Cloaking via anomalous localized resonance for doubly complementary media in the finite frequency regime

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Abstract

Cloaking a source via anomalous localized resonance (ALR) was discovered by Milton and Nicorovici in [15]. A general setting in which cloaking a source via ALR takes place is the setting of doubly complementary media. This was introduced and studied in [20] for the quasistatic regime. In this paper, we study cloaking a source via ALR for doubly complementary media in the finite frequency regime. To this end, we establish the following results: 1) Cloaking a source via ALR appears if and only if the power blows up; 2) The power blows up if the source is "placed" near the plasmonic structure; 3) The power remains bounded if the source is far away from the plasmonic structure. Concerning the analysis, on one hand we extend ideas from [20] and on the other hand we add new insights into the problem. This allows us not only to overcome difficulties related to the finite frequency regime but also to obtain new information on the problem. In particular, we are able to characterize the behaviour of the fields far enough from the plasmonic shell as the loss goes to 0 for an **arbitrary source** outside the core-shell structure in the doubly complementary media setting.

1 Introduction and statement of the main results

1.1 Introduction

Negative index materials (NIMs) were first investigated theoretically by Veselago in [30]. The existence of such materials was confirmed by Shelby, Smith, and Schultz in [29]. The study of NIMs has attracted a lot of attention in the scientific community thanks to their many possible applications. One of the appealing ones is cloaking. There are at least three ways to do cloaking using NIMs. The first one is based on plasmonic structures introduced by Alu and Engheta in [3]. The second one uses the concept of complementary media. This was suggested by Lai et al. in [13] and confirmed theoretically in [21] for related schemes (see also [26]). The last one is based on the concept of ALR discovered by Milton and Nicorovici in [15]. In this paper, we concentrate on the last method.

Cloaking a source via ALR was discovered by Milton and Nicorovici in [15]. Their work has its root from [27] (see also [14]) where the localized resonance was observed and established for constant symmetric plasmonic structures in the two dimensional quasistatic regime. More

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precisely, in [15], the authors studied core-shell plasmonic structures in which a circular shell has permittivity $-1 - i\delta$ while its complement has permittivity 1 where δ denotes the loss of the material in the shell [1]. Let r_1 and r_2 be the inner and the outer radius of the shell. They showed that there is a critical radius $r_* := (r_2^3 r_1^{-1})^{1/2}$ such that a dipole is not seen, after the normalization of the power [2], by an observer away from the core-shell structure, hence it is cloaked, if and only if the dipole is within distance r_* of the shell. Moreover, the power $E_\delta(u_\delta)$ of the field u_δ, which is defined in (1.5), blows up as the loss δ goes to 0. In [15], the authors also investigated a single dipole source in the finite frequency regime outside the slab lens of coefficient -1. Two key features of this phenomenon are: 1) the localized resonance, i.e., the fields blow up in some regions and remain bounded in some others as the loss goes to 0; 2) the connection between the localized resonance and the blow up of the power as the loss goes to 0.

Cloaking a source via ALR has been mainly studied in the quasistatic regime. In [7], Bouchitte and Schweizer proved that a small circular inclusion of radius $\gamma(\delta)$ (with $\gamma(\delta) \to 0$ fast enough) is cloaked by the core-shell plasmonic structure mentioned above in the two dimensional quasistatic regime if the inclusion is located within distance r_* of the shell; otherwise it is visible. Concerning the second feature on cloaking a source via ALR, the blow up of the power was studied for a more general setting in the two dimensional quasistatic regime by Ammari et al. in [5] and Kohn et al. in [12]. More precisely, they considered non-radial core-shell structures in which the shell has permittivity $-1 - i\delta$ and its complement has permittivity 1. In [5], Ammari et al. dealt with arbitrary shells and provided a characterization of sources for which the power blows up via the information of the spectral decomposition of a Neumann-Poincaré type operator. In [12], Kohn et al. considered core-shell structures in which the outer boundary of the shell is round but the inner is not and established the blow up of the power for some class of sources using a variational approach. A connection between the blow up of the power and the localized resonance depends on the geometry and property of plasmonic structures, see [25] (and also [22]) for a discussion on this. Cloaking a source via ALR in some special three dimensional geometry was studied in [4]. Motivated by the concept of reflecting complementary media suggested and studied in [17] and results mentioned above, in [20] we studied cloaking a source via ALR for a general core shell structure of doubly complementary media property (see Definition 1.2) in the quasistatic regime[3]. More precisely, we established the following three properties for doubly complementary media:

P1) Cloaking a source via ALR appears if and only if the power blows up.

P2) The power blows up if the source is located "near" the shell.

P3) The power remains bounded if the source is far away from the shell.

Using these results, we extended various results mentioned previously. Moreover, we were able to obtain schemes to cloak an arbitrary source concentrating on an arbitrary smooth bounded

[1] In fact, in [15] and in other works, the authors consider the permittivity $-1 + i\delta$ instead of $-1 - i\delta$; but this point is not essential.

[2] More details on the normalization process are given later.

[3] Roughly speaking, the plasmonic shell is not only complementary with a part of the complement of the core shell but also complement to a part of the core.

manifold of codimension 1 placed in an arbitrary medium via ALR; the cloak is independent of the source. The analysis in [20] is on one hand based on the reflecting techniques initiated in [17], the removing localized singularity technique introduced in [18, 21] to deal with the localized resonance. On the other hand, it is based on new observations on the Cauchy problems and the separation of variables technique for a general shell introduced there. The implementation of this technique is an ad-hoc part of [20].

In this paper, we study cloaking a source via ALR for the finite frequency regime. More precisely, we establish Properties P1), P2) and P3) for doubly complementary media in the finite frequency regime. As a consequence, we are also able to obtain schemes to cloak **a generic source** concentrating on **the boundary of a smooth bounded open subset of** $\mathbb{R}^d$ placed in an **arbitrary medium** via ALR; the cloak is independent of the source (see Proposition 5.1 in Section 5). Concerning the analysis, on one hand we extend ideas from [20] and on the other hand we add various new insights into the problem. This allows us 1) to overcome difficulties related to the finite frequency regime such as the use of the maximum priniciple, 2) to shorten the approach in [20], and more importantly 3) to obtain new information on cloaking a source via ALR. In particular, we can characterize the behaviour of the fields far enough from the plasmonic shell as the loss goes to 0 for **arbitrary** sources outside the core-shell structure in the doubly complementary media setting (Theorem 1.1). This fact is interesting in itself and new to our knowledge. Cloaking arbitrary objects via ALR is considered in [24].

1.2 Statement of the main results

Let $k > 0$, let A be a (real) uniformly elliptic symmetric matrix defined in $\mathbb{R}^d$ ($d \geq 2$), and let Σ be a real function defined in $\mathbb{R}^d$ such that it is bounded below and above by positive constants. Assume that

$$A(x) = I, \quad \Sigma(x) = 1 \text{ for large } |x|, \tag{1.1}$$

and [4]

$$A \text{ is piecewise } C^1. \tag{1.2}$$

Let $\Omega_1 \subset\subset \Omega_2 \subset\subset \mathbb{R}^d$ be smooth bounded simply connected open subsets of $\mathbb{R}^d$, and set, for $\delta \geq 0$,

$$s_\delta(x) = \begin{cases} -1 - i\delta & \text{in } \Omega_2 \setminus \Omega_1, \\ 1 & \text{in } \mathbb{R}^d \setminus (\Omega_2 \setminus \Omega_1). \end{cases} \tag{1.3}$$

For $f \in L^2_{\rm c}(\mathbb{R}^d)$ with $\operatorname{supp} f \cap \Omega_2 = \emptyset$ and $\delta > 0$, let $u_\delta \in H^1_{loc}(\mathbb{R}^d)$ be the unique outgoing solution to

$$\operatorname{div}(s_\delta A \nabla u_\delta) + k^2 s_0 \Sigma u_\delta = f \text{ in } \mathbb{R}^d. \tag{1.4}$$

Here and in what follows

$$L^2_{\rm c}(\mathbb{R}^d) := \big\{ f \in L^2(\mathbb{R}^d) \text{ with compact support} \big\}.$$

[4]This assumption is used for various uniqueness statements obtained by the unique continuation principle.

For $R > 0$ and $x \in \mathbb{R}^d$, we will denote $B(x,R)$ the open ball in $\mathbb{R}^d$ centered at x and of radius R; when $x = 0$, we simply denote $B(x,R)$ by B_R. Recall that a function $u \in H^1_{loc}(\mathbb{R}^d \setminus B_R)$ for some $R > 0$ which is a solution to the equation $\Delta u + k^2 u = 0$ in $\mathbb{R}^d \setminus B_R$ is said to satisfy the outgoing condition if

$$\partial_r u - iku = o(r^{\frac{1-d}{2}}) \text{ as } r = |x| \to +\infty.$$

The power $E_\delta(u_\delta)$, or more precisely the power dissipated in the medium, is defined by (see, e.g., [15])

$$E_\delta(u_\delta) = \delta \int_{\Omega_2 \setminus \Omega_1} |\nabla u_\delta|^2. \tag{1.5}$$

The normalization of u_δ is $v_\delta = c_\delta u_\delta$ which is the unique outgoing solution in $H^1_{loc}(\mathbb{R}^d)$ of

$$\operatorname{div}(s_\delta A \nabla v_\delta) + k^2 s_0 \Sigma v_\delta = f_\delta \text{ in } \mathbb{R}^d, \tag{1.6}$$

where

$$f_\delta = c_\delta f,$$

and c_δ is the normalization constant such that

$$E_\delta(v_\delta) = \delta \int_{\Omega_2 \setminus \Omega_1} |\nabla v_\delta|^2 = 1. \tag{1.7}$$

In this paper, we establish properties P1), P2), and P3) for (A, Σ) of doubly complementary property. Before giving the definition of doubly complementary media for a general core-shell structure in the finite frequency regime, let us recall the definition of reflecting complementary media introduced in [17, Definition 1].

