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CLOAKING AN ARBITRARY OBJECT VIA ANOMALOUS LOCALIZED RESONANCE: THE CLOAK IS INDEPENDENT OF THE OBJECT*

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Abstract. In this paper, we present various schemes of cloaking an arbitrary object via anomalous localized resonance and provide their analysis in two and three dimensions. This is a way to cloak an object using negative index materials in which the cloaking device is independent of the object. As a result, we show that in the two dimensional quasi-static regime an annular plasmonic structure of coefficient -1 cloaks small but finite size objects nearby. We also discuss its connections with superlensing and cloaking using complementary media. In particular, we confirm the possibility that a lens can act like a cloak and conversely. This possibility was raised about a decade ago in the literature.

Key words. cloaking, superlensing, complementary media, localized resonance, three-sphere inequality, conformal maps

AMS subject classifications. 78A40, 78M30, 35A15

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1. Introduction.

Negative index materials (NIMs) were investigated theoretically by Veselago in [29] and the existence of such materials was confirmed by Shelby, Smith, and Schultz in [28]. The study of NIMs has attracted a lot of attention in the scientific community thanks to their many potential applications. Mathematically, the study of NIMs faces two difficulties. First, the equations modeling NIMs have sign changing coefficients; hence the ellipticity and the compactness are lost, in general. Second, the localized resonance, i.e., the fields blow up in some regions and remain bounded in some others as the loss goes to 0, might appear.

Three known applications of NIMs are superlensing and cloaking using complementary media and cloaking a source via anomalous localized resonance (ALR). Superlensing using complementary media was suggested by Veselago in [29] for a slab lens (a slab of index -1) using the ray theory. Later, cylindrical lenses in the two dimensional quasistatic regime, the Veselago slab lens and cylindrical lenses in the finite frequency regime, and spherical lenses in the finite frequency regime were studied by Nicorovici, McPhedran, and Milton in [23], Pendry in [24, 25], and Ramakrishna and Pendry in [26], respectively, for dipoles sources. Superlensing using complementary media for arbitrary objects in the acoustic and electromagnetic settings was mathematically established by Nguyen in [14, 18] for related schemes. Cloaking using complementary media was suggested and investigated numerically by Lai et al. in [7]. This was mathematically established for related schemes by Nguyen in [15] for the quasi-static regime and later extended by Nguyen and Nguyen in [21] for the finite frequency regime. Cloaking a source via ALR was discovered by Milton and Nicorovici in [9] for constant radial symmetric plasmonic structures in the two dimensional quasi-static regime. Their work has its root from [23] (see also [22, 8]) where the localized resonance was observed and established for such a setting. Later, cloaking a source via ALR was studied by Milton et al. in [10], Bouchitte and Schweizer in [4], Ammari

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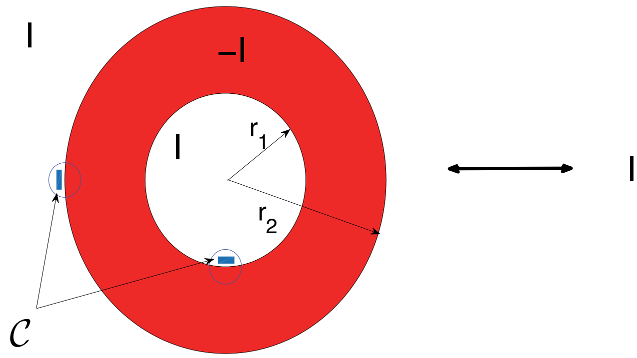

FIG. 1. *In the two dimensional quasi-static regime, the plamonic structure of coefficient $-I$ in $B_{r_2} \setminus B_{r_1}$ (the red region) cloaks arbitrary objects located nearby in $\mathcal{C}$. The medium on the left is equivalent to the homogeneous medium I on the right for sources far away from the plasmonic structure.*

et al. in [2, 3], Kohn et al. in [6], Nguyen and Nguyen in [20] in which special structures were considered due to the use of the separation of variables or the blowup of the power was investigated via spectral theory or variational method. In [12, 13, 19], Nguyen investigated cloaking a source via ALR for a class of complementary media called the class of doubly complementary media for a general core-shell structure. In these works, the blowup of the power, the localized resonance, and the cloaking effect are studied. The reader can find a recent survey on the mathematical aspects for NIMs in [17].

In this paper, we add to the list of applications of NIMs a new one, namely, cloaking *an arbitrary object via ALR*. More precisely, we propose various schemes for this type of cloaking and provide the mathematical analysis for them.

In what follows, given $R > 0$ and $x \in \mathbb{R}^d$, we denote $B(x, R)$ the open ball in $\mathbb{R}^d$ centered at x and of radius R; when $x = 0$, we simply denote $B(x, R)$ by B_R. In $\mathbb{R}^d$, we also denote I the $(d \times d)$ identity matrix.

The first result in this paper, whose proof is given in section 3.1, confirms that an annular plasmonic structure of coefficient $-I$ cloaks small but finite size objects close to it in the quasi-static regime (see Figure 1). More precisely, we have the following.

THEOREM 1.1. *Let $d = 2$, $0 < r_0 < r_1 < r_2 < R_0$, $x_1 \in \partial B_{r_1}$, and $x_2 \in \partial B_{r_2}$. Set $r_3 = r_2^2/r_1$ and $\mathcal{C} = (B(x_1, r_0) \cap B_{r_1}) \cup (B(x_2, r_0) \cap (B_{r_3} \setminus B_{r_2}))$ and let a_c be a symmetric uniformly elliptic matrix-valued function defined in $\mathcal{C}$. Define*

$$A_c = \begin{cases} a_c & \text{in } \mathcal{C}, \\ I & \text{otherwise}, \end{cases} \quad \text{and} \quad s_\delta = \begin{cases} -1 - i\delta & \text{in } B_{r_2} \setminus B_{r_1}, \\ 1 & \text{otherwise}. \end{cases} \tag{1.1}$$

Given $f \in L^2(\mathbb{R}^2)$ with $\operatorname{supp} f \subset B_{R_0} \setminus B_{r_3}$ and $\int_{\mathbb{R}^2} f = 0$, let $u_\delta, \hat{u} \in W^1(\mathbb{R}^2)$ be, respectively, the unique solution to the equations

$$\operatorname{div}(s_\delta A_c \nabla u_\delta) = f \text{ in } \mathbb{R}^2 \tag{1.2}$$

and

$$\Delta \hat{u} = f \text{ in } \mathbb{R}^2. \tag{1.3}$$

For any $0 < \gamma < 1/2$, there exists $r_0(\gamma) > 0$, which depends only on r_1 and r_2, such that if $r_0 < r_0(\gamma)$ then, for $R > 0$,

$$\|u_\delta - \hat{u}\|_{H^1(B_R \setminus B_{r_3})} \le C_R \delta^\gamma \|f\|_{L^2(\mathbb{R}^2)}, \tag{1.4}$$

where C_R is a positive constant independent of f, δ, r_0, x_1, and x_2.

Here, and in what follows,

$$W^1\left(\mathbb{R}^2\right) := \left\{ u \in H^1_{_{loc}}\left(\mathbb{R}^2\right); \nabla u \in L^2\left(\mathbb{R}^2\right) \text{ and } \frac{u(x)}{\ln(2+|x|)\sqrt{1+|x|^2}} \in L^2\left(\mathbb{R}^2\right) \right\}.$$

For an observer outside B_{r_3}, the medium $s_0 A_c$ in B_{r_3} looks like I in B_{r_3}: the object a_c in $\mathcal{C}$ is cloaked. No condition is imposed on a_c; any symmetric uniformly elliptic matrix a_c in $\mathcal{C}$ is allowed. The cloak $-I$ in $B_{r_2} \setminus B_{r_1}$ is *independent* of the object. It is interesting to note that the plasmonic structure $-I$ in $B_{r_2} \setminus B_{r_1}$ is used in lens devices using NIMs; see [23, 14].

In the two dimensional finite frequency regime, we obtain the following result whose proof is given in section 3.1.

THEOREM 1.2. *Let $d = 2$, $0 < r_0 < r_1 < r_2 < R_0$, $x_1 \in \partial B_{r_1}$, and $x_2 \in \partial B_{r_2}$. Set $r_3 = r_2^2/r_1$ and $\mathcal{C} = (B(x_1, r_0) \cap B_{r_1}) \cup (B(x_2, r_0) \cap (B_{r_3} \setminus B_{r_2}))$. Let a_c be a symmetric uniformly elliptic matrix-valued function and let σ_c be a bounded real function both defined in $\mathcal{C}$ such that a_c is piecewise C^1 and σ_c is bounded below by a positive constant. Define*

$$\left(A_c, \Sigma_c\right) = \begin{cases} \left(a_c, \sigma_c\right) & \text{in } \mathcal{C}, \\ \left(I, r_2^4/|x|^4\right) & \text{in } B_{r_2} \setminus B_{r_1}, \\ \left(I, r_3^2/r_1^2\right) & \text{in } B_{r_1} \setminus \mathcal{C}, \\ \left(I, 1\right) & \text{otherwise}, \end{cases} \quad \text{and} \quad s_\delta = \begin{cases} -1 - i\delta & \text{in } B_{r_2} \setminus B_{r_1}, \\ 1 & \text{otherwise}. \end{cases} \tag{1.5}$$

Given $f \in L^2(\mathbb{R}^2)$ with $\operatorname{supp} f \subset B_{R_0} \setminus B_{r_3}$, let $u_\delta, \hat{u} \in H^1_{_{loc}}(\mathbb{R}^2)$ be, respectively, the unique outgoing solutions to the equations

$$\operatorname{div}(s_\delta A_c \nabla u_\delta) + k^2 s_0 \Sigma_c u_\delta = f \text{ in } \mathbb{R}^2 \tag{1.6}$$

and

$$\Delta \hat{u} + k^2 \hat{u} = f \text{ in } \mathbb{R}^2. \tag{1.7}$$

For any $0 < \gamma < 1/2$, there exists $r_0(\gamma) > 0$, which depends only on r_1 and r_2, such that if $r_0 < r_0(\gamma)$ then

$$\|u_\delta - \hat{u}\|_{H^1(B_R \setminus B_{r_3})} \le C_R \delta^\gamma \|f\|_{L^2} \tag{1.8}$$

for some positive constant C_R independent of f, δ, r_0, x_1, and x_2.

Recall that a solution $u \in H^1_{_{loc}}(\mathbb{R}^d \setminus B_R)$ ($d \ge 2$) of the equation $\Delta u + k^2 u = 0$ in $\mathbb{R}^d \setminus B_R$ for some $R > 0$ is said to satisfy the outgoing condition if

$$\partial_r u - iku = o(r^{\frac{1-d}{2}}) \text{ as } r = |x| \to +\infty.$$

For an observer outside B_{r_3}, the medium $(s_0 A, s_0 \Sigma)$ in B_{r_3} looks like $(I, 1)$ in B_{r_3}: the object (a_c, σ_c) in $\mathcal{C}$ is cloaked. No condition other than the standard ones is imposed on a_c and σ_c. The cloak is independent of the object. It is interesting to note that the structure $(-I, -r_2^4/|x|^4)$ in $B_{r_2} \setminus B_{r_1}$ can be used in lens devices using NIMs; see [26, 14].

Remark 1.1. The constant $r_0 = r_0(\gamma)$ in Theorems 1.1 and 1.2 depends only on r_1 and r_2. The constant r_0 obtained from the proofs is small because of the use of various conformal maps (see section 3.1).

In section 2, we show in two and three dimensions that for a class of doubly complementary media, introduced in [13, 19] (see Definition 2.2 in section 2), an arbitrary object can disappear if it is small and located close to the plasmonic structure (see Theorem 2.1 in section 2). Therefore, a medium in this class becomes a cloaking device which is independent of the object.

Cloaking an object via ALR is related to but different from cloaking using complementary media suggested in [7] and mathematically established in [15, 21] for related schemes. Both types of cloaking use the concept of complementary media to design cloaking devices. Nevertheless, in the cloaking using the complementary media approach, one cloaks an object by using its complementary medium to cancel its effect on light; hence the cloaking device depends on the object.

Cloaking an object via ALR and cloaking a source via ALR share some similar figures but have some different characteristics. In the two dimensional quasi-static regime, one can use the plasmonic structure $-I$ in $B_{r_2} \setminus B_{r_1}$ as a cloaking device in both settings. More generally, doubly complementary media are used as a cloaking device in both types of cloaking (see [13, 19] and section 2). Nevertheless, concerning cloaking a source via ALR, the cloaking effect is *relative* in the sense that the source is cloaked after being renormalized so that the power remains finite (see, e.g., [13]). Concerning cloaking an object via ALR, the cloaking effect is *not relative* in the sense that one does not need to renormalize the power. In fact, the power is finite in the setting of cloaking an object via ALR considered here (see Remarks 2.5 and 3.1).

