each* $k \in \kappa$*, the map* $f_{(k,\cdot)}(\cdot)\colon \lambda \times \mathrm{X} \to \mathrm{X}$ *is a Borel enumeration of* $\mathscr{R}$*-classes inside given* $\mathscr{S}$*-classes.*

Before giving the proof of that proposition, let us immediately observe the motivating corollary.

Corollary 3.1.5. — *Under the same hypotheses and notations as in Proposition 3.1.4, the inclusion* $\mathscr{R} \subseteq \mathscr{S}$ *is tilable.*

Proof (of Corollary 3.1.5). — Let $\mathrm{B} = \lambda$ with the discrete topology and define a map $\Phi\colon \mathscr{R} \times \mathrm{B} \to \mathscr{S}$ by

$$\Phi(x, y, \ell) = (x, f_{k(x,y),\ell}(x)),$$

where the functions $f_{k,\ell}$ are those given by Proposition 3.1.4 and $k(x, y)$ is the (unique) element j of κ such that $y = f_{j,\ell'}(x)$ for some ℓ'. The map Φ is obviously a bijection, so that we only need to check that it is Borel, hence that $k\colon \mathscr{R} \to \kappa$ is Borel. But for any $j \in \kappa$, we have

$$\begin{aligned} k^{-1}(\{j\}) &= \{(x, y) \in \mathscr{R} \mid \exists \ell'\colon\ y = f_{j,\ell'}(x)\} \\ &= \bigcup_{\ell'\in\lambda} \mathrm{graph}(f_{j,\ell'}), \end{aligned}$$

hence k is Borel since the graph of a Borel function is a Borel set. ■

Let us now attack Proposition 3.1.4. The proof is strongly inspired by Kechris's proof of Lusin's theorem (as can be found e.g. in [**Sri98**, Theorem 5.8.11]), which relies on Kechris's σ-ideal machinery. The latter was originally given for various classes of sets in [**Kec73**, Theorem 4.1.1], but see [**Sri98**, Theorem 5.8.4] for a Borel down-to-earth version entirely suitable for our purposes.

Proof (of Proposition 3.1.4). — For the sake of readability, let us say that a function $f\colon \kappa\times\lambda \to \mathscr{S}_x$ *scours* (the class of) x if

- for each $\ell \in \lambda$, the image of $f(\cdot,\ell)$ is a *subset* of the $\mathscr{R}$-class of $f(0,\ell)$;
- for each $\ell \neq \ell' \in \lambda$, $f(0,\ell)$ and $f(0,\ell')$ are not in the same $\mathscr{R}$-class.

Observe that the set $\widehat{\mathrm{F}}$ of pairs (x,f), where f is function that scours x, is a Borel subset of $\mathrm{X}\times\mathrm{X}^{\kappa\times\lambda}$ (in particular, the set $\widehat{\mathrm{F}}_x$ of functions that scour x is a Borel subset of $\mathrm{X}^{\kappa\times\lambda}$). Indeed, we can write $\widehat{\mathrm{F}}$ as the intersection of the following three Borel sets:

$$\{(x,f)\ \mid\ (x,f_{0,0})\in\mathscr{S}\}\,,$$
$$\bigcap_{\substack{k,k'\in\kappa\\ \ell\in\lambda}} \mathrm{X}\times\{f\ \mid\ (f_{k,\ell},f_{k',\ell})\in\mathscr{R}\}\,,$$
$$\bigcap_{\substack{\ell,\ell'\in\lambda\\ \ell\neq\ell'}} \mathrm{X}\times\{f\ \mid\ (f_{0,\ell},f_{0,\ell'})\notin\mathscr{R}\}\,.$$

We will need more: the set

$$\mathrm{F}=\Big\{(x,f)\in\mathrm{X}\times\mathrm{X}^{\kappa\times\lambda}\ \Big|\ f \text{ is a bijection from } \kappa\times\lambda \text{ to } \mathscr{S}_x \text{ that scours } x\Big\}$$

is a Borel set. As the set J of injective maps is obviously a G_δ subset of $\mathrm{X}^{\kappa\times\lambda}$, all amounts to showing that surjectivity of scouring functions is a Borel condition. To do so, observe that

$$\mathrm{S}=\Big\{(x,f,y)\in\mathrm{X}\times\mathrm{X}^{\kappa\times\lambda}\times\mathrm{X}\ \Big|\ (x,y)\in\mathscr{S} \text{ and } \forall(k,\ell)\in\kappa\times\lambda,\ \ y\neq f_{k,\ell}\Big\}$$

is a Borel set (as an intersection of $\mathscr{S}$ and a G_δ set) and that its sections $\mathrm{S}_{(x,f)}$ are countable since $\mathscr{S}$ has countable classes. Hence its projection S' on $\mathrm{X}\times\mathrm{X}^{\kappa\times\lambda}$ is Borel by Lusin's countable-to-one theorem (see [**Sri98**, Theorem 4.12.3]). Therefore, $\mathrm{F}=(\widehat{\mathrm{F}}\cap(\mathrm{X}\times\mathrm{J}))\setminus\mathrm{S}'$ is Borel.

Now the maps $f_{k,\ell}$ of the statement can be produced as a Borel section $\mathrm{X}\to\mathrm{X}^{\kappa\times\lambda}$ of the set F. To show that such a section exists, we will use the above-mentioned Kechris's σ-ideal machinery.

For each $x\in\mathrm{X}$, endow $\mathscr{S}_x$ with the discrete topology and let $\mathrm{T}_x\subset\mathscr{S}_x^{\kappa\times\lambda}$ be the subset of injective functions that scour x. Observe that T_x is a closed subspace, hence Polish, admitting F_x as a dense subset. Define then $\mathcal{I}_x$ to be the set of subsets $\mathrm{I}\subseteq\mathrm{X}^{\kappa\times\lambda}$ such that $\mathrm{I}\cap\mathrm{F}_x$ is meager in T_x. Each $\mathcal{I}_x$ is obviously a σ-ideal (that is, $\mathcal{I}_x$ is stable by countable unions, by subsets and is not the whole $X^{\kappa\times\lambda}$).

Moreover, $F_x \notin \mathcal{I}_x$. Indeed, for $f \in T_x$, we have

$$f \in F_x \iff \forall y \in \mathscr{S}_x \, \exists (k,\ell) \in \kappa\times\lambda : \; y = f_{k,\ell},$$

so that F_x is G_δ, hence comeager. By Baire's Category Theorem, it cannot be also meager in T_x.

Lastly, let us show that $x \mapsto \mathcal{I}_x$ is Borel on Borel, that is, for any Borel set $A \subseteq X \times X^{\kappa\times\lambda}$, the set $\{x \in X \mid A_x \in \mathcal{I}_x\}$ is Borel. There are general tools to prove that sets of the form $\{x \in X \mid A_x \text{ is meager in } Y\}$ are Borel when Y is a Polish space and $A \subseteq X \times Y$. Our problem here is that there is no *common* Y suitable for all x, as each A_x lives in T_x. However, the T_x's are homeomorphic to each other: the strategy of the proof is then to use a common model space $\mathcal{N}$.

Let $\mathcal{N}$ be the set of functions $\kappa\times\lambda \to \kappa\times\lambda$ whose second coordinate of the image does not depend on the first coordinate of the variable, that is

$$\mathcal{N} = \{\varphi \colon \kappa\times\lambda \to \kappa\times\lambda \mid \forall k,k' \in \kappa \, \forall \ell \in \lambda, \, \pi_2(\varphi(k,\ell)) = \pi_2(\varphi(k',\ell))\}$$

where π_2 is the projection onto the second factor. Observe that $\mathcal{N}$ is a closed subspace of $(\kappa\times\lambda)^{\kappa\times\lambda}$ (where κ and λ are endowed with the discrete topology), hence Polish. This space will be our common model for T_x. More precisely, each scouring bijection $f\colon \kappa\times\lambda \to \mathscr{S}_x$ defines a homeomorphism ψ_f between $\mathcal{N}$ and T_x via $\psi_f(\alpha) = f\circ\alpha$. The only maybe nontrivial point is the surjectivity. But the map $T_x \to (\kappa\times\lambda)^{\kappa\times\lambda} \colon g \mapsto f^{-1}\circ g$ has range in $\mathcal{N}$ since the fact that f and g scour x precisely ensures that the second coordinate of $(f^{-1}\circ g)(k,\ell)$ does not depend on k. Hence ψ_f is surjective.

Consider now, for any Borel subset A of $X \times X^{\kappa\times\lambda}$, the set

$$P = \left\{(x,f,\alpha) \in X \times X^{\kappa\times\lambda} \times \mathcal{N} \;\middle|\; (x, f\circ\alpha) \in A\right\}.$$

Since the composition $(f,\alpha) \mapsto f\circ\alpha$ is continuous, the set P is Borel. Therefore, the set

$$Q = \left\{(x,f) \in X \times X^{\kappa\times\lambda} \;\middle|\; P_{(x,f)} \text{ is meager in } \mathcal{N}\right\},$$

is also Borel (see e.g. [**Sri98**, Proposition 3.5.18]).

We can now get back these facts from $\mathcal{N}$ to the T_x's. As ψ_f is a homeomorphism for scouring bijections f, we see that A_x is meager in T_x if and only if $\psi_f^{-1}(A_x)$ is meager in

$\mathcal{N}$ for some (equivalently, all) scouring bijections f. In particular,

$$\begin{aligned} \mathrm{A}_x \in \mathcal{I}_x \;&\Leftrightarrow\; \exists f \in \mathrm{X}^{\kappa\times\lambda} : (x, f) \in \mathrm{Q} \cap \mathrm{F} \\ &\Leftrightarrow\; \forall f \in \mathrm{X}^{\kappa\times\lambda} : (x, f) \in \mathrm{Q} \cup \mathrm{F}^{\mathrm{c}}. \end{aligned}$$

As we already saw that F is Borel, the first line shows that $\{x \in \mathrm{X} \mid \mathrm{A}_x \in \mathcal{I}_x\}$ is analytic, the second one, that it is coanalytic. Hence it is Borel by Suslin's theorem.

So the map $x \mapsto \mathcal{I}_x$ is Borel on Borel and we can apply Kechris's theorem [**Sri98**, Theorem 5.8.4] to get a Borel section $\mathrm{X} \to \mathrm{X}^{\kappa\times\lambda}$ of F, as required. ■

Remark 3.1.6. — Notice the following particular cases of the above proposition.

- If $\mathscr{R}$ is the trivial relation Δ, then we recover a weak form of Lusin's theorem: a Borel countable equivalence relation is the countable union of Borel graphs[7].
- The Borel maps $\{f_{0,\ell} \colon \mathrm{X} \to \mathrm{X}\}_{\ell\in\lambda}$ enumerate a transverse set of the $\mathscr{R}$-equivalence classes inside the $\mathscr{S}$-classes. The existence of such an enumeration was proved by I. Epstein [**Eps08**, Lemma 2.2.2].

3.1.D Measured equivalence relations

As explained in the introduction of this section, the "closedness" of a subrelation will be expressed by measure-theoretical means. The spaces on which we consider an equivalence relation (and a fortiori a group action) will then be endowed with some measure. But the equivalence relation itself needs to be enriched with a measure, as explained in the following definition.

Definition 3.1.7. — *A* measured equivalence relation $(\mathscr{R}, m)$ *is a Borel equivalence relation on a standard measured space* (X, ν) *endowed with a Borel σ-finite measure* m *such that:*

1. *the measure m is equivalent to its image $\check{m}$ under the flip;*
2. *the measure ν is a pseudo-image of m via the projection onto the first coordinate;*
3. *in the disintegration $m = \int_{\mathrm{X}} m_x \,\mathrm{d}\nu$ along the projection onto the first coordinate, the class of fiber measure $[m_x]$ is ν-almost surely $\mathscr{R}$-invariant, that is, for ν-almost all x and y, we have $[m_x] = [m_y]$ whenever $(x, y) \in \mathscr{R}$ and m_x, m_y are viewed as measures on the class $\mathscr{R}_x$.*

[7] Lusin's theorem is actually valid for any Borel subset of $\mathrm{X} \times \mathrm{Y}$ with countable fibers. It is however unclear how Proposition 3.1.4 could be stated in that generality. We should at least require the fibers of $\mathscr{S}$ to be covered by the fibers of $\mathscr{R}$.

For more background on pseudo-images of measures, disintegration of σ-finite measures and fiber measures, see Appendix A.

Remark 3.1.8. — In view of Condition 1, a counterpart to Condition 3 holds for the disintegration along the projection onto the second coordinate.

Remark 3.1.9. — Condition 2 allows us to restrict our attention to conull sets. More precisely, if X_0 is a conull subset of X, then the restriction of $\mathscr{R}$ to X_0, when endowed with the restriction of the measure m, is a measured equivalence relation on the standard measured space $(X_0, \nu|_{X_0})$.

Remark 3.1.10. — Our definition differs slightly from the standard one, which usually requires only the equivalence relation (as well as the space on which it is defined) to be endowed with a measure *class* with the above properties (cf. [**Moo82**, Section 2.1]). The latter definition is indeed slightly more natural, partly because we deal in any case with pseudo-images in Condition 2, partly because there is a natural choice of a class in many examples, whereas no natural choice of measures in this class (see Examples 3.1.12 and 3.1.13 below). For our applications, it will however be useful to single out a specific member of this class. We could have defined, more pedantically, a "pointed measured equivalence relation" as a triple $(\mathscr{R}, \mathrm{M}, m)$ where $(\mathscr{R}, \mathrm{M})$ is a measured equivalence relation in the standard sense (i.e., M is a measure class) and m is a measure in the class M.

Remark 3.1.11. — Hahn proved [**Hah78**, Corollary 3.13] that if $(\mathscr{R}, m)$ is a measured equivalence relation, then there exists an equivalent measure m' such that the invariance of fiber measure *classes* (Condition 3) becomes an invariance of fiber measures, that is, $m'_x = m'_y$ for almost all $(x, y) \in \mathscr{R}$. Moreover, such a measure is unique up to an $\mathscr{R}$-invariant choice $X \to \mathbf{R}_+^*$ of dilatation factors of the first coordinate [**Hah78**, Corollary 3.14]; in particular, it is unique up to a multiplicative constant for ergodic relations.

Once again, the countable equivalence relations and the orbit equivalence relation of Polish locally compact groups are structured enough to be considered as measured equivalence relations.

Example 3.1.12 (discrete measured equivalence relation). — Let $\mathscr{R}$ be a Borel countable equivalence relation on a σ-finite standard measured space (X, ν). Assume that ν is quasi-invariant, that is, the $\mathscr{R}$-saturation of a null set is a null set (equivalently, the class of ν is invariant under the action of the full group[8] of $\mathscr{R}$). Then $\mathscr{R}$ is a measured equivalence relation when endowed with the integral m of the counting measure on the fibers over ν.

[8] The full group of a relation $\mathscr{R}$ is the group of all Borel automorphisms of X whose graph is contained in $\mathscr{R}$.

More precisely, we define the *left counting measure* m_{l} by

$$m_{\mathrm{l}}(\mathrm{A}) = \int_{\mathrm{X}} \#\{y \in \mathrm{X} \mid (x,y) \in \mathrm{A}\}\ \mathrm{d}\nu(x)$$

and the *right counting measure* m_{r} by the analogous formula with fibers of the right coordinate:

$$m_{\mathrm{r}}(\mathrm{A}) = \int_{\mathrm{X}} \#\{x \in \mathrm{X} \mid (x,y) \in \mathrm{A}\}\ \mathrm{d}\nu(y),$$

where A is any Borel subset of $\mathscr{R}$. These measures are obviously Borel and σ-finite (as ν itself is σ-finite and $\mathscr{R}$ is the countable union of graphs of Borel functions by Lusin's theorem). The equivalence between m_{l} and $\check{m}_{\mathrm{l}} = m_{\mathrm{r}}$ follows from quasi-invariance of ν and the fact that ν is a pseudo-image of m_{l} follows from the latter and the countability of the fibers. Equivalence of the fiber measures is obvious from the definition (and is actually an equality). See [**FM77**, Theorem 2] for details.

If, moreover, the measure ν is invariant (that is, under the action of the full group of $\mathscr{R}$), then m_{l} is actually equal to m_{r} [**FM77**, Corollary 1 of Proposition 2.2].

Example 3.1.13 (Polish locally compact groups). — Let G be a Polish locally compact group with a Borel essentially free action on a σ-finite standard measured space (X, ν) and let $\mathscr{R}$ be the induced orbit equivalence relation on X. Let m_{G} be a left Haar measure on G. By identifying G × X with $\mathscr{R}$ via either $(g,x) \mapsto (x,gx)$ or $(g,x) \mapsto (gx,x)$ and by endowing G with m_{G} or $\check{m}_{\mathrm{G}}$, we get four different Borel measures on $\mathscr{R}$. Thanks to Fubini's theorem, they can be described by

$$\begin{aligned}
m_{\mathrm{ll}}(\mathrm{A}) &= \int_{\mathrm{X}} m_{\mathrm{G}}(\{g \in \mathrm{G} \mid (x,gx) \in \mathrm{A}\})\ \mathrm{d}\nu(x),\\
m_{\mathrm{lr}}(\mathrm{A}) &= \int_{\mathrm{X}} m_{\mathrm{G}}(\{g \in \mathrm{G} \mid (gx,x) \in \mathrm{A}\})\ \mathrm{d}\nu(x),\\
m_{\mathrm{rl}}(\mathrm{A}) &= \int_{\mathrm{X}} \check{m}_{\mathrm{G}}(\{g \in \mathrm{G} \mid (x,gx) \in \mathrm{A}\})\ \mathrm{d}\nu(x),\\
m_{\mathrm{rr}}(\mathrm{A}) &= \int_{\mathrm{X}} \check{m}_{\mathrm{G}}(\{g \in \mathrm{G} \mid (gx,x) \in \mathrm{A}\})\ \mathrm{d}\nu(x).
\end{aligned}$$

We can call these measures the *left-left, left-right, right-left and right-right natural measures* on $\mathscr{R}$. These measure are σ-finite thanks to the essential freeness of the action[9]. Of course, m_{ll} is the image of m_{lr} under the flip, and similarly for m_{rl} and m_{rr}. The fiber measure m_x on $\mathscr{R}_x$ of, say, m_{ll} is exactly the image of m_{G} through the orbit map $g \mapsto gx$, hence is equal to $\Delta_{\mathrm{G}}(h)m_y$, where $y = hx$ and m_y is the fiber measure on $\mathscr{R}_y = \mathscr{R}_x$. The fact that ν is a pseudo-image of all these measures follows from Lemma A.2.1. See [**Ram71**, Theorem 4.3] and [**Hah78**, Example 2.5] for more details on this example and in particular for

[9] It would have been enough to assume, for instance, that almost all stabilisers are contained in some given compact set.

the invariance of the class under the flip.

For the following lemma (where the link with the previous chapter finally starts disclosing itself), recall from Section 3.1.B that an orbit equivalence relation induced by an action of a group G is itself endowed with two commuting actions of G.

Lemma 3.1.14 (Measured equivalence relation and amplification). — *Let* G *be a Polish locally compact group with an essentially free measure-preserving Borel action on a standard measured space* (X, ν)*. Let* m_G *be a left Haar measure on* G *and* $\mathscr{R}$ *be the orbit equivalence relation induced by* G*. Then* $\mathscr{R}$*, endowed with its left-left (resp. right-left) natural measure and its* G*-action on the left, is an amplification of* (G, m_G) *(resp. of* $(G, \check{m}_G)$*), which is finite if and only if* ν *is finite. A similar statement holds for the left-right and right-right measures with the* G*-action on the right.*

Proof. — Indeed, the map

$$(G, m_G) \times (X, \nu) \to (\mathscr{R}, m_{ll}) \colon (g, x) \mapsto (x, gx)$$

is a Borel G-equivariant bijection between conull sets (by essential freeness), hence an isomorphism (see e.g. [**Sri98**, Proposition 4.5.1]). Measures are preserved by the very definition of the left-left natural measure. Similar maps yield the analogous statements for the other measures. ■

3.1.E Measured tilings

Now comes at last the measure-theoretical analogue of a closed subgroup.

Definition 3.1.15. — *Let* $(\mathscr{R}, m_{\mathscr{R}})$ *and* $(\mathscr{S}, m_{\mathscr{S}})$ *be two measured equivalence relations on a standard measured space* (X, ν) *such that* $\mathscr{R} \subseteq \mathscr{S}$*. A* measured tiling (B, τ, Φ) *of the inclusion* $(\mathscr{R}, m_{\mathscr{R}}) \subseteq (\mathscr{S}, m_{\mathscr{S}})$ *is a tiling* (B, Φ) *of the inclusion of the restrictions* $\mathscr{R}_0 \subseteq \mathscr{S}_0$ *to some conull subset* X_0 *of* X*, such that* (B, τ) *is a standard measured space and* $\Phi_*(m_{\mathscr{R}} \times \tau) = m_{\mathscr{S}}$*.*

A less formal wording would be to say that a measured tiling is an almost everywhere defined tiling that preserves the measure. Recall from Remark 3.1.9 that the restriction of $\mathscr{R}$ and $\mathscr{S}$ to X_0 can be naturally considered as measured equivalence relations.

As usual, our examples are provided by countable and orbit equivalence relations.

Example 3.1.16. — Let $\mathscr{R} \subseteq \mathscr{S}$ be two Borel countable equivalence relations on a Polish space X, with a tiling (Φ, B). Assume that X is endowed with a σ-finite Borel measure ν that is quasi-invariant relatively to $\mathscr{S}$ (hence also to $\mathscr{R}$). Then $(\Phi, \mathrm{B}, \#)$ is a measured tiling of $(\mathscr{R}, m_{\mathscr{R}}) \subseteq (\mathscr{S}, m_{\mathscr{S}})$, where $m_{\mathscr{R}}$ and $m_{\mathscr{S}}$ are the (say, left) counting measures naturally associated to $\mathscr{R}$ and $\mathscr{S}$ as in Example 3.1.12. Indeed, observe first that B must be countable as Φ produces bijections between $\mathscr{R}_x \times \mathrm{B}$ and $\mathscr{S}_x$, which are both countable. Hence $(\mathrm{B}, \#)$ is indeed a standard measured space. Moreover, Φ preserves the measure: for any Borel set $\mathrm{A} \subseteq \mathscr{S}$, we have, by Fubini's theorem,

$$\begin{aligned}(m_{\mathscr{R}} \times \#)(\Phi^{-1}(\mathrm{A})) &= \sum_{b \in \mathrm{B}} m_{\mathscr{R}}(\pi_{\mathscr{R}}(\Phi^{-1}(\mathrm{A}) \cap (\mathscr{R} \times \{b\}))) \\ &= \sum_{b \in \mathrm{B}} \int_{\mathrm{X}} \left|\Phi^{-1}(\mathrm{A}) \cap (\{x\} \times \mathscr{R}_x \times \{b\})\right| \mathrm{d}\nu(x) \\ &= \int_{\mathrm{X}} \left|\Phi^{-1}(\mathrm{A}) \cap (\{x\} \times \mathscr{R}_x \times \mathrm{B})\right| \mathrm{d}\nu(x) \\ &= \int_{\mathrm{X}} |\mathrm{A} \cap (\{x\} \times \mathscr{S}_x)| \, \mathrm{d}\nu(x) = m_{\mathscr{S}}(\mathrm{A}),\end{aligned}$$

where the last line is due to the fact that Φ is a bijection that sends $\{x\} \times \mathscr{R}_x \times \mathrm{B}$ to $\{x\} \times \mathscr{S}_x$.

In the above example, we did not need to go to a conull inclusion $\mathscr{R}_0 \subseteq \mathscr{S}_0$. But this flexibility is needed for group actions.

Example 3.1.17. — Let G be a Polish locally compact group with a Borel measure-preserving essentially free action on a standard measured space (X, ν) and $\mathrm{H} \leqslant \mathrm{G}$ a closed subgroup. Let $\mathscr{R}$, $\mathscr{S}$ be the orbit equivalence relations induced by H and G, respectively, endowed with their (say, left-left) natural measures $m_{\mathscr{R}}$, $m_{\mathscr{S}}$ (cf. Example 3.1.13). Then there exists a measured tiling of the inclusion $(\mathscr{R}, m_{\mathscr{R}}) \subseteq (\mathscr{S}, m_{\mathscr{S}})$. Indeed, if we restrict our attention (and our relations) to the conull subset X_0 of points with trivial stabilisers, we can find a tiling $(\mathrm{G}/\mathrm{H}, \Phi)$ by Lemma 3.1.3. Thanks to Ripley's theorem (Proposition 1.2.8), the quotient space G/H can be endowed with a measure τ such that $(\mathrm{G}/\mathrm{H}, \tau, \Phi)$ is a measured tiling.

We record the following very easy lemma for later use.

Lemma 3.1.18. — *A finite product of (measured) tilings is a (measured) tiling of the product relation.*

More precisely, let $\mathscr{R}_i \subseteq \mathscr{S}_i$ be Borel equivalence relations on the Polish space X_i with tilings (Φ_i, B_i), for $i = 1, 2$. Then $(\Phi_1 \times \Phi_2, \mathrm{B}_1 \times \mathrm{B}_2)$ is a tiling of the inclusion of product *relations* $\mathscr{R}_1 \times \mathscr{R}_2 \subseteq \mathscr{S}_1 \times \mathscr{S}_2$ *defined on the product space* $X_1 \times X_2$*. Moreover, if all the relations are measured and* τ_i *is a measure on* B_i *such that* (Φ_i, B_i, τ_i) *is a measured tiling, then the tiling* $(\Phi_1 \times \Phi_2, B_1 \times B_2, \tau_1 \times \tau_2)$ *is measured, when the product relations are endowed with the product measures.*

Proof. — It is completely obvious that $\Phi_1 \times \Phi_2$ is a Borel isomorphism between

$$(\mathscr{R}_1 \times \mathscr{R}_2,\ \mathcal{B}(\mathscr{R}_1) \otimes \mathcal{B}(\mathscr{R}_2)) \times (B_1 \times B_2,\ \mathcal{B}(B_1) \otimes \mathcal{B}(B_2)) \quad \text{and} \quad (\mathscr{S}_1 \times \mathscr{S}_2,\ \mathcal{B}(\mathscr{S}_1) \otimes \mathcal{B}(\mathscr{S}_2)),$$

that preserves the product measures if there are some. But, since B_i, $\mathscr{R}_i$ and $\mathscr{S}_i$ are standard Borel spaces, the Borel structures commute with the product (see for instance Lemma D.1.1), namely: $\mathcal{B}(\mathscr{R}_1) \otimes \mathcal{B}(\mathscr{R}_2) = \mathcal{B}(\mathscr{R}_1 \times \mathscr{R}_2)$, etc. Hence $\Phi_1 \times \Phi_2$ is indeed a (measured) tiling. ∎

Finally our efforts are rewarded: suitable measured tilings produce tychomorphisms.

Proposition 3.1.19. — *Let* H *and* G *be two Polish locally compact groups with essentially free measure-preserving actions on the same probability space* (X, ν)*. Let* $(\mathscr{R}, m_{\mathscr{R}})$ *and* $(\mathscr{S}, m_{\mathscr{S}})$ *be their respective induced measured orbit equivalence relations, where* $m_{\mathscr{R}}$ *is the left-left natural measure and* $m_{\mathscr{S}}$ *the right-right natural measure (cf. Example 3.1.13). Assume that* $\mathscr{R} \subseteq \mathscr{S}$*. If there is a measured tiling of* $(\mathscr{R}, m_{\mathscr{R}}) \subseteq (\mathscr{S}, m_{\mathscr{S}})$*, then there is a tychomorphism from* H *to* G*.*

Proof. — By Lemma 3.1.14, $(\mathscr{S}, m_{\mathscr{S}})$, endowed with the right G-action, is a finite amplification of $(G, \check{m}_G)$ and $(\mathscr{R}, m_{\mathscr{R}})$, endowed with the left H-action, is also a (finite) amplification of (H, m_H). Moreover, as $\mathscr{R} \subseteq \mathscr{S}$, the H-action on the first coordinate of $X \times X$ also preserves $\mathscr{S}$, and commutes with the right G-action. The existence of measured tiling for $(\mathscr{R}, m_{\mathscr{R}}) \subseteq (\mathscr{S}, m_{\mathscr{S}})$ precisely says that $(\mathscr{S}, m_{\mathscr{S}})$ is an H-amplification of $(\mathscr{R}, m_{\mathscr{R}})$—the H-invariance being granted by the fact that the tiling is constant on the first coordinate—hence of (H, m_H). In other words, $(\mathscr{S}, m_{\mathscr{S}})$ is a tychomorphism from H to G. ∎

Remark 3.1.20. — As we explain in Section 3.1.A, all this work was needed to recover for equivalence relations some concept of closedness. For tychomorphisms of *discrete* groups, this can therefore be bypassed by the following concrete translation of the above proposition, which can be easily checked concretely, without the abstract framework developed here:

Let G *and* H *be two discrete countable groups with essentially free measure-preserving actions on the same probability space. If the* H*-orbits are almost surely contained in the* G*-orbits, then there is a tychomorphism from* H *to* G*.*

3.2 Tychomorphisms to nonamenable groups

In this section, we will use percolation tools and equivalence relations to prove the following important source of examples of tychomorphisms.

Theorem 3.2.1. — *Let* G *be a locally compact second-countable group. If* G *is nonamenable, then there is a tychomorphism from any free group of countable rank to* G.

This theorem is a generalisation for locally compact groups of the Gaboriau–Lyons theorem. In its original form [**GL09**, Theorem 1], the latter states that for any countable discrete nonamenable group G, we can find, inside the orbit equivalence relation of the Bernoulli shift $[0, 1]^{\mathrm{G}}$, an equivalence relation induced by an essentially free action of a nonabelian free group. This can be seen as a measure-theoretical solution to the von Neumann problem and, as explained in Remark 3.1.20, produces a tychomorphism from a nonabelian free group to G.

Two proofs are given in [**GL09**], each one shrewdly using percolation on Cayley graphs and a lemma of Hjorth [**Hjo06**]. Thanks to Cayley–Abels graphs (a generalization of Cayley graphs for some locally compact groups), we can still use these ideas for the locally compact case, but several technical shackles have to be dealt with:

- Deep results about the structure of locally compact groups have to be invoked in order to reach groups where Cayley–Abels graphs are useful. As usual, we will quickly get rid of the connected case thanks to the power of Lie theory.
- The percolation results we use must hold for graphs that are more general than Cayley graphs.
- A mere equivalence subrelation is not enough to produce a tychomorphism.

It turns out that the percolation tools we need *have* been proved for suitable general graphs, so we will not elaborate more on the second problem. The third problem leads us to consider measured tilings of subrelations, this will appear later in the proof. Let us first see how we can use structure-theoretical results for locally compact groups to deduce Theorem 3.2.1 from the following more technical statement.

Theorem 3.2.2. — *Let* G *be a nonamenable unimodular compactly generated locally compact second-countable group and* K *a compact open subgroup of* G*.*

Then there is a tychomorphism from the free group of rank 2 *to* G/ Core(K)*.*

As for the hypotheses, we recall that totally disconnected groups have whole bases made of compact open subgroups (Proposition 1.2.10). Moreover, for discrete groups, the assumptions boil down to G being finitely generated and K can be chosen trivial.

Proof (of Theorem 3.2.1 assuming Theorem 3.2.2). — As the free group F_2 of rank 2 contains free subgroups of rank r for any $0 \leqslant r \leqslant \aleph_0$, it is enough to prove Theorem 3.2.1 for the rank $r = 2$ thanks to Lemma 2.4.1. We shall distinguish two cases, according to the amenability of the connected component.

If the connected component G^o is nonamenable, then G actually contains a discrete nonabelian free subgroup, hence a fortiori admits a tychomorphism from F_2 by Example 2.2.2. Indeed, thanks to the solution of Hilbert's fifth problem, the quotient group $Q = G^o/\operatorname{Ramen}(G)$ is a semisimple Lie group of positive **R**-rank. It therefore contains a closed subgroup of rank one, for which a standard ping-pong argument as in [**Har00**, II.B] (with hyperbolic elements acting on the boundary of its symmetric space) provides a discrete nonabelian free subgroup.

If the connected component G^o is amenable, then G/G^o is nonamenable and totally disconnected. Let $G_1 < G/G^o$ be the kernel of the modulus of G/G^o; it is a unimodular group (see Corollary 1.2.5). Moreover, G_1 is still totally disconnected, locally compact and nonamenable (as G/G^o is an extension of **R** by G_1). Consider now the direct family of subgroups of G_1 that are generated by a compact neighborhood of the identity. Its limit is G_1, hence we can find among the latter a subgroup G_2 which is nonamenable, and moreover still unimodular since it is open in G_1. Therefore, G_2 satisfies all the assumptions of Theorem 3.2.2 and the latter yields a tychomorphism from F_2 to $G_2/\operatorname{Core}(K)$ for any compact open subgroup K of G_2. Thence we get such a tychomorphism to G_2 by Lemma 2.4.3, to G/G^o by Lemma 2.4.1 and finally to G by Lemma 2.4.3 again. ■

The rest of this chapter is devoted to the proof Theorem 3.2.2. We first recall what Cayley–Abels graphs are and where to find the necessary material on percolation.

3.2.A Background on graphs

Cayley graphs of finitely generated groups are probably the most prominent tools of geometric group theory. It turns out that a similar tool can be built for locally compact groups admitting a compact open subgroup.

Let thus G be a locally compact group and $K < G$ a compact open subgroup. Choose moreover a subset $S \subseteq G$ such that $S = S^{-1}$ and $S = KSK$ (observe that, starting from an arbitrary subset S, the subset $S' = K(S \cup S^{-1})K$ will satisfy these conditions). The associated *Cayley–Abels graph* $\mathfrak{g}(G, K, S)$ is the graph with vertex set G/K and edge set $\{(gK, gsK) \mid g \in G,\ s \in S\}$. For any integer $n \geqslant 1$, the associated *Cayley–Abels graph of order n* $\mathfrak{g}^n(G, K, S)$ is the multigraph with vertex set G/K and edge multiset given by paths of length n in $\mathfrak{g}(G, K, S)$ (i.e., there are as many edges in $\mathfrak{g}^n(G, K, S)$ between two vertices gK and $g'K$ as there are paths of length n in $\mathfrak{g}(G, K, S)$ connecting them).

Observe that $\mathfrak{g}^1(G, K, S) = \mathfrak{g}(G, K, S)$. When G is a discrete group and S a (finite) generating set, $\mathfrak{g}(G, \{1\}, S)$ is the usual Cayley graph. The following properties are obvious and reminiscent of the discrete case.

Lemma 3.2.3. — *Let* G *be a locally compact group,* $K < G$ *a compact open subgroup,* $S \subseteq G$ *a subset satisfying* $S = S^{-1}$ *and* $S = KSK$, $n \geqslant 1$, *and* $\mathfrak{g} = \mathfrak{g}^n(G, K, S)$ *the associated Cayley–Abels graph.*

1. $\mathfrak{g}$ *is connected if and only if* S *generates* G.
2. $\mathfrak{g}$ *is locally finite if and only if* S *is compact.*
3. *The group* G *acts by automorphisms on* $\mathfrak{g}$ *and transitively on its vertex set.*
4. *The stabilizer in* G *of a vertex* sK *is the subgroup* sKs^{-1}.
5. *The core of the action of* G *is the core of* K *in* G. *In particular, the action is faithful if and only if* K *has trivial core.* □

The proof of Theorem 3.2.2 will use percolation on those Cayley–Abels graphs, as the original Gaboriau–Lyons theorem did on Cayley graphs. A gentle introduction to percolation on graphs can be found in [**Pete**, Chapter 12]; for a thorough study, see [**LP16**]. For the convenience of the reader, we give in Table 3.1 a chart to the main results that are needed for the proof of Theorem 3.2.2. We will also rely on the concept of *cost* of an equivalence relation, for which we refer to [**Gab10b**] and [**Gab00**].

	Concepts	General Background	Specific results we need
	Unimodularity of graphs	[**LP16**, Section 8.2]	Link with unimodularity of groups (p. 279)
	Spectral radius of graphs	[**Woe00**, Chapter II]	Link with amenability (Section 12 and Corollary 12.12)
Percolation on graphs	Critical probabilities p_c and p_u	[**HP99**]	Theorem 1.2
	Minimal spanning forests	[**LP16**, Chapter 11]	Link with $p_c < p_u$ (Proposition 11.7)
	Number of ends of a cluster	[**HP99**]	Theorem 6.1 (proof in [**LS99**, Proposition 3.10])
	Indistinguishability of infinite clusters	[**LS99**]	Theorem 3.3 and Remark 3.4

Table 3.1 – Some results from percolation on graphs.
Numbering in the last column refers to the work cited in the middle column.

3.2.B Detailed proof

We shall now embark on the proof of Theorem 3.2.2. The general strategy is the following. We will somehow "decompose" the group G in a compact part (the subgroup K of the statement) and a countable part (its coset space), and use percolation theory on the latter to get equivalence relation induced by a free group. We then paste back the compact part in a way to get a measured tiling of relations, hence a tychomorphism.

Observe first that all the properties of G and K are preserved if we replace them by their images in the quotient $G/\operatorname{Core}(K)$ (which is legitimate by the lifting property, cf. Lemma 2.4.3), hence we can as well assume that K has trivial core. Let S be a compact generating set of G, containing the identity. Up to replacing S by KSK (which is still compact), we can assume that $S = KSK$. Let n be an integer to be chosen three lines below and consider now the Cayley–Abels graph $\mathfrak{g} = \mathfrak{g}^n(G, K, S)$. By hypothesis, $\mathfrak{g}(G, K, S)$ is a unimodular nonamenable graph, hence its spectral radius ρ satisfies $0 < \rho < 1$. As $\mathfrak{g}$ has spectral radius ρ^n, we can and do choose n such that $\rho^n < 1/9$.

Let E be the edge set of $\mathfrak{g}$. Observe that E is countable as $\mathfrak{g}(G, K, S)$ has a countable set of vertices (namely, G/K) and is locally finite. Therefore, the compact space $[0, 1]^E$ is metrizable and its subspace X made of injective maps is G_δ. Moreover, the G-action on $[0, 1]^E$ (defined by the action on the edge set) leaves X invariant and the induced action on X is free as the action on E is faithful. Let ν be the (Radon) probability on X given by the restriction of the product of Lebesgue measures on $[0, 1]$. Observe that this measure is invariant and ergodic under the G-action (for ergodicity, compare with [**KT08**, Proposition 2.1]). Let $Y \subset X$ be a Borel fundamental domain for the action of K on X (which exists since K is compact and G acts continuously, cf. for instance [**Sri98**, Theorem 5.4.3]). By definition, there is thus a Borel isomorphism $K \times Y \cong KY = X$. We use this isomorphism to disintegrate the measure ν as a product measure of the normalised Haar measure on K and a Borel probability measure η on Y.

Now come the equivalence relations. Let $\mathscr{Q}$ be the orbit equivalence relation of the G-action on X and $\mathscr{R}$ be its restriction to Y. Thus for $x, x' \in X$ (resp. $y, y' \in Y$), we have $x\mathscr{Q}x'$ (resp. $y\mathscr{R}y'$) if and only if there is some $g \in G$ such that $x' = gx$ (resp. $y' = gy$). As Y is a fundamental domain, the relation $\mathscr{Q}$, when viewed on $K \times Y$, decomposes as a product $\mathscr{T}_K \times \mathscr{R}$, where $\mathscr{T}_K$ is the fully transitive relation on K (namely, $\mathscr{T}_K = K \times K$). In particular, $\mathscr{R}$ has countable equivalence classes. Observe moreover that $\mathscr{Q}$ and $\mathscr{R}$ preserve the measures ν and η, respectively.

For any parameter $0 \leqslant p \leqslant 1$, an element $x \in X$ defines a subgraph of $\mathfrak{g}$ by choosing all edges $e \in E$ such that $x(e) \leqslant p$. We will denote this as "the graph $x \leqslant p$". This yields a new relation, the *cluster equivalence subrelation*[10] $\mathscr{Q}_p^{\mathrm{cl}} \subseteq \mathscr{Q}$, by declaring $x\mathscr{Q}_p^{\mathrm{cl}}x'$ if and only if there is some $g \in \mathrm{G}$ such that $x' = g^{-1}x$ and that the vertices $g\mathrm{K}$ and $e\mathrm{K}$ are in the same connected component of the graph $x \leqslant p$. We write $\mathscr{R}_p^{\mathrm{cl}}$ for the restriction of $\mathscr{Q}_p^{\mathrm{cl}}$ to Y; once again, $\mathscr{Q}_p^{\mathrm{cl}}$ decomposes as the product $\mathscr{T}_{\mathrm{K}} \times \mathscr{R}_p^{\mathrm{cl}}$.

Finally, let the percolation processes come into play. For a given parameter p, we can view the graphs $x \leqslant p$ as a random variable on the probability space (X, ν) with values in the space of subgraphs of $\mathfrak{g}$. This is an instance of a G-invariant p-Bernoulli bond percolation on $\mathfrak{g}$ *with scenery*, cf. [**LS99**, Remark 3.4]. In particular, this process has indistinguishable infinite clusters.

We choose now a parameter p such that, for this percolation process, there are ν-almost surely infinitely many infinite clusters, each one having moreover uncountably many ends. For such a parameter to exist, it suffices to show that the critical probability p_{c} ("first apparition of an infinite cluster") is strictly below the critical uniqueness probability p_{u} ("merging of all infinite clusters into a single one"). And for this condition to hold, it suffices to prove that the wired and free minimal forests are distinct processes on $\mathfrak{g}$. This is the case in our context. Indeed, the wired minimal forest has expected degree 2 [**LP16**, Theorem 11.11] but the free minimal forest has expected degree > 2 by [**Tho16**, Theorem 1], since the spectral radius of $\mathfrak{g}$ is $< 1/9$.

Let therefore $\mathrm{X}_\infty \subseteq \mathrm{X}$ be the set of points belonging to an infinite $\mathscr{Q}_p^{\mathrm{cl}}$-class: by our choice of p, this set is non-null. Moreover, the restriction of $\mathscr{Q}_p^{\mathrm{cl}}$ to X_∞ is ergodic (see the proof of Proposition 5 in [**GL09**], which applies without change to our setting thanks to the indistinguishability of clusters). Write $\mathrm{Y}_\infty = \mathrm{Y} \cap \mathrm{X}_\infty$; once again, we have $\mathrm{X}_\infty = \mathrm{K}\mathrm{Y}_\infty$. In particular, $\eta(\mathrm{Y}_\infty) > 0$ and the restriction of $\mathscr{R}_p^{\mathrm{cl}}$ to Y_∞ is η-ergodic. This moreover shows that $\mathscr{R}_p^{\mathrm{cl}}$ also has η-almost surely infinitely many infinite clusters, each one having uncountably many ends.