Definition 1.1 (Reflecting complementary media)**.** Let $\Omega_1 \subset\subset \Omega_2 \subset\subset \Omega_3 \subset\subset \mathbb{R}^d$ be smooth bounded simply connected open subsets of $\mathbb{R}^d$. The media (A, Σ) in $\Omega_3 \setminus \Omega_2$ and $(-A, -\Sigma)$ in $\Omega_2 \setminus \Omega_1$ are said to be *reflecting complementary* if there exists a diffeomorphism $F : \Omega_2 \setminus \bar{\Omega}_1 \to \Omega_3 \setminus \bar{\Omega}_2$ such that $F \in C^1(\bar{\Omega}_2 \setminus \Omega_1)$,

$$(F_* A, F_* \Sigma) = (A, \Sigma) \text{ for } x \in \Omega_3 \setminus \Omega_2, \tag{1.8}$$

$$F(x) = x \text{ on } \partial \Omega_2, \tag{1.9}$$

and the following two conditions hold: 1) There exists an diffeomorphism extension of F, which is still denoted by F, from $\Omega_2 \setminus \{x_1\} \to \mathbb{R}^d \setminus \bar{\Omega}_2$ for some $x_1 \in \Omega_1$; 2) There exists a diffeomorphism $G : \mathbb{R}^d \setminus \bar{\Omega}_3 \to \Omega_3 \setminus \{x_1\}$ such that $G \in C^1(\mathbb{R}^d \setminus \Omega_3)$, $G(x) = x$ on $\partial \Omega_3$, and $G \circ F : \Omega_1 \to \Omega_3$ is a diffeomorphism if one sets $G \circ F(x_1) = x_1$.

Here and in what follows, if $\mathcal{T}$ is a diffeomorphism, a and σ are a matrix-valued function and a complex function, we use the following standard notations

$$\mathcal{T}_* a(y) = \frac{D\mathcal{T}(x) a(x) \nabla \mathcal{T}(x)^T}{|\det \nabla \mathcal{T}(x)|} \quad \text{ and } \quad \mathcal{T}_* \sigma(y) = \frac{\sigma(x)}{|\det \nabla \mathcal{T}(x)|} \quad \text{ where } x = \mathcal{T}^{-1}(y). \tag{1.10}$$

Conditions (1.8) and (1.9) are the main assumptions in Definition 1.1. The key point behind this requirement is roughly speaking the following property: if $u_0 \in H^1(\Omega_3 \setminus \Omega_1)$ is a solution of $\operatorname{div}(s_0 A \nabla u_0) + k^2 s_0 \Sigma u_0 = 0$ in $\Omega_3 \setminus \Omega_1$ and if u_1 is defined in $\Omega_3 \setminus \Omega_2$ by $u_1 = u_0 \circ F^{-1}$, then $\operatorname{div}(A \nabla u_1) + k^2 \Sigma u_1 = 0$ in $\Omega_3 \setminus \Omega_2$, $u_1 - u_0 = A\nabla(u_1 - u_0) \cdot \nu = 0$ on $\partial \Omega_2$ by Lemma 2.2, a change of variables formula. Here and in what follows, ν denotes the outward unit vector on the boundary of a smooth bounded open subset of $\mathbb{R}^d$. Hence $u_1 = u$ in $\Omega_3 \setminus \Omega_2$ by the unique continuation principle, see, e.g., [28]. Conditions 1) and 2) are mild assumptions. Introducing G makes the analysis more accessible, see [17, 18, 21, 26] and the analysis presented in this paper.

Remark 1.1. Let $d = 2$, $A = I$, $0 < r_1 < r_2 < +\infty$ and set $r_3 = r_2^2/r_1$. Letting F be the Kelvin transform with respect to ∂B_{r_2}, i.e., $F(x) = r_2^2 x/|x|^2$ and $\Omega_i = B_{r_i}$, one can verify that in the quasistatic regime the core-shell structures considered by Milton and Nicorovici in [15] and by Kohn et al. in [12] have the reflecting complementary property.

Remark 1.2. The class of reflecting complementary media has played an important role in other applications of NIMs such as cloaking and superlensing using complementary media see [18, 21, 26].

We are ready to introduce the concept of doubly complementary media for the finite frequency regime.

Definition 1.2. The medium $(s_0 A, s_0 \Sigma)$ is said to be *doubly complementary* if for some $\Omega_2 \subset\subset \Omega_3$, (A, Σ) in $\Omega_3 \setminus \Omega_2$ and $(-A, -\Sigma)$ in $\Omega_2 \setminus \Omega_1$ are reflecting complementary, and

$$F_* A = G_* F_* A = A \quad \text{ and } \quad F_* \Sigma = G_* F_* \Sigma = \Sigma \text{ in } \Omega_3 \setminus \Omega_2, \tag{1.11}$$

for some F and G coming from Definition 1.1 (see Figure 1).

The reason for which media satisfying (1.11) are called doubly complementary media is that $(-A, -\Sigma)$ in $\Omega_2 \setminus \Omega_1$ is not only complementary to (A, Σ) in $\Omega_3 \setminus \Omega_2$ but also to (A, Σ) in $(G \circ F)^{-1}(\Omega_3 \setminus \overline{\Omega_2})$ (a subset of Ω_1) (see [19]). The key property behind Definition 1.2 is as follows. Assume that $u_0 \in H^1_{loc}(\mathbb{R}^d)$ is a solution of (1.4) with $\delta = 0$ and $f = 0$ in Ω_2. Set $u_1 = u_0 \circ F^{-1}$ and $u_2 = u_1 \circ G^{-1}$. Then u_1, u_2 satisfy the equation $\operatorname{div}(A\nabla \cdot) + k^2 \Sigma \cdot = 0$ in $\Omega_3 \setminus \Omega_2$, $u_0 - u_1 = A\nabla u_0 \cdot \nu - A \nabla u_1 \cdot \nu = 0$ on $\partial\Omega_2$, and $u_1 - u_2 = A \nabla u_1 \cdot \nu - A \nabla u_2 \cdot \nu = 0$ on $\partial \Omega_3$ by Lemma 2.2 (two Cauchy's problems appear, one for (u_0, u_1) and one for (u_1, u_2)). This implies relations between u_0, u_1, and u_2.

Remark 1.3. Taking $d = 2$, $A = I$ and $r_3 = r_2^2/r_1$, and letting F and G be the Kelvin transform with respect to ∂B_{r_2} and ∂B_{r_3}, one can verify that the core-shell structures considered by Milton and Nicorovici in [15] have the doubly complementary property. It is worth noting that one requires no information of A outside B_{r_3} and inside $B_{r_1^2/r_2}$ in the definition of doubly complementary media. More examples on doubly complementary media with quite simple formulas are given in Section 2.3.

Remark 1.4. Given (A, Σ) in $\mathbb{R}^d$ and $\Omega_1 \subset \Omega_2 \subset\subset \mathbb{R}^d$, it is not easy in general to verify whether or not $(s_0 A, s_0 \Sigma)$ is doubly complementary. Nevertheless, given $\Omega_1 \subset \Omega_2 \subset\subset \Omega_3 \subset\subset$

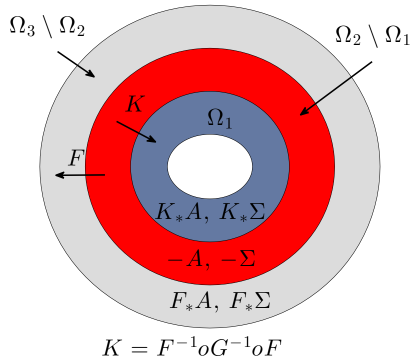

Figure 1: $(s_0A, s_0\Sigma)$ is doubly complementary: $(-A, -\Sigma)$ in $\Omega_2 \setminus \Omega_1$ (the red region) is complementary to $(F_*A, F_*\Sigma)$ in $\Omega_3 \setminus \Omega_2$ (the grey region) and $(K_*A, K_*\Sigma)$ with $K = F^{-1} \circ G^{-1} \circ F$ in $K(B_{r_2} \setminus B_{r_1})$ (the blue grey region).

$\mathbb{R}^d$ and (A, Σ) in $\Omega_3 \setminus \Omega_2$, it is quite easy to choose (A, Σ) in Ω_2 such that $(s_0A, s_0\Sigma)$ is doubly complementary. One just needs to choose diffeomorphisms F and G as in Definition 1.1 and define $(A, \Sigma) = (F_*^{-1}A, F_*^{-1}\Sigma)$ in $\Omega_2 \setminus \Omega_1$ and $(A, \Sigma) = (F_*^{-1}G_*^{-1}A, F^{-1} * G_*^{-1}\Sigma)$ in $F^{-1} \circ G^{-1}(\Omega_3 \setminus \Omega_2)$. This idea is used in Section 5 when we discuss cloaking sources in an arbitrary medium.

The first result of this paper is the following theorem which reveals the behavior of u_δ for a general source f with support outside Ω_2.

Theorem 1.1. *Let $d \geq 2$, $k > 0$, $0 < \delta < 1$, $f \in L^2_c(\mathbb{R}^d)$ with $\operatorname{supp} f \cap \Omega_2 = \emptyset$, and let $u_\delta \in H^1_{loc}(\mathbb{R}^d)$ be the unique outgoing solution of (1.4). Assume that $(s_0A, s_0\Sigma)$ is doubly complementary. Then*

$$u_\delta \to \hat{u} \text{ weakly in } H^1_{loc}(\mathbb{R}^d \setminus \Omega_3), \tag{1.12}$$

where $\hat{u} \in H^1_{loc}(\mathbb{R}^d)$ is the unique outgoing solution of

$$\operatorname{div}(\hat{A}\nabla\hat{u}) + k^2\hat{\Sigma}\hat{u} = f \text{ in } \mathbb{R}^d. \tag{1.13}$$

Here

$$(\hat{A}, \hat{\Sigma}) := \begin{cases} (A, \Sigma) & \text{in } \mathbb{R}^d \setminus \Omega_3, \\ (G_*F_*A, G_*F_*\Sigma) & \text{in } \Omega_3. \end{cases} \tag{1.14}$$

Using Theorem 1.1, one can establish the equivalence between the blow up of the power and the cloaking a source via ALR as follows. Suppose that the power blows up, i.e.,

$$\lim_{n\to\infty} \delta_n \|\nabla u_{\delta_n}\|^2_{L^2(\Omega_2\setminus\Omega_1)} = +\infty.$$

Then, by Theorem 1.1, $v_{\delta_n} \to 0$ in $\mathbb{R}^d \setminus \Omega_2$ since $c_{\delta_n} \to 0$; the localized resonance takes place. The source $c_{\delta_n} f$ is not seen by observers far away from the shell: the source is cloaked. If the power $E_{\delta_n}(u_{\delta_n})$ remains bounded, then the source is not cloaked since $u_{\delta_n} \to \hat{u}$ weakly in $H^1_{loc}(\mathbb{R}^d \setminus B_{r_3})$ and $\hat{u} \in H^1_{loc}(\mathbb{R}^d)$ is the unique outgoing solution to (1.13).