Milton and Nicorovici in [9] questioned whether or not the structure $-I$ in $B_{r_2} \setminus B_{r_1}$ would cloak small objects nearby in the two dimensional quasi-static regime. Various numerical simulations of this effect were reported by Bruno and Lintner in [5], even though the cloaking effect is still mysterious and questionable. In this paper, the guess of Milton and Nicorovici is confirmed (Theorem 1.1) and similar phenomena are observed and analyzed in the finite frequency regime in both two and three dimensions (Theorems 1.2 and 2.1). As mentioned, the plasmonic structures used in Theorem 1.1 and 1.2 can be used in lens devices using NIMs. More generally, it is established in [14] that doubly complementary media can act like lenses. In this paper, we showed that they might become a cloak for small but finite size objects nearby. Thus the results presented here show that the modification given in [14] from the suggestions in [9, 26] is necessary to ensure the superlensing property, otherwise, a lens can become a cloak (see section 3.2). In the same spirit, we show as well that it is necessary to modify the scheme of cloaking using complementary media suggested in [7] as was done in [15] (see section 3.3).

The analysis in this paper is on one hand based on the use of reflecting and removing singularity techniques introduced in [14, 15] and on the other hand involves a new type of three-sphere inequality with "partial data," and the use of conformal maps in two dimensions.

The rest of the paper is organized as follows. In section 2, we present a scheme of cloaking an object via ALR for a class of doubly complementary media and provide their mathematical analysis. The main result of this section is Theorem 2.1. In section 3, we first provide the proofs of Theorems 1.1 and 1.2 in section 3.1. We then discuss the validity of various schemes on superlensing and cloaking using complementary media in sections 3.2 and 3.3.

2. A class of doubly complementary media acting as a cloaking device.

Let $k > 0$ and let A be a real uniformly elliptic symmetric matrix-valued function and Σ be a real function bounded below and above by positive constants both defined in $\mathbb{R}^d$ ($d = 2, 3$), i.e., $A(x)$ is symmetric and for some $\Lambda \ge 1$

$$\Lambda^{-1}|\xi|^2 \le \langle A(x)\xi, \xi\rangle \le \Lambda|\xi|^2 \quad \text{and} \quad \Lambda^{-1} \le \Sigma(x) \le \Lambda \text{ for a.e. } x \in \mathbb{R}^d. \tag{2.1}$$

Assume that

$$A(x) = I, \quad \Sigma(x) = 1 \text{ for large } |x|, \tag{2.2}$$

and

$$A \text{ is piecewise } C^1. \tag{2.3}$$

The assumption (2.3) is used for the uniqueness of outgoing solutions. In what follows, in the case $k > 0$, any matrix-valued function considered is assumed to satisfy (2.3).

Let $\Omega_1 \subset\subset \Omega_2 \subset\subset \mathbb{R}^d$ be smooth bounded simply connected open subsets of $\mathbb{R}^d$, and set, for $\delta \ge 0$,

$$s_\delta(x) = \begin{cases} -1 - i\delta & \text{in } \Omega_2 \setminus \Omega_1, \\ 1 & \text{in } \mathbb{R}^d \setminus (\Omega_2 \setminus \Omega_1). \end{cases} \tag{2.4}$$

Given $f \in L^2(\mathbb{R}^d)$ with compact support and $\operatorname{supp} f \cap \Omega_2 = \emptyset$, and $\delta > 0$, let $u_\delta \in H^1_{loc}(\mathbb{R}^d)$ be the unique outgoing solution to

$$\operatorname{div}(s_\delta A \nabla u_\delta) + k^2 s_0 \Sigma u_\delta = f \text{ in } \mathbb{R}^d. \tag{2.5}$$

In this section, we discuss the behavior of u_δ as $\delta \to 0$ when the medium inherits the doubly complementary property. To this end, we first recall the definition of reflecting complementary media and doubly complementary media introduced in [11] and [13, 19], respectively.

DEFINITION 2.1 (reflecting complementary media). *Let $\Omega_1 \subset\subset \Omega_2 \subset\subset \Omega_3 \subset\subset \mathbb{R}^d$ be smooth bounded simply connected open subsets of $\mathbb{R}^d$. The media (A, Σ) in $\Omega_3 \setminus \Omega_2$ and $(-A, -\Sigma)$ in $\Omega_2 \setminus \Omega_1$ are said to be* reflecting complementary *if there exists a diffeomorphism $F : \Omega_2 \setminus \bar{\Omega}_1 \to \Omega_3 \setminus \bar{\Omega}_2$ such that $F \in C^1(\bar{\Omega}_2 \setminus \Omega_1)$,*

$$(F_*A, F_*\Sigma) = (A, \Sigma) \textit{ for } x \in \Omega_3 \setminus \Omega_2, \tag{2.6}$$

$$F(x) = x \textit{ on } \partial\Omega_2, \tag{2.7}$$

and the following two conditions hold: (1) *There exists a diffeomorphism extension of F, which is still denoted by F, from $\Omega_2 \setminus \{x_1\} \to \mathbb{R}^d \setminus \bar{\Omega}_2$ for some $x_1 \in \Omega_1$;* (2) *there exists a diffeomorphism $G : \mathbb{R}^d \setminus \bar{\Omega}_3 \to \Omega_3 \setminus \{x_1\}$ such that $G \in C^1(\mathbb{R}^d \setminus \Omega_3)$, $G(x) = x$ on $\partial\Omega_3$, and $G \circ F : \Omega_1 \to \Omega_3$ is a diffeomorphism if one sets $G \circ F(x_1) = x_1$.*

Here and in what follows, if $\mathcal{T}$ is a diffeomorphism and a and σ are, respectively, a matrix-valued function and a complex function, we use the following standard notations

$$\mathcal{T}_*a(y) = \frac{D\mathcal{T}(x)a(x)\nabla\mathcal{T}(x)^T}{|\det \nabla\mathcal{T}(x)|} \quad \text{and} \quad \mathcal{T}_*\sigma(y) = \frac{\sigma(x)}{|\det \nabla\mathcal{T}(x)|}, \quad \text{where } x = \mathcal{T}^{-1}(y). \tag{2.8}$$

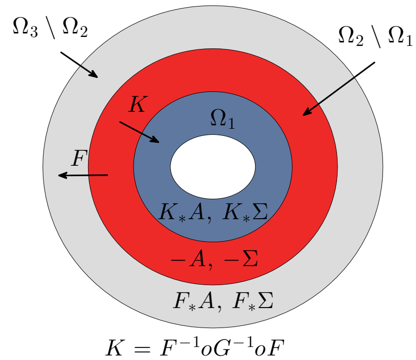

FIG. 2. *$(s_0A, s_0\Sigma)$ is doubly complementary: $(-A, -\Sigma)$ in $\Omega_2 \setminus \Omega_1$ (the red region) is complementary to $(F_*A, F_*\Sigma)$ in $\Omega_3 \setminus \Omega_2$ (the grey region) and $(K_*A, K_*\Sigma)$ in $K(B_{r_2} \setminus B_{r_1})$ (the blue grey region) with $K = F^{-1} \circ G^{-1} \circ F$.*

Conditions (2.6) and (2.7) are the main assumptions in Definition 2.1. The key point behind this requirement is roughly speaking the following property: if $u_0 \in H^1(\Omega_3 \setminus \Omega_1)$ is a solution of $\operatorname{div}(s_0 A\nabla u_0) + k^2 s_0 \Sigma u_0 = 0$ in $\Omega_3 \setminus \Omega_1$ and if u_1 is defined in $\Omega_3 \setminus \Omega_2$ by $u_1 = u_0 \circ F^{-1}$ then $\operatorname{div}(A\nabla u_1) + k^2 \Sigma u_1 = 0$ in $\Omega_3 \setminus \Omega_2$, $u_1 - u_0 = A\nabla(u_1 - u_0) \cdot \nu = 0$ on $\partial\Omega_2$ (see, e.g., Lemma 2.5). Here and in what follows, ν denotes the outward unit vector on the boundary of a smooth bounded open subset of $\mathbb{R}^d$. Hence $u_1 = u$ in $\Omega_3 \setminus \Omega_2$ by the unique continuation principle. Conditions (1) and (2) are mild assumptions. Introducing G makes the analysis more accessible; see [11, 14, 15, 21] and the analysis presented in this paper.

We are ready to recall the definition of doubly complementary media.

DEFINITION 2.2. *The medium $(s_0A, s_0\Sigma)$ is said to be* doubly complementary *if for some $\Omega_2 \subset\subset \Omega_3$, (A, Σ) in $\Omega_3 \setminus \Omega_2$ and $(-A, -\Sigma)$ in $\Omega_2 \setminus \Omega_1$ are reflecting complementary, and*

$$F_*A = G_*F_*A = A \quad \text{and} \quad F_*\Sigma = G_*F_*\Sigma = \Sigma \text{ in } \Omega_3 \setminus \Omega_2 \tag{2.9}$$

for some F and G coming from Definition 2.1.

The reason for which media satisfying (2.9) are called doubly complementary media is the following fact: $(-A, -\Sigma)$ in $B_{r_2} \setminus B_{r_1}$ is not only complementary to (A, Σ) in $\Omega_3 \setminus \Omega_2$ but also to (A, Σ) in $(G \circ F)^{-1}(\Omega_3 \setminus \overline{\Omega_2})$, a subset of Ω_1 (see Figure 2). The key property behind Definition 2.2 is as follows. Assume that $u_0 \in H^1_{loc}(\mathbb{R}^d)$ is a solution of (2.5) with $\delta = 0$ and $f = 0$ in Ω_3. Set $u_1 = u_0 \circ F^{-1}$ and $u_2 = u_1 \circ G^{-1}$. Then u_0, u_1, u_2 satisfy the equation $\operatorname{div}(A\nabla \cdot) + k^2\Sigma \cdot = 0$ in $\Omega_3 \setminus \Omega_2$, $u_0 - u_1 = A\nabla u_0 \cdot \nu - A\nabla u_1 \cdot \nu = 0$ on $\partial\Omega_2$, and $u_1 - u_2 = A\nabla u_1 \cdot \nu - A\nabla u_2 \cdot \nu = 0$ on $\partial\Omega_3$ (see, e.g., Lemma 2.5). This implies $u_0 = u_1 = u_2$ in $\Omega_3 \setminus \Omega_2$.

Taking $d = 2$ and $r_3 = r_2^2/r_1$, and letting F and G be the Kelvin transforms with respect to ∂B_{r_2} and ∂B_{r_3}, one can verify that the media considered in Theorems 1.1 and 1.2 with $r_0 = 0$ are of doubly complementary property. Theorems 1.1 and 1.2 reveal that doubly complementary media might cloak small but finite size arbitrary objects nearby.

Remark 2.1. Given (A, Σ) in $\mathbb{R}^d$ and $\Omega_1 \subset \Omega_2 \subset\subset \mathbb{R}^d$, it is not easy in general to verify whether or not $(s_0A, s_0\Sigma)$ is doubly complementary. Nevertheless, given $\Omega_1 \subset\subset$

$\Omega_2 \subset\subset \Omega_3 \subset\subset \mathbb{R}^d$ and (A, Σ) in $\Omega_3 \setminus \Omega_2$, it is quite easy to choose (A, Σ) in Ω_2 such that $(s_0 A, s_0 \Sigma)$ is doubly complementary. One just needs to choose diffeomorphisms F and G as in Definition 2.1 and define $(A, \Sigma) = (F_*^{-1} A, F_*^{-1} \Sigma)$ in $\Omega_2 \setminus \Omega_1$ and $(A, \Sigma) = (F_*^{-1} G_*^{-1} A, F^{-1} * G_*^{-1} \Sigma)$ in $F^{-1} \circ G^{-1}(\Omega_3 \setminus \Omega_2)$.

The following result established in [19, Theorem 1.1] (see also [11, Theorem 2 and Corollary 2]) provides an interesting property of doubly complementary media.

Proposition 2.1. *Let $d = 2, 3$, $k > 0$, $0 < \delta < 1$, $f \in L^2(\mathbb{R}^d)$ with compact support and* $\operatorname{supp} f \cap \Omega_3 = \emptyset$, *and let $U_\delta \in H^1_{loc}(\mathbb{R}^d)$ be the unique outgoing solution of* (2.5). *Assume that $(s_0 A, s_0 \Sigma)$ is doubly complementary. Then*

$$U_\delta \rightharpoonup \hat{u} \text{ weakly in } H^1_{loc}\left(\mathbb{R}^d \setminus \Omega_3\right), \tag{2.10}$$

where $\hat{u} \in H^1_{loc}(\mathbb{R}^d)$ is the unique outgoing solution of

$$\operatorname{div}(\hat{A} \nabla \hat{u}) + k^2 \hat{\Sigma} \hat{u} = f \text{ in } \mathbb{R}^d. \tag{2.11}$$

Here

$$(\hat{A}, \hat{\Sigma}) := \begin{cases} (A, \Sigma) & \text{in } \mathbb{R}^d \setminus \Omega_3, \\ (G_* F_* A, G_* F_* \Sigma) & \text{in } \Omega_3. \end{cases} \tag{2.12}$$

The cloaking property for a class of doubly complementary media is given in the following.

