Analytic twists of modular forms and applications

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Abstract

We are interested in the study of non-correlation of Fourier coefficients of Maass forms against a wide class of real analytic functions. In particular, the class of functions we are interested in should be thought of as some archimedean analogs of Frobenius trace functions.
In the first part of the thesis, we give an axiomatic definition for this class, and prove that these functions satisfy properties similar to that of Frobenius trace functions. In particular, we prove non-correlation statements analogous to those given by Fouvry, Kowalski and Michel for algebraic trace functions.
In the second part of the thesis, we establish the existence of large values of Hecke-Maass L-functions with prescribed argument. In studying these problems, one encounters sums of Fourier coefficients of Maass forms against real oscillatory functions. In some cases, one can prove that these functions satisfy the axioms discussed previously.

Key words: Automorphic forms, trace functions, exponential sums, L-functions.

Résumé

Nous nous intéressons à l'étude de non-corrélation de coefficients de Fourier de formes de Maass avec une grande classe de fonctions analytiques réelles. En particulier, la classe de fonctions à laquelle nous nous intéressons devrait être considérée comme analogue archimédien des fonctions traces de Frobenius.

Dans la première partie de la thèse, nous donnons une définition axiomatique de cette classe de fonctions, et démontrons que ces fonctions satisfont des propriétés similaires à celles des fonctions trace de Frobenius. En particulier, nous démontrons des résultats de non-corrélations analogues aux résultats de Fouvry, Kowalski et Michel dans le cas des fonctions trace algébriques.

Dans la seconde partie de la thèse, nous démontrons l'existence de grandes valeurs de fonctions L de Hecke-Maass avec angle prédéterminé. En étudiant ce problème, nous devons estimer des sommes de coefficients de Fourier de formes de Maass avec des fonctions réelles et oscillantes. Dans certains cas, nous pouvons montrer que ces fonctions satisfont les axiomes définis précédemment.

Mots clefs : Formes automorphes, fonctions traces, sommes exponentielles, fonctions L.

Contents

Introduction

This thesis investigates archimedean analogs of Frobenius trace functions from an analytic viewpoint. Namely, we define a class of real analytic, oscillatory, functions that exhibit properties similar to that of Frobenius trace functions (e.g. square-root cancelations, stability under Fourier transform ...). The main motivation to this problem is to give an archimedean analog to a non-correlation result of Fouvry, Kowalski and Michel [FKM15a], for sums of Fourier coefficients of modular forms against trace functions. These problems arise naturally in problems in analytic number theory (see e.g. [Hou16, FKM15b, CFH+14]). In the second part of the thesis, we study the existence of large values of Hecke-Maass L-functions, in which twisted sums of Fourier coefficients of Maass forms by real analytic oscillatory functions arise.

0.1 Non-correlation

The main question we are interested in is that of understanding how certain oscillatory functions interact with Fourier coefficients of Maass forms. In the following section, we recall well-known facts about Maass forms and their Fourier expansion. In particular, we see that they are essentially bounded and oscillatory.

Let f be a Maass form. Saying that a bounded analytic function $F : \mathbb{R}_{>0} \to \mathbb{C}$ does not correlate with Fourier coefficients of f, $(\rho_f(n))_{n\geq 1}$, is a way to measure to which extent $F(n) \neq \overline{\rho_f(n)}$. More precisely, we make the following definition.

Definition 0.1. *Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be two sequences of (essentially) bounded complex numbers. We say that (a_n) does not correlate with (b_n) if for all $A \geq 1$ and $x > 1$, we have*

$$\sum_{n\leq x} a_n b_n \ll_A x(\log x)^{-A}.$$

Non-correlation statements have deep consequences in number theory. For instance non-correlation between the constant sequence $(1)_{n\in\mathbb{N}}$ and $(\mu(n))_{n\in\mathbb{N}}$, the Möbius function, is equivalent to the existence of zero-free regions for the Riemann zeta function, proving the Prime Number Theorem. Moreover, stronger non-correlation statements with power savings would imply stronger zero-free regions towards the Riemann hypothesis. Similarly, non-correlation of $(\rho_f(n))$ and $(\mu(n))$ is equivalent to the existence of zero-free regions of the L-function associated to f.

0.2 Non-correlation in the context of trace functions

Frobenius trace functions, $K:\mathbb{F}_p \to \mathbb{C}$, are highly oscillatory functions that arise from algebraic geometric considerations. We will define trace functions in Chapter 4, in which we will see that in particular they exhibit strong quasi-orthogonality relations as a consequence of the Riemann Hypothesis for algebraic varieties over finite fields, due to Deligne [Del74]. In [FKM15a] Fouvry, Kowalski and Michel establish a non-correlation statement for Fourier coefficients of modular forms twisted by Frobenius trace functions.

Theorem 0.1 (F-K-M)**.** *Let f be a Hecke-Maass form, p be a prime number and V a smooth compactly supported function on $[1/2,2]$, such that $x^j V^{(j)}(x) \ll 1$, for all $j \geq 0$. Let K be an isotypic trace function of modulus p, then*

$$\sum_n \rho_f(n)K(n)V\left(\frac{n}{p}\right) \ll p^{1-\delta},$$

for any $\delta < 1/8$, where the implied constant depends only on f, δ and on the conductor of the trace function.

Examples of trace functions include Dirichlet characters of conductor p and Hyper-Kloosterman sums: for $m \geq 2$,

$$\mathrm{Kl}_m(n;p) := p^{\frac{1-m}{2}} \sum_{x_1\cdots x_m \equiv n \,(\mathrm{mod}\, p)} e\left(\frac{x_1+\cdots+x_m}{p}\right).$$

In the special case of a Dirichlet character, χ, Theorem 0.1 is essentially equivalent to subconvexity results of Burgess type for $L(f\otimes\chi, 1/2)$, which was already obtained by Bykovski and Blomer-Harcos [Byk96, BH08].

0.3 Analytic trace functions

The archimedean analog of Dirichlet characters can be thought of as functions of the form x^{it} for some $t \in \mathbb{R}$, which are the continuous homomorphisms $\mathbb{R}_{>0} \to \mathbb{C}^1$. In particular, a statement of the form (0.1) in the case $K(n) = n^{it}$ is essentially equivalent to subconvexity results for $L(f, 1/2+it)$, for which we have even better bounds [Goo82].

In this thesis, we give an axiomatic definition of a family of real analytic functions, $K_t:\mathbb{R}_{>0} \to \mathbb{C}$, indexed on a large real parameter t, that we call analytic trace functions of conductor t. In analogy to Theorem 0.1, we prove the following theorem.

Theorem 0.2. *Let $K_t:\mathbb{R}_{>0}\to\mathbb{C}$ be an analytic trace function. Let f and V be as in Theorem 0.1. We have*

$$\sum_n \rho_f(n)K_t(n)V\left(\frac{n}{t}\right) \ll t^{1-\delta},$$

for any $\delta < 1/8$ and where the implicit constant depends only on f, δ and on $||K_t||_\infty$.

In particular, Theorem 0.2 covers the special case of Bessel functions of any rank, which should be thought of as archimedean analogs of Hyper-Kloosterman sums.

As a corollary to Theorem 0.2, we give an ergodic theoretical interpretation in terms of equidistribution of twisted horocycle flows in analogy to that in [FKM15a].

Theorem 0.3. *Let K_t be an analytic trace function. Let f be a Hecke-Maass form, and V be a smooth real valued function with compact support in $[1/2, 5/2]$ such that $V^{(j)}(x) \ll 1$, for all $j \geq 0$. We then have for any $\delta > 0$,*

$$\frac{1}{\beta-\alpha}\int_{\alpha}^{\beta} f(x+iy)K_{1/y}\left(\frac{x}{y}\right)V(x)\mathrm{d}x \to 0,$$

uniformly as $y \to 0$ so long as $\beta - \alpha$ remains bigger than $y^{1/8-\delta}$.

0.4 Large values of $L(f, 1/2+it)$ with prescribed argument

The resonance method [Sou08, Hou16] is a technique to deduce the existence of exceptionally large values of L-functions from computations of weighted moments. Using the resonance method, we establish the existence of large values of the Hecke L-function, $L(f, 1/2+it)$, with prescribed argument. Namely, we prove the following theorem.

Theorem 0.4. *For any $\eta < 1$, any sufficiently large $T \in \mathbb{R}$ and any $\theta \in \mathbb{R}/\mathbb{Z}$, there exists $t \in [\frac{T}{2}, 2T]$ such that*

$$\frac{1}{2\pi}\arg L\left(f, \frac{1}{2}+it\right) \equiv \theta \mod \mathbb{Z}, \textit{and } \log\left|L\left(f, \frac{1}{2}+it\right)\right| \geq (\eta + o(1))\sqrt{\frac{\log T}{\log\log T}}.$$

Hough [Hou16] proved a similar statement for the Riemann zeta function. We therefore treat the case of an L-function of degree 2, for which the arguments are substantially more involved.

Moreover, Hough [Hou16] studies the case of Dirichlet L-functions. He proves that for any $\delta > 0$ and $\theta \in \mathbb{R}/\mathbb{Z}$ and for all sufficiently large prime q, there exists a non-principal Dirichlet character, $\chi \mod q$, such that

$$\left\|\frac{1}{2\pi}\arg L\left(\chi, \frac{1}{2}\right) - \theta\right\|_{\mathbb{R}/\mathbb{Z}} \leq \delta, \text{and } \log\left|L\left(\chi, \frac{1}{2}\right)\right| \gg \sqrt{\frac{\log q}{\log\log q}}.$$

In proving this result, Hough requires estimates on sums of the divisor function twisted by Hyper-Kloosterman sums, and uses Theorem 0.1. In our setting, one is led to estimating certain sums of Fourier coefficients of Maass forms against real analytic, oscillatory, functions. These should be seen as being archimedean analogs of the sums appearing in [Hou16]. In order to prove Theorem 0.4, we actually do not require Theorem 0.2, as in the ranges we consider, these oscillatory functions are manageable. We however note that in more interesting ranges, one can show that these functions are analytic trace functions.

0.5 Outline of the Thesis

In Chapter 1, we will recall some background material on Maass forms needed in the subsequent chapters. We also discuss the stationary phase method, that will be crucial in our understanding of analytic trace functions.

Chapters 2 and 3 are essentially preprints [Peya, Peyb] and contain the proofs of the Theorems of sections 0.3 and 0.4 respectively.

In Chapter 4 we define Frobenius trace functions and discuss some analogies with analytic trace functions. We then show how analytic trace functions defined in Chapter 2 appear in the problem of large values of L-functions discussed in Chapter 3. We conclude the Chapter and Thesis by giving some insight on possible further work. For instance, given that our definition of analytic trace functions lack geometric considerations, certain notions, analogous to those of Frobenius trace functions, have yet to become apparent. We also give a direction in which to further our work on large values of L-functions, in which the role of analytic trace functions becomes more apparent.

1 Preliminaries

1.1 Notation

We will let $f(x) \ll g(x)$, $f(x) \gg g(x)$ and $f(x) = O(g(x))$ denote the usual Vinogradov symbols. We emphasize that for us $f(x) \ll g(x)$ will be taken to mean exactly that $f(x) = O(g(x))$. The notation $f(x) \asymp g(x)$ will be used to mean that both $f(x) \ll g(x)$ and $g(x) \ll f(x)$ hold. The notation $f(x) \sim g(x)$ will be taken to mean that $\lim_{x\to\infty} f(x)/g(x) = 1$. We will write $f(x) = o(g(x))$ to mean that $\lim_{x\to\infty} f(x)/g(x) \to 0$. We also follow the convention that any ϵ appearing in the Thesis is defined to be an arbitrarily small unspecified positive real number, that might vary from one line to the other. Whenever we encounter a zero-free region for an L-function, we will take the k-th root of L in that region to be the one defined so that $L^{1/k} \to 1$ as $L \to 1$ with $s \to \infty, s \in \mathbb{R}$. The function $e(\cdot)$ will always represent the complex exponential $\exp(2\pi i \cdot)$. The notation $\bar{a} \pmod q$ will always be used to denote the multiplicative inverse of a modulo q.

1.2 Maass forms

Let $\mathbb{H} := \{z \in \mathbb{C} | \Im(z) > 0\}$ denote the upper half plane. A (cuspidal) Maass form with respect to $\mathrm{SL}_2(\mathbb{Z})$ is a function, $f : \mathbb{H} \to \mathbb{C}$ such that

- It satisfies the periodicity condition, $f(\gamma z) = f(z)$, for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, where the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}$ is given by Möbius transformations, i.e.
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z := \frac{az+b}{cz+d}.$$
- It is an eigenfunction of the Laplacian, $\Delta = -y^2 \left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{\mathrm{d}^2}{\mathrm{d}y^2}\right)$.
- It satisfies the following growth condition at the cusp,
$$f(x+iy) \ll e^{-2\pi y}.$$

Any Maass form, f, admits a Fourier expansion of the form
$$f(z) = \sum_{n\neq 0} \rho_f(n)|n|^{-1/2} W_{it_f}(4\pi|n|y)e(nx),$$

where $1/4 + t_f^2$ denotes its Laplace eigenvalue and W_{it} is a Whittaker function,

$$W_{it}(y) := \frac{e^{-y/2}}{\Gamma\left(\frac{1}{2}+it\right)} \int_0^\infty e^{-x} x^{it-\frac{1}{2}} \left(1+\frac{x}{y}\right)^{it-\frac{1}{2}} \mathrm{d}x.$$

The Fourier coefficients, $\rho_f(n)$, are normalized so that by Rankin-Selberg,

$$\sum_{|n|\le x} |\rho_f(n)|^2 \sim c_f x,$$

for some constant c_f depending on f (see [Iwa02, p. 110]). Moreover, the Fourier coefficients oscillate substantially, as

$$\sum_{n\le x} \rho_f(n) \ll x^{2/5}$$

holds, where the implied constant may depend on f (see [HI89]).

We define the Hecke operators $(T_n)_{n\ge1}$ acting on the space of Maass forms by

$$(T_n f)(z) = \frac{1}{\sqrt{n}} \sum_{ad=n} \sum_{0\le b<d} f\left(\frac{az+b}{d}\right).$$

A Maass form that is also an eigenfunction for all the Hecke operators will be called a Hecke-Maass form. We associate to f the sequence of Hecke-eigenvalues $(\lambda_f(n))_{n\ge1}$. We further note that $\lambda_f(n) \in \mathbb{R}$, for all $n \ge 1$, as well as the following realtion between Fourier coefficients and Hecke eigenvalues [Gol15]:

$$\rho_f(n) = \rho_f(1)\lambda_f(n), \forall n \ge 1.$$

We define the associated L-function,

$$L(f,s) := \sum_n \frac{\lambda_f(n)}{n^s} = \prod_p (1-\alpha_p p^{-s})^{-1}(1-\beta_p p^{-s})^{-1},$$

where α_p, β_p are given via $\alpha_p + \beta_p = \lambda_f(p)$ and $\alpha_p\beta_p = 1$.

Let $\iota : \mathbb{H} \to \mathbb{H}$ denote the antiholomorphic involution $\iota(x+iy) = -x+iy$. A Maass form satisfying

$$f\circ\iota = f, \text{or } f\circ\iota = -f,$$

will be called either *even* or *odd* accordingly. Given that the Hecke operators T_n, the involution ι and the Laplacian Δ all commute with each other [Gol15], we may simultaneously diagonalize the space of Maass forms, and thus only consider even or odd Hecke-Mass forms.

Proposition 1.1 (Functional Equation)**.** *Let f be a Hecke-Maass form as defined above. Let $\xi = 0$ if f is even, 1 if f is odd. Let*

$$\Lambda(f,s) := \pi^{-s}\Gamma\left(\frac{s+\xi+it_f}{2}\right)\Gamma\left(\frac{s+\xi-it_f}{2}\right)L(f,s),$$

be the completed L-function. Then $\Lambda(f,s)$ has analytic continuation to all s and satisfies the functional equation

$$\Lambda(f,s) = (-1)^{\xi}\Lambda(1-s,f).$$

Proof. See [Gol15]. $\square$

We will also need the Approximate Functional Equation for $L(f,s)$, that we quote from [IK04].

Proposition 1.2 (Approximate Functional Equation)**.** *Let f be a Hecke-Mass form as defined and with same notation as above. Let $\xi = 0$ if f is even, 1 if f is odd. Let $G(u)$ be any function which is holomorphic and bounded in the strip $-4 < \Re(u) < 4$, even, and normalized by $G(0) = 1$. Let $X > 0$. Then for s in the strip $0 \le \sigma \le 1$ we have*

$$L(f,s) = \sum_n \frac{\lambda_f(n)}{n^s} V_s\left(\frac{n}{X}\right) + \epsilon(f,s) \sum_n \frac{\lambda_f(n)}{n^{1-s}} V_{1-s}(nX),$$

where $V_s(y)$ is a smooth function defined by

$$V_s(y) = \frac{1}{2\pi i} \int_{(3)} y^{-u} G(u) \pi^{-u} \frac{\Gamma\left(\frac{s+u+\xi+it_f}{2}\right)\Gamma\left(\frac{s+u+\xi-it_f}{2}\right)}{\Gamma\left(\frac{s+\xi+it_f}{2}\right)\Gamma\left(\frac{s+\xi-it_f}{2}\right)} \mathrm{d}u$$

and

$$\epsilon(f,s) = \xi \pi^{-1+2s} \frac{\Gamma\left(\frac{1-s+\xi+it_f}{2}\right)\Gamma\left(\frac{1-s+\xi-it_f}{2}\right)}{\Gamma\left(\frac{s+\xi+it_f}{2}\right)\Gamma\left(\frac{s+\xi-it_f}{2}\right)}.$$

We note that in applications we often take $G(u) = e^{u^2}$. We also note the following proposition from [IK04].

Proposition 1.3. *Suppose $\Re(s) \ge 3\alpha > 0$. Then the derivatives of $V_s(y)$ satisfy*

$$y^a V_s^{(a)}(y) \ll \left(1 + \frac{y}{\sqrt{(|s+it_f|+3)(|s-it_f|+3)}}\right)^{-A},$$

$$y^a V_s^{(a)}(y) = \delta_a + O\left(\left(\frac{y}{\sqrt{(|s+it_f|+3)(|s-it_f|+3)}}\right)^{\alpha}\right)$$

where $\delta_0 = 1$, $\delta_a = 0$ if $a > 0$ and the implied constants depend only on α, a, and A.

Remark 1.1. *Combining these two proposition, we obtain that $L(f, \sigma + it)$ in the critical strip may be written as two sums of length roughly t.*

1.3 Stationary phase integrals

Throughout the thesis, we will need several stationary phase lemmas to estimate oscillatory integrals. In particular, we will regularly be faced with a special kind of oscillatory integral which we now define. Let W be any smooth real valued function, with support in $[a,b] \subset (0,\infty)$, and such that $W^{(j)}(x) \ll_{a,b,j} 1$. We then define

$$W^\dagger(r,s) := \int_0^\infty W(x) e(-rx) x^{s-1} \mathrm{d}x, \tag{1.1}$$

where $r \in \mathbb{R}$ and $s \in \mathbb{C}$. Munshi gives in [Mun15] estimations and asymptotics for $W^{\dagger}$, however we will also need a slightly more precise version of this asymptotic. To this purpose, we quote from [BKY13] a version of the stationary lemma.