Remark 1.5. It follows from (1.10) that if $(s_0 A, s_0 \Sigma)$ is a doubly complementary medium then (A, Σ) is not piecewise constant; hence the separation of variables method is out of reach for this setting in general.

In comparison with [20, Theorem 1.1], Theorem 1.1 in this paper is stronger: no conditions on the blow up rate of the power are required. The proof of Theorem 1.1 is in the spirit of [20]. Nevertheless, we add two important ingredients. The first one is on the blow up rate of the power of u_δ in (2.21) which is derived in this paper instead of being assumed previously. The second one is on the removing localized singularity technique. In this paper, we are able to construct in a simple and robust way the singular part of u_δ which is necessary to be removed. This helps us to avoid the ad-hoc separation of variables method for a general shell developed and implemented in [20]. The construction of the removing term comes from a remark of Etienne Sandier. The author would like to thank him for it. To our knowledge, Theorem 1.1 is new and is the first result providing the connection between the blow up of the power and the invisibility of a source in the finite frequency regime. A numerical simulation from [10] illustrating Theorem 1.1 is given in Section 2.3.

Concerning the blow up of the power, we can prove the following result which holds for a large class of media in which the reflecting complementary property holds only locally.

Proposition 1.1. *Let $d \geq 2$ and $k > 0$. Assume that there exists a diffeomorphism $F : \Omega_2 \setminus \Omega_1 \to \Omega_3 \setminus \Omega_2$ for some $\Omega_2 \subset\subset \Omega_3 \subset \mathbb{R}^d$ such that $F \in C^1(\bar{\Omega}_2 \setminus \Omega_1)$, $F(x) = x$ on $\partial \Omega_2$, and*

$$(A, \Sigma) = (F_* A, F_* \Sigma) \text{ in } D \text{ where } D := B(x_0, R_0) \cap (\Omega_3 \setminus \Omega_2)$$

for some $x_0 \in \partial \Omega_2$ and $R_0 > 0$. Let $f \in L^2_{\rm c}(\mathbb{R}^d)$ and assume that A is Lipschitz in $\overline{D}$. There exists $0 < r_0 < R_0$, independent of f, such that if
$\Big(D_1 := D \cap B(x_0, r_0)$ and there is no solution $v \in H^1(D_1)$ to the Cauchy problem

$$\operatorname{div}(A \nabla v) + k^2 \Sigma v = f \text{ in } D_1 \quad \text{and} \quad v = A \nabla v \cdot \nu = 0 \text{ on } \partial D_1 \setminus \partial B(x_0, r_0)\Big)$$

then

$$\limsup_{\delta \to 0} \delta \int_{\Omega_2 \setminus \Omega_1} |\nabla u_\delta|^2 = +\infty,$$

where $u_\delta \in H^1_{loc}(\mathbb{R}^d)$ is the unique outgoing solution of (1.4).

Property P2) is understood in the sense of Proposition 1.1. Some conditions on the source are necessarily imposed since for sources of the form $\operatorname{div}(A \nabla \varphi) + k^2 \Sigma \varphi$ with smooth φ and $\operatorname{supp} \varphi \subset \mathbb{R}^d \setminus \Omega_3$, the corresponding solution is φ, which is bounded, and the power remains finite and even goes to 0. Note that $(s_0 A, s_0 \Sigma)$ is not required to be doubly complementary in Proposition 1.1. Proposition 1.1 is inspired from [22, Lemma 10] which has its root from [20]. More quantitative conditions on the blow-up of the power are presented in Proposition 1.3

where $\Omega_2 \setminus \Omega_1 = B_{r_2} \setminus B_{r_1}$ and $(A, \Sigma) = (I, 1)$ in $\Omega_3 \setminus \Omega_2$, and the medium is doubly complementary.

Concerning the boundedness of the power, we have the following result, which implies Property P3).

Proposition 1.2. *Let $d \geq 2$, $k > 0$, $0 < \delta < 1$, and $f \in L^2_c(\mathbb{R}^d)$, and let $u_\delta \in H^1(\mathbb{R}^d)$ be the unique solution (1.4). Assume that $(s_0 A, s_0 \Sigma)$ is a doubly complementary medium and* $\operatorname{supp} f \cap \Omega_3 = \emptyset$. *We have, for $R > 0$,*

$$\|u_\delta\|_{H^1(B_R)} \leq C_R \|f\|_{L^2},$$

for some positive constant C_R independent of f and δ.

Proposition 1.2 is a consequence of [17, Corollary 2 and Theorem 1]. A more general version of Proposition 1.2 is given in Lemma 4.1 in Section 4.1. The conclusion of Proposition 1.2 is somehow surprising and requires the doubly complementary property since in general $\|u_\delta\|_{H^1(B_R)}$ can be blown up with the order $1/\delta$ for some $R > 0$ (see [25, Theorem 2]). The blow up rate $1/\delta$ is the worst case possible (see Lemma 2.1).

In the case $\Omega_j = B_{r_j}$ for $j = 2, 3$, $(A, \Sigma) = (I, 1)$ in $\Omega_3 \setminus \Omega_2$ and $d = 2, 3$, more quantitative estimates on the blow up and the boundedness of the power are given in the following

Proposition 1.3. *Let $d = 2, 3$, $k > 0$, and $f \in L^2_c(\mathbb{R}^d)$, and let $u_\delta \in H^1(\mathbb{R}^d)$ be the unique solution of (1.4). Assume that $(s_0 A, s_0 \Sigma)$ is a doubly complementary medium, $\Omega_2 = B_{r_2}$ and $\Omega_3 = B_{r_3}$ for some $0 < r_2 < r_3$, and $(A, \Sigma) = (I, 1)$ in $B_{r_3} \setminus B_{r_2}$. We have*

1. *If there exists $w \in H^1(B_{r_0} \setminus B_{r_2})$ for some $r_0 > \sqrt{r_2 r_3}$ with the properties*

$$\Delta w + k^2 w = f \text{ in } B_{r_0} \setminus B_{r_2} \quad \text{and} \quad w = \partial_r w = 0 \text{ on } \partial B_{r_2},$$

then

$$\limsup_{\delta \to 0} \delta \|u_\delta\|^2_{H^1(B_{r_3})} < +\infty.$$

2. *If there does* **not** *exist $v \in H^1(B_{r_0} \setminus B_{r_2})$ for some $r_0 < \sqrt{r_2 r_3}$ with the properties*

$$\Delta v + k^2 v = f \text{ in } B_{r_0} \setminus B_{r_2} \quad \text{and} \quad v = \partial_r v = 0 \text{ on } \partial B_{r_2},$$

then

$$\liminf_{\delta \to 0} \delta \|\nabla u_\delta\|^2_{L^2(B_{r_3} \setminus B_{r_2})} = +\infty.$$

This proposition is in the spirit of [20, Theorems 1.2 and 1.3] (inspired by [5]). One only assumes that $(A, \Sigma) = (I, 1)$ in $B_{r_3} \setminus B_{r_2}$ and $(s_0 A, s_0 \Sigma)$ is doubly complementary. (A, Σ) can be arbitrary outside of B_{r_3}: the separation of variables method is out of reach here. The proof of the first statement of Proposition 1.3 is based on a kind of removing singularity technique and has roots from [20]. A key point is the construction of the auxiliary function W_δ in (4.17). The proof of the second statement is based on an observation on a Cauchy problem in [20] and involves a three spheres inequality.

As a consequence of Proposition 1.3 and Theorem 1.1, one obtains new (non-trivial) variants and generalizations of the result of Milton and Nicorovici in the finite frequency regime in both two and three dimensions; note that (A, Σ) can be arbitrary outside B_{r_3}.

We finally point out that the stability of the Helmholtz equation with sign changing coefficients was studied by the integral method, the pseudo differential operator theory, and the T-coercivity approach in [1, 9, 11] and references therein, and was recently unified and extended in [22] via the use of the reflecting technique and the study of Cauchy's problems. It was also shown in [22] that the complementary property is necessary for the appearance of resonance.

The paper is organized as follows. The proof of Theorem 1.1 is given in Section 2. In this section, we also provide various examples of doubly complementary media with quite simple formulas and numerical simulations illustrating Theorem 1.1 (section 2.3). Sections 3 and 4 are devoted to the proofs of Propositions 1.1 and 1.3 respectively. Finally, in Section 5, we present schemes of cloaking to cloak a general class of sources via ALR in an arbitrary medium for the finite frequency regime.

2 Proof of Theorem 1.1

This section containing three subsections is organized as follows. In the first subsection, we present a lemma on the stability of (1.4) and recall a change of variables formula from [17] which is used repeatedly in this paper. The proof of Theorem 1.1 is given in the second subsection. In the last subsection, we present various examples of doubly complementary media with quite simple formulas and present a simulation illustrating Theorem 1.1.

2.1 Preliminaries

The main result of this section is the following lemma, which implies the stability of (1.4) and is used repeatedly in this paper.

Lemma 2.1. *Let $d \geq 2$, $k > 0$, $\delta_0 > 0$, $R_0 > 0$, $g \in H^{-1}(\mathbb{R}^d)$ [5] with support in B_{R_0}. For $0 < \delta < \delta_0$, there exists a unique outgoing solution $v_\delta \in H^1_{loc}(\mathbb{R}^d)$ to the equation*

$$\operatorname{div}(s_\delta A \nabla v_\delta) + k^2 s_0 \Sigma v_\delta = g \text{ in } \mathbb{R}^d. \tag{2.1}$$

Moreover,

$$\|v_\delta\|^2_{H^1(B_R)} \leq \frac{C_R}{\delta} \Big| \int g \bar{v}_\delta \Big| + C_R \|g\|^2_{H^{-1}}, \tag{2.2}$$

for some positive constant C_R independent of g and δ.

Proof. We only establish (2.2). The uniqueness of v_δ follows from (2.2). The existence of v_δ can be derived from the uniqueness of v_δ by using the limiting absorption principle, see, e.g., [22]. Without loss of generality, one may assume that (1.1) holds for $|x| \geq R_0$ and

[5] $H^{-1}(\mathbb{R}^d)$ denotes the dual space of $H^1(\mathbb{R}^d)$.