Theorem 2.1. *Let $d = 2, 3$, $0 < s < r_1 < 6r_1 < r_2 < r_3 < R_0$. Set*

$$H_t = \left\{x' \in \mathbb{R}^{d-1};\ |x'| < s\right\} \times [t, +\infty) \quad \text{for } t \in \mathbb{R}.$$

Assume that $(s_0 A, s_0 \Sigma)$ is doubly complementary with $\Omega_1 = B_{r_1}$, $\Omega_2 = B_{r_2} \setminus H_{2r_1}$, and $\Omega_3 = B_{r_3}$. Let a_c be a symmetric uniformly elliptic matrix-valued function and σ_c be a real function bounded above and below by positive constants both defined in $H_{2r_1} \setminus H_{3r_1}$. Assume that A is Lipschitz in $B(z, r_1)$ with $z = (0, \dots, 0, 4r_1)$. Define

$$(A_c, \Sigma_c) = \begin{cases} (a_c, \sigma_c) & \text{in } H_{2r_1} \setminus H_{3r_1}, \\ (A, \Sigma) & \text{otherwise.} \end{cases}$$

Let $k > 0$, $0 < \delta < 1$, $f \in L^2(\mathbb{R}^d)$ with $\operatorname{supp} f \subset B_{R_0} \setminus B_{r_3}$ *and let $u_\delta \in H^1_{loc}(\mathbb{R}^d)$ be the unique outgoing solution of*

$$\operatorname{div}(s_\delta A_c \nabla u_\delta) + k^2 s_0 \Sigma_c u_\delta = f \text{ in } \mathbb{R}^d. \tag{2.13}$$

For any $0 < \gamma < 1/2$, there exists a positive constant m depending only on γ, r_1, and the elliptic and the Lipschitz constant of A in $B(z, r_1)$ such that if $r_1 > ms$ then

$$\|u_\delta - \hat{u}\|_{H^1(B_R \setminus B_{r_3})} \le C_R \delta^\gamma \|f\|_{L^2} \tag{2.14}$$

for some positive constant C_R independent of f and δ, where $\hat{u} \in H^1_{loc}(\mathbb{R}^d)$ is the unique outgoing solution of (2.11).

The geometry of the cloak is given in Figure 3.

For an observer outside B_{r_3}, the medium $(s_0 A, s_0 \Sigma)$ in B_{r_3} looks like $(\hat{A}, \hat{\Sigma})$ in B_{r_3}, which is independent of (a_c, σ_c) in $\mathcal{C}$: the object (a_c, σ_c) in $\mathcal{C}$ is cloaked. For

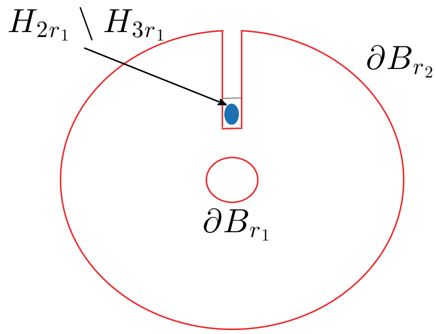

FIG. 3. *The cloaking device has the doubly complementary property and contains the plasmonic structure in the region $\Omega_2 \setminus B_{r_1}$ which has the red boundary. The plamonic structure is charaterized by $(F_*^{-1}I, F_*^{-1}1)$ for some diffeomorphism $F : \Omega_2 \setminus B_{r_1} \to B_{r_3} \setminus \bar{\Omega}_2$. The region $H_{2r_1} \setminus H_{3r_1}$ is the rectangle containing the blue object. Any object inside this region (for example, the blue one) is cloaked.*

example, by choosing the medium (A, Σ) in Ω_1 in such a way that $G_*F_*A = I$ and $G_*F_*\Sigma = 1$ in Ω_3, i.e., $A = F_*^{-1}G_*^{-1}I$ and $\Sigma = F_*^{-1}G_*^{-1}1$ in Ω_1 then $(\hat{A}, \hat{\Sigma}) = (I, 1)$: the object (a_c, σ_c) in $H_{r_1} \setminus H_{2r_1}$ disappears in comparison with the homogeneous medium $(I, 1)$.

Remark 2.2. The assumption that A is Lipschitz in $B(z, r_1)$ is required for the use of three-sphere inequalities.

The rest of this section containing two subsections is devoted to the proof of Theorem 2.1. Some useful lemmas are presented in the first subsection and the proof of Theorem 2.1 is given in the second subsection.

2.1. Some useful lemmas. We first state a three-sphere inequality which is an immediate consequence of [21, Theorem 2].

LEMMA 2.1. *Let $d = 2, 3$, $c_1, c_2 > 0$, $0 < R_1 < R_2 < R_3$, and let a be a Lipschitz uniformly elliptic symmetric matrix-valued function defined in B_{R_3}, and $v \in H^1(B_{R_3} \setminus B_{R_1})$. Assume that*

$$|\operatorname{div}(a\nabla v)| \le c_1|\nabla v| + c_2|v| \quad \text{a.e. in } B_{R_3} \setminus B_{R_1}. \tag{2.15}$$

There exists a constant $q \ge 1$, depending only on the elliptic and the Lipschitz constants of a, such that

$$\|v\|_{\mathbf{H}(\partial B_{R_2})} \le C\|v\|^{\alpha}_{\mathbf{H}(\partial B_{R_1})}\|v\|^{1-\alpha}_{\mathbf{H}(\partial B_{R_3})}, \quad \text{where} \quad \alpha := \frac{R_2^{-q} - R_3^{-q}}{R_1^{-q} - R_3^{-q}}, \tag{2.16}$$

where, for a smooth manifold Γ of $\mathbb{R}^d$ with or without boundary,[1]

$$\|v\|_{\mathbf{H}(\Gamma)} := \|v\|_{H^{1/2}(\Gamma)} + \|a\nabla v \cdot \nu\|_{H^{-1/2}(\Gamma)}. \tag{2.17}$$

Here C is a positive constant depending on the elliptic and the Lipschitz constants of a, and the constants c_1, c_2, R_1, R_2, R_3, but independent of v.

[1] In this paper, $H^{-1/2}(\Gamma)$ denotes the dual space of $H_0^{1/2}(\Gamma)$, the completion of $C_c^1(\Gamma)$ in $H^{1/2}(\Gamma)$, with the corresponding norm. Note that if Γ has no boundary, then $H_0^{1/2}(\Gamma) = H^{1/2}(\Gamma)$.

Remark 2.3. In Lemma 2.1, the constant q is independent of c_1 and c_2 and there is no requirement on the smallness of R_1, R_2, and R_3. This is different from the standard three-sphere inequalities obtained previously in the literature (see, e.g., [1]) and plays an important role in our analysis.

Applying Lemma 2.1, we can establish the following.

LEMMA 2.2. *Let $d = 2, 3$, $0 < s < R$, and set $D = \{x' \in \mathbb{R}^{d-1}; |x'| < s\} \times (-6R/7, 6R/7)$. Let a and σ be, respectively, a Lipschitz symmetric uniformly elliptic matrix-valued function and a real bounded function both defined in B_R. Assume that $D \subset B_R$ and let $f \in L^2(D)$ and $v \in H^1(D)$ be such that*

$$\operatorname{div}(a\nabla v) + \sigma v = f \text{ in } D.$$

Define $\Gamma = \{(x', x_d) \in \partial D;\ |x_d| \le \frac{5}{6}R\}$. For any $0 < \alpha < 1$, there exists a constant $m > 1$ depending only on the Lipschitz and the elliptic constants of a such that if $R > ms$ then, for some positive constant $C_{s,R}$ independent of v and f,

$$\|v\|_{H^1(B_{2s}\cap D)} \le C_{s,R}\big(\|v\|_{\mathbf{H}(\Gamma)} + \|f\|_{L^2(D)}\big)^{\alpha}\big(\|v\|_{H^1(D)} + \|f\|_{L^2(D)}\big)^{1-\alpha}. \tag{2.18}$$

Remark 2.4. Lemma 2.2 can be considered as a three-sphere–inequality-type with partial data since no information of v on $\partial D \setminus \Gamma$ is required. Lemma 2.2 plays an important role in the proof of Theorem 2.1.

Proof. We first consider the case $f = 0$. Fix $\varphi \in C^1_{\rm c}(\mathbb{R}^d)$ such that $\varphi = 1$ on $B_{2R/3}$ and $\operatorname{supp}\varphi \subset B_{3R/4}$. Let $w \in H^1(B_{4R/5} \setminus \Gamma)$ be such that

$$\operatorname{div}(a\nabla w) + \sigma w = 0 \text{ in } B_{4R/5} \setminus \Gamma, \quad a\nabla w \cdot \nu + iw = 0 \text{ on } \partial B_{4R/5},$$
$$[w] = \varphi v, \quad [a\nabla w \cdot \nu] = \varphi a \nabla v \cdot \nu \text{ on } \Gamma.$$

Here and in what follows, on ∂D, $[u] := u|_{\mathbb{R}^d\setminus D} - u|_D$ for an appropriate function u; similar notation is used for $[a\nabla u \cdot \nu]$ on ∂D. We have

$$\|w\|_{H^1(B_{4R/5}\setminus\Gamma)} \le C\big(\|\varphi v\|_{H^{1/2}(\partial\Gamma)} + \|\varphi a\nabla v\cdot\nu\|_{H^{-1/2}(\Gamma)}\big). \tag{2.19}$$

From now in this proof, C denotes a positive constant independent of v. The uniqueness of w follows from the definition of w; here the boundary condition $a\nabla w\cdot\nu + iw = 0$ on $\partial B_{4R/5}$ is used. The existence and the estimate of w follow from the uniqueness of w and the Fredholm theory. Since

$$\|\varphi v\|_{H^{1/2}(\Gamma)} + \|\varphi a\nabla v\cdot\nu\|_{H^{-1/2}(\Gamma)} \le C\|v\|_{\mathbf{H}(\Gamma)},$$

it follows from (2.19) that

$$\|w\|_{H^1(B_{4R/5}\setminus\Gamma)} \le C\|v\|_{\mathbf{H}(\Gamma)}. \tag{2.20}$$

Define

$$V = \begin{cases} w + v & \text{in } D,\\ w & \text{in } B_{4R/5}\setminus D. \end{cases} \tag{2.21}$$

Note that

$$\|v\|_{\mathbf{H}(\Gamma)} \le C_{s,R}\|v\|_{H^1(D)}. \tag{2.22}$$

Since $\varphi = 1$ in $B_{2R/3}$, it follows from (2.20), (2.21), and (2.22) that $V \in H^1(B_{2R/3})$,

$$\|V\|_{H^1(B_{2R/3})} \le C\|v\|_{H^1(D)}, \tag{2.23}$$

and

$$\operatorname{div}(a\nabla V) + \sigma V = 0 \text{ in } B_{2R/3}. \tag{2.24}$$

Set $z = (R/\sqrt{m}, 0, \dots, 0)$, $R_1 = (R/\sqrt{m}) - s$, $R_2 = (R/\sqrt{m}) + 4s$, and $R_3 = R/3$. Applying Lemma 2.1, we have

$$\|V\|_{\mathbf{H}\left(\partial B(z,R_2)\right)} \le C\|V\|^{\alpha}_{\mathbf{H}\left(\partial B(z,R_1)\right)}\|V\|^{1-\alpha}_{\mathbf{H}\left(\partial B(z,R_3)\right)} \tag{2.25}$$

with

$$\alpha = \frac{R_2^{-q} - R_3^{-q}}{R_1^{-q} - R_3^{-q}} \tag{2.26}$$

for some $q > 1$ independent of v, R, m, and s. From (2.24), we derive from Lemma 2.3 below that, for some positive constant $C > 1$,

$$C^{-1}\|V\|_{H^1\left(B(z,r)\right)} \le \|V\|_{\mathbf{H}\left(\partial B(z,r)\right)} \le C\|V\|_{H^1\left(B(z,r)\right)} \tag{2.27}$$

with $r = R_1,\ R_2$, and R_3. It follows from (2.25) that

$$\|V\|_{H^1\left(B(z,R_2)\right)} \le C\|V\|^{\alpha}_{H^1\left(B(z,R_1)\right)}\|V\|^{1-\alpha}_{H^1\left(B(z,R_3)\right)}. \tag{2.28}$$

Note that α can be chosen arbitraly close to 1 by taking $R > ms$ and large m in (2.26). The conclusion in the case $f = 0$ follows from (2.20), (2.23), and (2.28) by the definition of V in (2.21); in particular, $V = w$ in $B(z, R_1)$.