Lemma 1.1. *Let* $0 < \delta < 1/10$, *and* $X, Y, V, V_1, Q > 0, Z := Q + X + Y + V_1 + 1$, *and assume that*

$$Y \geq Z^{3\delta}, V_1 \geq V \geq \frac{QZ^{\delta/2}}{Y^{1/2}}.$$

Suppose that w *is a smooth function on* $\mathbb{R}$ *with support on an interval* $[a, b]$ *of finite length* V_1, *satisfying*

$$w^{(j)}(t) \ll_j XV^{-j},$$

for all $j \geq 0$. *Suppose that* h *is a smooth function on* $[a, b]$, *such that there exists a unique point* t_0 *in the interval such that* $h'(t_0) = 0$, *and furthermore that*

$$h''(t) \gg \frac{Y}{Q^2}, h^{(j)}(t) \ll_j \frac{Y}{Q^j}, \qquad \text{for } j = 1, 2, 3, \cdots, t \in [a, b].$$

Then, the integral defined by

$$I := \int_{-\infty}^{\infty} w(t) e^{ih(t)} \mathrm{d}t$$

has an asymptotic expansion of the form

$$I = \frac{e^{ih(t_0)}}{\sqrt{h''(t_0)}} \sum_{n \leq 3\delta^{-1}A} p_n(t_0) + O_{A,\delta}(Z^{-A}),$$

and

$$p_n(t_0) := \frac{\sqrt{2\pi} e^{\pi i/4}}{n!} \left(\frac{i}{2h''(t_0)} \right)^n G^{(2n)}(t_0), \tag{1.2}$$

where A *is arbitrary, and*

$$G(t) := w(t) e^{iH(t)}; H(t) = h(t) - h(t_0) - \frac{1}{2} h''(t_0)(t - t_0)^2. \tag{1.3}$$

Furthermore, each p_n *is a rational function in* $h', h'', \cdots$, *satisfying*

$$\frac{\mathrm{d}^j}{\mathrm{d}t_0^j} p_n(t_0) \ll_{j,n} X \left(V^{-j} + Q^{-j} \right) \left((V^2 Y/Q^2)^{-n} + Y^{-n/3} \right). \tag{1.4}$$

We want to extract the first five terms in the asymptotic expansion, in order to have a small enough error term that will be easy to deal with. We therefore compute

$$p_0(t_0) = \sqrt{2\pi} e(1/8) w(t_0),$$

and

$$G'(t) = w'(t) e^{iH(t)} + i w(t) H'(t) e^{iH(t)},$$
$$G''(t) = e^{iH(t)} (w''(t) + 2i w'(t) H'(t) + i w(t) H''(t) - w(t) H'(t)^2).$$

We now see that $H(t_0) = 0$, while

$$H'(t) = h'(t) - h''(t_0)(t - t_0),$$

and

$$H''(t) = h''(t) - h''(t_0).$$

Hence, we see that also $H'(t_0), H''(t_0) = 0$. We therefore have

$$p_1(t_0) = \sqrt{2\pi}e(1/8)\frac{i}{2h''(t_0)}w''(t_0).$$

Noting that only the terms that don't contain $H^{(i)}$ for $i = 0, 1, 2$ survive, and that $H^{(j)}(t) = h^{(j)}(t)$ for $j \geq 3$, we have

$$G^{(4)}(t_0) = w^{(4)}(t_0) + 4i w'(t_0)h^{(3)}(t_0) + i w(t_0)h^{(4)}(t_0),$$

and thus

$$p_2(t_0) = -\frac{\sqrt{2\pi}e\left(\frac{1}{8}\right)}{8h''(t_0)^2}(w^{(4)}(t_0) + 4i w'(t_0)h^{(3)}(t_0) + i w(t_0)h^{(4)}(t_0)).$$

In general, $G^{(2n)}(t_0)$ is a linear combination of terms of the form

$$w^{(\nu_0)}(t_0)H^{(\nu_1)}(t_0)\cdots H^{(\nu_l)},$$

where $\nu_0 + \cdots + \nu_l = 2n$.
We now wish to use these in the context of the study of $W^{\dagger}(r, s)$, where we write $s = \sigma + i\beta \in \mathbb{C}$. We may thus use the lemma above with

$$w(x) = W(x)x^{\sigma-1},$$

and

$$h(x) = -2\pi r x + \beta \log x.$$

Then,

$$h'(x) = -2\pi r + \frac{\beta}{x}, \text{ and } h^{(j)}(x) = (-1)^{j-1}(j-1)!\frac{\beta}{x^j}, \tag{1.5}$$

for $j \geq 2$. The unique stationary point is given by

$$x_0 = \frac{\beta}{2\pi r}.$$

We now let

$$\check{W}(x) := x^{1-\sigma}\sum_{n=0}^{5} p_n(x),$$

and claim it is non-oscillatory in the following sense.

Lemma 1.2. *Suppose that there exists a constant $c > 0$ such that for any $r \in \mathbb{R}$, $\beta \geq c$. Then for all $j \geq 0$, and $x \in [a, b]$,*

$$\check{W}^j(x) \ll_{\sigma,j,a,b} 1.$$

Proof. We compute

$$\check{W}^{(j)}(x) = \sum_{l=0}^{j} \begin{pmatrix} j \\ l \end{pmatrix} (x^{1-\sigma})^{(j-l)} \sum_{n=0}^{5} p_n^{(l)}(x).$$

Now, it is clear that $(x^{1-\sigma})^{(j-l)} \ll_{j,\sigma,a,b} 1$, and so we just need to control the derivatives of each p_n. Since w is a product of a power of x with W and $W^{(j)}(x) \ll_j 1$, we can easily see that $p_0(x) \ll_{j,\sigma,a,b} 1$. Now

$$h''(x_0) = -\frac{\beta}{x^j},$$

and since $\beta \gg 1$, by the same argument as for p_0, it is clear that $p_1(x) \ll 1$. We may apply the same reasoning for p_2, and more generally for any p_n, since (1.5) implies the higher derivatives of h don't grow compared to the powers of h'' in the denominator. □

We may now give the following result for $W^\dagger(r,s)$.

Lemma 1.3. *Let $r \in \mathbb{R}$ and $s = \sigma + i\beta \in \mathbb{C}$, such that $x_0 = \frac{\beta}{2\pi r} \in [a/2, 2b]$. Then,*

$$W^\dagger(r,s) = \frac{\sqrt{2\pi}e(1/8)}{\sqrt{-\beta}} \left(\frac{\beta}{2\pi r}\right)^{\sigma} \left(\frac{\beta}{2\pi e r}\right)^{i\beta} \check{W}\left(\frac{\beta}{2\pi r}\right) + O(\min\{|\beta|^{-5/2}, |r|^{-5/2}\}).$$

Proof. This is a direct application of Lemma 1.1 with $X = V = Q = 1$, $Y = \max\{|\beta|, |r|\}$, $V_1 = b - a$, using the above computations as well as (1.4). □

We also quote from [Mun15] the following lemma.

Lemma 1.4.

$$W^\dagger(r,s) = O_{a,b,\sigma,j}\left(\min\left\{\left(\frac{1+|\beta|}{|r|}\right)^j, \left(\frac{1+|r|}{|\beta|}\right)^j\right\}\right).$$

This Lemma follows from the following version of the stationary phase lemma without stationary point from [BKY13].

Lemma 1.5. *Let $Y \geq 1$, let $X, Q, U, R > 0$, and suppose that w is a smooth function with support on $[\alpha, \beta]$ satisfying*

$$w^{(j)}(t) \ll_j XU^{-j}.$$

Suppose that h is a smooth function on $[\alpha, \beta]$ such that

$$|h'(t)| \geq R$$

for some $R > 0$, and such that

$$h^{(j)}(t) \ll_j YQ^{-j}, \text{ for } j = 2, 3, \cdots.$$

Then the integral I defined by

$$I = \int_{-\infty}^{\infty} w(t) e^{ih(t)} \mathrm{d}t$$

satisfies

$$I \ll_A (\beta - \alpha) X[(QR/\sqrt{Y})^{-A} + (RU)^{-A}].$$

1.4 Summation formulae

We start this section by recalling the Poisson summation formula. Let $f \in L^1(\mathbb{R})$. We define the Fourier transform of f by

$$\hat{f}(y) := \int_{\mathbb{R}} f(x)e(-xy)\mathrm{d}x.$$

Proposition 1.4 (Poisson Summation Formula)**.** *Suppose that both f, $\hat{f}$ are in $L^1(\mathbb{R})$ and have bounded variation. Then*

$$\sum_{m\in\mathbb{Z}} f(m) = \sum_{n\in\mathbb{Z}} \hat{f}(n)$$

where both series converge absolutely.

Proof. See [IK04]. □

This formula is particularly interesting when estimating a long sum with "conductor smaller than the square of its length". This concept which is not formulated rigorously will be very useful to us, and we therefore seek via the following example to motivate it. Suppose therefore that V is a smooth real valued function, with support in $[a,b] \subset (0,\infty)$, and such that $V^{j)}(x) \ll_{a,b,j} 1$. Let $t \in \mathbb{R}$ and $N \in \mathbb{N}$, and suppose we wish to estimate

$$\begin{aligned}
\sum_m V\left(\frac{m}{N}\right) m^{it} &= \sum_n \int_{\mathbb{R}} V\left(\frac{x}{N}\right) x^{it} e(-xn)\mathrm{d}x \\
&= N^{1+it} \sum_n \int_{\mathbb{R}} V(y) y^{it} e(-Nyn)\mathrm{d}y \\
&= N^{1+it} \sum_n V^{\dagger}(Nn, it+1), \qquad (1.6)
\end{aligned}$$

by Poisson and where $V^{\dagger}$ is given by (1.1). We therefore see from Lemma 1.4 that the sum is negligible unless $m \ll \frac{t}{N}$. Therefore, we reduced the problem of estimating a sum of length N to that of estimating a sum of length t/N.

We now recall the Voronoi summation formula for Maass forms, which should be thought of as a GL_2 version of (1.6). We quote from [KMV02] the following formula.

Lemma 1.6. *Let g be a Hecke-Maass form over $\mathrm{SL}_2(\mathbb{Z})$ and spectral parameter t_g. Let F be a smooth function rapidly decaying at infinity, which vanishes in a neighborhood of the origin. Then, for $(a,c) = 1$, we have*

$$\sum_{n\geq 1} \rho_g(n) e\left(\frac{an}{c}\right) F(n) = \frac{1}{c} \sum_{\pm} \sum_{n\geq 1} \rho_f(\mp n) e\left(\pm \frac{n\bar{a}}{c}\right) V^{\pm}\left(\frac{n}{c^2}\right),$$

where

$$\begin{aligned}
V^{-}(y) &= \int_0^{\infty} F(x) J_g(4\pi\sqrt{xy})\mathrm{d}x \\
V^{+}(y) &= \int_0^{\infty} F(x) K_g(4\pi\sqrt{xy})\mathrm{d}x,
\end{aligned}$$

and

$$J_g(x) = -\frac{\pi}{\sin(\pi i t_g)} \left(J_{2it_g}(x) - J_{-2it_g}(x) \right),$$

and

$$K_g(x) = 4\cos(\pi i t_g) K_{2it_g}(x).$$

We now use [EMOT54, p. 326, 331] that

$$K_{2ir}(x) = \frac{1}{4}\frac{1}{2\pi i}\int_{(\sigma')} \left(\frac{x}{2}\right)^{-s} \Gamma\left(\frac{s}{2}+ir\right)\Gamma\left(\frac{s}{2}-ir\right) \mathrm{d}s, \qquad |\Re(2ir)| < \sigma'$$
$$J_{2ir}(x) = \frac{1}{2}\frac{1}{2\pi i}\int_{(\sigma')} \left(\frac{x}{2}\right)^{-s} \frac{\Gamma(s/2+ir)}{\Gamma(1-s/2+ir)} \mathrm{d}s, \qquad -\Re(2ir) < \sigma' < 1,$$

and define

$$\gamma_-(s) = \frac{-\pi}{4\pi i \sin(\pi i t_g)} \left\{ \frac{\Gamma(s/2+it_g)}{\Gamma(1-s/2+it_g)} - \frac{\Gamma(s/2-it_g)}{\Gamma(1-s/2-it_g)} \right\}$$
$$\gamma_+(s) = \frac{4\cos(\pi i t_g)}{8\pi i} \Gamma\left(\frac{s}{2}+it_g\right)\Gamma\left(\frac{s}{2}-it_g\right)$$

to deduce that for any $0 < \sigma' < 1$,

$$V^-(y) = \int_0^\infty F(x) \int_{(\sigma')} (2\pi\sqrt{xy})^{-s} \gamma_-(s) \mathrm{d}s \mathrm{d}x,$$

and

$$V^+(y) = \int_0^\infty F(x) \int_{(\sigma')} (2\pi\sqrt{xy})^{-s} \gamma_+(s) \mathrm{d}s \mathrm{d}x.$$

We conclude this section by giving a heuristic by means of an example. The idea being that the Voronoi summation formula "takes a sum of length N and conductor t to a sum of length t^2/N". Suppose therefore that V is a smooth real valued function, with support in $[a,b] \subset (0,\infty)$, and such that $V^{j)}(x) \ll_{a,b,j} 1$. Let $t \in \mathbb{R}$, c and $N \in \mathbb{N}$, and suppose we wish to estimate

$$\sum_{n\geq 1} \rho_g(n) e\left(\frac{n}{c}\right) V\left(\frac{n}{N}\right) n^{it}.$$

We estimate in this case

$$\begin{aligned} V^\pm(y) &= \int_0^\infty V\left(\frac{x}{N}\right) x^{it} \int_{(\sigma')} (2\pi\sqrt{xy})^{-s} \gamma_\pm(s) \mathrm{d}s \mathrm{d}x \\ &= N^{1+it} \int_{(\sigma')} (2\pi\sqrt{Ny})^{-s} \gamma_\pm(s) \int_0^\infty V(x) x^{it-s/2} \mathrm{d}x \mathrm{d}s. \end{aligned}$$

By a stationary phase analysis, the inner integral is negligible unless writing $s = \sigma' + i\tau$, we have $t \leq \tau \leq 3t$. We therefore reduced the problem to estimating up to negligible error for $x \in [a,b]$,

$$\int_{\mathbb{R}} (2\pi\sqrt{Nyx})^{-\sigma'-i\tau} \gamma_\pm(\sigma'+i\tau) W(\tau) \mathrm{d}\tau,$$

where W is a smooth real valued function with compact support in $[t/2, 4t]$ and satisfying $\tau^j W^{(j)}(\tau) \ll_j 1$. Restricting our attention to the first term of γ_- and using Stirling's formula, we obtain

$$\begin{aligned}&\int_{\mathbb{R}} W(\tau)(2\pi\sqrt{Nyx})^{-\sigma'-i\tau}\left|t_g+\frac{\tau}{2}\right|^{\frac{\sigma'-1}{2}}\left|t_g-\frac{\tau}{2}\right|^{\frac{1-\sigma'}{2}}\left|\frac{t_g+\frac{\tau}{2}}{e}\right|^{i(t_g+\frac{\tau}{2})}\left|\frac{t_g-\frac{\tau}{2}}{e}\right|^{i(\frac{\tau}{2}-t_g)}\mathrm{d}\tau\\ =&\int_{\mathbb{R}} g(\tau)e(f(\tau))\mathrm{d}\tau,\end{aligned}$$

where

$$g(\tau) = W(\tau)\left|t_g+\frac{\tau}{2}\right|^{\frac{\sigma'-1}{2}}\left|t_g-\frac{\tau}{2}\right|^{\frac{1-\sigma'}{2}},$$

and

$$2\pi f(\tau) = -\frac{\tau}{2}\log(4\pi^2 Nyx) + \left(t_g+\frac{\tau}{2}\right)\log\left|\frac{t_g+\frac{\tau}{2}}{e}\right| + \left(\frac{\tau}{2}-t_g\right)\log\left|\frac{\frac{\tau}{2}-t_g}{e}\right|.$$

We compute

$$2\pi f'(\tau) = -\frac{1}{2}\log(4\pi^2 Nyx) + \frac{1}{2}\log\left|\frac{t_g+\frac{\tau}{2}}{e}\right| + \frac{1}{2}\log\left|\frac{\frac{\tau}{2}-t_g}{e}\right| + 1.$$

We see that $f'(\tau) \gg 1$ unless $y \ll \tau^2/N \ll t^2/N$. The integral would then be negligible by Lemma 1.5, and thus Voronoi summation transformed a sum of length N to a sum of length t^2/N.

2 Analytic twists of modular forms

2.1 Introduction

In this chapter we are interested in sums of Fourier coefficients of GL_2 Maass forms against a certain class of oscillatory functions. The type of oscillatory functions we consider can be thought as archimedean analogs of trace functions studied in [FKM15a]. Our main result gives a non-correlation statement between Fourier coefficients of Maass forms against a family of functions, $K_t : \mathbb{R}_{>0} \to \mathbb{C}$, depending on a large real parameter t.

2.1.1 Setup

We let throughout f be a fixed cuspidal Maass Hecke eigenform for $\mathrm{SL}_2(\mathbb{Z})$, and denote by $1/4 + t_f^2$ the associated eigenvalue of the Laplacian. The form f admits a Fourier expansion

$$f(z) = \sum_{n \neq 0} \rho_f(n)|n|^{-1/2} W_{it_f}(4\pi|n|y)e(nx),$$

where W_ν is a Whittaker function. The Fourier coefficients, $\rho_f(n)$, are normalized so that by Rankin-Selberg,

$$\sum_{n \leq X} |\rho_f(n)|^2 \asymp X. \tag{2.1}$$

We moreover know that the Fourier coefficients oscillate substantially. For example, the following estimate

$$\sum_{n \leq x} \rho_f(n)e(\alpha n) \ll_f x^{1/2+\epsilon} \tag{2.2}$$

holds for any $\epsilon > 0$ uniformly for all $\alpha \in \mathbb{R}$ (see [Iwa02] Theorem 8.1). In order to understand better the oscillatory nature of the Fourier coefficients, we make the following definition.

Definition 2.1. *Let $(K(n))_{n\in\mathbb{N}}$ be a bounded sequence of complex numbers. We say that $(K(n))$ does* not *correlate with $(\rho_f(n))$ if we have*

$$\sum_{n \leq x} \rho_f(n)K(n) \ll_{f,A} x(\log x)^{-A},$$

for all $A \geq 1, x > 1$.

For example, (2.2) gives a non-correlation statement for the additive twist $K(n) = e(\alpha n)$ with a power saving of $1/2-\epsilon$. Another important example of non-correlation arises when $K(n) = \mu(n)$, the Möbius function, in which case non-correlation is an incarnation of the Prime Number Theorem (see [FG14] for a general result combining this and additive twists). Obtaining power saving statements against the Möbius function would be equivalent to proving a strong zero-free region towards the Riemann Hypothesis for the L-function attached to f. We give here a final example, which will be the main motivation for our work: let p be a prime number and let K be an isotypic trace function of conductor p, then [FKM15a] gives a non-correlation result for $(K(n))$ with a power saving of $1/8-\epsilon$.

We will study non-correlation against a family of functions $(K_t)_{t\in\mathbb{R}}$,

$$K_t : \mathbb{R}_{>0} \to \mathbb{C},$$

where t is a parameter which we will let grow to infinity.

Definition 2.2. *A family of smooth functions $(K_t)_{t\in\mathbb{R}}, K_t : \mathbb{R}_{>0} \to \mathbb{C}$ is called a family of analytic trace functions if there exist real numbers $a < b, b > 0$ and a family of analytic functions $(M_t(s))_{t\in\mathbb{R}}$ in the strip $a < \Re(s) < b$, such that the following conditions hold uniformly for $a < \Re(s) < b$.*

1. *The following integral converges for any $a < \sigma < b$,*

$$\frac{1}{2\pi i}\int_{(\sigma)} M_t(s)x^{-s}\mathrm{d}s, \tag{2.3}$$

and is equal to $K_t(x)$ for all $x \in \mathbb{R}_{>0}, t \in \mathbb{R}$.

2. *There exist constants c_1, c_2 depending on the family $(K_t)_{t\in\mathbb{R}}$, independent of t, such that we may write $M_t(\sigma + i\nu) = g_t(\sigma + i\nu)e(f_t(\sigma + i\nu))$, in such a way that for all $x \in [t, 2t]$, the following*

$$g_t^{(j)}(\sigma + i\nu) \ll_j \nu^{\sigma-1/2-j} \qquad \forall j \geq 0, \tag{2.4}$$

holds, as well as the following conditions on f_t.

   (a) *Whenever $|\nu| \leq c_1 t$ or $|\nu| \geq c_2 t$, we have*

$$\left| f_t'(\sigma + i\nu) - \frac{1}{2\pi}\log(x)\right| \gg 1, \tag{2.5}$$

   where the implicit constant does not depend on t.

   (b) *When $c_1 t \leq |\nu| \leq c_2 t$, either (2.5) holds, or we have*

$$f_t''(\sigma + i\nu) \gg (1 + |\nu|)^{-1}, \tag{2.6}$$

   while for all $\epsilon > 0$, $j \geq 0$,

$$f_t^{(j)}(\sigma + i\nu) \ll_{j,\epsilon} (1 + |\nu|)^{1+\epsilon-j}, \tag{2.7}$$

   where all the implicit constants do not depend on t.

(c) Finally, we require that

$$f_t''(\sigma+i\nu)-\frac{1}{2\pi\nu}\gg(1+|\nu|)^{-1}, \tag{2.8}$$

whenever $c_1t\le|\nu|\le c_2t$, and where the implicit constant does not depend on t.