We can now rely on [**GL09**]. The relation $\mathscr{R}_p^{\mathrm{cl}}$ has nontrivial finite cost (same proof as for Proposition 12 in loc. cit.) and therefore contains a subrelation $\mathscr{F}$ which is produced by a measure-preserving, ergodic, and almost surely free action of the free group F_2 of rank 2 on the space (Y, η) (Propositions 13 and 14 therein). We can extend this action to a (non-ergodic) action on $\mathrm{X} \cong \mathrm{K} \times \mathrm{Y}$ by letting F_2 act trivially on the factor K.

In order to finish our proof, we only need to endow our equivalence relations with measures in such a way to produce tychomorphisms.

- We first endow the countable relations $\mathscr{F}$ and $\mathscr{R}$ with the usual counting left measures (cf. Example 3.1.12). By ergodicity, $\mathscr{F}$ has index λ in $\mathscr{R}$ for some cardinal λ. Moreover, Δ_{Y} has index $\aleph_0$ in $\mathscr{F}$ since the latter is induced by an almost surely free action of an infinite countable group. Therefore, there is tiling of the inclusion

[10] This kind of equivalence subrelations was considered in [**Gab05**, 1.2].

$\mathscr{F} \subseteq \mathscr{R}$ by Corollary 3.1.5, hence also a measured tiling by Example 3.1.16.

- As for relations on K, endow the trivial relation $\Delta_K \cong K$ with the normalised Haar measure of K and the fully transitive relation $\mathscr{T}_K = K \times K$ with the product of the latter with itself. (Equivalently, Δ_K is endowed with the measure of Example 3.1.12 for the action of the trivial group, and $\mathscr{T}_K$ is endowed with any of the measures of Example 3.1.13 for the action of K on itself.) It is easy to check that there is a measured tiling of the inclusion $\Delta_K \subseteq \mathscr{T}_K \cong \Delta_K \times K$.

By Lemma 3.1.18, there is therefore a measured tiling of the inclusion $\Delta_K \times \mathscr{F} \subseteq \mathscr{T}_K \times \mathscr{R}$ of relations on $K \times Y \cong X$. But $\Delta_K \times \mathscr{F}$ is nothing but the orbit equivalence relation of the F_2-action on X, and $\mathscr{T}_K \times \mathscr{R} \cong \mathscr{Q}$ is the orbit equivalence relation of the G-action. Therefore, by Proposition 3.1.19, there is a tychomorphism from the free group F_2 to G. This concludes the proof of Theorem 3.2.2.

3.2.C Another proof

We briefly indicate here another strategy, due to Bowen, Hoff, and Ioana, to prove Theorem 3.2.1 with other means. Percolation (and Hjorth's lemma) is still a key tool, but the solution to Hilbert's fifth problem may be avoided. The idea is to start with the Bernoulli equivalence relation of a nonamenable group G and then to use the *cross section equivalence relation* [**KPV15**, §4] to get a nonamenable *countable* subrelation $\mathscr{R}$ and then to try to apply the original Gaboriau–Lyons theorem on $\mathscr{R}$. Unfortunately, it is not known whether *any* nonamenable countable equivalence relation contains a subrelation induced by a free action of a free group (cf. the question in [**GL09**, p. 539])—the original Gaboriau–Lyons theorem deals with specific relations coming from Bernoulli actions. This difficulty is bypassed in [**BHI15**] by replacing $\mathscr{R}$ by a Bernoulli extension of it (Theorem A in loc. cit.) and then by observing that the latter is still a cross section of some equivalence relation induced by a free action of G. See [**BHI15**] and in particular Section 8 therein for details.

3.2.D Remark on orbit equivalence

The reader acquainted with measure equivalence may have noticed that we have actually used in this chapter the orbit equivalence counterpart of tychomorphisms, what we could call "orbit subvalence". For countable groups H and G, let us say that H is *orbit subvalent* to G if both groups admit essentially free measure-preserving actions on a probability space X such that the orbit equivalence relation induced by H is contained in that induced by G (almost surely). A corresponding definition for locally compact Polish groups can be given with measured equivalence relations, but let us simplify this remark by considering only countable groups.

The link between orbit equivalence and measure equivalence of countable groups is clear and well-known (see [**Fur11**, Theorem 2.5]), and Section 3.1 gave a "subgroup" version of one direction of that link.

The Gaboriau–Lyons theorem can be rephrased by saying that nonabelian free groups are orbit subvalent to any nonamenable group. That is, for G nonamenable, we can find two essentially free measure-preserving actions of G and F_2 on a probability space X such that $\mathscr{R}_{F_2} \subseteq \mathscr{R}_G$. Having in mind the possible decomposition of a tychomorphism as a subgroup of a measure-equivalent group (Problem 1, p. 33), it is tempting to try to enlarge F_2 (inside the group of automorphisms of X) into a bigger group H such that $\mathscr{R}_H = \mathscr{R}_G$. This is indeed possible: simply add to F_2 enough automorphisms of X coming from G. Unfortunately, it is unclear how to add sufficiently *few* automorphisms in order to keep the essential freeness of the actions, which we would need to say that H is measure-equivalent to G. The problem of producing *free* actions of groups to produce a given equivalence relation was already raised by Feldman and Moore [**FM77**, p. 292].

4 BEYOND CLASSICAL INDUCTION

UNDERSTANDING A GROUP through its actions on vector spaces is a method as old as powerful. A central tool is the possibility to *induce* a representation of a subgroup to a given ambient group. We will see in this chapter how this technique can still be carried off for tychomorphisms. This will be used in the next two chapters to produce interesting actions of nonamenable groups.

We first survey in Section 4.1 general facts about linear and affine representations. Section 4.2 introduces a tool to build non-isometric representations. Section 4.3 then explains the induction techniques available for tychomorphisms.

4.1 Linear and affine representations of groups

Let G be a group and V a vector space. Recall that a *linear representation* or *linear action* of G on V is a morphism $\pi\colon G \to GL(V)$ of G into the general linear group of V. An *affine representation* or *affine action* of a group G on V is given by a morphism $\alpha\colon G \to Aff(V)$ into the affine group of V.

We made the following convention. If V is a *topological* vector space, we consider that the groups GL(V) and Aff(V) are made of linear or affine *homeomorphisms.* In particular, a representation on a topological vector space is by default assumed to be an action *by* continuous transformations. No confusion should arise from this convention: namely, we will consider either topological vector spaces *or* vector spaces on which no topology has been introduced, but we will never consider, for group actions, the "abstract" vector space underlying a topological vector space.

We emphasize that we however do *not* assume by default any further continuity on our representations. Such continuity conditions will always be made explicit (we will introduce them in Section 4.1.C).

Thanks to the semi-direct decomposition $\mathrm{Aff}(\mathrm{V}) = \mathrm{GL}(\mathrm{V}) \ltimes \mathrm{V}$, we can write $\alpha = \pi + b$, where $\pi\colon \mathrm{G} \to \mathrm{GL}(\mathrm{V})$ is called the *linear part* and $b\colon \mathrm{G} \to \mathrm{V}$ the *translation part* or the *cocycle*[1,2]. It is a routine exercise to check that π is again a morphism of G into GL(V) (we call it the *underlying linear representation* or *action*) and that b satisfies a *cocycle identity* with respect to π, namely:

$$b(gh) = \pi(g)b(h) + b(g) \qquad \text{for all } g, h \in \mathrm{G}.$$

Observe that $b(g)$ is also the image of zero under the action of $\alpha(g)$.

The main interest of this language is to translate geometric properties of the action into algebraic properties of the cocycle. For instance, one easily verifies that an affine action α has a fixed point $x \in \mathrm{V}$ if and only if the cocycle is the *coboundary of* $-x$, i.e.satisfies

$$b(g) = \pi(g)(-x) + x \qquad \text{for all } g \in \mathrm{G}.$$

Conversely, any coboundary satisfies the cocycle identity, that is why such cocycles are also called *trivial*.

4.1.A Abstract constructions

We give here three basic constructions to deduce new representations starting with other ones, for the same group.

Apart from the above semi-direct decomposition, there is another general method to build affine representations from linear ones and vice versa. Let $\alpha = \pi + b$ be an affine action of G on a vector space V. We define a linear representation ρ_α on the direct sum $\mathrm{V} \oplus \mathbf{R}$ as

$$\rho_\alpha(g) = \begin{pmatrix} \pi(g) & b(g) \\ 0 & 1 \end{pmatrix}.$$

Geometrically, this amounts to extending linearly on $\mathrm{V} \oplus \mathbf{R}$ the action given on its affine 1-codimensional subspace $\mathrm{V} \times \{1\}$, thanks to conjugating by a homothety of ratio given

[1] There is a slight but common abuse of notation, both in the sign + and in the use of the same symbol to denote a vector and a translation by this vector. It should be read "apply π, then translate by b". This is righteous by looking at the image of a point v under the action of g, which is indeed $\alpha(g)v = \pi(g)v + b(g)$.

[2] No confusion should arise between these *affine* cocycles and the *measurable* cocycles of Section 2.3: the former only have group elements in their arguments, whereas the latter also have an element of a measure space.

by (the absolute value of) the second coordinate. Observe that ρ_α has invariant vectors if and only if α has a fixed point or π has invariant vectors.

In the other direction, if π is a linear representation on V that leaves an affine subspace W invariant, then the deduced action on W is affine. The linear and translation parts of the latter depend on a choice of a base point $w_0 \in W$ and are then given, respectively, by $\pi(g)(w - w_0)$ and $\pi(g)(w_0)$.

These constructions also hold in a topological setting (endowing $V \oplus \mathbf{R}$ with the product topology and W with the subspace topology) or with complex vector spaces.

We now show how to build representations on the tensor product and on the dual space. The latter are *linear* objects, hence we avoid considering their interplay with affine representations.

Let π and ρ be two linear representations of G on vector spaces V and W. We define the *tensor product* of π and ρ, written $\pi \otimes \rho$, as the linear representation of G on the vector space $V \otimes W$ which linearly extends the formula

$$(\pi \otimes \rho)(g)(v \otimes w) = (\pi(g)v) \otimes (\rho(g)w) \qquad (v \in V, w \in W).$$

Remark that, even if we had started with *topological* vector spaces V and W, the tensor product $\pi \otimes \rho$ is a priori only a representation on the abstract vector space $V \otimes W$, since there is no *unique* topology for tensor products, not even in the Banach case (but see the proof of Proposition 4.3.9 below for the usefulness of tensor products for some continuity considerations).

Let now π be a linear representation of G on a *topological* vector space V. We define the *contragredient representation* π' on the topological dual V' by $\pi'(g) = \pi(g^{-1})'$, that is

$$(\pi'(g)\lambda)(v) = \lambda(\pi(g^{-1})v) \qquad (\lambda \in V', v \in V).$$

Observe that this is nothing but an instance of the general deduction, starting from an action on a set X, of a new action on a set of functions defined on X.

4.1.B Boundedness conditions

From now on, assume V is a Banach space. There are three possible interactions between an action on V and the distance induced by the norm: the distance can be preserved; it can be stretched and shrunk, up to some given factor; or it can be indefinitely stretched and shrunk (i.e., there are points arbitrarily close to each other that are sent to points arbitrarily far from each other, and vice versa). These behaviours are spotted by the operator norm. For affine actions, the translation part plays no role in these interactions, which motivates the following definition.

Definitions 4.1.1. — *A linear representation* π *of a group* G *on a Banach space is called*

- isometric *if* $\|\pi(g)\| = 1$ *for all* $g \in \mathrm{G}$*;*
- uniformly bounded *if* $\sup_g \|\pi(g)\| < \infty$*;*
- unbounded *if* $\sup_g \|\pi(g)\| = \infty$*.*

In the second case, we will sometimes be more precise and say that π *is* uniformly bounded by some constant C *if* $\sup_g \|\pi(g)\| \leqslant \mathrm{C}$*.*

An affine action α *is called* isometric, uniformly bounded *or* unbounded *when its underlying linear representation is so.*

On the terminology. — Isometric representations are indeed made of isometric operators, as $\|x\| = \|\pi(g^{-1}g)x\| \leqslant \|\pi(g)x\| \leqslant \|x\|$ (this trick would not work for semigroups for instance). When the Banach space V is a complex Hilbert space, the name *unitary* representation is more common, since surjective linear isometries are indeed unitary[3].

As continuous operators between Banach spaces are automatically uniformly continuous, another common name for uniformly bounded representations is *uniformly equicontinuous representations* (cf. [**Bou74**, x, § 2, n° 1, déf. 2]). We chose not to use the latter in order to avoid any confusion with continuity of the action (as a uniformly bounded representation is not necessarily a continuous representation). On the other hand, a drawback of our terminology is that a uniformly bounded *affine* representation may well have no bounded *affine* orbit at all (consider a Banach space, viewed as an additive group, acting on itself by translations). Actually, this pathological behavior can only come from the cocycle: the linear orbits behave better as shown in Lemma 4.1.4 below.

For general topological vector spaces, we also find the terminology *noncontracting* or *distal* (see [**NA67**, p. 444] or [**Bou81**, iv, appendice, exerc. 1] for the corresponding definition in the absence of a norm). However, these terms seem to be not that used when dealing with groups acting on Banach spaces.

Remark 4.1.2. — From a functional-analytic point of view, a uniformly bounded linear representation is morally isometric. Indeed, the formula $\|x\|' = \sup_g \|gx\|$ defines a new norm, equivalent to the former one, and preserved by the given action. Therefore, as far as only topological or analytic properties of the space V are at stake (for instance, being

[3] For some authors, the term "unitary representation" also includes some continuity. We do not follow this convention in this work.

reflexive or finite-dimensional), uniformly bounded actions behave as isometric ones. *However*, they may be quite different from a geometric point of view, as the new norm does not necessarily enjoy the same metric properties as the old ones. A pleasant exception is the uniform convexity, which is preserved by this process thanks to [**BFGM**, Proposition 2.3].

Remark 4.1.3. — In any case, to know when a uniformly bounded representation on a *Hilbert* space could be equivalent to an isometric one on another *Hilbert* structure is a deep problem, known as the *Dixmier problem.* We will briefly review it in Chapter 6.

As uniformly bounded representations are more tractable than unbounded ones, it is useful to be able to spot them without precisely knowing the operator norms. This is done by the following lemma.

Lemma 4.1.4. — *Let* G *be a group,* V *a Banach space, and* $\alpha = \pi + b$ *an affine action of* G *on* V*. Consider the following statements.*

(1) all affine orbits are bounded;

(1') the set of points whose affine orbit is bounded is nonmeager;

(2) all linear orbits are bounded;

(2') the set of points whose linear orbit is bounded is nonmeager;

(3) the representation is uniformly bounded.

The following implications hold: (1) ⇔ (1') ⇒ (2) ⇔ (2') ⇔ (3). Moreover, all the statements are equivalent if *the cocycle* b *is bounded.*

Proof. — The implications (1) ⇒ (1') and (3) ⇒ (2) ⇒ (2') are obvious. From the decomposition $\alpha = \pi + b$ follows (1) ⇒ (2), and this implication can be reversed if b is bounded. This leaves us two implications to prove: (2') ⇒ (3) and (1') ⇒ (1), for which it will be convenient to also prove directly (1') ⇒ (3).

(2') ⇒ (3) This is simply the Banach–Steinhaus uniform boundedness principle.

(1') ⇒ (3) This is a mere adaptation of the Banach–Steinhaus principle. We spell it out as this "affine" version is not that common. Let D be the set of points whose affine orbit is bounded and write it as the countable union of D_n, where

$$\mathrm{D}_n = \left\{ x \in \mathrm{V} \;\middle|\; \sup_{g \in \mathrm{G}} \| \alpha(g)x \| \leqslant n \right\}.$$

As D is nonmeager, some D_n must contain an open ball $\mathrm{B}(x_0, \varepsilon)$ of radius ε around

x_0. Now for any $g \in G$ and $y \in V$ such that $\| y \| < \varepsilon$, we have

$$\begin{aligned} \| \pi(g)y \| &= \| \pi(g)(y + x_0) - \pi(g)(x_0) \| \\ &= \| \alpha(g)(y + x_0) - \alpha(g)(x_0) \| \leqslant 2n. \end{aligned}$$

Hence the representation is uniformly bounded (by $2n\varepsilon^{-1}$).

(1') ⇒ (1) We already know by the previous step that the representation is uniformly bounded, hence in particular all linear orbits are bounded. Therefore, we only need to show that the cocycle is bounded. But this follows from the equality $b(g) = \alpha(g)x - \pi(g)x$ applied to any $x \in V$ with a bounded affine orbit. ■

Remark 4.1.5. — This lemma implies in particular that if a representation is *not* uniformly bounded, then the set of points whose affine (resp. linear) orbit is unbounded is comeager, hence *dense*.

4.1.C Continuity assumptions

It is essential to settle on the meaning of a "continuous" representation, as most groups we will consider carry some nontrivial topology. Various continuity conditions can be considered but fortunately, most of them are equivalent for locally compact groups. In any case, recall that, by definition, all our groups always act *by* continuous transformations, that is, each element of the group individually acts as a homeomorphism. The continuity assumptions studied in this section hence really concern the continuity of the action as a whole.

We recall from our general terminology (Section 1.1) that a representation of a topological group is *jointly continuous* if the action map is continuous and *orbitally continuous* if all orbit maps are continuous. Obviously, an affine representation is jointly (resp. orbitally) continuous if and only if its linear part is so and its cocycle is continuous.

On the terminology. — The orbital continuity of a linear representation π is nothing but the continuity of the morphism $\pi\colon G \to GL(V)$, when $GL(V)$ is endowed with the strong operator topology[4]. It is therefore sometimes called *strong continuity*.

Of course, joint continuity implies orbital continuity. In general, this implication cannot be reversed (examples will be given below), but the goal of this section is to show that, for the representations we will consider later, these continuity requirements coincide. For

[4]Beware however that the latter is in general *not* compatible with the group structure of GL(V), cf. [**Dix96**, chap. I, § 3, n° 1].

that purpose, we need one more definition. A linear representation π on a Banach space is said to be *locally bounded* if the operator norm is locally bounded, that is, if any $g \in G$ admits a neighborhood on which $\| \pi(\cdot) \|$ is bounded. Observe that, as G acts in any case *by* bounded transformations, it is enough to check that the identity admits a neighborhood on which the operator norm is bounded. The local boundedness condition can be rephrased by saying that for any converging net $g_j \to g$ (equivalently, for any converging net $g_j \to e$), the net $\| \pi(g_j) \|$ is eventually bounded .

The first easy but fundamental lemma is the following.

Lemma 4.1.6. — *A linear representation is jointly continuous if and only if it is both orbitally continuous and locally bounded.*

Proof. — By joint continuity, there are some $\delta > 0$ and an identity neighborhood $U \subset G$ such that $\| \pi(g)v \| < 1$ whenever $g \in U$ and $\| v \| < \delta$. In particular, $\| \pi(g) \| \leqslant \delta^{-1}$ for any $g \in U$.

Conversely, let (g_j, v_j) be a net converging to (g, v) in $G \times V$ and consider

$$\| \pi(g_j)v_j - \pi(g)v \| \leqslant \| \pi(g_j) \| \| v_j - v \| + \| \pi(g_j)v - \pi(g)v \|.$$

Both terms of the right-hand side go to zero: the first one thanks to local boundedness and the second one by orbital continuity. ■

Corollary 4.1.7. — *Joint and orbital continuities agree for (linear or affine) representations of Baire or first-countable groups.*

Proof. — The affine case follows from the linear one and, by the above lemma, the latter requires only to show that an orbitally continuous linear representation π is automatically locally bounded for these groups.

If the group G is Baire, we write G as an increasing union $\bigcup_n G_n$, where G_n is the set of g such that $\| \pi(g) \| \leqslant n$. Observe that each G_n is closed: indeed, it is the intersection over unit vectors $v \in V$ of the sets $\{g \in G \mid \| \pi(g)v \| \leqslant n\}$. Hence, by the Baire property, some G_n has an interior point g. Therefore, any $h \in G$ admits $hg^{-1}G_n$ as a neighborhood on which the operator norm of π is bounded by $n\| \pi(hg^{-1}) \|$.

When G is first countable, observe that local boundedness can be checked on sequences instead of nets and that, for the former, eventual and actual boundedness agree. Let thus $(g_n)_n$ be a converging sequence. By orbital continuity, $\pi(g_n)v$ is convergent for any $v \in V$, hence bounded. Therefore, $\| \pi(g_n) \|$ is bounded by Banach–Steinhaus uniform boundedness principle. ■

We can henceforth speak unambiguously of a *continuous* representation whenever we consider:

- locally bounded representations (in particular, uniformly bounded representations),
- or representations of Baire or first-countable groups.

The above corollary applies in particular to locally compact groups and to Polish groups. For more general groups, an orbitally continuous representation, even on a separable Hilbert space, need not be locally bounded (hence not jointly continuous), as shown by the following family of examples.

Example 4.1.8. — Let U be any normed space and G be the underlying additive group of U, endowed with the weak topology. Let $V = \mathbf{R} \oplus_2 U^*$ be the ℓ^2-direct sum of the reals with the (strong) dual of U. In particular, V is a separable Hilbert space whenever U is so.

Now consider the representation of G on V given by $\pi(g)(t,\lambda) = (t+\lambda(g),\lambda)$. It is orbitally continuous by the very definition of the weak topology. However, the operator norm satisfies $1+\|g\|_U^2 \leqslant \|\pi(g)\|^2 \leqslant (1+\|g\|_U)^2$, hence π is locally bounded if and only if U is finite-dimensional.

We record now a very easy fact that allows to lighten even further the checking of continuity. We will say in short that a linear representation is *orbitally continuous on a subset* X of the Banach space V if all the orbit maps associated to points in X are continuous.

Lemma 4.1.9. — *Let π be a linear representation of a topological group* G *on a Banach space* V *and* X *be a subset of* V. *If π is orbitally continuous on* X, *then so it is on the linear span of* X. *If moreover π is locally bounded, then continuity also holds for the closure of* X, *hence for its closed linear span.*

In particular, a locally bounded representation is jointly continuous as soon as it is orbitally continuous on a dense subset.

Proof. — The only non-completely obvious statement is that continuity propagates to the closure. Let $x \in \overline{X}$ and g_j be a net converging to g in G. For any $y \in X$, consider the following inequality, which follows from the triangle inequality:

$$\|\pi(g_j)x - \pi(g)x\| \leqslant (\|\pi(g_j)\| + \|\pi(g)\|)\,\|x-y\| + \|\pi(g_j)y - \pi(g)y\|.$$

This gives the required convergence of $\pi(g_j)x$ to $\pi(g)x$: choose first y close enough to x to make the first term of the right-hand side (eventually) arbitrarily small thanks to local boundedness, and then choose j far enough to shrink the second term. ∎

Remark 4.1.10. — A further continuity requirement would be to ask for the representation π to be *norm-continuous* or *uniformly continuous*, i.e.continuous when $\mathrm{GL}(V)$ is endowed with the operator norm topology. It is actually a very strong condition. As an illustration, consider that the regular representation of $\mathbf{R}$ fails to satisfy it. This condition also firmly restricts the possible structure of the given group. For instance, A.I. Shtern showed that such a representation factorizes through a Lie quotient [**Sht08**, Theorem 1]. Even among connected real Lie groups, the existence of a faithful uniformly continuous representation characterizes linear groups, by a result of D. Luminet and A. Valette [**LV94**, Theorem A]. Therefore we shall not make any use of this stronger continuity requirement in this work.

Remark 4.1.11. — In some contexts, the continuity of a representation is already ensured by some *measurability* of the orbit maps. More details on this practical tool in Section D.4.

Remark 4.1.12. — In Chapter 5, we will consider representations on general topological vector spaces. In that framework, alas, orbital and joint continuity may disagree even for compact groups. Therefore, for representations on non-Banach spaces, we will only consider orbital continuity. For more information on the interplays between joint and orbital continuity outside the Banach world, see Appendix C.

4.1.D Separability of the linear space

In Banach space theory, separability is one of the most palatable assumptions (for one instance of that fact, relevant to our work, see Appendix D). It is thence valuable to know that, for group considerations, this assumption often comes for free.

Proposition 4.1.13. — *Let* G *be a* σ-compact *group and* π *be an orbitally continuous linear representation of* G *on a Banach space* V*. Then the closed linear span of any orbit is a separable subspace.*

Proof. — It is enough to show that any orbit is separable. By continuity of the orbit map, each orbit is itself a σ-compact set. But V is a metric space, hence its compact and σ-compact subsets are separable. ∎

This result dates back to Kakutani and Kodaira, who proved it for the left regular representation of a locally compact group in [**KK44**, Satz 5] (their proof is surprisingly intricate and relies both on the Haar measure and on the regular representation).

Remark 4.1.14. — Of course, σ-compactness is a necessary assumption. For instance, if G is any discrete uncountable group, then the regular representation of G on the nonseparable space $\ell^2(G)$ (see the next section) admits a *cyclic* vector (that is, a vector whose orbit spans a dense subspace). Actually, there even exist groups such that *no* nontrivial orbit spans a separable subspace, whatever the representation on a Hilbert space, see Remark 5.2.17.

Remark 4.1.15. — The isometry group O(V) of a separable Banach space admits a left-invariant metric which defines the strong operator topology (an example of such a metric is given by $d(T, S) = \sum_n 2^{-n} \| Tx_n - Sx_n \|$, where $\{x_n\}$ is a countable dense subset of the unit sphere of V). Hence a faithful continuous isometric representation of a group G on a separable Banach space endows G with a continuous left-invariant metric, that is, with a metric group topology coarser than the initial one. This fact was used by Kakutani and Kodaira to prove their celebrated theorem (Proposition 1.2.1), thanks to the following two observations:

- the regular representation of a locally compact group contains subrepresentations with arbitrary small kernel;
- the topology induced on a locally compact group G by the strong operator topology of $O(L^2(G))$ is the initial topology (in other words, the regular representation $\lambda\colon G \to (O(L^2(G)), \mathrm{SOT})$ is a homeomorphism onto its image).

Both facts follow from this easy exercise: if f is a continuous square-integrable function on G, then $\| \lambda(g)f - f \|_2 < \sqrt{2} \| f \|_2$ implies that $g \in UU^{-1}$, where U is the support of f. (Alternatively, for the second claim: any continuous surjective homomorphism from a σ-compact locally compact group is open, cf. [**CH16**, Corollary 2.D.6].)

Remark 4.1.16. — A Banach space that is not isomorphic to a Hilbert space admits a closed subspace which is not complemented (cf. [**LT71**, Theorem 1]) and moreover an invariant complemented subspace does not necessarily admit an invariant complement[5]. Thus Proposition 4.1.13 does not, in general, yield a decomposition of V into a direct sum of invariant separable subspaces. Such a decomposition is however possible for Hilbert spaces through a transfinite induction, namely: any continuous unitary representation of a σ-compact group on a Hilbert space is the Hilbert direct sum of continuous unitary representations on separable Hilbert spaces[6].

Remark 4.1.17. — Of course, the proof yields more generally: *if a σ-compact group acts orbitally continuously on a metrizable space, then the closure of any orbit is separable.*

[5] A concrete example is given in Example 2.29 of [**BFGM**].
[6] The transfinite induction step is explained in details in [**HR79**, Theorems 21.14 and 21.13].

4.1.E Regular representations of locally compact groups

Let G be a locally compact group, endowed with the class of Haar measures. As G acts non-singularly on itself, the linear representations λ and ρ of G on the space $\mathscr{L}(G)$ of all (Haar) measurable functions defined by

$$(\lambda(g)f)(x) = f(g^{-1}x) \quad \text{and} \quad (\rho(g)f)(x) = f(xg), \quad \text{where} \ \ g, x \in G, f \in \mathscr{L}(G),$$

descends to the quotient L(G) of classes of measurable functions that agree locally almost everywhere. We shall use the same letters λ and ρ to denote the resulting representations on L(G) and call them respectively the *left and right regular representations.*

From now on, we choose some left Haar measure m_G on G and consider the left regular representation λ. By the very invariance of m_G under left translations, the representation λ preserves the subspaces $L^p(G)$ and the norm on it. Once again, we will keep the same letter λ for the deduced representations and still call them *(left) regular representation.*

Lemma 4.1.18. — *For $1 \leqslant p < \infty$, the left regular representation of* G *on* $L^p(G)$ *is continuous.*

Proof. — By Lemma 4.1.9, it is enough to check the continuity of orbit maps associated to elements of the dense subset $\mathscr{C}_c(G)$ of continuous functions with compact support. Let $f \in \mathscr{C}_c(G)$. As its support is compact, f is right-uniformly continuous. Hence for any $\varepsilon > 0$, there is an identity neighborhood U of G such that $|f(x) - f(y)| < \varepsilon$ whenever $xy^{-1} \in U$. But this implies that $\| \pi(g)f - f \|_p \leqslant \varepsilon |\operatorname{supp} f|^{1/p}$ for $g \in U^{-1}$, which is the required orbital continuity. ∎

Remark 4.1.19. — As for $L^\infty(G)$, the deduced representation is not continuous (consider the orbit map of an indicator function of some set of positive measure). The orbit maps are however *weakly-∗ continuous* (trivially, as the representation is contragredient to the regular representation on $L^1(G)$).

Remark 4.1.20. — Lemma 4.1.18 yields another proof of Lemma 1.2.3. Indeed the function f therein is equal to $\| \mathbb{1}_{gE \triangle E} \|_1 = \| \lambda(g)\mathbb{1}_E - \mathbb{1}_E \|_1$, hence continuous.

Remark 4.1.21. — The results of this section are of course still valid when restricting the representations to some closed subgroup H of G. More generally, we could similarly consider the representations of a locally compact group G deduced from a continuous action of G on a locally compact space X endowed with an invariant Radon measure, and get the same continuity results, with the same proof. The invariance of the measure is not even needed (at least for Polish groups), as long as the measure class is preserved (see [**Büh11**, Appendix D]). We shall however not need this generality.

4.1.F Lagniappe: weakly continuous representations

The results of this section are not needed in the rest of this work.

As Banach spaces are naturally endowed with a weak topology, a third natural continuity requirement can be considered: a representation is said *weakly continuous* if all its orbit maps are continuous with respect to the weak topology. Once again, an affine representation is weakly continuous if and only if its underlying linear representation is so and its cocycle is continuous with respect to the weak topology. To complete the picture given in Section 4.1.C, we show here that not much is to be gained from this condition.

Lemma 4.1.22. — *A weakly continuous linear representation of a Baire or first-countable group is locally bounded.*

Proof. — Same proof as for Corollary 4.1.7: the sets G_n are still closed by the Hahn–Banach theorem, and weakly convergent sequences are still bounded. ∎

For locally compact groups, we have more.

Lemma 4.1.23. — *Joint, orbital and weak continuities agree for linear and affine representations of* locally compact *groups.*

Proof. — By the previous lemma and Lemma 4.1.6, it is enough to prove that, for locally compact groups,

1. a weakly continuous linear representations is orbitally continuous;
2. a weakly continuous cocycle associated to an orbitally continuous linear representation is also continuous with respect to the norm topology.

We only spell out the second point, since the first one has a similar proof that can easily be found in the literature (it seems to have been first published by de Leeuw and Glicksberg in [**LG65**, Theorem 2.8], see also the theorem of page 89 in [**Lyu88**, Chap. 3, Sec. 1]).

So let b be a weakly continuous cocycle associated to an orbitally continuous linear representation π. Observe first that continuity at any point of G will follow from continuity at identity $e \in G$, thanks to the cocycle identity and to the orbital continuity of π. Let A be a relatively compact identity neighborhood of G. For any $h \in G$, define the following

(Pettis) integral

$$b_{\mathrm{A}}(h) = \int_{\mathrm{G}} \mathbb{1}_{\mathrm{A}}(g) b(hg)\, \mathrm{d}g \quad \in \mathrm{V},$$

which exists thanks to weak continuity[7]. As a map from V to V, each operator $\pi(h)$ is also weakly continuous, hence the cocycle identity yields

$$b_{\mathrm{A}}(h) = \pi(h) b_{\mathrm{A}}(e) + |\mathrm{A}| b(h).$$

In particular, the continuity of b at e will follow from the continuity of $b_{\mathrm{A}} \colon \mathrm{G} \to \mathrm{V}$ at e.

Let h_j be a net converging to $e \in \mathrm{G}$. We may assume that $h_j \in \mathrm{A}$. We now compute:

$$\begin{aligned} \| \, b_{\mathrm{A}}(h_j) - b_{\mathrm{A}}(e) \, \| &= \left\| \int_{\mathrm{G}} \Big(\mathbb{1}_{\mathrm{A}}(h_j^{-1} g) - \mathbb{1}_{\mathrm{A}}(g) \Big)\, b(g)\, \mathrm{d}g \right\| \\ &\leqslant \mathrm{C} \Big\| \, \mathbb{1}_{h_j \mathrm{A}} - \mathbb{1}_{\mathrm{A}} \Big\|_1, \end{aligned}$$

where C is the supremum of $\| \, b(g) \, \|$ over the relatively compact set AA (which exists since b is weakly continuous and weakly compact sets are bounded)[8]. Now the sought convergence follows from $\|\mathbb{1}_{h_j \mathrm{A}} - \mathbb{1}_{\mathrm{A}}\|_1 = |h_j \mathrm{A} \triangle \mathrm{A}| \to 0$ (cf. Lemma 1.2.3). ∎

In general, however, a weakly continuous representation is not necessarily orbitally continuous. An example has been provided by Megrelishvili of an abelian topological group (homeomorphic to **Q** as a topological space) that acts isometrically and weakly continuously, but not orbitally continuously, on the Banach space $\mathscr{C}_{\mathrm{b}}([0,1]^2)$, see [**Meg01b**, Remark 2.9(3)] (easier examples, but for much huger groups, are contained in Remark 4.1.24 below). For Polish groups, some positive results can be established by completely different methods (cf. Example D.5.3).

To conclude this section, we observe that, for any topological group, the gap from weak continuity to orbital continuity can be filled if the Banach space is nice enough to enable some nontrivial connections between weak and norm topologies. Besides finite-dimensional spaces, the easiest examples are *Kadec* spaces, that is, spaces for which weak and norm

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[7]We quickly recall how the Pettis integral is built in the present case. The element $b_{\mathrm{A}}(h)$ is defined as an element of V'^*, the algebraic dual of the topological dual of V, by

$$\langle b_{\mathrm{A}}(h), w \rangle = \int_{\mathrm{G}} \langle \mathbb{1}_{\mathrm{A}}(g) b(hg), w \rangle \, \mathrm{d}g \qquad (w \in \mathrm{V}').$$

We now need to show that this linear form, which is obviously continuous for the norm topology on V', is also weakly-$*$ continuous, so that it actually belongs to V. For our particular setting, this can be found e.g. in [**Lyu88**, p. 2–4].

[8]Do not be fooled by the apparent simplicity of this computation: the Pettis integral is not subadditive in norm (for the very reason that, in general, the *norm* of the integrand is not integrable). The point here is that the term $b(g)$ is multiplied by an integrable *scalar* function, namely $\mathbb{1}_{h_j \mathrm{A}} - \mathbb{1}_{\mathrm{A}}$.

topologies agree on the unit sphere[9]. Examples of Kadec spaces include Hilbert spaces, superreflexive Banach spaces and more generally locally uniformly convex Banach spaces (see [**DGZ93**, Chap. II, Prop. 1.4]). The very definition of this property implies that weak, orbital and joint continuities agree for *isometric* linear representations on *Kadec* spaces, whatever the group.

But we cannot remove the isometry assumption. Indeed, by a celebrated theorem of Kadec, *every* separable Banach space admits an equivalent locally uniformly convex norm (see [**DGZ93**, Chap. II, Theorem 2.6]). In particular, the above-mentioned example of Megrelishvili can be seen, up to renorming, as a uniformly bounded representation on a Kadec space that is weakly continuous but not orbitally continuous.

Remark 4.1.24. — Weakly compact sets of a Banach space are in general not metrizable or separable (consider for instance the unit ball of $\ell^2(\mathrm{I})$, for I an uncountable set), hence we cannot expect Proposition 4.1.13 to hold also for weakly continuous representations. And indeed, let V be a non-separable reflexive Banach space and G be its additive group endowed with the weak topology. By the Alaoglu–Bourbaki theorem, G is σ-compact (and even compactly generated—though not locally compact since V is infinite-dimensional). The natural affine action by translations of G on V is, by definition, weakly continuous, but there is only one orbit, V itself, which is not separable. By linearizing this affine action (cf. Section 4.1.A), we get a similar linear counterexample. (Incidentally, we observe that these representations cannot be orbitally continuous, since precisely the conclusion of Proposition 4.1.13 does not hold.)

4.2 Moderateness

In order to build induced representations in the next section, we will need to control coarsely how the groups grow "at infinity". Such a control is trivial on a finitely generated group: the size of the balls (with respect to the word metric associated to a finite generating set) grows at most exponentially. We sketch here how this bound can still be established for general σ-compact locally compact groups (Section 4.2.A). Its usefulness for group representations is shown in Section 4.2.B.

[9] This property is also occasionally called the Kadec–Klee property. Unfortunately, the latter name is sometimes used only for the weaker sequential version (which is also known as the Radon–Riesz property or property (H) . . .). The name "Kadec" is sometimes spelled "Kadets".

4.2.A Moderate lengths and measures

Definition 4.2.1. — *A* length *on a group* G *is a function* $\ell\colon \mathrm{G} \to \mathbf{R}_+$ *such that*

(i) $\ell(g) = \ell(g^{-1})$ *for all* $g \in \mathrm{G}$,

(ii) $\ell(gh) \leqslant \ell(g) + \ell(h)$ *for all* $g, h \in \mathrm{G}$.

When G *is a locally compact group, a length* ℓ *is* moderate *if moreover*

(iii) the ball $\mathrm{B}(r) = \ell^{-1}([0, r])$ *is compact for all* $r \geq 0$,

(iv) for any Haar measure m_{G} *there is* $\mathrm{C} \geq 1$ *such that* $m_{\mathrm{G}}(\mathrm{B}(r)) \leq \mathrm{C}^r$ *for all* $r \geq 1$.

We do not require any kind of compatibility between the length of a topological group and its topology, even when the group is metrizable. Observe however that a *moderate* length is at least lower semi-continuous by (iii), hence Borel.

Example 4.2.2. — If G is a locally compact group generated by some compact symmetric subset K with nonempty interior (for instance if G is finitely generated), then the word length ℓ_{K} relatively to K is a moderate length. Recall that ℓ_{K} is defined as

$$\ell_{\mathrm{K}}(g) = \min\{n \in \mathbf{N} \mid \exists k_1, \ldots, k_n \in \mathrm{K} : \ g = k_1 \ldots k_n\}.$$

(with $\ell_{\mathrm{K}}(e) = 0$). Each ball $\mathrm{B}(r) = \ell_{\mathrm{K}}^{-1}([0, r]) = \mathrm{K}^{\lfloor r \rfloor}$, where $\lfloor r \rfloor$ is the greatest integer $\leqslant r$, hence is compact. Moreover, as the interior of K is nonempty, K^2 is covered by m translates of K, hence for any n, K^{n+1} is also covered by m translates of K^n. Moderateness follows easily from the latter fact.

The existence of moderate lengths was proved by U. Haagerup and A. Przybyszewska. As we are not concerned with the compatibility between the length and the topology, we can state their result for σ-compact groups.

Proposition 4.2.3. — *Any locally compact σ-compact group admits a moderate length.*

On the proof. — Let H be an open compactly generated subgroup of the locally compact σ-compact group G. By Example 4.2.2, we can endow H itself with a moderate length; moreover, the coset space G/H is countable. Choose some set of representatives $s_0 = e, s_1, \ldots$. The idea is now to consider the weighted length on G, where each element $s_i h$ is given a weight depending on s_i and on the length of h. Explicit computations are made in [**HP06**, Theorem 5.3]. The result is proved there for second-countable groups in order to get a compatibility with the topology of G but the proof yields also non-compatible moderate length on σ-compact groups (alternatively, we could pull back a moderate length from a second-countable quotient by a compact kernel, thanks to the Kakutani–Kodaira theorem). □

Remark 4.2.4. — On the other hand, σ-compactness is a necessary condition by (iii), so that the existence of a moderate length is actually a characterization of σ-compactness.

Remark 4.2.5. — It follows immediately from the definition that the restriction of a moderate length to any open subgroup is still moderate. In particular, we can also provide a moderate length for any countable group by embedding it in a finitely generated group [**HNN49**, Theorem IV]. However, this shortcut cannot be used for locally compact σ-compact groups, since they do not necessarily embed into compactly generated locally compact groups [**CC14**, Theorem 1.3]—even not, by the way, as abstract (i.e., non-closed) subgroups.

In order to give some context to moderate lengths, we recall that there is a natural interplay between lengths and (pseudo)metrics on groups[10]. If ℓ is a length on a group G, then the function d defined by $d(x, y) = \ell(x^{-1}y)$ is a left-invariant pseudometric on G. Reciprocally, if d is a left-invariant pseudometric, then the function ℓ defined by $\ell(g) = d(e, g)$ is a length on G. If the length satisfies moreover Condition (iii), then the associated pseudometric is proper. Conversely, a lower semicontinuous proper metric produces a length satisfying Condition (iii).

The central result about metrics on group is the Birkhoff–Kakutani theorem: a topological group is first countable if and only if it admits a left-invariant compatible metric. On a metric *locally compact* group, balls of sufficiently small radius must be relatively compact. It is natural to wonder when it is possible to have relative compactness for all balls (that is, a *proper* metric). Second countability is obviously a necessary condition, Struble proved that it is also sufficient.

Remark 4.2.6. — Roughly speaking, Struble's strategy [**Str74**] is to build inductively compact "pseudoballs" U_s indexed by the positive real numbers, starting with the compact balls of sufficiently small radius of a left-invariant compatible metric (as given by the Birkhoff–Kakutani theorem) and then adding a countable cover of compact neighborhoods (as given by second countability). The length of an element g is then given as the smallest s such that U_s contains g.

Unfortunately, the metric built by Struble does not produce a moderate length, for the relation between the balls U_{2^n} and $\mathrm{U}_{2^{n+1}}$ given at the end of page 219 in [**Str74**] does not let us hope for more than a superexponential bound of the form $m_{\mathrm{G}}(\mathrm{B}(r)) \leqslant$

[10]We refer to the book [**CH16**] for more details on metrics on topological groups.

C^{r^2}. And indeed, Struble's construction starting with the usual word metric on a free group of rank $d \geqslant 2$ produces another metric for which the balls of radius 2^n have at least $\exp((2^n)^2 \ln C)$ elements (where $C = \sqrt{2d-1}$). To our knowledge, the first general construction of moderate lengths was given by Haagerup and A. Przybyszewska.

We emphasize again that our concern will be on the growth of the Haar measure of the balls, hence on their large-scale behavior. That's why we are not concerned with the compatibility of the length with the topology, since the non-metrizability of the latter is a small-scale pathology by the Kakutani–Kodaira theorem (Proposition 1.2.1).