We next consider the case $f \in L^2(D)$. Let $v_1 \in H^1(D)$ be such that

$$\operatorname{div}(a\nabla v_1) + \sigma v_1 = f \text{ in } D \quad \text{ and } \quad a\nabla v_1 \cdot \nu + iv_1 = 0 \text{ on } \partial D.$$

Then

$$\|v_1\|_{H^1(D)} \le C\|f\|_{L^2} \quad \text{ and } \quad \|v_1\|_{\mathbf{H}(\partial D)} \le C\|f\|_{L^2(D)}.$$

The conclusion in the general case now follows from the case $f = 0$ by applying the previous case for $v - v_1$. The proof is complete. ∎

In the proof of Lemma 2.2, we use the following result.

LEMMA 2.3. *Let $d = 2, 3$, $R > 0$, a and σ be, respectively, a Lipschitz symmetric uniformly elliptic matrix-valued function and a real bounded function both defined in B_R. Let $v \in H^1(B_R)$ be such that*

$$\operatorname{div}(a\nabla v) + \sigma v = 0 \textit{ in } B_R.$$

Then, for some positive constant $C > 1$, independent of v,

$$C^{-1}\|v\|_{H^1(B_R)} \le \|v\|_{\mathbf{H}(\partial B_R)} \le C\|v\|_{H^1(B_R)}. \tag{2.29}$$

Proof. The second inequality of (2.29) is a consequence of the trace theory. It remains to establish the first inequality of (2.29). We first prove that

$$\|v\|_{L^2(B_R)} \le C\|v\|_{\mathbf{H}(\partial B_R)} \tag{2.30}$$

by contradiction. In this proof, C denotes a positive constant independent of v and n. Suppose that this is not true. There exists some sequence $(v_n) \subset H^1(B_R)$ such that

$$\operatorname{div}(a\nabla v_n) + \sigma v_n = 0 \text{ in } B_R \tag{2.31}$$

and

$$n\|v_n\|_{\mathbf{H}(\partial B_R)} \le \|v_n\|_{L^2(B_R)} = 1. \tag{2.32}$$

Multiplying the equation of v_n (2.31) by $\bar{v}_n$ (the conjugate of v_n), integrating in B_R, and using (2.32), we obtain

$$\int_{B_R} |\nabla v_n|^2 \le C \int_{B_R} |v_n|^2 + C.$$

Thus (v_n) is bounded in $H^1(B_R)$. Without loss of generality, one may assume that (v_n) converges to v weakly in $H^1(B_R)$ and strongly in $L^2(B_R)$ for some $v \in H^1(B_R)$. It follows from (2.31) and (2.32) that

$$\operatorname{div}(a\nabla v) + \sigma v = 0 \text{ in } B_R \quad \text{ and } \quad \|v\|_{\mathbf{H}(\partial B_R)} = 0.$$

By the unique continuation principle, we have $v = 0$ in B_R. This contradicts the fact $\|v\|_{L^2(B_R)} = \lim_{n\to+\infty} \|v_n\|_{L^2(B_R)} = 1$. Hence (2.30) is established.

The first inequality of (2.29) is now a consequence of (2.30) after multiplying the equation of v by $\bar{v}$ and integrating in B_R. ∎

We next recall a stability result of (2.5) established in [19, Lemma 2.1] (see [21, Lemma 7] for a variant in a bounded domain).

LEMMA 2.4. *Let $d = 2, 3$, $k > 0$, $\delta_0 > 0$, $R_0 > 0$, $f \in L^2(\mathbb{R}^d)$ with support in B_{R_0}. For $0 < \delta < \delta_0$, there exists a unique outgoing solution $u_\delta \in H^1_{_{loc}}(\mathbb{R}^d)$ of (2.5). Moreover,*

$$\|u_\delta\|^2_{H^1(B_R)} \le \frac{C_R}{\delta}\Big|\int f\bar{u}_\delta\Big| + C_R\|f\|^2_{L^2} \tag{2.33}$$

for some positive constant C_R independent of f and δ.

We end this section by recalling the following change of variables formula [11, Lemma 2] which is used several times in this paper.

LEMMA 2.5. *Let $D_1 \subset\subset D_2 \subset\subset D_3$ be three smooth bounded open subsets of $\mathbb{R}^d$. Let $a \in [L^\infty(D_2 \setminus D_1)]^{d\times d}$, $\sigma \in L^\infty(D_2 \setminus D_1)$, and let $\mathcal{T}$ be a bijective from $D_2 \setminus \bar{D}_1$ onto $D_3 \setminus \bar{D}_2$ such that $\mathcal{T} \in C^1(\bar{D}_2 \setminus D_1)$ and $\mathcal{T}^{-1} \in C^1(\bar{D}_3 \setminus D_2)$. Assume that $u \in H^1(D_2 \setminus D_1)$ and set $v = u \circ \mathcal{T}^{-1}$. Then*

$$\operatorname{div}(a\nabla u) + \sigma u = f \text{ in } D_2 \setminus D_1$$

for some $f \in L^2(D_2 \setminus D_1)$ if and only if

$$\operatorname{div}(\mathcal{T}_* a\nabla v) + \mathcal{T}_*\sigma v = \mathcal{T}_* f \text{ in } D_3 \setminus D_2. \tag{2.34}$$

Assume in addition that $\mathcal{T}(x) = x$ *on* ∂D_2*. Then*

$$v = u \quad \text{and} \quad \mathcal{T}_* a \nabla v \cdot \nu = -a \nabla u \cdot \nu \text{ on } \partial D_2. \tag{2.35}$$

Recall that $\mathcal{T}_* a$, $\mathcal{T}_* \sigma$, and $\mathcal{T}_* f$ are defined in (2.8).

2.2. Proof of Theorem 2.1. The proof uses the reflecting and removing localized singularity techniques introduced in [14, 15]. Applying Lemma 2.4, we have

$$\|u_\delta\|_{H^1(B_R)} \le C\Big(\delta^{-1/2} \|f\|_{L^2}^{1/2} \|u_\delta\|_{L^2(\operatorname{supp} f)}^{1/2} + \|f\|_{L^2}\Big). \tag{2.36}$$

Here and in what follows in this proof, C denotes a positive constant independent of δ and f. Set

$$S = H_{2r_1} \setminus H_{3r_1}.$$

Define

$$u_{1,\delta} = u_\delta \circ F^{-1} \text{ in } \mathbb{R}^d \setminus \Omega_2 \quad \text{and} \quad u_{2,\delta} = u_{1,\delta} \circ G^{-1} \text{ in } B_{r_3},$$

where F and G come from the definition of doubly complementary media. Applying Lemma 2.5, we obtain, since $G_* F_* A = F_* A = A$ in $B_{r_3} \setminus \Omega_2$ (the medium is of doubly complementary property),

$$u_{1,\delta} = u_\delta, \quad (1 + i\delta) A \nabla u_{1,\delta} \cdot \nu = A \nabla u_\delta \big|_{B_{r_3} \setminus \Omega_2} \cdot \nu \text{ on } \partial \Omega_2 \setminus \partial S, \tag{2.37}$$

$$u_{2,\delta} = u_{1,\delta}, \quad \text{and} \quad A \nabla u_{2,\delta} = (1 + i\delta) A \nabla u_{1,\delta} \big|_{B_{r_3} \setminus \Omega_2} \cdot \nu \text{ on } \partial B_{r_3}. \tag{2.38}$$

This implies

$$\|u_{1,\delta} - u_\delta\|_{\mathbf{H}(\partial \Omega_2 \setminus \partial S)} \le C\delta \|u_\delta\|_{H^1(B_{r_3})} \tag{2.39}$$

and

$$\|u_{2,\delta} - u_{1,\delta}\|_{\mathbf{H}(\partial B_{r_3})} \le C\delta \|u_\delta\|_{H^1(B_{r_3})}. \tag{2.40}$$

Applying Lemma 2.5, we have, since $F_* A = A$ and $F_* \Sigma = \Sigma$ in $B_{r_3} \setminus \Omega_2$ and $f = 0$ in B_{r_3},

$$\operatorname{div}\big((1 + i\delta) A \nabla u_{1,\delta}\big) + k^2 \Sigma u_{1,\delta} = 0 \text{ in } B_{r_3} \setminus \Omega_2,$$

which yields

$$\operatorname{div}(A \nabla u_{1,\delta}) + k^2 \Sigma u_{1,\delta} = \left(1 - \frac{1}{1 + i\delta}\right) k^2 \Sigma u_{1,\delta} = -\frac{i\delta}{1 + i\delta} k^2 \Sigma u_{1,\delta} \text{ in } B_{r_3} \setminus \Omega_2. \tag{2.41}$$

Recall that

$$\operatorname{div}(A \nabla u_\delta) + k^2 \Sigma u_\delta = 0 \text{ in } (B_{r_3} \setminus \Omega_2) \setminus S. \tag{2.42}$$

Set $\alpha = \gamma + 1/2 < 1$. Applying Lemma 2.2 with $D = \big\{x' \in \mathbb{R}^{d-1}; |x'| < s\big\} \times (-6r_1/7, 6r_1/7)$ and $v(x) = u_{1,\delta}(x + z) - u_\delta(x + z)$ for $x \in D$, we derive from (2.39), (2.41), and (2.42) that if $r_1 > ms$ and m is sufficiently large then

$$\|u_{1,\delta} - u_\delta\|_{H^1\big((H_{22r_1/7} \setminus H_{34r_1/7}) \cap B(z, 2s)\big)} \le C\delta^\alpha \|u_\delta\|_{H^1(B_{r_3})}. \tag{2.43}$$

Define $O_2 = B_{r_2} \setminus H_{4r_1}$ and $O = B_{r_3} \setminus O_2$, and set

$$\mathcal{U}_\delta = \begin{cases} u_\delta & \text{in } \mathbb{R}^d \setminus B_{r_3}, \\ u_{2,\delta} + u_\delta - u_{1,\delta} & \text{in } O, \\ u_{2,\delta} & \text{in } O_2. \end{cases}$$

Then $\mathcal{U}_\delta \in H^1\big(B_R \setminus \partial O\big)$ for all $R > 0$ and U_δ is an outgoing solution of the equation

$$\operatorname{div}(\hat{A}\nabla \mathcal{U}_\delta) + k^2 \hat{\Sigma} \mathcal{U}_\delta = f \text{ in } \mathbb{R}^d \setminus \partial O.$$

Note that $\hat{A}$ is uniformly elliptic and $\hat{\Sigma}$ is bounded above and below by positive constants. It follows that

$$\|\mathcal{U}_\delta\|_{H^1(B_R\setminus\partial O)} \le C_R\Big(\|[\mathcal{U}_\delta]\|_{H^{1/2}(\partial O)} + \|[\hat{A}\nabla\mathcal{U}_\delta\cdot\nu]\|_{H^{-1/2}(\partial O)}\Big). \tag{2.44}$$

By the definition of $\mathcal{U}_\delta$, we have

$$\begin{aligned}\|[\mathcal{U}_\delta]\|_{H^{1/2}(\partial O_2)} &+ \|[\hat{A}\nabla\mathcal{U}_\delta\cdot\nu]\|_{H^{-1/2}(\partial O_2)} \\ &= \|u_\delta - u_{1,\delta}\|_{H^{1/2}(\partial O_2)} + \|A\nabla(u_\delta - u_{1,\delta})\cdot\nu\|_{H^{-1/2}(\partial O_2)}\end{aligned}$$

and

$$\begin{aligned}\|[\mathcal{U}_\delta]\|_{H^{1/2}(\partial B_{r_3})} &+ \|[\hat{A}\nabla\mathcal{U}_\delta\cdot\nu]\|_{H^{-1/2}(\partial B_{r_3})} \\ &= \|u_{2,\delta} - u_{1,\delta}\|_{H^{1/2}(\partial B_{r_3})} + \|A\nabla(u_{2,\delta} - u_{1,\delta})\cdot\nu\|_{H^{-1/2}(\partial B_{r_3})}.\end{aligned}$$

Since $\partial O = \partial B_{r_3} \cup \partial O_2$, we derive from (2.39), (2.40), and (2.43) that

$$\|[\mathcal{U}_\delta]\|_{H^{1/2}(\partial O)} + \|[\hat{A}\nabla\mathcal{U}_\delta\cdot\nu]\|_{H^{-1/2}(\partial O)} \le C\delta^\alpha \|u_\delta\|_{H^1(B_{r_3})}. \tag{2.45}$$