Remark 2.1. *Throughout the paper, we will abuse notation and say that K_t is an analytic trace function when it arises as part of such a family.*

Remark 2.2. *Conditions (2.3) - (2.7) guarantee by means of stationary phase that the integral representation is concentrated around multiplicative character of conductor t. Condition (2.8) ensures that we avoid functions such as $e(x)$, as motivated in Section 2.4.*

Remark 2.3. *By the properties of the Mellin transform, we note that if $K_t(x)$ is an analytic trace function, then for any constant $\alpha\in\mathbb{R}_{>0}$, we have that $K_t(\alpha x)$ is also an analytic trace function.*

Remark 2.4. *We note that in interesting examples, in conjunction with condition (2.5), we will also have some stationary points in the region $c_1t\le|\nu|\le c_2t$, guaranteeing that $||K_t||_\infty\asymp1$.*

Remark 2.5. *We note that in practice, we may always ensure that condition (2.3) holds, by studying $K_t(x)V\left(\frac{x}{t}\right)$, where V is a smooth compactly supported function in $[\frac{1}{2},2]$. In that case, $M_t(s)$ is given by $\int_0^\infty K_t(x)x^{s-1}\mathrm{d}x$, and the integral in (2.3) converges absolutely.*

We give here some examples of analytic trace functions (see Section 2.4 for proofs).

Example 2.1. *Let J_{it} denote the usual Bessel function of order it (see [EMOT81, p. 4]). The normalized J-Bessel function of order it,*

$$F_{it}(x):=t^{1/2}\Gamma\left(\frac{1}{2}+it\right)J_{it}(x),$$

is an analytic trace function of conductor t.

This should be thought of as an archimedean analog of Kloosterman sums. We now give as a second example that of higher rank Bessel functions as appearing in [Qi15], in analogy to hyper-Kloosterman sums.

Example 2.2. *For any $n\ge3$, the n-th rank Bessel function of order t,*

$$J_{n,t}:=\frac{t^{\frac{n-1}{2}}}{2\pi in}\int_{(\frac{1}{4})}\Gamma\left(\frac{s-int}{n}\right)\Gamma\left(\frac{s}{n}+\frac{it}{n-1}\right)^{n-1}e\left(\frac{s}{4}\right)x^{-s}\mathrm{d}s,$$

is an analytic trace function.

We will study sums of the shape

$$S(t):=\sum_n\rho_f(n)K_t(n)V\left(\frac{n}{t}\right),$$

where K_t is an analytic trace function and V is a smooth function supported in $[1,2]$ and such that $V^{(j)}(x)\ll_j1$. For convenience we also normalize V so that $\int V(y)\mathrm{d}y=1$. We will show in Section 2.2 that any analytic trace function, K_t, satisfies $||K_t||_\infty \ll 1$, so that by Cauchy-Schwarz and (2.1), we have that

$$S(t) \ll t.$$

Our main result improves on that bound.

Theorem 2.1. *Let $K_t : \mathbb{R} \to \mathbb{C}$ be an analytic trace function. We have*

$$S(t) \ll t^{1-1/8+\epsilon},$$

where the implicit constant depends only on f, ϵ and on $||K_t||_\infty$.

Remark 2.6. *For simplicity we have studied the case where $n \asymp t$. We note that for $N \leq t$, one may study similarly*

$$Z(N) := \sum_n \rho_f(n) K_t(n) V\left(\frac{n}{N}\right).$$

If for $x \asymp N$, conditions (2.5) - (2.8) hold (which is the case in practice), we may show that

$$Z(N) \ll t^{1/2+\epsilon} N^{3/8},$$

which improves on the trivial bound so long as $N \gg t^{4/5+\epsilon}$.

Our bound has an application to the geometric question of equidistribution of horocycle flows with respect to a twisted signed measure. Let us recall that for every continuous compactly supported function f on $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$, we have

$$\int_0^1 f(x+iy)\mathrm{d}x \to \mu(\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H})^{-1} \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}} f(z)\mathrm{d}\mu(z),$$

as $y \to 0$, where $\mu(z) = \frac{\mathrm{d}x\mathrm{d}y}{y^2}$ denotes the hyperbolic measure (see [Zag81]). In [Str04] Strömbergsson gives a similar result by restricting to subsegments of hyperbolic length $y^{-1/2-\delta}$, i.e. that for any $\delta > 0$ and f as above,

$$\frac{1}{\beta-\alpha}\int_\alpha^\beta f(x+iy)\mathrm{d}x \to \mu(\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H})^{-1} \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}} f(z)\mathrm{d}\mu(z),$$

uniformly as $y \to 0$ so long as $\beta - \alpha$ remains bigger than $y^{1/2-\delta}$. We use Theorem 2.1 to give the following twisted version of Strömbergsson's result, which is analogous to what is proven in [FKM15a] for horocycles twisted by Frobenius trace functions.

Theorem 2.2. *Let $(K_t)_{t\in\mathbb{R}}$ be a family of analytic trace functions. Let f be a Maass form on $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$, and V be a smooth real valued function with compact support in $[\frac{1}{2}, \frac{5}{2}]$ such that $V^{(j)}(x) \ll 1$, for all $j \geq 0$. We then have for any $\delta > 0$,*

$$\frac{1}{\beta-\alpha}\int_\alpha^\beta f(x+iy)K_{1/y}\left(\frac{x}{y}\right)V(x)\mathrm{d}x \to 0,$$

uniformly as $y \to 0$ so long as $\beta - \alpha$ remains bigger than $y^{1/8-\delta}$.

2.1.2 Outline of proof of Theorem 2.1

We will show in Section 2.2 that our definition of analytic trace function implies that we may essentially write

$$K_t(x) = \frac{1}{2\pi}\int_{v\asymp t} g_t(\sigma+iv)e(f_t(\sigma+iv))x^{-\sigma-iv}\mathrm{d}v.$$

Interchanging order of summation and integration, we may therefore write

$$S(t) = \frac{1}{2\pi}\int_{v\asymp t} g_t(\sigma+iv)e(f_t(\sigma+iv))\sum_{n=1}^{\infty}\rho_f(n)n^{-\sigma-iv}V\left(\frac{n}{t}\right)\mathrm{d}v.$$

We then adapt the circle method of Munshi, as in [Mun15], allowing us to write the inner sum essentially as

$$\frac{1}{K}\int_K^{2K}\sum_{q\asymp Q}\sum_{\substack{a\asymp Q\\(a,q)=1}}\frac{1}{aq}\sum_{n\asymp t}\rho_f(n)n^{iv}e\left(\frac{n\bar{a}}{q}-\frac{nx}{aq}\right)\sum_{m\asymp t}m^{-i(v+\upsilon)}e\left(-\frac{m\bar{a}}{q}+\frac{mx}{aq}\right)\mathrm{d}\upsilon,$$

where $K\le t$ is a parameter that will ultimately be chosen optimally to be $K=t^{1/2}$, and $Q=(t/K)^{1/2}$. We may now apply Poisson summation to the m-sum, and Voronoi summation to the n-sum to arrive at the following expression for $S(t)$,

$$\sum_{n\ll K}\frac{\rho_f(n)}{\sqrt{n}}\sum_{q\asymp Q}\sum_{\substack{(m,q)=1\\1\le|m|\ll q}}e\left(\frac{n\bar{m}}{q}\right)\int_{-K}^{K}\int_{v\asymp t}n^{-i\tau/2}g(q,m,\tau,v)e(f(q,m,\tau,v))\mathrm{d}v\mathrm{d}\tau,$$

where g is a non-oscillatory amplitude function of size K and f is a well understood phase. In particular, we note that (2.8) implies that $f''(q,m,\tau,v)\gg|v|^{-1}$, so that we may use second derivative bounds for multivariable integrals and save in the integral. Applying the Cauchy-Schwarz inequality to get rid of the Fourier coefficients, and using the second derivative bound to save $(Kt)^{1/2}$ in the integral, we arrive at

$$\begin{aligned}S(t) &\ll Kt^{1/4}\left(\sum_{q,q'\asymp Q}\sum_{m,m'\asymp Q}\left(\frac{Q^{-2}}{K^{1/2}}+\sum_{\substack{n\asymp t\\ n\equiv q\overline{m'}-q'\overline{m}\mod qq'}}\frac{1}{K^{3/2}|n|^{1/2}}\right)\right)^{1/2}\\ &\ll K^{1/4}t^{3/4}+\frac{t}{K^{1/4}},\end{aligned}$$

which upon taking $K=t^{1/2}$ gives the desired result.

2.2 Analysis of K_t

In this section, we analyse further the integral representation of K_t. We make a partition of unity in the integral: let $\mathscr{I}=\{0\}\cup_{j\ge0}\{\pm\left(\frac{4}{3}\right)^j\}$, such that for each $l\in\mathscr{I}$, we take a smooth function $W_l(x)$ supported in $[\frac{3l}{4},\frac{4l}{3}]$ for $l\ne0$ and such that

$$x^kW_l^{(k)}(x)\ll_k 1,$$

for all $k \geq 0$. for $l = 0$, take $W_0(x)$ supported in $[-2,2]$ with $W_0^{(k)}(x) \ll_l 1$. and such that $1 = \sum_{l \in \mathscr{I}} W_l(x)$. We then let for any $i \in \mathscr{I}$,

$$I_{l,t}(x) := \frac{1}{2\pi} \int_{\mathbb{R}} g_t(\sigma + iv) e(f_t(\sigma + iv)) x^{-\sigma - iv} W_l(v) \mathrm{d}v.$$

We prove the following result.

Lemma 2.1. *Let K_t be an analytic trace function. We have, for $x \in [t, 2t]$, and any $\epsilon > 0$,*

$$K_t(x) = \sum_{Supp(W_l) \subset [\pm t^{1-\epsilon}, \pm t^{1+\epsilon}] \cup [-t^{\epsilon}, t^{\epsilon}]} I_{l,t}(x) + O(t^{-1000}).$$

Moreover, we also have

$$\max_{x \in [t,2t]} |K_t(x)| \ll 1.$$

Proof. Condition (2.3) implies that we may write

$$K_t(x) = \frac{1}{2\pi i} \int_{(\sigma)} M_t(s) x^{-s} \mathrm{d}s = \frac{1}{2\pi} \int_{\mathbb{R}} g_t(\sigma + iv) e(f_t(\sigma + iv)) x^{-\sigma - iv} \mathrm{d}v, \tag{2.9}$$

for any $\sigma \in [a, b]$. We now wish to run a stationary phase argument to localise the integral around the points without too much oscillation. If $l \ll t^{\epsilon}$ for some small $0 < \epsilon < \sigma/(1/2 + \sigma)$, then

$$I_{i,t}(x) \ll t^{\epsilon + \epsilon(\sigma - 1/2) - \sigma} = o(1),$$

as long as we take $\sigma > 0$. We now fix such an ϵ and look at l such that $\mathrm{Supp}(W_l) \subset [\pm t^{\epsilon}, \pm\infty)$, and look at

$$x^{\sigma} I_{l,t}(x) = \int_{\mathbb{R}} g_t(\sigma + iv) W_l(v) e\Big(f_t(\sigma + iv) - \frac{v}{2\pi} \log(x)\Big) \mathrm{d}v,$$

for $x \in [t, 2t]$. We now compute a few derivatives, in order to apply stationary phase arguments. We have by (2.4)

$$(g_t(\sigma + iv) W_l(v))^{(j)}(v) \ll_j i^{\sigma - 1/2 - j}, \qquad \forall j \geq 0,$$

while by (2.5)

$$f_t'(\sigma + iv) - \frac{\log(x)}{2\pi} \gg 1,$$

if $v \not\asymp t$ and by (2.7)

$$f_t^{(j)}(\sigma + iv) \ll l^{1 + \epsilon/2 - j}.$$

Therefore, in the case that $v \not\asymp t$, we may use Lemma 1.1 (with $X = l^{\sigma - 1/2}, U = l, \beta - \alpha = 3l/2, R = 1, Y = l^{1+\epsilon/2}$ and $Q = l$), to deduce that

$$I_{l,t}(x) \ll_A l^{-A},$$

for any $A > 0$.

In the case that $v \asymp t$, we use the second derivative bound for oscillatory integrals along with (2.6) to deduce that

$$I_{l,t}(x) \ll 1.$$

□

To conclude this section we note that the case where $\mathrm{Supp}(W_l) \subset [-t^{\epsilon}, t^{\epsilon}]$ can be handled as follows. Since V is a smooth compactly supported function, it admits a Mellin transform,

$$\tilde{V}(s) = \int_0^{\infty} V(x)x^{s-1}\mathrm{d}x,$$

that decays very rapidly in vertical strips. One can thus write for any $\alpha \in \mathbb{R}$,

$$V(x) = \int_{(\alpha)} \tilde{V}(s)x^{-s}\mathrm{d}s.$$

Using this, we write for any $\sigma \geq 0$,

$$\begin{aligned}\sum_{n=1}^{\infty} \rho_f(n) I_{l,t}(n) V\left(\frac{n}{t}\right) &= \int_{\mathbb{R}} M_t(\sigma + iv) W_l(v) \sum_{n=1}^{\infty} \rho_f(n) n^{-\sigma - iv} V\left(\frac{n}{t}\right) \mathrm{d}v \\ &= \int_{\mathbb{R}} \int_{(\alpha)} M_t(\sigma + iv) W_l(v) \tilde{V}(s) t^s L(f, \sigma + iv + s) \mathrm{d}s \mathrm{d}v \\ &\ll t^{1/2+\epsilon},\end{aligned}$$

by the rapid decay of $\tilde{V}$.

We will therefore only focus on the cases where the support of W_l is close to t. This may be interpreted as the fact that the spectral decomposition of any analytic trace function, K_t, concentrates around multiplicative characters of conductor t.

2.3 Proof of Theorem 2.1

Following Munshi [Mun15] we adapt Kloosterman's version of the circle method along with a conductor dropping mechanism. We quote here the following proposition in [IK04].

Proposition 2.1. *Let*

$$\delta(n) = \begin{cases} 1 & \text{if } n = 0; \\ 0 & \text{otherwise.} \end{cases}$$

Then, for any real number $Q \geq 1$, we have

$$\delta(n) = 2\Re \int_0^1 \sideset{}{^*}\sum_{1 \leq q \leq Q < a \leq q+Q} \frac{1}{aq} e\left(\frac{n\bar{a}}{q} - \frac{nx}{aq}\right) \mathrm{d}x.$$

In particular, we will use this proposition with $Q := (t/K)^{1/2}$, where $t^{\epsilon'} < K < t^{1-\epsilon'}$ (for some $\epsilon' > 0$) is a parameter to be chosen optimally later. We let

$$S_l(t) := \sum_{n=1}^{\infty} \rho_f(n) I_{l,t}(n) V\left(\frac{n}{t}\right),$$

and note that in order to bound non-trivially $S(t)$, it is sufficient to do so for $S_l(t)$, for l such that $\mathrm{Supp} W_l \subset [\pm t^{1-\epsilon}, \pm t^{1+\epsilon}]$, as follows from the previous section. We may thus write

$$\begin{aligned} S_l(t) &= \sum_{n=1}^{\infty} \rho_f(n) I_{l,t}(n) V\left(\frac{n}{t}\right) \\ &= \frac{1}{K} \int_{\mathbb{R}} V\left(\frac{v}{K}\right) \sum_{\substack{n,m=1 \\ n=m}}^{\infty} \rho_f(n) I_{l,t}(m) \left(\frac{n}{m}\right)^{iv} V\left(\frac{n}{t}\right) U\left(\frac{m}{t}\right) \mathrm{d}v \\ &= S_l^+(t) + S_l^-(t),\end{aligned}$$

where U is a smooth functions supported in $[1/2, 5/2]$, with $U(x) = 1$ for $x \in \mathrm{Supp}(V)$ and $U^{(j)} \ll_j 1$, and

$$S_l^{\pm}(t) = \frac{1}{K}\int_0^1 \int_{\mathbb{R}} V\left(\frac{v}{K}\right) \sideset{}{^*}\sum_{1\le q\le Q<a\le Q+q} \frac{1}{aq}$$
$$\times \sum_{n,m=1}^{\infty} \rho_f(n) n^{iv} I_{l,t}(m) m^{-iv} e\left(\pm\frac{(n-m)\bar{a}}{q} \mp \frac{(n-m)x}{aq}\right) V\left(\frac{n}{t}\right) U\left(\frac{m}{t}\right) \mathrm{d}v\mathrm{d}x.$$

We will now describe the analysis for $S_l^+(t)$ (the analysis for $S_l^-(t)$ being completely analogous).

2.3.1 Summation formulae

We start with the m-sum, which we split into congruence classes mod q, and after applying Poisson summation, we obtain

$$\sum_{m=1}^{\infty} I_{l,t}(m) m^{-iv} U\left(\frac{m}{t}\right) e\left(-\frac{m\bar{a}}{q}\right) e\left(\frac{mx}{aq}\right)$$
$$= \sum_{\substack{m\in\mathbb{Z} \\ m\equiv \bar{a}\,(\mathrm{mod}\, q)}} \frac{t^{1-\sigma-iv}}{2\pi} \int_{\mathbb{R}} t^{-i\nu} M_t(\sigma+i\nu) W_l(\nu) U^{\dagger}\left(\frac{t(ma-x)}{aq}, 1-\sigma-i(\nu+v)\right) \mathrm{d}\nu.$$

We now note that since $|\nu| \in [t^{1-\epsilon}, t^{1+\epsilon}]$, we may as in [Mun15] use Lemma 1.4 to deduce that only the contribution from $1 \le |m| \ll q t^{\epsilon}$ is non-negligible. We take a dyadic subdivision to obtain the following.