We now explain how moderate lengths can be used to build nice measures on the groups.

Definition 4.2.7. — *A* moderate measure[11] *on a locally compact group* G *is a probability measure* μ *in the same measure class as the Haar measures and such that*

(i) for all $g \in G$*, the Radon–Nikodym derivative* $dg\mu/d\mu$ *is essentially bounded on* G*,*

(ii) the map $g \mapsto \| dg\mu/d\mu \|_\infty$ *is locally bounded on* G.

The point of this definition is that moderate measures yields interesting linear representations of G. This feature, at the core of the moderate induction techniques of Sections 4.3.D and 4.3.E, will be given an easy illustration in Section 4.2.B below. Let us first show how to build moderate measures.

Proposition 4.2.8. — *Let* ℓ *be a moderate length on a locally compact group* G *and* m_G *be a left Haar measure. Then the measure* μ *defined by*

$$d\mu(x) = kD^{-\ell(x)} dm_G(x)$$

is moderate when $D > 1$ *is large enough and* k *is an appropriate normalization constant.*

Proof. — As moderate lengths are Borel, the given formula makes sense and defines a measure in the same class as the Haar measures. Moreover, this measure is finite as soon as $D > C$, for C as in Definition 4.2.1(iv), hence an appropriate choice of k will turn it into a probability measure. Finally, the Radon–Nikodym derivative $dg\mu/d\mu$ is bounded above by $D^{\ell(g)}$ thanks to Conditions (i) and (ii) of Definition 4.2.1. So in order to check the local boundedness condition of moderate measures, we only need to check that ℓ is locally bounded.

[11] For measures on Hausdorff topological spaces, there exists another notion of moderateness, cf. [**Bou69**, § 1, n° 9, Définition 12]. Fortunately, the latter is void for Radon measures on σ-compact locally compact spaces, so that no confusion is to be feared with our terminology.

As the ℓ-balls $B(r)$ are closed, Baire's theorem implies that some ball $B(s)$ contains an interior point x_0. Since $B(s)x_0^{-1}B(r) \subset B(r+2s)$ by Definition 4.2.1(ii), the interior of the ℓ-balls $B(r)$ still cover the whole group. Hence any compact set in G is contained in some ball $B(r)$ for r large enough, from which it follows that ℓ is locally bounded. ■

Remark 4.2.9. — Actually, the above construction yields more than needed: the measure μ enjoys likewise the boundedness conditions of Definition 4.2.7 with respect to the right translations (the bounds are the same up to the multiplication by the modulus of G, but the latter is continuous, hence locally bounded). Observe also that the Radon–Nikodym derivative $d\mu/dm_G$, is here a *bounded* function in $L^1(G, m_G)$.

Corollary 4.2.10. — *Every locally compact σ-compact group admits a moderate measure.*

Proof. — This is an immediate consequence of Propositions 4.2.3 and 4.2.8. ■

Remark 4.2.11. — As for lengths, if a locally compact group G admits a moderate measure μ, it has to be σ-compact. Indeed, assume G is not σ-compact and let U be the subgroup generated by some compact identity neighborhood of G. The Haar measure of U and of all its cosets are positive, hence the same holds for the measure μ. But U has to be of uncountable index in G, as G is not σ-compact. The measured space G would then have an uncountable partition into sets of positive μ-measure, which is impossible for finite measures.

4.2.B Moderately regular representations

Let us now have a look at the space $L^p(G, \mu)$ of integrable scalar functions with respect to a moderate measure. If $p = \infty$, nothing new occurs, since $L^\infty(G, \mu) = L^\infty(G)$ depends only on the measure class. We will therefore focus on $1 \leqslant p < \infty$. Recall from Section 4.1.E that we have defined the left regular representation λ of G on the space $L(G)$ as the representation given by precomposition by left translation. The point of moderate measures is to have the following result.

Proposition 4.2.12. — *Let* G *be a locally compact σ-compact group and* μ *a moderate measure on* G. *Let* $1 \leqslant p < \infty$. *Then the left regular representation* λ *of* G *on* $L(G)$ *preserves* $L^p(G, \mu)$ *and acts on the latter by continuous transformations. Moreover, the deduced representation on* $L^p(G, \mu)$ *is continuous.*

Proof. — Let $f \in L^p(G, \mu)$ and $g \in G$. A straightforward computation yields

$$\| \lambda(g)f \|^p_{L^p(G,\mu)} \leqslant \| f \|^p_{L^p(G,\mu)} \left\| \frac{dg^{-1}\mu}{d\mu} \right\|_\infty,$$

hence $\lambda(g)$ preserves $L^p(G, \mu)$ and is a bounded operator on the latter thanks to Condition (i) of Definition 4.2.7. Moreover, the restriction of λ to $L^p(G, \mu)$ is a locally bounded representation by Condition (ii).

We then only need to check the continuity of orbit maps associated to a set E with dense linear span (cf. Lemmas 4.1.6 and 4.1.9). We choose for E the set of all indicator functions of measurable sets A with finite *Haar* measure. Let g_α be a net in G converging to g. As μ and any Haar measure m_G are two σ-finite measures in the same class, the $L^p(\mu)$-convergence of $\lambda(g_\alpha)\mathbb{1}_A$ to $\lambda(g)\mathbb{1}_A$ is equivalent to the $L^p(m_G)$-convergence of $\lambda(g_\alpha)\mathbb{1}_A$ to $\lambda(g)\mathbb{1}_A$. But the latter is exactly given by the continuity of the left regular representation (cf. Lemma 4.1.18 or alternatively only Lemma 1.2.3). ■

We call the continuous representation of G on $L^p(G, \mu)$ defined by Proposition 4.2.12 the *moderately (left) regular representation* associated to μ.

The interesting feature of $L^p(G, \mu)$ is that it contains $L^\infty(G)$ as μ is a probability measure. This will be useful to transfer some knowledge from the latter to the former. Indeed, despite being quite unwieldy from a Banach point of view, L^∞ spaces arise naturally in many contexts, for instance in bounded cohomology.

Remark 4.2.13. — In the course of the proof, we have seen that the operator norm of $\lambda(g)$ is bounded above by $\| dg^{-1}\mu/d\mu \|^{1/p}_\infty$. Actually, this is the exact value. Indeed, choose a Radon–Nikodym derivative φ of $dg^{-1}\mu/d\mu$ and consider the set

$$A = \left\{ x \in G \;\middle|\; \varphi(x) > \left\| \frac{dg^{-1}\mu}{d\mu} \right\|_\infty - \varepsilon \right\}$$

for some given $\varepsilon > 0$, which is of positive measure by definition of the essential supremum norm. Then an easy change of variable yields

$$\| \lambda(g)\mathbb{1}_A \|^p \geqslant \left(\left\| \frac{dg^{-1}\mu}{d\mu} \right\|_\infty - \varepsilon \right) \| \mathbb{1}_A \|^p,$$

hence $\| \lambda(g) \| = \| dg^{-1}\mu/d\mu \|^{1/p}_\infty$ by choosing ε arbitrarily small.

Remark 4.2.14. — It would have been too strong to require a global bound in the definition of a moderate measure, that is, $g \mapsto \|dg\mu/d\mu\|_\infty$ to be bounded on the whole G by some constant C. Assume for simplicity that G is countable and write $f \in \ell^1(G)$ for the Radon–Nikodym of μ with respect to the counting measure. By the cocycle relation for Radon–Nikodym derivatives, the global bound becomes also $C^{-1} \leqslant \frac{gf}{f}$ for all g. This inequality forces the counting measure to be finite, as, for any $x \in G$,

$$\frac{1}{f(x)} = \frac{1}{f(x)} \sum_{g \in G} f(gx) = \sum_{g \in G} \frac{f(gx)}{f(x)} \geqslant \sum_{g \in G} C^{-1}.$$

In particular, the moderately regular representation is not uniformly bounded, unless G is finite.

This also shows that the logarithm of the Radon–Nikodym derivative $\mathrm{d}g\mu/\mathrm{d}\mu$ of a moderate measure μ, which is indeed a cocycle (relatively to the regular representation on L(G)) lying in the space of all bounded measurable functions thanks to the moderateness assumptions, cannot usefully be a *bounded* cocycle.

4.3 Moderate and measured inductions

4.3.A Heuristic: the ideas behind classical induction

The main idea of induction of representations is to build a representation of a group G by twisting a representation of a subgroup H by the coset space G/H. Frobenius introduced successfully this technique in 1898 for finite groups. Very loosely speaking, the induced representation is made of maps from G/H to the initial H-space (or, equivalently, of H-equivariant maps from G).

The generalization of Frobenius's theory to unitary representations of *locally compact* groups has been initiated by G. Mackey in the 1950's [**Mac51**, **Mac52**, **Mac53**]. A number of technical difficulties arise in this context, to say the least. One of them is that we need to keep track of the relationship between the moduli of the whole group and of the (closed) subgroup. Fortunately, the latter is plain in two cases: when the subgroup is open or normal. In this context, Δ_H is nothing but the restriction of Δ_G to H. Moreover, the coset space is endowed with a G-invariant measure (the counting measure when H is open and the quotient Haar measure when H is normal).

Let us then restrict our attention to one of these two tamable cases. Let H be an open or normal (closed) subgroup of a locally compact group G and $(\pi, \mathcal{H})$ be a unitary representation of H on some Hilbert space $\mathcal{H}$. We will further assume that $\mathcal{H}$ is separable and G is second countable, to avoid other technical issues.

There are two commuting H-actions on the space $\mathscr{L}(\mathrm{G}, \mathcal{H})$ of measurable functions from G to $\mathcal{H}$: the precomposition by right translation of h and the postcomposition by $\pi(h)$. Moreover, these two actions commute with the G-action given by precomposition by left translations. All these actions preserve classes of almost everywhere equal functions, hence can be viewed as actions on the quotient space $\mathrm{L}(\mathrm{G}, \mathcal{H})$. We can therefore consider the G-space $\mathrm{L}(\mathrm{G}, \mathcal{H})^{\mathrm{H}}$ of functions that are invariant for the diagonal H-action (equivalently, these functions are H-equivariant). Pointwise, this reads, for $f \colon \mathrm{G} \to \mathcal{H}$,

$$\pi(h)f(xh) = f(x)$$

for every $h \in \mathrm{H}$ and almost every $x \in \mathrm{G}$. As the representation is unitary, the norm $\| f \|_{\mathcal{H}}$ is an H-invariant function, hence can be viewed as a function defined on the quotient space G/H. We can therefore define the *induced representation space* $\mathbf{ind}_{\mathrm{H}}^{\mathrm{G}}\mathcal{H}$ as

$$\left\{ f \in \mathrm{L}(\mathrm{G}, \mathcal{H})^{\mathrm{H}} \;\middle|\; \| f \|_{\mathcal{H}} \in \mathrm{L}^2(\mathrm{G}/\mathrm{H}) \right\}$$

(where G/H is endowed with a G-invariant measure) and the *induced representation* $\mathbf{ind}_{\mathrm{H}}^{\mathrm{G}}\pi$ as

$$(\mathbf{ind}_{\mathrm{H}}^{\mathrm{G}}\pi(g)f)(x) = f(g^{-1}x) \qquad (f \in \mathbf{ind}_{\mathrm{H}}^{\mathrm{G}}\mathcal{H}).$$

Thanks to the H-invariance of the functions we considered, this representation indeed remembers, to some extent, the initial representation of H as well as the embedding of H into G. We can see that more precisely by *choosing* some Borel section $\chi \colon \mathrm{G}/\mathrm{H} \to \mathrm{H}$ and see that the above representation is equivalent to the representation on the whole space $\mathrm{L}^2(\mathrm{G}/\mathrm{H}, \mathcal{H})$, where the G-action is now the precomposition “twisted” by an appropriate cocycle, namely:

$$(g \cdot f)(x\mathrm{H}) = \pi(\chi(x\mathrm{H})^{-1} g \chi(g^{-1}x\mathrm{H})) f(g^{-1}x\mathrm{H}) \qquad (f \in \mathrm{L}^2(\mathrm{G}/\mathrm{H}, \mathcal{H}),\ x\mathrm{H} \in \mathrm{G}/\mathrm{H},\ g \in \mathrm{G}).$$

When H is open, this remembrance is actually very sharp. Indeed, in that case, G/H is discrete, so the space $\mathrm{L}^2(\mathrm{G}/\mathrm{H}, \mathcal{H})$ *contains* a copy of $\mathcal{H}$ (namely, the maps supported on the trivial coset H). This subspace is invariant for the restriction of $\mathbf{ind}_{\mathrm{H}}^{\mathrm{G}}\pi$ to H; moreover, the latter is nothing but the initial representation π if the section was chosen so that $\chi(\mathrm{H}) = e$. In general, such a perfect memory of the initial representation is not to be expected but, as we shall see, we can still hope to transfer some properties of π to the induced representation.

Now look again at the above representation. Two key observations lead to different modifications of Mackey’s induction.

- If we had imposed an L^2-condition on the whole space $L(G, \mathcal{H})$, we could not have found nontrivial H-invariant functions. Hence the representation π is assumed to be unitary only to be able *to define an* L^2*-condition that is not incompatible with* H-*invariance.*

- In the space $L(G, \mathcal{H})$ used to define the induced representation space, G *is not used as a group, but as a* $(G \times H)$*-measured space.*

If we want to drop the isometry assumption on the representation π, the first observation suggests to use a global L^2-condition but with respect to moderate measures on G, as bounded functions on probability spaces are trivially square-integrable. If we want to induce from groups that are not true subgroups of G, the second point hints towards tychomorphisms. We will therefore discuss three kinds of inductions:

1. measured induction, that deals with inducing isometric representations through tychomorphisms;

2. moderate induction, that deals with inducing non-isometric representations of subgroups;

3. moderate measured induction, that mixes the above two strategies in order to induce non-isometric representations through tychomorphisms.

We point out a slight flaw in our terminology. Since closed subgroups are examples of tychomorphisms, classical Mackey's induction and moderate induction will indeed turn out to be special cases of their measured versions. On the other hand, the Haar measure is not moderate (unless for compact groups), so that classical Mackey's induction and measured induction are *not* special cases of their respective moderate variants. This asymmetry is reflected by the point of view we will take:

- the measured induction will be considered for linear representations (as in the above discussion), and each cocycle will be individually induced;

- the moderate inductions will be considered directly for affine representations.

This difference is motivated in part by our applications, which are of linear nature for the measured induction (Chapter 6) but of affine nature for the moderate induction (Chapter 5). Another reason is, as we shall see, a difficulty inherent in moderate inductions: the non-triviality of the linear part is hard to show without having a good cocycle at hand.

The necessary background on measurability in Banach spaces is given in Section 4.3.B (more on that topic in Appendix D, we give here only the bare minimum we will need). We then explain the measured, moderate, and moderate measured inductions in Sections 4.3.C, 4.3.D, and 4.3.E, respectively. These sections run more or less parallel: a definition of a suitable $(G \times H)$-space of measurable maps, a proof that its subspace of

H-invariant maps is closed, and then a proof that some properties of the initial representation are transferred to the induced one. The arguments for each step are similar from one section to another, but however technically different between the moderate and the non-moderate cases. Moreover, for the moderate inductions, some easier arguments and stronger results can be given if H is a true subgroup of G. For the sake of readability, we have therefore chosen to expand the arguments for each induction instead of looking for some abstruse common framework. Each section is therefore readable independently of the other ones.

Remark 4.3.1. — The induction results we present here bring interesting results when we have at our disposal a *bounded nontrivial* cocycle associated to the initial linear representation. These cocycles cannot exist for uniformly bounded representations on reflexive Banach space (see Corollary 5.2.2). Hence the reader should keep in mind that our induction techniques will be ultimately applied either to non-reflexive Banach spaces or to unbounded representations.

4.3.B Measurability in Banach spaces: the bare minimum

As we explained in the last section, we will work with measurable maps from some $(\mathrm{G}\times\mathrm{H})$-measure space S to a Banach space V. We need to be more precise and careful, because the sum of two norm-measurable maps need not be norm-measurable! To avoid such a scandal, we will restrict our attention to two more tractable σ-algebras.

- The Borel σ-algebra of a *separable* Banach space V. In this case, the Borel σ-algebra $\mathcal{B}(\mathrm{V}\times\mathrm{V})$ is equal to the product σ-algebra $\mathcal{B}(\mathrm{V})\otimes\mathcal{B}(\mathrm{V})$, hence the sum of two measurable maps is again measurable (for the convenience of the reader, the proof of these two facts are recalled in Section D.1).
- The *scalar weak-$*$ σ-algebra* of a dual Banach space $\mathrm{V}=\mathrm{V}_\flat'$, that is, the smallest σ-algebra on V such that all evaluation maps coming from $\mathrm{V}_\flat$ are measurable. (Observe that this σ-algebra depends a priori on a choice of a predual $\mathrm{V}_\flat$.) In this case, a map $f\colon \mathrm{S}\to\mathrm{V}$ is measurable (*scalarly weakly-$*$ measurable* if we need to be more precise) if and only if all the scalar maps $f(\cdot)(w)$ are measurable ($w\in\mathrm{V}_\flat$). In particular, the sum of two measurable maps is again measurable.

The first measurability will be used for moderate inductions, the second one for the measured induction. Let us now explain what spaces of measurable maps we will consider in each case. Let V be a separable Banach space and S a measure space. We will write $\mathscr{L}(\mathrm{S},\mathrm{V})$ for the vector space of all measurable functions f from S to V, and $\mathrm{L}(\mathrm{S},\mathrm{V})$ for its quotient space of almost everywhere equal functions. For $1\leqslant p<\infty$, a measurable function f is called *Bochner* L^p*-integrable*, or simply L^p*-integrable* if the scalar function $\| f \|^p$ is integrable, and its *(Bochner)* L^p*-seminorm* is, as expected:

$$\| f \|_p = \left(\int_S \| f(s) \|^p \, ds \right)^{\frac{1}{p}} .$$

This seminorm does not depend on the choice of a function in its class and defines a complete norm on the spaces of classes of equivalent L^p-integrable functions; this space is called the *Bochner–Lebesgue space* and is written $L^p(S, V)$.

For $p = \infty$, we define the seminorm by

$$\| f \|_\infty = \operatorname*{ess\,sup}_{s \in S} \| f(s) \|$$

and have, similarly, the *Bochner–Lebesgue* space of essentially bounded functions, written $L^\infty(S, V)$.

For a Bochner L^1-integrable function $f \in L^1(S, V)$, we can define the *Bochner integral* of f, written $\int_S f$, exactly as for the Lebesgue integral, using approximations of f by simple functions (which is possible thanks to the separability of V). This definition readily shows that the Bochner integral commutes with bounded operators on V.

The Bochner integral shares many properties with the usual Lebesgue integral, such as the subadditivity in norm or the σ-additivity over the subsets of S. There are however two major differences with the finite-dimensional case. The first one is the problem of the measurability of the sum alluded to in the beginning of this section; we got rid of it by assuming the target space V to be separable. The second one is the possibility to represent continuous (vector-valued) measures by integrable functions, namely the Radon–Nikodym theorem. The validity of the latter in the Bochner setting depends on the target Banach space V. But we shall not need this theorem, so that the reader unfamiliar with Bochner integral may safely consider it, for the present work, as a "Lebesgue integral with norms". And the unfamiliar but sceptical reader can find more about the Bochner integral e.g. in [**DU77**, Chapter II] or [**SY05**, Chapter 1].

Remark 4.3.2. — A satisfactory theory of Bochner integrable functions, on general Banach spaces, requires only the functions to be scalarly weakly measurable and essentially separably valued (that is, the image of some conull set is separable), cf. [**DU77**, Theorem ii.1.2, p. 42]. All the results of this chapter would be equally valid in that setting, but the subsequent applications to group representations motivated us to simplify the presentation by assuming the whole space to be separable (cf. Proposition 4.1.13).

Let now V be a dual Banach space (with some chosen predual $V_\flat$) and S a measure space. We write $\mathscr{L}_{w*}(S, V)$ for the space of scalarly weakly-$*$ measurable maps from S to V, and $L_{w*}(S, V)$ for its quotient of classes of almost everywhere equal functions. We will consider their respective subspaces $\mathscr{L}^\infty_{w*}(S, V)$ and $L^\infty_{w*}(S, V)$ of essentially bounded maps. The latter is a Banach space for the essential supremum norm. Moreover, it is isometric to the dual of $L^1(S, V_\flat)$ [**ITIT69**, VII.4].

Our favorite examples of (non-reflexive) dual Banach spaces are $L^\infty(\mu)$ for a σ-finite measure μ, like the Lebesgue measure, and $\mathscr{B}(\mathcal{H})$, the space of all bounded operators on a Hilbert space $\mathcal{H}$ (whose predual is the space of all trace-class operators).

Remark 4.3.3. — Although it will not play a role for us, we mention in passing that the scalar weak-$*$ σ-algebra is particularly tractable when the predual $V_\flat$ is separable (cf. Section D.2).

In the rest of this chapter, we will always be in one of these two situations: measurable maps to separable Banach spaces or scalarly weakly-$*$ measurable maps to a dual Banach space. G *and* H *will denote two locally compact second-countable groups. The Banach space* V *will be endowed with a linear representation* π *of* H *(or an affine representation* $\alpha = \pi + b$*), which is* not *assumed to be continuous. The measure space* S *will be either* G *with its Haar measure, or a tychomorphism from* H *to* G.

4.3.C Measured induction

In this section, we sketch how we can induce *isometric* representations without having a true subgroup, thanks to tychomorphisms. This *measured induction* has been developed in details for measure equivalence between countable groups in [**MS06**, Section 4]; it is used there for its fruitful interplay with bounded cohomology. The generalisation for tychomorphisms between countable groups is already pointed out in [**Mon06**, Section 5] and actually the proofs of [**MS06**, Section 4] can be carried through for general tychomorphisms, mutatis mutandis. So we only explain in this section the very basic case we will need in Chapter 6, namely the induction of a linear representation and of its associated cocycles.

Let Σ be a tychomorphism from H to G. Let π a linear representation of H on a dual Banach space V (with some chosen predual $V_\flat$). We consider the space $\mathscr{L}_{w*}(\Sigma, V)$ of all scalarly weakly-$*$ measurable maps from Σ to V, endowed with two actions of H given by precomposition on Σ and postcomposition (by π) on V, as well as with the action of G given by precomposition on Σ. Observe that all these actions commute with each other (since the given actions on Σ commute). These actions preserve the class of almost everywhere equal functions (the actions on Σ are not singular), and we can therefore consider them on the quotient space $\mathrm{L}_{\mathrm{w}*}(\Sigma, \mathrm{V})$. An element $f \in \mathrm{L}_{\mathrm{w}*}(\Sigma, \mathrm{V})$ is H-invariant for the diagonal action if for any $h \in \mathrm{H}$,

$$\pi(h)f(h^{-1}s) = f(s)$$

for almost all $s \in \Sigma$.

The L^∞*-measurably induced representation* $\mathrm{L}^\infty\Sigma\mathbf{ind}_\mathrm{H}^\mathrm{G}\mathrm{V}$ is the space $\mathrm{L}^\infty_{\mathrm{w}*}(\Sigma, \mathrm{V})^\mathrm{H}$ of H-invariant essentially bounded weakly-$*$ measurable maps from Σ to V, endowed with its G-action. The representation of G on the latter is given by the usual precomposition:

$$(\mathrm{L}^\infty\Sigma\mathbf{ind}_\mathrm{H}^\mathrm{G}\pi(g)f)(s) = f(g^{-1}s) \qquad (f \in \mathrm{L}^\infty\Sigma\mathbf{ind}_\mathrm{H}^\mathrm{G}\mathrm{V}).$$

This representation is an isometric one on a normed space. If we start with a *uniformly bounded* representation π, then the space $\mathrm{L}^\infty_{\mathrm{w}*}(\Sigma, \mathrm{V})^\mathrm{H}$ is necessarily nonzero. In general, if the representation π is not uniformly bounded, then this space may be zero or not. We will see in Proposition 4.3.5 that if π admits some nice cocycles, then $\mathrm{L}^\infty_{\mathrm{w}*}(\Sigma, \mathrm{V})^\mathrm{H}$ is nonzero. However, there does not seem to be an easy characterization of those representations that produce nonzero L^∞-measurably induced representations.

We first check that, trivial or not, this space is in any case a Banach space.

Lemma 4.3.4. — *The subspace* $\mathrm{L}^\infty_{\mathrm{w}*}(\Sigma, \mathrm{V})^\mathrm{H}$ *is closed.*

Proof. — Let f_n be a sequence in $\mathrm{L}^\infty_{\mathrm{w}*}(\Sigma, \mathrm{V})^\mathrm{H}$ converging to some $f \in \mathrm{L}^\infty_{\mathrm{w}*}(\Sigma, \mathrm{V})$. Let $h \in \mathrm{H}$. By definition of the essential supremum, the sequence f_n converges a fortiori almost everywhere to f. Let therefore Σ' be a measurable subset of full measure on which f_n converges pointwise to f and, in addition, let Σ_n be measurable subsets of full measure such that $\pi(h)f_n(h^{-1}s) = f_n(s)$ for any $s \in \Sigma_n$. Finally, let

$$\Sigma'' = \Sigma' \cap h\Sigma' \cap \bigcap_n \Sigma_n,$$

it is a set of full measure (H acts non-singularly on Σ) and, for any $s \in \Sigma''$, we have

$$f_n(s) \to f(s), \qquad f_n(h^{-1}s) \to f(h^{-1}s), \quad \text{and} \quad \pi(h)f_n(h^{-1}s) = f_n(s),$$

hence $\pi(h)h \cdot f = f$ almost everywhere, that is, $f \in \mathrm{L}^\infty_{\mathrm{w}*}(\Sigma, \mathrm{V})^\mathrm{H}$. ■

We now proceed to transfer properties from π to $\mathrm{L}^\infty\Sigma\mathbf{ind}_\mathrm{H}^\mathrm{G}\pi$. We observe that the measurability requirement on the cocycle in the below proposition is in particular fulfilled for continuous cocycles.

Proposition 4.3.5. — *Let* G *and* H *be locally compact second-countable groups. Let* Σ *be a tychomorphism from* H *to* G *and choose some retraction* $\chi\colon \Sigma \to \mathrm{H}$*. Let* π *be a representation of* H *on a dual Banach space* $\mathrm{V} = \mathrm{V}_\flat'$*. Let* $\mathrm{W} = \mathrm{L}^\infty\Sigma\mathbf{ind}_\mathrm{H}^\mathrm{G}\mathrm{V}$ *be the* L^∞*-measurably induced representation space and* $\rho = \mathrm{L}^\infty\Sigma\mathbf{ind}_\mathrm{H}^\mathrm{G}\pi$ *the induced representation of* G *on* W*.*

Let b *be a cocycle for the representation* π*, that we require to be essentially bounded and scalarly weakly-$*$ measurable as a map* $\mathrm{H} \to \mathrm{V}$*. For any* $g \in \mathrm{G}$*, we consider the map* $\beta(g)$ *defined (almost everywhere) as*

$$\beta(g)\colon \Sigma \to \mathrm{V}\colon s \mapsto b(\chi(g^{-1}s)) - b(\chi(s)).$$

1. *The map* $\beta(g)$ *is essentially bounded, scalarly weakly-$*$ measurable, and* H*-invariant (that is,* $\beta(g) \in \mathrm{W}$*).*

2. *The map* $\beta\colon g \mapsto \beta(g)$ *defines a bounded cocycle for the representation* ρ*.*

3. *Assume moreover that* π *is contragredient to some representation* $\pi_\flat$ *on* $\mathrm{V}_\flat$*. Then the cocycle* β *is trivial (if and) only if* b *is trivial.*

Proof. — 1. As G acts non-singularly on Σ and χ is measurable, the map $\beta(g)$ is the sum of two essentially bounded scalarly weakly-$*$ measurable maps, hence enjoys the same properties. Let $h \in \mathrm{H}$, we have, thanks to the H-invariance of χ,

$$\begin{aligned}\beta(g)(hs) &= b(\chi(g^{-1}hs)) - b(\chi(hs))\\ &= \pi(h)(b(\chi(g^{-1}s)) + b(h) - \pi(h)(b(\chi(s))) - b(h)\\ &= \pi(h)(\beta(g)(s)),\end{aligned}$$

for almost every $s \in \Sigma$. Hence the class of $\beta(g)$ lies in W.

2. For $g_1, g_2 \in \mathrm{G}$, we have

$$\begin{aligned}\beta(g_1g_2)(s) &= b(\chi((g_1g_2)^{-1}s)) - b(\chi(s))\\ &= \Big(b(\chi(g_2^{-1}g_1^{-1}s)) - b(\chi(g_1^{-1}s))\Big) + \Big(b(\chi(g_1^{-1}s)) - b(\chi(s))\Big)\\ &= \beta(g_2)(g_1^{-1}s) + \beta(g_1)(s),\end{aligned}$$

for almost every $s \in \Sigma$. Hence β defines a cocycle for ρ, which is obviously bounded by twice the supremum norm of b.

3. Assume that β is trivial and let $f \in \mathrm{W}$ be such that $\beta(g) = \rho(g)f - f$. By definition of β, this means that the function $\beta \circ \chi - f$ is a G-invariant element of $\mathrm{L}^\infty_{\mathrm{w}*}(\Sigma, \mathrm{V})$. It therefore defines an element φ of $\mathrm{L}^\infty_{\mathrm{w}*}(\mathrm{X}, \mathrm{V})$, where (X, θ) is an inert part for the G-action on Σ. We may and shall assume that $\theta(\mathrm{X}) = 1$.

Observe that, for any $h \in h$, we have $\varphi(h \cdot x) - \pi(h)\varphi(x) = b(h)$ almost everywhere, since f is H-invariant. We would like now to integrate this equality over the H-invariant measure θ in order to show that b is trivial. This could be done in a general framework of weakly-$*$ integration (the so-called *Gelfand–Dunford integral*, see e.g. [**Bou59**, VI, §1, n° 4]); instead, we use here an ad hoc argument. Since φ is essentially bounded and scalarly weakly-$*$ measurable, we can define a linear form v on $V_\flat$ by

$$\langle v, w\rangle = \int_X \langle \varphi(x), w\rangle \, d\theta(x) \qquad (w \in V_\flat). \tag{4.1}$$

As $|\langle \varphi(x), w\rangle| \leqslant \| \varphi \|_\infty \| w \|_{V_\flat}$ almost everywhere, the linear form v is actually continuous, hence belongs to the *topological* dual of $V_\flat$, namely V. Since π is contragredient to $\pi_\flat$, we also have, for all $w \in V_\flat$,

$$\begin{aligned}\langle \pi(h)v, w\rangle &= \langle v, \pi_\flat(h)w\rangle \\ &= \int_X \langle \varphi(x), \pi_\flat(h)w\rangle \, d\theta(x) \\ &= \int_X \langle \pi(h)\varphi(x), w\rangle \, d\theta(x).\end{aligned}$$

On the other hand, H preserves the measure θ (see Section 2.3), hence we may replace $\varphi(x)$ by $\varphi(h \cdot x)$ in the integral (4.1) and get

$$\langle v - \pi(h)v, w\rangle = \langle b(h), w\rangle.$$

for all $w \in V_\flat$. Therefore $b(h) = v - \pi(h)v$, for any $h \in H$: the cocycle b is trivial. ■

Remark 4.3.6. — A bounded cocycle for a *uniformly bounded* representation on a *reflexive* Banach space is always trivial (see Corollary 5.2.2 below). The above proposition is therefore interesting either for uniformly bounded representations on non-reflexive spaces or for non-uniformly bounded representations.

In the former case, we can rephrase the induction in the language of bounded cohomology: the above proof shows that there is a continuous injective map from $H^1_b(H, V)$ to $H^1_b(G, L^\infty\Sigma\mathbf{ind}_H^G V)$. Actually, much more is true: such a map exists at any degree, and is independent of the choice of a retraction χ. See [**MS06**, Section 4.3].

If we had started with a nice representation space for H, for instance a Hilbert space, we may be quite disappointed to get an L^∞_{w*}-space as an induced representation. Following the classical Mackey induction, we can try to define an L^p-measurably induced representation (p finite) as follows. Let now V be a separable Banach space. The space $\mathscr{L}(\Sigma, V)$ of norm-measurable maps from Σ to V is similarly endowed with three commuting actions, two of H and one of G, that preserve the classes of almost everywhere equal functions, hence can be considered on the quotient. Let as above $\mathrm{L}(\Sigma, \mathrm{V})^{\mathrm{H}}$ be the space of class of functions that are invariant under the diagonal action; that is, the class of measurable functions $f\colon \Sigma \to \mathrm{V}$ such that

$$\pi(h)f(h^{-1} \cdot s) = f(s)$$

for every $h \in \mathrm{H}$ and almost every $s \in \Sigma$. If we require now π to be *isometric*, the norm $\| f \|_{\mathrm{V}}$ is H-invariant, hence can be considered as a measurable map from (Y, η). The L^p*-measurably induced representation space* $\mathrm{L}^p\Sigma\mathbf{ind}_{\mathrm{H}}^{\mathrm{G}}\mathrm{V}$ is then the space

$$\left\{ f \in \mathrm{L}(\Sigma, \mathrm{V})^{\mathrm{H}} \;\middle|\; \| f \|_{\mathrm{V}} \in \mathrm{L}^p(\mathrm{Y}, \eta) \right\}$$

endowed with the norm given by

$$\| f \|^p = \int_{\mathrm{Y}} \| f \|_{\mathrm{V}}^p \,\mathrm{d}\eta.$$

The L^p*-measurably induced linear representation* $\mathrm{L}^p\Sigma\mathbf{ind}_{\mathrm{H}}^{\mathrm{G}}\pi$ on this space is defined by

$$(\mathrm{L}^p\Sigma\mathbf{ind}_{\mathrm{H}}^{\mathrm{G}}\pi(g)f)(x) = (\Delta_{\mathrm{G}}(g))^{\frac{1}{p}} f(g^{-1}x) \qquad (f \in \mathrm{L}^p\Sigma\mathbf{ind}_{\mathrm{H}}^{\mathrm{G}}\mathrm{V}).$$

One can easily check that all these definitions are legitimate, namely: they do not depend on a choice of an inert part Y and the representation is well-defined on the class of almost everywhere equal functions. The factor $(\Delta_{\mathrm{G}}(g))^{\frac{1}{p}}$ ensures an isometric representation.

Unfortunately, as far as affine actions are concerned, this L^p variant is hard to handle. Indeed, for the “formal” cocycle $\beta(g)$ of Proposition 4.3.5 to actually lie in $\mathrm{L}^p\Sigma\mathbf{ind}_{\mathrm{H}}^{\mathrm{G}}\mathrm{V}$, we would need to know some integrability of the map $b \circ \chi$. The easiest way to get the latter would be to work with a measure equivalence instead of a tychomorphism (so that the H-inert part has finite measure) and a bounded cocycle b. But the latter would be trivial for an isometric representation on a reflexive Banach space (see Remark 4.3.1), hence the cocycle $\beta(g)$ would also be trivial… To produce new affine actions on a Hilbert space, we will therefore need the moderate variant of the induction, explained in Sections 4.3.D and 4.3.E.

Remark 4.3.7. — The problem in the above discussion is specific to 1-cocycles (that correspond to affine actions): cocycles of higher degree may be bounded and nontrivial even for isometric representations on Hilbert spaces. Monod and Shalom were able to induce such 2-cocycles, see [**MS06**, Proposition 4.4].

Remark 4.3.8. — We note that, as in the classical case, we have a “twisted” model for the induced representation. Choose an inert part (Y, η) for H-action on the tychomorphism

Σ. We can view the space $L^\infty\Sigma\mathbf{ind}_H^G V$ as the space $L^\infty_{w*}(Y, V)$, endowed with the following twisted action of G:

$$(L^\infty\Sigma\mathbf{ind}_H^G\pi(g)f)(y) = \pi(\beta(g^{-1}, y)^{-1})f(g^{-1}y) \qquad (f \in L^\infty\Sigma\mathbf{ind}_H^G V).$$

A similar formula holds for the L^p version (with a $(\Delta_G(g))^{\frac{1}{p}}$ factor). These models have the advantage to identify the representation space with a genuine L^∞_{w*} or L^p space, which can be technically useful (as in the proof of Theorem 6.1.5 in Chapter 6). However, the isometries between these models and the above one depend on a choice of an inert factor.

4.3.D Moderate induction

We give here the moderate variant of classical induction for subgroups. It will be slightly more convenient to induce directly an affine representation, rather than to induce a linear representation and then each of its cocycles separately. The reason is the following. For measured induction, we could always induce a linear representation, possibly getting a trivial space. We then used nice cocycles to get some nontriviality results about that space. For moderate induction, the corresponding construction could yield an *empty* space—hence, strictly speaking, not a vector space—unless we have some nice cocycles. So one might as well directly start with an affine representation associated to some nice cocycle.

Let H be a closed subgroup of a second-countable locally compact group G. Let $\alpha = \pi + b$ an affine representation of H on a separable Banach space V. We consider, as in the classical case, the space $\mathscr{L}(G, V)$ of all norm-measurable maps from G to V, where G is viewed as a measurable space (for the class of Haar measures). This space is endowed with three actions: a linear G-action given by precomposition by *left* translations, a similar linear H-action given by *right* translations, and an *affine* H-action given by postcomposition by α. All these actions commute with each other and preserve classes of almost everywhere equal functions. These actions are therefore well-defined on the quotient space $L(G, V)$ and we can consider $L(G, V)^H$, the affine G-space of classes invariant for the diagonal H-action. More precisely, a function class $f \in L(G, V)$ is in $L(G, V)^H$ if and only if, for all $h \in H$,

$$\alpha(h)f(xh) = f(x)$$

for almost all $x \in G$.

To get a Banach space, we will further introduce another G-subspace of $L(G, V)$. Let μ be a moderate measure on G (Corollary 4.2.10). For any $1 \leqslant p \leqslant \infty$, consider the space $L^p((G, \mu), V)$ of Bochner L^p-integrable functions. (If $p = \infty$, this space depends only the measure class; for p finite, it depends on the choice of μ.) To avoid lengthy notation, we will write G_μ for (G, μ) (hence $L^p(G_\mu, V)$ for $L^p((G, \mu), V)$).

We first observe the following generalization of Proposition 4.2.12.

Proposition 4.3.9. — *The linear representation of* G *on* $L(G, V)$ *preserves* $L^p(G_\mu, V)$ *and* G *acts on the latter by continuous transformations. Moreover, the deduced representation on* $L^p(G_\mu, V)$ *is continuous if* p *is finite.*

Proof. — The case $p = \infty$ is immediate since G-action then preserves the norm of $L^\infty(G, V)$. Let then p be finite, $f \in L^p(G_\mu, V)$ and $g \in G$. We observe, as in the scalar case, that

$$\| g \cdot f \|_{L^p(G_\mu, V)} \leqslant \left\| \frac{dg^{-1}\mu}{d\mu} \right\|_\infty^{\frac{1}{p}} \| f \|_{L^p(G_\mu, V)}.$$

By definition of a moderate measure, this shows that G indeed preserves $L^p(G_\mu, V)$ and acts on it by continuous transformations. Moreover, the deduced representation on the latter space is locally bounded.

We therefore only need to check orbital continuity on a dense linear subspace. We can argue as in Proposition 4.2.12 and work with indicator functions (that is, measurable maps $G_\mu \to V$ with only one nonzero value) whose support has a finite Haar measure. Alternatively, we can observe that, on the dense subspace $L^p(G_\mu) \otimes V$, the representation is nothing but the tensor representation of the moderately regular representation and of the trivial one, which is then orbitally continuous by Proposition 4.2.12. ∎

We can now define the L^p*-moderately induced representation*, $L^p\mathbf{ind}_H^{G,\mu}V$, as the intersection

$$L(G, V)^H \cap L^p(G_\mu, V) = \{f \in L^p(G_\mu, V) \mid \forall h \in H,\ \alpha(h)h \cdot f = f\}.$$

What is *not* obvious is the fact that this space is nonempty and closed in $L^p(G_\mu, V)$, this is addressed by the first half of the next proposition. However, granted these two facts, this space obviously becomes a Banach space endowed with an affine representation of G, since it is the intersection of two G-invariant subspaces of $L(G, V)$, an affine one and a linear one. Moreover, this affine representation is continuous for p finite and isometric for $p = \infty$, since it is a subrepresentation of a representation with the same properties.

Proposition 4.3.10. — *Let* G *be a second-countable locally compact group endowed with a moderate measure* μ*. Let* H *be a closed subgroup of* G *and* α *an affine representation of* H *on a separable Banach space* V*, whose cocycle* b *is a measurable map from* H *to* V*. Let* $W = L^p\mathbf{ind}_H^{G,\mu}V$ *be the* L^p*-moderately induced representation space, endowed with its* G*-action.*

1. *The space* W *is closed in* $L^p(G_\mu, V)$.
2. *If* V *contains a bounded orbit, then so does* W *(in particular,* W *is non-empty).*
3. *If* W *contains a fixed point, then so does* V.
4. *Assume* G *is* discrete *and* p *is finite. If* V *contains a relatively compact orbit, then so does* W.

Proof. — 1. We will give a general argument below (Theorem 4.3.12). Here, an ad hoc shortcut is available. Assume that the moderate measure μ also enjoys some bound for the right translates, that is

$$\frac{d\mu(xg)}{d\mu(x)} \leqslant C_g,$$

for some constant C_g independent of x. (Observe that this is indeed the case for the moderate measures we built in Proposition 4.2.8.) Then the action of H on L(G, V) given by precomposition by right translations actually *preserves* $L^p(G_\mu, V)$, since

$$\| h \cdot f \|^p \leqslant C_h \| f \|^p$$

(for $f \in L(G_\mu, V)$ with p finite; for $p = \infty$, the norm is in any case H-invariant). Therefore, for any $h \in H$, the map $f \mapsto \alpha(h) h \cdot f$ is continuous, hence W is closed.

2. Up to conjugating, we may assume that the cocycle b itself is bounded. Choose a right H-equivariant Borel map $\chi\colon G \to H$ (as yielded by an obvious modification of the left equivariant one given in Example 2.1.4). Define now, for any $g \in G$, the following measurable map

$$\beta(g)\colon G \to V\colon x \mapsto b(\chi(g^{-1}x)^{-1}).$$

As b is bounded, then so are the maps $\beta(g)$, hence in particular they belong to $L^p(G_\mu, V)$ for any p. Moreover, thanks to the cocycle identity of b, we have, for any $g, x \in G$ and $h \in H$,

$$\begin{aligned}\beta(g)(xh) = b(\chi(g^{-1}xh)^{-1}) &= b(h^{-1}\chi(g^{-1}x)^{-1}) \\ &= \pi(h^{-1})b(\chi(g^{-1}x)^{-1}) + b(h^{-1}) \\ &= \alpha(h^{-1})\beta(g)(x).\end{aligned}$$

Hence the maps $\beta(g)$ belong to W. Moreover, we obviously have $\beta(g) = g \cdot \beta(e)$, and all maps $\beta(g)$ are bounded by $\| b \|_\infty$, hence W contains a bounded orbit.