$\Omega_2 \subset\subset B_{R_0}$. We begin with establishing (2.2) with $R = R_0$ by contradiction. Assume that (2.2) with $R = R_0$ is not true. Then there exists $(g_\delta) \subset H^{-1}(\mathbb{R}^d)$ such that $\operatorname{supp} g_\delta \subset\subset B_{R_0}$,

$$\|v_\delta\|_{H^1(B_{R_0})} = 1 \text{ and } \frac{1}{\delta}\Big|\int g_\delta \bar{v}_\delta\Big| + \|g_\delta\|_{H^{-1}}^2 \to 0, \tag{2.3}$$

as $\delta \to \hat{\delta} \in [0, \delta_0]$, where $v_\delta \in H^1_{loc}(\mathbb{R}^d)$ is the unique solution to the equation

$$\operatorname{div}(s_\delta A \nabla v_\delta) + k^2 s_0 \Sigma v_\delta = g_\delta \text{ in } \mathbb{R}^d. \tag{2.4}$$

In fact, by contradiction these properties only hold for a sequence $(\delta_n) \to \hat{\delta}$. However, for simplicity of notation, we still use δ instead of δ_n to denote an element of such a sequence. We only consider the case $\hat{\delta} = 0$; the case $\hat{\delta} > 0$ follows similarly. Since (see e.g., [16, Lemma 2.3]), for $R > R_0$,

$$\|v_\delta\|_{H^1(B_R \setminus B_{R_0})} \le C_R \|v_\delta\|_{H^{1/2}(\partial B_{R_0})}, \tag{2.5}$$

for some positive constant C_R independent of δ and g_δ, and $\Delta v_\delta + k^2 v_\delta = 0$ in $\mathbb{R}^d \setminus B_{R_0}$, without loss of generality, one may assume that (v_δ) converges to v_0 strongly in $L^2_{loc}(\mathbb{R}^d)$, weakly in $H^1_{loc}(\mathbb{R}^d)$, and strongly in $H^2(B_{R_0+2} \setminus B_{R_0})$ for some $v_0 \in H^1_{loc}(\mathbb{R}^d)$. Then, by (2.3), we obtain

$$\operatorname{div}(s_0 A \nabla v_0) + k^2 s_0 \Sigma v_0 = 0 \text{ in } \mathbb{R}^d. \tag{2.6}$$

Since v_δ satisfies the outgoing condition, it follows that v_0 also satisfies the outgoing condition. Multiplying (2.4) by $\bar{v}_\delta$ and integrating on B_R with $R \ge R_0$, we have

$$\int_{B_R} s_\delta \langle A \nabla v_\delta, \nabla v_\delta \rangle \, dx - \int_{B_R} k^2 s_0 \Sigma |v_\delta|^2 \, dx = -\int_{B_R} g_\delta \bar{v}_\delta \, dx + \int_{\partial B_R} \partial_r v_\delta \bar{v}_\delta. \tag{2.7}$$

Letting $\delta \to 0$, by (2.3), we obtain, for $R \ge R_0$,

$$\Im\Big(\int_{\partial B_R} \partial_r v_0 \bar{v}_0\Big) = 0. \tag{2.8}$$

Since v_0 satisfies the outgoing condition, it follows from Rellich's lemma that $v_0 = 0$ in $\mathbb{R}^d \setminus B_{R_0}$ [6]. Using (2.6) and the fact that $v_0 \in H^1_{loc}(\mathbb{R}^d)$, we derive from the unique continuation principle that

$$v_0 = 0 \text{ in } \mathbb{R}^d. \tag{2.9}$$

Letting $R \to \infty$, considering the imaginary part in (2.7), and using (2.3), we obtain

$$\|\nabla v_\delta\|_{L^2(\Omega_2 \setminus \Omega_1)} \to 0 \text{ as } \delta \to 0. \tag{2.10}$$

Since $v_\delta \to v_0 = 0$ strongly in $H^2(B_{R_0+2} \setminus B_{R_0})$, it follows that

$$\lim_{\delta \to 0} \int_{\partial B_{R_0+1}} \partial_r v_\delta \bar{v}_\delta = 0.$$

[6] In the case $\hat{\delta} > 0$, instead of (2.8), we obtain $\Im\Big(\int_{\partial B_R} \partial_r v_{\hat{\delta}} \bar{v}_{\hat{\delta}}\Big) \le 0$. This also implies that $v_{\hat{\delta}} = 0$ by Rellich's lemma. The rest of the proof works well for the case $\hat{\delta} > 0$.

Considering the real part of (2.7) with $R = R_0 + 1$, we derive from (2.10) that

$$\|v_\delta\|_{H^1(B_{R_0+1})} \to 0 \text{ as } \delta \to 0.$$

We have a contradiction by (2.3). Hence (2.2) holds for $R = R_0$.

The conclusion now follows from (2.5). $\square$

We end this subsection by stating a change of variables formula which is a consequence of [17, Lemma 2].

Lemma 2.2. *Let $D_1 \subset\subset D_2 \subset\subset D_3$ be three smooth bounded open subsets of $\mathbb{R}^d$. Let $a \in [L^\infty(D_2 \setminus D_1)]^{d\times d}$, $\sigma \in L^\infty(D_2 \setminus D_1)$ and let $\mathcal{T}$ be a diffeomorphism from $D_2 \setminus \bar{D}_1$ onto $D_3 \setminus \bar{D}_2$. Assume that $u \in H^1(D_2 \setminus D_1)$ and set $v = u \circ \mathcal{T}^{-1}$. Then*

$$\operatorname{div}(a\nabla u) + \sigma u = f \textit{ in } D_2 \setminus D_1,$$

for some $f \in L^2(D_2 \setminus D_1)$, if and only if

$$\operatorname{div}(\mathcal{T}_* a\nabla v) + \mathcal{T}_*\sigma v = \mathcal{T}_* f \textit{ in } D_3 \setminus D_2. \tag{2.11}$$

Assume in addition that $\mathcal{T}(x) = x$ on ∂D_2. Then

$$v = u \quad \textit{and} \quad \mathcal{T}_* a\nabla v \cdot \nu = -a\nabla u \cdot \nu \textit{ on } \partial D_2. \tag{2.12}$$

Recall that $\mathcal{T}_*a$, $\mathcal{T}_*\sigma$, and $\mathcal{T}_*f$ are given in (1.10). Here and in what follows, when we mention a diffeomorphism $\mathcal{T} : \Omega \to \Omega'$ for two smooth open subsets Ω, Ω' of $\mathbb{R}^d$, we mean that $\mathcal{T}$ is a diffeomorphism, $\mathcal{T} \in C^1(\bar{\Omega})$, and $\mathcal{T}^{-1} \in C^1(\bar{\Omega}')$.

2.2 Proof of Theorem 1.1

Define

$$u_{1,\delta} = u_\delta \circ F^{-1} \text{ in } \mathbb{R}^d \setminus \Omega_2$$

and

$$u_{2,\delta} = u_{1,\delta} \circ G^{-1} \text{ in } \Omega_3.$$

It follows from (1.11) and Lemma 2.2 that

$$\operatorname{div}(A\nabla u_{1,\delta}) + k^2\Sigma u_{1,\delta} + i\delta \operatorname{div}\big(A\nabla u_{1,\delta}\big) = \operatorname{div}(A\nabla u_{2,\delta}) + k^2\Sigma u_{2,\delta} = 0 \text{ in } \Omega_3 \setminus \Omega_2. \tag{2.13}$$

$$u_\delta - u_{1,\delta} = 0 \text{ on } \partial\Omega_2, \quad A\nabla u_\delta\big|_{\Omega_3\setminus\Omega_2} \cdot \nu - (1 + i\delta)A\nabla u_{1,\delta} \cdot \nu = 0 \text{ on } \partial\Omega_2, \tag{2.14}$$

$$u_{1,\delta} - u_{2,\delta} = 0 \text{ on } \partial\Omega_3, \quad \text{and} \quad (1 + i\delta)A\nabla u_{1,\delta}\big|_{\Omega_3\setminus\Omega_2} \cdot \nu - A\nabla u_{2,\delta} \cdot \nu = 0 \text{ on } \partial\Omega_3. \tag{2.15}$$

Set

$$\hat{u}_\delta = \begin{cases} u_\delta & \text{in } \mathbb{R}^d \setminus \Omega_3, \\ u_\delta - (u_{1,\delta} - u_{2,\delta}) & \text{in } \Omega_3 \setminus \Omega_2, \\ u_{2,\delta} & \text{in } \Omega_2. \end{cases} \tag{2.16}$$

It follows from (2.13), (2.14), and (2.15) that $\hat{u}_\delta \in H^1_{loc}(\mathbb{R}^d)$ is the unique outgoing solution of

$$\begin{cases} \operatorname{div}(\hat{A}\nabla \hat{u}_\delta) + k^2 \hat{\Sigma} \hat{u}_\delta = f & \text{in } \mathbb{R}^d \setminus (\partial\Omega_2 \cup \partial\Omega_3), \\ \hat{A}\nabla \hat{u}_\delta \big|_{\mathbb{R}^d\setminus\Omega_3} \cdot \nu - \hat{A}\nabla \hat{u}_\delta \big|_{\Omega_3} \cdot \nu = -i\delta A \nabla u_{1,\delta}\big|_{\Omega_3\setminus\Omega_2} \cdot \nu & \text{on } \partial\Omega_3, \\ \hat{A}\nabla \hat{u}_\delta \big|_{\Omega_3\setminus\Omega_2} \cdot \nu - \hat{A}\nabla \hat{u}_\delta \big|_{\Omega_2} \cdot \nu = i\delta A \nabla u_{1,\delta}\big|_{\Omega_3\setminus\Omega_2} \cdot \nu & \text{on } \partial\Omega_2. \end{cases} \tag{2.17}$$