Lemma 2.2.

$$S_l^+(t) = \frac{t^{1-\sigma}}{K} \sum_{1\le C\le (t/K)^{1/2}} S_l(t,C) + O(t^{-1000}),$$

where C runs over dyadic integers and

$$S_l(t,C) = \frac{1}{2\pi}\int_{\mathbb{R}}\int_0^1\int_{\mathbb{R}} M_t(\sigma+i\nu) W_l(\nu) t^{-i(v+\nu)} V\left(\frac{v}{K}\right) \sum_{C<q\le 2C} \sum_{\substack{(m,q)=1 \\ 1\le |m|\ll q t^{\epsilon}}} \frac{1}{aq}$$
$$\times U^{\dagger}\left(\frac{t(ma-x)}{aq}, 1-\sigma-i(v+\nu)\right) \sum_{n=1}^{\infty} \rho_f(n) n^{iv} e\left(\frac{nm}{q}\right) e\left(-\frac{nx}{aq}\right) V\left(\frac{n}{t}\right) \mathrm{d}v\mathrm{d}x\mathrm{d}\nu$$

and $a = a_Q(m,q)$ is the unique multiplicative inverse of m $\mod q$ *in $(Q, q+Q]$.*

We wish to use the Voronoi summation on the n-sum. We apply Lemma 1.6 to

$$\sum_{n\ge 1} \rho_f(n) e\left(\frac{nm}{q}\right) n^{iv} e\left(\frac{-nx}{aq}\right) V\left(\frac{n}{t}\right) = \frac{t^{1+iv}}{q} \sum_{\pm}\sum_{n\ge 1} \rho_f(\mp n) e\left(\pm\frac{n\bar{m}}{q}\right) I(n,q,v,x),$$

where

$$I(n,q,v,x) = \int_{(\sigma')} \left(\frac{2\pi\sqrt{nt}}{q}\right)^{-s} \gamma_{\pm}(s) \int_0^{\infty} y^{iv} e\left(\frac{-tyx}{aq}\right) V(y) y^{-s/2} \mathrm{d}y\mathrm{d}s$$
$$= \int_{(\sigma')} \left(\frac{2\pi\sqrt{nt}}{q}\right)^{-s} \gamma_{\pm}(s) V^{\dagger}\left(\frac{tx}{aq}, 1+iv-s/2\right) \mathrm{d}s.$$

By Stirling's formula:

$$\Gamma(\sigma' + it)$$
$$= \sqrt{2\pi}\exp\left(\frac{-\pi|t|}{2}\right)|t|^{\sigma'-1/2}\left|\frac{t}{e}\right|^{it}\exp(\operatorname{sign}(t)i\pi(\sigma'-1/2)/2)(1+O(|t|^{-1})),$$

for $|t| \ge 1$ and bounded σ', we deduce that

$$\gamma_\pm(\sigma' + i\tau) \ll 1 + |\tau|^{\sigma'-1}.$$

Now, by Lemma 1.4,

$$V^\dagger\left(\frac{tx}{aq}, 1 + iv - s/2\right) \ll \min\left\{1, \left(\frac{(Kt)^{1/2}}{|v - \tau/2|q}\right)^j\right\}.$$

Thus, shifting the contour to $\sigma' = M$ a large positive integer and taking $j = M+1$ for instance, we see that if $n \gg Kt^\epsilon$, then the integral is negligible (by splitting the integral into a box around $|v - \frac{\tau}{2}|q \le (Kt)^{1/2}$ and its complement). In the remaining range, we study this more closely. We shift our contour to $\sigma = 1$ (the γ_+ contribution is trivial, so we only consider γ_-), and note that

$$\gamma_-(i\tau + 1) = \left(\frac{|\tau|}{2e}\right)^{i\tau}\Phi_-(\tau),$$

where $\Phi'_-(\tau) \ll |\tau|^{-1}$. We thus have

$$I(n,q,v,x) = \frac{qi}{2\pi\sqrt{nt}}\sum_{J\in\mathscr{J}}\int_{\mathbb{R}}\left(\frac{2\pi\sqrt{nt}}{q}\right)^{-i\tau}\gamma_\pm(i\tau+1)V^\dagger\left(\frac{tx}{aq}, \frac{1}{2} + i(v - \tau/2)\right)W_J(\tau)\mathrm{d}\tau$$
$$+ O(t^{-1000}),$$

where $\mathscr{J}$ is a collection of $O(\log t)$ integers such that $J \in \mathscr{J}$ if and only if

$$\operatorname{Supp} W_J \subset [-(tK)^{1/2}t^\epsilon/C, (tK)^{1/2}t^\epsilon/C].$$

We have proven the following:

Lemma 2.3.

$$S_l(t,C) = \frac{iKt^{1/2}}{4\pi^2}\sum_\pm\sum_{J\in\mathscr{J}}\sum_{n\ll Kt^\epsilon}\frac{\rho_f(\mp n)}{\sqrt{n}}\sum_{C<q\le 2C}\sum_{\substack{(m,q)=1\\1\le|m|\ll qt^\epsilon}}\frac{e\left(\pm\frac{n\bar{m}}{q}\right)}{aq}I^*_\pm(q,m,n)$$
$$+ O(t^{-10000}),$$

where

$$I^*_\pm(q,m,n) = \int_{\mathbb{R}^2} M_t(\sigma + i\nu)W_l(\nu)t^{-i\nu}\left(\frac{2\pi\sqrt{nt}}{q}\right)^{-i\tau}\gamma_\pm(i\tau+1)I^{**}(q,m,\tau,\nu)W_J(\tau)\mathrm{d}\tau\mathrm{d}\nu,$$

and

$$I^{**}(q,m,\tau,\nu) = \int_0^1\int_{\mathbb{R}} V(v)V^\dagger\left(\frac{tx}{aq}, i\left(kv - \frac{\tau}{2}\right) + \frac{1}{2}\right)$$
$$\times U^\dagger\left(\frac{t(ma - x)}{aq}, 1 - \sigma - i(Kv + \nu)\right)\mathrm{d}v\mathrm{d}x.$$

In the next two subsections we evaluate $I^{**}(q,m,\tau,\nu)$.

2.3.2 Analysis of the integrals

We apply lemma 1.3 to

$$
\begin{aligned}
&U^{\dagger}\left(\frac{t(ma-x)}{aq}, 1-\sigma-i(Kv+\nu)\right)\\
&= e\left(\frac{1}{8}\right)\left(\frac{Kv+\nu}{2\pi}\right)^{1/2-\sigma}\left(\frac{aq}{t(x-ma)}\right)^{1-\sigma}\left(\frac{(Kv+\nu)aq}{2\pi et(x-ma)}\right)^{-i(Kv+\nu)}\\
&\qquad\qquad \times \check{U}\left(\frac{(Kv+\nu)aq}{2\pi t(x-ma)}\right)+O(t^{-5/2}).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
I^{**}(q,m,\tau,\nu) =c_1 \int_0^1\int_{\mathbb{R}} &V(v)V^{\dagger}\left(\frac{tx}{aq}, i\left(Kv-\frac{\tau}{2}\right)+\frac{1}{2}\right)(Kv+\nu)^{1/2-\sigma}\left(\frac{aq}{t(x-ma)}\right)^{1-\sigma}\\
&\times\left(\frac{(Kv+\nu)aq}{2\pi et(x-ma)}\right)^{-i(Kv+\nu)}\check{U}\left(\frac{(Kv+\nu)aq}{2\pi t(x-ma)}\right)\mathrm{d}v\mathrm{d}x+O(t^{-5/2}),
\end{aligned}
$$

for some constant c_1. We now use lemma 5 of [Mun15] to

$$
\begin{aligned}
V^{\dagger}\left(\frac{tx}{aq}, i(Kv-\tau/2)+\frac{1}{2}\right) &= \frac{(aq)^{1/2}e(-\frac{1}{8})}{(tx)^{1/2}}\left(\frac{(Kv-\frac{\tau}{2})aq}{2e\pi tx}\right)^{i(Kv-\frac{\tau}{2})} V\left(\frac{(Kv-\tau/2)aq}{2\pi tx}\right)\\
&\quad + O\left(\min\left\{|Kv-\tau/2|^{-3/2}, \left(\frac{tx}{aq}\right)^{-3/2}\right\}\right).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
I^{**}(q,m,\tau,\nu) =c_2 \int_0^1\int_{\mathbb{R}} &V(v)V\left(\frac{(Kv-\frac{\tau}{2})aq}{2\pi tx}\right)\left(\frac{(Kv-\frac{\tau}{2})aq}{2e\pi tx}\right)^{i(Kv-\frac{\tau}{2})}\check{U}\left(\frac{(kv+\nu)aq}{2\pi t(x-ma)}\right)\\
&\left(\frac{aq}{t}\right)^{\frac{3}{2}-\sigma}\frac{(\nu+Kv)^{\frac{1}{2}-\sigma}}{(x-ma)^{1-\sigma}}\left(\frac{(Kv+\nu)aq}{2\pi et(x-ma)}\right)^{-i(Kv+\nu)}\mathrm{d}v\frac{\mathrm{d}x}{x^{1/2}}+E+O(t^{-\frac{5}{2}}),
\end{aligned}
$$

for some constant c_2 and where E comes from the error term of $V^{\dagger}$ which we will now describe. We first note that since $V^{\dagger}\left(\frac{tx}{aq}, i(Kv-\tau/2)+\frac{1}{2}\right)$ does not depend on ν, neither does the error term, and therefore we may perform the ν-integral without losing control of the phase, before plugging absolute values. We thus estimate

$$
\begin{aligned}
&\int_{\mathbb{R}} M_t(\sigma+i\nu)W_l(\nu)t^{-i\nu}(Kv+\nu)^{1/2-\sigma}\left(\frac{(Kv+\nu)aq}{2\pi et(x-ma)}\right)^{-i(Kv+\nu)}\check{U}\left(\frac{(Kv+\nu)aq}{2\pi t(x-ma)}\right)\mathrm{d}\nu\\
&=\int_{\mathbb{R}} g(\nu)e(f(\nu))\mathrm{d}\nu,
\end{aligned}
$$

where, temporarily, we define

$$
g(\nu) = g_t(\sigma+i\nu)W_l(\nu)(Kv+\nu)^{1/2-\sigma}\check{U}\left(\frac{(Kv+\nu)aq}{2\pi t(x-ma)}\right),
$$

and

$$
2\pi f(\nu) = 2\pi f_t(\sigma+i\nu)-\nu\log t-(Kv+\nu)\log\left|\frac{(Kv+\nu)aq}{2\pi et(x-ma)}\right|.
$$

We have

$$
2\pi f''(\nu) = 2\pi f_t''(\sigma+i\nu)-\frac{1}{Kv+\nu} \gg \nu^{-1},
$$

by (2.8). Noting that $g(v) \ll 1$, and $\int |g'(v)| \ll t^{\epsilon}$, we may use the second derivative bound for oscillatory integrals (see [Sri65], Lemma 5) to deduce that

$$\int_{\mathbb{R}} g(v)e(f(v))\mathrm{d}v \ll t^{1/2+\epsilon}. \tag{2.10}$$

Our error term, E, therefore satisfies

$$\begin{aligned}&\int_{\mathbb{R}} M_t(\sigma+iv)W_l(v)t^{-iv}E\mathrm{d}v\\ &\ll t^{\sigma-1/2+\epsilon}\int_0^1\int_1^2 \min\left\{\left|Kv-\frac{\tau}{2}\right|^{-3/2},\left(\frac{tx}{aq}\right)^{-3/2}\right\}\mathrm{d}v\mathrm{d}x.\end{aligned}$$

This integral is the same than the one appearing in [Mun15], where it is proved that

$$\int_0^1\int_1^2 \min\left\{\left|Kv-\frac{\tau}{2}\right|^{-3/2},\left(\frac{tx}{aq}\right)^{-3/2}\right\}\mathrm{d}v\mathrm{d}x \ll \frac{1}{K^{3/2}}\min\left\{1,\frac{10K}{|\tau|}\right\}t^{\epsilon}.$$

Moreover, we note that

$$\int_{\mathbb{R}} g_t(\sigma+iv)W_l(v)t^{-5/2} \ll t^{-2+\sigma},$$

and thus (keeping in mind that $t^{\epsilon} < K < t^{1-\epsilon}$),

$$\int_{\mathbb{R}} M_t(\sigma+iv)W_l(v)t^{-iv}(E+O(t^{-5/2}))\mathrm{d}v \ll \frac{t^{\sigma+\epsilon}}{t^{1/2}K^{3/2}}\min\left\{1,\frac{10K}{|\tau|}\right\}.$$

We now treat the main term. Let $\delta' > 0$ to be determined later and examine the contribution from $x < 1/K^{1-\delta'}$. Using (2.10) and that $u^{\alpha}\check{U}(u), v^{\alpha}V(v) \ll 1$, for all $\alpha \in \mathbb{R}$, (and thus $t(x-ma)(aq)^{-1} \gg t^{1-\epsilon}$), we estimate

$$\begin{aligned}&\left(\frac{aq}{t}\right)^{1/2}\int_0^{K^{\delta'-1}}\int_{\mathbb{R}} V(v)V\left(\frac{(Kv-\frac{\tau}{2})aq}{2\pi tx}\right)\left(\frac{aq}{t(x-ma)}\right)^{1-\sigma}\left|\int_{\mathbb{R}} g(v)e(f(v))\mathrm{d}v\right|\mathrm{d}v\frac{\mathrm{d}x}{x^{1/2}}\\ &\ll t^{\epsilon}\int_0^{K^{\delta'-1}}\int_{Kv-\frac{\tau}{2}\asymp\frac{tx}{aq}} V(v)\frac{t^{\sigma}}{t^{1/2}(Kv-\frac{\tau}{2})^{1/2}}\mathrm{d}v\mathrm{d}x \ll \frac{t^{1/2+\sigma+\epsilon}}{K^{3-\epsilon}aq},\end{aligned}$$

upon taking $\delta' = 2\epsilon/3$. We now look at the contribution from $x \in [K^{\delta'-1}, 1]$. We now reset temporarily

$$g(v) = (v+Kv)^{1/2-\sigma}\left(\frac{aq}{t(x-ma)}\right)^{1-\sigma}V(v)V\left(\frac{(Kv-\frac{\tau}{2})aq}{2\pi tx}\right)\check{U}\left(\frac{(Kv+v)aq}{2\pi t(x-ma)}\right),$$

and

$$f(v) = \frac{Kv-\frac{\tau}{2}}{2\pi}\log\left(\frac{(Kv-\frac{\tau}{2})aq}{2e\pi tx}\right)-\frac{Kv+v}{2\pi}\log\left(\frac{(Kv+v)aq}{2\pi et(x-ma)}\right).$$

Then,

$$f'(v) = -\frac{K}{2\pi}\log\left(\frac{(v+Kv)x}{(Kv-\frac{\tau}{2})(x-ma)}\right), f^{(j)}(v) = -\frac{(j-2)!(-K)^j}{2\pi(v+Kv)^{j-1}}+\frac{(j-2)!(-K)^j}{2\pi(Kv-\frac{\tau}{2})^{j-1}},$$

for $j \geq 2$, and the stationary point is given by

$$v_0 = -\frac{(2\nu+\tau)x - \tau ma}{2Kma}.$$

Now, since $\nu \gg t^{1-\epsilon}$, we have that in the support of the integral,

$$f^{(j)} \asymp \frac{tx}{aq}\left(\frac{Kaq}{tx}\right)^j,$$

for $j \geq 2$, and

$$g^{(j)}(v) \ll t^{-1/2+\epsilon}\left(1+\frac{Kaq}{tx}\right)^j,$$

for $j \geq 0$. Moreover, we can write

$$f'(v) = \frac{K}{2\pi}\log\left(1+\frac{K(v_0-v)}{\nu+Kv}\right) - \frac{K}{2\pi}\log\left(1+\frac{K(v_0-v)}{Kv-\tau/2}\right),$$

and note that in the support of the integral we have $0 \leq Kv - \tau/2 \ll tx/aq \ll K^{1/2}t^{1/2}$. It follows that if $v_0 \notin [.5,3]$, then in the support of the integral we have

$$|f'(v)| \gg K\min\left\{1, \frac{Kaq}{tx}\right\}.$$

We now use Lemma 1.5 with

$$X = t^{-1/2+\epsilon}, U(=V) = \min\left\{1, \frac{tx}{Kaq}\right\}, \qquad R = K\min\left\{1, \frac{Kaq}{tx}\right\},$$
$$Y = \frac{tx}{aq}, \qquad Q = \frac{tx}{Kaq},$$

so that, choosing $K > t^{1/3+\epsilon}$,

$$\begin{aligned}\int_{\mathbb{R}} g(v)e(f(v))\mathrm{d}v &\ll t^{-1/2+\epsilon}\left[\left(\left(\frac{tx}{aq}\right)^{1/2}\min\left\{1,\frac{Kaq}{tx}\right\}\right)^{-A} + K^{-A}\right] \\ &\ll t^{-1/2+\epsilon}\left[(t^{3\epsilon/4})^{-A} + (K^{\delta'})^{-A} + K^{-A}\right] \ll t^{-B},\end{aligned}$$

for any $B > 0$. In the case where $v_0 \in [.5,3]$, we will use Lemma 1.1, with $\delta = 1/100$, $A = 10000\delta'^{-1}$ and the same X, Y, V and Q as above. We have

$$\int_{\mathbb{R}} g(v)e(f(v))\mathrm{d}v = \frac{e(f(v_0))}{\sqrt{2\pi f''(v_0)}}\sum_{n=0}^{300A} p_n(v_0) + O_{\delta'}\left(\left(\frac{tx}{aq}\right)^{-A}\right)$$

where p_n is given by (1.2). Now, since $x \in [K^{\delta'-1}, 1]$, we have $tx/aq \gg K^{\delta'}$, and therefore the error term is negligible.

2.3.3 Contribution from $n \geq 1$ terms

We find that

$$f(v_0) = -\frac{\nu+\tau/2}{2\pi}\log\left(-\frac{(\nu+\tau/2)q}{2e\pi tm}\right),$$

and

$$f''(v_0) = \frac{K^2(ma)^2}{2\pi(\nu+\tau/2)(x-ma)x}, \tag{2.11}$$

and

$$f^{(j)}(v_0) = \frac{(j-2)!(-K)^j(ma)^{j-1}((x-ma)^{j-1}+(-x)^{j-1})}{2\pi(\nu+\tau/2)^{j-1}(ma-x)^{j-1}x^{j-1}}. \tag{2.12}$$

We also find

$$\begin{aligned} g(v_0) = {} & \left(\frac{tm}{(\nu+\frac{\tau}{2})q}\right)^{\sigma}\left(\frac{aq}{t}\right)\left(\frac{-(\nu+\frac{\tau}{2})}{(x-ma)ma}\right)^{1/2} \\ & \times V\left(\frac{\tau}{2K}-\frac{(\nu+\frac{\tau}{2})x}{Kma}\right)\check{U}\left(\frac{-(\nu+\frac{\tau}{2})q}{2\pi tm}\right)V\left(-\frac{(\nu+\frac{\tau}{2})q}{m2\pi t}\right). \end{aligned} \tag{2.13}$$

We wish to keep the term $n = 0$ and show that the terms with $n \geq 1$ can be absorbed into an error term. We thus look to bound

$$\int_{\mathbb{R}} M_t(\sigma+i\nu)W_l(\nu)t^{-i\nu}\frac{e(f(v_0))}{\sqrt{f''(v_0)}}p_n(v_0)\mathrm{d}\nu = \int_{\mathbb{R}} \tilde{g_n}(\nu)e(\tilde{f}(\nu))\mathrm{d}\nu,$$

where

$$\tilde{g_n}(\nu) := \frac{\sqrt{2\pi}(x-ma)^{1/2}x^{1/2}}{Kma}g_t(\sigma+i\nu)W_l(\nu)(\nu+\tau/2)^{1/2}p_n\left(-\frac{(2\nu+\tau)x-\tau ma}{2Kma}\right),$$

and

$$\tilde{f}(\nu) := f_t(\sigma+i\nu) - \frac{\nu}{2\pi}\log t - \frac{\nu+\frac{\tau}{2}}{2\pi}\log\left(-\frac{(\nu+\frac{\tau}{2})q}{2e\pi tm}\right).$$

We compute

$$\tilde{f}'(\nu) = f_t'(\sigma+i\nu) - \frac{\log t}{2\pi} - \frac{1}{2\pi}\log\left(-\frac{(\nu+\frac{\tau}{2})q}{2e\pi tm}\right) - \frac{1}{2\pi},$$

and

$$\tilde{f}''(\nu) = f_t''(\sigma+i\nu) - \frac{1}{2\pi(\nu+\frac{\tau}{2})}.$$

In order to estimate the size of $\tilde{g_n}$, we estimate first

$$p_1(v_0) \ll \frac{g''(v_0)}{f''(v_0)} \ll \frac{XQ^2}{V^2Y}, \qquad p_2(v_0) \ll \frac{XQ^4}{V^4Y^2} + \frac{XQ}{VY} + \frac{X}{Y},$$

while, by (1.4), for $n \geq 3$ we have

$$p_n(v_0) \ll X\left(\left(\frac{V^2Y}{Q^2}\right)^{-n} + Y^{-n/3}\right).$$

We now distinguish two cases. If $x \leq \frac{Kaq}{t}$, then $V = Q = \frac{tx}{Kaq}$, and thus

$$p_n(v_0) \ll \frac{X}{Y},$$

for all $n \geq 1$, since $Y = \frac{tx}{aq} \gg K^{\delta'}$. We then show by (1.4) that

$$\tilde{g_n}'(v) \ll \frac{(x-ma)^{1/2}x^{1/2}(v+\frac{\tau}{2})^{1/2}}{Kmav^{3/2-\sigma}}\frac{X}{Y},$$

so that by the second derivative bound for oscillatory integrals (using that $q \asymp m$, by the support of $\check{U}$),

$$\int_{\mathbb{R}} \tilde{g_n}(v)e(f(v))\mathrm{d}v \ll \frac{t^{\sigma+\epsilon}(aq)^{1/2}}{Ktx^{1/2}}.$$

Therefore the total contribution from this part is dominated by

$$\left(\frac{aq}{t}\right)^{1/2}\int_{K^{\delta'-1}}^{1}\frac{t^{\sigma+\epsilon}(aq)^{1/2}}{Ktx}\mathrm{d}x \ll \frac{t^{\sigma+\epsilon}}{K^2t^{1/2}}.$$

For $x > \frac{Kaq}{t}$, we have $V = 1$, and so

$$p_n(v_0) \ll \frac{t^{1/2+\epsilon}x}{K^2aq}.$$

In this region, we first pass the x integral inside the v-integral, and since the phase does not depend on x, the same analysis holds, replacing $\tilde{g_n}(v)$ by

$$\hat{g_n}(v) := \left(\frac{aq}{t}\right)^{1/2}\int_{\max\{K^{-1+\delta'},Kaq/t\}}^{1}\frac{1}{\sqrt{x}}\tilde{g_n}(v)\mathrm{d}x.$$