3. Assume that $\xi \in W$ is fixed by G. As G acts transitively on itself, this forces ξ to be almost everywhere constant. Let $v \in V$ be the essentially unique value taken by

ξ. As $\xi \in W$, we must have $\alpha(h)v = v$ for all $h \in H$, hence V contains a fixed point.

4. Assume that the cocycle b has a relatively compact image and consider again the functions $\beta(g)$ built above. We will show that the orbit $\{\beta(g)\}$ is relatively compact in W. As the maps $\beta(g)$ share a common relatively compact range (namely, $b(H)$), we only need to show that, for any $\lambda \in V'$, the set $\{\lambda \circ \beta(g) \mid g \in G\}$ is relatively compact in $\ell^p(G_\mu)$ (cf. [**Nee14**, Lemma 2]). But $\{\lambda \circ \beta(g)\}$ is actually a bounded set in $\ell^\infty(G)$, hence is relatively compact in $\ell^p(G_\mu)$ since the inclusion operator $\ell^\infty(G) \hookrightarrow \ell^p(G_\mu)$ is compact for $1 \leqslant p < \infty$ (the measure μ is finite and atomic). ■

4.3.E Moderate measured induction

We will now extend the moderate induction to the case where we have a tychomorphism instead of a true subgroup. The strategy is very similar to the subgroup case, but needs some measure-theoretical arguments to become rigorous.

Let Σ be a tychomorphism from H to G. Let $\alpha = \pi + b$ an affine representation of H on a separable Banach space V. We consider the space $\mathscr{L}(\Sigma, V)$ of all measurable maps from Σ to V. This space is endowed with three actions: two linear actions of G and H given by precomposition on Σ and a *affine* action of H given by postcomposition by α. All these actions commute with each other and preserve classes of almost everywhere equal functions (since the actions of G and H on Σ commute and are non-singular). These actions are therefore well-defined on the quotient space $L(\Sigma, V)$ and we can consider $L(\Sigma, V)^H$, the affine G-space of classes invariant for the diagonal H-action. More precisely, a function class $f \in L(\Sigma, V)$ is in $L(\Sigma, V)^H$ if and only if, for all $h \in H$,

$$\alpha(h) f(h^{-1}s) = f(s)$$

for almost all $s \in \Sigma$.

To get a Banach space, we will further introduce another G-subspace of $L(\Sigma, V)$. Let μ be a moderate measure on G (Corollary 4.2.10). We first transfer this measure to Σ, as follows. We choose once and for all some G-equivariant isomorphism

$$(\Sigma, m) \simeq (G, \check{m}_G) \times (X, \theta)$$

where $\check{m}_G$ is a right Haar measure on G and (X, θ) is some standard probability space. By pushing the product measure $\mu \times \theta$ through that isomorphism, we get a probability m_μ on Σ, in the same class as m. We can now consider, for any $1 \leqslant p \leqslant \infty$, the space $L^p((\Sigma, m_\mu), V)$ of Bochner L^p-integrable functions. (If $p = \infty$, this space depends only the measure class; for p finite, it depends on the choice of μ.) To avoid lengthy notation, we will write Σ_μ for (Σ, m_μ) (hence $L^p(\Sigma_\mu, V)$ for $L^p((\Sigma, m_\mu), V)$).

We first observe the following generalization of Proposition 4.2.12.

Proposition 4.3.11. — *The linear representation of* G *on* $L(\Sigma, V)$ *preserves* $L^p(\Sigma_\mu, V)$ *and* G *acts on the latter by continuous transformations. Moreover, the deduced* G-*representation on* $L^p(\Sigma_\mu, V)$ *is continuous if p is finite.*

Proof. — Let $f \in L^p(\Sigma_\mu, V)$ and $g \in G$. The case $p = \infty$ is obvious (and the representation is even isometric), since G acts non-singularly on Σ. For p finite, we check, by Fubini's theorem:

$$\begin{aligned} \| g \cdot f \|^p_{L^p(\Sigma_\mu, V)} &= \int_\Sigma \| g \cdot f \|^p_V \, dm_\mu \\ &= \int_{G \times X} \| f(g', x) \|^p_V \frac{dg^{-1}\mu}{d\mu}(g') \, d\mu(g') \, d\theta(x) \\ &\leqslant \left\| dg^{-1}\mu / d\mu \right\|_\infty \| f \|^p_{L^p(\Sigma_\mu, V)} \end{aligned}$$

Hence G preserves $L^p(\Sigma_\mu, V)$ by the very definition of a moderate measure, and it acts on the latter by continuous linear maps.

The above computation showed that the representation is locally bounded, thanks to moderateness. We then only need to prove orbital continuity on a dense subset (Lemmas 4.1.6 and 4.1.9). Using the amplification $\Sigma \simeq G \times X$, we can find in $L^p(\Sigma_\mu, V)$ the dense subspace $L^p(G_\mu) \otimes L^p(X) \otimes V$, on which λ is given by the tensor representation of the moderately regular representation (which is continuous, cf. Proposition 4.2.12) and of the trivial ones, hence is orbitally continuous. ■

We can now define the L^p-*moderately measurably induced representation*, $L^p\Sigma\mathbf{ind}_H^{G,\mu}V$, as the intersection

$$L(\Sigma, V)^H \cap L^p(\Sigma_\mu, V) = \{ f \in L^p(\Sigma_\mu, V) \mid \forall h \in H, \ \alpha(h) h \cdot f = f \} .$$

Once again, it is *not* obvious that this space is nonempty and closed in $L^p(\Sigma_\mu, V)$, this is addressed by the first two thirds of the next theorem. However, granted these two facts, this action is indeed an affine representation of G on a Banach space, since it is the intersection of two G-invariant subspaces of $L(\Sigma, V)$, an affine one and a linear one. Moreover, this affine representation is continuous for p finite and isometric for $p = \infty$, since it is a subrepresentation of a representation with the same properties.

Theorem 4.3.12. — *Let* G *and* H *be locally compact second-countable groups. Let* Σ *be a tychomorphism from* H *to* G*,* μ *be a moderate measure on* G *and* α *an affine action of* H *on a separable Banach space* V*, whose cocycle* b *is a measurable map from* H *to* V*. Let* $W = L^p\Sigma\mathbf{ind}_H^{G,\mu}V$ *be the* L^p *moderate measured induced representation space* $(1 \leqslant p \leqslant \infty)$*, endowed with its* G*-action.*

1. *The space* W *is closed in* $L^p(\Sigma_\mu, V)$*.*
2. *If* V *contains a bounded orbit in* V*, then so does* W *(in particular,* W *is nonempty).*
3. *If* W *has a fixed point, then so does* V*.*

Proof. — 1. Let f_n be a sequence of functions in W, converging in L^p to a function f in $L^p(\Sigma_\mu, V)$, and let $h \in H$. If $p = \infty$, the sequence f_n a fortiori converges almost everywhere to f. If p is finite, then a subsequence of f_n converges almost everywhere to f by Riesz's theorem[12]. So in any case we may assume, up to extracting, that f_n converges almost everywhere to f; let Σ' be a measurable subset of full measure on which f_n converges pointwise to f. In addition, let Σ_n be a measurable subset of full measure such that $\alpha(h)f_n(h^{-1}s) = f_n(s)$ for any $s \in \Sigma_n$. Finally, let

$$\Sigma'' = \Sigma' \cap h\Sigma' \cap \bigcap_n \Sigma_n,$$

it is a set of full measure (H acts non-singularly on Σ) and, for any $s \in \Sigma''$, we have

$$f_n(s) \to f(s), \qquad f_n(h^{-1}s) \to f(h^{-1}s), \quad \text{and} \quad \alpha(h)f_n(h^{-1}s) = f_n(s),$$

hence $\alpha(h)h \cdot f = f$ almost everywhere. This holds for any h, so $f \in W$.

2. Up to conjugating, we may assume that the cocycle b itself is bounded. Choose some H-equivariant retraction $\chi\colon \Sigma \to H$ (cf. Section 2.1) and define now, for any $g \in G$, the following measurable map

$$\beta(g)\colon \Sigma \to V\colon s \mapsto b(\chi(g^{-1}s)).$$

As b is bounded, then so are the maps $\beta(g)$, hence in particular they belong to $L^p(\Sigma_\mu, V)$. Moreover, thanks to the cocycle identity of b, we have, for any $g \in G$,

[12] Riesz's theorem for Bochner integral follows by applying the classical scalar result to the sequence of scalar functions $\| f_n - f \|$. For a proof of the scalar case, see [**Bog07**, Theorem 2.2.5].

$h \in \mathrm{H}$, and almost all $s \in \Sigma$,

$$\begin{aligned}\beta(g)(h^{-1}s) &= b(\chi(g^{-1}h^{-1}s)) \\ &= b(h^{-1}\chi(g^{-1}s)) \\ &= \pi(h^{-1})b(\chi(g^{-1}s)) + b(h^{-1}) \\ &= \alpha(h^{-1})\beta(g)(s).\end{aligned}$$

Hence the maps $\beta(g)$ belong to W. Moreover, we obviously have $\beta(g) = g \cdot \beta(e)$, and all maps $\beta(g)$ are bounded by $\| b \|_\infty$, hence W contains a bounded orbit.

3. Assume that G fixes some point in W. As G acts transitively on itself, this defines a measurable map $\varphi\colon \mathrm{X} \to \mathrm{V}$, which is H-equivariant. Let

$$v = \int_{\mathrm{X}} \varphi(x)\, \mathrm{d}\theta(x) \quad \in \mathrm{V}$$

For any $h \in \mathrm{H}$, the Bochner integral commutes with the bounded linear operator $\pi(h)$, hence the above average commutes with the affine operator $\alpha(h)$ (recall that X is a probability space). Since φ is H-equivariant and H preserves the measure θ, the point v is then fixed by H. ■

Remark 4.3.13. — In the above proof, behold the fact that W is closed *although* H does not (a priori) preserve the space $\mathrm{L}^p(\Sigma_\mu, \mathrm{V})$. Indeed, the trick we used in Proposition 4.3.10 to prove the invariance by H used the fact that the Radon–Nikodym derivative of the translate $h\mu$ with respect to the moderate measure μ can be controlled by a bounded factor. But for tychomorphisms, an estimate of that derivative would require a sharp control on the cocycle $\mathrm{H} \times \mathrm{X} \to \mathrm{G}$.

Remark 4.3.14. — For discrete groups, in comparison to the induction with respect to a genuine subgroup (Proposition 4.3.10), we lost the statement about relatively compact orbits. There is indeed no reason to expect $\{\beta(g)\}$ to be relatively compact in $\mathrm{L}^p(\Sigma_\mu, \mathrm{V})$ as soon as the measure space Σ is non-discrete. On the other hand, we saw in Proposition 2.6.1 that a discrete tychomorphism basically reduces itself to a genuine subgroup.

5 ACTIONS ON CONVEX SETS

To which extent does a group unveil itself through its affine actions? In particular, can we recover some algebraic or topological features from these geometric actions? These broad questions will be approached in this chapter through fixed-point properties.

The first section reviews some general properties of actions on compact convex sets and in particular how they are related to an ambient locally convex space. The second section then studies the interplay between fixed-point properties and some group properties. The goal is to reach a new characterization of amenability, Theorem 5.2.21, for which induction techniques through tychomorphisms will be used. But in order to give some context to that theorem, we first review other fixed-point properties that are interesting per se (such as characterizations of finite or compact groups as well as the Day–Rickert characterization of amenability), hence the first two thirds of this chapter can be read independently.

5.1 Canonical equivariant embedding of compact convex sets

On a vector space V, a finite sum

$$x = \sum_i \lambda_i x_i \qquad (\lambda_i \in \mathbf{R},\ x_i \in \mathrm{V})$$

is the common form of four geometrically different kinds of points:

- a linear combination (general case),
- an affine combination ($\sum_i \lambda_i = 1$),
- a conical combination ($\lambda_i \geqslant 0$),
- or a convex combination ($\sum_i \lambda_i = 1$ and $\lambda_i \geqslant 0$).

This sum will respectively belong to the linear span, the affine span, the conical hull or the convex hull of the x_i's. Consequently, a transformation $\mathrm{T}\colon \mathrm{V} \to \mathrm{W}$ to another vector space can be compelled in four ways to preserve the spatial structure of vector spaces, namely, applying the condition "the image of a combination is the corresponding combination of the images" to one of the four cases above. The first one is the usual linearity; the second one is called affinity. It is easy to check that an affine transformation is the composition of a linear map and a translation (therefore, an affine map is linear if and only if it preserves the origin). The other two cases don't have a specific name because, as easily proved, preserving conical combinations is enough to be linear and preserving convex combination is enough to be affine[1].

Our focus in this chapter will be on (nonempty) convex subsets $\mathrm{K} \subseteq \mathrm{V}$ and transformations $\mathrm{T}\colon \mathrm{K} \to \mathrm{K}$. We will similarly call T affine if it preserves convex combinations. A first and easy observation, recalled in the next lemma, is that the convex structure of K viewed abstractly is not richer than the convex structure of K as a subset. Let the reader be relieved: these somewhat trite observations will soon become more exciting, when topology comes into play.

Lemma 5.1.1. — *Let* K *be a convex subset of a vector space* V *and* $\mathrm{V_K}$ *the affine span of* K *in* V*. There is an isomorphism between*

- *the semigroup of affine transformations from* K *to* K
- *and the semigroup of affine transformations from* $\mathrm{V_K}$ *to* $\mathrm{V_K}$ *that preserve* K*.*

Proof. — The restriction to K is obviously an injective morphism from the second semigroup into the first one. For surjectivity, observe that, whenever $x_i \in \mathrm{K}$ and $\sum_i \lambda_i = 1$, the point $\sum_i \lambda_i \mathrm{T}(x_i)$ depends only on the point $\sum_i \lambda_i x_i$ and not on the specific choice of combination, so that $\widetilde{\mathrm{T}}(\sum_i \lambda_i x_i) = \sum_i \lambda_i \mathrm{T}(x_i)$ is a well-defined affine map from $\mathrm{V_K}$ to $\mathrm{V_K}$ that extends T. ■

The *algebraic* structure of a vector space is a bit dull (since, up to the axiom of choice, it depends only on the dimension) and thus so are affine actions on vector space. Fortunately, the landscape is much more varied for *topological* vector spaces, thence our focus will be on topological groups acting (orbitally) continuously on some topological vector space V.

We will actually be even more interested in the case of an action on some convex subset K. However, caution is in order, because Lemma 5.1.1 is not valid as such for continuous transformations. Indeed, not only $\mathrm{V_K}$ may be not closed in V, but also the extension $\widetilde{\mathrm{T}}$

[1] As the name "convex" is then free, "convex transformation" usually has a different, weaker, meaning in analysis (a map to **R** whose epigraph is convex) or geometry (a maps that sends convex sets on convex sets).

built in the proof need not be continuous, so may not extend to the closure of V_K in V. This phenomenon already occurs in the simplest situation, where V is a separable Hilbert space, as shown below.

Example 5.1.2. — Let $\mathcal{H} = \ell^2(\mathbf{Z} \setminus \{0\})$ and consider the compact convex subset K of sequences $\mathbf{x} = (x_n)$ such that

$$|x_n| \leqslant n^{-1} \quad \text{and} \quad |x_{-n}| \leqslant n^{-2} \qquad \text{for all } n > 0.$$

Observe that K is total in $\mathcal{H}$ since it contains multiples of Dirac masses. Define on K the affine transformation $T\colon K \to K$ by

$$(T(\mathbf{x}))_n = nx_{-n} \quad \text{and} \quad (T(\mathbf{x}))_{-n} = \frac{x_n}{n} \qquad \text{for any } n > 0.$$

This is a continuous involution of K. However, any linear extension of T to the whole space $\mathcal{H}$ would need to send δ_{-n} to $n\delta_n$ for $n > 0$, which cannot be continuous.

Actually, a deeper reason prevents Lemma 5.1.1 to hold in a topological setting: many compact convex sets forget all about the locally convex space into which they live, as shown by the celebrated following results of V. Klee [**Kle55**, (1.1) on p. 31].

Proposition 5.1.3 (Klee). — *Let* K *be a compact convex subset of a locally convex space* V*. Assume that there exists a countable set of continuous affine maps* $K \to \mathbf{R}$ *that separate points of* K*. Then* K *is affinely homeomorphic to a compact convex subset of a Hilbert space.*

(The affine maps are not assumed to be defined on the whole space V.)

Proof. — Let $\{f_n\}_{n\geqslant 1}$ be a countable set of continuous affine maps that separate the points of K. Up to rescaling, we may assume that the maximum of $|f_n|$ on K is n^{-1}. Then the map

$$T\colon K \to \ell^2(\mathbf{N})\colon k \mapsto (f_1(k), f_2(k), f_3(k), \dots)$$

is affine, continuous, and injective. Hence it induces a homeomorphism of K onto its image. ■

Corollary 5.1.4 (Klee). — *Compact convex subsets with a countable separating set of continuous linear forms are characterized, up to homeomorphisms, by their algebraic dimension.*

Proof. — Keller proved that compact convex subsets of *Hilbert* spaces are characterized, up to homeomorphisms, by their algebraic dimension (see [**Kel31**, § 6 and § 4] or e.g. [**BP75**, Chapter III, § 3] for a more modern exposition). But, by Proposition 5.1.3, the Hilbert assumption is not a restriction. ■

Klee's results apply in particular for any compact convex subset of a space with separable dual. In general, we cannot unfortunately weaken the assumption into a mere separability of the compact convex set (see Remark 5.1.6 below). But less surprise arises in normed spaces, where this assumption is automatically satisfied, as shown by the following example.

Example 5.1.5. — Any compact convex subset K of a *normed* space admits a countable separating family of continuous affine maps. Indeed, such a set is separable and so contained in a multiple of the (usually non-compact) balanced closed convex hull C of some countable set of points x_n of norm 1 (and we may assume the family $\{x_n\}$ to be algebraically free). As we are in a normed space, C can be equivalently described as the set of all points of the form $\sum_n \lambda_n x_n$, where λ_n are real numbers such that $\sum_n |\lambda_n| \leqslant 1$. For all natural numbers i, j, choose by the Hahn–Banach theorem a continuous linear form $f_{i,j}$ of norm 1 such that

$$\begin{aligned} f_{i,j}(x_i) &= 1 \\ f_{i,j}(x_n) &= 0 \quad \text{for all } 0 \leqslant n \leqslant j,\ n \neq i. \end{aligned}$$

The countable family $\{f_{i,j}\}$ separates the points of C (hence of K). Indeed, otherwise we could find (by linearity) a sequence λ_n of real numbers such that $\sum_n |\lambda_n| \leqslant 2$ and

$$x = \sum_n \lambda_n x_n \neq 0 \quad \text{but} \quad f_{i,j}(x) = 0 \quad \text{for all} \quad i, j.$$

We would then have, for any $j \geqslant i$,

$$|\lambda_i| \leqslant \sum_{n>j} |\lambda_n|.$$

Letting j tend to infinity, we would have $\lambda_i = 0$ for all i, a contradiction.

Remark 5.1.6. — We cannot drop the "dual-separability" hypothesis in Proposition 5.1.3, since a compact set in a Hilbert space admits such a countable separating family (indeed, it must lie in a separable Hilbert subspace, since it is itself separable). We can however ask if Corollary 5.1.4 still holds without such a hypothesis. The answer is, by far, negative. For instance, J. Lindenstrauss gave in [**Lin72**, p. 269, Remark 1] an example of a weakly compact convex set of a Banach space which is not homeomorphic to its closed balanced convex hull. More recently, an example of two equivalent norms on a non-separable Hilbert space such that their respective unit balls are not homeomorphic in their weak topology was given in [**Avi07**, Theorem 4]. Even more striking is the result of [**Avi08**]: for any uncountable cardinal κ, there are 2^{κ} non-homeomorphic weakly compact convex sets of weight[2] κ in the Hilbert space of dimension κ.

Getting back to Klee's result, we could also ask whether any compact convex set is affinely homeomorphic to a *weakly* compact convex subset of a Hilbert or maybe of a Banach space. Once again, the answer is negative. Indeed, in a weakly compact (convex or not) subset of a Banach space, the points that are G_δ form a dense subset [**Lin72**, Theorem 3.8]. However, there are compact convex sets without any G_δ point (for instance, $[-1,1]^{I} \subset \mathbf{R}^{I}$, when I is uncountable and $\mathbf{R}^{I}$ is endowed with the product topology). Alternatively, weakly compact sets in Banach spaces are weakly sequentially compact by the Eberlein–Šmulian theorem, but there are compact convex sets in general locally convex spaces that are not sequentially compact (we will give a whole family of them in Lemma 5.2.26).

The only positive result around these ideas is the fact that a weakly compact set C of a Banach space is affinely homeomorphic to a weakly compact set of $c_0(X)$ for some discrete (maybe uncountable) space X (convexity plays no role here), see [**AL68**, Main Theorem]. Moreover, contrary to what happens for Klee's result, the homeomorphism extends to a continuous linear injection of the closed subspace spanned by C into $c_0(X)$.

The moral of Proposition 5.1.3 is that, for many compact convex sets, affine homeomorphisms of K are unlikely to admit an extension to a given ambient space, since K may be embedded into topologically very different vector spaces. This raises a question: is there at least a *canonical* topological vector space V such that K equivariantly embeds into V? More precisely, we are looking for a space V such that the affine homeomorphisms of K are precisely the restrictions to K of affine homeomorphisms of V that preserve K.

The answer is given by the following result. Apart maybe from the last statement about topological groups, it seems to be a common folklore in convexity of the sixties and seventies.

Proposition 5.1.7. — *Let* K *be a nonempty compact convex subset of a locally convex space* V*. Let* $\mathscr{A}$ *be the Banach space of all affine continuous maps* $K \to \mathbf{R}$*, endowed with the supremum norm, and* $\mathscr{A}'$ *its dual, endowed with the weak-$*$ topology. Let* K' *be the subset of* $\mathscr{A}'$ *made of all the positive linear functionals that send* $\mathbb{1}_K$ *to* 1*.*

1. *The evaluation map*
$$\mathrm{ev}\colon K \to \mathscr{A}' : k \mapsto [f \mapsto f(k)]$$

[2] The *weight* of a topological space is the smallest cardinal of a base of open sets.

induces an affine homeomorphism between K *and* K′.

2. *Any continuous affine transformation* $\mathrm{T}\colon \mathrm{K} \to \mathrm{K}$, *when viewed as a map from* K′ *to* K′ *via the above identification, extends uniquely as a continuous linear map* $\widetilde{\mathrm{T}}\colon \mathscr{A}' \to \mathscr{A}'$.

   In particular, there is an abstract isomorphism between the semigroup of all affine transformations of K *and the semigroup of all affine transformations of* $\mathscr{A}'$ *that preserve* K *(and thence also between their respective subgroup of affine homeomorphisms). Any affine action of an abstract group on* K *admits a unique linear extension on* $\mathscr{A}'$.

3. *Let* G *be a topological group acting jointly continuously on* K *by affine transformations. Then the extension of the action to* $\mathscr{A}'$ *is orbitally continuous.*

Proof. — 1. The evaluation map is obviously affine, injective (as the elements of V′ separate points of V and define, by restriction, elements of $\mathscr{A}$) and continuous (by the very definition of the weak-$*$ topology). As K is compact, we automatically get a homeomorphism between K and its image; let us show that the latter is K′. Let $\mu \in \mathrm{K}'$; by the Hahn–Banach theorem, we can extend μ to a positive linear functional $\widetilde{\mu}$ on $\mathscr{C}(\mathrm{K})$. The latter is a probability measure on K (as $\widetilde{\mu}(\mathbb{1}_{\mathrm{K}}) = \mu(\mathbb{1}_{\mathrm{K}}) = 1$), hence admits a barycenter k defined by the property that

$$f(k) = \widetilde{\mu}(f)$$

for any $f \in \mathscr{A}$ (see Section B.5). But $\widetilde{\mu}(f) = \mu(f)$ for such f, hence the above equation precisely means that $\mu = \mathrm{ev}(k)$.

2. Any continuous affine transformation $\mathrm{T}\colon \mathrm{K} \to \mathrm{K}$ defines a transformation on $\mathscr{A}$ by precomposition, hence also a transformation $\widetilde{\mathrm{T}}$ on $\mathscr{A}'$, still by precomposition: to be precise, for any $\lambda \in \mathscr{A}'$ and $f \in \mathscr{A}$, we have $\widetilde{\mathrm{T}}(\lambda)(f) = \lambda(f \circ \mathrm{T})$. Observe that $\widetilde{\mathrm{T}}$ is still continuous and affine. Moreover, for $k \in \mathrm{K}$, we have

$$\widetilde{\mathrm{T}}(\mathrm{ev}(k))(f) = (f \circ \mathrm{T})(k) = \mathrm{ev}(\mathrm{T}(k))(f),$$

hence $\widetilde{\mathrm{T}}$ indeed extends T when the latter is viewed as a map from K′ to K′. Unicity of continuous *linear* extensions of T is obvious from the fact that the cone spanned by K′, namely the positive linear functionals, admits K′ as a base and generates $\mathscr{A}'$ [**AT07**, Thereom 2.40].

3. Let G be a topological group with a jointly continuous affine action on K. The associated action on $\mathscr{A}$ defined by precomposition as above is then orbitally continuous (cf. Proposition C.2.2 and Lemma C.2.5). Hence the contragredient representation on $\mathscr{A}'$ is weakly-$*$ continuous. ■

In particular, getting back to Remark 5.1.6, we observe that any compact convex set is affinely homeomorphic to a weakly-$*$ compact convex set of a dual Banach space. But this probably hints more towards the wildness of the weak-$*$ topology than towards the tamability of compact convex sets.

Remark 5.1.8. — The space V was supposed locally convex only to ensure that there are enough affine maps $K \to \mathbf{R}$ to separate the points of K. Recall that an infinite-dimensional non-locally convex topological vector space may well have a trivial dual [**Bou65**, IV, §6, exerc. 13.d].

Remark 5.1.9. — Proposition 5.1.7 showed that the topological convex structure of a compact convex subset K of a space V, whereas defined as the restriction of that of V, is in fact intrinsic. This suggests the possibility to define *abstractly* a compact convex set, without using an ambient vector space. See for instance [**Alf71**, p. 81–82] for an abstract definition of a compact convex set as a pair (K, A) of a compact space K and a suitable subspace A of $\mathscr{C}(K)$.

5.2 Affine fixed-point properties

Let us now focus on the following broad question: what groups, if any, can act affinely on some class of locally convex spaces or of convex subsets *without* having a global fixed point? The reason why we want to exclude a fixed point is because it witnesses a "false" affinity: the affine action is only the conjugation of a linear one.

Broadly speaking, two opposite phenomena occur for fixed-point properties. Either the class of locally convex spaces or affine actions thereon is so rigid that any group will have a fixed point with respect to them or, on the contrary, this class is so broad that only very small groups (such as finite or compact ones) will enjoy the corresponding fixed-point property. For group theory, the most interesting properties are therefore those that somewhat interpolate between these two extremes, namely, that delineate some interesting class of groups. A sharp understanding of fixed-point properties requires thus to have constantly in mind the slight assumptions that force us to fall in one of the former extremes, hence we will first review them.

Let us start by general fixed-point properties, that is: under some assumptions on the affine action, any group must have a fixed point. The most important theorem in that area is Ryll-Nardzewski's theorem.

Proposition 5.2.1 (Ryll-Nardzewski). — *Let* G *be an abstract group of weakly continuous affine transformations on some locally convex space* E*. Assume that* G *preserves some nonempty weakly compact convex set* K *and that the action is* distal *on* K*, namely, that for any $x \neq y \in$ K, the closure of $\{g \cdot x - g \cdot y \mid g \in \mathrm{G}\}$ avoids the origin. Then there is a* G*-fixed point in* K*.*

On the proof. — A probabilistic proof was given in [**RN67**], while a more geometric one can be found in [**NA67**]. For other references on this cornerstone theorem, see [**GD03**, p. 196]. □

The strength of Ryll-Nardzewski's theorem is to allow to play with two topologies: the weak topology to have many compact sets and the initial topology to make distality easier to be satisfied (observe by the way that continuous affine transformations are automatically weakly continuous). This flexibility is illustrated in the following corollary.

Corollary 5.2.2. — *Let* G *be an abstract group with an affine uniformly bounded action on a reflexive Banach space. If there is a bounded orbit, then there is a fixed point.*

Proof. — The result follows directly from Ryll-Nardzewski's theorem: uniform boundedness ensures distality and the closed convex hull of the bounded orbit is a nonempty weakly compact convex subset, as the ambient space is reflexive. ■

For non-reflexive Banach spaces, it may be hard to find weakly compact convex sets, and in particular boundedness is not sufficient to get relative weak compactness. The next fixed-point theorem addresses this issue for *L-embedded* Banach spaces, that is, those that are an ℓ^1-direct factor in their biduals. This encompasses notably all $\mathrm{L}^1(\mu)$ for σ-finite measure and, more generally, all predual of von Neumann algebras. For group actions, we need to handle this geometric notion with care: it is not invariant by renorming (hence we cannot consider as equivalent isometric and uniformly bounded actions) and it does not pass to closed subspaces (hence we cannot restrict our attention to the closed span of an orbit). See [**HWW**] and especially Chapter 3 therein for more information about L-embedded spaces.

Proposition 5.2.3. — *Let* E *be an* L*-embedded Banach space. Any group of isometries of* E *preserving some bounded nonempty set* A *has a fixed point in* E*.*

On the proof. — The idea is to observe that the Chebyshev centers[3] of A in the bidual E″ (which exist by the Alaoglu–Bourbaki theorem) actually all lie in E thanks to L-

[3]That is, the centers of closed balls circumscribed to A.

embeddedness. A fixed point is then given by the Ryll-Nardzewski theorem applied to the set of Chebyshev centers in E. See [**BGM12**] for details. □

Remark 5.2.4. — Observe that the fixed point may well be outside the closed convex hull, or even the closed affine span, of A! Consider for instance the regular representation of a non-compact locally compact group G on $L^1(G)$: its only fixed point is zero, however there are bounded orbits whose closed affine span avoids zero (e.g. the orbit of any function whose integral is not zero). The reason behind that phenomenon is the fact that Chebyshev centers of A can avoid the closed affine span of A (see [**BGM12**, 3.C]), and usually avoids its closed convex hull when E is a Banach space of dimension at least three [**Kle60**, Corollary 2].

We will now switch our attention to contexts for which the presence of a fixed point depends on the nature of the group. We will focus on three classes of groups: finite, compact, and amenable ones. Each class admits *characterizations* via some fixed-point property (at least inside some class of groups). See Proposition 5.2.6 and Corollary 5.2.13 for finite groups, Propositions 5.2.7 and 5.2.18 for compact groups, and Proposition 5.2.20 and Theorem 5.2.21 for amenability.

Remark 5.2.5. — Let $\mathcal{P}$ be some group property (e.g., finiteness, compactness, or amenability) and $\mathcal{F}$ a fixed-point property on some class of affine actions on locally convex vector spaces (see below for examples). When proving that $\mathcal{F}$ characterizes $\mathcal{P}$, we often are in a situation where the implication $\mathcal{P} \Rightarrow \mathcal{F}$ is quite general, but the implication $\mathcal{F} \Rightarrow \mathcal{P}$ only uses one specific action. Hence we automatically get another equivalence of these properties with any formally weaker property $\mathcal{F}'$ that encompasses the specific action needed to prove $\mathcal{P}$.

5.2.A Finiteness

When qualifying discrete groups, *finite* and *trivial* are often considered as synonyms from the geometers' point of view. The following proposition shows that this linguistic equivalence also holds for topological groups.

Proposition 5.2.6. — *Let* G *be a topological group. Then the following assertions are equivalent.*

1. G *is finite.*
2. *Any affine action of* G *on a vector space has a fixed point.*

3. *Any orbitally continuous affine action of* G *on a topological vector space has a fixed point.*

4. *Any orbitally continuous affine action of* G *on a locally convex space has a fixed point.*

Proof. — If G is finite and has an affine action on a vector space V, then for any $x \in \mathrm{V}$, the affine combination

$$\frac{1}{|\mathrm{G}|}\sum_{g\in \mathrm{G}} \alpha(g)x$$

is fixed by G.

On the other hand, of course, we only need to prove that the last fixed-point property implies finiteness. Let then G be an infinite topological group. Consider the space E of *all* maps from $\mathscr{C}(\mathrm{G}, \mathbf{R})$ to $\mathbf{R}$, endowed with the topology of pointwise convergence (that is, E is a product of $\mathscr{C}(\mathrm{G}, \mathbf{R})$-many copies of $\mathbf{R}$, hence a locally convex space). There is a natural (non orbitally continuous) linear representation π of G on E deduced from precomposition on $\mathscr{C}(\mathrm{G}, \mathbf{R})$. Let G' be the subset of E made by the evaluations δ_g for $g \in \mathrm{G}$:

$$\delta_g \colon \mathscr{C}(\mathrm{G}, \mathbf{R}) \to \mathbf{R} \colon f \mapsto f(g).$$

Observe that the map $g \mapsto \delta_g$ is continuous. Let H be the (non-closed) subspace of E vectorially spanned by G'. So H is isomorphic to $\mathbf{R}[\mathrm{G}]$, the free vector space of basis G, endowed with the locally convex topology induced by the inclusion $\mathbf{R}[\mathrm{G}] \subset \mathrm{E}$ that linearly extends $g \mapsto \delta_g$. Moreover, the subspace H is invariant for the representation π, and the restriction $\pi|_{\mathrm{H}}$ is orbitally continuous. Now consider the affine invariant subspace H_0 made of the maps that send the constant function $\mathbb{1}_{\mathrm{G}}$ to 1 (equivalently, H_0 is the *affine* subspace spanned by G'). The deduced affine action of G on H_0 has no fixed point: indeed, such a fixed point should be an affine combination of evaluations at points in some finite set $\mathrm{S} \subset \mathrm{G}$, and this subset should be invariant for the action of G on itself by translation, which is impossible if G is infinite. ■

The interesting part of the above proposition is that we are able to recover finiteness by topological means *without* assuming any discreteness[4]. We are not aware of an affine fixed-point property characterizing discrete groups among topological groups.

[4]Actually, despite our standing assumptions, it is not even needed to assume G to be Hausdorff—if we are willing, of course, to let the topological vector spaces considered in Proposition 5.2.6 to be themselves non-Hausdorff.

5.2.B Compactness

Let us now have a look at the next class of "trivial" topological groups, namely compact groups. As shown by Proposition 5.2.6, we need to restrict our attention to some particular locally convex spaces if we want to associate a fixed-point property to compact groups. We will say that a locally convex space is a *Krein space* if the closed convex hull of any compact set is still compact. We refer to Appendix B for more background on this class of spaces, which includes Banach spaces endowed with their norm, weak or weak-$*$ topology.

We need another technical notion before investigating the fixed-point properties for compact groups. We will say that an action is *slightly continuous* if there exists a point whose associated orbit map is continuous. The prototypical examples are given by spaces of measures on a topological group G (for the natural action by precomposition): for instance, the orbit maps of Dirac masses are continuous with respect to the narrow topology (that is, the topology of pointwise convergence when measures are considered as functionals on the space of bounded continuous functions), but this is a priori not the case for a general measure. This mild condition was considered by Day in [**Day64**] during his study of fixed-point properties for amenability (cf. Section 5.2.C).

Proposition 5.2.7. — *Let* G *be a topological group and consider the following properties.*

1. *The group* G *is compact.*
2. *Any slightly continuous affine action of* G *on a Krein space has a fixed point.*
3. *The group* G *is precompact.*

Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3)*. In particular, compactness is equivalent to* (2) *for complete groups.*

Proof. — (1) $\Rightarrow$ (2). Let G be a compact group with a slightly continuous affine action $\alpha = \pi + b$ on a Krein locally convex space E and let $\varphi\colon \mathrm{G} \to \mathrm{E}\colon g \mapsto \alpha(g)x$ be a continuous orbit map associated to some point $x \in \mathrm{E}$. By continuity, the image C of φ (that is, the orbit of x) is compact. The image μ of the normalised Haar measure m_{G} of G through the orbit map φ is then a probability measure on a compact subset of a locally convex space. By the Krein property, the barycenter c_μ of this measure, defined by

$$c_\mu = \int_{\mathrm{G}} \alpha(g)x \;\mathrm{d}m_{\mathrm{G}}(g),$$

is an element of E (cf. Section B.5). Moreover, the barycenter is fixed by any $h \in \mathrm{G}$, as

$$\begin{aligned}\alpha(h)c_\mu &= \int_{\mathrm{G}} \pi(h)(\alpha(g)x)\ \mathrm{d}m_{\mathrm{G}}(g) + b(h)\\ &= \int_{\mathrm{G}} \alpha(h)\alpha(g)x\ \mathrm{d}m_{\mathrm{G}}(g)\\ &= \int_{\mathrm{G}} \alpha(g)x\ \mathrm{d}m_{\mathrm{G}}(h^{-1}g) = c_\mu,\end{aligned}$$

where the commutation of $\pi(h)$ with the integral is allowed as $\pi(h)$ is continuous [**Bou65**, III, § 3, n° 2, proposition 2].

(2) $\Rightarrow$ (3). Assume that G is a (Hausdorff) topological group with the fixed-point property of the statement. The idea is to use the fixed-point property on some space of measures in order to get an invariant probability on the group, the existence of which implies precompactness. As we made no a priori topological assumption on the group, this idea cannot be made rigorous without technical care. In order to not bury an otherwise simple proof under lengthy topological considerations, the concepts of measure theory we will need are actually defined and discussed in Section B.6.

So we consider the space $\mathscr{M}_\tau(\mathrm{G})$ of Borel τ-*additive* measures on G, endowed with the narrow topology, which is a Krein space (cf. Proposition B.6.1). The natural action of G on $\mathscr{M}_\tau(\mathrm{G})$ is slightly continuous and preserves the affine closed subspace E made of (signed) measures of total mass one. By hypothesis, there is a fixed point in E, that we may assume positive since the linear action of G on $\mathscr{M}_\tau(\mathrm{G})$ preserves the order lattice. In other words, there exists an invariant probability measure m on G, which is Borel and τ-additive. Let U be any open set in G; we will show that a finite number of translates of U covers G. Let V be another open set such that $\mathrm{VV}^{-1} \subseteq \mathrm{U}$. As $m(\mathrm{V}) > 0$ (cf. Lemma B.6.2) but $m(\mathrm{G}) = 1$, there exists a maximal finite set $\{x_1\mathrm{V}, \dots, x_n\mathrm{V}\}$ of *disjoint* translates of V. By maximality, for any $g \in \mathrm{G}$, there exists at least one x_i such that $g\mathrm{V} \cap x_i\mathrm{V} \neq \emptyset$, and therefore $g \in x_i\mathrm{VV}^{-1} \subseteq x_i\mathrm{U}$. Hence G is covered by the translates $x_1\mathrm{U}, \dots, x_n\mathrm{U}$. ■

Remark 5.2.8. — Without the Krein property, the barycenter built in the above proof lies a priori only in E'^*, the algebraic dual of the topological dual of E. It is however still a fixed point for the natural extension of the affine action α to E'^*. For instance, if we consider the action built in the proof of Proposition 5.2.6 for a compact group G, the space H (or H_0) considered there is not Krein: the set G' is a compact set whose closed convex hull is not compact. Actually, the barycenter of the pushforward to G' of the Haar measure in G is precisely the Haar measure (seen as a map $\mathscr{C}(\mathrm{G}, \mathbf{R}) \to \mathbf{R}$, that is, as an element of E—which is a Krein space all right).

Remark 5.2.9. — The implication (2) $\Rightarrow$ (3) cannot be reversed. Indeed, let G be a countable precompact but non-compact group (for instance, a countably infinite dense subgroup of a compact group). Such a group cannot carry an invariant probability measure m as yielded by the fixed-point property (2).

More generally, any precompact non-compact group G that is a Baire subset[5] of its completion $\widehat{G}$ will fail to enjoy the fixed-point property (2). Indeed, the measure m built in the above proof can be extended to a Haar probability measure $\widehat{m}$ on $\widehat{G}$ defined on the Baire σ-algebra of $\widehat{G}$ (cf. e.g. [**Hal50**, §62, Theorem H]). *If* G is Baire in $\widehat{G}$ and $G \neq \widehat{G}$, there would therefore be a contradiction between the facts that G has full measure and that it has disjoint cosets.

Actually, we suspect the fixed-point property (2) to characterize compactness, but were unable to prove it without further assumption on the embedding of G inside its completion[6].

Problem 2. — Let G be a topological group. Assume that any slightly continuous affine action of G on a Krein locally convex space has a fixed point. Must G be compact?

Remark 5.2.10. — Nguyen Van The and Pestov proved [**NVTP**, Theorem 5.8] that precompact groups are also characterized by the fixed-point property for isometric affine actions on Banach spaces. Observe that their trick to get a fixed-point property for *precompact* groups, using a similar property for compact groups, requires the action morphism $\alpha\colon G \to \mathrm{Aff}(E)$ to actually lie in some complete topological subgroup of Aff(E) (in their case, the subgroup of affine isometries, endowed with the pointwise convergence, which is indeed a complete topological group if E is a Banach space). The extension of α to $\widehat{G}$ follows then from the fact that a continuous group homomorphism is uniformly continuous. The reason why we cannot similarly extend the above fixed-point property (2) to precompact groups is the following: even if we succeed in defining some subgroup of Aff(E) that contains the translations and is a topological subgroup for the pointwise convergence, it will not be complete unless E itself is so, a property that is not necessarily fulfilled for Krein spaces (cf. Section B.4).

We know by Proposition 5.2.6 that the Krein assumption in the above proposition is necessary, namely that any infinite compact group admits a fixed-point free affine action on a non-Krein space. As a normed space is Krein if and only if it is complete (Lemma B.4.1),

[5] A subset of a topological space X is called a *Baire subset* if it belongs to the coarsest σ-algebra such that all continuous functions $X \to \mathbf{R}$ are measurable (the *Baire σ-algebra*). (If X is locally compact σ-compact, the Baire σ-algebra is also generated by the compact G_δ subsets of X.) From the point of view of measure theory, the Baire subsets are the "well-behaved" Borel sets, as shown by the current paragraph.