Here we used the fact that $(\hat{A}, \hat{\Sigma}) = (A, \Sigma)$ in $\Omega_3 \setminus \Omega_2$. By Lemma 2.1, we have, for $R > 0$,

$$\|u_\delta\|_{H^1(B_R)} \le C_R \delta^{-1} \|f\|_{L^2}. \tag{2.18}$$

It follows from (2.17) and Lemma 2.1 again that, for $R > 0$,

$$\|\hat{u}_\delta\|_{H^1(B_R)} \le C_R \|f\|_{L^2}.$$

As a consequence, we have, for $R > 0$,

$$\|u_\delta\|_{H^1(B_R\setminus\Omega_3)} \le C_R \|f\|_{L^2}. \tag{2.19}$$

First fix $R > 0$ such that $\Omega_3 \subset\subset B_R$ and then fix $x_0 \in B_R \setminus \Omega_3$ and $r_0 > 0$ such that $B(x_0, r_0) \subset B_R \setminus \Omega_3$. We have, from (2.19),

$$\|u_\delta\|_{L^2(B(x_0,r_0))} \le C_R \|f\|_{L^2}. \tag{2.20}$$

Using (2.18), (2.20), and the fact that $\operatorname{div}(A\nabla u_\delta) + k^2 \Sigma u_\delta = f$ in $B_R \setminus \Omega_2$, one is able to derive from a three spheres inequality that $\|u_\delta\|_{L^2(B_R\setminus\Omega_2)}$ is much smaller than $\delta^{-1}\|f\|_{L^2}$, which is the order of an upper bound of $\|u_\delta\|_{H^1(B_R\setminus\Omega_2)}$. Indeed, applying [2, Theorem 5.3] to u_δ in $B_R \setminus \Omega_2$ with $\varepsilon = C\|f\|_{L^2(B_R\setminus\Omega_2)}$ for some positive constant C large enough so that [2, (1.29)] holds with $F = 0$ (the largeness of C depends only on R and Ω_2), we obtain from (2.18) and (2.20) that, for $0 < \delta < 1/2$,

$$\|u_\delta\|_{L^2(B_R\setminus\Omega_2)} \le C\delta^{-1}\|f\|_{L^2} / \ln^\mu(1/\delta),$$

for some positive constants C and μ, independent of f and δ (recall that R and r_0 are fixed); which implies

$$\lim_{\delta\to 0} \delta \|u_\delta\|_{L^2(B_R\setminus\Omega_2)} = 0. \tag{2.21}$$

Using (2.2) in Lemma 2.1, we get from (2.21) that

$$\lim_{\delta\to 0} \delta \|u_\delta\|_{H^1(B_R)} = 0.$$

From (2.17) we have, for $R > 0$,

$$\|\hat{u}_\delta - \hat{u}\|_{H^1(B_R)} \to 0 \text{ as } \delta \to 0.$$

The proof is complete. □

Remark 2.1. One of the key points in the proof is the definition of $\hat{u}_\delta$ in (2.16) after introducing $u_{1,\delta}$ and $u_{2,\delta}$ as in [20]. In $\Omega_3 \setminus \Omega_2$, we remove $u_{1,\delta} - u_{2,\delta}$ from u_δ. The removing term is the singular part of u_δ in $\Omega_3 \setminus \Omega_2$. The way of defining the removing term is intrinsic and more robust than the one in [20], which is based on the separation of variables for a general shell developed there. As seen from there, the removing term becomes more and more singular when one approaches $\partial\Omega_2$. The idea of removing the singular term was inspired by the study of the Ginzburg-Landau equation in the work of Bethuel, Brezis, and Helein in [6]. Another new important point in the proof is to establish (2.21). This is obtained by first proving that u_δ is bounded outside Ω_3 (this is again based on the behaviour of $\hat{u}_\delta$) and then applying a three spheres inequality.

2.3 Some examples of doubly complementary media and a numerical simulation

We first present some examples of doubly complementary media with quite simple formulas. Let $0 < r_1 < r_2$ and $\alpha, \beta > 1$ be such that $\alpha\beta - \alpha - \beta = 0$. Set $r_3 = r_2^\alpha / r_1^{\alpha-1}$, $r_0 = r_1^\alpha / r_2^{\alpha-1}$, and $m = r_3/r_1 = (r_2/r_1)^\alpha$, and define $\Omega_j = B_{r_j}$ for $j = 1, 2, 3$. Assume that

$$A, \Sigma = \begin{cases} I, 1 & \text{in } B_{r_3} \setminus B_{r_2}, \\ A_1, \Sigma_1 & \text{in } B_{r_2} \setminus B_{r_1}, \\ m^{d-2} I, m^d & \text{in } B_{r_1} \setminus B_{r_0}, \end{cases} \tag{2.22}$$

where

$$A_1,\ \Sigma_1 = \frac{r_2^\alpha}{r^\alpha} \left[\frac{1}{\alpha - 1} e_r \otimes e_r + (\alpha - 1)\Big(e_\theta \otimes e_\theta + e_\theta \otimes e_\varphi \Big) \right],\ (\alpha - 1)\frac{r_2^{3\alpha}}{r^{3\alpha}} \text{ if } d = 3, \tag{2.23}$$

and

$$A_1,\ \Sigma_1 = \frac{1}{\alpha - 1} e_r \otimes e_r + (\alpha - 1) e_\theta \otimes e_\theta,\ (\alpha - 1)\frac{r_2^{2\alpha}}{r^{2\alpha}} \text{ if } d = 2. \tag{2.24}$$

Here the spherical and the polar coordinates are used. Considering $F(x) = r_2^\alpha x/|x|^\alpha$ and $G(x) = r_3^\beta x/|x|^\beta$ and noting that $G \circ F = mI$, one can verify that $(s_0 A, s_0 \Sigma)$ is doubly complementary [18] (see also [23] for the details), $(\hat{A}, \hat{\Sigma}) = (I, 1)$ in $B_{r_3} \setminus B_{r_2}$.

We next present simulations illustrating Theorem 1.1. These are taken from the joint work with Droxler and Hesthaven in [10], where we present various simulations illustrating cloaking and superlensing properties of NIMs in the two and three dimensional acoustic settings. The author thanks them for letting him present some simulations here. Consider the two dimensional finite frequency regime with $k = 1$. Set

$$r_0 = 1/2, \quad r_1 = \sqrt{2}/2, \quad r_2 = 1, \quad r_3 = \sqrt{2}, \quad \text{and} \quad r_4 = \sqrt{2} + 1,$$

and define $\Omega_j = B_{r_j}$ for $j = 1, 2, 3$. Consider

$$A, \Sigma = \begin{cases} I, 1 & \text{in } B_{r_4} \setminus B_{r_2}, \\ I, r_2^4/|x|^4 & \text{in } B_{r_2} \setminus B_{r_1}, \\ I, 4 & \text{in } B_{r_1} \setminus B_{r_0}, \\ 2I, 2 & \text{in } B_{r_0}. \end{cases}$$

Then $(s_0 A, s_0 \Sigma)$ is doubly complementary by taking $\alpha = \beta = 2$. Since $G \circ F = 2I$, one can verify that

$$\hat{A}, \hat{\Sigma} = \begin{cases} I, 1 & \text{in } B_{r_4} \setminus B_{r_2}, \\ 2I, 1/2 & \text{in } B_{r_2}. \end{cases}$$

In Figure 2, we present a simulation of $\hat{u}$, the unique solution in $H^1_0(B_{r_4})$ of the equation $\operatorname{div}(\hat{A}\nabla \hat{u}) + k^2 \hat{\Sigma} \hat{u} = f$ in B_{r_4} where $f = 5$ in D and 0 otherwise and D is the small (pink) region visible on the figure. The real part of u_δ, the unique solution in $H^1_0(B_{r_4})$ to the equation $\operatorname{div}(s_\delta A \nabla u_\delta) + k^2 s_\delta \Sigma u_\delta = f$ in B_{r_4} with $\delta = 5 * 10^{-5}$, is given in Figure 3. One

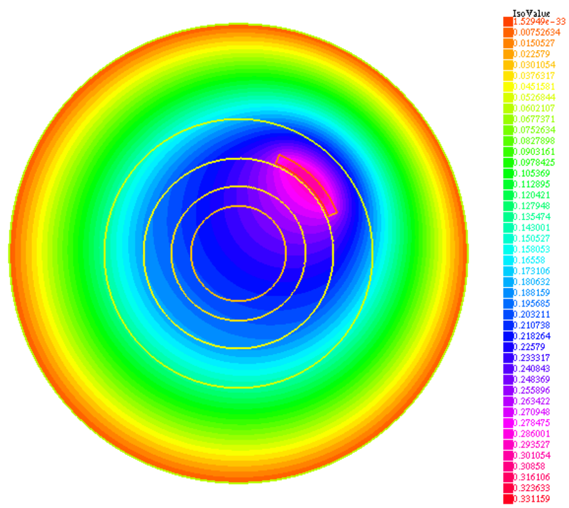

Figure 2: Simulation of $\hat{u}$.

easily sees from these simulations that the real part of u_δ and $\hat{u}$ are close outside B_{r_3}. This is consistent with the prediction given in Theorem 1.1. Note that u_δ blows up in $B_{r_3} \setminus B_{r_1}$; hence the removing term $u_{1,\delta} - u_{2,\delta}$ is necessary for the boundedness of $\hat{u}_\delta$. [7].

3 Proof of Proposition 1.1

We prove Proposition 1.1 by contradiction. Assume that

$$\limsup_{\delta \to 0} \delta \|\nabla u_\delta\|^2_{L^2(\Omega_2 \setminus \Omega_1)} < +\infty. \tag{3.1}$$

Since $\operatorname{div}(A \nabla u_\delta) + k^2 s_0 s_\delta^{-1} \Sigma u_\delta = 0$ in $\Omega_2 \setminus \Omega_1$ and Σ is bounded above and below by positive constants, it follows from a compactness argument that

$$\|u_\delta\|_{L^2(\Omega_2 \setminus \Omega_1)} \le C \|\nabla u_\delta\|_{L^2(\Omega_2 \setminus \Omega_1)}.$$

[7] The simulations are done for a bounded domain but this point is not essential.

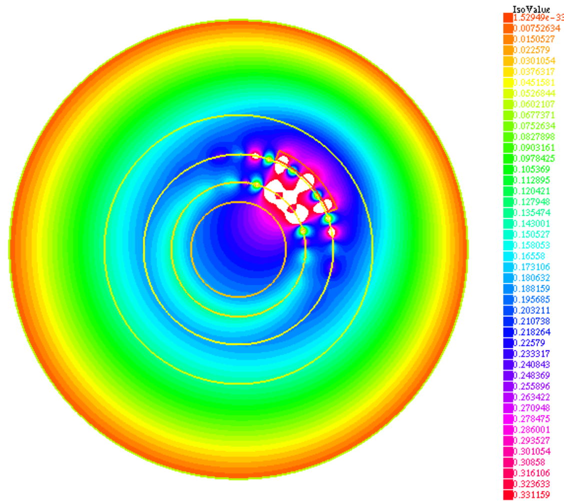

Figure 3: Simulation of u_δ with $\delta = 5 \times 10^{-5}$.