We have, using that $m \asymp q$,

$$\hat{g_n}(v) \ll \left(\frac{aq}{t}\right)^{1/2}\tilde{g_n}(v) \ll \frac{t^{\sigma+\epsilon}}{K^3aq}.$$

In order to control $\hat{g_n}'(v)$, we will first execute the x-integral, using integration by parts. Looking at the definition of p_n, we note that it is a rational function in $f''(v_0), f'''(v_0), \cdots, g(v_0), g'(v_0)\cdots$ and will describe what the terms of p_n depending on x look like. We first recall that by (1.2) and (1.3),

$$p_n(v_0) = \frac{\sqrt{2\pi}e^{\pi i/4}}{n!}\left(\frac{i}{2\tilde{f}''(v_0)}\right)^n G^{(2n)}(v_0),$$

where $G^{(2n)}(v_0)$ is a linear combination of elements of the form

$$\hat{g}_n^{(l_0)}(v_0)\tilde{f}^{(l_1)}(v_0)\cdots\tilde{f}^{(l_j)}(v_0),$$

where $l_0+\cdots+l_j = 2n$. Using (2.11), (2.12) and (2.13), we therefore have that those terms of p_n depending on x are of the shape

$$x^i(x-ma)^{j+1/2}V^{(l)}\left(\frac{\tau}{2K}-\frac{(v+\frac{\tau}{2})x}{Kma}\right),$$

for some $i, j \geq 1$ and $l \geq 0$. We thus compute

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}\nu}\int_{\max\{K^{-1+\delta'},Kaq/t\}}^{1} x^{i-1/2}(x-ma)^{j+1/2}V^{(l)}\left(\frac{\tau}{2K}-\frac{(\nu+\tau/2)x}{Kma}\right)\mathrm{d}x\\
&=\frac{\mathrm{d}}{\mathrm{d}\nu}\left(\left[x^{i-1/2}(x-ma)^{j+1/2}\frac{-Kma}{t\nu+\tau/2}V^{(l-1)}\left(\frac{\tau}{2K}-\frac{(\nu+\tau/2)x}{Kma}\right)\right]_{\max\{K^{-1+\delta'},Kaq/t\}}^{1}\right.\\
&\left.+\int_{\max\{K^{-1+\delta'},Kaq/t\}}^{1}(x^{i-1/2}(x-ma)^{j+1/2})'\frac{Kma}{t\nu+\tau/2}V^{(l-1)}\left(\frac{\tau}{2K}-\frac{(\nu+\tau/2)x}{Kma}\right)\right)\mathrm{d}x\\
&\ll\frac{(ma)^{j+1/2}}{\nu+\frac{\tau}{2}}.
\end{aligned}$$

These calculations show that

$$\int_{\mathbb{R}}\left|\frac{\mathrm{d}}{\mathrm{d}\nu}\hat{g_n}(\nu)\right|\mathrm{d}\nu \ll t^{\epsilon}\hat{g_n} \ll \frac{t^{\sigma+\epsilon}}{K^3aq},$$

and by the second derivative bound for oscillatory integrals,

$$\int_{\mathbb{R}}\hat{g_n}(\nu)e(\tilde{f}(\nu))\mathrm{d}\nu \ll \frac{t^{1/2+\sigma+\epsilon}}{K^3aq},$$

which is the same bound we obtained for $x \in (0, K^{\delta'-1})$. We therefore obtain

$$\left(\frac{aq}{t}\right)^{1/2}\int_0^1\int_{\mathbb{R}} g(\upsilon)e(f(\upsilon))\mathrm{d}\upsilon\frac{\mathrm{d}x}{x^{1/2}} = \left(\frac{aq}{t}\right)^{1/2}\int_{K^{-1+\delta'}}^{1}\frac{g(\upsilon_0)e(f(\upsilon_0)+1/8)}{x^{1/2}\sqrt{f''(\upsilon_0)}}\mathrm{d}x + E^*,$$

where E^* is an error term such that

$$\int_{\mathbb{R}} M_t(\sigma+i\nu)W_l(\nu)t^{-i\nu}E^*\mathrm{d}\nu \ll \frac{t^{1/2+\sigma+\epsilon}}{aqK^3}.$$

Now, plugging in the value for υ_0, we get that the leading term above reduces to

$$\begin{aligned}
&c_3\frac{\nu+\frac{\tau}{2}}{K}\left(\frac{-q}{mt}\right)^{3/2}V\left(\frac{-(\nu+\frac{\tau}{2})q}{2\pi mt}\right)\left(-\frac{(\nu+\frac{\tau}{2})q}{2e\pi tm}\right)^{-i(\nu+\tau/2)}\left(\frac{tm}{(\nu+\frac{\tau}{2})q}\right)^{\sigma}\\
&\times\check{U}\left(-\frac{(\nu+\frac{\tau}{2})q}{2\pi tm}\right)\int_{K^{-1+\delta'}}^{1}V\left(\frac{\tau}{2K}-\frac{(\nu+\frac{\tau}{2})x}{Kma}\right)\mathrm{d}x,
\end{aligned}$$

for some absolute constant c_3. Set

$$B(C,\tau,\nu) = t^{-5/2}+E+E^*,$$

and note that

$$\int_{-\frac{(tK)^{1/2}t^{\epsilon}}{C}}^{\frac{(TK)^{1/2}t^{\epsilon}}{C}}\int_{\mathbb{R}} M_t(\sigma+i\nu)W_l(\nu)t^{-i\nu}B(C,\tau,\nu)\mathrm{d}\nu\mathrm{d}\tau \ll \frac{t^{\sigma+\epsilon}}{t^{1/2}K^{1/2}}\left(1+\frac{t}{C^2K^{3/2}}\right). \tag{2.14}$$

We may now derive from these computations the following:

Lemma 2.4. *We have*

$$I^{**}(q,m,\tau,\nu) = I_1(q,m,\tau,\nu) + I_2(q,m,\tau,\nu),$$

where

$$\begin{aligned} I_1(q,m,\tau,\nu) = {} & \frac{c_4}{(\nu+\frac{\tau}{2})^{1/2}K}\left(-\frac{(\nu+\frac{\tau}{2})q}{2\pi etm}\right)^{3/2-i(\nu+\frac{\tau}{2})} V\left(-\frac{(\nu+\frac{\tau}{2})q}{2\pi mt}\right)\left(\frac{tm}{(\nu+\frac{\tau}{2})aq}\right)^{\sigma} \\ & \times \check{U}\left(-\frac{(\nu+\frac{\tau}{2})q}{2\pi mt}\right)\int_{K^{-1+\delta'}}^{1} V\left(\frac{\tau}{2K}-\frac{(\nu+\frac{\tau}{2})x}{Kma}\right)\mathrm{d}x, \end{aligned}$$

where c_4 is an absolute constant, and

$$I_2(q,m,\tau,\nu) := I^{**}(q,m,\tau,\nu) - I_1(q,m,\tau,\nu) = B(C,\tau,\nu).$$

Consequently from Lemma 2.3 we arrive at:

Lemma 2.5. *We have*

$$S_l(t,C) = \sum_{J\in\mathscr{J}}\left\{S_{1,J}(t,C) + S_{2,J}(t,C)\right\} + O(t^{-1000}),$$

where

$$\begin{aligned} S_{r,J}(t,C) = {} & \frac{iKt^{1/2}}{4\pi^2}\sum_{\pm}\sum_{n\ll Kt^{\epsilon}}\frac{\rho_f(\mp n)}{\sqrt{n}} \\ & \sum_{C<q\le 2C}\sum_{\substack{(m,q)=1\\ 1\le|m|\ll qt^{\epsilon}}}\frac{e\left(\pm\frac{n\bar{m}}{q}\right)}{aq} I_{r,J,\pm}(q,m,n), \end{aligned}$$

and

$$I_{r,J,\pm}(q,m,n) = \int_{\mathbb{R}^2} M_t(\sigma+i\nu)W_l(\nu)t^{-i\nu}\left(\frac{2\pi\sqrt{nt}}{q}\right)^{-i\tau}\gamma_{\pm}(i\tau+1)I_r(q,m,\tau,\nu)W_J(\tau)\mathrm{d}\tau\mathrm{d}\nu.$$

2.3.4 Application of Cauchy and Poisson I

We will estimate here

$$\tilde{S}_2(t,C) := \sum_{J\in\mathscr{J}} S_{2,J}(t,C).$$

Taking a dyadic subdivision and using the bound $|\gamma_{\pm}(i\tau+1)| \ll 1$, we get

$$\begin{aligned} \tilde{S}_2(t,C) \ll {} & Kt^{1/2}\int_{\frac{-(tK)^{1/2}t^{\epsilon}}{C}}^{\frac{(tK)^{1/2}t^{\epsilon}}{C}}\sum_{\pm}\sum_{\substack{1\le L\ll Kt^{\epsilon}\\ L\text{ dyadic}}}\sum_{n\in\mathbb{Z}}\frac{|\rho_f(\mp n)|}{\sqrt{n}}U\left(\frac{n}{L}\right) \\ & \times\left|\sum_{C<q\le 2C}\sum_{\substack{(m,q)=1\\ 1\le|m|\ll qt^{\epsilon}}}\frac{e\left(\pm\frac{n\bar{m}}{q}\right)}{aq^{1-i\tau}}B(C,\tau)\right|\mathrm{d}\tau, \end{aligned}$$

where

$$B(C,\tau) := \int_{\mathbb{R}} M_t(\sigma+i\nu)W_l(\nu)t^{-i\nu}B(C,\tau,\nu)\mathrm{d}\nu$$
$$\ll t^{\sigma+\epsilon}\left(\frac{1}{t^{1/2}K^{3/2}}\min\left\{1,\frac{10K}{|\tau|}\right\}+\frac{1}{CK^{5/2}}\right)$$

By Cauchy and Rankin-Selberg, we get

$$\tilde{S}_2(t,C) \ll Kt^{1/2+\epsilon}\int_{-\frac{(tK)^{1/2}t^\epsilon}{C}}^{\frac{(tK)^{1/2}t^\epsilon}{C}}\sum_{\pm}\sum_{\substack{1\le L\ll Kt^\epsilon\\ L\text{ dyadic}}} L^{1/2}\left[S_{2,\pm}(t,C,L,\tau)\right]^{1/2}\mathrm{d}\tau,$$

where

$$S_{2,\pm}(t,C,L,\tau) = \sum_{n\in\mathbb{Z}}\frac{1}{n}U\left(\frac{n}{L}\right)\left|\sum_{C<q\le 2C}\sum_{\substack{(m,q)=1\\ 1\le|m|\ll qt^\epsilon}}\frac{e\left(\pm\frac{n\bar{m}}{q}\right)}{aq^{1-i\tau}}B(C,\tau)\right|^2.$$

Opening the absolute square and interchanging the order of summation, we obtain

$$S_{2,\pm}(t,C,L,\tau) = \sum_{C<q,q'\le 2C}\sum_{\substack{(m,q)=1\\ 1\le|m|\ll qt^\epsilon}}\sum_{\substack{(m',q')=1\\ 1\le|m'|\ll q't^\epsilon}}\frac{|B(C,\tau)|^2}{aa'q^{1-i\tau}q'^{1+i\tau}}T,$$

where

$$T := \sum_{n\in\mathbb{Z}}\frac{1}{n}U\left(\frac{n}{L}\right)e\left(\pm\frac{n\bar{m}}{q}\mp\frac{n\bar{m}'}{q'}\right).$$

Splitting in congruence classes mod qq' and applying Poisson summation, we get

$$T = \sum_{n\in\mathbb{Z}}\delta_{\pm q'\bar{m}\mp q\bar{m}'+n\equiv 0\,(\mathrm{mod}\,qq')}\int_{\mathbb{R}}\frac{1}{y}U(y)e\left(-\frac{Lny}{qq'}\right).$$

We may now truncate the n-sum to $n\ll C^2t^\epsilon/L$, for otherwise the oscillatory integral is negligibly small. We may therefore estimate

$$S_{2,\pm}(t,C,L,\tau) \ll \sum_{C<q,q'\le 2C}\sum_{\substack{(m,q)=1\\ 1\le|m|\ll qt^\epsilon}}\sum_{\substack{(m',q')=1\\ 1\le|m'|\ll q't^\epsilon}}\sum_{\substack{n\ll\frac{C^2t^\epsilon}{L}\\ n\equiv\pm q\bar{m}'\mp q'\bar{m}\,(\mathrm{mod}\,qq')}}\frac{K|B(C,\tau)|^2}{tC^2}$$
$$\ll \frac{t^\epsilon C^3K|B(C,\tau)|^2}{tL}.$$

Thus, by (2.14), we have

$$\tilde{S}_2(t,C) \ll \sum_{\substack{1\le L\ll Kt^\epsilon\\ L\text{ dyadic}}} C^{3/2}K^{3/2}t^\epsilon\int_{-\frac{(tK)^{1/2}t^\epsilon}{C}}^{\frac{(tK)^{1/2}t^\epsilon}{C}}|B(C,\tau)|\mathrm{d}\tau$$
$$\ll t^{\sigma+\epsilon}\left(\frac{C^{3/2}K}{t^{1/2}}+\frac{t^{1/2}}{C^{1/2}K^{1/2}}\right).$$

The contribution of $S_2(t,C)$ to $S_l^+(t)$ is therefore bounded by

$$t^\epsilon\left(\frac{t^{5/4}}{K^{3/4}}+\frac{t^{3/2}}{K^{3/2}}\right).$$

Upon taking $K=t^{1/2}$, we note that this is bounded by $t^{1-1/8+\epsilon}$.

2.3.5 Application of Poisson and Cauchy II

The analysis for $S_{1,J}$ is more delicate as we need to exploit some cancelation coming from both the ν and τ integrals. The idea is to use Cauchy and Rankin-Selberg as before, but keeping the integrals over τ and ν inside. We may bound

$$S_{1,J}(t,C) \ll Kt^{1/2}\sum_{\pm}\sum_{\substack{1\le L\ll Kt^{\epsilon}\\ L \text{ dyadic}}} L^{1/2}\left[S_{1,J,\pm}(t,C,L)\right]^{1/2},$$

where

$$S_{1,J,\pm}(t,C,L)=\sum_{n\in\mathbb{Z}}\frac{1}{n}U\left(\frac{n}{L}\right)\Bigg|\int_{\mathbb{R}}\int_{\mathbb{R}} M_t(\sigma+i\nu)W_l(\nu)t^{-i\nu}(2\pi\sqrt{nt})^{-i\tau}$$
$$\times\gamma_{\pm}(i\tau+1)\sum_{C<q\le 2C}\sum_{\substack{(m,q)=1\\1\le|m|\ll qt^{\epsilon}}}\frac{e\left(\pm\frac{n\bar{m}}{q}\right)}{aq^{1-i\tau}}I_1(q,m,\tau,\nu)W_J(\tau)\mathrm{d}\tau\mathrm{d}\nu\Bigg|^2.$$

Opening the absolute square and interchanging the order of summation, we find that $S_{1,J,\pm}(t,C,L)$ is given by

$$\int_{\mathbb{R}^4} M_t(\sigma+i\nu)\overline{M_t(\sigma+i\nu')}W_l(\nu)W_l(\nu')t^{i(\frac{\tau'-\tau}{2}+\nu'-\nu)}\gamma_{\pm}(1+i\tau)\overline{\gamma_{\pm}(1+i\tau')}W_J(\tau)W_J(\tau')$$
$$\sum_{C<q,q'\le 2C}\sum_{\substack{(m,q)=1\\1\le|m|\ll qt^{\epsilon}}}\sum_{\substack{(m',q')=1\\1\le|m'|\ll q't^{\epsilon}}}\frac{I_1(q,m,\tau,\nu)\overline{I_1(q',m',\tau',\nu')}}{aa'(2\pi)^{i(\tau-\tau')}q^{1-i\tau}q'^{1+i\tau'}}\mathscr{T}'\mathrm{d}\tau\mathrm{d}\tau'\mathrm{d}\nu\mathrm{d}\nu',$$

where

$$\mathscr{T}'=\sum_{n\in\mathbb{Z}}\frac{1}{n^{1+i\frac{\tau-\tau'}{2}}}U\left(\frac{n}{L}\right)e\left(\pm\frac{n\bar{m}}{q}\mp\frac{n\bar{m}'}{q'}\right).$$

Applying Poisson summation, similarly to the previous section, we obtain

$$\mathscr{T}'=\frac{L^{i\frac{\tau'-\tau}{2}}}{qq'}\sum_{n\in\mathbb{Z}}\delta\pm(n,m,m',q,q')U^{\dagger}\left(\frac{nL}{qq'},-i\frac{\tau-\tau'}{2}\right),$$

where

$$\delta_{\pm}(n,m,m',q,q')=qq'\delta_{\pm q'\bar{m}\mp q\bar{m}'+n\equiv 0\,(\mathrm{mod}\,qq')}.$$

Since $|\tau-\tau'|\ll (tK)^{1/2}t^{\epsilon}/C$ and $q,q'\asymp C$, we have by Lemma 1.4 that if $|n|\gg C(tK)^{1/2}t^{\epsilon}/L$, then the contribution is negligibly small.

Lemma 2.6. *The sum $S_{1,J,\pm}(t,C,L)$ is dominated by the sum*

$$\frac{K}{tC^2}\sum_{C<q,q'\le 2C}\sum_{\substack{(m,q)=1\\1\le|m|\ll qt^{\epsilon}}}\sum_{\substack{(m',q')=1\\1\le|m'|\ll q't^{\epsilon}}}\sum_{\substack{|n|\ll C(tK)^{1/2}t^{\epsilon}/L\\ n\equiv\pm q\bar{m}'\mp q'\bar{m}\,(\mathrm{mod}\,qq')}}|\mathscr{K}_{\pm}|+O(t^{-1000}),$$

where

$$\mathscr{K}_{\pm}=\int_{\mathbb{R}^4}M_t(\sigma+i\nu)\overline{M_t(\sigma+i\nu')}W_l(\nu)W_l(\nu')t^{i(\nu'-\nu)}\frac{(4\pi^2tL)^{-i\frac{\tau-\tau'}{2}}}{q^{-i\tau}q'^{i\tau'}}W_J(\tau)W_J(\tau')$$
$$\gamma_{\pm}(i\tau+1)\overline{\gamma_{\pm}(i\tau'+1)}I_1(q,m,\tau,\nu)\overline{I_1(q',m',\tau',\nu')}U^{\dagger}\left(\frac{nL}{qq'},i\frac{\tau'-\tau}{2}\right)\mathrm{d}\tau\mathrm{d}\tau'\mathrm{d}\nu\mathrm{d}\nu'.$$

We are thus only left with understanding $\mathscr{K}_{\pm}$. Writing out explicitly $I_1(q,m,\tau,\nu)$, we obtain

$$\mathscr{K}_{\pm} = \frac{|c_4|^2}{K^2}\int_{\mathbb{R}^4} W_J(q,m,\tau,\nu)\overline{W_J(q',m',\tau',\nu')}e(f_t(\sigma+i\nu)-f_t(\sigma+i\nu'))$$
$$\times\, t^{i(\nu'-\nu)}U^{\dagger}\left(\frac{nL}{qq'},i\frac{\tau'-\tau}{2}\right)\left(-\frac{(\nu+\frac{\tau}{2})q}{2\pi etm}\right)^{-i(\nu+\frac{\tau}{2})}\left(-\frac{(\nu'+\frac{\tau'}{2})q'}{2\pi etm'}\right)^{i(\nu'+\frac{\tau'}{2})}$$
$$\times\,\gamma_{\pm}(1+i\tau)\overline{\gamma_{\pm}(1+i\tau')}\frac{(4\pi^2 tL)^{i\frac{\tau'-\tau}{2}}}{q^{-i\tau}q'^{i\tau'}}\mathrm{d}\tau\mathrm{d}\tau'\mathrm{d}\nu\mathrm{d}\nu',$$

where

$$W_J(q,m,\tau,\nu) = g_t(\sigma+i\nu)\frac{W_l(\nu)W_J(\tau)}{(\nu+\frac{\tau}{2})^{1/2}}\left(-\frac{(\nu+\frac{\tau}{2})q}{2\pi etm}\right)^{3/2}V\left(-\frac{(\nu+\frac{\tau}{2})q}{2\pi tm}\right)$$
$$\times\left(\frac{tm}{(\nu+\frac{\tau}{2})q}\right)^{\sigma}\check{U}\left(-\frac{(\nu+\frac{\tau}{2})q}{2\pi mt}\right)\int_{K^{\sigma'-1}}^{1}V\left(\frac{\tau}{2K}-\frac{(\nu+\frac{\tau}{2})x}{Kma}\right)\mathrm{d}x.$$