[6] There are precompact groups that are not Borel in their completion. Let $\{e_\alpha\}$ be a basis for $\mathbf{R}$ as a vector space over $\mathbf{Q}$, such that $e_{\alpha_0} = 1$ for some index α_0. Choose any $\alpha \neq \alpha_0$ and consider the projection π_α of $\mathbf{R}$ onto $\mathbf{Q}e_\alpha$. Then it induces a (non-continuous) group epimorphism $\overline{\pi_\alpha}\colon \mathbf{S}^1 \to \mathbf{Q}$. The kernel H of this epimorphism is a subgroup of countably infinite index in $\mathbf{S}^1$. It therefore cannot be meagre nor open, hence it is not Borel (cf. [**Bou74**, IX, §6, n° 8, Lemme 9]), however it is dense in $\mathbf{S}^1$.

it is natural to wonder if such an action can always be given on a normed space. The answer is positive.

Proposition 5.2.11. — *Let* G *be an infinite compact group. Then* G *admits a continuous affine isometric action on a prehilbertian space without any fixed point.*

Proof. — Thanks to Corollary 1.2.2, we may assume that G is metrizable. Let d be a compatible metric on G and consider the function

$$f\colon \mathrm{G} \to \mathbf{R}\colon g \mapsto \sqrt{d(e,g)}.$$

Clearly, f is a continuous nonnegative function on G. We will show that the algebraic span H of the orbit $\mathrm{G}\cdot f \subset \mathscr{C}(\mathrm{G})$ does not contain any nonzero constant function. Therefore, endowing H with the usual prehilbertian norm coming from $\mathscr{C}(\mathrm{G}) \hookrightarrow \mathrm{L}^2(\mathrm{G})$ and considering its intersection with the invariant affine subspace of functions of integral one, we will get an affine isometric action on a prehilbertian space without any fixed point. This action will be orbitally continuous since any element of H (or of $\mathscr{C}(\mathrm{G})$) is uniformly continuous, and orbital and joint continuities agree for isometric actions.

Let then $\varphi = \sum_{k=1}^{n} t_k g_k f$ be a linear combination of translates of f (so $t_k \in \mathbf{R}$ and $g_k \in \mathrm{G}$). Without loss of generality, we may assume that all the g_k's are distinct. As G cannot be a discrete group, the scalar 0 is not an isolated point in the image of the metric d. Therefore, the function $g_k f$ is *not* Lipschitz at g_k, *but* is Lipschitz everywhere else. Consequently, the sum φ cannot be Lipschitz at $g_1, \dots, g_n$, hence in particular cannot be constant. ■

Remark 5.2.12. — Let us focus on a subpart of the above proof, namely finding a (right-uniformly) continuous (bounded) function f such that the algebraic span of the orbit $\mathrm{G}\cdot f$ avoids the constant. For that argument, we only used compactness to rule out discreteness (so that 0 is not an isolated point of the image of the distance). Indeed, we can always ask a compatible right-invariant metric to be bounded, and the triangle inequality ensures that $d(e,\cdot)$ is always right-uniformly continuous. On the other hand, it is trivial to produce such a function f for discrete groups (for instance, f can be a Dirac mass). Therefore, we observe the following: *for any group* G *with an infinite metrizable quotient (in particular for any infinite locally compact* σ*-compact group), there exists a right-uniformly continuous bounded function f such that the algebraic span of* $\mathrm{G}\cdot f$ *avoids nonzero constant functions.* Moreover, if we do not pass to a quotient, the argument even yields the fact that the family $\{g\cdot f \mid g \in \mathrm{G}\}$ is algebraically free.

As a consequence of Proposition 5.2.11, we get the following strengthening of Proposition 5.2.6.

Corollary 5.2.13. — *Let* G *be a topological group. Then* G *is finite if and only if any continuous affine isometric action of* G *on a normed space has a fixed point.*

Proof. — We already know that finite groups enjoy the fixed-point property of the statement. Reciprocally, let G be a topological group with this fixed-point property. The latter encompasses in particular Banach spaces, hence G must be precompact by the result of Nguyen Van The and Pestov [**NVTP**]. The result then follows from Proposition 5.2.11 since the restriction of an orbitally continuous fixed-point free action to a dense subgroup stays fixed-point free. ■

Despite the above results, a non-precompact group may well enjoy very strong fixed-point properties on some large class of spaces. This phenomenon actually occurs in the opposite direction of compactness, namely that the group is too "big" to be representable in nice spaces. We give two examples to illustrate that point.

Example 5.2.14 (Bergman groups). — A (discrete) group is said to have the *Bergman property* if every isometric action on a metric space has bounded orbits. Bergman property implies finiteness for countable groups, but there are uncountable examples such as Sym(Ω) for any infinite set Ω [**Ber06**, Theorem 6] or F^I for any finite perfect group F and (infinite) set I [**Cor06**, Theorem 4.1]. It is not hard to see that *any affine action of a Bergman group on a reflexive Banach space* E *has a fixed point.* Indeed, the set $\mathcal{N}$ of equivalent norms on E is itself a metric space, with distance

$$d(\|\cdot\|_1, \|\cdot\|_2) = \sup_{x\neq 0} \left|\log \frac{\| x \|_1}{\| x \|_2}\right|.$$

A linear representation π of a group G on E produces an isometric action on $\mathcal{N}$ (given by $g \cdot \| x \| = \| \pi(g)x \|$). The orbits in $\mathcal{N}$ are therefore bounded by the Bergman property, which precisely means that π is uniformly bounded. We can thus choose an equivalent invariant norm on E. In other words, up to renorming, any affine action of a Bergman group is isometric. Hence it has bounded orbits, which in turns implies a fixed point if the space is reflexive thanks to Ryll-Nardzewski's theorem (Corollary 5.2.2).

There is a natural extension of the Bergman property to topological groups, namely that any *continuous* isometric action on a metric space has bounded orbits. The above argument to deduce from that a fixed-point property on reflexive Banach spaces as above does not work in that topological setting (since the deduced action on the space $\mathcal{N}$ is not continuous unless π was uniformly continuous). However, this fixed-point property still holds for *Polish* groups with Bergman property. See Proposition 1.30 in [**Ros13**] for the proof, as well as [**Ros09**] for more about the topological Bergman property and for Polish examples.

Remark 5.2.15. — As we saw in the above example, up to renorming, any affine action of a Bergman group on a Banach space can be assumed to be isometric (and hence has bounded orbits). The strength of Ryll-Nardzewski's theorem is to impose only topological conditions on the space (that is, conditions invariant by renorming). The other general fixed-point theorem we saw, Proposition 5.2.3, relies unfortunately on a geometric condition (that is, depending on the norm). This raises a natural question.

Problem 3. — Let E be an L-embedded space and G be a group with a uniformly bounded affine action on E. If G preserves some bounded nonempty subset of E, must G have a fixed point?

Example 5.2.16 (Underrepresented groups). — Let $G = \mathrm{Homeo}_+[0,1]$ be the group of orientation-preserving homeomorphisms of the interval, endowed with the compact-open topology. It is a Polish group with the topological Bergman property (since it is a closed subgroup of $\mathrm{Homeo}[0,1]$, which is Roelcke-precompact—see [**Ros13**, p. 1623] for details), hence any orbitally continuous affine action on a reflexive Banach space has a fixed point by the above example. Actually, more is true: any orbitally continuous affine action of G on a reflexive Banach space is *trivial* (that is, fixes any point)! Indeed, Megrelishvili proved that this group admits no nontrivial isometric linear representations in reflexive Banach spaces [**Meg01a**], but we already know by the Bergman property that we can assume the action to be isometric. Hence the linear part of the affine action is trivial. As the affine action has a fixed point (again by the Bergman property), the cocycle must also be trivial.

Remark 5.2.17. — Let G_d be the group $\mathrm{Homeo}_+[0,1]$ with the discrete topology. Thanks to automatic continuity (see [**RS07**, Theorem 4]), any linear isometric representation of G_d on a *separable* Hilbert space would actually be continuous for the compact-open topology of $\mathrm{Homeo}_+[0,1]$, and therefore trivial by the above example. Hence, for any linear isometric representation of G_d on a Hilbert space V and any vector $\xi \in V$, we have the following dichotomy: either ξ is invariant or its orbit spans a non-separable space.

The interesting part of Propositions 5.2.6 and 5.2.7 is to avoid a priori topological assumptions on the group, thanks to the large class of spaces considered. On the other hand, the above examples show that we cannot characterize "smallness" (i.e., finiteness, (pre)compactness) with actions on nice spaces such as reflexive Banach spaces. However, if we ruled out from the beginning topologically "huge" groups—for instance, if we require a priori the group to be σ-compact—, then the situation is much closer to the intuition (namely, that a non-compact group should admit some fixed-point free action even on nice spaces), as shown by the following proposition.

Proposition 5.2.18. — *Let* G *be a locally compact σ-compact group. Then the following properties are equivalent.*

1. G *is compact.*
2. *Any orbitally continuous affine action of* G *on a Krein locally convex space has a fixed point.*
3. *Any continuous affine isometric action of* G *on a Banach space has a fixed point.*
4. *Any continuous affine isometric action of* G *on a strictly convex reflexive Banach space has a fixed point.*
5. *Any continuous affine action of* G *on a Hilbert space has a fixed point.*

Beware that there is no isometry assumption in the last fixed-point property. The fixed-point property for isometric action on Hilbert spaces is known as property (FH), and is equivalent to property (T) for locally compact σ-compact groups [**BHV08**, Theorem 2.12.4].

On the proof. — Compactness implies the first fixed-point property by Proposition 5.2.7, which in in turn implies all the other fixed-point properties. Of course, it is enough to prove (4) $\Rightarrow$ (1) and (5) $\Rightarrow$ (1), but some other direct implications are available.

1. (2) $\Rightarrow$ (1). A locally compact group is complete [**Bou71**, III, §3, n° 3, cor. 1], hence the result follows from Proposition 5.2.7.
2. (3) $\Rightarrow$ (1) If G is locally compact, then its regular representation on $L^1(G)$ leaves invariant the affine closed subspace E of functions of sum one. Invariant vectors for the regular representations of a σ-compact group are (almost everywhere) constant functions, which can live in E if and only if G is compact.
3. (4) $\Rightarrow$ (1) A. Przybyszewska and U. Haagerup proved in [**HP06**, Theorem 6.5] that any locally compact second-countable group G admits a proper affine isometric action on the strictly convex reflexive Banach space $\bigoplus_{n=1}^{\infty} L^{2n}(G)$. In particular, locally compact σ-compact non-compact groups admit a fixed-point free affine isometric action on a strictly convex reflexive Banach space, by the Kakutani–Kodaira theorem (see Proposition 1.2.1).
4. (5) $\Rightarrow$ (1) This implication was proved for locally compact Polish groups by C. Rosendal (see [**Ros13**, Theorem 1.4]), from which the result for σ-compact groups follows by the Kakutani–Kodaira theorem. Rosendal's proof reduces itself to the case of *compactly generated* groups [**Ros13**, Theorem 1.45], thanks to the fact that property (5) implies in particular that G has property (FH), hence is compactly gen-

erated [**BHV08**, Corollary 2.4.2]. He then proved that compactly generated locally compact Polish groups admit a *proper* affine continuous action on a Hilbert space. □

Remark 5.2.19. — Another proof for Rosendal's result (5) $\Rightarrow$ (1) is available. As noted, property (5) implies in particular property (FH), which is equivalent to property (T) for locally compact σ-compact groups. We will see below (Theorem 5.2.21) that it also implies (again for locally compact σ-compact groups) amenability. The conjunction of both implies compactness [**BHV08**, Theorem 1.1.6].

5.2.C Amenability

Recall that a topological group is amenable if it admits a left-invariant mean on the space of right-uniformly continuous functions. A mean is a way to "average" a function; as averaging was at the core of the fixed-point properties studied above (via a genuine average for finite groups or via the barycenter for compact groups), it is not surprising that amenability can be characterized as a fixed-point property. This was done by Day for discrete groups [**Day61**] and Rickert [**Ric67**, Theorem 4.2] for topological ones[7].

Proposition 5.2.20 (Day–Rickert). — *A topological group* G *is amenable if and only if any jointly continuous affine action of* G *on a nonempty compact convex set* K *has a fixed point.*

Proof. — Let G be a topological group with the fixed-point property of the statement. Let K be the set of means on G, that is,

$$K = \{\mathfrak{m} \colon \mathscr{C}_{\mathrm{rucb}}(G) \to \mathbf{R} \mid \mathfrak{m}(\mathbb{1}_G) = 1 \quad \text{and} \quad \mathfrak{m} \geqslant 0\}.$$

Endowed with the weak-$*$ topology of $\mathscr{C}_{\mathrm{rucb}}(G)'$, K is a compact convex set[8]. The usual linear representation of G on $\mathscr{C}(G)$ leaves $\mathscr{C}_{\mathrm{rucb}}(G)$ invariant, hence there is a well-defined affine action of G on K by precomposition. Observe that a fixed point for this action is nothing but an invariant mean. We then only need to show that this action is jointly

[7] Day actually considered a stronger requirement than amenability, namely the existence of an invariant mean on the space of all bounded continuous functions. Day claimed the characterization of the latter with a fixed-point property, but Rickert pointed out that the representation of G on $\mathscr{C}_{\mathrm{b}}(G)$ is usually not continuous. He then gave the right equivalence, namely Proposition 5.2.20. For a fixed-point property characterizing the existence of an invariant mean on $\mathscr{C}_{\mathrm{b}}(G)$, see [**Day64**] and for more information about this strengthening of amenability, see [**GH17**].

[8] A positive linear functional on $\mathscr{C}_{\mathrm{rucb}}(G)$ is continuous, since $|f| \leqslant \| f \| \mathbb{1}_G$, hence $|\mathfrak{m}(f)| \leqslant \| f \| \mathfrak{m}(\mathbb{1}_G)$ for any $f \in \mathscr{C}_{\mathrm{rucb}}(G)$. This holds more generally, see [**AT07**, Corollary 2.33].

continuous. Let $(g_\alpha, \mathfrak{m}_\alpha)$ be a net in $G \times K$ converging to $(g, \mathfrak{m})$. *As* all means have norm 1, we can write, for any $f \in \mathscr{C}_{\mathrm{rucb}}(G)$

$$|\mathfrak{m}_\alpha(g_\alpha f) - \mathfrak{m}(gf)| \leqslant \| g_\alpha f - gf \| + |\mathfrak{m}_\alpha(gf) - \mathfrak{m}(gf)|.$$

The first term goes to zero as f is right-uniformly continuous and the second term goes to zero as $\mathfrak{m}_\alpha$ converges to $\mathfrak{m}$.

Reciprocally, let G be an amenable group with a jointly continuous affine action on nonempty compact convex set K. Choose any point $x \in K$. As the orbit map $G \to K\colon g \mapsto gx$ is right uniformly continuous (see Lemma C.2.4), it defines by pre-composition an equivariant continuous positive linear map $\varphi\colon \mathscr{C}(K) \to \mathscr{C}_{\mathrm{rucb}}(G)$. The adjoint $\varphi'\colon \mathscr{C}_{\mathrm{rucb}}(G)' \to \mathscr{C}(K)'$ is therefore also equivariant, continuous, and positive, hence it maps means on G to means on K. But a mean on a compact set is nothing but a Radon probability measure by Riesz's representation theorem; it therefore admits a unique barycenter. Finally, the map $\mathscr{P}_1(K) \to K$ that assigns to a probability measure its barycenter is equivariant by the very definition of a barycenter (see Section B.5). To sum up, we got an equivariant map $M_1(G) \to K$ from the means on G to K: the image of any invariant mean is therefore a fixed point. ∎

As we saw in Proposition 5.1.7, any jointly continuous affine action on a compact convex set K extends to an orbitally continuous affine action on a canonical locally convex space containing K. On the other hand, if G is locally compact, then any orbitally continuous action on a compact space is jointly continuous (Lemma C.1.9). Therefore, for locally compact groups, we get another characterization of amenability: G *is amenable if and only if, for any orbitally continuous affine action of* G *on a locally convex space* V *preserving some nonempty compact convex subset* K*, there is a fixed point in* K. As noted in Remark 5.2.5, we can in the latter fixed-point property restrict our attention to any class of locally convex spaces that encompasses the dual of $\mathscr{C}_{\mathrm{rucb}}(G)$. Hence *a locally compact group* G *is amenable if and only if, for any orbitally continuous affine action of* G *on the dual* V *of a Banach space, preserving some nonempty weakly-∗ compact convex* K*, there is a fixed point in* K. Observe moreover that we can equivalently require K to be only some bounded set: on the one hand, weakly-∗ compact sets are bounded; on the other hand, the weak-∗ closed convex hull of an invariant bounded set is weakly-∗ compact (by Corollary B.3.4 and the Alaoglu–Bourbaki theorem) and still invariant (as G acts by continuous affine maps).

Can we go further? What happens to a Day-like property when restricted to even nicer spaces V, such as Banach or even Hilbert spaces? Such a property certainly cannot characterize amenability for general locally compact groups: recall from Example 5.2.14 that discrete uncountable groups, amenable or not, may well enjoy a very strong fixed-point property on Hilbert spaces (without additional hypothesis on the preservation of some bounded set). However, as usual, the landscape is less wild for σ-compact groups (compare with Proposition 5.2.18).

Theorem 5.2.21. — *Let* G *be a locally compact* σ*-compact group. Then* G *is amenable if and only if every continuous affine action of* G *on a separable Hilbert space with a bounded orbit has a fixed point.*

As alluded to in the discussion preceding the theorem, we could as well rephrase Theorem 5.2.21 by replacing "with a bounded orbit" by "preserving a nonempty weakly compact set". Moreover, a continuous affine action on a locally convex space can also be viewed as an orbitally continuous affine action on the same space but endowed with the weak topology. Therefore, thanks to Day–Rickert characterization of amenability, any amenable group enjoys the fixed-point property of the statement. Our task is thus to prove the converse, namely that any locally compact σ-compact nonamenable group admits a continuous affine action on a separable Hilbert space with a bounded orbit but without any fixed point. The proof will be given below, on p. 115, but first several remarks are in order.

Remark 5.2.22. — Let α be an affine action on a Hilbert space with a bounded orbit but without fixed point, as given for nonamenable locally compact σ-compact groups by Theorem 5.2.21. Then the linear part of α cannot be uniformly bounded (and a fortiori not isometric). Otherwise, up to renorming, we could assume that α is an isometric action on a Banach space isomorphic to a Hilbert one; in particular, on a reflexive Banach space. The presence of a bounded orbit, that is, of a nonempty weakly compact convex subset, would then force the action to have a fixed point by Ryll-Nardzewski's theorem. By Lemma 4.1.4, we then observe that our actions exhibit an interesting dynamical behavior: they have both a bounded orbit and a dense (even comeager) subset of points with unbounded orbits!

(The use of Ryll-Nardzewski's theorem and of Lemma 4.1.4 relies on the completeness of the space: for instance, the actions on non-complete prehilbertian spaces built in Proposition 5.2.11 have no fixed point but all their orbits are bounded.)

Remark 5.2.23. — The use of Day–Rickert characterization of amenability to prove the "only if" direction of Theorem 5.2.21 furthermore enables us to find a fixed point in the closed convex hull of any bounded orbit. To be more precise, for a locally compact σ-compact group G, the following three properties are equivalent:

1. G is amenable;

2. every continuous affine action of G on a separable Hilbert space with a bounded orbit has a fixed point inside the closed convex hull of any bounded orbit;
3. every continuous affine action of G on a separable Hilbert space with a bounded orbit has a fixed point.

Day–Rickert characterization proves that amenability implies the first fixed-point property, which in turn obviously implies the second one. We will prove that the latter implies amenability. Observe that the equivalence between the two fixed-point properties is not a priori obvious, since fixed points may in general occur outside the affine span of a bounded invariant set (cf. Remark 5.2.4). We do not know by the way if they stay equivalent without σ-compactness.

Remark 5.2.24. — We already observed that σ-compactness is an essential assumption, since, for instance, the nonamenable uncountable discrete group Sym(Ω) has a fixed point for any affine action on a Hilbert space (see Example 5.2.14). Likewise, Theorem 5.2.21 cannot hold in a (non-locally compact) Polish setting, since there are Polish nonamenable groups with the Bergman property, like Homeo($\mathbf{S}^n$) [**Ros09**, Theorem 7.2].

Remark 5.2.25. — In Theorem 5.2.21, continuity of the action can be understood indifferently as joint or orbital continuity, since both agree for locally compact groups acting on Hilbert spaces (Corollary 4.1.7). Likewise, the separability of the Hilbert space is a feature we can get for free thanks to Proposition 4.1.13.

Doomed attempts

Our proof of Theorem 5.2.21 is rather indirect and intricate: we will use the tools of Chapters 3 and 4. Before going further, it may be useful to pause and see how some strategies of more direct proofs fail.

A first strategy, along the lines of Remark 5.2.5, would be to try to view the set of means $\mathrm{M}_1(\mathrm{G})$ as a weakly compact convex set of a Hilbert space. Such a strategy is doomed to failure, as shown by the following lemma (which is purely topological, no linearity is required).

Lemma 5.2.26. — *Let* G *be a non-compact locally compact group. There is no weakly continuous injective function from* $\mathrm{M}_1(\mathrm{G})$ *to any Banach space.*

Proof. — Assume for a contradiction that there is such a function. By compactness, $\mathrm{M}_1(\mathrm{G})$ would therefore be homeomorphic to a weakly compact set in a Banach space. But such a set must be weakly sequentially compact thanks to the Eberlein–Šmulian theorem. So we only need to prove that $\mathrm{M}_1(\mathrm{G})$ is not sequentially compact.

Let m be a Haar measure on G and let K_n be a sequence of nested compact sets in G (with nonempty interior), such that $(m(\mathrm{K}_n))$ tend to infinity (we do not need the sequence to exhaust G, that is, G is not assumed σ-compact). Consider the sequence $(\mathfrak{m}_n)$ of means on G defined by

$$\mathfrak{m}_n(f) = \frac{1}{m(\mathrm{K}_n)} \int_{\mathrm{K}_n} f(x)\, \mathrm{d}m(x) \qquad (f \in \mathrm{L}^\infty(\mathrm{G})).$$

We will show that this sequence admits no convergent subsequence. Let $(\mathfrak{m}_{n_j})$ be a subsequence; up to extraction, we may assume that

$$m(\mathrm{K}_{n_{j+1}}) \geqslant (j+1) m(\mathrm{K}_{n_j})$$

for any j.

Consider now the following measurable set:

$$\mathrm{A} = \bigcup_{j=0}^{\infty} \left(\mathrm{K}_{n_{2j+1}} \setminus \mathrm{K}_{n_{2j}} \right).$$

An easy computation shows that

$$\begin{aligned} \mathfrak{m}_{n_{2j+1}}(\mathbb{1}_\mathrm{A}) &\geqslant 1 - m(\mathrm{K}_{n_{2j}}) m(\mathrm{K}_{n_{2j+1}})^{-1} \geqslant 1 - (2j)^{-1} \xrightarrow{j \to \infty} 1, \\ \mathfrak{m}_{n_{2j}}(\mathbb{1}_\mathrm{A}) &\leqslant \ m(\mathrm{K}_{n_{2j-1}}) m(\mathrm{K}_{n_{2j}})^{-1} \qquad \leqslant (2j)^{-1} \quad \xrightarrow{j \to \infty} 0, \end{aligned}$$

hence the subsequence $(\mathfrak{m}_{n_j})$ cannot converge. ■

So the natural compact convex set without fixed point that Day–Rickert characterization associates to any nonamenable group cannot be used to prove Theorem 5.2.21. Maybe we could, for locally compact σ-compact groups, be luckier and find another compact convex set without fixed point that (weakly) embeds into a Hilbert space (and then transport there its G-structure)? For instance, one to which Klee's result applies (Proposition 5.1.3)? Even in that case, we would be stuck: an affine action on a total compact convex set need not extend to an ambient Hilbert space (Example 5.1.2). And in the yet luckier case where the action does extend to the whole space, we would still need to exclude the possibility to have a fixed point outside the initial weakly compact convex set (see Remark 5.2.4 and Problem 8)!

Now that we have put Theorem 5.2.21 in its right context, we can attack its proof. We will appeal to tools from *bounded cohomology*. For a comprehensive treatment of the latter, see [**Mon01**]; for an overview, see [**Mon06**]; for an quick introduction, see [**BI09**, §2].

The reader is warned however that Theorem 5.2.21 does not fall like an overripe fruit from some big theorem in bounded cohomology (even if amenability can indeed be characterized via bounded cohomology), because this cohomology theory is definable only for uniformly bounded representations.

For the reader queasy in front of bounded cohomology, we have unfolded the functorial part of the proof into an equivalent elementary argument in Remark 5.2.28. The latter is undoubtedly less illuminating but at least more pedestrian.

Proof (of Theorem 5.2.21). — Recall that our goal is to build, for any locally compact σ-compact nonamenable group G, an affine action on a Hilbert space without fixed point but with a bounded orbit. Of course, it suffices to build such an action for some quotient of G, hence we may assume that G is second countable thanks to the Kakutani–Kodaira theorem (since the quotient of a nonamenable group by a compact subgroup stays nonamenable). Such a group admits both a moderate measure (Corollary 4.2.10) and a tychomorphism from any free group of countable rank (Theorem 3.2.1). Hence, thanks to the induction techniques of Theorem 4.3.12, we only need to build the sought action for some free group. And, once again, it suffices to build such an action for some quotient of a free group.

To sum up: Theorem 5.2.21 will be proved as soon as we have found *one* countable group with *one* affine action on a Hilbert space without fixed point but with a bounded orbit!

Up to conjugating the action by some translation, we may assume that the sought bounded orbit is the orbit of zero. Hence our task is to find a *nontrivial bounded cocycle* associated to some linear representation π on a Hilbert space.

Let G be a discrete countable group admitting a (nonzero) class ω in degree-two cohomology with real coefficients and assume that ω can be represented by a *bounded* cocycle, i.e. that it lies in the image of the comparison map

$$\mathrm{H}^2_{\mathrm{b}}(\mathrm{G}, \mathbf{R}) \longrightarrow \mathrm{H}^2(\mathrm{G}, \mathbf{R}).$$

A well of examples of such groups lies in hyperbolic geometry: we can choose for G the fundamental group of a hyperbolic closed surface and for ω its fundamental class (cf. [**Thu78**, §6]). Equip G with some moderated measure μ (in our case, G is finitely generated, so that can be done easily as in Example 4.2.2) and define the spaces V and E as the quotients of $\ell^2(\mathrm{G}, \mu)$ and $\ell^\infty(\mathrm{G})$ by the subspace of constant functions, respectively. We endow $\ell^2(\mathrm{G}, \mu)$ and $\ell^\infty(\mathrm{G})$ with the usual linear representations by precomposition (which is well-defined for the former, see Proposition 4.2.12); these representations pass to their quotients V and E since the subspace of constant functions is invariant.

Behold now the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbf{R} & \longrightarrow & \ell^\infty(\mathrm{G}) & \longrightarrow & \mathrm{E} & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathbf{R} & \longrightarrow & \ell^2(\mathrm{G},\mu) & \longrightarrow & \mathrm{V} & \longrightarrow & 0 \end{array}$$

The rows are exact and the vertical arrows are G-equivariant linear injective maps, of norm $\leqslant 1$. The idea is to get from this diagram long exact sequences in cohomology. As bounded cohomology is ill-defined for non-uniformly bounded representations, we apply the bounded cohomology functor to the first row and the ordinary cohomology functor to the second row. The naturality of the comparison maps and of the long exact sequences produces another commutative diagram, which reads as follows since $\mathrm{H}^n_{\mathrm{b}}(\mathrm{G},\ell^\infty(\mathrm{G}))$ vanishes for all $n \geqslant 1$ [**Mon01**, Propositions 4.4.1 and 7.4.1]:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{H}^1_{\mathrm{b}}(\mathrm{G},\mathrm{E}) & \xrightarrow{\sim} & \mathrm{H}^2_{\mathrm{b}}(\mathrm{G},\mathbf{R}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & \mathrm{H}^1(\mathrm{G},\mathrm{V}) & \longrightarrow & \mathrm{H}^2(\mathrm{G},\mathbf{R}) & \longrightarrow & \cdots \end{array}$$

We now only have to let ω be chased across this diagram. As ω is nonzero and lies in the image of the comparison map, there is a nontrivial bounded cocycle $b\colon \mathrm{G} \to \mathrm{E}$ which stays nontrivial when seen as a cocycle from G to V. Moreover, it is still bounded in V since the injection map $\iota\colon \mathrm{E} \to \mathrm{V}$ is bounded. Hence $\iota \circ b\colon \mathrm{G} \to \mathrm{V}$ is a nontrivial bounded cocycle in some Hilbert space, as required. ■

Remark 5.2.27. — We already noticed in Remark 5.2.22 that the points in V with an unbounded orbit should form a dense subset of V. The action built in the above proof for the surface groups (or for nonabelian free groups that surjects on them, that is those of rank at least 4) has moreover a dense (but meager) subset of points with a *bounded* orbit: the image in V of points in E. Indeed, the injection from E to V is equivariant and bounded, but the representation on E is isometric.

Remark 5.2.28. — We can avoid the functorial argument as follows. Represent the cocycle ω above in the inhomogeneous model of cohomology, that is, as a bounded map $c\colon \mathrm{G}^2 \to \mathbf{R}$ such that

$$c(y,z) - c(xy,z) + c(x,yz) - c(x,y) = 0 \qquad (\text{for all } x,y,z \in \mathrm{G}).$$

Then $(\iota \circ b)(g)$ is given as the class (modulo the constant functions) of the map $x \mapsto c(x^{-1},g)$. We can check directly that this is indeed a bounded cocycle for the linear representation on V, which is trivial only if c is so.

5.2.D Some questions

Theorem 5.2.21 and above all its proof raise several interesting questions. All of them seem open even for discrete groups.

The first question that comes in mind, especially by comparison with the other theorems of this chapter, is of course: can we have a simpler proof? (We already excluded some avenues, see Lemma 5.2.26 and the discussion following it.)

Problem 4. — Is there an elementary direct proof of Theorem 5.2.21, that would in particular not rely on Theorem 3.2.1?

A related problem is the following.

Problem 5. — Is there a statement analogous to Theorem 5.2.21 for semigroups?

Indeed, the induction through Theorem 3.2.1 itself is unlikely to have a semigroup counterpart, because amenable countable groups may well have nonabelian free sub*semi*groups (consider for instance the "$ax+b$" group, that is, the group of affine transformations of $\mathbf{R}$, with rational coefficients). Observe by the way that the Day–Rickert fixed-point property is indeed a characterization of left amenability for discrete semigroups (it is actually one of the very few characterizations of amenability that survive when considering amenability of semigroups).

Looking closely at the proof of Theorem 5.2.21, we see that the specific action built for the surface groups are actually better than expected: it even preserves a (nonempty) *norm*-compact subset. Indeed, the bounded orbit comes from a bounded subset of $\ell^\infty(\mathrm{G})/\mathbf{R}$, hence is relatively compact in $\ell^2(\mathrm{G}, \mu)/\mathbf{R}$ since the injection $\ell^\infty(\mathrm{G}) \hookrightarrow \ell^2(\mathrm{G}, \mu)$ is a compact operator (μ is a finite atomic measure).

We know by Proposition 4.3.10 that moderate induction for subgroups of discrete groups preserves the relative compactness of the cocycle. Hence we can likewise produce such actions for discrete groups containing a nonabelian free subgroup (or any subgroup that surjects to a compact hyperbolic surface group). However, moderate induction for tychomorphisms or for subgroups of (non-discrete) locally compact groups will unfortunately loose that relative compactness, because the measures involved are non-atomic (see Proposition 2.6.1). Hence the following problem arises:

Problem 6. — Let G be a locally compact σ-compact group (or even a discrete countable group) with the following property: any continuous affine action of G on a Hilbert space preserving some nonempty compact set has a fixed point. Must G be amenable?

Of course, the compact set can be assumed to be convex by Mazur's theorem, and all amounts to finding an orbit with a relatively compact image. There does not seem to be many tools to handle cocycles with relatively compact image (in contrast to those with bounded image), which motivates the following vague problem.

Problem 7. — Is there a useful theory of *compact* cohomology (that is, whose cocycles have relatively compact image)?

By "useful", we think about the usual tools of cohomology: long exact sequences, induction techniques, existence of suitable resolutions for computations,... A good interplay with tychomorphisms or measure equivalence would be great, but maybe too fanciful.

Getting back to a more concrete level, we notice that not much seems to be known about non-uniformly bounded actions[9], and Theorem 5.2.21 shows that their behavior can be very wild. We were unable to answer the following seemingly simple question (compare with the end of Remark 5.2.23).

Problem 8. — Let G be an abstract group of affine transformations on a Hilbert space. Assume that G preserves some bounded set A and has, somewhere, a fixed point. Must G also have a fixed point in the closed convex hull of A?

If the action is isometric, then the projection of the fixed point on the closed convex hull of A will be fixed, since projections in Hilbert spaces are defined metrically. Likewise, if the action is uniformly bounded, we can assume that it is isometric with respect to an equivalent uniformly convex norm (see Remark 4.1.2) and use a similar projection argument. On the other hand, if the action is isometric but the space is not uniformly convex enough, fixed points may occur outside the closed convex hull of an invariant bounded set (Remark 5.2.4).

Lastly, we emphasize that the affine fixed-point properties studied in this chapter focus on only two kinds of actions: either general or preserving some compact convex subset. A third interesting—and perhaps overlooked for a long time—case is given by affine actions preserving some proper convex cones. The typical example of such an action is given by a cone of nonnegative measures (such an invariant cone made a cameo appearance in the proof of Proposition 5.2.7). We refer to [**Mon17**] for more information on fixed-point property for convex cones.

[9] We have to mention here, of course, the impressive works of V. Lafforgue on the Baum–Connes conjecture and the strengthened property (T) [**Laf02**, **Laf08**, **Laf09**, **Laf10**, **Laf12**].

6 AROUND THE DIXMIER PROBLEM FOR LOCALLY COMPACT GROUPS

ANALYSIS AND ITS OFFSPRING (functional analysis, harmonic analysis, operator algebras,. . .) are particularly efficient to study *unitary* representations. General representations are more untractable—and their geometric behavior can be rather unexpected, as we have seen in the preceding chapter. It is therefore natural to wonder whether a given general representation could be *unitarizable*, that is, actually equivalent to a unitary representation. The answer turns out to depend on the group and to be related with amenability. We investigate here this question for locally compact groups.

The goal of this chapter is to prove Theorem 6.1.5, which was originally proved by Monod and Ozawa for discrete groups. Background on unitarizability is given in Section 6.1. Preparatory material for the proof of Theorem 6.1.5 is explained in the next three sections: link with bounded cohomology in Section 6.2, stability by open subgroups in Section 6.3, and representations of permutational wreath products in Section 6.4. Theorem 6.1.5 is finally proved in Section 6.5. We end this chapter by some discussions about the class of unitarizable groups in Section 6.6.

In this chapter, contrary to the rest of this work, Hilbert spaces are complex, as customary for analysis.

6.1 Unitarizable groups and Dixmier's problem

*For details about the results of this section, as well as precise credits and references, we refer to the nice survey of G. Pisier [**Pis05**] as well as to his book on similarity problems [**Pis01**].*

A *unitary representation* is a linear representation lying in the subgroup $\mathrm{U}(\mathcal{H})$ of the general linear group $\mathrm{GL}(\mathcal{H})$ of a Hilbert space. Observe that this definition is not "natural", in the sense not "invariant under automorphisms of $\mathrm{GL}(\mathcal{H})$". It therefore makes sense to consider as "morally unitary" a representation that lies in a conjugate of $\mathrm{U}(\mathcal{H})$.

Definition 6.1.1. — *A linear representation* π *of a group* G *into a Hilbert space* $\mathcal{H}$ *is called* unitarizable *if there is an invertible operator* $\mathrm{S}\colon \mathcal{H} \to \mathcal{H}$ *such that the representation* $\mathrm{S}^{-1}\pi\mathrm{S}$ *is a unitary representation.*

By $\mathrm{S}^{-1}\pi\mathrm{S}$, we mean the representation defined by $(\mathrm{S}^{-1}\pi\mathrm{S})(g) = \mathrm{S}^{-1}\pi(g)\mathrm{S}$. It follows immediately from the definition that a unitarizable representation is uniformly bounded (by $\| \mathrm{S}^{-1} \| \| \mathrm{S} \|$). When does the converse hold? As the answer depends strongly on the group, the following definition will be convenient.

Definition 6.1.2. — *A topological group is said* unitarizable *if all its continuous uniformly bounded representations on a Hilbert space are unitarizable.*

Observe that, as we are dealing with uniformly bounded representations, continuity can be understood indifferently as joint or orbital continuity (cf. Lemma 4.1.6).

Before going further, it is important to remark that, from a geometric point of view, the unitarizability of a representation is a renorming problem. Indeed, if the representation π is unitarizable, then the norm on $\mathcal{H}$ defined by $\| x \|' = \| \mathrm{S}^{-1}x \|$ is an equivalent Hilbert norm which is invariant under π. On the other hand, if $\| \cdot \|'$ is an equivalent Hilbert norm on $\mathcal{H}$, then there is an isometry S from $(\mathcal{H}, \| \cdot \|)$ to $(\mathcal{H}, \| \cdot \|')$. Therefore, if π is unitary for $\| \cdot \|'$, then $\mathrm{S}^{-1}\pi\mathrm{S}$ is unitary for $\| \cdot \|$. Consequently, we can say that a representation is unitarizable if and only if there is an equivalent invariant *Hilbert* norm. Analogously, a representation is uniformly bounded if and only if there is an equivalent invariant (Banach) norm. Hence the question of the unitarizability of a representation can be paraphrased as "*if we can find an equivalent invariant norm, can we find a Hilbert one?*".

In the negative direction, the first example of a uniformly bounded but non-unitarizable representation was given for the group $\mathrm{SL}_2(\mathbf{R})$ in 1955 [**EM55**]. Several other explicit examples were found in the 1980's for nonabelian free groups. We will observe below (Proposition 6.3.1) that unitarizability passes to open subgroups, hence any discrete group containing a nonabelian free subgroup is not unitarizable.

In the positive direction, Szőkefalvi-Nagy showed in 1947 [**SzN47**] that uniformly bounded representations of the integers are unitarizable. This was generalized a few years later independently by Day [**Day50**, Theorem 8] and Dixmier [**Dix50**, Théorème 6] for continuous representations of amenable groups. We recall their argument[1].

Lemma 6.1.3. — *Let* G *be a topological group. If* G *is amenable, then* G *is unitarizable.*

Proof. — Let π be a uniformly bounded continuous representation of G on some Hilbert space $\mathcal{H}$ (whose scalar product will be written $\langle\,,\,\rangle$) and let $\mathfrak{m}$ be a right-invariant mean on the space of left-uniformly continuous bounded (complex-valued) functions $\mathscr{C}_{\mathrm{lucb}}(\mathrm{G})$. For any $x, y \in \mathcal{H}$, define the map

$$\psi_{x,y}\colon \mathrm{G} \to \mathbf{C}\colon g \mapsto \langle \pi(g)x, \pi(g)y \rangle.$$

Observe that $\psi_{x,y}$ is a left uniformly continuous bounded map since π is continuous and uniformly bounded. Moreover,

$$h \cdot \psi_{x,y} = \psi_{\pi(h)x,\pi(h)y}$$

(recall that the action on $\mathscr{C}_{\mathrm{lucb}}(\mathrm{G})$ is by right translation of the argument). On the other hand, it is straightforward to check that, for any $g \in \mathrm{G}$, the map $(x, y) \mapsto \psi_{x,y}(g)$ is a scalar product on $\mathcal{H}$, and this family of scalar products is uniformly equivalent to the original one: there is a constant C such that

$$\mathrm{C}^{-1}\langle x, x\rangle \leqslant \psi_{x,x}(g) \leqslant \mathrm{C}\langle x, x\rangle$$

for any $x \in \mathcal{H}$ and $g \in \mathrm{G}$ (of course, this constant C is the square of a bound for $\| \pi(g) \|$).

Putting all these observations together, we can therefore define a new scalar product on $\mathcal{H}$, equivalent to the original one and invariant via π, by the formula

$$(x, y) \mapsto \mathfrak{m}(\psi_{x,y}). \qquad \blacksquare$$

Prompted by his result, Dixmier asked the following question in [**Dix50**, p. 221] (see also [**Day50**, p. 291]).

[1] Dixmier and Day used invariant means on the whole space of bounded continuous functions on the group, but their proof requires only a mean on the subspace of bounded left-uniformly continuous functions (compare with Footnote 7 on page 110). Day attributes his Theorem 8 in loc. cit. to Kaplansky.

Problem 9 (Dixmier). — Is there a locally compact unitarizable nonamenable group?

The problem was considered originally by Dixmier for general topological groups, although it actually attracted even more interest for discrete groups. However, it admits a trivial positive answer for non-locally compact groups (see the next example), since such groups may well have no uniformly bounded representation at all. This pathology cannot appear for locally compact groups, that are always endowed with at least their regular representation.

Example 6.1.4 (Polish unitarizable nonamenable groups). — Let $G = \mathrm{Homeo}_+(\mathbf{S})$ be the Polish group of orientation-preserving homeomorphisms of the circle (endowed with the compact-open topology). This group is not amenable since it cannot preserve any probability measure on the circle. Choose a base point $*$ in $\mathbf{S}$ and consider the stabilizer G_* in G; it is isomorphic to the group $\mathrm{Homeo}_+[0,1]$. Any uniformly bounded representation of G on a Hilbert space should vanish on G_* (see Example 5.2.16), hence on the whole G since G is simple (see e.g. [**Ghy01**, Theorem 4.3]). Therefore, any uniformly bounded representation of G is trivial, hence in particular unitarizable.

For the same reason, any topologically simple group that contains $\mathrm{Homeo}_+(\mathbf{S})$ (or $\mathrm{Homeo}_+[0,1]$) is unitarizable. This enables to produce many examples of Polish unitarizable nonamenable groups, thanks to D. Epstein's simplicity criterion [**Eps70**, Theorem 1.1][2]. For instance, the groups $\mathrm{Homeo}(\mathbf{M})$, where $\mathbf{M}$ is a compact manifold of dimension at least two, are also unitarizable and not amenable[3].

So far, Dixmier's problem for locally compact or even discrete groups is still open. Finding a unitarizable nonamenable group is notoriously difficult for at least three reasons. Firstly, as usual, everything works smoothly in the realm of connected groups, for which unitarizability implies amenability (cf. [**Pis05**, Remark 0.8]). Hence all the powerful Lie theoretical tools for studying representations are probably useless for Dixmier's problem.

Secondly, a discrete unitarizable group cannot contain a nonabelian free subgroup and there is a tremendous lack of tractable examples of nonamenable groups without free subgroups. Perhaps the best candidates for an answer to the Dixmier problem are to be found among Monod's Frankenstein groups [**Mon13**], as they enjoy several weaker forms of amenability.

Thirdly, two recent results show that slightly strengthening unitarizability is enough

[2] Epstein's criterion yields simplicity of the derived subgroup. But even for non-perfect groups, this argument yields unitarizability of the whole group, since the quotient by the derived subgroup is abelian. See Lemma 6.6.1 below for more details on that step.