We derive from (3.1) that

$$\limsup_{\delta \to 0} \delta \|u_\delta\|^2_{H^1(\Omega_2 \setminus \Omega_1)} < +\infty. \tag{3.2}$$

Since, for $R > 0$,

$$\|u_\delta\|_{H^1(B_R)} \le C \|u_\delta\|_{H^1(\Omega_2 \setminus \Omega_1)}$$

see, e.g., [22, Lemmas 1 and 3], it follows that, for $R > 0$,

$$\limsup_{\delta \to 0} \delta \|u_\delta\|^2_{H^1(B_R)} < +\infty.$$

Define

$$u_{1,\delta} = u_\delta \circ F^{-1} \text{ in } \Omega_3 \setminus \Omega_2$$

and set

$$v_\delta = u_{1,\delta} - u_\delta \text{ in } D.$$

We then obtain

$$\operatorname{div}(A \nabla v_\delta) + k^2 \Sigma v_\delta = g_\delta \text{ in } D, \tag{3.3}$$

$$v_\delta = 0 \text{ on } D \cap \partial\Omega_2 \quad \text{ and } \quad A \nabla v_\delta \cdot \nu = h_\delta \text{ on } \partial D \cap \partial \Omega_2. \tag{3.4}$$

Here

$$g_\delta = -i\delta \operatorname{div}\big(A \nabla u_{1,\delta}\big) - f = \frac{i\delta}{1 + i\delta} k^2 \Sigma u_{1,\delta} - f \text{ in } D$$

and

$$h_\delta = i\delta \nabla u_{1,\delta} \cdot \nu \text{ on } \partial D \cap \partial \Omega_2.$$

It is clear from (3.2) that

$$\|g_\delta + f\|_{L^2(D)} + \|h_\delta\|_{H^{-1/2}(\partial D \cap \partial \Omega_2)} \le C \delta^{1/2}, \tag{3.5}$$

for some positive constant C independent of δ. Using (3.3), (3.4), and (3.5), and applying [22, Lemma 10], we have

$$\limsup_{\delta\to 0} \delta^{1/2}\|v_\delta\|_{H^1(D)} = +\infty :$$

which contradicts (3.2). The proof is complete. □

4 Proof of Proposition 1.3

This section containing two subsections is devoted to the proof of Proposition 1.3. In the first subsection, we present two lemmas used in the proof of Proposition 1.3 and the proof of Proposition 1.3 is given in the second subsection.

4.1 Two useful lemmas

In this subsection, we establish two lemmas which are used in the proof of Proposition 1.3. The first one is a more general version of Proposition 1.2.

Lemma 4.1. *Let $d \ge 2$, $k > 0$, $0 < \delta < 1$, $f \in L^2_c(\mathbb{R}^d)$, $g \in H^{1/2}(\partial\Omega_3)$, and $h \in H^{-1/2}(\partial\Omega_3)$. Assume that $(s_0 A, s_0\Sigma)$ is doubly complementary, $\operatorname{supp} f \cap \Omega_3 = \emptyset$, and $V_\delta \in \bigcap_{R>0} H^1(B_R \setminus \partial\Omega_3)$ is the unique outgoing solution of*

$$\begin{cases} \operatorname{div}(s_\delta A \nabla V_\delta) + k^2 s_0 \Sigma V_\delta = f \text{ in } \mathbb{R}^d \setminus \partial\Omega_3, \\ [V_\delta] = g \quad \text{and} \quad [A\nabla V_\delta \cdot \nu] = h \text{ on } \partial\Omega_3. \end{cases} \tag{4.1}$$

Then, for $R > 0$,

$$\|V_\delta\|_{H^1(B_R\setminus\partial\Omega_3)} \le C_R\big(\|f\|_{L^2(\Omega)} + \|g\|_{H^{1/2}(\partial\Omega_3)} + \|h\|_{H^{-1/2}(\partial\Omega_3)}\big),$$

for some positive constant C_R independent of δ, f, g, and h.

Here and in what follows in this paper, we denote $[v] = v\big|_{ext} - v\big|_{int}$ and $[M\nabla v \cdot \nu] = M\nabla v \cdot \nu\big|_{ext} - M\nabla v\cdot\nu\big|_{int}$ on $\partial\Omega$ for a smooth bounded open subset Ω of $\mathbb{R}^d$, for a matrix-valued function M, and for an appropriate function v.

Proof. The proof has its roots from [17] (see also [20]) and the key point is to construct a solution V_0 of (4.1) for $\delta = 0$. Let $\hat V \in \bigcap_{R>0} H^1(B_R \setminus \partial\Omega_3)$ be the unique outgoing solution to

$$\begin{cases} \operatorname{div}(\hat A \nabla \hat V) + k^2 \hat\Sigma \hat V = f \text{ in } \mathbb{R}^d \setminus \partial\Omega_3, \\ [\hat V] = g \quad \text{and} \quad [\hat A \nabla \hat V \cdot \nu] = h \text{ on } \partial\Omega_3, \end{cases}$$

where $(\hat A, \hat\Sigma)$ is defined in (1.14). We obtain, for $R > 0$,

$$\|\hat V\|_{H^1(B_R\setminus\partial\Omega_3)} \le C_R\big(\|f\|_{L^2} + \|g\|_{H^{1/2}(\partial\Omega_3)} + \|h\|_{H^{-1/2}(\partial\Omega_3)}\big), \tag{4.2}$$

for some positive constant C_R independent of f, g, and h. Define V_0 in $\mathbb{R}^d$ as follows

$$V_0 = \begin{cases} \hat{V} & \text{in } \mathbb{R}^d \setminus \Omega_2, \\ \hat{V} \circ F & \text{in } \Omega_2 \setminus \Omega_1, \\ \hat{V} \circ G \circ F & \text{in } \Omega_1. \end{cases} \tag{4.3}$$

Applying Lemma 2.2, we derive from (1.11) that $V_0 \in \bigcap_{R>0} H^1(B_R \setminus (\partial\Omega_3 \cup \partial\Omega_1))$ is an outgoing solution to

$$\operatorname{div}(s_0 A \nabla V_0) + k^2 s_0 \Sigma V_0 = f \text{ in } \mathbb{R}^d \setminus (\partial\Omega_3 \cup \partial\Omega_1).$$

Applying Lemma 2.2 again, one obtains from the definition of V_0 and $\hat{V}$ that

$$[V_0] = g \quad \text{and} \quad [A\nabla V_0 \cdot \nu] = h \text{ on } \partial\Omega_3.$$

and

$$[V_0] = 0 \quad \text{and} \quad [A\nabla V_0 \cdot \nu] = 0 \text{ on } \partial\Omega_1.$$

Hence $V_0 \in \bigcap_{R>0} H^1(B_R \setminus \partial\Omega_3)$ is an outgoing solution of (4.1) with $\delta = 0$. Set

$$W_\delta = V_\delta - V_0 \text{ in } \mathbb{R}^d. \tag{4.4}$$

Then $W_\delta \in H^1_{loc}(\mathbb{R}^d)$ is the unique solution to

$$\operatorname{div}(s_\delta A \nabla W_\delta) + k^2 s_0 \Sigma W_\delta = -\operatorname{div}\left(i\delta A \nabla V_0 \mathbb{1}_{\Omega_2 \setminus \Omega_1}\right) \text{ in } \mathbb{R}^d.$$

Here and in what follows, for a subset D of $\mathbb{R}^d$, $\mathbb{1}_D$ denotes its characteristic function. Applying Lemma 2.1, we have

$$\|W_\delta\|_{H^1(B_R)} \le C_R \|V_0\|_{H^1(\Omega_2 \setminus \Omega_1)}. \tag{4.5}$$

The conclusion now follows from (4.2), (4.3), (4.4), and (4.5). □

Before stating the second lemma, we recall some properties of the Bessel and Neumann functions and the spherical Bessel and Neumann functions of large orders. We first introduce, for $n \ge 0$,

$$\hat{J}_n(r) = 2^n n! J_n(r) \quad \text{and} \quad \hat{Y}_n(r) = -\frac{\pi}{2^n (n-1)!} Y_n(r), \tag{4.6}$$

and

$$\hat{j}_n(t) = 1 \cdot 3 \cdots (2n+1) j_n(t) \quad \text{and} \quad \hat{y}_n = -\frac{y_n(t)}{1 \cdot 3 \cdots (2n-1)}, \tag{4.7}$$

where J_n and Y_n are the Bessel and Neumann functions and j_n and y_n are the spherical Bessel and Neumann functions of order n respectively. Then (see, e.g., [8, (3.57), (3.58), (2.37) and (2.38)]), one has, as $n \to +\infty$,

$$\hat{J}_n(t) = t^n \big[1 + O(1/n)\big], \quad \hat{Y}_n(t) = t^{-n} \big[1 + O(1/n)\big], \tag{4.8}$$

$$\hat{j}_n(r) = r^n\big[1 + O(1/n)\big] \quad \text{and} \quad \hat{y}_n(r) = r^{-n-1}\big[1 + O(1/n)\big]. \tag{4.9}$$

Using (4.8) and (4.9), we can now implement the analysis in the quasistatic regime developed in [20] to the finite frequency regime in this section.

We are ready to state the second lemma which is on a three spheres inequality for the homogeneous Helmholtz equation in two and three dimensions.

Lemma 4.2. *Let $d = 2, 3$, k, $R > 0$, and let $v \in H^1(B_R)$ be a solution to the equation $\Delta v + k^2 v = 0$ in B_R. Then, for $0 < R_1 < R_2 < R_3 \le R$,*

$$\|v\|_{H^1(B_{R_2})} \le C_{R,k}\|v\|^{\alpha}_{H^1(B_{R_1})}\|v\|^{1-\alpha}_{H^1(B_{R_3})},$$

where $\alpha = \ln(R_3/R_2)/\ln(R_3/R_1)$ and $C_{R,k}$ is a positive constant independent of R_1, R_2, R_3, and v.

Remark 4.1. The case $k = 0$ is well-known and first noted by Hadamard in two dimensions. A recent discussion on three spheres inequalities for second order elliptic equations and their applications for cloaking using complementary media is given in [26].