We note in passing the following estimates

$$\frac{\mathrm{d}}{\mathrm{d}\tau}W_J(q,m,\tau,\nu) \ll \frac{|\nu|^{\sigma-1}}{|\tau|}, \tag{2.15}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}\nu}W_J(q,m,\tau,\nu) \ll |\nu|^{\sigma-2}. \tag{2.16}$$

We first analyse the case $n=0$; it will be sufficient to consider

$$\int_{\mathbb{R}^2} W_J(q,m,\tau,\nu)\overline{W_J(q',m',\tau',\nu')}e(f_t(\sigma+i\nu)-f_t(\sigma+i\nu'))\,t^{i(\nu'-\nu)}$$
$$\times\left(-\frac{(\nu+\frac{\tau}{2})q}{2\pi etm}\right)^{-i(\nu+\frac{\tau}{2})}\left(-\frac{(\nu'+\frac{\tau'}{2})q'}{2\pi etm'}\right)^{i(\nu'+\frac{\tau'}{2})}\mathrm{d}\nu\mathrm{d}\nu'$$
$$=\int_{\mathbb{R}^2} W_J(q,m,\tau,\nu)\overline{W_J(q',m',\tau',\nu')}e(f(\nu,\nu'))\mathrm{d}\nu\mathrm{d}\nu',$$

where we temporarily define

$$f(\nu,\nu') = f_t(\sigma+i\nu)-f_t(\sigma+i\nu')+\frac{\nu'-\nu}{2\pi}\log t$$
$$-\frac{\nu+\frac{\tau}{2}}{2\pi}\log\left(-\frac{(\nu+\frac{\tau}{2})q}{2\pi etm}\right)+\frac{\nu'+\frac{\tau'}{2}}{2\pi}\log\left(-\frac{(\nu'+\frac{\tau'}{2})q'}{2\pi etm'}\right).$$

We compute

$$\frac{\mathrm{d}f}{\mathrm{d}\nu} = f_t'(\sigma+i\nu)-\frac{\log t}{2\pi}-\frac{1}{2\pi}\log\left(-\frac{(\nu+\frac{\tau}{2})q}{2\pi etm}\right)-\frac{1}{2\pi},$$
$$\frac{\mathrm{d}f}{\mathrm{d}\nu'} = -f_t'(\sigma+i\nu')+\frac{\log t}{2\pi}+\frac{1}{2\pi}\log\left(-\frac{(\nu'+\frac{\tau'}{2})q'}{2\pi etm'}\right)+\frac{1}{2\pi}.$$

and thus

$$\frac{\mathrm{d}^2 f}{\mathrm{d}v\mathrm{d}v'} = 0,$$

while by (2.8), we have

$$\frac{\mathrm{d}^2 f}{\mathrm{d}v^2} = f_t''(\sigma + iv) - \frac{1}{2\pi(v + \frac{\tau}{2})} \gg |v|^{-1},$$

and

$$\frac{\mathrm{d}^2 f}{\mathrm{d}v'^2} = -f_t''(\sigma + iv') + \frac{1}{2\pi(v' + \frac{\tau'}{2})} \gg |v'|^{-1}.$$

We also note that by (2.16), we have

$$\mathrm{Var}(W_J(q,m,\tau,v)\overline{W_J(q',m',\tau',v')}) \ll t^{2\sigma-2+\epsilon}.$$

We now have by the second derivative bound for oscillatory integrals in multivariables (see [Sri65]) that

$$\int_{\mathbb{R}^2} W_J(q,m,\tau,v)\overline{W_J(q',m',\tau',v')}e(f(v,v'))\mathrm{d}v\mathrm{d}v' \ll t^{2\sigma-1+\epsilon}. \tag{2.17}$$

By integration by parts, if $|\tau-\tau'| \gg t^\epsilon$, then $U^\dagger\left(0, i\frac{\tau-\tau'}{2}\right)$ is negligibly small. The contribution from $n=0$ to $\mathcal{K}_\pm$ is thus bounded by

$$K^{-2}\int\int_{\substack{|\tau-\tau'|\ll t^\epsilon \\ |\tau|,|\tau'|\asymp J}} t^{2\sigma-1+\epsilon} \ll \frac{t^{2\sigma+\epsilon}}{Ct^{1/2}K^{3/2}}.$$

We now treat the case $n \neq 0$. We have by Lemma 5 of [Mun15] that

$$\begin{aligned} U^\dagger\left(\frac{nL}{qq'}, -i\frac{\tau-\tau'}{2}\right) &= \frac{c_5}{(\tau'-\tau)^{1/2}} U\left(\frac{(\tau'-\tau)qq'}{4\pi nL}\right)\left(\frac{(\tau'-\tau)qq'}{4\pi enL}\right)^{-i(\tau-\tau')/2} \\ &\quad + O\left(\min\left\{\frac{1}{|\tau-\tau'|^{3/2}}, \frac{C^3}{(|n|L)^{3/2}}\right\}\right), \end{aligned}$$

for some constant c_5 (which depends on the sign of n). In order to bound the error term, we use (2.17) to see that the contribution is bounded by

$$\frac{t^{2\sigma-1+\epsilon}}{K^2}\int_{[J,2J]^2} \min\left\{\frac{1}{|\tau-\tau'|^{3/2}}, \frac{C^3}{(|n|L)^{3/2}}\right\}.$$

We first estimate

$$\frac{t^{2\sigma-1+\epsilon}}{K^2}\int_{\substack{[J,2J]^2 \\ |\tau-\tau'|\leq |nL|/C^2}} \frac{C^3}{(|n|L)^{3/2}}\mathrm{d}\tau\mathrm{d}\tau' \ll \frac{t^{2\sigma-1+\epsilon}CJ}{K^2(|n|L)^{1/2}} \ll \frac{t^{2\sigma-1/2+\epsilon}}{K^{3/2}(|n|L)^{1/2}},$$

and then

$$\begin{aligned} \frac{t^{2\sigma-1+\epsilon}}{K^2}\int_{\substack{[J,2J]^2 \\ |\tau-\tau'|> |nL|/C^2}} \frac{1}{|\tau-\tau'|^{3/2}}\mathrm{d}\tau\mathrm{d}\tau' &\ll \frac{Ct^{2\sigma-1+\epsilon}}{K^2(|nL|)^{1/2}}\int_{[J,2J]^2} \frac{1}{|\tau-\tau'|^{1-\epsilon}}\mathrm{d}\tau\mathrm{d}\tau' \\ &\ll \frac{CJt^{2\sigma-1+\epsilon}}{K^2(|nL|)^{1/2}} \ll \frac{t^{2\sigma-1/2+\epsilon}}{K^{3/2}(|nL|)^{1/2}}. \end{aligned}$$

We thus set
$$B^*(C,0) = \frac{t^{2\sigma+\epsilon}}{K^{3/2}Ct^{1/2}},$$
and for $n \neq 0$,
$$B^*(C,n) = \frac{t^{2\sigma+\epsilon}}{K^{3/2}t^{1/2}(|n|L)^{1/2}}.$$

We now consider the main term. As noted in Section 4.1, the contribution from γ_+ is simpler, and thus we will only focus on γ_-. We first note that by Fourier inversion, we have
$$\left(\frac{4\pi nL}{(\tau'-\tau)qq'}\right)^{1/2} U\left(\frac{(\tau'-\tau)qq'}{4\pi nL}\right) = \int_{\mathbb{R}} U^\dagger\left(r,\frac{1}{2}\right) e\left(r\frac{(\tau'-\tau)qq'}{4\pi nL}\right)\mathrm{d}r.$$
Pulling out the oscillation from the γ_- factors, we conclude that for some constant c_6 (depending on the sign of n), we have
$$\begin{aligned}\mathcal{K}_- =& \frac{c_6}{K^2}\left(\frac{qq'}{|n|L}\right)^{1/2}\int_{\mathbb{R}} U^\dagger\left(r,\frac{1}{2}\right)\int_{\mathbb{R}^4} g(\tau,\tau',\nu,\nu')e(f(\tau,\tau',\nu,\nu',r))\mathrm{d}\tau\mathrm{d}\tau'\mathrm{d}\nu\mathrm{d}\nu'\mathrm{d}r\\ &+ O(B^*(C,n)),\end{aligned}$$
where
$$\begin{aligned}f(\tau,\tau',\nu,\nu',r) =& f_t(\sigma+i\nu) - f_t(\sigma+i\nu') + \frac{\nu'-\nu}{2\pi}\log t + \frac{\tau}{2\pi}\log\left(\frac{|\tau|}{e}\right) - \frac{\tau'}{2\pi}\log\left(\frac{|\tau'|}{e}\right)\\ &+ \frac{\tau'-\tau}{4\pi}\log\left(\frac{(\tau'-\tau)4\pi tqq'}{en}\right) - \frac{\nu+\frac{\tau}{2}}{2\pi}\log\left(-\frac{(\nu+\frac{\tau}{2})q}{2\pi etm}\right) + \frac{\tau}{2\pi}\log q\\ &- \frac{\tau'}{2\pi}\log q' + \frac{\nu'+\frac{\tau'}{2}}{2\pi}\log\left(-\frac{(\nu'+\frac{\tau'}{2})q'}{2\pi etm'}\right) + \frac{r(\tau'-\tau)qq'}{4\pi nL},\end{aligned}$$
and
$$g(\tau,\tau',\nu,\nu') = W_J(q,m,\tau,\nu)\overline{W_J(q',m',\tau',\nu')}\Phi_-(\tau)\overline{\Phi_-(\tau')}.$$
We will use the second derivative bound for multivariable oscillatory integrals as can be found in [Sri65] and hence compute
$$\begin{aligned}&2\pi\frac{\mathrm{d}^2 f}{\mathrm{d}\tau^2} = \frac{1}{\tau} - \frac{1}{4(\nu+\frac{\tau}{2})} + \frac{1}{2(\tau'-\tau)},\ 2\pi\frac{\mathrm{d}^2 f}{\mathrm{d}\tau\mathrm{d}\tau'} = \frac{1}{2(\tau-\tau')},\ 2\pi\frac{\mathrm{d}^2 f}{\mathrm{d}\tau\mathrm{d}\nu} = -\frac{1}{2(\nu+\frac{\tau}{2})},\\ &2\pi\frac{\mathrm{d}^2 f}{\mathrm{d}\tau'^2} = -\frac{1}{\tau'} + \frac{1}{4(\nu'+\frac{\tau'}{2})} + \frac{1}{2(\tau'-\tau)},\ 2\pi\frac{\mathrm{d}^2 f}{\mathrm{d}\tau'\mathrm{d}\nu'} = \frac{1}{2(\nu'+\frac{\tau'}{2})},\\ &\frac{\mathrm{d}^2 f}{\mathrm{d}\nu^2} = f_t''(\sigma+i\nu) - \frac{1}{2\pi(\nu+\frac{\tau}{2})},\ \frac{\mathrm{d}^2 f}{\mathrm{d}\nu'^2} = \frac{1}{2\pi(\nu'+\frac{\tau'}{2})} - f_t''(\sigma+i\nu'),\end{aligned}$$
while
$$\frac{\mathrm{d}^2 f}{\mathrm{d}\tau\mathrm{d}\nu'} = \frac{\mathrm{d}^2 f}{\mathrm{d}\tau'\mathrm{d}\nu} = \frac{\mathrm{d}^2 f}{\mathrm{d}\nu\mathrm{d}\nu'} = 0.$$
Computing the minors of the Hessian matrix , we see from [Sri65, Lemma 5] that for D a box in R^4,
$$\int_D e(f(\tau,\tau',\nu,\nu'))\mathrm{d}\tau\mathrm{d}\tau'\mathrm{d}\nu\mathrm{d}\nu' \ll t^\epsilon Jt, \tag{2.18}$$

where we used $r_1 = r_2 = J^{-1/2}$ and $r_3 = r_4 = t^{-1/2}$ as can be seen from our calculations of the second derivatives and that $\tau, \tau' \in [J, \frac{4}{3}J]$. Using (2.15) and (2.16), we compute the total variation, using that $t^{1-\epsilon} \ll |v| \ll t^{1+\epsilon}$:

$$\begin{aligned}
\operatorname{Var}(g(\tau,\tau',v,v')) &:= \int_{\mathbb{R}^4} \left| \frac{\mathrm{d}g}{\mathrm{d}\tau \mathrm{d}\tau' \mathrm{d}v \mathrm{d}v'} \right| \mathrm{d}\tau \mathrm{d}\tau' \mathrm{d}v \mathrm{d}v' \\
&\ll \int_{\mathbb{R}^4} \frac{|v|^{\sigma-2}|v'|^{\sigma-2}}{|\tau||\tau'|} J t^{1+\epsilon} \mathrm{d}\tau \mathrm{d}\tau' \mathrm{d}v \mathrm{d}v' \\
&\ll t^{2\sigma-2+\epsilon}.
\end{aligned} \tag{2.19}$$

By integration by parts, we note that by (2.18) and (2.19), we have

$$\begin{aligned}
&\int_{\mathbb{R}^4} g(\tau,\tau',v,v') e(f(\tau,\tau',v,v',r)) \mathrm{d}\tau \mathrm{d}\tau' \mathrm{d}v \mathrm{d}v' \\
&\ll \int_{\mathbb{R}^4} \left| \frac{\mathrm{d}g}{\mathrm{d}\tau \mathrm{d}\tau' \mathrm{d}v \mathrm{d}v'} \right| J t^{1+\epsilon} \mathrm{d}\tau \mathrm{d}\tau' \mathrm{d}v \mathrm{d}v' \\
&\ll J t^{2\sigma-1+\epsilon}.
\end{aligned}$$

Then, integrating trivially over r and using the rapid decay of Fourier transforms, we arrive at the following result:

Lemma 2.7. *We have*

$$\mathcal{K}_- \ll B^*(C,n).$$

We now write

$$S_{1,J,-}(t,C,L) = S^\flat_{1,J,-}(t,C,L) + S^\sharp_{1,J,-}(t,C,L),$$

where $S^\flat_{1,J,-}(t,C,L)$ corresponds to $n = 0$ contribution, while $S^\sharp_{1,J,-}(t,C,L)$ corresponds to the $n \neq 0$ frequencies. We first estimate

$$\begin{aligned}
S^\flat_{1,J,-}(t,C,L) &\ll \frac{K}{tC^2} \sum_{C<q,q'\le 2C} \sum_{\substack{(m,q)=1 \\ 1\le |m| \ll q t^\epsilon}} \sum_{\substack{(m',q')=1 \\ 1\le |m'| \ll q' t^\epsilon}} \delta_{-q\overline{m'}+q'\overline{m}\equiv 0 \, (\mathrm{mod}\, qq')} \frac{t^{2\sigma+\epsilon}}{K^{3/2} C t^{1/2}} \\
&\ll \frac{t^{2\sigma+\epsilon}}{t^{3/2} K^{1/2}}.
\end{aligned}$$

Taking a dyadic subdivision, we estimate

$$
\begin{aligned}
S^{\sharp}_{1,J,-}(t,C,L) &\ll \frac{K}{tC^2}\sum_{C<q,q'\le 2C}\ \sum_{\substack{(m,q)=1\\ 1\le|m|\ll qt^{\epsilon}}}\ \sum_{\substack{(m',q')=1\\ 1\le|m'|\ll q't^{\epsilon}}}\ \sum_{\substack{1\le|n|\ll\frac{C(tK)^{\frac12}t^{\epsilon}}{L}\\ n\equiv -q\overline{m'}+q'\overline{m}\,(\mathrm{mod}\, qq')}} B^*(C,n)\\
&\ll \frac{K}{tC^2}\sum_{C<q,q'\le 2C}\ \sum_{\substack{(m,q)=1\\ 1\le|m|\ll qt^{\epsilon}}}\ \sum_{\substack{(m',q')=1\\ 1\le|m'|\ll q't^{\epsilon}}}\ \sum_{\substack{1\le|n|\ll\frac{C(tK)^{\frac12}t^{\epsilon}}{L}\\ n\equiv q'\overline{m}-q\overline{m'}\,(\mathrm{mod}\, qq')}} \frac{t^{2\sigma+\epsilon}}{K^{3/2}t^{1/2}(|n|L)^{\frac12}}\\
&\ll \frac{t^{2\sigma+\epsilon}}{t^{3/2}K^{1/2}C^2L^{1/2}}\sum_{\substack{H\le\frac{C(tK)^{1/2}t^{\epsilon}}{L}\\ H\ \text{Dyadic}}}\ \sum_{C<q,q'\le 2C}\ \sum_{\substack{(m,q)=1\\ 1\le|m|\ll qt^{\epsilon}}}\\
&\times \sum_{\substack{(m',q')=1\\ 1\le|m'|\ll q't^{\epsilon}}}\ \sum_{\substack{H<|n|\le 2H\\ n\equiv -q\overline{m'}+q'\overline{m}\,(\mathrm{mod}\, qq')}} H^{-1/2}\\
&\ll \frac{t^{2\sigma+\epsilon}}{t^{3/2}K^{1/2}C^2L^{1/2}}\sum_{\substack{H\le\frac{C(tK)^{1/2}t^{\epsilon}}{L}\\ H\ \text{Dyadic}}} H^{-1/2}\sum_{C<q,q'\le 2C}\ \sum_{H<|n|\le 2H}\\
&\times \sum_{\substack{1\le|m|\ll qt^{\epsilon}\\ (m,q)=1}}\ \sum_{\substack{1\le|m'|\ll q't^{\epsilon}\\ (m',q')=1}} \delta_{-q\overline{m'}+q'\overline{m}\equiv n\,(\mathrm{mod}\, qq')}.
\end{aligned}
$$

We let $d=(q,q')$ and notice that looking at the congruence condition above modulo q implies that $q'\overline{m}\equiv n \mod q$, which in turn implies that d divides n. We let $q_0 := q/d$, $q_0' = q'/d$ and $n_0 := n/d$, so that

$$
n_0 \equiv q_0'\overline{m}\,(\mathrm{mod}\, q_0),\ \text{and}\ n_0\equiv q_0\overline{m'}\,(\mathrm{mod}\, q_0').
$$

We may thus bound

$$
\begin{aligned}
S^{\sharp}_{1,J,-}(t,C,L) &\ll \frac{t^{2\sigma+\epsilon}}{t^{3/2}K^{1/2}C^2L^{1/2}}\sum_{\substack{H\le\frac{C(tK)^{1/2}t^{\epsilon}}{L}\\ H\ \text{Dyadic}}} H^{-1/2}\sum_{C<q,q'\le 2C}\ \sum_{\frac{H}{d}<n_0\le\frac{2H}{d}}\\
&\times \sum_{\substack{1\le|m|\ll qt^{\epsilon}\\ (m,q)=1}}\ \sum_{\substack{1\le|m'|\ll q't^{\epsilon}\\ (m',q')=1}} \delta_{q_0'\overline{m}\equiv n_0\,(\mathrm{mod}\, q_0)}\delta_{q_0\overline{m'}\equiv n_0\,(\mathrm{mod}\, q_0')}\\
&\ll \frac{t^{2\sigma+\epsilon}}{t^{3/2}K^{1/2}C^2L^{1/2}}\sum_{\substack{H\le\frac{C(tK)^{1/2}t^{\epsilon}}{L}\\ H\ \text{Dyadic}}} H^{-1/2}\sum_{C<q,q'\le 2C}\ \sum_{\frac{H}{d}<n_0\le\frac{2H}{d}} t^{\epsilon}d^2\\
&\ll \frac{t^{2\sigma+\epsilon}(tK)^{1/4}}{t^{3/2}K^{1/2}C^{3/2}L}\sum_{C<q,q'\le 2C} d\\
&\ll \frac{t^{2\sigma+\epsilon}(tK)^{1/4}}{t^{3/2}K^{1/2}C^{3/2}L}\sum_{d\le 2C}\ \sum_{\frac{C}{d}\le q_0,q_0'\le\frac{2C}{d}} d\\
&\ll \frac{t^{2\sigma+\epsilon}}{tK^{1/2}L}.
\end{aligned}
$$

We conclude that

$$
S_{1,J,-}(t,C,L) \ll t^{2\sigma+\epsilon}\left(\frac{1}{t^{3/2}K^{1/2}}+\frac{1}{tK^{1/2}L}\right).
$$

The same bound holds for $S_{1,J,+}(t,C,L)$, via the same analysis, so that

$$\begin{aligned} S_{1,J}(t,C) &\ll K t^{\sigma+1/2+\epsilon} \sum_{\substack{1\le L\ll Kt^{\epsilon} \\ L \text{ Dyadic}}} \left(\frac{L^{1/2}}{t^{3/4}K^{1/4}} + \frac{1}{t^{1/2}K^{1/4}}\right) \\ &\ll t^{\sigma+\epsilon}\left(\frac{K^{5/4}}{t^{1/4}} + K^{3/4}\right). \end{aligned}$$

The same bound holds for all values of J. Since there are $O(\log t)$ many terms, we can sum over them without worsening the bound, and so the same bound holds for $\hat{S}_1(t,C) := \sum_J S_{1,J}(t,C)$. Thus the total contribution of $\hat{S}_1(t,C)$ to $S_l^+(t)$ is bounded by

$$\frac{t^{1+\epsilon}}{K}\left(\frac{K^{5/4}}{t^{1/4}} + K^{3/4}\right) \ll t^{\epsilon}\left(t^{3/4}K^{1/4} + \frac{t}{K^{1/4}}\right).$$

Choosing $K = t^{1/2}$, we obtain

$$S_l^+(t) \ll t^{1-1/8+\epsilon}.$$

2.4 Examples

In this section, we study some examples of analytic trace functions to motivate the analogy with Frobenius trace functions studied in [FKM15a]. The analog of Kloosterman sums is given in the following example.