It follows from (2.36) and (2.45) that, for $R > R_0$,

$$\|\mathcal{U}_\delta\|_{H^1(B_R\setminus\partial O)} \le C_R\delta^\alpha\Big(\delta^{-1/2}\|\mathcal{U}_\delta\|^{1/2}_{L^2(B_R\setminus B_{r_3})}\|f\|^{1/2}_{L^2} + \|f\|_{L^2}\Big) + C_R\|f\|_{L^2},$$

since $\operatorname{supp} f \subset B_R \setminus B_{r_3}$. Since $\alpha = \gamma + 1/2 > 1/2$, we obtain

$$\|\mathcal{U}_\delta\|_{H^1(B_R\setminus\partial O)} \le C_R\|f\|_{L^2}. \tag{2.46}$$

Since $\operatorname{supp} f \cap B_{r_3} = \emptyset$, we derive from (2.36) that, for $R > 0$,

$$\|u_\delta\|_{H^1(B_R)} \le C_R\delta^{-1/2}\|f\|_{L^2}, \tag{2.47}$$

which yields, by (2.45),

$$\|[\mathcal{U}_\delta]\|_{H^{1/2}(\partial O)} + \|[\hat{A}\nabla\mathcal{U}_\delta\cdot\nu]\|_{H^{-1/2}(\partial O)} \le C\delta^{\alpha-1/2}\|f\|_{L^2}. \tag{2.48}$$

Hence $\mathcal{U}_\delta$ is bounded in $H^1(B_R \setminus \partial O)$. Without loss of generality, one may assume that $\mathcal{U}_\delta \rightharpoonup \mathcal{U}$ weakly in $H^1(B_R \setminus \partial O)$ as $\delta \to 0$ for any $R > 0$; moreover, $\mathcal{U} \in H^1_{loc}(\mathbb{R}^d)$ is the unique outgoing solution to the equation

$$\operatorname{div}(\hat{A}\nabla\mathcal{U}) + k^2\hat{\Sigma}\mathcal{U} = f \text{ in } \mathbb{R}^d.$$

Hence $\mathcal{U} = \hat{u}$ in $\mathbb{R}^d$. Since the limit is unique, the convergence holds for the whole family $(\mathcal{U}_\delta)$ as $\delta \to 0$. In other words, $\mathcal{U}_\delta \to \hat{u}$ in $H^1_{loc}(\mathbb{R}^d)$.

To obtain the rate of convergence, let us consider the equation of $U_\delta - \hat{u}$. We have

$$\operatorname{div}\big(\hat{A}\nabla(\mathcal{U}_\delta - \hat{u})\big) + k^2\hat{\Sigma}(\mathcal{U}_\delta - \hat{u}) = 0 \text{ in } \mathbb{R}^d \setminus \partial O.$$

As in (2.44), we have

$$\|\mathcal{U}_\delta - \hat{u}\|_{H^1(B_R\setminus\partial O)} \le C_R\Big(\|[\mathcal{U}_\delta]\|_{H^{1/2}(\partial O)} + \|[\hat{A}\nabla\mathcal{U}_\delta\cdot\nu]\|_{H^{-1/2}(\partial O)}\Big). \tag{2.49}$$

The conclusion now follows from (2.48) and (2.49). ∎

Remark 2.5. The power of u_δ defined by $\int_{\Omega_2\setminus\Omega_1}\delta|\nabla u_\delta|^2$ is finite in the setting considered in Theorem 2.1 by (2.47).

Remark 2.6. It is showed in [16] that resonance takes place for reflecting complementary media. Various conditions on the stability of the Helmholtz equations with sign changing coefficients were given there and the necessity of the reflecting complementary property of media for the occurrence of the resonance was discussed.

3. Cloaking an object via ALR in two dimensions and related problems.

This section containing three subsections is organized as follows. The proofs of Theorems 1.1 and 1.2 are given in the first subsection. The second subsection is on superlensing using complementary media. More precisely, we show that a modification on the superlensing schemes suggested in [23, 24, 25] is necessary, otherwise a lens can become a cloaking device (see Proposition 3.1). The last subsection is on cloaking using complementary media. We prove here that a modification on the schemes proposed in [7] is necessary to achieve cloaking (see Proposition 3.2).

In this section, we use the complex notations for several places, for example, the polar coordinate $z = re^{i\theta}$ is used and for $x \in \mathbb{R}$, $B(x,R)$ means $B(z,R)$ with $z = x$.

3.1. Proofs of Theorems 1.1 and 1.2.

We first state and prove a variant of Lemma 2.2.

LEMMA 3.1. *Let* $0 < R \le R_1$, $m \in \mathbb{N}$, *and define* $D = T(B_R \setminus B(-R_1, R_1))$ *where* $T(z) = z^{1/m}$.[2] *Set* $\hat{D} = \{z \in D;\ \frac{1}{10}R^{1/m} < |z| < R^{1/m}\}$ *and* $\Gamma = \{z \in \partial D;\ \frac{1}{10}R^{1/m} < |z| < R^{1/m}\}$. *Let* σ *be a real bounded function defined in* $\hat{D}$, $f \in L^2(\hat{D})$, *and* $v \in H^1(\hat{D})$ *be such that*

$$\Delta v + \sigma v = f \text{ in } \hat{D}.$$

For any $0 < \alpha < 1$, *there exists* $m_0 \in \mathbb{N}$ *depending only on* R *such that if* $m > m_0$ *then, for some neighborhood* D_1 *of* $\{z \in \bar{D};\ |z| = \frac{1}{2}R^{1/m}\}$ *and for some positive constant* C *both independent of* v *and* f,

$$\|v\|_{H^1(D_1\cap D)} \le C\big(\|v\|_{\mathbf{H}(\Gamma)} + \|f\|_{L^2(\hat{D})}\big)^\alpha\big(\|v\|_{H^1(\hat{D})} + \|f\|_{L^2(\hat{D})}\big)^{1-\alpha}.$$

Proof. The proof of Lemma 3.1 is similar to the one of Lemma 2.2. For the convenience of the reader, we present the proof. We first consider the case $f = 0$

[2]The following definition of $T(z)$ is used in this paper: for $z = re^{i\theta}$ with $r \ge 0$, $\theta \in (-\pi,\pi)$, we define $T(z) = r^{1/m}e^{i\theta/m}$.

in $\hat{D}$. For the simplicity of notations, set $\hat{R} = R^{1/m}$. We first assume that m is sufficiently large such that $1/2 < \hat{R} < 2$; this is possible since $\lim_{m\to+\infty} R^{1/m} = 1$. Fix $\varphi \in C^1(\mathbb{R}^2)$ such that $\varphi = 1$ in $B_{\hat{R}/4}$ and $\operatorname{supp}\varphi \subset B_{7\hat{R}/24}$, and define $\varphi_m(z) = \varphi(z - \hat{R}/2)$. Set $O = B(\hat{R}/2, \hat{R}/3)$, the disk centered at $\hat{R}/2$ and of radius $\hat{R}/3$, and extend σ by 1 in $O \setminus \hat{D}$. We still denote the extension of σ by σ for simplicity of notations. Let $w \in H^1(O \setminus \Gamma)$ be the unique solution of the system

$$\Delta w + \sigma w = 0 \text{ in } O \setminus \Gamma, \quad \partial_\nu w + iw = 0 \text{ on } \partial O,$$
$$[w] = \varphi_m v, \quad \text{ and } \quad [\partial_\nu w] = \varphi_m \partial_\nu v \text{ on } \Gamma.$$

We have

$$\|w\|_{H^1(O\setminus\Gamma)} \le C\big(\|\varphi_m v\|_{H^{1/2}(\Gamma)} + \|\varphi_m \partial_\nu v\|_{H^{-1/2}(\Gamma)}\big).$$

Here and in what follows C denotes a positive constant independent of v. The uniqueness of w follows from the definition of w; here the boundary condition $a\nabla w\cdot\nu + iw = 0$ on ∂O is used. The existence and the estimate of w follow from the uniqueness of w and the Fredholm theory. Since

$$\|\varphi_m v\|_{H^{1/2}(\Gamma)} + \|\varphi_m \partial_\nu v\|_{H^{-1/2}(\Gamma)} \le C\|v\|_{\mathbf{H}(\Gamma)},$$

it follows that

$$\|w\|_{H^1(O\setminus\Gamma)} \le C\|v\|_{\mathbf{H}(\Gamma)}. \tag{3.1}$$

Define

$$V = \begin{cases} w + v & \text{in } \hat{D}, \\ w & \text{in } O \setminus \hat{D}. \end{cases} \tag{3.2}$$

Then $V \in H^1(O)$,

$$\Delta V + \sigma V = 0 \text{ in } O,$$

and

$$\|V\|_{H^1(O)} \le C\|v\|_{H^1(\hat{D})}. \tag{3.3}$$

Set $z = \hat{R}/2 + i\hat{R}/\sqrt{m}$, $R_1 = \hat{R}/\sqrt{m} - 2\pi\hat{R}/m$, $R_2 = \hat{R}/\sqrt{m} + 2\pi\hat{R}/m$, and $R_3 = \hat{R}/4$. Applying Lemma 2.1, we have

$$\|V\|_{H^1\left(B(z,R_2)\right)} \le C\|V\|^{\alpha}_{H^1\left(B(z,R_1)\right)}\|V\|^{1-\alpha}_{H^1\left(B(z,R_3)\right)}, \tag{3.4}$$

where

$$\alpha = \frac{R_2^{-q} - R_3^{-q}}{R_1^{-q} - R_3^{-q}}$$

for some $q > 1$ independent of v, $\hat{R}$, and m. Here we use the fact that, by Lemma 2.3,

$$C^{-1}\|V\|_{H^1\left(B(z,r)\right)} \le \|V\|_{\mathbf{H}\left(\partial B(z,r)\right)} \le C\|V\|_{H^1\left(B(z,r)\right)}$$

with $r = R_1, R_2$, and R_3. The conclusion in the case $f = 0$ follows from (3.1), (3.3), (3.4), and the definition of V in (3.2) by noting that $B(z, R_1) \subset O \setminus \hat{D}$ (hence $V = w$

in $B(z, R_1)$), $B(z, R_2)$ contains a neighborhood of $\{z \in \bar{D};\ |z| = \hat{R}/2\}$, and α can be taken aribitrary close to 1 if m is large enough.

The conclusion in the general case follows from the case $f = 0$ by applying the result in the case $f = 0$ for $v - v_1$, where $v_1 \in H^1(\hat{D})$ is the unique solution of the system

$$\Delta v_1 + \sigma v_1 = f \text{ in } \hat{D} \quad \text{ and } \quad \partial_\nu v_1 + i v_1 = 0 \text{ on } \partial \hat{D}.$$

The proof is complete. ∎

As a consequence of Lemma 2.2, we have the following.

COROLLARY 3.1. *Let $0 < r < R \le R_1$, and define $\mathcal{D} = B_R \setminus B(-R_1, R_1)$. Set $\widetilde{\mathcal{D}} = \{z \in \mathcal{D}; r < |z| < R\}$. Let σ be a real bounded function defined in $\widetilde{\mathcal{D}}$, $f \in L^2(\widetilde{\mathcal{D}})$, and $v \in H^1(\widetilde{\mathcal{D}})$ be such that*

$$\Delta v + \sigma v = f \ \text{ in } \widetilde{\mathcal{D}}.$$

For any $0 < \alpha < 1$, there exist two positive constants $r_0(\alpha) < R_0(\alpha) < R$ depending only on R such that if $r < r_0(\alpha)$, then, for some neighborhood $\mathcal{D}_1$ of $\{z \in \overline{\mathcal{D}};\ |z| = R_0(\alpha)\}$ and for some positive constant C, both independent of v and f,

$$\|v\|_{H^1(\mathcal{D}_1 \cap \mathcal{D})} \le C\big(\|v\|_{\mathbf{H}(\Gamma_1)} + \|f\|_{L^2(\widetilde{\mathcal{D}})}\big)^\alpha \big(\|v\|_{H^1(\widetilde{\mathcal{D}})} + \|f\|_{L^2(\widetilde{\mathcal{D}})}\big)^{1-\alpha},$$

where $\Gamma_1 = \{z \in \partial \mathcal{D};\ r_0(\alpha) < |z| < R\}$.