Proof. By rescaling, one can assume that $k = 1$. We consider the case $d = 2$ and $d = 3$ separately. We first give the proof in two dimensions. Since $\Delta v + v = 0$ in B_R, one can represent v in the form

$$v = \sum_{n=0}^{\infty}\sum_{\pm} a_{n,\pm}\hat{J}_n(|x|)e^{\pm in\theta} \text{ in } B_R,$$

for $a_{n,\pm} \in \mathbb{C}$ $(n \ge 0)$ with $a_{0,+} = a_{0,-}$ where $\hat{J}_n$ is defined in (4.6). Note that, for $0 < r \le R$,

$$\|v\|^2_{H^1(B_r)} \sim \sum_{n=0}^{\infty}\sum_{\pm}\|a_{n,\pm}\hat{J}_n(|x|)e^{\pm in\theta}\|^2_{H^1(B_r)} \tag{4.10}$$

Here and in what follows in this section, $a \lesssim b$ means that $a \le Cb$ for some positive constant C independent of n and δ, $a \sim b$ means that $a \lesssim b$ and $b \lesssim a$. On the other hand, for each n, there exists a constant $C_n > 1$ such that

$$C_n^{-1}|a_{n,\pm}|^2 r^2 \le \|a_{n,\pm}\hat{J}_n(|x|)e^{\pm in\theta}\|^2_{H^1(B_r)} \le C_n|a_{n,\pm}|^2 r^2. \tag{4.11}$$

The conclusion now follows from (4.8), (4.10) and (4.11) after applying Hölder's inequality.

The proof in three dimensions follows similarly. In this case v can be represented in the form

$$v = \sum_{n=0}^{\infty}\sum_{-n}^{n} a_m^n \hat{j}_n(|x|)Y_m^n(\hat{x}) \quad \text{in } B_R,$$

for $a_m^n \in \mathbb{C}$ and $\hat{x} = x/|x|$ where Y_m^n is the spherical harmonic function of degree n and of order m. The conclusion is now a consequence of (4.9) after applying Hölder's inequality as in the two dimensional case. The details are left to the reader. □

4.2 Proof of Proposition 1.3

The proof is in the same spirit as the one of [20, Theorems 1.2 and 1.3] and is divided into two steps. By rescaling, one can assume that $k = 1$.

Step 1: Proof of the first statement. Define

$$u_{1,\delta} = u_\delta \circ F^{-1} \text{ in } \mathbb{R}^d \setminus B_{r_3},$$

and

$$u_{2,\delta} = u_{1,\delta} \circ G^{-1} \text{ in } B_{r_3}.$$

Let $\phi \in H^1(B_{r_3} \setminus B_{r_2})$ be the unique solution to

$$\Delta \phi + \phi = f \text{ in } B_{r_3} \setminus B_{r_2}, \quad \phi = 0 \text{ on } \partial B_{r_2}, \quad \text{and} \quad \partial_r \phi - i\phi = 0 \text{ on } \partial B_{r_3}, \tag{4.12}$$

and set

$$W = w - \phi \text{ in } B_{r_0} \setminus B_{r_2}.$$

Then $W \in H^1(B_{r_3} \setminus B_{r_2})$ satisfies

$$\Delta W + W = 0 \text{ in } B_{r_0} \setminus B_{r_2}, \quad W = 0 \text{ on } \partial B_{r_2}, \quad \text{and} \quad \partial_r W = -\partial_r \phi \text{ on } \partial B_{r_2}. \tag{4.13}$$

We now consider the case $d = 2$ and $d = 3$ separately.

Case 1: $d = 2$. Since $\Delta W + W = 0$ in $B_{r_3} \setminus B_{r_2}$, one can represent W as follows

$$W = \sum_{n=0}^{\infty} \sum_{\pm} \big[a_{n,\pm} \hat{J}_n(|x|) + b_{n,\pm} \hat{Y}_n(|x|)\big] e^{\pm in\theta} \quad \text{in } B_{r_3} \setminus B_{r_2}, \tag{4.14}$$

for $a_{n,\pm}, b_{n,\pm} \in \mathbb{C}$ $(n \geq 0)$ with $a_{0,+} = a_{0,-}$ and $b_{0,+} = b_{0,-}$ where $\hat{J}_n$ and $\hat{Y}_n$ are defined in (4.6). Using (4.8) and the fact that $W = 0$ on ∂B_{r_2}, we derive that, for large n,

$$|b_{n,\pm}| \sim |a_{n,\pm}| r_2^{2n}. \tag{4.15}$$

It follows from (4.8) and (4.15) that, for some $N > 0$ independent of W,

$$\|W\|^2_{H^1(B_{r_0} \setminus B_{r_2})} \sim \sum_{n=0}^{N} \sum_{\pm} (|a_{n,\pm}|^2 + |b_{n,\pm}|^2) + \sum_{n=N+1}^{\infty} \sum_{\pm} n |a_{n,\pm}^2| r_0^{2n} < +\infty. \tag{4.16}$$

We also assume that (4.15) holds for $n > N$. One of the keys in the proof is the construction of $W_\delta \in H^1(B_{r_3} \setminus B_{r_2})$ which is defined as follows

$$W_\delta = \sum_{n=0}^{\infty} \sum_{\pm} \frac{1}{1 + \xi_n} \big[a_{n,\pm} \hat{J}_n(|x|) + b_{n,\pm} \hat{Y}_n(|x|)\big] e^{\pm in\theta} \text{ in } B_{r_3} \setminus B_{r_2}, \tag{4.17}$$

where

$$\xi_n = \begin{cases} 0 & \text{if } 0 \leq n \leq N, \\ \delta^{1/2} (r_3/r_0)^n & \text{if } n \geq N+1. \end{cases} \tag{4.18}$$

From the definition of W_δ, we have

$$\Delta W_\delta + W_\delta = 0 \text{ in } B_{r_3} \setminus \bar{B}_{r_2}, \quad W_\delta = 0 \text{ on } \partial B_{r_2}, \tag{4.19}$$

and

$$\|W_\delta\|^2_{H^1(B_{r_3}\setminus B_{r_2})} \sim \sum_{n=0}^{N}\sum_{\pm}(|a_{n,\pm}|^2 + |b_{n,\pm}|^2) + \sum_{n=N+1}^{\infty}\sum_{\pm}\frac{n|a_{n,\pm}|^2}{1+\xi_n^2} r_3^{2n}. \tag{4.20}$$

From the definition of ξ_n in (4.18), we have

$$\frac{n|a_{n,\pm}|^2}{1+\xi_n^2} r_3^{2n} \le \delta^{-1} n |a_{n,\pm}|^2 r_0^{2n}. \tag{4.21}$$

A combination of (4.16), (4.20), and (4.21) yields

$$\|W_\delta\|_{H^1(B_{r_3}\setminus B_{r_2})} \le C\delta^{-1/2}. \tag{4.22}$$

Here and in what follows, C denotes a positive constant independent of n and δ. Let $W_{1,\delta} \in H^1_{loc}(\mathbb{R}^2)$ be the unique outgoing solution to

$$\begin{cases} \operatorname{div}(s_\delta A \nabla W_{1,\delta}) + s_0 \Sigma W_{1,\delta} = 0 \text{ in } \mathbb{R}^2 \setminus \partial B_{r_2}, \\ [s_\delta A \nabla W_{1,\delta} \cdot \nu] = (-1 - i\delta) h_\delta \text{ on } \partial B_{r_2}, \end{cases}$$

where

$$h_\delta = -\partial_r(\phi + W_\delta) \text{ on } \partial B_{r_2},$$

and let $W_{2,\delta} \in H^1_{loc}(\mathbb{R}^2 \setminus \partial B_{r_3})$ be the unique outgoing solution to

$$\begin{cases} \operatorname{div}(s_\delta A \nabla W_{2,\delta}) + s_0 \Sigma W_{2,\delta} = f \mathbb{1}_{\mathbb{R}^2 \setminus B_{r_3}} \text{ in } \mathbb{R}^2 \setminus \partial B_{r_3}, \\ [W_{2,\delta}] = \phi + W_\delta \quad \text{and} \quad [A\nabla W_{2,\delta} \cdot \nu] = \partial_r \phi + \partial_r W_\delta \text{ on } \partial B_{r_3}. \end{cases}$$

From (4.12), (4.19), and the fact $(A, \Sigma) = (I, 1)$ in $B_{r_3} \setminus B_{r_2}$, we have

$$u_\delta - (\phi + W_\delta)\mathbb{1}_{B_{r_3}\setminus B_{r_2}} = W_{1,\delta} + W_{2,\delta} \text{ in } \mathbb{R}^2. \tag{4.23}$$

Using (4.13) and (4.17), we obtain, on ∂B_{r_2},

$$h_\delta = -\partial_r(\phi + W_\delta) = \partial_r(W - W_\delta) = \partial_r\left(\sum_{n=N+1}^{\infty}\sum_{\pm}\frac{\xi_n}{1+\xi_n}\big[a_{n,\pm}\hat{J}_n(|x|) + b_{n,\pm}\hat{Y}_n(|x|)\big]e^{\pm in\theta}\right).$$

It follows from (4.15) that

$$\|h_\delta\|^2_{H^{-1/2}(\partial B_{r_2})} \lesssim \sum_{n=N+1}^{\infty}\sum_{\pm}\frac{n|\xi_n|^2}{1+|\xi_n|^2}|a_{n,\pm}|^2 r_2^{2n}. \tag{4.24}$$

From the definition of ξ_n in (4.18) and the fact that $r_0 > \sqrt{r_2 r_3}$, we derive that

$$\frac{n|\xi_n|^2}{1+|\xi_n|^2} r_2^{2n} \le \delta n r_0^{2n}. \tag{4.25}$$

A combination of (4.24) and (4.25) yields

$$\|h_\delta\|_{H^{-1/2}(\partial B_{r_2})} \le C\delta^{1/2}\|W\|_{H^{1/2}(\partial B_{r_0})} \le C\delta^{1/2}.$$

Applying Lemma 2.1, we have

$$\|W_{1,\delta}\|_{H^1(\Omega)} \le (C/\delta)\delta^{1/2} = C\delta^{-1/2}. \tag{4.26}$$

On the other hand, from (4.22) and Lemma 4.1, we obtain

$$\|W_{2,\delta}\|_{H^1(B_{r_3}\setminus B_{r_3})} \le C\delta^{-1/2}. \tag{4.27}$$

The conclusion in the case $d = 2$ now follows from (4.22), (4.23), (4.26), and (4.27).