[3] The inclusion $\mathrm{Homeo}_+(\mathbf{S}) \hookrightarrow \mathrm{Homeo}(\mathbf{M})$ can be seen by choosing a circle in $\mathbf{M}$ such that a sufficiently small neighborhood V is homeomorphic to $\mathbf{S} \times \mathbf{B}$, where $\mathbf{B}$ is the ball of dimension $\dim(\mathbf{M}) - 1$.

to imply amenability (actually, to be equivalent to it), hence the gap between amenability and unitarizability, if it exists, is narrow.

Quantitative strengthening Pisier showed that, if a locally compact group G is unitarizable, then there is a natural number $d \geqslant 2$ such that the operator S in Definition 6.1.1 can always be chosen so that

$$\|S\|\|S^{-1}\| \leqslant \left(\sup_{g\in G} \|\,\pi(g)\,\|\right)^d .$$

This suggests a hierarchy of unitarizable groups, ranked according to $d(G)$, the smallest d such that the above holds. Pisier then showed that (infinite) amenable groups are precisely those groups for which $d(G) = 2$ (see [**Pis05**, Theorem 1.1] and [**Pis98**, §3])[4].

Structural strengthening Monod and Ozawa showed that if G is a discrete unitarizable group such that $G \wr A$ is also unitarizable, where A is some infinite countable abelian group, then G is amenable (see [**MO10**, Theorem 1]).

Pisier's quantitative result holds for locally compact groups. The purpose of this chapter is to give a locally compact version of the Monod–Ozawa theorem, namely the following.

Theorem 6.1.5. — *Let* G *be any locally compact group. For any infinite abelian discrete group* A*, the following assertions are equivalent.*

(i) The group G *is amenable;*

(ii) The locally compact group $A \wr_{G/O} G$ *is unitarizable, where* $O < G$ *is a suitable open subgroup.*

Remark 6.1.6. — The meaning of "suitable" in the second assertion will become clear during the proof; loosely speaking, it will be the preimage of a compact open subgroup of some totally disconnected quotient. Examples for two extreme cases are:

- For discrete groups, O can be chosen to be any finite group with trivial core. For instance, O can be trivial, in which case we recover the original Monod–Ozawa theorem.

[4] As a consequence of this characterization of amenability, a positive answer to Problem 9 is equivalent to the non-collapsing of the hierarchy $d(G)$.

- For connected groups, the only open subgroup is $O = G$ and the wreath product boils down to the direct product of A and G. At any rate, unitarizability and amenability are equivalent for connected groups (cf. [**Pis05**, Remark 0.8]).

6.2 Unitarizability and bounded cohomology

In order to use the full power of induction methods, we will first translate unitarizability problem as a bounded cohomology problem.

Let π be a *unitary* representation of a topological group G into a (complex) Hilbert space $\mathcal{H}$. A *derivation* (relative to π) is a map $D\colon G \to \mathscr{B}(\mathcal{H})$ satisfying the following Leibniz rule:

$$D(gh) = D(g)\pi(h) + \pi(g)D(h) \qquad \text{for all } g, h \in G.$$

A derivation is *inner* if there is an operator $T\colon \mathcal{H} \to \mathcal{H}$ such that $D = \pi T - T\pi$. The cohomological smell is vivid and indeed, a map D is a derivation if and only if the map

$$D\pi^*\colon G \to \mathscr{B}(\mathcal{H})\colon g \mapsto D(g)\pi(g)^*$$

is a cocycle for the Banach G-module $\mathscr{B}(\mathcal{H})$, where the action is given by

$$g \cdot S = \pi(g) S \pi(g)^* \qquad g \in G, S \in \mathscr{B}(\mathcal{H}).$$

In particular, the space of all bounded derivation, modulo the subspace of inner derivation, is canonically isomorphic to the first bounded cohomology group $H^1_b(G, \mathscr{B}(\mathcal{H}))$.

Going back to derivations, we define new operators on the direct sum $\mathcal{H} \oplus \mathcal{H}$ as

$$\pi_D(g) = \begin{pmatrix} \pi(g) & D(g) \\ 0 & \pi(g) \end{pmatrix} \qquad \text{for all } g \in G.$$

It is straightforward to check that $\pi_D\colon G \to \mathscr{B}(\mathcal{H})$ is a morphism if (and only if) D is a derivation relative to π. Moreover, the operator norms satisfy

$$\Big(1 + \| D(g) \|^2_{\mathscr{B}(\mathcal{H})}\Big)^{\frac{1}{2}} \leqslant \| \pi_D(g) \|_{\mathscr{B}(\mathcal{H}\oplus\mathcal{H})} \leqslant 1 + \| D(g) \|_{\mathscr{B}(\mathcal{H})} \qquad (g \in G),$$

hence π_D is uniformly bounded if and only if D is a bounded map.

Now comes the unitarizability translation[5].

[5]This well-known fact is easy to find in the literature in an operator algebra form (e.g., [**Pau86**, Propo-

Lemma 6.2.1. — *The representation π_D is unitarizable if and only if* D *is inner.*

Proof. — If $D = \pi T - T\pi$ for some operator T, then the operator $S = \begin{pmatrix} \mathrm{Id} & T \\ 0 & \mathrm{Id} \end{pmatrix} \in \mathscr{B}(\mathcal{H} \oplus \mathcal{H})$, whose inverse is $\begin{pmatrix} \mathrm{Id} & -T \\ 0 & \mathrm{Id} \end{pmatrix}$, is a unitarizer for π_D, as

$$S\pi_D S^{-1} = \begin{pmatrix} \pi & 0 \\ 0 & \pi \end{pmatrix}.$$

Conversely, assume S is a unitarizer for π_D and write $A = SS^*$. As $S^{-1}\pi_D S$ is unitary, we have

$$S^{-1}\pi_D(g^{-1})S = S^*\pi_D(g)^*(S^{-1})^* \qquad \text{for all } g \in G,$$

from which we deduce the relation

$$\pi_D(g)A = A\pi_D(g^{-1})^*.$$

Expanding this equality in matrix form and reading the $(2,2)$- and $(1,2)$-coefficients, we get in particular

$$\pi(g)A_{22} = A_{22}\pi(g) \tag{6.1}$$

$$\pi(g)A_{12} + D(g)A_{22} = A_{12}\pi(g) \tag{6.2}$$

(writing, as usual, $A = (A_{ij})$).

But A_{22} is itself an invertible operator on $\mathcal{H}$. Indeed, as A is positive and invertible, there is some $\delta > 0$ such that $\langle Ax, x\rangle \geqslant \delta \| x \|^2$ for all $x \in \mathcal{H} \oplus \mathcal{H}$. The same holds for A_{22}, hence the latter is injective and open and the claim follows from the open mapping theorem. Therefore, we can rewrite relations (6.1) and (6.2) as

$$D(g) = (A_{12}A_{22}^{-1})\pi(g) - \pi(g)(A_{12}A_{22}^{-1}) \qquad \text{for all } g \in G,$$

hence D is inner. ■

To sum up: a uniformly bounded non-unitarizable representation is associated to each bounded non-inner derivation, and the latter corresponds to a representative of a nontrivial class in $H^1_b(G, \mathscr{B}(\mathcal{H}))$. All this translation, being elementary, works smoothly for any topological group—but is especially useful for locally compact groups, since the latter enjoy more powerful bounded cohomology tools).

We record for later use that non-unitarizability of groups containing a nonabelian free

sition 8.9], [**Pis01**, Lemma 4.5],...), we spell out here its group representation form for the convenience of the reader.

subgroup can be established by this construction.

Proposition 6.2.2. — *Let* G *be a discrete group containing a nonabelian free subgroup. Then there is a non-inner bounded derivation* $D\colon G \to \mathscr{B}(\ell^2(G))$ *relative to the left regular representation* λ *on* $\ell^2(G)$*. In particular,* λ_D *is uniformly bounded but not unitarizable and* $H^1_b(G, \mathscr{B}(\ell^2(G)))$ *does not vanish.*

On the proof. — This is essentially contained in Lemma 2.7(ii) and in the proof of Theorem 2.1 of [**Pis01**]. Observe that the derivation D is built in loc. cit. with the usual trick as an inner derivation over another space (namely, $\mathscr{B}(\ell^1(G))$) that happens to also lie in (and be bounded on) $\mathscr{B}(\ell^2(G))$, but where it is not inner anymore. □

Remark 6.2.3. — As a by-product, we observe that all Banach G-modules $\mathscr{B}(\mathcal{H})$ (for the above-mentioned action) have trivial first bounded cohomology groups for unitarizable groups. Recall that amenability of locally compact groups is also characterized as the vanishing of bounded cohomology groups of positive degree for all dual Banach modules (see Theorem 2.5 and the middle of p. 35 in [**Joh72**]) and that $\mathscr{B}(\mathcal{H})$ is the dual of the space of trace-class operators. This suggests the following problem.

Problem 10. — Find a bounded cohomology characterization of unitarizability (at least for locally compact groups).

6.3 Unitarizability and induction

We show now that unitarizability passes to open subgroups.

Proposition 6.3.1. — *Let* G *be a topological subgroup. If* G *is unitarizable, then so are all its open subgroups.*

Proof. — Let H be an open subgroup of G and π be a continuous uniformly bounded representation of H on a Hilbert space $\mathcal{H}$. Let $\sigma\colon G/H \to G$ be a section of the projection, such that $\sigma(H) = e$, and let ρ be the representation of G on $\ell^2(G/H, \mathcal{H})$ defined by

$$\rho(g)f(x) = \pi(\beta(g^{-1}, x)^{-1})f(g^{-1}x),$$

where $\beta\colon G \times G/H \to H$ is the cocycle determined by σ, that is, $\beta(g, x) = \sigma(gx)^{-1}g$. Observe that β is continuous since H is open, hence ρ is a continuous uniformly bounded representation of G. There is therefore an invertible operator S such that $S\rho S^{-1}$ is unitary.

Let $\mathcal{H}_0$ be the copy of $\mathcal{H}$ in $\ell^2(G/H, \mathcal{H})$ corresponding to functions supported on the trivial coset. As $\sigma(H) = e$, the restriction of ρ to H preserves $\mathcal{H}_0$, and produces on the latter the starting representation π. Writing T for the isomorphism between $\mathcal{H}_0$ and $S\mathcal{H}_0$ defined by S, we can again identify $T\pi T^{-1}$ with a restriction of the unitary representation $S\rho S^{-1}$. Since $T\pi T^{-1}$ is nothing but the representation π transported on $S\mathcal{H}_0$, this shows that π is unitarizable. ■

At this point, a strategy to attack Dixmier's problem arises. Could we generalize Proposition 6.3.1 to measured induction, so that the Gaboriau–Lyons theorem would yield that any unitarizable group is amenable? This strategy was proposed in [**Mon06**, Problem N]. Unfortunately, it does not seem that a representation is unitarizable whenever the measurably induced representation is so, as the latter does not exactly contain the former (contrary to induction with respect to open subgroups as above). However, N. Monod and O. Ozawa were still able to use this strategy in [**MO10**] to deduce, in the discrete case, that amenability of G follows from unitarizability of suitable extensions of G by abelian groups. The generalization of Gaboriau and Lyons's theorem to locally compact groups allows to generalize Monod–Ozawa result for locally compact groups.

6.4 Permutational wreath products and their representations

We give here some very basic facts about wreath products. Let G and H be two groups and assume G acts on a set X. The *permutational wreath product* of G and H over X is the group

$$H \wr_X G = \left(\bigoplus_X H\right) \rtimes G,$$

where the action of G on $\bigoplus_X H$ is given by permutations on X (recall that $\bigoplus_X H$ is the group of finitely supported functions from X to H). We will omit the index X in the case where X = G and G acts on itself by left translation. We will only work with transitive G-sets, hence in particular $H \wr_X G$ is generated by a copy of H and a copy of G.

We will consider the case where G is endowed with a nontrivial topology, but H is discrete. In that setting, we require that the action on the discrete set X is continuous, that is, that stabilisers are open. This ensures that the action on $\bigoplus_X H$ is also continuous, hence the permutational wreath product is naturally endowed with a group topology (cf. [**Bou71**, III, §2, n° 10]). Moreover, $H \wr_X G$ is locally compact as soon as G is so.

The definition of semidirect products yields the following easy lemma.

Lemma 6.4.1. — *Let* A *and* B *be two subgroups of a group* C *and* X *be a transitive* A*-set. Let* $x_0 \in X$ *be some base point and write* $K = \mathrm{Stab}_A(x_0)$*. Assume that*

1. *the subgroups* K *and* B *commute;*
2. *the subgroups* B *and* B^a *commute for every* $a \in A$.

Then there is a unique surjection from the permutational wreath product $B \wr_X A$ *onto the subgroup generated in* C *by* A *and* B*, whose restrictions to the subgroups* A *and* B' *of* $B \wr_X A$ *are the identity maps, where* $B' \simeq B$ *is the subgroup of maps* $X \to B$ *whose support is contained in* $\{x_0\}$.

Proof. — Let $N = \bigoplus_X B$ and write elements of N as functions from X to B with finite support. Choose a representative set $\mathcal{R}$ for the cosets of K in A. We first build a morphism $\psi \colon N \to \langle A, B\rangle$ as

$$\psi(f) = \sum_{a \in \mathcal{R}} f(ax_0)^a \qquad \text{for } f \in N.$$

This finite sum is well defined, as B commutes with B^a for every $a \in A$, and is a morphism for the same reason. Moreover, it does not depend on the choice of $\mathcal{R}$ as K and B commute. This independence yields $\psi(a \cdot f) = a\psi(f)a^{-1}$ for every $a \in A$. Therefore, the map $\varphi \colon B \wr_X A \to \langle A, B\rangle$ defined as $\varphi((f,a)) = \psi(f)a$ ($f \in N$, $a \in A$) is a morphism: indeed,

$$\begin{aligned}\varphi((f,a)(f',a')) &= \varphi((f(a\cdot f'), aa')) = \psi(f(a\cdot f'))aa' \\ &= \psi(f)a\psi(f')a' = \varphi((f,a))\varphi((f',a')).\end{aligned}$$

Lastly, φ is surjective, as A and B are trivially in its image. ∎

Note that, even if this surjection does not depend on the choice of a representative set, it does depend on the choice of the base point. If φ' is the surjection built from another base point $x_1 = a_1x_0$, then we have $\varphi'(f,a) = \varphi(a_1^{-1}fa_1, a)$. When there is a canonical choice of a base point, for instance when X is already given as a quotient G/K, we will implicitly use it for the surjection of the above lemma.

Applied to representations, Lemma 6.4.1 gives the useful following tool.

Corollary 6.4.2. — *Let* (π, E) *and* (ρ, E) *be representations of* G *and* H *into the same Banach space* E*, respectively. Let* G *act continuously and transitively on some set* X *and write* $K = \mathrm{Stab}_G(x_0)$ *for the stabiliser of some base point* $x_0 \in X$*. Assume that the representation* ρ *commutes with* $\pi|_K$ *and with* $\rho^{\pi(g)}$ *for every* $g \in G$*. Then there is a unique representation* $\rho \wr_X \pi$ *of* $H \wr_X G$ *into* E *whose restrictions to the subgroups* G *and* H' *are* π *and* ρ*, respectively, where* $H' \simeq H$ *is the subgroup of maps* $X \to H$ *whose support* *is contained in* $\{x_0\}$*.*

If moreover π *and* ρ *are isometric, then so is* $\rho \wr_X \pi$*.*

If G *is endowed with a topology and* H *is discrete, then* $\rho \wr_X \pi$ *is orbitally continuous whenever* π *is so.*

Proof. — The existence and uniqueness of the representation $\rho \wr_X \pi$ is the previous lemma. If $\pi(G)$ and $\rho(H)$ lie in the isometry group of E, then so does the subgroup they generate. For the continuity statement, observe that, representing elements of $N = \bigoplus_X H$ as finitely supported maps, $(\rho \wr_X \pi)(f, g) = \widetilde{\rho}(f)\pi(g)$, where $\widetilde{\rho}$ is the extension of ρ to N (i.e. the map ψ in the previous lemma). Orbital continuity is then ensured thanks to discreteness of H. ■

6.5 Proof of Theorem 6.1.5

We now undertake the proof of Theorem 6.1.5, which was proved for discrete groups in [**MO10**].

As amenability is stable by extension and implies unitarizability, the direction (i) $\Rightarrow$ (ii) is straightforward. Hence we shall assume that G is nonamenable and prove that the permutational wreath product $A \wr_{G/O} G$ is non-unitarizable.

We first claim that we can suppose that G is σ-compact and A countable (so that $A \wr_{G/O} G$ is also σ-compact). Indeed, if G_0 is some open σ-compact subgroup of G and A_0 some infinite countable subgroup of A, then for any open subgroup $O < G_0$, the subgroup $A_0 \wr_{G_0/O} G_0$ is open in $A \wr_{G/O} G$ (observe that O is also open in G). As unitarizability passes to open subgroup by Proposition 6.3.1, our claim is proved.

We now distinguish two cases according to whether the connected component G^o is amenable.

Assume first that G^o is not amenable. Then, upon replacing G by an open subgroup (which we may thanks to Proposition 6.3.1), we can assume by Proposition 1.3.1 that G has a non-compact connected simple Lie quotient. By Remark 0.8 in [**Pis05**], this quotient is then non-unitarizable, hence so are G and any wreath product of the form $A \wr_X G$, since the latter admits G as a quotient.

Assume now that G^o is amenable. Then $G_1 = G/\operatorname{Ramen}(G)$ is a totally disconnected nonamenable σ-compact group. Moreover, any compact open subgroup $K < G_1$ has trivial core. Choose such a K and write O for the preimage of K in G. Observe that $\mathrm{A} \wr_{\mathrm{G}_1/\mathrm{K}} \mathrm{G}_1$ is a quotient of $\mathrm{A} \wr_{\mathrm{G}/\mathrm{O}} \mathrm{G}$ because Ramen(G) acts trivially on G/O. It is therefore enough to prove the non-unitarizability of $\mathrm{A} \wr_{\mathrm{G}_1/\mathrm{K}} \mathrm{G}_1$. We will write $\mathrm{N} = \bigoplus_{\mathrm{G}_1/\mathrm{K}} \mathrm{A}$, so that the latter wreath product is $\mathrm{N} \rtimes \mathrm{G}_1$.

By Theorem 3.2.1, there is a tychomorphism Σ from the free group F_2 to G_1. Let

$$(\Sigma, m) \cong (\mathrm{F}_2, \#) \times (\mathrm{Y}, \nu)$$

be the decomposition of Σ as an amplification of F_2 and write β for the corresponding cocycle $\mathrm{F}_2 \times \mathrm{Y} \to \mathrm{G}_1$, as in Section 2.3. We may and shall assume that the space Y is atomless, since otherwise the group G_1 would be a discrete subgroup with a nonabelian free subgroup (Proposition 2.6.1), hence G_1, as its wreath product, would be non-unitarizable by Proposition 6.3.1.

Recall from Proposition 6.2.2 that the non-unitarizability of F_2 follows from the non-vanishing of $\mathrm{H}^1_{\mathrm{b}}(\mathrm{F}_2, \mathscr{B}(\mathrm{V}))$, where $\mathrm{V} = \ell^2(\mathrm{F}_2)$ is endowed with the left regular representation λ. By L^∞-measured induction (Proposition 4.3.5), this yields the non-vanishing of $\mathrm{H}^1_{\mathrm{b}}(\mathrm{G}_1, \mathrm{E})$, where

$$\mathrm{E} = \mathrm{L}^\infty_{\mathrm{w}*}(\mathrm{Y}, \nu; \mathscr{B}(\ell^2(\mathrm{F}_2))).$$

Recall from Remark 4.3.8 that the action of G_1 on E is given by twisting the usual action by translation in the argument by the action of the cocycle on $\mathscr{B}(\mathrm{F}_2)$, and the latter is itself a conjugation of the representation λ. In symbols ($g \in \mathrm{G}$, $\varphi \in \mathrm{E}$, $y \in \mathrm{Y}$)[6]:

$$(g \cdot \varphi)(y) = \lambda(\beta(g^{-1}, y)^{-1})\varphi(g^{-1} \cdot y)\lambda(\beta(g^{-1}, y)).$$

Our goal is now to reach from E a non-vanishing result in bounded cohomology for a $(\mathrm{A} \wr_{\mathrm{G}_1/\mathrm{K}} \mathrm{G}_1)$-module of the form $\mathscr{B}(\mathcal{H})$.

Consider the L^2-measured induced representation σ of V (cf. Section 4.3.C, p. 85), given by

$$(\sigma(g)(f))(y) = \Delta_{\mathrm{G}_1}^{-\frac{1}{2}}(g)\lambda(\beta(g^{-1}, y)^{-1})f(g^{-1} \cdot y),$$

for $g \in \mathrm{G}_1$ and $f \in \mathrm{W} = \mathrm{L}^2\Sigma\mathbf{ind}_{\mathrm{F}_2}^{\mathrm{G}_1}\mathrm{V} = \mathrm{L}^2(\mathrm{Y}, \mathrm{V})$. Conjugation by σ gives a G_1-representation on $\mathscr{B}(\mathrm{W})$, that we will denote by π. This representation preserves the subalgebra E of $\mathscr{B}(\mathrm{W})$ and produces on it the former L^∞-measured induced representation. Indeed, writing M_φ for the operator in $\mathscr{B}(\mathrm{W})$ corresponding to the multiplication

[6]Throughout the proof, we alleviate the description of the actions by using a "pointwise" definition. The reader can check that these descriptions always make sense (i.e., do not depend on a choice of a representative in the corresponding class of functions and do not wrongfully permute "for all" and "for almost all" quantifiers).

by a function $\varphi \in E$, we get, for $f \in W$ and $y \in Y$,

$$\begin{aligned}(\sigma(g)M_{\varphi}\sigma(g)^*)(f)(y) &= \Delta_{G_1}^{-\frac{1}{2}}(g)\lambda(\beta(g^{-1},y)^{-1})\left[(M_{\varphi}\sigma(g^{-1})(f)(g^{-1}\cdot y)\right] \\ &= \Delta_{G_1}^{-\frac{1}{2}}(g)\lambda(\beta(g^{-1},y)^{-1})\varphi(g^{-1}\cdot y)\left[(\sigma(g^{-1})(f)(g^{-1}\cdot y)\right] \\ &= \lambda(\beta(g^{-1},y)^{-1})\varphi(g^{-1}\cdot y)\lambda(\beta(g,g^{-1}\cdot y)^{-1})f(g\cdot(g^{-1}\cdot y)) \\ &= (g\cdot\varphi)(y)f(y) \\ &= (M_{g\cdot\varphi}f)(y).\end{aligned}$$

We now consider the space $\mathscr{B}(W)$ as a $L^{\infty}(Y)$-module, where the action of $L^{\infty}(Y)$ is given by pointwise multiplication:

$$\varphi\cdot T = M_{\varphi}T \qquad (\varphi \in L^{\infty}(Y),\ T \in \mathscr{B}(W)). \tag{6.3}$$

A similar computation to the above one (for the invariance of E) shows that this structure is compatible with the actions of G_1 on $\mathscr{B}(W)$ and on $L^{\infty}(Y)$ (the latter being given by precomposition on Y), in the following sense:

$$\pi(g)\varphi\pi(g^{-1})T = (g\varphi)\cdot T \qquad (T \in \mathscr{B}(W)). \tag{6.4}$$

In particular, as $L^{\infty}(Y)$ is a commutative algebra, we see that *any representation of* A *into the unitary group of* $L^{\infty}(Y)$ *commutes with its conjugate by* π. Therefore, in order to have a representation of the wreath product $A \wr_{G_1/K} G_1$, we only need to find a representation ρ of A into the unitary group of $L^{\infty}(Y)$ that commutes with $\pi|_K$ (cf. Corollary 6.4.2), that is, that lies into the unitary group of $L^{\infty}(Y)^K$.

Lemma 6.5.1. — *For any compact subgroup* K' *of* G_1*, the space* $L^{\infty}(Y)^{K'}$ *can be canonically identified with* $L^{\infty}(K'\backslash Y)$*, where* $K'\backslash Y$ *is the set-theoretical quotient.*

Proof. — Of course, the identification consists in viewing elements of $L^{\infty}(Y)^{K'}$ as functions invariant on the K'-orbits. The only problem is to endow the set-theoretical quotient $K'\backslash Y$ with a structure of a standard Borel space so that this identification makes sense.

This is possible because the action of G_1 on Y can, without loss of generality, be considered as a continuous one on an invariant Borel subset of a Polish space (a result established in [**Var63**, Theorem 3.2] for Polish locally compact groups; a stronger statement for all Polish groups can be found in [**BK96**, Theorem 5.2.1]). By compactness of K', the K'-orbits are therefore closed, hence the equivalence relation induced by K' is smooth (see [**Kan08**, Proposition 7.2.1(iv)]), which exactly means that the set-theoretical quotient $K'\backslash Y$ inherits a standard Borel structure from Y. ■

So, in particular, the representation ρ we are looking for lies into the unitary group of $L^\infty(K\backslash Y)$. We will need more.

Lemma 6.5.2. — *There exists a representation of* A *into the unitary group of* $L^\infty(K\backslash Y)$ *that generate* $L^\infty(K\backslash Y)$ *as a von Neumann algebra.*

Proof. — This lemma being concerned only with the measure class of $K\backslash Y$, we can as well find a suitable representation into $L^\infty(Z)$ for any standard measure space Z. Let us write Z as the disjoint union of an atomless part Z_1 and a purely atomic part Z_2: by considering the diagonal representation, we only need to find suitable representations into $L^\infty(Z_1)$ and $L^\infty(Z_2)$. For the atomless part, let $Z_1 = \widehat{A}$ be the Pontryagin dual of A. As the latter is countable and infinite, Z_1 is a compact metrizable infinite group, so it is an atomless standard measure space when endowed with its Haar measure. The Fourier transform yields an isomorphism of von Neumann algebras between $L^\infty(Z_1)$ and the group von Neumann algebra $vN(A)$. But the latter is, by definition, generated by the left regular representation of A on $\ell^2(A)$.

For the purely atomic part, we consider the product of $|Z_2|$ distinct characters of A. This is a morphism into the product of $|Z_2|$ copies of the circle group, that is, a representation into the unitary group of $\ell^\infty(Z_2)$. The von Neumann algebra $\mathscr{L}$ generated by its image is a subalgebra of $\ell^\infty(Z_2)$, which corresponds to an equivalence relation on Z_2: two points are equivalent if elements of $\mathscr{L}$ cannot separate them. Since the characters we chose are distinct, this equivalence relation is the equality, hence $\mathscr{L} = \ell^\infty(Z_2)$, which finishes the proof. ■

Let thus ρ be such a representation of A. So far, we have a representation $\rho \wr_{G_1/K} \pi$ of the wreath product $A \wr_{G_1/K} G_1$ on the Banach space $\mathscr{B}(W)$. In order to use the non-vanishing result in bounded cohomology that we know about the subspace E of $\mathscr{B}(W)$, we need to recover E from the action of the wreath product. For the following lemma, recall that $N = \bigoplus_{G_1/K} A$.

Lemma 6.5.3. — *The space* $\mathscr{B}(W)^N$ *of* N*-invariants vectors is precisely* E.

Proof. — In view of the definition of the action on $\mathscr{B}(W)$ and of its compatibility with the actions of G_1 (Formulæ (6.3) and (6.4)), the subspace of N-invariant vectors is the commutant of the union of the $g(L^\infty(Y)^K)$ $(= L^\infty(Y)^{gKg^{-1}})$ when g runs among G_1 (or, equivalently, among a representative set for G_1/K). On the other hand, the subspace E is itself the commutant of $L^\infty(Y) \otimes \mathbf{C}\,\mathrm{id}_V$ (see [**Tak02**, Theorem IV.5.9]). So all we have to show is that the von Neumann algebra generated by all the $L^\infty(Y)^{gKg^{-1}}$ is the whole $L^\infty(Y)$.

Thanks to Lemma 6.5.1, for any two compact subgroups K' and K'', the von Neumann algebra generated by $L^\infty(Y)^{K'}$ and $L^\infty(Y)^{K'}$ is $L^\infty(Y)^{K'\cap K''}$. Moreover, whenever K' is open, we have a conditional expectation from $L^\infty(Y)$ to $L^\infty(K'\backslash Y)$ which is given by the normalised integration over K'. Therefore, the net $L^\infty(Y)^{K'}$, where K' runs among the intersections of finitely many conjugates of K, is an inverse system. Moreover, the index set admits a cofinal sequence, since K has countably many conjugates (recall that G_1 is second countable). Therefore, we can apply the convergence theorem for martingales (see e.g. [**Kal97**, Theorem 6.23]), which implies that the algebra generated by all the $L^\infty(Y)^{gKg^{-1}}$ is weak-$*$ dense in the von Neumann subalgebra $B \leqslant L^\infty(Y)$ of maps that are measurable with respect to the common refinement of all the partitions arising as $Y \twoheadrightarrow K'\backslash Y$ (in other words, measurable with respect to the smallest σ-algebra making all projections $Y \twoheadrightarrow K'\backslash Y$ measurable). Since K has trivial core (we have from the beginning got rid of all compact normal subgroups), this common refinement is, thanks again to Lemma 6.5.1, trivial: the subalgebra B is the whole $L^\infty(Y)$. ∎

We finally have gathered all the tools to conclude the proof of Theorem 6.1.5. As N is an amenable normal subgroup of $A \wr_{G_1/K} G_1$, we have an isomorphism between

$$H^1_b(N \rtimes G_1, \mathscr{B}(W)) \qquad \text{and} \qquad H^1_b(G_1, \mathscr{B}(W)^N)$$

(see [**Mon01**, Corollary 7.5.10]). The latter space is itself isomorphic to $H^1_b(G_1, E)$ by Lemma 6.5.3, hence nonzero. But the non-vanishing of $H^1_b(N \rtimes G_1, \mathscr{B}(W))$ implies in particular that $N \rtimes G_1 = A \wr_{G_1/K} G_1$ is non-unitarizable. Thus ends the proof of Theorem 6.1.5.

Remark 6.5.4. — It is instructive to scrutinize the proof in order to understand why Monod and Ozawa's strategy did not completely solve Dixmier's problem. The only bounded cohomology statement we get about the nonamenable group G_1 via the measured induction is the non-vanishing of

$$H^1_b(G_1, L^\infty_{w*}(Y, \mathscr{B}(V))).$$

There is a priori no reason why this particular G_1-module could be identified with $\mathscr{B}(\mathcal{H})$ for some unitary G_1-module $\mathcal{H}$, hence contradicting the only bounded cohomology necessary condition for unitarizability of our knowledge. However, as we saw, this module *can* be identified to $\mathscr{B}(\mathcal{H})^N$ for some representation of $N \rtimes G_1$, hence yielding the non-unitarizability of this bigger group.

This difficulty to identify the module $L^\infty_{w*}(Y, \mathscr{B}(V))$ also motivates our desire to have refined bounded cohomology obstruction to unitarizability (Problem 10).

6.6 The class of unitarizable groups

To close this chapter, let us review a few results and problems about the class of unitarizable groups.

We already know that the class of unitarizable groups is stable by open subgroups, and the stability by quotient is obvious. What about other stability properties:

- closed subgroups;
- direct limit;
- direct products, semi-direct products or even extensions?

To our knowledge, the answers for these stability properties are unknown. Of course, if unitarizability is equivalent to amenability, then all these properties would hold. Conversely, if one of these does not hold, then this must imply the existence of a unitarizable nonamenable group. Hence Dixmier's problem is closely related to the robustness of unitarizability.

The standard way to prove the stability under closed subgroups would be to induce representations of the latter. But non-open subgroups are of measure zero in a locally compact group, hence the initial representation is completely "diluted" in the induced representation.

Problem 11. — Let G be a locally compact unitarizable group. Are all closed subgroups of G also unitarizable? Is it at least true that G does not contain a discrete nonabelian free subgroup?

Observe that these problems (contrary to the rest of this thesis) admit more or less elementary positive answers for discrete groups, since all subgroups are open there. By the way, the local compactness restriction in the statement is not a mere trace of our topological habits: indeed, discrete nonabelian free subgroups can be found inside an amenable non-locally compact group [**Har73**, Proposition 1.(iii)], even inside a Polish one with a biinvariant compatible metric [**CT16**, Theorem 4.2].

Regarding these gaps in our understanding of closed subgroups, it is actually surprising that tychomorphisms and measured induction can actually yield some interesting results about unitarizability!

The stability under direct limits seems to be a problem equivalent to Dixmier's one. Indeed, if there are unitarizable nonamenable groups, then Pisier's hierarchy [**Pis05**, Theorem 4.3] suggests that there should exist unitarizable groups that are "less and less" unitarizable, and a limit of such groups would not be unitarizable[7]. It is however known that unitarizability is a countable (or σ-countably) property, that is: if G is a discrete (resp. locally compact) group whose all countable (resp. closed σ-compactly generated) subgroups of G are unitarizable, then so is G. See Corollary 0.11 and the following Remark in [**Pis05**].

As for extension, it is not even known whether the direct product of two unitarizable groups is unitarizable. We have however the following straightforward generalisation of Day's and Dixmier's original result. Recall that a closed subgroup H of a topological group G is *coamenable* if there is a G-invariant mean on $\mathscr{C}_{\mathrm{lucb}}(\mathrm{H}\backslash\mathrm{G})$ (we endow the quotient $\mathrm{H}\backslash\mathrm{G}$ with the quotient of the left uniformity on G). Two particular extreme cases to have in mind is when G is amenable (then all its subgroups are coamenable) or when H is normal (in which case H is coamenable if and only if the quotient is amenable).

Lemma 6.6.1. — *Let* G *be a topological group and* $\mathrm{H} < \mathrm{G}$ *a unitarizable coamenable closed subgroup. Then* G *is unitarizable.*

In particular, a semidirect product $\mathrm{H} \rtimes \mathrm{Q}$ *of an amenable group* Q *acting on a unitarizable group* H *is unitarizable.*

Proof. — Let π be a uniformly bounded representation of G on a Hilbert space $\mathcal{H}$. As H is unitarizable, there is an equivalent scalar product $\langle\ ,\ \rangle$ on $\mathcal{H}$ for which the restriction $\pi|_{\mathrm{H}}$ is unitary. For any $x, y \in \mathcal{H}$, there is therefore a well-defined map

$$\psi_{x,y} : \mathrm{H}\backslash\mathrm{G} \to \mathbf{C} : \mathrm{H}g \mapsto \langle \pi(g)x, \pi(g)y \rangle.$$

We can now argue exactly as in Lemma 6.1.3: for any G-invariant mean $\mathfrak{m}$ on $\mathscr{C}_{\mathrm{lucb}}(\mathrm{H}\backslash\mathrm{G})$, the map

$$(x, y) \mapsto \mathfrak{m}(\psi_{x,y})$$

is a G-invariant equivalent scalar product on G. ■

Note that the roles of Q and H in the previous lemma matter: actually, if they could be swapped, then Dixmier's problem would be solved. More precisely: *if* $\mathrm{Q} \ltimes \mathrm{H}$ *is unitarizable whenever* Q *is a unitarizable discrete (resp. locally compact) group acting continuously on an amenable discrete group* H*, then all unitarizable discrete (resp. locally compact) groups are amenable.* Indeed, this would in particular apply to the case where $\mathrm{H} = \bigoplus_{\mathrm{X}} \mathrm{A}$, where A is abelian and X is a transitive G-set, hence Q would be amenable by Theorem 6.1.5.

Lastly, as Pisier's and Monod–Ozawa results suggest that unitarizability is at least

[7] This is only a heuristic: Pisier's hierarchy may as well stabilize.

"close" to amenability, a first step towards a better understanding of Dixmier's problem is to show how unitarizability is incompatible with some strong negation of amenability.

Problem 12. — Assume that a locally compact group is both Kazhdan and unitarizable. Must the group be compact?

Recall that this is indeed the case for Kazhdan amenable groups [**BHV08**, Theorem 1.1.6]. Another parallel with amenability is the fact that this mutual exclusion does not hold beyond the locally compact case: the Polish unitarizable groups of Example 6.1.4 are also Kazhdan (for the same trivial lack of representations).

On a related note, I. Epstein and Monod have proved that the non-vanishing of the first ℓ^2-Betti number (another strong negation of amenability) implies non-unitarizability for residually finite groups [**EM09**, Theorem 1.1].

7 PERSPECTIVES

SEVERAL NATURAL PROBLEMS are raised by our work. Some of them were already stated and motivated in the text (for the convenience of the reader, we list them below). We would like here to discuss three more vague and indefinite questions, related to tychomorphisms, induction, and fixed-point properties.

Tychomorphisms

Tychomorphisms are to measure equivalence what monomorphisms are to isomorphisms. Can this analogy be made more precise? In other words, is there a suitable categorical framework encompassing measure equivalence, where tychomorphisms are injective arrows? Hints for the existence of such a category are numerous:

- functorial interplay with bounded cohomology via measured induction;
- analogy between measure equivalence and quasi-isometry;
- some rigidity results for measure equivalence [**Fur99**, **Kid10**] look like the determination of a "group of automorphisms" in that putative category.

For these categorical considerations, it is likely that some equivalence relations on tychomorphisms have to be considered: if Σ is a tychomorphism from H to G, then $\Sigma \times Z$, where Z is any standard probability space with trivial actions of G and H, is another tychomorphism that carry actually the same information as Σ.

Induction

The reader fond of Mackey's induction was probably left unsatisfied with Chapter 4. Many questions can be asked: pick any book on Mackey's induction and look how much still holds for measured induction, and how much can be translated for moderate inductions. We can also ask for a more cohomological framework into which these inductions take place. However, we feel that such a vast research program should be first motivated by more (concrete or expected) applications of these induction techniques.

Fixed-point properties

Many, many fixed-point properties for groups are studied in the literature. The relationships between them or with other group properties is often more or less clear for discrete groups. For topological groups, however, the picture is more blurred since several continuity conditions can be imposed on the considered actions: at the very least, we get two a priori different properties by considering joint or orbital continuity. Sometimes, abstract considerations can show that these formally different properties are actually equivalent: for instance, for isometric actions on Banach spaces, joint and orbital continuities agree, full stop. (This partially motivated us to look for other, more refined, abstract arguments for equivalence of different continuity conditions for group actions, see Appendix C.) We suspect however than the following phenomenon occurs: whenever no abstract argument of that kind is available, these properties are indeed distinct, unless they are restricted to a nice class of topological groups (for instance locally compact σ-compact groups or Polish groups).

So here is a vague program: choose any fixed-point property, define as many variants as possible by playing with continuity conditions, and see when they agree. Besides filling the gap in our understanding of fixed-point properties, we believe that this maybe thankless task may help to understand "wild" topological groups (by which we mean non-locally compact σ-compact, or non-Polish).

List of problems

We list here, with a few keywords, the problems explicitly stated in the text (one of them will actually appear in Appendix B).

A INTEGRATION AND DISINTEGRATION OF MEASURES

THE PURPOSE OF THIS SMALL APPENDIX is to check that disintegration of measures is indeed available in the setting of measured equivalence relations, so that Definition 3.1.7 makes sense. This technical verification is probably trivial for measure theorists but we were unable to locate in their abundant literature a disintegration theorem that encompasses exactly that setting. It is likely that a more direct argument is available.

Terminology in measure theory differ from one reference book to another; here are the definitions we use. For this appendix, all measures are σ-additive functions defined on some σ-algebra of subsets and taking values in $[0, +\infty]$.

- A measure is called *Borel* if it is defined on the σ-algebra of Borel subsets of some topological space.
- A Borel measure is *locally finite* if each point admits a neighborhood of finite measure.
- A Borel measure is *inner regular* if it can be approximated via compact sets, that is, for any Borel set B, the measure $m(\mathrm{B})$ is the supremum of the measures $m(\mathrm{K})$ for all compact sets K contained in B.
- A *Radon* measure is a Borel measure which is locally finite and inner regular.
- Two measures are *equivalent* if they are defined on the same σ-algebra and have the same null sets.

For σ-finite measures, the Radon–Nikodym theorem asserts that equivalent measures can be retrieved by their density; but for general measures, this equivalence relation is loose. A somewhat trivial example is given by the fact that, for any measure m, we can define a boring equivalent measure m' via

$$m'(\mathrm{A}) = \begin{cases} \infty & \text{if } m(\mathrm{A}) > 0 \\ 0 & \text{otherwise.} \end{cases}$$

To avoid cumbersome notations, we will always drop the notation of the σ-algebra on which our measures are defined, since the same space will not be endowed with different σ-algebras. Consequently, we will write in short "m is a measure on X" when m is defined on the σ-algebra of subsets attached to X; when X is a topological space, this will always be the Borel σ-algebra.

A.1 Pseudo-images of measures

If we have a measurable map p from a measured space (T, τ) to a measurable space X, we can always define the *image measure* $p(\tau)$ on X by $p(\tau)(\mathrm{A}) = \tau(p^{-1}(\mathrm{A}))$ for any measurable subset A of X. However, if τ is not finite, its image measures will often take only two values, 0 and ∞. We therefore need to introduce the following concept.

Definition A.1.1. — *Let* (T, τ) *and* (X, ν) *be two measured spaces and* $p\colon \mathrm{T} \to \mathrm{X}$ *a measurable map. We say that* ν *is a* pseudo-image *of* τ *via* p *if there are equivalent measures* $\tau' \sim \tau$ *and* $\nu' \sim \nu$ *such that* $\nu' = p(\tau')$.

Lemma A.1.2. — *Let* (T, τ) *be a measured space,* X *a measurable space and* $p\colon \mathrm{T} \to \mathrm{X}$ *a measurable map. Assume that* τ *is* σ*-finite.*

- *There exists a probability measure on* X *which is a pseudo-image of* τ.
- *A measure* ν *on* X *is a pseudo-image of* τ *via* p *if and only if, for any measurable set* $\mathrm{A} \subseteq \mathrm{X}$,

$$\tau(p^{-1}(\mathrm{A})) = 0 \quad \textit{if and only if} \quad \nu(\mathrm{A}) = 0. \tag{A.1}$$

Proof. — Choose (thanks to σ-finiteness) a probability τ' equivalent to τ. Its image $p(\tau')$ is therefore a probability on X which is a pseudo-image of τ.

Necessity in the second point is obvious. Reciprocally, if (A.1) holds, then ν is equivalent to $p(\tau')$, hence is a pseudo-image of τ. ■

A.2 Integration of measures

Let (X, ν) be a measured space and $\{m_x\}_{x\in \mathrm{X}}$ be a family of measures on a space T. The formula
$$m(\mathrm{A}) = \int_{\mathrm{X}} m_x(\mathrm{A})\ \mathrm{d}\nu(x) \tag{A.2}$$
defines a new measure on T, called the integral of the measures m_x over ν. This construction makes sense for any A in the σ-algebra $\mathcal{A}$ of sets of T that are m_x-measurable for ν-almost all x, and such that $x \mapsto m_x(\mathrm{A})$ is ν-measurable. In practice, the m_x's often share a common σ-algebra of measurable sets.