Proof. Define $T(z) = z^{1/m}$ for some $m \in \mathbb{N}$. Set $v = u \circ T^{-1}$, $D = T(\widetilde{\mathcal{D}})$, and $\hat{D} = \{z \in D;\ \frac{1}{10} R^{1/m} < |z| < R^{1/m}\}$. Then $v \in H^1(\hat{D})$ satisfies the equation

$$\Delta v + \tilde{\sigma} v = \tilde{f} \text{ in } \hat{D},$$

where, by Lemma 2.5,

$$\tilde{\sigma} v = T_* \sigma v / c \text{ and } \tilde{f} = T_* f / c$$

for some positive constant c such that $T_* I = cI$ (such a constant c exists since T is a conformal map) if $r^{1/m} < \frac{1}{10} R^{1/m}$. Set

$$\Gamma = \left\{ z \in \partial D;\ \frac{1}{10} R^{1/m} < |z| < R^{1/m} \right\}.$$

By Lemma 2.2, for m large enough, we have

$$\|v\|_{H^1(D_1 \cap D)} \le C\big(\|v\|_{\mathbf{H}(\Gamma)} + \|f\|_{L^2(\hat{D})}\big)^\alpha \big(\|v\|_{H^1(\hat{D})} + \|f\|_{L^2(\hat{D})}\big)^{1-\alpha} \tag{3.5}$$

for some neighborhood D_1 of $\{z \in \bar{D};\ |z| = \frac{1}{2} R^{1/m}\}$ if $r^{1/m} < \frac{1}{10} R^{1/m}$. Fix such an m. One can now choose $r_0(\alpha) = \frac{1}{10^m} R$, $R_0(\alpha) = \frac{1}{2^m} R$, and $\mathcal{D}_1 = T^{-1}(D_1)$, and the conclusion follows from (3.5). ∎

A variant of Lemma 3.1 is the following.

LEMMA 3.2. *Let $0 < R \le R_1$, $m \in \mathbb{N}$, and define $D = T(B_R \cap B(R_1, R_1))$, where $T(z) = z^{1/m}$. $\hat{D} = \{z \in D;\ \frac{1}{10} R^{1/m} < |z| < R^{1/m}\}$ and $\Gamma = \{z \in \partial D;\ \frac{1}{10} R^{1/m} < |z| < R^{1/m}\}$. Let σ be a real bounded function defined in $\hat{D}$, $f \in L^2(\hat{D})$, and $v \in H^1(\hat{D})$ be such that*

$$\Delta v + \sigma v = f \ \text{ in } \hat{D}.$$

For any $0 < \alpha < 1$, *there exists* $m_0 \in \mathbb{N}$ *depending only on* R *such that if* $m > m_0$ *then, for some neighborhood* D_1 *of* $\{z \in \bar{D};\ |z| = \frac{1}{2}R^{1/m}\}$ *and for some positive constant* C *both independent of* v *and* f,

$$\|v\|_{H^1(D_1\cap D)} \le C\big(\|v\|_{\mathbf{H}(\Gamma)} + \|f\|_{L^2(\hat{D})}\big)^{\alpha}\big(\|v\|_{H^1(\hat{D})} + \|f\|_{L^2(\hat{D})}\big)^{1-\alpha}.$$

Proof. The proof of Lemma 3.2 is similar to the one of Lemma 3.1. The details are left to the reader. ∎

As a consequence of Lemma 3.1, we obtain the following.

COROLLARY 3.2. *Let* $0 < r < R \le R_1$ *and define* $\mathcal{D} = B_R \cap B(R_1, R_1)$. *Set* $\widetilde{\mathcal{D}} = \{z \in \mathcal{D};\ r < |z| < R\}$. *Let* σ *be a real bounded function defined in* $\widetilde{\mathcal{D}}$, $f \in L^2(\widetilde{D})$, *and* $v \in H^1(\widetilde{\mathcal{D}})$ *be such that*

$$\Delta v + \sigma v = f \ \text{ in } \widetilde{\mathcal{D}}.$$

For any $0 < \alpha < 1$, *there exist two positive constants* $r_0(\alpha) < R_0(\alpha) < R$ *depending only on* R *such that if* $r < r_0(\alpha)$ *then, for some neighborhood* $\mathcal{D}_1$ *of* $\{z \in \overline{\mathcal{D}};\ |z| = R_0(\alpha)\}$ *and for some positive constant* C *both independent of* v *and* f,

$$\|v\|_{H^1(\mathcal{D}_1\cap\mathcal{D})} \le C\big(\|v\|_{\mathbf{H}(\Gamma_1)} + \|f\|_{L^2(\widetilde{\mathcal{D}})}\big)^{\alpha}\big(\|v\|_{H^1(\widetilde{\mathcal{D}})} + \|f\|_{L^2(\widetilde{\mathcal{D}})}\big)^{1-\alpha},$$

where $\Gamma_1 = \{z \in \partial\mathcal{D};\ r_0(\alpha) < |z| < R\}$.

Proof. The proof of Corollary 3.2 is similar to the one of Corollary 3.1. The details are left to the reader. ∎

We are ready to present the following proof.

Proof of Theorem 1.2. The proof of Theorem 1.2 is in the same spirit of the one of Theorem 2.1. Applying Lemma 2.4, we have

$$\|u_\delta\|_{H^1(B_R)} \le C\Big(\delta^{-1/2}\|f\|_{L^2}^{1/2}\|u_\delta\|_{L^2(\operatorname{supp} f)}^{1/2} + \|f\|_{L^2}\Big). \tag{3.6}$$

Here and in what follows in this proof, C denotes a positive constant independent of δ and f.

Let F and G be the Kelvin transform with respect to ∂B_{r_2} and ∂B_{r_3}. Define

$$u_{1,\delta} = u_\delta \circ F^{-1} \text{ in } \mathbb{R}^2 \setminus B_{r_2} \quad \text{ and } \quad u_{2,\delta} = u_{1,\delta} \circ G^{-1} \text{ in } B_{r_3}.$$

Set

$$S = (B_{r_3} \setminus B_{r_2}) \cap \Big(B(x_2, r_0) \cup (G \circ F)\big(B(x_1, r_0) \cap B_{r_1}\big)\Big).$$

By Lemma 2.5, we have

$$u_{1,\delta} = u_\delta \quad \text{ and } \quad (1 + i\delta)\partial_\nu u_{1,\delta} = \partial_\nu u_\delta \text{ on } \partial B_{r_2} \setminus \partial S \tag{3.7}$$

and

$$u_{2,\delta} = u_{1,\delta} \quad \text{ and } \quad \partial_\nu u_{2,\delta} = (1 + i\delta)\partial_\nu u_{1,\delta} \text{ on } \partial B_{r_3} \setminus \partial S. \tag{3.8}$$

Combining (3.6), (3.7), and (3.8) yields

$$\|u_{1,\delta} - u_\delta\|_{\mathbf{H}(\partial B_{r_2}\setminus \partial S)} \le C\delta\|u_\delta\|_{H^1(B_{r_3})} \tag{3.9}$$

and

$$\|u_{2,\delta} - u_{1,\delta}\|_{\mathbf{H}(\partial B_{r_3} \setminus \partial S)} \leq C\delta \|u_\delta\|_{H^1(B_{r_3})}. \tag{3.10}$$

Applying Lemma 2.5, we have

$$\operatorname{div}\big((1+i\delta)\nabla u_{1,\delta}\big) + k^2 u_{1,\delta} = 0 \text{ in } B_{r_3} \setminus B_{r_2}$$

and

$$\Delta u_{2,\delta} + k^2 u_{2,\delta} = 0 \text{ in } B_{r_3} \setminus (G \circ F)\big(B(x_1, r_0) \cap B_{r_1}\big).$$

We derive that

$$\Delta u_{1,\delta} + k^2 u_{1,\delta} = \Big(1 - \frac{1}{1+i\delta}\Big) k^2 u_{1,\delta} = -\frac{i\delta}{1+i\delta} k^2 u_{1,\delta} \text{ in } B_{r_3} \setminus B_{r_2}. \tag{3.11}$$

Recall that

$$\Delta u_\delta + k^2 u_\delta = 0 \text{ in } (B_{r_3} \setminus B_{r_2}) \setminus \mathcal{C}. \tag{3.12}$$

Denote $x_3 \in \partial B_{r_3}$ the image of x_1 by F and set $\alpha = \gamma + 1/2$. We claim that there exist three positive constants $r_0(\alpha), R_2(\alpha), R_3(\alpha)$, depending only on r_2 and r_3, with $r_0(\alpha) < R_2(\alpha) = R_2$, $r_0(\alpha) < R_3(\alpha) = R_3$, such that, if $r_0 < r_0(\alpha)$ then for some neighborhood D_2 of $\{z \in \mathbb{R}^2 \setminus B_{r_2};\ |z - x_2| = R_2\}$ and some neighborhood D_3 of $\{z \in \bar{B}_{r_3};\ |z - x_3| = R_3\}$,

$$\|u_{1,\delta} - u_\delta\|_{H^1\left(D_2 \cap (B_{r_3} \setminus B_{r_2})\right)} \leq C\delta^\alpha \|u_\delta\|_{H^1(B_{r_3})} \tag{3.13}$$

and

$$\|u_{2,\delta} - u_{1,\delta}\|_{H^1\left(D_3 \cap (B_{r_3} \setminus B_{r_2})\right)} \leq C\delta^\alpha \|u_\delta\|_{H^1(B_{r_3})}. \tag{3.14}$$

We first derive (3.13) from Corollary 3.1. For simplicity of notations, assume that $x_2 = (|x_2|, 0)$. Applying Corollary 3.1 for $u_{1,\delta}(\cdot - x_2) - u_\delta(\cdot - x_2)$, $R_1 = r_2$, and $R = \min\{r_2, r_3 - r_2\}$, we obtain assertion (3.13). Assertion (3.14) can be obtained similarly by using Corollary 3.2.

Set

$$O_2 = B_{r_2} \cup \{|z - x_2| < R_2\}, \quad O_3 = B_{r_3} \setminus \{|z - x_3| < R_3\}, \quad \text{and} \quad O = O_3 \setminus O_2.$$

It follows from (3.9) and (3.13) that

$$\|u_{1,\delta} - u_\delta\|_{\mathbf{H}(\partial O_2)} \leq C\delta^\alpha \|u_\delta\|_{H^1(B_{r_3})} \tag{3.15}$$

and from (3.10) and (3.14) that

$$\|u_{2,\delta} - u_{1,\delta}\|_{\mathbf{H}(\partial O_3)} \leq C\delta^\alpha \|u_\delta\|_{H^1(B_{r_3})}. \tag{3.16}$$

Define

$$\mathcal{U}_\delta = \begin{cases} u_\delta & \text{in } \mathbb{R}^2 \setminus O_3, \\ u_{2,\delta} - (u_{1,\delta} - u_\delta) & \text{in } O, \\ u_{2,\delta} & \text{in } O_2. \end{cases} \tag{3.17}$$

Then $\mathcal{U}_\delta \in H^1\big(B_R \setminus \partial O\big)$ for all $R > 0$ and U_δ is an outgoing solution of the equation

$$\operatorname{div}(\hat{A}\nabla \mathcal{U}_\delta) + k^2 \hat{\Sigma} \mathcal{U}_\delta = f \text{ in } \mathbb{R}^2 \setminus \partial O$$

by the definition of $(\hat{A}, \hat{\Sigma})$ and Lemma 2.5. This implies

$$\|\mathcal{U}_\delta\|_{H^1(B_R \setminus \partial O)} \le C\Big(\|[\mathcal{U}_\delta]\|_{H^{1/2}(\partial O)} + \|[\hat{A}\nabla \mathcal{U}_\delta \cdot \nu]\|_{H^{-1/2}(\partial O)} + \|f\|_{L^2} \Big). \tag{3.18}$$

From the definition of $\mathcal{U}_\delta$ in (3.17), we derive from (3.15) and (3.16) that

$$\|[\mathcal{U}_\delta]\|_{H^{1/2}(\partial O)} + \|[\hat{A}\nabla \mathcal{U}_\delta \cdot \nu]\|_{H^{-1/2}(\partial O)} \le C\delta^\alpha \|u_\delta\|_{H^1(B_{r_3})}. \tag{3.19}$$

Combining (3.18) and (3.19) and using (3.6) yield, for large $R > 0$ such that $\operatorname{supp} f \subset B_R$,

$$\|\mathcal{U}_\delta\|_{H^1(B_R \setminus \partial O)} \le C_R \delta^\alpha \Big(\delta^{-1/2} \|\mathcal{U}_\delta\|^{1/2}_{L^2(B_R \setminus B_{r_3})} \|f\|^{1/2}_{L^2} + \|f\|_{L^2} \Big) + C\|f\|_{L^2},$$

which implies, since $\alpha > 1/2$,

$$\|\mathcal{U}_\delta\|_{H^1(B_R \setminus \partial O)} \le C_R \|f\|_{L^2}. \tag{3.20}$$

By (3.6), we obtain

$$\|u_\delta\|_{H^1(B_R)} \le C_R \delta^{-1/2} \|f\|_{L^2}. \tag{3.21}$$

We derive from (3.19) that

$$\|[\mathcal{U}_\delta]\|_{H^{1/2}(\partial O)} + \|[\hat{A}\nabla \mathcal{U}_\delta \cdot \nu]\|_{H^{-1/2}(\partial O)} \le C\delta^{\alpha - 1/2} \|f\|_{L^2}. \tag{3.22}$$

From (3.20) and (3.22), without loss of generality, one may assume that $\mathcal{U}_\delta \rightharpoonup \mathcal{U}$ weakly in $H^1\big(B_R \setminus \partial O\big)$ as $\delta \to 0$ for any $R > 0$; moreover, $\mathcal{U} \in H^1_{loc}(\mathbb{R}^2)$ is the unique outgoing solution to the equation

$$\operatorname{div}(\hat{A}\nabla \mathcal{U}) + k^2 \hat{\Sigma} \mathcal{U} = f \text{ in } \mathbb{R}^2.$$

Hence $\mathcal{U} = \hat{u}$ in $\mathbb{R}^2$. Since the limit is unique, the convergence holds for the whole family $(\mathcal{U}_\delta)$ as $\delta \to 0$.