Proposition 2.2. *Let*

$$F_{it}(x) := t^{1/2}\Gamma\left(\frac{1}{2} + it\right) J_{it}(x)$$

be the normalized J-Bessel function of order t. Then, F_{it} is an analytic trace function.

Proof. By [EMOT54, p. 331], the Mellin inversion theorem holds for F_{it} and the Mellin transform is given by

$$M_{F,t}(s) := \int_0^{\infty} F_{it}(x)x^{s-1}\mathrm{d}x = t^{1/2}\Gamma\left(\frac{1}{2} + it\right)2^{s-1}\frac{\Gamma\left(\frac{s+it}{2}\right)}{\Gamma\left(1 + \frac{it-s}{2}\right)},$$

for any $0 < \sigma < 1$, where $s = \sigma + i\nu$. We will assume for simplicity that $t \ge 1$, the same argument holding also for negative t. In order to understand $M_{F,t}(\sigma + i\nu)$, we differentiate between three cases, using Stirling's formula for some of the Gamma factors. We first note that

$$\left|\Gamma\left(\frac{1}{2} + it\right)\right| = \sqrt{2\pi}\exp\left(-\frac{\pi t}{2}\right)(1 + O(|t|^{-1})). \tag{2.20}$$

First assume we are in the range where $|t \pm \nu| \ge 1$, then we may apply Stirling's formula to all the Gamma factors, and find that

$$M_{F,t}(s) = t^{1/2}g_{F,t}(s)e(f_{F,t}(s)),$$

where, up to a constant,

$$g_{F,t}(s)=\exp\left(\frac{\pi}{4}(|t-\nu|-|\nu+t|-2t)\right)|(\nu+t)(t-\nu)|^{\frac{\sigma-1}{2}}(1+O(\max\{t^{-1},|t\pm\nu|^{-1}\})),$$

and

$$2\pi f_{F,t}(s)=\frac{\nu+t}{2}\log\left|\frac{\nu+t}{2e}\right|+\frac{\nu-t}{2}\log\left|\frac{t-\nu}{2e}\right|+\nu\log 2.$$

We note that if $\nu\geq-\frac{t}{2}$, then $g_{F,t}(s)$ is negligible. We therefore only focus on the case where $\nu<-\frac{t}{2}$ and verify condition (2.5) for $f_{F,t}$. We thus compute

$$2\pi\frac{\mathrm{d}}{\mathrm{d}\nu}f_{F,t}(s)=\frac{1}{2}\log\left|\frac{t^2-\nu^2}{4e^2}\right|+1+\log 2.$$

Since we only consider $\nu\gg t$ by exponential decay of $g_{F,t}$ otherwise, we find that

$$\log\left|\frac{(t^2-\nu^2)^{1/2}}{x}\right|\ll 1,$$

may only occur if $\nu\asymp t$, for $x\in[t,2t]$.

On the other hand, if we are in the range $|t-\nu|<1$, then we may not apply Stirling's formula for the Gamma factor in the denominator. However, we will have that $|t+\nu|\gg t$, and thus by (2.20) and the exponential decay of Gamma factors, we get that the contribution is negligible. Finally, if we are in the range $|t+\nu|<1$, then the phase of $M_{F,t}(s)$ will be of the form

$$2\pi\tilde{f}_{F,t}(s):=\frac{\nu-t}{2}\log\left|\frac{t-\nu}{2e}\right|+\nu\log 2,$$

and so

$$2\pi\frac{\mathrm{d}}{\mathrm{d}\nu}\tilde{f}_{F,t}(s)-\log(x)\gg 1$$

in this region, and is thus negligible by integration by parts. Moreover, looking at $f_{F,t}$, there can be no stationary point in any region such that $\nu=-t+o(t)$.

We thus assume from now on that we are in the region where $|t\pm\nu|\gg t$, and $t\ll\nu\leq-t$, and will show that conditions (2.4), (2.6), (2.7) and (2.8) hold for $g_{F,t}(s)$ and $f_{F,t}(s)$. Indeed, in this region,

$$t^{1/2}g_{F,t}(s)=t^{1/2}|(\nu+t)(\nu-t)|^{\frac{\sigma-1}{2}}(1+O(t^{-1}))\ll t^{\sigma-1/2},$$

and thus

$$t^{1/2}\frac{\mathrm{d}^j}{\mathrm{d}\nu^j}g_{F,t}(s)\ll t^{\sigma-1/2-j},$$

for all $j\geq 0$, proving (2.4). We now compute

$$2\pi\frac{\mathrm{d}^2}{\mathrm{d}\nu^2}f_{F,t}(s)=\frac{\nu}{(\nu^2-t^2)}\gg\nu^{-1},$$

and thus

$$2\pi\frac{\mathrm{d}^j}{\mathrm{d}\nu^j}f_{F,t}(s)\ll_{j,\epsilon}\nu^{1+\epsilon-j},$$

for all $j \geq 0$, proving (2.6) and (2.7). Finally we look at

$$2\pi \frac{\mathrm{d}^2}{\mathrm{d}v^2} f_{F,t}(s) - \frac{1}{v} = \frac{t^2}{v(v^2 - t^2)} \gg v^{-1},$$

proving (2.8), concluding the proof that F_{it} is an analytic trace function.

□

Another interesting example is that of Bessel functions of high rank. These can be thought of as analogs to hyper-Kloosterman sums. We study here higher rank Bessel functions appearing in the Voronoi summation formulas in higher rank (as in [Qi15]).

Proposition 2.3. *For any $n \geq 3$, let*

$$J_{n,t} := \frac{t^{\frac{n-1}{2}}}{2\pi i n} \int_{(\frac{1}{4})} \Gamma\left(\frac{s - int}{n}\right) \Gamma\left(\frac{s}{n} + \frac{it}{n-1}\right)^{n-1} e\left(\frac{s}{4}\right) x^{-s} \mathrm{d}s.$$

Then $J_{n,t}$ is an analytic trace function.

Proof. Let

$$M_{J_{n,t}}(s) := \frac{t^{\frac{n-1}{2}}}{n} \Gamma\left(\frac{s - int}{n}\right) \Gamma\left(\frac{s}{n} + \frac{it}{n-1}\right)^{n-1} e\left(\frac{s}{4}\right),$$

with $s = \frac{1}{4} + iv$. We assume again for simplicity that $t > 1$ and want to show that $M_{J_{n,t}}$ satisfies all the conditions in Definition 2.2. As in the case of the Bessel function, we wish to use Stiriling's formula to understand the phase and amplitude of $M_{J_{n,t}}$. Again we distinguish three different cases. First assume we are in the range $|v - t| \geq n$ and $|(n-1)v + nt| \geq n(n-1)$. We may then apply Stirling's formula to both Gamma factors to obtain

$$M_{J_{n,t}}(s) = e\left(\frac{1}{8}\right) \frac{t^{\frac{n-1}{2}}}{n} g_{J_{n,t}}(s) e(f_{J_{n,t}}(s)),$$

where $g_{J_{n,t}}(s)$ is given by

$$\exp\left(-\frac{\pi(|v - nt| + |(n-1)v + nt| + nv)}{2n}\right) \left|\frac{v - nt}{n}\right|^{\frac{1}{4n} - \frac{1}{2}} \left|\frac{v}{n} + \frac{t}{n-1}\right|^{(n-1)(\frac{1}{4n} - \frac{1}{2})}$$
$$\times \left(1 + O\left((1 + |v - nt|)^{-1} + \left(1 + \left|\frac{v}{n} + \frac{t}{n-1}\right|\right)^{-1}\right)\right),$$

and

$$2\pi f_{J_{n,t}}(s) = \frac{(n-1)v + nt}{n} \log\left|\frac{v}{en} + \frac{t}{e(n-1)}\right| + \frac{v - nt}{n} \log\left|\frac{v - nt}{ne}\right|.$$

We note that if $v \geq -\frac{n}{2(n-1)} t$, then $g_{J_{n,t}}$ is negligible. We therefore only focus on the case where $v < -\frac{n}{2(n-1)} t$ and verify condition (2.5) for $f_{J_{n,t}}$. We thus compute

$$2\pi \frac{\mathrm{d}}{\mathrm{d}v} f_{Jn,t}(s) = \frac{n-1}{n} \log\left|\frac{v}{en} + \frac{t}{e(n-1)}\right| + \frac{1}{n} \log\left|\frac{v - nt}{ne}\right| + 1.$$

Since we only consider $v \gg t$ by exponential decay of $g_{J_{n,t}}$ otherwise, we find that

$$\log\left|\left(\frac{(n-1)v+nt}{n-1}\right)^{\frac{n-1}{n}}\frac{(v-nt)^{\frac{1}{n}}}{xn}\right| \ll 1,$$

may only occur if $v \asymp x$, for $x \asymp t$. Moreover, as in the Bessel function case, we see from this that in the two cases where we might not use Stirling's formula for one of the Gamma factors, either $g_{J_{n,t}}$ will be negligible, or the phase cannot vanish and the contribution is also negligible. We thus assume from now on that we are in the region where $|(n-1)v+nt|, |v-nt| \gg t$ and $t \ll v \leq -\frac{n}{(n-1)}t$, and will show that conditions (2.4), (2.6), (2.7) and (2.8) hold for $g_{J_{n,t}}(s)$ and $f_{J_{n,t}}(s)$. Indeed, in this region,

$$t^{\frac{n-1}{2}} g_{J_{n,t}}(s) = t^{\frac{n-1}{2}}\left|\frac{v-nt}{n}\right|^{\frac{1}{4n}-\frac{1}{2}}\left|\frac{v}{n}+\frac{t}{n-1}\right|^{(n-1)(\frac{1}{4n}-\frac{1}{2})}(1+O(t^{-1})) \ll t^{\frac{1}{4}-\frac{1}{2}},$$

and thus

$$t^{\frac{n-1}{2}}\frac{\mathrm{d}^j}{\mathrm{d}v^j} g_{J_{n,t}}(s) \ll t^{\frac{1}{4}-\frac{1}{2}-j},$$

for all $j \geq 0$, proving (2.4). We now compute

$$2\pi\frac{\mathrm{d}^2}{\mathrm{d}v^2} f_{J_{n,t}}(s) = \frac{(n-1)v+nt(2-n)}{(v-nt)((n-1)v+nt)} \gg v^{-1},$$

since $v<0$, and thus

$$2\pi\frac{\mathrm{d}^j}{\mathrm{d}v^j} f_{J_{n,t}}(s) \ll_{j,\epsilon} v^{1+\epsilon-j},$$

for all $j \geq 0$, proving (2.6) and (2.7). Finally, we look at

$$2\pi\frac{\mathrm{d}^2}{\mathrm{d}v^2} f_{J_{n,t}}(s) - \frac{1}{v} = \frac{nt^2}{v(v-nt)((n-1)v+nt)} \gg v^{-1},$$

proving (2.8), concluding the proof that $J_{n,t}$ is an analytic trace function. □

We end this section with an example motivating condition (2.8). Namely, we study $e(x)$ in the range $x \in [t, 2t]$ and show that it satisfies all the conditions to be an analytic trace function, besides (2.8). By Mellin inversion, we thus have

$$\begin{aligned} V\left(\frac{x}{t}\right)e(x) &= \frac{1}{2\pi}\int_{\mathbb{R}} t^{iv}V^{\dagger}(-t, iv)x^{-iv}\mathrm{d}v \\ &:= \frac{1}{2\pi}\int_{\mathbb{R}} M_{e,t}(iv)x^{-iv}\mathrm{d}v, \end{aligned}$$

where

$$M_{e,t}(iv) = t^{iv}V^{\dagger}(-t, iv).$$

We first note that by Lemma 1.4, we may assume that $v \asymp t$, for otherwise $V^{\dagger}(-t, iv)$ is negligible. We now use Lemma 5 in [Mun15] to write in this region

$$M_{e,t}(iv) = g_{e,t}(iv)e(f_{e,t}(iv)),$$

where, up to a constant,

$$g_{e,t}(i\nu) = \nu^{-1/2} V\left(-\frac{\nu}{2\pi t}\right)(1 + O(\nu^{-3/2})),$$

and

$$f_{e,t}(i\nu) = \frac{\nu}{2\pi} \log\left(-\frac{\nu}{2\pi e}\right).$$

One now verifies that

$$g_{e,t}^{(j)}(i\nu) \ll_j \nu^{-1/2-j},$$

for all $j \geq 0$. We compute

$$f'_{e,t}(i\nu) = \frac{1}{2\pi} \log\left(-\frac{\nu}{2\pi e}\right) + \frac{1}{2\pi},$$

and

$$f_{e,t}^{(j)}(i\nu) = \frac{(-1)^j}{2\pi \nu^{j-1}},$$

for $j \geq 2$. We thus have that $f_{e,t}$ satisfies (2.6), (2.7), and the only condition not satisfied is (2.8). Given that our results should generalise to holomorphic forms as well as Eisenstein series, this example illustrates the necessity of condition (2.8), since the divisor function, $d(n)$, correlates with additive characters [Tit86, Theorem 7.15].

2.5 Horocycle twists

In this section, we prove Theorem 2.2. We thus let $K_t : \mathbb{R}_{>0} \to \mathbb{C}$ be an analytic trace function, and f be a Maass form as in the previous sections. Let $[\alpha, \beta] \subset [1,2]$ and V be a smooth compactly supported function in $[\frac{1}{2}, \frac{5}{2}]$, such that $x^j V^j(x) \ll_j 1$. We study

$$\int_\alpha^\beta f(x+iy) K_{1/y}\left(\frac{x}{y}\right) V(x)\mathrm{d}x = \sum_{n \neq 0} \frac{\rho_f(n)}{|n|^{1/2}} W_{it_f}(4\pi|n|y) \int_\alpha^\beta K_{1/y}\left(\frac{x}{y}\right) e(nx) V(x)\mathrm{d}x.$$

The proof of the theorem will then follow from the following proposition.

Proposition 2.4. *Let K_t be an analytic trace function. Then there exists an analytic trace function, $\tilde{K}_t(x)$, such that the Fourier transform,*

$$\hat{K}_t(x) := t^{1/2} \int_1^2 K_t(tu) V(u) e(-xu)\mathrm{d}u,$$

satisfies

$$\hat{K}_t(x) = \tilde{K}_t(x) + O(t^{-1/2}).$$

Proof. We have

$$\begin{aligned}\int_1^2 K_t(tu)V(u)e(-xu)\mathrm{d}u &= \frac{1}{2\pi i}\int_{(\sigma)} M_t(s) \int_1^2 (tu)^{-s} V(u) e(-xu)\mathrm{d}u\mathrm{d}s \\ &= \frac{1}{2\pi i}\int_{(\sigma)} M_t(s) t^{-s} V^\dagger(x, 1-s)\mathrm{d}s.\end{aligned}$$

We note that by the properties of $M_t(s)$, discussed in Section 2.2, it is sufficient to consider $v \asymp t$, such that for some $x \in [t, 2t]$,

$$f_t'(\sigma + iv) - \frac{\log x}{2\pi} = o(1), \tag{2.21}$$

for otherwise by repeated integration by parts, the integral is negligible. By Lemma 5 of [Mun15], we may write

$$V^\dagger(x, 1-\sigma-iv) = \frac{\sqrt{2\pi}e(1/8)}{\sqrt{v}} V\left(-\frac{v}{2\pi x}\right)\left(-\frac{v}{2\pi x}\right)^{1-\sigma}\left(-\frac{v}{2\pi e x}\right)^{-iv} + O(|v|^{-3/2}).$$

We thus have that the main term of $\hat{K}_t(x)$ is

$$\frac{e(1/8)t^{1/2-\sigma}}{\sqrt{2\pi}i}\int_{(\sigma)} M_t(\sigma+iv)W(v)\frac{t^{-iv}}{\sqrt{v}} V\left(-\frac{v}{2\pi x}\right)\left(-\frac{v}{2\pi x}\right)^{1-\sigma}\left(-\frac{v}{2\pi e x}\right)^{-iv} \mathrm{d}v,$$

where W is a smooth compactly supported function such that $W^{(j)}(v) \ll_j v^{-j}$, and supported only whenever (2.21) holds. We may thus rewrite the main term as

$$\frac{1}{2\pi i}\int_{(1-\sigma)} \tilde{M}_{t,x}(1-\sigma+iv)x^{\sigma-1-iv}\mathrm{d}v,$$

where up to a constant,

$$\tilde{M}_{t,x}(1-\sigma+iv) = t^{1/2-\sigma+iv}M_t(\sigma-iv)W(-v)V\left(\frac{v}{2\pi x}\right)v^{1/2-\sigma+iv}(2\pi e)^{-iv}.$$

We write

$$\tilde{M}_{t,x}(1-\sigma+iv) = \tilde{g}_{t,x}(1-\sigma+iv)e(\tilde{f}_t(1-\sigma+iv)),$$

where

$$\tilde{g}_{t,x}(1-\sigma+iv) = t^{1/2-\sigma}W(-v)g_t(\sigma-iv)V\left(\frac{v}{2\pi x}\right)v^{1/2-\sigma},$$

and

$$\tilde{f}_t(1-\sigma+iv) = \frac{v}{2\pi}\log(tv) + f_t(\sigma-iv).$$

We compute

$$\frac{\mathrm{d}}{\mathrm{d}v}\tilde{f}_t(1-\sigma+iv) - \frac{1}{2\pi}\log x = \frac{1}{2\pi}\log\left(\frac{tv}{2\pi x}\right) - f_t'(\sigma-iv),$$

and note that if $\frac{v}{2\pi x} \not\in [\frac{1}{2}, \frac{5}{2}]$, then by (2.21), we have that (2.5) holds, so that by repeated integration by parts the integral in that region is negligible. We may therefore write

$$\int_{(1-\sigma)} \tilde{M}_{t,x}(1-\sigma+iv)x^{\sigma-1-iv}\mathrm{d}v = \int_{(1-\sigma)} \tilde{M}_t(1-\sigma+iv)x^{\sigma-1-iv}\mathrm{d}v + O(t^{-100}),$$

where $\tilde{M}_t(1-\sigma+iv) = \tilde{g}_t(1-\sigma+iv)e(\tilde{f}(1-\sigma+iv))$, and

$$\tilde{g}_t(1-\sigma+iv) = t^{1/2-\sigma}W(-v)g_t(1-\sigma+iv)v^{1/2-\sigma}.$$

In the range $v \asymp t$, we have

$$\tilde{g}_t^{(j)}(1-\sigma+iv) \ll t^{1/2-\sigma-j},$$

and therefore $\tilde{g}_t$ satisfies condition (2.4). We moreover have

$$\frac{\mathrm{d}^2}{\mathrm{d}v^2}\tilde{f}_t(1-\sigma+iv) = \frac{1}{2\pi v} + f_t''(\sigma - iv) \gg v^{-1},$$

by (2.8) and thus (2.6) is satisfied for $\tilde{f}_t$. Moreover, by direct computation, we see that since (2.7) holds for f_t, it also holds for $\tilde{f}_t$. By (2.6), we have

$$\tilde{f}_t''(1-\sigma+iv) - \frac{1}{2\pi v} = f_t''(\sigma-iv) \gg v^{-1},$$

so that (2.8) holds for $\tilde{f}_t$. □

We deduce Theorem 2.2 from Proposition 2.4. We first note that the exponential decay of W_{it_f} restricts n to the range $|n| \ll y^{-1}$. Keeping in mind that the Fourier transform is negligible unless $n \asymp y^{-1}$, we only need to show that

$$\frac{1}{\beta-\alpha}\sum_{n\asymp y^{-1}} \frac{\rho_f(n)}{y^{1/2}|n|^{1/2}} y^{1/2}\int_\alpha^\beta K_{1/y}\left(\frac{x}{y}\right) e(nx)V(x)\mathrm{d}x \to 0,$$

as $y \to 0$. However, by Fourier inversion, we have

$$\begin{aligned} y^{-1/2}\int_\alpha^\beta K_{1/y}\left(\frac{x}{y}\right) e(nx)V(x)\mathrm{d}x &= \int_\alpha^\beta \int_{\mathbb{R}} \hat{K}_{1/y}(z)e(zx)e(nx)\mathrm{d}z\mathrm{d}x \\ &= \int_{\mathbb{R}} \hat{K}_{1/y}(z+n)\int_\alpha^\beta e(zx)\mathrm{d}x\mathrm{d}z \\ &= \frac{1}{2\pi i}\int_{\mathbb{R}} \hat{K}_{1/y}(z+n)\frac{e(\beta z)-e(\alpha z)}{z}\mathrm{d}z. \end{aligned}$$

Now by Proposition 2.4 and the properties of analytic trace functions, we must have $z+n \asymp y^{-1}$, for otherwise $\tilde{K}_{1/y}(z+n)$ is negligible. We may thus apply Theorem 2.1 to conclude that

$$\frac{1}{\beta-\alpha}\sum_{n\asymp y^{-1}} \frac{\rho_f(n)}{y^{1/2}|n|^{1/2}} y^{1/2}\int_\alpha^\beta K_{1/y}\left(\frac{x}{y}\right) e(nx)V(x)\mathrm{d}x \ll \frac{y^{1/8-\epsilon}}{\beta-\alpha},$$

proving Theorem 2.2.