Observe that, whenever f is m-integrable, Formula (A.2) becomes
$$\int_{\mathrm{T}} f\ \mathrm{d}m = \int_{\mathrm{X}} \left(\int_{\mathrm{T}} f\ \mathrm{d}m_x \right)\ \mathrm{d}\nu(x).$$

This construction is technically smooth, but, even when starting with very nice measures ν and m_x, some care is in order to produce a non-boring measure m. Indeed, observe that if we start with (T, τ) any measured space (for instance, τ can be a Radon probability), $\mathrm{X} =]0, 1]$ endowed with the Lebesgue measure and m_x defined by $x^{-1}\tau$, then m is the measure equivalent to τ that is infinite on every set of τ-positive measure.

In concrete situations, we are often given a map $p\colon \mathrm{T} \to \mathrm{X}$ and the m_x are concentrated on the fibers, that is, $m_x(\mathrm{T} \setminus p^{-1}\{x\}) = 0$. The following lemma is a straightforward corollary of Lemma A.1.2.

Lemma A.2.1. — *Let (X, ν) be a measured space, $(\mathrm{T}, \mathcal{A})$ be a measurable space, $p\colon \mathrm{T} \to \mathrm{X}$ a measurable map and $\{m_x\}$ a family of measures defined on $\mathcal{A}$, concentrated on the fibers of p, and such that $x \mapsto m_x(\mathrm{A})$ is ν-measurable for any $\mathrm{A} \in \mathcal{A}$. If the integral m of the measures m_x over ν is σ-finite, then ν is a pseudo-image of m.*

An important specific case is that of a product space $\mathrm{T} = \mathrm{X} \times \mathrm{Y}$ with $p = p_{\mathrm{X}}$ the projection onto the first factor. If θ is a measure on Y, then we can defined measures concentrated on the fibers by
$$m_x(\mathrm{A}) = \theta(p_{\mathrm{Y}}(\mathrm{A} \cap (\{x\} \times \mathrm{Y}))),$$
where p_{Y} is the projection onto Y. If ν and θ are σ-finite, then the integrated measure $m = \int_{\mathrm{X}} m_x\ \mathrm{d}\nu$ is exactly the product measure $\nu \times \theta$, by Fubini's theorem.

Of course, integration of measures is a generalisation of integration of functions, in the following sense. If $\mathrm{T} = \mathrm{X}$ and f is a locally integrable function, then the measure m defined by the density f, that is, $\mathrm{d}m = f\mathrm{d}\nu$, is exactly the integration of the measures $m_x = f(x)\delta_x$ over ν.

A.3 Disintegration of measures

It is natural to wonder whether and how, given a measure m on a space T, we can write m as an integral of other measures as in Formula (A.2). After all, this is precisely the content of the Radon–Nikodym theorem for the very specific case of equivalent σ-finite measures on the same space ($\mathrm{T} = \mathrm{X}$, the projection is the identity).

More precisely, we are interested in the following problem. Let (T, m) be a measured space, X a measurable space and $p\colon \mathrm{T} \to \mathrm{X}$ a measurable map, can we find a measure ν on X and a family of measures $\{m_x\}_{x\in\mathrm{X}}$ on T such that:

1. the measure ν is a pseudo-image of m via p,
2. each measure m_x is concentrated on the fiber $p^{-1}\{x\}$,
3. the measure m is the integral of $\{m_x\}$ over ν?

If so, we say that the family $\{m_x\}$ is a *disintegration* of the measure m over ν. Such a disintegration is *unique* if, for any other disintegration $\{m'_x\}$, we have $m_x = m'_x$ for ν-almost all x.

It is immediate that the existence as well as the uniqueness of a disintegration do not change if we replace ν by an equivalent measure ν' that admits a density with respect to ν. In particular, when m is σ-finite, we can say that m *admits a (unique) disintegration along* p if it admits a (unique) disintegration over ν for some (or any) σ-finite pseudo-image ν of m via p.

Disintegrating a measure turns out to be quite technically involved, mainly because the image of a nice measure is rarely nice itself. Leaving alone the so-called “catastrophe of image measures” [**Sch73**, p. 30], this issue arises even in the simplest daily-life situations: consider for instance the image of the Lebesgue measure on $\mathbf{R}^2$ through the projection on any line. Therefore, the strongest disintegration results are only available either when a nice image measure is already given (see [**Fre06**, 452]), when dealing with measures behaving well under direct images (for instance, Radon probabilities, see [**Bog07**, Section 10.6]), or if the map p has fibers “small” enough to produce nontrivial measures (see [**Bou69**, § 2, n$^{\mathrm{o}}$ 7]). Unfortunately, measured equivalence relations fall, strictly speaking, in none of these situations, but fortunately, they do share some common features with these tractable cases.

We will restrict the disintegration problem to the case where T and X are topological spaces (hence the measure m is Borel) and p is continuous. This setting may seem rather non-measure-theoretical; it is however largely enough for our needs, since we will ultimately be interested in cases where T is a Borel subset of a product of two standard Borel spaces and p is the projection on one factor.

Proposition A.3.1. — *Let* (T, μ) *be a metrizable space endowed with a Radon finite measure,* X *a topological space, and* $p\colon \mathrm{T} \to \mathrm{X}$ *a continuous map. Then there exists a unique disintegration of* μ *over* $p(\mu)$*.*

Proof. — See [**Bou69**, IX, §2, n° 7]. ■

We now turn our attention to σ-finite measures. For the following result, recall that a topological space is *Radon* if any Borel finite measure defined on it is inner regular (hence Radon). See [**Bou69**, IX, §3, n° 3], [**Bog07**, Section 7.14(vii)] or [**Fre06**, 434F] for more background on this class of spaces, which notably includes all Polish spaces.

Proposition A.3.2. — *Let* (T, m) *be a Radon metrizable space endowed with a Borel* σ-*finite measure,* X *a topological space and* $p\colon \mathrm{T} \to \mathrm{X}$ *be a continuous map. The measure* m *admits a unique disintegration along* p*.*

Proof. — Choose some σ-finite pseudo-image ν of the measure m via p. Let A_n be a nondecreasing sequence of Borel subsets of T of finite measure, such that $\mathrm{T} = \bigcup \mathrm{A}_n$. Define the Borel measure m^n on T via $m^n(\mathrm{B}) = m(\mathrm{B} \cap \mathrm{A}_n)$. Each measure m^n is a Radon finite measure since T is a Radon space, and for any Borel set B, the nondecreasing sequence $m^n(\mathrm{B})$ converges to $m(\mathrm{B})$.

Existence. Let $m^n = \int_\mathrm{X} m^n_x \, \mathrm{d}\nu(x)$ be a disintegration for each m^n (Proposition A.3.1). As the restriction of m^{n+1} to A_n is m^n, the uniqueness of disintegration yields that, for any x in a subset $\mathrm{X}_0 \subseteq \mathrm{X}$ of full measure, the restriction of m^{n+1}_x to A_n is m^n_x. In particular, for any $x \in \mathrm{X}_0$ and Borel subset B, the sequence $m^n_x(\mathrm{B})$ is nondecreasing. Let $m_x(\mathrm{B})$ be its limit. Clearly, m_x is a Borel measure on T, supported on the fiber of x. For x outside X_0, define simply m_x to be the zero measure. Thanks to B. Levi's monotone convergence theorem, $m(\mathrm{B})$ is equal to $\int_\mathrm{X} m_x(\mathrm{B}) \, \mathrm{d}\nu(x)$ for any Borel set B, hence $\{m_x\}$ is a disintegration of m.

Uniqueness. Let $\{m_x\}$ and $\{m'_x\}$ be two disintegrations of m over ν. For each n, the restriction of these disintegrations to A_n are therefore disintegrations of the measure m^n.

By uniqueness of disintegrations of the latter, we get, for ν-almost all x,

$$m_x(\mathrm{A}_n \cap \mathrm{B}) = m'_x(\mathrm{A}_n \cap \mathrm{B})$$

for all n and all Borel sets B. Letting n tend to infinity, we get the equality of m_x and m'_x. ■

The following application shows that the third condition of Definition 3.1.7 makes sense.

Corollary A.3.3. — *Let* (X, ν) *be* σ*-finite standard measure space and* Y *be another standard Borel space. Let* B *be a Borel subset of* $\mathrm{X} \times \mathrm{Y}$*, endowed with a* σ*-finite Borel measure* m*. If* ν *is a pseudo-image of* m *along the projection onto the first factor, then* m *has a unique disintegration over* ν*.*

Proof. — The space B is standard (i.e., a Borel subset of a Polish space), since both X and Y are so. Therefore, it is metrizable and Suslin [**Bou74**, IX, § 6, n° 3, prop. 10], hence Radon [**Bou69**, IX, §3, n° 3, proposition 3]: Proposition A.3.2 applies. ■

B KREIN SPACES

O God, I could be bounded in a nutshell and count myself a king of infinite space, were it not that I have bad dreams.

W. Shakespeare, *Hamlet*, II.2
(from 1623 *first Folio*)

LOCALLY CONVEX SPACES for which the closed convex hull of a compact set is still compact occurred naturally in the study of affine actions of compact groups (Section 5.2.B), and are fundamental for weak integration. Standard reference books on topological vector spaces, although they contain almost all facts on these spaces, do not seem to have named them; our terminology honours M. Krein, who proved that the closed convex hull of a weakly compact set in a separable Banach space is again weakly compact.

The first two sections review basic tools about duality and completeness of locally convex spaces. Section B.3 then surveys the class of Krein spaces and examples are provided in Section B.4. The last two sections are more directly related to the fixed-point property of compact groups: the link between the Krein property and barycenters is sketched in Section B.5 and the space needed to show that the fixed-point property considered in Proposition 5.2.7 implies precompactness is explained in Section B.6.

This appendix uses freely various results on topological vector spaces and relies mostly on Bourbaki's book [**Bou81**]. For a gentle introduction to the topic, we recommend the neat and small [**RR73**]. A self-contained, readable, and surprisingly short survey on the central notion of duality of vector spaces can be found in [**Sim11**, Chapter 5].

B.1 Topologies consistent with the canonical duality

Let E be a locally convex space and E′ its dual. The *canonical duality* is the application

$$E \times E' \to \mathbf{R} \colon (x, \lambda) \mapsto \lambda(x).$$

A locally convex topology $\mathscr{T}$ on E is *consistent* with the canonical duality if the dual of $(E, \mathscr{T})$ is again E′. Trivial examples include the given topology on E (that we will sometimes called the *initial* topology, to distinguish it from the other topologies we will associate on the same space) and the weak topology.

In other words, topologies consistent with the canonical duality are those that do not change the dual space. In particular, any topology consistent with the canonical duality is *finer* than the weak topology. As a consequence, closed *convex* sets and bounded sets are the same for all topologies consistent with the canonical duality.

Remarkably, a theorem of Mackey asserts that there exists also a *finest* topology consistent with the canonical duality. Called the *Mackey topology* and noted $\tau(E, E')$, it can be defined as the topology (on E) of uniform convergence on convex $\sigma(E', E)$-compact subsets of E′ (when E is viewed as a subset of E'^*). For our purposes in this appendix, it is however enough to know that the Mackey topology $\tau(E, E')$ is the finest topology consistent with the canonical duality and hence that any other topology consistent with the canonical duality (in particular, the initial one on E) is weaker than $\tau(E, E')$ and finer than $\sigma(E, E')$.

For many usual locally convex spaces, the Mackey topology is the same as the initial topology. This is the case for any barrelled or bornological space, in particular, for all normed spaces.

When juggling with different topologies, the following "bootstrap" result for completeness is particularly useful.

Lemma B.1.1. — *Let* E *be a locally convex space and* $\mathscr{T}_1$ *and* $\mathscr{T}_2$ *be two locally convex topologies on* E *consistent with the canonical duality. Assume that* $\mathscr{T}_1$ *is finer than* $\mathscr{T}_2$*. Then any subset* A *of* E *which is complete for* $\mathscr{T}_2$ *is also complete for* $\mathscr{T}_1$*.*

Proof. — Since $\mathscr{T}_1$ and $\mathscr{T}_2$ have the same closed convex sets, there is an identity neighborhood base for $\mathscr{T}_1$ made of sets that are also closed for $\mathscr{T}_2$. In particular, the $\mathscr{T}_1$-uniform structure of A admits a vicinity base made of $\mathscr{T}_2$-closed sets and the result follows from [**Bou71**, II, §3, n° 3, cor. de la prop. 7]. ■

In particular, with respect to completeness, the weakest requirement is to be complete for the Mackey topology. (Recall that, without further assumption, a uniform structure finer that a complete one is not necessarily itself complete, cf. [**Bou71**, II, § 3, exerc. 2].)

B.2 Quasi-complete spaces

The purpose of this section is to recall some results about quasi-complete spaces. Our interest in this weakening of completeness dwells in the fact that it produces many examples of Krein spaces, see Proposition B.3.6.

Definition B.2.1. — *A locally convex space is* quasi-complete *if any bounded closed subset is complete.*

Complete locally convex spaces are trivially quasi-complete. On the other hand, as any Cauchy *sequence* in a locally convex space is bounded [**Bou81**, III, §1, n° 2, cor. de la prop. 2], quasi-complete spaces are also sequentially complete.

Quasi-completeness is quite a weak property (see [**Bou81**, IV, Tableau 1, p. 75]), we will review a few properties of these spaces.

Lemma B.2.2. — *Closed subspaces, arbitrary products and topological direct sums of quasi-complete spaces are quasi-complete. A strict inductive limit of a sequence of quasi-complete spaces is also quasi-complete.*

Proof. — See [**Bou81**, III, §1, n° 6, prop. 9]. ■

Lemma B.2.3. — *Let* E *be a locally convex space and* $\mathscr{T}_1$ *and* $\mathscr{T}_2$ *be two locally convex topologies on* E *consistent with the canonical duality. Assume that* $\mathscr{T}_1$ *is finer than* $\mathscr{T}_2$*. If* E *is quasi-complete for* $\mathscr{T}_2$*, then it is also quasi-complete for* $\mathscr{T}_1$*.*

Proof. — The result follows from Lemma B.1.1. ■

In particular, if E is quasi-complete for its weak topology, it is also quasi-complete for its initial topology and for its Mackey topology.

Of interest for producing examples is the following lemma, which encompasses in particular the dual of a Banach space endowed with its weak-$*$ topology. We recall that the *weak dual* of a locally convex space E is its dual E′ endowed with the topology of pointwise convergence (that is, the $\sigma(E', E)$-topology) and the *strong dual* is its dual endowed with the topology of uniform convergence on bounded sets.

Lemma B.2.4. — *The strong and the weak duals of a barrelled space are quasi-complete.*

Proof. — A bounded set in the dual of a barrelled space is equicontinuous, hence complete if closed. See [**Bou81**, III, §4, n°2, cor. 5] for details. ■

B.3 Krein spaces

We now come to the main concept of this appendix.

Definition B.3.1. — *A locally convex space* E *is a* Krein space *(or has the* Krein property*) if, for any compact subset* A *of* E*, the closed convex hull of* A *is compact.*

Equivalently, we can require the closed *balanced* convex hull of A to be compact, since the latter is nothing but the closed convex hull of $A \cup -A$. We will only consider locally convex spaces, but the definition makes sense for any topological vector space[1]. The prominent examples of Krein spaces are Banach spaces endowed with their norm, weak, or any weak-∗ topology, as we shall see later. Let us first review some general facts about Krein spaces.

The Krein property is intimately related to a lack of completeness in the space, as shows the following lemma.

Lemma B.3.2 (Mazur). — *The convex hull of a precompact set is precompact.*

Proof. — See [**Bou81**, II, § 4, n° 2, prop. 3]. ■

In particular, complete spaces are Krein. More generally:

Corollary B.3.3. — *A quasi-complete locally convex space is Krein.*

Proof. — We apply Lemma B.3.2, relying either on the fact that precompact sets are

[1] And even more generally for any topological space with some "convex" structure, such as uniquely geodesic spaces.

bounded [**Bou81**, III, § 1, n° 2, prop. 2], or more elementarily on the fact that closed convex hulls of bounded sets are bounded [**Bou81**, III, § 1, n° 2, prop. 1]. ■

Corollary B.3.4. — *The strong and weak duals of a barrelled space are Krein.*

Proof. — Such spaces are quasi-complete by Lemma B.2.4. Alternatively, we can directly see that for A a (strongly or weakly) compact set in the dual, its balanced closed convex hull C is bounded. Hence the polar of C is a barrel, so the bipolar (which contains C) is compact by the Alaoglu–Bourbaki theorem. ■

Let E be a locally convex space. Endowing E with a weaker topology, two opposite phenomena occur in relation to the Krein property: there are more compact sets on which to test the definition, but it is easier for the closed convex hull to be compact. The former seems to overshadow the latter for topologies consistent with the canonical duality; more precisely, we have the following lemma.

Lemma B.3.5. — *Let* E *be a locally convex space and* $\mathscr{T}_1$ *and* $\mathscr{T}_2$ *be two locally convex topologies on* E *consistent with the canonical duality. Assume that* $\mathscr{T}_1$ *is finer than* $\mathscr{T}_2$*. If* E *is Krein for* $\mathscr{T}_2$*, then it is also Krein for* $\mathscr{T}_1$*.*

In particular, if E *is Krein for the weak topology, then it is also Krein for any topology consistent with the canonical duality (in particular, for the initial and Mackey topologies).*

Proof. — Let A be a $\mathscr{T}_1$-compact set and C be its $\mathscr{T}_1$-closed convex hull. The set A is a fortiori $\mathscr{T}_2$-compact; moreover, C is also its $\mathscr{T}_2$-closed convex hull since the topologies are consistent with the canonical duality. Hence C is $\mathscr{T}_2$-compact by hypothesis. It is therefore in particular $\mathscr{T}_2$-complete, hence $\mathscr{T}_1$-complete by Lemma B.1.1 and thus $\mathscr{T}_1$-compact by Lemma B.3.2. ■

We do not know if we can go “downwards”, namely:

Problem 13. — Is there a Krein locally convex space that is not Krein for its weak topology?

In any case, such a space must be very incomplete. More precisely, we have the following deep and fundamental result of Krein.

Proposition B.3.6 (Krein). — *A quasi-complete locally convex space is Krein for the weak topology.*

In particular, the lightest completeness requirement we can make on a locally convex space, namely being quasi-complete for the Mackey topology, is enough to imply the Krein property for all topologies consistent with the canonical duality.

Krein proved the above proposition for separable Banach spaces [**Kre37**, Théorème 4] and the generalisation for all Banach spaces follows by the Eberlein–Šmulian theorem. Putting all ingredients of Krein's proof into their general locally convex form, we get the above result (see [**Bou81**, IV, §5, n° 5]).

However, quite elegantly, the most general result relating completeness and compactness for different topologies, the below theorem of Grothendieck, requires only to know Krein's theorem in its original form, that is, that Banach spaces are weakly Krein.

Proposition B.3.7 (Grothendieck). — *Let* A *be a subset of a locally convex space* E *and* C *be its closed convex hull. Then* C *is compact if and only if* A *is relatively compact and* C *is complete for the Mackey topology.*

In particular, a locally convex space is Krein if and only if the closed convex hull of any compact set is complete for the Mackey topology.

Proof. — The conditions are obviously necessary, by Lemma B.1.1.

Reciprocally, let A be a relatively compact set with a Mackey complete closed convex hull C. Write E_τ and E_σ for the space E endowed with its Mackey or weak topology, respectively. Recall that, by considering a family of seminorms defining the topology, any locally convex space is a (non-closed) subspace of a product of Banach spaces; let $\{E_i\}$ be such a family of Banach spaces for the space E_τ. As the weak topology is compatible with arbitrary subspaces and products [**Bou81**, II, §6, n$^{\text{os}}$ 5 et 6], we get the following commutative diagram.

$$\begin{array}{ccc} E_\tau & \longrightarrow & \prod_i E_i \\ \downarrow & & \\ E & & \downarrow \\ \downarrow & & \\ E_\sigma & \longrightarrow & \prod_i (E_i)_\sigma \end{array}$$

The projections A_i of the set A on the Banach spaces E_i are relatively weakly compact, hence their closed convex hulls C_i are weakly compact by the original Krein theorem. Since C is complete in E_τ, it is a fortiori a *closed* subset of $\prod_i E_i$, hence also a weakly closed subset since it is convex.

In particular, C, a weakly closed subset of the product $\prod_i \mathrm{C}_i$, is weakly compact. Therefore, C is also weakly complete, hence complete for the initial topology. Thus C is compact by Lemma B.3.2. ■

Remark B.3.8. — Actually, in the above result, we only need to assume that A is relatively *countably* compact, thanks to the Eberlein–Šmulian theorem (which then also needs to be known only for Banach spaces). Grothendieck used this fact to deduce various compactness results in functional analysis, see [**Gro50**, **Gro52**].

So Problem 13 asks whether, in a space endowed with its Mackey topology, the family of closed convex hulls of compact sets can be significantly smaller than the family of closed convex hulls of *weakly* compact sets, so that all elements of the former are complete without all elements of the latter being so for all that.

We close this section by observing the following stability properties of Krein spaces.

Lemma B.3.9. — *Closed subspaces, products, direct sums, projective limits and strict inductive limits of Krein spaces are Krein.*

Proof. — The stability under closed subspaces is trivial and the one under products follows from Tychonov's theorem. We get then stability under projective limits as a special case.

As for direct sums and strict inductive limits, the result follows from the fact that any bounded set dwells inside, respectively, a finite direct sum (hence a product) or an element of the sequence, cf. [**Bou81**, III, §1, n° 4]. ■

B.4 Examples and counterexamples

By Corollaries B.3.3 and B.3.4 and Proposition B.3.6, Banach spaces are Krein for their norm, weak-$*$ and weak topologies. In general, the non-Banach locally convex spaces that attract attention in functional analysis (for instance semi-reflexive spaces, Montel spaces, duals of Fréchet spaces,...) are at least quasi-complete, hence Krein.

By considering fixed-point properties of both Propositions 5.2.6 and 5.2.7, we indirectly see that the space H built in the proof of Proposition 5.2.6 is not Krein when G is an infinite compact group. This can also be checked directly: the set G' in H is compact, but its closed convex hull C is not complete (choose a net in C converging in $\mathbf{R}^{\mathscr{C}(\mathrm{G},\mathbf{R})}$ to the normalised Haar measure, the latter does not belong to H if G is infinite).

Let us have a closer look at the link between the Krein property and completeness. As we saw in Proposition B.3.6, the Krein property is guaranteed as soon as the *much finer* Mackey topology is quasi-complete. It is therefore trivial to produce Krein spaces that are not themselves quasi-complete. For instance, the space c_0 of real-valued sequences converging to zero is weakly Krein (as any Banach space) but is not even weakly sequentially complete (consider the sequence $x_n = \sum_{k=1}^n \delta_k$, which is weakly-$*$ convergent in the bidual ℓ^∞, hence is weakly Cauchy in c_0, but is not weakly convergent).

However, as Grothendieck's result showed, the Krein property is equivalent to a "certain form" of completeness. For metrizable spaces, this equivalence is limpid.

Lemma B.4.1. — *Let* E *be a metrizable locally convex space. Then the following are equivalent.*

1. E *is complete.*
2. E *is weakly Krein.*
3. E *is Krein.*

Proof. — Any complete space is weakly Krein (Proposition B.3.6) and weakly Krein spaces are Krein for their initial topology (Lemma B.3.5). So let E be a metrizable Krein space and let (x_n) be a Cauchy sequence in E. As any Cauchy sequence is precompact, the sequence (x_n) lies inside the closed balanced convex hull C of some sequence converging to zero, by the Banach–Dieudonné theorem [**Bou81**, IV, §3, n° 5, cor. 1]. Hence C is compact by the Krein property and therefore complete, so (x_n) is convergent. ∎

Remark B.4.2. — The above proof yields a general heuristic to find a compact set A in a normed space whose closed convex hull C is not compact: A can always be assumed to be a convergent sequence and C must contain some non-convergent Cauchy sequence. For instance, in c_{00}, in the closed convex hull of the compact set $A = \{0\} \cup \{k^{-1}\delta_k \mid k > 0\}$, we can find the non-convergent Cauchy sequence $x_n = \sum_{k=1}^n \lambda_k \delta_k$, where (λ_k) is any sequence of positive integers such that $\sum_{k>0} k\lambda_k = 1$.

As a consequence of the above lemma, we observe that a barrelled space need not be Krein. Indeed, any Baire space is barrelled, but there are non-complete Baire normed spaces [**Bou81**, III, §4, exerc. 4].

B.5 Barycenters

For this section, "probability" always means a positive Radon measure of total mass one. We recall that there is a perfect correspondence between signed finite Radon measures on a compact spaces A and continuous linear forms on $\mathscr{C}(\mathrm{A})$.

Definition B.5.1. — *Let* μ *be a probability on a compact subset* A *of a locally convex space* E. *The* barycenter *of* μ *is the point* b_μ *of* E'^* *(the algebraic dual of the topological dual) defined by*

$$b_\mu = \int_{\mathrm{A}} x \,\mathrm{d}\mu(x).$$

A barycenter is called concrete *if it belongs to* E.

The definition of the barycenter means that, for any $\lambda \in \mathrm{E}'$, we have

$$b_\mu(\lambda) = \int_{\mathrm{A}} \lambda(x)\,\mathrm{d}\mu(x)$$

and, of course, $b_\mu(\lambda) = \lambda(b_\mu)$ if the barycenter is concrete.

Caveat lector! Following Bourbaki, we choose to consider that the barycenter always exist, but only as a point in E'^*. We will then be interested in spotting the situations where this point actually lies in E, that is, is concrete. Other authors (e.g. [**Fre06**, 461]) reserve the terminology "barycenter" for the latter case (hence, in their statements, "let μ be a probability with barycenter x" is not just a notation but really a nonvoid assumption).

Lemma B.5.2. — *The closed convex hull of a compact set* C *is the set of* concrete *barycenters of probabilities supported on* C.

Proof. — See [**Bou65**, IV, § 7, n° 1, prop. 1]. ■

Lemma B.5.3. — *The barycenter of a probability supported on a compact* convex *set is concrete.*

Proof. — See [**Bou65**, IV, § 7, n° 1, cor. de la prop. 1]. ■

Of course, a probability supported on a compact set is also supported on its closed convex hull, hence concreteness of barycenters is related to Krein spaces. The above lemmas show that, in a Krein space, all barycenters are concrete. In the other direction, we have the following result.

Proposition B.5.4. — *Let* E *be a locally convex space. The following conditions are equivalent.*

1. E *is Krein for its weak topology.*

2. E *contains the barycenter of any probability supported on a weakly compact subset.*

Proof. — The implication $1 \Rightarrow 2$ follows from Lemma B.5.3.

Let us prove $2 \Rightarrow 1$. The space E'^* is a closed subspace of $\mathbf{R}^{\mathrm{E}'}$ endowed with the product topology[2], hence is Krein (Lemma B.3.9). The induced topology on its (non-closed) subspace E is precisely the weak topology. Let A be a weakly compact subset of E. The closed convex hull C of A inside E'^*, which is compact by the Krein property, is also equal to the closed convex hull of A inside E, since the latter contains all barycenters. Hence C is also a weakly compact subset of E. ■

Another way to formulate the above proof is to observe that the map $b \colon \mathscr{P}_1(\mathrm{A}) \to \mathrm{E}'^*$ that assigns a barycenter to a probability, has range in E by hypothesis and is continuous when $\mathscr{P}_1(\mathrm{A})$ is endowed with the vague topology. As the latter is compact, the range of b, which is the closed convex hull of A by Lemma B.5.2, is also compact.

Remark B.5.5. — We could have defined similarly the barycenter for any positive finite Radon measure. By homothety, we would have gotten similar results for the closed convex cone generated by C.

Remark B.5.6. — For the sake of completeness, let us mention that there is a rich interplay between barycenters and extreme points, starting with the following easy characterization: a point x of a compact convex set C is extreme if and only if the Dirac measure on x is the only probability supported on C whose barycenter is x (see [**Bou65**, IV, § 7, n° 2, corollaire de la proposition 1]). By the Krein–Milman theorem, a compact convex set C is the closed convex hull of its extreme points, hence a natural question arises: is any point of C the barycenter of some probability supported on the extreme points? (Lemma B.5.2 above does not provide a trivial answer, as the set of extreme points of a compact convex set is not necessarily closed, see [**Bou81**, II, §7, exerc. 11].) The answer is known to be positive if C is metrizable, or if we allow the probability to be Baire instead of Borel. For more on this topic, see [**Phe01**].

[2]The induced topology on E'^* is nothing but the $\sigma(\mathrm{E}'^*, \mathrm{E}')$-topology.

B.6 Spaces of measures

We will now describe in details the Krein space used in Proposition 5.2.7. In this section, X will denote a Hausdorff completely regular space. Equivalently, this means that the topology on X is induced by a Hausdorff uniform structure (however, different uniform structures can yield the same topology).

Consider the vector space $\mathscr{M}(X)$ of Borel finite measures[3] on X. The most common topology on $\mathscr{M}(X)$ is the so-called *narrow topology*, which is the topology given by the duality between $\mathscr{M}(X)$ and $\mathscr{C}_b(X)$, the space of bounded continuous functions on X—the pairing between a measure μ and a function f is the integral of f over μ, which exists since μ is finite and f is bounded and measurable, see e.g. [**Bog07**, Theorem 2.5.1(iii)].

Beware that this topology is very strong from the point of view of group actions. Let G be a topological group with an orbitally continuous action on X by homeomorphisms (for instance, X = G and G acts on itself by translations): the deduced representation of G on $\mathscr{M}(X)$ may fail to be orbitally continuous for the narrow topology. Indeed, if g_α converges to g, then $g_\alpha \cdot f$ converges pointwise to $g \cdot f$ for any $f \in \mathscr{C}_b(X)$, but the pointwise convergence of *nets* is in general not enough to deduce a convergence in mean. However, this representation is always *slightly continuous*, that is, there are *some* points with an associated continuous orbit map (namely, the Dirac masses).

Actually, the space $\mathscr{M}(X)$ is too big for our purposes. We recall that a Borel (signed) measure m is called τ-*additive* if, for any *nondecreasing* net of *open* sets U_α, we have

$$|m|\left(\bigcup_\alpha U_\alpha\right) = \lim_\alpha |m|(U_\alpha).$$

The space of all Borel τ-additive finite measures will be written $\mathscr{M}_\tau(X)$. To perceive the actual mildness of this technical continuity requirement, we observe a few easy facts:

- The point of that definition is to be able to define the support of a measure. Indeed, open sets of null measure form a directed set, hence their union is still of null measure in presence of τ-additivity.
- A standard compactness argument shows that inner regular measures (in particular, Radon measures) are τ-additive.
- Any Borel measure on a second-countable topological space is τ-additive. Indeed, for such spaces, the arbitrary union $\bigcup_\alpha U_\alpha$ can be replaced by a countable one.

[3]For this section, all measures are allowed to be signed (that is, taking values in $\mathbf{R}$), since we need to consider vector spaces of measures.

- If a topological group G acts orbitally continuously on X by homeomorphisms, then $\mathscr{M}_\tau(\mathrm{X})$ is an invariant subspace; since it contains the Dirac masses, the deduced representation is still slightly continuous for the narrow topology.

The relevance of τ-additivity for fixed-point properties lies in the following result.

Proposition B.6.1. — *The space* $\mathscr{M}_\tau(\mathrm{X})$ *is Krein for the narrow topology.*

On the proof. — The idea is to associate to each probability *on* a compact subset A of $\mathscr{M}_\tau(\mathrm{X})$ its barycenter. The latter is actually concrete, that is, is a τ-additive Borel measure. This therefore yields a continuous map from the space of probabilities on A to $\mathscr{M}_\tau(\mathrm{X})$ (cf. the discussion after Proposition B.5.4). Details can be found in [**Bog07**, Theorem 8.10.5]. □

Lastly, we record the following straightforward generalisation of the fact that open subsets of a locally compact group have positive Haar measure (we needed it in the proof of Proposition 5.2.7).

Lemma B.6.2. — *Let* m *be a nonzero positive Borel τ-additive measure on a topological group* G*. Assume that* m *is invariant. If* V *is any nonempty open set of* G*, then* $m(\mathrm{V}) > 0$.

Proof. — If $m(\mathrm{V}) = 0$, then all open sets $g\mathrm{V}$ have null measure. By τ-additivity, their union, which is the whole G, also has null measure, hence m should be trivial. ■

Remark B.6.3. — Considered individually, Radon measures are of course more pleasant than τ-additive ones. But, as a whole, the space of Radon measures is hard to handle, because the Radon condition is not closed for the natural topologies we can consider on a space of measures. In particular, it is unclear whether it could be Krein for the narrow topology in all generality[4].

[4] For some remarkable topological spaces, the so-called *Prohorov spaces*, the compactness for the narrow topology of a set of Radon measures can be translated into an "equicontinuity" condition that makes straightforward the checking of the Krein property. This class includes notably locally compact spaces and Polish spaces, see [**Bou69**, IX, §5, n° 5] and [**Bog07**, Section 8.10(ii)]. But our goal in Proposition 5.2.7 was to get a Krein space without any topological assumption on the group, so that Prohorov spaces are of little help for us.

C CONTINUOUS ACTIONS ON UNIFORM SPACES

Bref un éclairage qui non seulement obscurcit mais brouille par-dessus le marché.

S. Beckett, *Le Dépeupleur*

Given an action of a topological group G on a topological space X, there are two natural continuity conditions: the joint continuity (continuity of the action map $G \times X \to X$) and the orbital continuity (continuity of all the orbit maps $G \to X$). The latter is usually strictly weaker. However, it is a common folklore knowledge that these two conditions are equivalent in many contexts, for instance for isometric actions on metric spaces or for linear representations of locally compact groups on Banach spaces. The deep reason behind these equivalences is the preservation of some uniform structure.

Throughout this appendix, G will denote a topological group, X a *uniform* space and α an action of G on X. We always endow uniform spaces with the topology induced by their uniformity. However, an action will a priori be only a morphism $\alpha\colon G \to \mathrm{Homeo}(X)$; that is, we require G to act by continuous transformations, not necessarily by uniformly continuous ones. As we shall not deal with more than one action at a time, we will simplify our notations by writing gx or $g \cdot x$ for $\alpha(g)(x)$. We will write $\circ$ for the "transitivity" composition law on subsets of $X \times X$, that is, $E \circ F = \{(x, z) \mid \exists y\colon (x, y) \in E \text{ and } (y, z) \in F\}$. It will be convenient to use the same notation for elements of $X \times X$, that is, to write $(x, z) = (x, y) \circ (y, z)$.

From the point of view of applications, the joint continuity is often the most useful. On the other hand, orbital continuity is more natural from the intrinsic topological point of view of the space: indeed, it is equivalent to require the map $\alpha\colon G \to \mathrm{Homeo}(X) \subset$

$\mathscr{C}_s(X, X)$ to be continuous when $\mathscr{C}_s(X, X)$ is the space of all continuous transformations of X, endowed with the topology of pointwise convergence[1]. Of course, joint continuity would be automatic if the morphism α were continuous for the topology of uniform convergence (cf. [**Bou74**, x, §1, n° 5, prop. 9]), but such a condition would be too restrictive to be useful, as illustrated by Remark 4.1.10.

C.1 The exact gap between orbital and joint continuities

We first need to introduce two technical definitions.

Definition C.1.1. — *An action* α *of a topological group* G *on a uniform space* X *is said* semilocally equicontinuous *at* $(g_0, x_0) \in G \times X$ *if for any vicinity* E *of* X*, there are a neighborhood* U *of* g_0 *and a neighborhood* V *of* x_0 *such that for any* $h \in U$,

$$\alpha(h)(V) \subseteq E[hx_0].$$

The action is said semilocally equicontinuous *if it is semilocally equicontinuous at* (g_0, x_0) *for any* $(g_0, x_0) \in G \times X$.

Definition C.1.2. — *An action* α *of a topological group* G *on a uniform space* X *is said* locally equicontinuous *if for any* $x \in X$ *and* $g \in G$*, there is a neighborhood* U *of* g *such that the family of maps* $\{\alpha(h) \colon X \to X \mid h \in U\}$ *is equicontinuous at* x.

Remark C.1.3. —

- In other words, a locally equicontinuous action is a semilocally equicontinuous action where the neighborhood U can be chosen independently of the vicinity E. But the neighborhood U may in any case depend on the point x.
- Observe that a semilocally equicontinuous action is necessarily an action *by* continuous transformations.
- In the language of nets, the local equicontinuity becomes: for any $x \in X$, any converging net $g_j \to g$ in G is eventually equicontinuous at x.
- Obviously, it is enough to check the condition of any of the above definitions for E belonging to a vicinity base; in particular, we may require E to be symmetric or closed. We will use this fact without further notice.

[1] Even if α actually lies in the subset Homeo(X), we prefer here to put emphasis on $\mathscr{C}_s(X, X)$, since the topology of pointwise convergence is usually not compatible with the group structure of Homeo(X).

- If G acts by *uniformly* continuous maps, then it is enough to check the conditions when g is the identity.

The display of quantifiers in the definition of semilocal equicontinuity is justified by its relevance for continuity of actions, as shown by the following lemma.

Lemma C.1.4. — *Let α be an action of a topological group* G *on a uniform space* X*. The following assertions are equivalent.*

1. *The action is jointly continuous.*

2. *The action is semilocally equicontinuous and orbitally continuous.*

Proof. — **1 $\Rightarrow$ 2.** The continuity of the orbit maps follows easily from that of the action map. Let us then show that a jointly continuous action is semilocally equicontinuous.

Let $x \in \mathrm{X}$, $g \in \mathrm{G}$ and E be a vicinity of X. Let F be a symmetric vicinity such that $\mathrm{F} \circ \mathrm{F} \subseteq \mathrm{E}$. By continuity of the action map $\mathrm{G} \times \mathrm{X} \to \mathrm{X}$, there are neighborhoods U of g and V of x such that $\mathrm{U} \cdot \mathrm{V} \subseteq \mathrm{F}[gx]$. Therefore, we have, for any $h \in \mathrm{U}$ and $y \in \mathrm{V}$,

$$(hy, hx) = (hy, gx) \circ (gx, hx) \in \mathrm{F} \circ \mathrm{F} \subseteq \mathrm{E}.$$

Hence $\alpha(h)(\mathrm{V}) \subseteq \mathrm{E}[hx]$ for any $h \in \mathrm{U}$.

2 $\Rightarrow$ 1. Let us show that the action map is continuous at (g, x) for any $(g, x) \in \mathrm{G} \times \mathrm{X}$. Let W be a neighborhood of $gx \in \mathrm{X}$ and choose some symmetric vicinities E and F such that $\mathrm{E}[gx] \subseteq \mathrm{W}$ and $\mathrm{F} \circ \mathrm{F} \subseteq \mathrm{E}$. By orbital continuity, there is a neighborhood U_1 of g such that $\mathrm{U} \cdot x \subseteq \mathrm{F}[gx]$. By semilocal equicontinuity, there is another neighborhood U_2 of g and a neighborhood V of x such that $\alpha(h)(\mathrm{V}) \subseteq \mathrm{F}[hx]$ for any $h \in \mathrm{U}_2$. Let $\mathrm{U} = \mathrm{U}_1 \cap \mathrm{U}_2$, for any $h \in \mathrm{U}$ and $y \in \mathrm{V}$, we have

$$(hy, gx) = (hy, hx) \circ (hx, gx) \in \mathrm{F} \circ \mathrm{F} \subseteq \mathrm{E}.$$

Therefore, $\mathrm{U} \cdot \mathrm{V} \subseteq \mathrm{E}[gx] \subseteq \mathrm{W}$, hence the action map is continuous. ■

Remark C.1.5. — Actually, the proof yields the following refined equivalence, for any $(g_0, x_0) \in \mathrm{G} \times \mathrm{X}$.

1. The action map $\mathrm{G} \times \mathrm{X} \to \mathrm{X} \colon (g, x) \mapsto gx$ is continuous at (g_0, x_0).

2. The action is semilocally equicontinuous at (g_0, x_0) and the orbit map $\mathrm{G} \to \mathrm{X} \colon g \mapsto gx_0$ is continuous at g_0.

This detailed equivalence is indeed valuable, because if the action map is continuous at

(g_0, x_0) for some (g_0, x_0), then it is actually continuous at (g, x_0) for any g, since each $\alpha(g)$ is continuous. Therefore, *an orbitally continuous action is jointly continuous if and only if, for any $x_0 \in X$, there is* some $g_0 \in G$ *for which the action is semilocally equicontinuous at* (g_0, x_0). Compare this with the fact that, without orbital continuity, the possibility to check the semilocal equicontinuity condition only at (g_0, x_0) requires each $\alpha(g)$ to be *uniformly* continuous.

Corollary C.1.6. — *Joint and orbital continuities agree for isometric actions on metric spaces.*

Proof. — Such actions are automatically locally equicontinuous (actually, even (globally) uniformly equicontinuous): in the definition, choose $U = G$ and $V = E[x]$. ■

Corollary C.1.7 (cf. Lemma 4.1.6). — *Joint and orbital continuities agree for locally bounded linear representations on normed spaces.*

Proof. — By local boundedness, there is an identity neighborhood U of G such that the family $\{\alpha(g) \mid g \in U\}$ is equicontinuous (at every point). ■

The following lemmata show that when the group or the space is nice enough, joint and orbital continuities agree. All amounts to showing that, in these contexts, orbital continuity is enough to imply semilocal equicontinuity, or even local equicontinuity.

Lemma C.1.8. — *Joint and orbital continuities agree for locally compact groups acting (linearly or affinely) on barrelled locally convex spaces.*

Proof. — Let $\alpha = \pi + b$ be an affine orbitally continuous representation of a locally compact group G on a barrelled space X. Let us show that α is locally equicontinuous. Let $g \in G$ and choose any compact neighborhood U of g. As $b\colon G \to X$ is the orbit map of the zero vector, the linear representation $\pi = \alpha - b$ is also orbitally continuous. The set $\pi(U)$ is therefore a compact subset of $\mathscr{L}(E, E)$ endowed with the topology of pointwise convergence, hence is equicontinuous since X is barrelled (cf. [**Bou81**, III, § 4, n° 2, théorème 1]). On the other hand, the set of all translations of X is *uniformly* equicontinuous, hence $\alpha(U)$ is also equicontinuous. ■

Lemma C.1.9 (Ellis). — *Joint and orbital continuities agree for locally compact groups acting on locally compact spaces.*

On the proof. — Observe first that, considering Alexandroff's one-point compactification, we may assume that the space X on which the group G acts is actually compact. Using several clever reductions, Ellis showed [**Ell57**, Lemma 9] that the orbital continuity for the locally compact group G implies the joint continuity for all (possibly non-discrete) *countable* subgroups of G, hence the induced topology on the latter is finer than the topology of compact convergence [**Bou74**, x, §3, n° 4, cor. 1 du théorème 3]. Therefore, any compact identity neighborhood V in G is countably compact, hence precompact, for the topology of compact convergence. By Ascoli's theorem, V is thus equicontinuous. See [**Ell57**] or [**Bou74**, x, §3, exerc. 21 à 24] for more details. □

Corollary 4.1.7 showed that joint and orbital continuities of linear representations on Banach spaces agree for Baire or first-countable groups. This was the trace of two very different more general phenomena.