By considering $\mathcal{U}_\delta - \hat{u}$, one obtains the rate of the convergence as in the proof of Theorem 2.1. The proof is complete. ∎

Remark 3.1. The power of u_δ defined by $\int_{\Omega_2 \setminus \Omega_1} \delta |\nabla u_\delta|^2$ is finite in the setting considered in Theorem 1.1 by (3.21).

Proof of Theorem 1.1. The proof of Theorem 1.1 is similar to the one of Theorem 1.2 though even simpler. We sketch here the main steps. We have

$$\|v_\delta\|^2_{H^1(B_R)} \le \frac{C_R}{\delta} \left| \int f \bar{v}_\delta \right| + C_R \|f\|^2_{L^2}.$$

This implies

$$\|u_\delta\|_{H^1(B_R)} \le C\Big(\delta^{-1/2} \|f\|^{1/2}_{L^2} \|u_\delta\|^{1/2}_{L^2(\operatorname{supp} f)} + \|f\|_{L^2} \Big). \tag{3.23}$$

Let F and G be the Kelvin transform with respect to ∂B_{r_2} and ∂B_{r_3}. Define

$$u_{1,\delta} = u_\delta \circ F^{-1} \text{ in } \mathbb{R}^2 \setminus B_{r_2} \quad \text{ and } \quad u_{2,\delta} = u_{1,\delta} \circ G^{-1} \text{ in } B_{r_3}.$$

Let O_2, O_3, and O be as in the proof of Theorem 1.2 and define U_δ by

$$\mathcal{U}_\delta = \begin{cases} u_\delta & \text{in } \mathbb{R}^2 \setminus O_3, \\ u_{2,\delta} - (u_{1,\delta} - u_\delta) & \text{in } O, \\ u_{2,\delta} & \text{in } O_2. \end{cases} \tag{3.24}$$

As in the proof of Theorem 1.2, one can prove that $\mathcal{U}_\delta \rightharpoonup \hat{u}$ weakly in $H^1(B_R \setminus \partial O)$ as $\delta \to 0$ for any $R > 0$; moreover, $\hat{u} \in H^1_{loc}(\mathbb{R}^2)$ is the unique outgoing solution to the equation

$$\operatorname{div}(\hat{A} \nabla \hat{u}) = f \text{ in } \mathbb{R}^2.$$

By considering the equation for $\mathcal{U}_\delta - \hat{u}$, one obtains the rate of convergence. ∎

3.2. A remark on superlensing using complementary media. In this section, we revisit the construction of lenses using complementary media proposed in [14] which has roots from [23, 25, 26]. We first consider the quasi-static regime. To magnify M times the region B_{τ_0} of coefficient a (a uniformly elliptic symmetric matrix-valued function) for some $\tau_0 > 0$ and $M > 1$, following [23, 25, 26] one puts a lens in $B_{r_2} \setminus B_{\tau_0}$ whose medium is characterized by matrix $-I$ with $r_2^2/\tau_0^2 = M$. The lens construction in [14] is related to but different from this. Our lens contains **two parts**. The first one is given by

$$-I \quad \text{in } B_{r_2} \setminus B_{r_1} \tag{3.25}$$

and the second one is

$$I \quad \text{in } B_{r_1} \setminus B_{\tau_0}. \tag{3.26}$$

Here r_1 and r_2 are such that

$$M\tau_0 = r_2 \quad \text{ and } \quad r_3/r_1 = M, \quad \text{ where } r_3 := r_2^2/r_1. \tag{3.27}$$

Other choices for the first and second layers are possible via the concept of complementary media (see [14]). Given $f \in L^2(\mathbb{R}^2)$ with compact support such that $\int_{\mathbb{R}^2} f = 0$, and $\operatorname{supp} f \cap B_{r_3} = \emptyset$, we showed in [14] that

$$u_\delta \to \tilde{u} \text{ in } H^1_{loc}(\mathbb{R}^2 \setminus \bar{B}_{r_3}) \text{ as } \delta \to 0.$$

Here u_δ and $\tilde{u}$ are the unique solutions in $W^1(\mathbb{R}^2)$ of the equations

$$\operatorname{div}(s_\delta A \nabla u_\delta) = f \text{ in } \mathbb{R}^2 \quad \text{ and } \quad \operatorname{div}(\tilde{A} \nabla \tilde{u}) = f \text{ in } \mathbb{R}^2,$$

where

$$A(x) = \begin{cases} a(x) & \text{in } B_{\tau_0}, \\ I & \text{otherwise}, \end{cases} \qquad s_\delta = \begin{cases} -1 - i\delta & \text{in } B_{r_2} \setminus B_{r_1}, \\ 1 & \text{otherwise}, \end{cases}$$

$$\tilde{A} = \begin{cases} a(x/M) & \text{in } B_{M\tau_0}, \\ I & \text{otherwise}. \end{cases}$$

The object in B_{τ_0} is magnified M times. In fact, in [14], we considered a bounded setting; nevertheless, the result stated here holds with a similar proof. In comparison with the schemes suggested previously, the second part in $B_{r_1} \setminus B_{\tau_0}$ is added. The second layer can be chosen thinner (see [14]). Nevertheless, the second layer in (3.26) is necessary. Indeed, we have, as a consequence of Theorem 1.1, the following proposition.

PROPOSITION 3.1. *Let $0 < r_0 < r_1 < r_2 < r_3$, $x_1 \in \partial B_{r_1}$, and let a_c be a symmetric uniformly elliptic matrix-valued function defined in $B(x_1, r_0) \cap B_{r_1}$. Define*

$$A_c = \begin{cases} a_c & \text{in } B(x_1, r_0) \cap B_{r_1}, \\ I & \text{otherwise}, \end{cases} \quad \text{and} \quad s_\delta = \begin{cases} -1 - i\delta & \text{in } B_{r_2} \setminus B_{r_1}, \\ 1 & \text{otherwise}. \end{cases} \tag{3.28}$$

Given $f \in L^2(\mathbb{R}^2)$ with compact support such that $\int_{\mathbb{R}^2} f = 0$ and $\operatorname{supp} f \cap B_{r_3} = \emptyset$ with $r_3 = r_2^2/r_1$, let $u_\delta \in W^1(\mathbb{R}^2)$ be the unique solution of $\operatorname{div}(s_\delta A_c \nabla u_\delta) = f$ in $\mathbb{R}^2$. There exists $r_* > 0$ depending only on r_1 and r_2, such that if $r_0 < r_*$ then

$$u_\delta \to \hat{u} \text{ in } H^1_{loc}(\mathbb{R}^2 \setminus \bar{B}_{r_3}),$$

where $\hat{u} \in W^1(\mathbb{R}^2)$ is the unique solution of $\Delta \hat{u} = f$ in $\mathbb{R}^2$.

For an observer outside B_{r_3}, the medium $s_0 A_c$ in B_{r_3} looks like I in B_{r_3}: the object a_c in $B(x_1, r_0) \cap B_{r_1}$ disappears. The superlensing device becomes a cloak for small objects close to it. A similar conclusion also holds for the finite frequency regime as a consequence of Theorem 1.2.

3.3. A remark on cloaking using complementary media. This section deals with cloaking using complementary media. We show that a modification proposed in [15] (see also [21]) from the suggestion in [7] on cloaking using complementary media is necessary: Without a modification, cloaking might be not achieved. We only consider here the two dimensional finite frequency regime. The two dimensional quasi-static case holds similarly.

We first describe how to cloak the region $B_{2r_2} \setminus B_{r_2}$ for some $r_2 > 0$ using complementary media as proposed in [15]. Assume that the medium in $B_{2r_2} \setminus B_{r_2}$ is characterized by a symmetric uniformly elliptic matrix-valued function a_c and a real function σ_c bounded above and below by positive constants. We only consider here the two dimensional case; even the construction in three dimensions is similar via the concept of (reflecting) complementary media. The idea suggested by Lai et al. in [7] is to construct a complementary media in $B_{r_2} \setminus B_{r_1}$ for some $0 < r_1 < r_2$. Our cloak proposed in [15] is related to but different from [7]. It consists of two parts. The first one, in $B_{r_2} \setminus B_{r_1}$, makes use of reflecting complementary media to cancel the effect of the cloaked region and the second one in B_{r_1} is to fill the space which "disappears" from the cancellation by the homogeneous medium $(I, 1)$. For the first part, we also modified the strategy in [7]. Instead of $B_{2r_2} \setminus B_{r_2}$, we consider $B_{r_3} \setminus B_{r_2}$ for some $r_3 > 0$ as the cloaked region in which the medium is given by

$$(a_1, \sigma_1) = \begin{cases} (a_c, \sigma_c) & \text{in } B_{2r_2} \setminus B_{r_2}, \\ (I, 1) & \text{in } B_{r_3} \setminus B_{2r_2}. \end{cases} \tag{3.29}$$

The complementary medium in $B_{r_2} \setminus B_{r_1}$ with $r_1 = r_2^2/r_3$ is

$$\left(-F^{-1}{}_*a_1, -F^{-1}{}_*\sigma_1\right),$$

where $F : B_{r_2} \setminus \bar{B}_{r_1} \to B_{r_3} \setminus \bar{B}_{r_2}$ is the Kelvin transform with respect to ∂B_{r_2}. Concerning the second part, the medium in B_{r_1} is given by

$$\left(I, r_3^2/r_1^2\right). \tag{3.30}$$

Set

$$(A, \Sigma) = \begin{cases} (a_1, \sigma_1) & \text{in } B_{r_3} \setminus B_{r_2}, \\ \left(F^{-1}{}_*a_1, F^{-1}{}_*\sigma_1\right) & \text{in } B_{r_2} \setminus B_{r_1}, \\ \left(I, r_3^2/r_1^2\right) & \text{in } B_{r_1}, \\ (I, 1) & \text{otherwise}, \end{cases} \tag{3.31}$$
$$\text{and} \quad s_\delta = \begin{cases} -1 - i\delta & \text{in } B_{r_2} \setminus B_{r_1}, \\ 1 & \text{otherwise.} \end{cases}$$

Given $f \in L^2(\mathbb{R}^2)$ with compact support such that $\operatorname{supp} f \cap B_{r_3} = \emptyset$, we showed in [21, Theorem 1] (see [15, Theorem 1] for the case $k = 0$) that, if r_3/r_2 is large enough,

$$u_\delta \to \tilde{u} \text{ in } H^1_{_{loc}}\left(\mathbb{R}^2 \setminus \bar{B}_{r_3}\right) \text{ as } \delta \to 0.$$

Here u_δ and $\tilde{u}$ are the unique outgoing solutions in $H^1_{_{loc}}(\mathbb{R}^2)$ of

$$\operatorname{div}(s_\delta A \nabla u_\delta) + s_0 k^2 \Sigma u_\delta = f \text{ in } \mathbb{R}^2 \quad \text{and} \quad \Delta \tilde{u} + k^2 \tilde{u} = f \text{ in } \mathbb{R}^2.$$

In fact, in [21], we considered a bounded setting, nevertheless, the result stated here holds with a similar proof. For an observer outside B_{r_3}, the medium $(s_0 A, s_0 \Sigma)$ acts like $(I, 1)$: cloaking is achieved.

The following proposition show that it is necessary to extend (a_c, σ_c) by $(I, 1)$ in $B_{r_3} \setminus B_{2r_2}$ in the construction of the cloaking device given above.