<u>Case 2:</u> $d = 3$. We represent W in the form

$$W = \sum_{n=0}^{\infty}\sum_{-n}^{n}\big[a_m^n \hat{j}_n(|x|) + b_m^n \hat{y}_n(|x|)\big]Y_m^n(\hat{x}) \quad \text{ in } B_{r_3}\setminus B_{r_0}, \tag{4.28}$$

for $a_m^n, b_m^n \in \mathbb{C}$ and $\hat{x} = x/|x|$, where $\hat{j}_n$ and $\hat{y}_n$ are defined in (4.7). Define $W_\delta \in H^1(B_{r_3}\setminus B_{r_2})$ by

$$W_\delta = \sum_{n=0}^{\infty}\sum_{-n}^{n}\frac{1}{1+\xi_n}\big[a_m^n \hat{j}_n(|x|) + b_m^n \hat{y}_n(|x|)\big]Y_m^n(\hat{x}) \text{ in } B_{r_3}\setminus B_{r_2},$$

where ξ_n is given by (4.18). The proof now follows similarly as in the case $d = 2$; however instead of using (4.8), we apply (4.9). The details are left to the reader.

Step 2: Proof of the second statement. Define $u_{1,\delta} = u_\delta \circ F$ and denote $u_{2^{-n}}$ and $u_{1,2^{-n}}$ by u_n and $u_{1,n}$. for notational ease. We prove by contradiction that

$$\limsup_{n\to+\infty} 2^{-n/2}\big(\|u_n\|_{H^1(B_{r_3}\setminus B_{r_2})} + \|u_{1,n}\|_{H^1(B_{r_3}\setminus B_{r_2})}\big) = +\infty. \tag{4.29}$$

Assume that

$$m := \sup_n 2^{-n/2}\big(\|u_n\|_{H^1(B_{r_3}\setminus B_{r_2})} + \|u_{1,n}\|_{H^1(B_{r_3}\setminus B_{r_2})}\big) < +\infty. \tag{4.30}$$

Define

$$v_n = u_n - u_{1,n} \text{ in } B_{r_3}\setminus B_{r_2} \quad \text{ and } \quad \phi_n = i2^{-n}\partial_r u_{1,n} \text{ on } \partial B_{r_2}.$$

Then, by Lemma 2.2, we obtain

$$\Delta v_n + v_n = f \text{ in } B_{r_3}\setminus B_{r_2}, \;\; v_n = 0 \text{ on } \partial B_{r_2}, \text{ and } \quad \partial_r v_n = \phi_n \text{ on } \partial B_{r_2}. \tag{4.31}$$

We claim that (v_n) is a Cauchy sequence in $H^1(B_{r_0}\setminus B_{r_2})$. Indeed, set

$$V_n = v_{n+1} - v_n \text{ in } B_{r_3}\setminus B_{r_2} \quad \text{ and } \quad \Phi_n = \phi_{n+1} - \phi_n \text{ on } \partial B_{r_2}.$$

We have

$$\Delta V_n + V_n = 0 \text{ in } B_{r_3}\setminus B_{r_2}, \quad V_n = 0 \text{ on } \partial B_{r_2}, \quad \text{ and } \quad \partial_r V_n = \Phi_n \text{ on } \partial B_{r_2}.$$

From (4.30), we derive that

$$\|V_n\|_{H^1(B_{R_2}\setminus B_{R_1})} \le Cm2^{n/2} \quad \text{and} \quad \|\Phi_n\|_{H^{1/2}(\partial B_{r_2})} \le Cm2^{-n/2}. \tag{4.32}$$

Let $U_n \in H^1(B_{r_3})$ be the unique solution of

$$\Delta U_n + U_n = 0 \text{ in } B_{r_3} \setminus \partial B_{r_2}, \quad [\partial_r U_n] = \Phi_n, \quad \text{and} \quad \partial_r U_n - iU_n = 0 \text{ on } \partial B_{r_3}.$$

We have

$$\|U_n\|_{H^1(B_{r_3})} \le C\|\Phi_n\|_{H^{-1/2}(\partial B_{r_2})}. \tag{4.33}$$

Applying Lemma 4.2 for $V_n \mathbb{1}_{B_{r_3}\setminus B_{r_2}} - U_n$ in B_{r_3}, we obtain from (4.33) that

$$\|V_n\|_{H^1(B_{r_0}\setminus B_{r_2})} \le C\Big(\|\Phi_n\|^{\alpha}_{H^{-1/2}(\partial B_{r_2})}\|V_n\|^{1-\alpha}_{H^1(B_{r_3}\setminus B_{r_2})} + \|\Phi_n\|_{H^{-1/2}(\partial B_{r_2})}\Big),$$

where $\alpha = \ln(r_3/r_0)/\ln(r_3/r_2) > 1/2$ since $r_0 < \sqrt{r_2 r_3}$. It follows from (4.32) that

$$\|V_n\|_{H^1(B_{r_0}\setminus B_{r_2})} \le Cm2^{-\beta n},$$

where $\beta = \big(2\alpha - 1\big)/2 > 0$. Hence (v_n) is a Cauchy sequence in $H^1(B_{r_0} \setminus B_{r_2})$. Let v be the limit of v_n in $H^1(B_{r_0} \setminus B_{r_2})$. It follows from (4.31) that

$$\Delta v + v = f \text{ in } B_{r_0} \setminus B_{r_2}, \quad v = 0 \text{ on } \partial B_{R_1}, \quad \partial_r v = 0 \text{ on } \partial B_r.$$

This contradicts the non-existence of v. Hence (4.29) holds. The proof is complete. □

5 Cloaking a source via anomalous localized resonance in the finite frequency regime

In this section, we describe how to use the theory discussed previously to cloak a source f concentrating on a bounded smooth manifold of codimension 1 in an arbitrary medium. We follow the strategy in [20]. Without loss of generality, one may assume that the medium is contained in $B_{r_3} \setminus B_{r_2}$ for some $0 < r_2 < r_3$ and characterized by a matrix-valued function a and a real bounded function σ. We assume in addition that a is Lipschitz and uniformly elliptic in $\overline{B_{r_3} \setminus B_{r_2}}$ and σ is bounded below by a positive constant. Let $f \in L^2(\partial\Omega)$ for some bounded smooth open subset $\Omega \subset\subset B_{r_3} \setminus B_{r_2}$. We also assume that $\Omega \subset\subset B(x_0, r_0)$ for some $r_0 > 0$ and $x_0 \in \partial B_{r_2}$ where r_0 is the constant coming from Proposition 1.1. Define $r_1 = r_2^2/r_3$. Let $F : B_{r_2} \setminus \{0\} \to \mathbb{R}^d \setminus B_{r_2}$ and $G : \mathbb{R}^d \setminus B_{r_3} \to B_{r_3} \setminus \{0\}$ be the Kelvin transform with respect to ∂B_{r_2} and ∂B_{r_3} respectively. Define

$$A, \Sigma = \begin{cases} a, \sigma & \text{in } B_{r_3} \setminus B_{r_2}, \\ F_*^{-1}a, F_*^{-1}\sigma & \text{in } B_{r_2} \setminus B_{r_1}, \\ F_*^{-1}G_*^{-1}a, F_*^{-1}G_*^{-1}\sigma & \text{in } B_{r_1} \setminus B_{r_1^2/r_2}, \\ I, 1 & \text{otherwise.} \end{cases} \tag{5.1}$$

It is clear that $(s_0 A, s_0 \Sigma)$ is doubly complementary. Applying Theorem 1.1 and Proposition 1.1, we obtain

Proposition 5.1. *Let $d \geq 2$, $\delta > 0$, and $\Omega \subset\subset D := B(x_0, r_0) \cap (B_{r_3} \setminus B_{r_2})$ be smooth and open, let $f \in L^2(\partial\Omega)$ and let u_δ and v_δ be defined by (1.4) and (1.6) where (A, Σ) is given in (5.1). Assume that $f \notin \mathcal{H}$ where*

$$\mathcal{H} := \Big\{ A\nabla v \cdot \nu \big|_{\partial\Omega}; v \in H^1_0(\Omega) \text{ is a solution of } \operatorname{div}(A\nabla v) + k^2 \Sigma v = 0 \text{ in } \Omega \Big\}.$$

There exists a sequence $(\delta_n) \to 0$ such that

$$\lim_{n\to\infty} E_{\delta_n}(u_{\delta_n}) = +\infty.$$

Moreover,

$$v_{\delta_n} \to 0 \text{ weakly in } H^1_{loc}(\mathbb{R}^d \setminus B_{r_3}).$$

Remark 5.1. It is worth noting from the definition of $\mathcal{H}$ that $\mathcal{H}$ has finite dimensions and for all positive k except for a discrete set, $\mathcal{H} = \{0\}$ by Fredholm's theory. Therefore, as a consequence of Proposition 5.1, for all positive frequency except a discrete set, and for all $f \in L^2(\partial\Omega)$, f is cloaked by the structure (5.1) after the normalization.

Proof. By Theorem 1.1 and Proposition 1.1, it suffices to prove that there is **no** $W \in H^1(D)$ such that

$$\operatorname{div}(A\nabla W) + k^2\Sigma = f \text{ in } D \quad \text{and} \quad W = A\nabla W \cdot \eta = 0 \text{ on } \partial D \cap \partial B_{r_2}.$$

In fact, Theorem 1.1 and Proposition 1.1 only deal with the case $f \in L^2$, however, the same results hold for f stated here and the proofs are unchanged. Suppose that this is not true, i.e., such a W exists. Since $\operatorname{div}(A\nabla W) + k^2\Sigma W = 0$ in $D \setminus \bar{\Omega}$ and $W = A\nabla W \cdot \nu = 0$ on $\partial D \cap \partial B_{r_2}$, it follows from the unique continuation principle that $W = 0$ in $D \setminus \bar{\Omega}$. Hence $W\big|_\Omega \in H^1_0(\Omega)$ is a solution of $\operatorname{div}(A\nabla W) + k^2\Sigma W = 0$ in Ω. We derive that $f = -A\nabla W \cdot \nu\big|_\Omega$ on $\partial\Omega$. This contradicts the fact that $f \notin \mathcal{H}$. The proof is complete. □

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