Lemma C.1.10. — *Joint and orbital continuities agree for Baire groups acting on first-countable uniform spaces*[2].

Proof. — Let G be a Baire group with an orbitally continuous action α on a first-countable space X and let $x_0 \in X$. We will show that there exists some $g_0 \in G$ such that the action is semilocally equicontinuous at (g_0, x_0) (cf. Remark C.1.5). Let $\{E_n\}$ be a countable base of symmetric vicinities such that $E_{n+1} \subseteq E_n$. Let E be any *closed* symmetric vicinity. Consider the sets

$$W_n = \{g \in G \mid \alpha(g)(E_n[x_0]) \subseteq E[gx_0]\}.$$

As each $\alpha(g)$ is continuous, the sets W_n form a nondecreasing cover of G. Moreover, each W_n is closed whenever α is orbitally continuous, since E is closed. Therefore, by Baire property, there is an index j such that W_j has an interior point g_0. This precisely means that α is semilocally equicontinuous at (g_0, x_0) (choose $U = W_j$ and $V = E_j[x_0]$ in the definition). ■

Lemma C.1.11. — *Joint and orbital continuities agree for first-countable groups acting (linearly or affinely) on barrelled locally convex spaces.*

Proof. — As in the proof of Lemma C.1.8, the affine case follows from the linear one since the set of all translations on a topological vector space is uniformly equicontinuous. Moreover, for first-countable groups, local equicontinuity can be checked on sequences instead of nets. But a converging *sequence* in a locally convex space is bounded. Hence if g_n converges to g in G and π is an orbitally continuous linear representation of G, then the set $\{\pi(g_n)\}$ is simply bounded, hence equicontinuous by barrelledness. ■

[2] A (Hausdorff) first-countable uniform space is nothing but a metric space. We keep however the former formulation in order to emphasize that the action need not be isometric.

As the previous results showed, local equicontinuity is, in many contexts, a "hidden" assumption which is implied by orbital continuity: in other words, the *topological* regularity of the action is enough to get some *uniform* regularity. Such a "rigidity" phenomenon should not be expected in general: after all, many different uniform structures can yield the same topological space. The following examples show indeed the sharpness of the implications proved in this appendix.

Example C.1.12. — We already saw in Example 4.1.8 a linear representation on a Hilbert space that is orbitally continuous but not locally bounded. But for orbitally continuous linear representations on normed spaces, semilocal equicontinuity, local equicontinuity and local boundedness are all equivalent, by the conjunction of Lemma C.1.4 and Lemma 4.1.6 (the completeness of the norm was not used in the latter). Hence the same examples produce orbitally continuous representations that are not semilocally equicontinuous.

Example C.1.13. — Let X be an infinite compact space and G be the group of homeomorphisms of X. Endowed with the topology of uniform convergence, G is a topological group [**Bou74**, X, §3, n° 5, prop. 11] whose natural action on X is jointly continuous. An identity neighborhood base for G is given by $\mathrm{U(E)} = \{h \in \mathrm{Homeo(X)} \mid \forall x \in \mathrm{X},\ (x, h(x)) \in \mathrm{E}\}$, where E runs among a base of vicinity for the (unique) uniform structure of X. These neighborhoods are usually never equicontinuous at any point: for instance, when $\mathrm{X} = [0, 1]$, any U(E) contains diffeomorphisms whose right derivative at zero can be arbitrarily large. Hence we see that a jointly continuous action, whereas always semilocally equicontinuous by Lemma C.1.4, need not be locally equicontinuous.

Observe that the argument used in Lemma 4.1.6, to deduce that local equicontinuity (or boundedness) follows from joint continuity in the case of linear representations on Banach spaces used the crucial fact that there exists a *bounded* identity neighborhood in E. The latter characterizes normed spaces amongst the locally convex spaces (cf. [**Bou81**, III, §1, n° 2, rem. 1]).

Without further assumptions (as in Lemmas C.1.8 or C.1.9), an orbitally continuous action *of* a compact group or *on* a compact space need not be jointly continuous[3]. For the latter case, an orbitally continuous, but non jointly continuous, action on the square has been given by Helmer in [**Hel80**, Example 13] (the group can be chosen abelian and homeomorphic (as a space) to **Q** endowed with its usual Euclidean topology). We give below an easy example of the former case.

Example C.1.14. — Let G be the circle group and $\mathrm{X} = \mathscr{C}(\mathrm{G})$ be the space of all continuous maps from G to **R**, endowed with the topology of pointwise convergence. Then the natural

[3] *« Il n'y a pas d'Ellis, hélas ! — C'est là qu'est l'os. »*

action of G on X is trivially orbitally continuous, but not jointly continuous. Indeed, for any $n \geqslant 1$, let f_n be the "triangle" continuous function from G to $\mathbf{R}$ defined by

$$f(e^{i\theta}) = \begin{cases} 0 & \text{if } \theta \leqslant 0 \text{ or } \theta \geqslant \frac{2}{n} \\ n\theta & \text{if } 0 \leqslant \theta \leqslant \frac{1}{n} \\ 2 - n\theta & \text{if } \frac{1}{n} \leqslant \theta \leqslant \frac{2}{n} \end{cases}$$

and let $g_n = \exp(-\frac{i}{n})$. Then (g_n, f_n) converges to $(1, \mathbf{0})$ in $G \times X$ (where $\mathbf{0}$ is the function that maps everything to zero), but $(g_n \cdot f_n)(1) = 1$ for any n. This phenomenon may even occur when the action is contragredient to a jointly continuous one (see Example C.2.6 below).

Group	Action	Space
	semilocally equicontinuous	
	by isometries	metric
	locally bounded and affine	normed
locally compact		locally compact
locally compact	affine	barrelled locally convex
first countable	affine	barrelled locally convex
Baire		first countable

Table C.1 – Cases where orbital and joint continuities agree. A blank cell means no specific assumption (i.e., a Hausdorff topological group acting by continuous transformations on a Hausdorff uniform space). In all but the first and the last cases, the action is actually locally equicontinuous.

C.2 Building new continuous representations

We will now consider another continuity requirement, much stronger than the above ones, but useful to build other continuous actions.

Definition C.2.1. — *An action* α *of a topological group* G *on a uniform space* X *is said* uniformly continuous *if for any* $g \in G$ *and any vicinity* E *of* X, *there is a neighborhood* U *of g such that for any $h \in \mathrm{U}$ and $x \in \mathrm{X}$,*

$$(hx, gx) \in \mathrm{E}.$$

In other words, an action α is uniformly continuous if the morphism $\alpha\colon \mathrm{G} \to \mathscr{C}(\mathrm{X},\mathrm{X})$ is continuous when $\mathscr{C}(\mathrm{X},\mathrm{X})$ is endowed with the topology of uniform convergence.

Of course, if the group acts by uniformly continuous transformations, it is enough to check the condition of the definition at the identity of G.

A uniformly continuous action is automatically jointly continuous (cf. [**Bou74**, x, §1, n° 6, prop. 9]) but this requirement is very strong: for instance a linear uniformly continuous representation would factor through a discrete quotient because of the last "for all $x \in \mathrm{X}$" quantifier[4]. However, this notion is useful to build some representations, as follows.

Proposition C.2.2. — *Let G be a topological group acting via α on a uniform space X. If the action is uniformly continuous, then the associated linear isometric representation π of X on $\mathscr{C}_{\mathrm{ucb}}(\mathrm{X})$ defined by*

$$(\pi(g)f)(x) = f(\alpha(g)^{-1}x)$$

is jointly continuous.

Proof. — We only need to show orbital continuity, since π is isometric. Let $f \in \mathscr{C}_{\mathrm{ucb}}(\mathrm{X})$, $g \in \mathrm{G}$ and $\varepsilon > 0$. The uniform continuity of f means that we can find some symmetric vicinity E such that $|f(x) - f(y)| \leqslant \varepsilon$ whenever $(x, y) \in \mathrm{E}$. But the uniform continuity of the action ensures that $(h^{-1}x, g^{-1}x) \in \mathrm{E}$ whenever h belongs to some neighborhood U of g. Hence $\| \pi(h)f - \pi(g)f \|_\infty \leqslant \varepsilon$ for those h, which ends the proof. ■

As the action of a topological group on itself given by translation on the *left* is uniformly continuous for the *right* uniformity, we immediately get the following corollary.

Corollary C.2.3. — *Let G be a topological group. The representation of G on $\mathscr{C}_{\mathrm{rucb}}(\mathrm{G})$, the space of bounded* right-*uniformly continuous functions of G, deduced from* left *translations in the argument, is jointly continuous.*

[4] There is a slight terminological conflict for representations on Banach spaces, for which "uniformly continuous" often means "continuous for the operator norm", see Remark 4.1.10. In the framework of this Appendix, a more proper name for the latter would be something like "boundedly uniformly continuous", since it amounts to the continuity of the morphism $\alpha\colon \mathrm{G} \to \mathscr{C}(\mathrm{X},\mathrm{X})$ when the latter is endowed with the topology of uniform convergence on bounded sets. But we shall not use that terminology, or even try to resolve this terminological conflict because, whatever the chosen definition, "uniform continuity" is far too strong for linear representations to be considered on a daily basis.

On the terminology. — The asymmetry in the above corollary motivates some authors to switch the definitions of right and left uniformly continuous functions.

Another useful continuity property satisfied by uniformly continuous action is the following.

Lemma C.2.4. — *If an action is uniformly continuous, then each orbit map is right-uniformly continuous.*

Proof. — Let $x \in X$ and E be a vicinity. Since the action is uniformly continuous, we can choose some identity neighborhood U such that $(y, uy) \in E$ for all $y \in X$ and $u \in U$. Therefore, for any $g, h \in G$ such that $gh^{-1} \in U$, we have $(hx, gx) = (hx, (gh^{-1})hx) \in E$. Hence the orbit map associated to x is right-uniformly continuous. ■

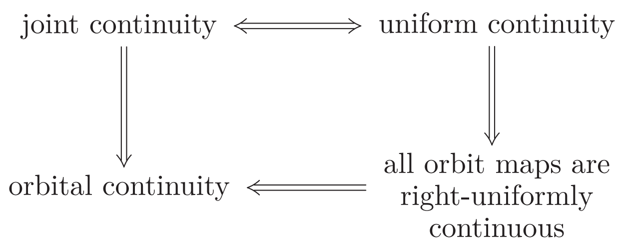
The next lemma shows that, fortunately, uniform continuity is easy to get when the space is compact.

Lemma C.2.5. — *An action on a compact space* X *is uniformly continuous if and only if it is jointly continuous.*

Proof. — We already observed that any uniformly continuous action is jointly continuous. Reciprocally, if the action is not uniformly continuous, we can find a vicinity E, a converging net $g_j \to g$ in G and a net of points x_j in X such that $(g_j x_j, g x_j)$ is *never* in E. By compactness, we can assume that x_j converges to some point x. Joint continuity would therefore imply that $(g_j x_j, g x_j)$ converges to (gx, gx), hence must eventually belong to the vicinity of E, a contradiction. ■

The above two lemmas and Ellis's theorem (Lemma C.1.9) yield a useful collapsing of continuity properties for compact spaces, summarized in Figure C.1.

Figure C.1 – Continuity properties for actions on compact spaces. All properties are equivalent when the group is locally compact.



We close this appendix by a discussion and an example explaining why we carefully avoided considering jointly continuous linear (or affine) representations for the fixed-point properties of Section 5.2. Actually, in light of the following, we argue that orbital continuity is indeed the *natural* continuity requirement to consider for linear representations. It is only a lucky coincidence that this natural continuity turns out to be much stronger in some cases (namely, when the space is a reflexive Banach space and either the group is locally compact or the action is uniformly bounded).

Let E be a locally convex space and E′ its dual, endowed with any polar topology[5]. The canonical duality $\mathrm{E} \times \mathrm{E}' \to \mathbf{R}$ is *separately* continuous by the very definition of the dual and of a polar topology. However, the canonical duality is *jointly* continuous if *and only if* E is a normed space and E′ is endowed with its dual norm[6].

This lack of joint continuity for the canonical duality has an important consequence for group actions: *the contragredient representation of a jointly continuous representation should not be expected to be jointly continuous*, even in the most palatable case of a compact group acting isometrically on a non-reflexive Banach space[7].

Example C.2.6. — Let G be an infinite compact metrizable group. Its natural isometric representation on $\mathscr{C}(\mathrm{G})$ is orbitally continuous (each element of $\mathscr{C}(\mathrm{G})$ is uniformly continuous), hence jointly continuous. However, the contragredient representation on the dual, $\mathscr{M}(\mathrm{G})$, is not jointly continuous. To check this fact, choose some $f \in \mathscr{C}(\mathrm{G})$ such that the family $\{g \cdot f\}_{g\in \mathrm{G}}$ is algebraically free (see Remark 5.2.12). Let g_n be a sequence of pairwise distinct elements of G converging to the identity. For any finite set of functions $\mathrm{F} \subset \mathscr{C}(\mathrm{G})$ and any n, define the element $h_{\mathrm{F},n}$ as any (for instance, the first) element g_k such that $k \geqslant n$ and $g_k \cdot f$ is not in the algebraic span of F. By the Hahn–Banach theorem, we can therefore find an element $m_{\mathrm{F},n}$ such that $m_{\mathrm{F},n}(g_{\mathrm{F},n} \cdot f) = 1$ but $m_{\mathrm{F},n}(f') = 0$ for any $f' \in \mathrm{F}$. Endowing the index set $\mathcal{P}_f(\mathscr{C}(\mathrm{G})) \times \mathbf{N}$ with the product order, we see that the net $h_{\mathrm{F},n}$ converges to the identity, the net $m_{\mathrm{F},n}$ converges to zero, but the net $h_{\mathrm{F},n}^{-1} m_{\mathrm{F},n}$ does not converge to zero in the weak topology.

[5] We recall that a *polar topology* is the topology of uniform convergence on elements of some covering family $\mathfrak{S}$ of bounded sets in E. In particular, a polar topology is always finer than the weak topology $\sigma(\mathrm{E}', \mathrm{E})$ (which corresponds to the case where $\mathfrak{S}$ is the family of singletons of E) and weaker than the topology of uniform convergence on all bounded sets.

[6] The “only if” part is a standard polar argument. If the canonical duality is jointly continuous, we can find balanced convex zero neighborhoods $\mathrm{U} \subset \mathrm{E}$ and $\mathrm{V} \subset \mathrm{E}'$ such that $|\langle \mathrm{U}, \mathrm{V}\rangle| \leqslant 1$. But V can be assumed to be the polar of some bounded set B in E, hence U is contained in the bipolar of B. Since the latter is the closed convex hull of $\mathrm{B} \cup \{0\}$, this shows that U is a *bounded* neighborhood, hence E is a normed space. This in turn implies that V absorbs the polar of any bounded set.

[7] On a reflexive space, the contragredient representation would actually be norm-continuous (by Lemma 4.1.23) and isometric, hence jointly continuous.

D MEASURABILITY IN BANACH SPACES

Alle Kunst ist Maß. Maß gegen Maß,
das ist alles[1].

W. LEHMBRUCK

THE BOREL STRUCTURE of a non-separable Banach space is highly pathological. This explains why we restricted our attention to separable Banach spaces for moderate inductions, and why Banach space integration relies on the weaker but better-behaved scalar measurability.

Section D.1 explains the first pathologies of the Borel σ-algebra generated by any topology on a large enough Banach space. The next two sections are more optimistic: Section D.2 introduces the scalar measurability, which is suitable for integration purposes, and Section D.3 shows that the Borel σ-algebra of the norm can be tamed if measurable maps are defined on standard spaces. Section D.4 explains how some measurability requirements for group representations are enough to get continuity. Section D.5 finally gives some examples distinguishing the different σ-algebras encountered in previous sections.

In this appendix, contrary to the rest of the thesis, the statements made about topological groups are intended to be applied ultimately to the additive group of a locally convex space, even of a Banach space with one of its natural topologies. We nonetheless state the results in their "group" form, since they have nothing to do with commutativity or scalar multiplication.

[1] *All art is measure. Measure against measure, that's all.* Quoted in P. WESTHEIM, *Wilhelm Lehmbruck: das Werk Lehmbrucks in 84 Abbildungen* (1920), p. 61.

D.1 Pathology of Borel σ-algebras

We recall that, given two σ-algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ on sets X_1 and X_2, the *product σ-algebra* $\mathcal{A}_1 \otimes \mathcal{A}_2$ is the coarsest σ-algebra on $X_1 \times X_2$ that makes the projections $\pi_i \colon X_1 \times X_2 \to (X_i, \mathcal{A}_i)$ measurable. In particular, a map f from a measurable space S to $X_1 \times X_2$ is measurable with respect to $\mathcal{A}_1 \otimes \mathcal{A}_2$ if and only if each $\pi_i \circ f$ is measurable with respect to $\mathcal{A}_i$. Equivalently, $\mathcal{A}_1 \otimes \mathcal{A}_2$ is the σ-algebra generated by measurable rectangles $A_1 \times A_2$, $A_i \in \mathcal{A}_i$. This σ-algebra is a natural object to consider for measure theory, since there is an obvious way to associate a measure to a rectangle $A_1 \times A_2$ if we are given measures on each A_i.

For a topological space X, we write $\mathcal{B}(X)$ for its *Borel σ-algebra*, that is, the σ-algebra generated by all open sets in X. Let us begin with the well-known link between the Borel σ-algebra of a product and the product of Borel σ-algebras.

Lemma D.1.1. — *Let* X *and* Y *be two topological spaces. Then* $\mathcal{B}(X) \otimes \mathcal{B}(Y) \subseteq \mathcal{B}(X \times Y)$. *If* X *and* Y *are second countable, we have* $\mathcal{B}(X) \otimes \mathcal{B}(Y) = \mathcal{B}(X \times Y)$.

Proof. — Since $\mathcal{B}(X) \otimes \mathcal{B}(Y)$ is the coarsest σ-algebra such that the projections are measurable whereas $\mathcal{B}(X \times Y)$ is the Borel σ-algebra of the coarsest topology such that the projections are continuous, the latter obviously contains the former.

Assume now that X and Y are second countable. Their product admits therefore a *countable* base $\mathfrak{B}$ made of products of open sets. Each element of $\mathfrak{B}$ trivially belongs to $\mathcal{B}(X) \otimes \mathcal{B}(Y)$, and $\mathfrak{B}$ generates $\mathcal{B}(X \times Y)$ by countability. This proves $\mathcal{B}(X \times Y) \subseteq \mathcal{B}(X) \otimes \mathcal{B}(Y)$, hence the equality. ■

The following corollary explains why it was so much convenient to assume our Banach spaces to be separable in Sections 4.3.D and 4.3.E (recall that, for a metric space, separability is equivalent to second countability)—and why fewer problems arise in finite dimension.

Corollary D.1.2. — *Let* S *be a measurable space and* G *be a second-countable topological group, endowed with its Borel σ-algebra. Let* f_1 *and* f_2 *be two measurable maps from* S *to* G. *The pointwise product* $f_1 f_2 \colon S \to G \colon s \mapsto f_1(s) f_2(s)$ *is also measurable.*

Proof. — The map $f_1 f_2$ is the composition

$$\begin{array}{ccccc} \mathrm{S} & \longrightarrow & \mathrm{G}\times\mathrm{G} & \longrightarrow & \mathrm{G} \\ s & \longmapsto & (f_1(s), f_2(s)) & \longmapsto & f_1(s)f_2(s). \end{array}$$

The first arrow is measurable with respect to $\mathcal{B}(\mathrm{G})\otimes\mathcal{B}(\mathrm{G})$ by definition of the product σ-algebra and the second arrow is measurable with respect to $\mathcal{B}(\mathrm{G}\times\mathrm{G})$ since the group multiplication is continuous. As G is second countable, these two σ-algebras coincide, hence $f_1 f_2$ is indeed measurable. ■

It is transparent from the above proof that the product of two measurable maps may fail to be measurable if the group is huge enough to have $\mathcal{B}(\mathrm{G})\otimes\mathcal{B}(\mathrm{G}) \neq \mathcal{B}(\mathrm{G}\times\mathrm{G})$. In order to produce simple examples, we will first establish three lemmas of independent interest about general σ-algebras.

Let us first recall the very useful "countable generation" lemma. The σ-algebra generated by some family of sets $\mathcal{F}$ is written $\sigma(\mathcal{F})$.

Lemma D.1.3. — *Let $\mathcal{F} \subseteq \mathscr{P}(\mathrm{X})$ be any collection of subsets of a set X. If $\mathrm{A} \in \sigma(\mathcal{F})$, then there is a countable $\mathcal{F}' \subseteq \mathcal{F}$ such that $\mathrm{A} \in \sigma(\mathcal{F}')$.*

Proof. — The collection

$$\{\mathrm{A} \subseteq \mathrm{X} \mid \exists \mathcal{F}' \subseteq \mathcal{F},\, \mathcal{F}' \text{ countable, such that } \mathrm{A} \in \sigma(\mathcal{F}')\}$$

is a σ-subalgebra of $\sigma(\mathcal{F})$ that obviously contains $\mathcal{F}$, hence which is equal to $\sigma(\mathcal{F})$. ■

Lemma D.1.4. — *Let $\mathcal{A}$ be a σ-algebra on a set X. Then the following are equivalent.*

1. *The diagonal Δ belongs to $\mathcal{A}\otimes\mathcal{A}$.*

2. *The σ-algebra $\mathcal{A}$* countably separates *the points of X, that is: there is a countable family $\{\mathrm{A}_n\}$ in $\mathcal{A}$ such that for any $x \neq y \in \mathrm{X}$, there is some n such that $x \in \mathrm{A}_n$ but $y \notin \mathrm{A}_n$.*

Proof. — Both directions are proved via a detour by equivalence relations. For any family $\mathcal{F}$ of subsets of X, define the "indistinguishability" equivalence relation $\mathscr{R}_{\mathcal{F}}$ via

$$x\mathscr{R}_{\mathcal{F}}y \quad \equiv \quad \forall \mathrm{A} \in \mathcal{F},\, (x \in \mathrm{A} \;\Leftrightarrow\; y \in \mathrm{A}).$$

In other words, $x\mathscr{R}_{\mathcal{F}}y$ if the family $\mathcal{F}$ cannot make a difference between x and y.

For any subset A of the product X × X, define the "right fiber" equivalence relation $\mathscr{R}_{\mathrm{A}}$ by equality of the right fibers, that is

$$x\mathscr{R}_{\mathrm{A}}y \quad \equiv \quad \{z \mid (z,x) \in \mathrm{A}\} = \{z \mid (z,y) \in \mathrm{A}\}\,.$$

We now proceed to the proof of the lemma. If $\mathcal{F}$ is a *countable* family of sets in $\mathcal{A}$, then $\mathscr{R}_{\mathcal{F}}$ belongs to $\mathcal{A} \otimes \mathcal{A}$, and if $\mathcal{F}$ separates the points, this precisely means that the relation $\mathscr{R}_{\mathcal{F}}$ is the equality relation, namely Δ. This proves (2) ⇒ (1).

Reciprocally, assume that $\Delta \in \mathcal{A} \otimes \mathcal{A}$. Since the latter is generated by rectangles, we can find two countable families $\{\mathrm{A}_n\}$ and $\{\mathrm{B}_n\}$ such that $\Delta \in \sigma(\{\mathrm{A}_n \times \mathrm{B}_n\})$. The fiber equivalence relation $\mathscr{R}_\Delta$ therefore contains the indistinguishability equivalence relation $\mathscr{R}_{\{\mathrm{B}_n\}}$. Since $\mathscr{R}_\Delta = \Delta$, this means that the family $\{\mathrm{B}_n\}$ separates the points. ■

Corollary D.1.5. — *Let $\mathcal{A}$ be a σ-algebra on the set* X. *If $\Delta \in \mathcal{A} \otimes \mathcal{A}$, then the cardinal of* X *is at most* $2^{\aleph_0}$.

Proof. — By the above lemma, we know that there is a countable family $\{\mathrm{A}_n\}$ of subsets of X that separates the points. So, for any $x \in \mathrm{X}$, we have

$$\{x\} = \bigcap_{n:x\in \mathrm{A}_n} \mathrm{A}_n \cap \bigcap_{n:x\notin \mathrm{A}_n} \mathrm{A}_n^{\mathrm{c}}.$$

Hence the map

$$\begin{aligned} \mathrm{X} &\to \mathscr{P}(\mathbf{N}) \\ x &\mapsto \{n \mid x \in \mathrm{A}_n\} \end{aligned}$$

is injective. ■

Now comes the promised result about a nonmeasurable product of two measurable maps.

Corollary D.1.6. — *Let* G *be a Hausdorff topological group. If* $|\mathrm{G}| > 2^{\aleph_0}$, *then there exist a measurable space* S *and two measurable functions from* S *to* G *whose pointwise product is not measurable.*

Proof. — We choose $\mathrm{S} = (\mathrm{G} \times \mathrm{G}, \mathcal{B}(\mathrm{G}) \otimes \mathcal{B}(\mathrm{G}))$. Let f_1 and f_2 be defined as

$$f_1(g,h) = g \qquad f_2(g,h) = h^{-1}.$$

These maps from S to G are indeed measurable, since the projections are measurable by definition of the product σ-algebra and the inversion is continuous, hence measurable with respect to the Borel σ-algebras. However, the pointwise product $f_1 f_2$ is not measurable, since the inverse image of the closed set $\{e\}$ is the diagonal, which does not belong to $\mathcal{B}(G) \otimes \mathcal{B}(G)$ by Corollary D.1.5. ■

Observe than the above corollary applies notably to the additive group of $\ell^\infty(I)$, for I any uncountable set, with any of its usual topologies. It also applies to any discrete group of large enough cardinality, showing that, for measurability considerations, "discrete" is not as trivial as for continuity considerations (the problem being, of course, than $\mathscr{P}(X) \otimes \mathscr{P}(X)$ is strictly smaller than $\mathscr{P}(X \times X)$ whenever $|X| > 2^{\aleph_0}$, as shown by Corollary D.1.5).

Remark D.1.7. — Since cardinal bounds appear for such simple arguments, the reader may have guessed (or feared) that undecidable questions are not far from the results presented in this appendix. Here is a first one: whether $\mathscr{P}(\mathbf{R}) \otimes \mathscr{P}(\mathbf{R})$ is the whole $\mathscr{P}(\mathbf{R} \times \mathbf{R})$ is independent of **ZFC** (the Zermelo–Fraenkel axioms and the axiom of choice). More precisely, for any cardinal κ, let R(κ) be the statement "$\mathscr{P}(\kappa) \otimes \mathscr{P}(\kappa) = \mathscr{P}(\kappa \times \kappa)$". It is not hard to prove R(κ) for $\kappa \leqslant \aleph_1$ (see e.g. [**Sri98**, Theorem 3.1.24]) and we have proved ¬R(κ) whenever $\kappa > 2^{\aleph_0}$ (Corollary D.1.5). This is it for **ZFC**. Assuming the continuum hypothesis, the picture is complete; but, for instance, $R(\aleph_2)$ is independent of **ZFC** and the negation of the continuum hypothesis [**Kun68**, § 12].

D.2 Scalar measurability

In order to avoid the pathologies of the Borel σ-algebras shown in the above section, we will now "linearize" the measurability of Banach spaces[2]. The *scalar weak* σ*-algebra* on a Banach space E, written $\mathcal{A}_{\mathrm{sw}}(E)$, is the coarsest σ-algebra such that all continuous linear forms are measurable. The interest of this σ-algebra is to linearize all questions concerning the measure. For instance, a map $f : S \to E$ is scalarly weakly measurable (that is, measurable with respect to the scalar weak σ-algebra) if and only if $\lambda \circ f : S \to \mathbf{R}$ is measurable for all $\lambda \in E$. This shows in particular that $\mathcal{A}_{\mathrm{sw}}(E) \otimes \mathcal{A}_{\mathrm{sw}}(E) = \mathcal{A}_{\mathrm{sw}}(E \times E)$ (since $(E \times E)' = E' \times E'$) and that the sum of any two scalarly weakly measurable maps is still scalarly weakly measurable by Corollary D.1.2 (since **R** is second countable), whatever their definition domain.

[2]The results of this section could be stated in the framework of duality of locally convex spaces, but we will stay in the Banach world for simplicity of exposition.

For a dual Banach space $\mathrm{E} = \mathrm{E}_\flat'$, we define similarly $\mathcal{A}_{\mathrm{sw}*}(\mathrm{E})$, the *scalar weak-$*$ σ-algebra*, as the coarsest σ-algebra such that all linear forms coming from $\mathrm{E}_\flat$ are measurable.

For a Banach space E, we will write $\mathcal{B}(\mathrm{E})$ and $\mathcal{B}_{\mathrm{w}}(\mathrm{E})$ for the Borel σ-algebras generated by the norm and weak topologies, respectively, as well as $\mathcal{B}_{\mathrm{w}*}(\mathrm{E})$ for the one generated by the weak-$*$ topology if we are given a predual for E. Since all continuous linear forms are weakly measurable, we have $\mathcal{A}_{\mathrm{sw}}(\mathrm{E}) \subseteq \mathcal{B}_{\mathrm{w}}(\mathrm{E}) \subseteq \mathcal{B}(\mathrm{E})$. Similarly, $\mathcal{A}_{\mathrm{sw}*}(\mathrm{E}) \subseteq \mathcal{B}_{\mathrm{w}*}(\mathrm{E}) \subseteq \mathcal{B}(\mathrm{E})$. We will see below (Section D.5) that all these inclusions may be strict, but let us first have a look at when they can agree.

It is well-known that $\mathcal{B}(\mathrm{E}) = \mathcal{B}_{\mathrm{w}}(\mathrm{E})$ for separable Banach spaces (see e.g. [**Mon01**, Lemma 3.3.3]). Actually, we have more:

Lemma D.2.1. — *Let* E *be a Banach space. If* E *is separable, then* $\mathcal{B}(\mathrm{E}) = \mathcal{A}_{\mathrm{sw}}(\mathrm{E})$.

Proof. — Let $\{x_n\}$ be a dense countable set in E and choose, by the Hahn–Banach theorem, linear forms φ_n of norm one and such that $\varphi_n(x_n) = \| x_n \|$. A simple approximation argument then shows that $\| x \| = \sup_n |\varphi_n(x)|$ for all $x \in \mathrm{E}$. In particular, for any $y \in \mathrm{E}$, the function $\| y - \cdot \|$ is the supremum of a countable family of $\mathcal{A}_{\mathrm{sw}}(\mathrm{E})$-measurable maps, hence is $\mathcal{A}_{\mathrm{sw}}(\mathrm{E})$-measurable itself. This shows that all balls belong to $\mathcal{A}_{\mathrm{sw}}(\mathrm{E})$; since they generate $\mathcal{B}(\mathrm{E})$ by second countability of E, we got the lemma. ■

By the same supremum argument as above, we can see that the norm is $\mathcal{A}_{\mathrm{sw}*}(\mathrm{E})$-measurable when E is endowed with a separable predual[3]. However, it does not follow that $\mathcal{A}_{\mathrm{sw}*}(\mathrm{E}) = \mathcal{B}(\mathrm{E})$, since the latter may not be generated by balls if E itself is not separable.

Remark D.2.2. — Edgar has proved [**Edg77**, Theorem 2.3] that $\mathcal{A}_{\mathrm{sw}}(\mathrm{E})$ agree with the *Baire σ-algebra* of the weak topology, that is, the coarsest σ-algebra such that all weakly continuous real-valued functions (linear or not) are measurable (and similarly for $\mathcal{A}_{\mathrm{sw}*}(\mathrm{E})$). Since the Baire σ-algebras are often less pathological than the Borel σ-algebras from the measure-theoretical point of view, it is not surprising that $\mathcal{A}_{\mathrm{sw}}(\mathrm{E})$ and $\mathcal{A}_{\mathrm{sw}*}(\mathrm{E})$ are convenient for Banach space integration.

[3] This fact can be useful for integration theory, since it allows to impose some integrability conditions on the function $\| f \|$ even if f is only assumed to be $\mathcal{A}_{\mathrm{sw}*}(\mathrm{E})$-measurable. This explains the omnipresence of separable preduals in induction techniques for bounded cohomology, see e.g. [**Mon01**, § 10.2].

D.3 Automatic separability

We saw that some separability is useful to avoid measurability problems. We show here that we can get that assumption for free if we start from a small enough measurable space. Recall that the cardinal bound of the next lemmas notably applies to the Borel σ-algebra of any second-countable space, by a standard transfinite induction argument (see e.g. [**Fol99**, Proposition 1.23]).

Lemma D.3.1. — *Let* $(\mathrm{S}, \mathcal{A})$ *be a measurable space with at most* $2^{\aleph_0}$ *measurable sets and* X *a metrizable space. Then any measurable function* $f\colon \mathrm{S} \to \mathrm{X}$ *has a separable range.*

Proof. — Assume that $f(\mathrm{S})$ is not separable. We can therefore find some $\varepsilon > 0$ and an uncountable set $\mathrm{C} \subset f(\mathrm{S})$ such that for any $c, c' \in \mathrm{C}$, we have $d(c, c') > 2\varepsilon$, where d is a compatible metric on X. Choose any point $x \in \mathrm{X}$ and define a function $g\colon \mathrm{S} \to \mathrm{X}$ as

$$g(s) = \begin{cases} c & \text{if there exists } c \in \mathrm{C} \text{ such that } d(f(s), c) < \varepsilon \\ x & \text{otherwise.} \end{cases}$$

Observe that the function g is measurable. Indeed, writing U for the union over $c \in \mathrm{C}$ of the open balls $\mathbf{B}(c, \varepsilon)$, we have, for any Borel subset B of X,

$$\begin{aligned} g^{-1}(\mathrm{B}) &= g^{-1}(\mathrm{B} \cap \mathrm{C}) \cup g^{-1}(\mathrm{B} \cap \{x\}) \\ &= f^{-1}(\mathrm{B} \cap \mathrm{U}) \cup f^{-1}(\mathrm{B} \cap \mathrm{U}^{\mathrm{c}}), \end{aligned}$$

which is measurable since f is so.

But this means that the preimage g^{-1} defines an injection

$$\begin{aligned} \mathscr{P}(\mathrm{C}) &\hookrightarrow \mathcal{A} \\ \mathrm{A} &\mapsto g^{-1}\left(\bigcup_{c \in \mathrm{A}} \mathbf{B}(c, \varepsilon)\right), \end{aligned}$$

which contradicts our cardinal bound on $\mathcal{A}$. ■

Corollary D.3.2. — *Let* S *be a measurable space with at most* $2^{\aleph_0}$ *measurable sets and* G *be a metrizable topological group, endowed with its Borel σ-algebra. The pointwise product of two measurable functions* $\mathrm{S} \to \mathrm{G}$ *is again measurable.*

Proof. — Let f_1, f_2 be two measurable maps from S to G. By Lemma D.3.1, there is a separable subgroup $\mathrm{G}' \leqslant \mathrm{G}$ such that f_1 and f_2 have values in G'. Since G' is separable and metrizable, it is second countable and we can therefore apply Corollary D.1.2 to conclude that the pointwise product $f_1 f_2$ is measurable. ■

D.4 From measurability to continuity

We review here how we can automatically get some continuity of group representations from the measurability of orbit maps. Since the latter often comes for free (for instance, if we build our representation from a non-singular action), this kind of tools is valuable[4].

The fundamental lemma is the following.

Lemma D.4.1. — *Let* G *be a Baire group and* π *be a uniformly bounded representation of* G *on a separable Banach space* V*. If all orbit maps are Borel, then the representation is continuous.*

On the proof. — The proof relies on three ideas. First, if a Borel set A of a Baire group is not meager, then AA^{-1} is an identity neighborhood (see [**Bou74**, IX, § 6, n° 8, lemme 9]). Second, if the preimage of an open set by some orbit map F were meager, then so would be all its translates; using Lindelöf's theorem, we could then write G as a countable union of meager sets, which is absurd. Last, if U is a neighborhood of $v \in \mathrm{V}$, then there is a smaller neighborhood W of v such that $(\mathrm{F}^{-1}(\mathrm{W}))(\mathrm{F}^{-1}(\mathrm{W}))^{-1} \subset \mathrm{F}^{-1}(\mathrm{U})$, thanks to uniform boundedness. See [**Mon01**, Proposition 1.1.3] for details (the proof is given there for isometric representation, which is not a restriction as the hypotheses and conclusion of the lemma are invariant under renorming). □

The results of the previous section allow to get rid of the separability assumption when the group is Polish.

Corollary D.4.2. — *Let* G *be a Polish group and* π *be a uniformly bounded linear representation of* G *on a Banach space* E*. If all orbit maps are Borel, then the representation is continuous.*

Proof. — Since π is uniformly bounded, we only need to prove orbital continuity (by Lemma 4.1.6), for which it is enough to show that the restriction of π to any subspace

[4] This strategy was used in [**GM17**] to prove the continuity of the moderately measured induced representations.

generated by an orbit is continuous. This follows from Lemma D.4.1 since orbits are separable by Lemma D.3.1. ■

Compare the above "automatic separability of orbits" of the above proof with Proposition 4.1.13.

In general, we cannot weaken the Borel measurability of the above lemmas by some scalar measurability (Example D.5.2 below). See however [**SvN50**, Theorem 2] for an interesting decomposition of scalarly measurable unitary representations of locally compact groups into the direct sum of a continuous representation and completely degenerate one.

D.5 Some examples

We conclude by some examples.

Example D.5.1. — We show here that the scalar σ-algebras $\mathcal{A}_{\mathrm{sw}}(\mathrm{E})$ and $\mathcal{A}_{\mathrm{sw}*}(\mathrm{E})$ can be strictly smaller than their Borel counterparts $\mathcal{B}_{\mathrm{w}}(\mathrm{E})$ and $\mathcal{B}_{\mathrm{w}*}(\mathrm{E})$. Let $\mathrm{S} = (\mathbf{R}, \mathcal{B}(\mathbf{R}))$ be the real line endowed with its usual Borel σ-algebra and let $\mathbf{R}_d$ be the real line but endowed with the counting measure. Let $\mathrm{E} = \ell^2(\mathbf{R}_d)$ and consider the map

$$\delta\colon \mathrm{S} \to \mathrm{E}\colon s \mapsto \delta_s.$$

Since any $\xi \in \mathrm{E}' = \mathrm{E}$ has a countable support, the map $\xi \circ \delta\colon s \mapsto \xi(s)$ also has a countable support, hence is measurable. This shows that δ is $\mathcal{A}_{\mathrm{sw}}(\mathrm{E})$-measurable. However, for *any* subset A of $\mathbf{R}$, we have

$$\mathrm{A} = \delta^{-1}\left(\bigcup_{s\in \mathrm{A}} \left\{\xi \in \mathrm{E} \;\middle|\; |\xi(s)| > \frac{1}{2}\right\}\right),$$

hence δ cannot be $\mathcal{B}_{\mathrm{w}}(\mathrm{E})$-measurable since the union of the right-hand side is weakly open.

Therefore, $\mathcal{A}_{\mathrm{sw}}(\mathrm{E}) \neq \mathcal{B}_{\mathrm{w}}(\mathrm{E})$ (and similarly for the weak-$*$ variants of these algebras, since E is reflexive).

Example D.5.2. — An adaptation of the above example shows that the representation of the group $\mathbf{R}$ on $\ell^2(\mathbf{R})$ (in other words, the left regular representation of the discrete group $\mathbf{R}_d$ but viewed as a representation of the Polish group $\mathbf{R}$), has all its orbit maps scalarly measurable. It is however obviously not continuous, so that Corollary D.4.2 cannot be extended to scalar measurability.

Looking back at Lemma D.2.1, we could ask if the Borel σ-algebras of the norm and the weak topologies could agree for non-separable Banach spaces.

Example D.5.3. — Edgar showed [**Edg77**, Theorem 1.1] that $\mathcal{B}(\mathrm{E}) = \mathcal{B}_{\mathrm{w}}(\mathrm{E})$ whenever E admits an equivalent Kadec norm. We recall from the end of Section 4.1.F that a norm is Kadec if the norm and the weak topologies agree on the unit sphere and that any separable Banach space admits an equivalent Kadec norm. By combining Edgar's result with Corollary D.4.2, we conclude in particular that any weakly continuous linear representation of a Banach space on a space which admits an equivalent Kadec norm must be orbitally continuous.

Example D.5.4. — On the other hand, Talagrand showed that $\mathcal{B}(\mathrm{E}) \neq \mathcal{B}_{\mathrm{w}}(\mathrm{E})$ for the space $\mathrm{E} = \ell^\infty(\mathbf{N})$ [**Tal78**, Théorème 3].

The last example is the most convincing reason to avoid working with the Borel σ-algebra of a non-separable Banach space for integration theory.

Example D.5.5. — Is the sum of two norm-measurable maps with values in $\ell^\infty(\mathbf{N})$ again norm-measurable? The answer is positive if we assume the continuum hypothesis, but otherwise independent of **ZFC** (and also of **ZFC** plus the negation of the continuum hypothesis). More generally, we have the following equivalence. For any Banach space E, let S(E) be the statement "the sum of any two norm-measurable maps is again norm-measurable". In [**Tal79**, Théorème 1], Talagrand showed[5] that S(E) is actually equivalent to the statement R(κ), "$\mathscr{P}(\kappa) \otimes \mathscr{P}(\kappa) = \mathscr{P}(\kappa \times \kappa)$", where $\kappa = \kappa(\mathrm{E})$ is the smallest cardinal such that E admits a dense subset of that cardinality. For $\mathrm{E} = \ell^\infty(\mathbf{N})$, we have $\kappa(\mathrm{E}) = 2^{\aleph_0}$, and Kunen showed that $\mathrm{R}(\aleph_1)$ holds but $\mathrm{R}(2^{\aleph_0})$ is independent of **ZFC** with the negation of the continuum hypothesis (see Remark D.1.7).

[5] Talagrand formulates S(E) as the measurability of the sum $\mathrm{E} \times \mathrm{E} \to \mathrm{E}$ with respect to the product σ-algebra $\mathcal{B}(\mathrm{E}) \otimes \mathcal{B}(\mathrm{E})$, but we can easily see (by an argument similar to the proof of Corollary D.1.6) that this is equivalent to our statement.

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INDEX OF NOTIONS

Boldface numbers refer to main definitions, italic ones to important uses of the notion. The abbreviation "n" stands for "footnote".

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