PROPOSITION 3.2. *Let $0 < r_0 < r_1 < r_2$ and $x_3 \in \partial B_{r_3}$ with $r_3 = r_2^2/r_1$. Let a_c be a symmetric uniformly elliptic matrix-valued function and σ_c be a positive function bounded below by a positive constant both defined in $B(x_3, r_0) \cap B_{r_3}$. Define (A, Σ) by* (3.31) *where*

$$(a_1, \sigma_1) = \begin{cases} (a_c, \sigma_c) & \text{in } B(x_3, r_0) \cap B_{r_3}, \\ (I, 1) & \text{in } (B_{r_3} \setminus B_{r_2}) \setminus B(x_3, r_0). \end{cases}$$

Given $f \in L^2(\mathbb{R}^2)$ with compact support and $\operatorname{supp} f \cap B_{r_3} = \emptyset$*, let $u_\delta \in H^1_{_{loc}}(\mathbb{R}^2)$ be the unique outgoing solution to the equation*

$$\operatorname{div}(s_\delta A \nabla u_\delta) + s_0 k^2 \Sigma u_\delta = f \text{ in } \mathbb{R}^2,$$

where (A, Σ) and s_δ are given by (3.31)*. There exists $r_* > 0$ depending only on r_1 and r_2 such that if $r_0 < r_*$ then*

$$u_\delta \to \hat{u} \text{ in } H^1_{_{loc}}\left(\mathbb{R}^2 \setminus \bar{B}_{r_3}\right). \tag{3.32}$$

Here $\hat{u} \in H^1_{_{loc}}(\mathbb{R}^2)$ is the unique outgoing solution to the equation

$$\operatorname{div}(\hat{A} \nabla \hat{u}) + k^2 \hat{\Sigma} \hat{u} = f \text{ in } \Omega \text{ where } (\hat{A}, \hat{\Sigma}) = \begin{cases} (a_c, \sigma_c) & \text{in } B(x_0, r_0) \cap B_{r_3}, \\ (I, 1) & \text{otherwise.} \end{cases} \tag{3.33}$$

As a consequence of Proposition 3.2, the object (a_c, σ_c) in $B(x_3, r_0) \cap B_{r_3}$ does not disappear: cloaking is *not* achieved.

Proof. The proof is similar to the one of Theorem 1.2 thought even simpler. We only sketch the proof. We have, by Lemma 2.4,

$$\|u_\delta\|_{H^1(B_R)} \le C_R\Big(\delta^{-1/2}\|f\|_{L^2}^{1/2}\|u_\delta\|_{L^2(\operatorname{supp} f)}^{1/2} + \|f\|_{L^2}\Big).$$

Here and in what follows in this proof, C denotes a positive constant independent of δ and f. Define

$$u_{1,\delta} = u_\delta \circ F^{-1} \text{ in } \mathbb{R}^2 \setminus B_{r_2} \quad \text{ and } \quad u_{2,\delta} = u_{1,\delta} \circ G^{-1} \text{ in } B_{r_3},$$

where F and G are the Kelvin transforms with respect to ∂B_{r_2} and ∂B_{r_3}, respectively. By Lemma 2.5, as in (3.9) and (3.10) we have

$$\|u_{1,\delta} - u_\delta\|_{\mathbf{H}(\partial B_{r_2})} \le C\delta\|u_\delta\|_{H^1(B_{r_3})} \tag{3.34}$$

and

$$\|u_{2,\delta} - u_{1,\delta}\|_{\mathbf{H}(\partial B_{r_3}\setminus B(x_3,r_0))} \le C\delta\|u_\delta\|_{H^1(B_{r_3})}. \tag{3.35}$$

Applying Lemma 2.5, we have

$$(1+i\delta)\Delta u_{1,\delta} + k^2 u_{1,\delta} = 0 \text{ in } (B_{r_3}\setminus B_{r_2})\setminus B(x_3, r_0)$$

and

$$\Delta u_{2,\delta} + k^2 u_{2,\delta} = 0 \text{ in } B_{r_3}.$$

We derive that

$$\Delta u_{1,\delta} + k^2 u_{1,\delta} = \left(1 - \frac{1}{1+i\delta}\right)k^2 u_{1,\delta} = -\frac{i\delta}{1+i\delta}k^2 u_{1,\delta} \text{ in } (B_{r_3}\setminus B_{r_2})\setminus B(x_3, r_0). \tag{3.36}$$

For simplicity of notations, we assume that $x_3 = (-|x_3|, 0)$. Applying Corollary 3.2 for $u_{1,\delta}(\cdot - x_3) - u_{2,\delta}(\cdot - x_3)$, $R = (r_3 - r_2)/2 < R_1 = r_3$, and $\alpha = 3/4$, there exist two positive constants $r_* < R_0 < (r_3 - r_2)/2$ such that if $r_0 < r_*$ then for some neighborhood D of $\{z \in \bar{B}_{r_3}; |z - x_3| = R_0\}$

$$\|u_{1,\delta} - u_{2,\delta}\|_{H^1(D\cap(B_{r_3}\setminus B_{r_2}))} \le C\delta^\alpha\|u_\delta\|_{H^1(B_{r_3})}.$$

Set $O = (B_{r_3}\setminus B_{r_2})\setminus\{z\,; |z - x_3| < R_0\}$ and define

$$\mathcal{U}_\delta = \begin{cases} u_{2,\delta} + u_\delta - u_{1,\delta} & \text{in } O,\\ u_{2,\delta} & \text{in } B_{r_2},\\ u_\delta & \text{otherwise.}\end{cases}$$

Then $\mathcal{U}_\delta \in H^1(B_R \setminus \partial O)$ for all $R > 0$ and U_δ is an outgoing solution of the equation

$$\operatorname{div}(\hat{A}\nabla\mathcal{U}_\delta) + k^2\hat{\Sigma}\mathcal{U}_\delta = f \text{ in } \mathbb{R}^2\setminus\partial O,$$

and, as in (3.19),

$$\|[\mathcal{U}_\delta]\|_{H^{1/2}(\partial O)} + \|[\hat{A}\nabla\mathcal{U}_\delta\cdot\nu]\|_{H^{-1/2}(\partial O)} \le C\delta^\alpha\|u_\delta\|_{H^1(B_{r_3})}.$$

It follows that, for large $R > 0$ such that $\operatorname{supp} f \subset B_R$,

$$\|\mathcal{U}_\delta\|_{H^1(B_R \setminus \partial O)} \le C_R \delta^\alpha \Big(\delta^{-1/2} \|\mathcal{U}_\delta\|_{L^2(B_R \setminus B_{r_3})}^{1/2} \|f\|_{L^2(\Omega)}^{1/2} + \|f\|_{L^2} \Big) + C \|f\|_{L^2}.$$

Since $\alpha = 3/4 > 1/2$, as in the proof of Theorem 1.2, one deduces that U_δ is bounded in $H^1(B_R \setminus \partial O)$,

$$\|[\mathcal{U}_\delta]\|_{H^{1/2}(\partial O)} + \|[\hat{A} \nabla \mathcal{U}_\delta \cdot \nu]\|_{H^{-1/2}(\partial O)} \to 0,$$

and $U_\delta \to \hat{u}$ in $H^1_{_{loc}}(\mathbb{R}^2 \setminus B_{r_3})$ as $\delta \to 0$. The details are left to the reader. ∎

Remark 3.2. It would be interesting to understand the cooperation and the combat of various cloaking devices using NIMs put together as raised in [27].

REFERENCES

[1] G. Alessandrini, L. Rondi, E. Rosset, and S. Vessella, *The stability for the Cauchy problem for elliptic equations*, Inverse Problems, 25 (2009), 123004.

[2] H. Ammari, G. Ciraolo, H. Kang, H. Lee, and G. W. Milton, *Spectral theory of a Neumann-Poincaré-type operator and analysis of cloaking due to anomalous localized resonance*, Arch. Ration. Mech. Anal., 218 (2013), pp. 667–692.

[3] H. Ammari, G. Ciraolo, H. Kang, H. Lee, and G. W. Milton, *Anomalous localized resonance using a folded geometry in three dimensions*, R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci., 469 (2013), 20130048.

[4] G. Bouchitté and B. Schweizer, *Cloaking of small objects by anomalous localized resonance*, Quart. J. Mech. Appl. Math., 63 (2010), pp. 437–463.

[5] O. P. Bruno and S. Lintner, *Superlens-cloaking of small dielectric bodies in the quasistatic regime*, J. Appl. Phys., 102 (2007), 12452.

[6] R. V. Kohn, J. Lu, B. Schweizer, and M. I. Weinstein, *A variational perspective on cloaking by anomalous localized resonance*, Comm. Math. Phys., 328 (2014), pp. 1–27.

[7] Y. Lai, H. Chen, Z. Zhang, and C. T. Chan, *Complementary media invisibility cloak that cloaks objects at a distance outside the cloaking shell*, Phys. Rev. Lett., 102 (2009), 093901.

[8] G. W. Milton, N. A. Nicorovici, R. C. McPhedran, and V. A. Podolskiy, *A proof of superlensing in the quasistatic regime, and limitations of superlenses in this regime due to anomalous localized resonance*, R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci., 461 (2005), pp. 3999–4034.

[9] G. W. Milton and N. A. Nicorovici, *On the cloaking effects associated with anomalous localized resonance*, R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci., 462 (2006), pp. 3027–3059.

[10] G. W. Milton, N. A. Nicorovici, R. C. McPhedran, K. Cherednichenko, and Z. Jacob, *Solutions in folded geometries, and associated cloaking due to anomalous resonance*, New J. Phys., 10 (2008), 115021.

[11] H.-M. Nguyen, *Asymptotic behavior of solutions to the Helmholtz equations with sign changing coefficients*, Trans. Amer. Math. Soc., 367 (2015), pp. 6581–6595.

[12] H.-M. Nguyen, *Cloaking via anomalous localized resonance. A connection between the localized resonance and the blow up of the power for doubly complementary media*, C. R. Math. Acad. Sci. Paris, 353 (2015), pp. 41–46.

[13] H.-M. Nguyen, *Cloaking via anomalous localized resonance for doubly complementary media in the quasistatic regime*, J. Eur. Math. Soc. (JEMS), 17 (2015), pp. 1327–1365.

[14] H.-M. Nguyen, *Superlensing using complementary media*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 32 (2015), pp. 471–484.

[15] H.-M. Nguyen, *Cloaking using complementary media in the quasistatic regime*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 33 (2016), pp. 1509–1518.

[16] H.-M. Nguyen, *Limiting absorption principle and well-posedness for the Helmholtz equation with sign changing coefficients*, J. Math. Pures Appl. (9), 106 (2016), pp. 342–374.

[17] H.-M. Nguyen, *Negative index materials and their applications: Recent mathematics progress.* (2016), Chin. Ann. Math. Ser. B, 38 (2017), pp. 601–628.

[18] H.-M. Nguyen, *Superlensing Using Complementary Media and Reflecting Complementary Media for Electromagnetic Waves*, Adv. Nonlinear Anal., to appear.

[19] H.-M. Nguyen, *Cloaking via anomalous localized resonance for doubly complementary media in the finite frequency regime*, J. Anal. Math., to appear.
[20] H.-M. Nguyen and H. L. Nguyen, *Complete resonance and localized resonance in plasmonic structures*, ESAIM Math. Model. Numer. Anal., 49 (2015), pp. 741–754.
[21] H.-M. Nguyen and H. L. Nguyen, *Cloaking using complementary media for the Helmholtz equation and a three spheres inequality for second order elliptic equations*, Trans. Amer. Math. Soc. Ser. B, 2 (2015), pp. 93–112.
[22] N. A. Nicorovici, R. C. McPhedran, and G. W. Milton, *Transport properties of a three-phase composite material: The square array of coated cylinders*, R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci., 442 (1993), pp. 599–620.
[23] N. A. Nicorovici, R. C. McPhedran, and G. W. Milton, *Optical and dielectric properties of partially resonant composites*, Phys. Rev. B (3), 49 (1994), pp. 8479–8482.
[24] J. B. Pendry, *Negative refraction makes a perfect lens*, Phys. Rev. Lett., 85 (2000), pp. 3966–3969.
[25] J. B. Pendry, *Perfect cylindrical lenses*, Opt. Express, 1 (2003), pp. 755–760.
[26] S. A. Ramakrishna and J. B. Pendry, *Spherical perfect lens: Solutions of Maxwell's equations for spherical geometry*, Phys. Rev. B (3), 69 (2004), 115115.
[27] R. C. McPhedran, N. A. Nicorovici, L. C. Botten, and G. W. Milton, *Cloaking by plasmonic resonance among systems of particles: Cooperation or combat*, C. R. Phys., 10 (2009), pp. 391–399.
[28] R. A. Shelby, D. R. Smith, and S. Schultz, *Experimental verification of a negative index of refraction*, Science, 292 (2001), pp. 77–79.
[29] V. G. Veselago, *The electrodynamics of substances with simultaneously negative values of ε and μ*, Uspekhi Fiz. Nauk, 92 (1964), pp. 517–526.