3 Large values of Hecke-Maass L-functions with prescribed argument

3.1 Introduction and Setup

The resonance method developed by Soundararajan [Sou08] allows the detection of large values of certain L-functions on the critical line. Building on this work, Hough [Hou16] proves the existence of large values of the Riemann zeta function on the critical line with prescribed argument. In this paper we extend the resonance method to find large values of Hecke-Maass L-functions on the critical line with prescribed argument. More precisely, we let f be an (even) Hecke-Maass eigenform for $\mathrm{SL}_2(\mathbb{Z})$, and denote by $1/4+r^2$ the associated eigenvalue of the Laplacian. We define the Hecke operators $(T_n)_{n\geq 1}$ acting on the space of Maass forms by

$$(T_n f)(z) = \frac{1}{\sqrt{n}} \sum_{ad=n} \sum_{0\leq b<d} f\left(\frac{az+b}{d}\right).$$

We associate to f the sequence of Hecke-eigenvalues $(\lambda_f(n))_{n\geq 1}$. We define the associated L-function,

$$L(f,s) := \sum_n \frac{\lambda_f(n)}{n^s} = \prod_p (1-\alpha_p p^{-s})^{-1}(1-\beta_p p^{-s})^{-1},$$

where α_p, β_p are given via $\alpha_p+\beta_p = \lambda_f(p)$ and $\alpha_p\beta_p = 1$. We prove the following theorem.

Theorem 3.1. *For any $\eta < 1$, any sufficiently large $T \in \mathbb{R}$ and any $\theta \in \mathbb{R}/\mathbb{Z}$, there exists $t \in [\frac{T}{2}, 2T]$ such that*

$$\frac{1}{2\pi} \arg L\left(f, \frac{1}{2}+it\right) \equiv \theta \mod \mathbb{Z}, \text{ and } \log\left|L\left(f,\frac{1}{2}+it\right)\right| \geq (\eta+o(1))\sqrt{\frac{\log T}{\log\log T}}.$$

We follow Hough's strategy [Hou16], namely we exploit sign changes of $L(f,s)$ by comparing the weighted signed moment and unsigned first moment, which we define in the next section. Several substantial complications, however, arise due to the fact that $L(f,s)$ is of degree 2. We may no longer exploit combinatorial arguments to handle sums of fractional divisor functions. We treat these sums by relating them to the symmetric square L-function, $L(\mathrm{sym}^2 f, s)$, and exploiting a zero-free region.

We note that the results presented also hold for holomorphic cusp forms, as they exhibit the same properties as those exploited for Maass forms. Moreover, we expect that the methods are flexible enough to carry over to the case of Maass forms of $\mathrm{SL}_n(\mathbb{Z})$ [1] , by some more elaborate calculations.

3.1.1 Outline of proof

Following [Hou16], we implement the resonance method developed in [Sou08]. We thus let T be a large real number and $\theta \in \mathbb{R}$ be a fixed angle. Let $\xi > 0$ be a small real number and let $N = T^{1-3\xi}$. We set $L = \sqrt{\log N \log\log N}$, and define the multiplicative function, $r(n)$, which is supported on square-free integers and defined at primes by

$$r(p) = \begin{cases} \frac{L}{\sqrt{p}\log p}, & \text{if } L^2 \le p \le \exp((\log L)^2) \\ 0, & \text{otherwise} \end{cases}. \tag{3.1}$$

We define a preliminary resonating polynomial,

$$R^*(s) = \sum_{n \le N} \frac{r(n)\lambda_f(n)}{n^s}.$$

We also introduce a short Dirichlet polynomial,

$$A_{1/2}(s) := \sum_{n \le T^\xi} \frac{d_{1/2}(n)\lambda_f(n)}{n^s},$$

where $d_{1/2}$ are the Dirichlet series coefficients for $\zeta^{1/2}$. In particular $d_{1/2}$ is multiplicative, non-negative, and is given at prime powers by

$$d_{1/2}(p^k) = \frac{1}{2^k k!} \prod_{i=1}^{k} (2i-1).$$

We define our resonating polynomial to be

$$R(s) = R^*(s) A_{1/2}\left(\frac{1}{2} + s\right) =: \sum_{n \le T^{1-2\xi}} \frac{a_n}{n^s}.$$

In order to prove Theorem 3.1 we compute weighted first moments of $L(f, \frac{1}{2} + it)$. Namely, we let

$$T_\theta := \left\{ t \in \mathbb{R} | \arg\left(L\left(f, \frac{1}{2} + it\right)\right) \equiv \theta \,(\mathrm{mod}\, \pi) \right\},$$

and letting $H = T/(\log T)^2$, we define

$$\omega_{T,\theta}(t) = \frac{|R(it)|^2}{\cosh\left(\frac{t-T}{H}\right)} \bigg/ NW,$$

where

$$NW := \sum_{t \in T_\theta} \frac{|R(it)|^2}{\cosh\left(\frac{t-T}{H}\right)}, \tag{3.2}$$

is the normalizing weight required to obtain a probability measure. Theorem 3.1 will be deduced from the following proposition.

[1] Either for self-dual forms, or in the case that the form satisfies Ramanujan-Petersson by a recent non-zero region due to Goldfeld and Li [GL].

Proposition 3.1. *We have*

$$\sum_{t\in T_\theta}\left|L\left(f,\frac{1}{2}+it\right)\right|\omega_{T,\theta}(t)\gg(\log T)^{\frac{3}{4}}\prod_p\left(1+\frac{r(p)\lambda_f^2(p)}{\sqrt{p}}\right),\tag{3.3}$$

and

$$\sum_{t\in T_\theta}L\left(f,\frac{1}{2}+it\right)\omega_{T,\theta}(t)\ll(\log T)^{\frac{1}{2}}\prod_p\left(1+\frac{r(p)\lambda_f^2(p)}{\sqrt{p}}\right).\tag{3.4}$$

We explain here the strategy that allows us to detect the angle of $L(f,s)$ thus allowing us to estimate these moments. Let

$$\Lambda(f,s)=L_\infty(s)L(f,s),$$

be the completed L-function of f, where

$$L_\infty(s):=\pi^{-s}\Gamma\left(\frac{s+ir}{2}\right)\Gamma\left(\frac{s-ir}{2}\right),$$

is the local factor at ∞. The L-function satisfies the functional equation:

$$\Lambda(f,s)=\Lambda(f,1-s).$$

We let

$$\Delta(s):=\frac{L\left(f,\frac{1}{2}+s\right)}{L\left(f,\frac{1}{2}-s\right)}=\frac{L_\infty\left(\frac{1}{2}-s\right)}{L_\infty\left(\frac{1}{2}+s\right)},$$

and observe that the points, t, such that $\arg(L(f,\frac{1}{2}+it))=\theta\,(\mathrm{mod}\,\pi)$ are the solution set of $\Delta(it)=e^{2i\theta}$. In particular, we note that T_θ is not empty. By the Residue Theorem, one may then express the moment as a contour integral of the form

$$\int_\Gamma L\left(f,\frac{1}{2}+s\right)R(s)R(-s)\frac{\Delta'(s)}{\Delta(s)-e^{2i\theta}}\frac{\mathrm{d}s}{\cos\left(\frac{s-iT}{H}\right)},$$

where Γ is an appropriate contour supported at height T. Expanding the L-function into its Dirichlet series we end up having to estimate sums of Hecke eigenvalues against certain arithmetic functions.

We end this section by showing how Theorem 3.1 follows from Proposition 3.1. By Proposition 3.1, we have

$$\begin{aligned}\sum_{\arg(L)=\theta}\left|L\left(f,\frac{1}{2}+it\right)\right|\omega_{T,\theta}(t)&=\frac{1}{2}\sum_{t\in T_\theta}\left(\left|L\left(f,\frac{1}{2}+it\right)\right|+e^{-i\theta}L\left(f,\frac{1}{2}+it\right)\right)\omega_{t,\theta}(t)\\&\gg(\log T)^{3/4}\prod_p\left(1+\frac{r(p)\lambda_f(p)^2}{\sqrt{p}}\right),\end{aligned}$$

so that

$$\max_{\substack{\frac{T}{2}\le t\le 2T\\ \arg(L)=\theta}}\left|L\left(f,\frac{1}{2}+it\right)\right|\gg(\log T)^{3/4}\prod_p\left(1+\frac{r(p)\lambda_f(p)^2}{\sqrt{p}}\right).$$

Theorem 3.1 now follows from

$$\log \prod_p \left(1 + \frac{r(p)\rho_f(p)^2}{\sqrt{p}}\right) \sim L \sum_{L^2 \le p \le \exp(\log^2 L)} \frac{\lambda_f(p)^2}{p \log p}$$
$$\sim \sqrt{(1-3\xi)\frac{\log T}{\log\log T}},$$

and letting $\xi \to 0$.

3.2 Preliminary lemmas

In order to estimate these moments, we will require some preliminary lemmas that we prove in this section.

Lemma 3.1. *Let T be large, and $1 \le m, n$ and assume $m < T^{2-\delta}$, and $\min(m,n) < T^{1-\delta}$ for some $\delta > 0$. We then have for any $\omega \in \mathbb{S}^1$ and for any $A > 0$,*

$$\int_{\Re(s)=\frac{1}{2}+\epsilon} \left(\frac{m}{n}\right)^s \frac{\Delta'(s)}{\Delta(s)} \frac{\Delta(s)}{1-\omega\Delta(s)} \frac{\mathrm{d}s}{\cos\left(\frac{iT\pm s}{H}\right)} = O_{\delta,A}(T^{-A}). \tag{3.5}$$

Letting

$$I_T := \int_{t\ge 20} \frac{-2\Delta'(it)/\Delta(it)}{\cosh\left(\frac{t-T}{H}\right)} \mathrm{d}t,$$

we also have,

$$\frac{1}{2\pi i}\int_{\Re(s)=\frac{1}{2}+\epsilon} \left(\frac{m}{n}\right)^s \frac{\Delta'(s)}{\Delta(s)} \frac{\mathrm{d}s}{\cos\left(\frac{iT\pm s}{H}\right)} = -\frac{\delta_{m=n}}{4\pi} I_T + O_{\delta,A}(T^{-A}). \tag{3.6}$$

Proof. We need some estimates about $\Delta(s)$. By Stirling's formula, we have that for $|t| \gg 1$,

$$|\Delta(\sigma+it)| \ll \left| \frac{\pi^{2(\sigma+it)}\Gamma\left(\frac{\frac{1}{2}-\sigma+i(r-t)}{2}\right)\Gamma\left(\frac{\frac{1}{2}-\sigma-i(r+t)}{2}\right)}{\Gamma\left(\frac{\frac{1}{2}+\sigma+i(r+t)}{2}\right)\Gamma\left(\frac{\frac{1}{2}+\sigma+i(t-r)}{2}\right)} \right| \ll_\sigma t^{-2\sigma}.$$

Writing $\Delta(s) = \pi^{2s}\Gamma_1\Gamma_2/(\Gamma_3\Gamma_4)$, we compute

$$\frac{\Delta'(it)}{\Delta(it)} = 2\log\pi + \frac{\Gamma_1'}{\Gamma_1}(it) + \frac{\Gamma_2'}{\Gamma_2}(it) - \frac{\Gamma_3'}{\Gamma_3}(it) - \frac{\Gamma_4'}{\Gamma_4}(it)$$
$$= -\frac{1}{2}\log\left(\frac{\frac{1}{16}+\frac{1}{4}((r+t)^2+(t-r)^2)+(t^2-r^2)^2}{16\pi^4}\right) + O(|t|^{-1+\epsilon}), \tag{3.7}$$

and thus also

$$\frac{\mathrm{d}^j}{\mathrm{d}t^j}\frac{\Delta'}{\Delta}(it) = O_j(|t|^{1-j}),$$

for $j \ge 2$. In order to prove (3.5), we push the line of integration rightwards to $\Re(s) = (A+1)/\delta + \delta'$, with $0 < \delta' < 1$ chosen so that the contour has a distance bounded from any pole of the integrand. In pushing the line rightwards as indicated above, the only poles we pass are counter-weighted by the hyperbolic cosine factor (since these poles can only occur for t bounded away from the real axis) and they therefore contribute a negligible amount. We are thus left with estimating

$$\begin{aligned}
&\int_{\Re(s)=(A+1)/\delta+\delta'} \left(\frac{m}{n}\right)^s \frac{\Delta'}{\Delta}(s) \frac{\Delta(s)}{1-\omega\Delta(s)} \frac{\mathrm{d}s}{\cos\left(\frac{iT\pm s}{H}\right)} \\
&\quad \ll \int_{\frac{T}{2}}^{2T} T^{(2-\delta)\left(\frac{A+1}{\delta}+\delta'\right)} \log(|t|) T^{-2\left(\frac{A+1}{\delta}+\delta'\right)} \mathrm{d}t + O(|T|^{-A}) \\
&\quad \ll T^{-A}.
\end{aligned}$$

In order to prove (3.6), we note that $\Delta(s)$ has no poles nor zeroes on $\Re(s) = 0$, and as before the only poles we might encounter are negligible, and we may thus shift our line of integration to $\Re(s) = 0$. By (3.7), the integral becomes

$$-\frac{1}{4\pi}\int_{\mathbb{R}} \left(\frac{m}{n}\right)^{it} \frac{\log\left(\frac{\frac{1}{16}+\frac{1}{4}((r+t)^2+(t-r)^2)+(t^2-r^2)^2}{16\pi^4}\right) + O(|t|^{-1+\epsilon})}{\cosh\left(\frac{T\pm t}{H}\right)} \mathrm{d}t.$$

If $m \neq n$, then by repeated integration by parts, the integral is negligible. The lemma follows. □

We note that I_T satisfies

$$I_T = \int_{t\geq 20} \frac{\log\left(\frac{\frac{1}{16}+\frac{1}{4}((r+t)^2+(t-r)^2)+(t^2-r^2)^2}{16\pi^4}\right) + O(|t|^{-1})}{\cosh\left(\frac{t-T}{H}\right)} \mathrm{d}t.$$

We recall that by the analog of Mertens' Theorem for Rankin-Selberg L-functions, there exists a constant, C, such that

$$\sum_{p\leq x} \frac{\lambda_f(p)^2}{p} = \log\log x + C + o(1),$$

and will use it without mention in the proof of the following lemmas.

Lemma 3.2. *For any* $|\alpha| \leq \frac{1}{(\log L)^3}$, *we have*

$$\begin{aligned}
&\log\prod_p (1 + r(p)^2\lambda_f(p)^2 p^\alpha) - \log\prod_p (1 + r(p)^2\lambda_f(p)^2) \\
&\quad \leq \alpha\left(\log N - (1+o(1))\frac{\log N \log\log\log N}{\log\log N}\right).
\end{aligned}$$

Proof. We write

$$
\begin{aligned}
&\log\prod_p(1+r(p)^2\lambda_f(p)^2p^\alpha)-\log\prod_p(1+r(p)^2\lambda_f(p)^2)\\
&\quad=\sum_{L^2\le p\le\exp(\log^2 L)}\log\left(1+\frac{r(p)^2\lambda_f(p)^2(p^\alpha-1)}{1+r(p)^2\lambda_f(p)^2}\right)\\
&\quad=\sum_{L^2\le p\le\exp(\log^2 L)}\frac{r(p)^2\lambda_f(p)^2(p^\alpha-1)}{1+r(p)^2\lambda_f(p)^2}\left(1+O\left(\frac{r(p)^2\lambda_f(p)^2(p^\alpha-1)}{1+r(p)^2\lambda_f(p)^2}\right)\right).
\end{aligned}
$$

Since,

$$
(p^\alpha-1)\frac{r(p)^2\lambda_f(p)^2}{1+r(p)^2\lambda_f(p)^2}\le p^\alpha-1\ll\alpha\log p\ll\frac{1}{\log L},
$$

we may bound the difference of logarithms by

$$
\begin{aligned}
&\sum_{L^2\le p\le\exp(\log^2 L)}r(p)^2\lambda_f(p)^2(p^\alpha-1)\left(1+O\left(\frac{1}{\log L}\right)\right)\\
&\quad=\alpha L^2\sum_{L^2\le p\le\exp(\log^2 L)}\frac{\lambda_f(p)^2}{p}\frac{1}{\log p}\left(1+O\left(\frac{1}{\log L}\right)\right)\\
&\quad=\alpha L^2\left(\frac{\log(\log^2 L)+C+o(1)}{\log^2 L}-\frac{\log\log L^2+C+o(1)}{\log L^2}\right.\\
&\qquad\qquad\left.+\int_{L^2}^{\exp(\log^2 L)}\frac{\log\log x+C+o(1)}{(\log x)^2x}\mathrm{d}x\right)\left(1+O\left(\frac{1}{\log L}\right)\right)\\
&\quad=\alpha L^2\left(\frac{1}{2\log L}+o\left(\frac{1}{\log L}\right)\right)\left(1+O\left(\frac{1}{\log L}\right)\right)\\
&\quad=\alpha\left(\log N-(1+o(1))\frac{\log N\log\log\log N}{\log\log N}\right).
\end{aligned}
$$

□

As a corollary, we deduce the following lemma.

Lemma 3.3. *For any integer $l\ge1$ and for any $Z>N\exp\left(-\frac{\log N}{(\log\log N)^2}\right)$,*

$$
\sum_{\substack{n<Z\\(n,l)=1}}r(n)^2\lambda_f(n)^2=\left(1+O\left(\exp\left(-\frac{L^2}{(\log L)^5}\right)\right)\right)\prod_{p\nmid l}(1+r(p)^2\lambda_f(p)^2).
$$

Proof. We use Rankin's trick to write

$$
\begin{aligned}
\sum_{\substack{n<Z\\(n,l)=1}}r(n)^2\lambda_f(n)^2&=\sum_{\substack{n=1\\(n,l)=1}}^{\infty}r(n)^2\lambda_f(n)^2-\sum_{\substack{n\ge Z\\(n,l)=1}}r(n)^2\lambda_f(n)^2\\
&=\prod_{p\nmid l}(1+r(p)^2\lambda_f(p)^2)+O(Z^{-\alpha}\prod_{p\nmid l}(1+p^\alpha r(p)^2\lambda_f(p)^2)).
\end{aligned}
$$

The result then follows immediately from Lemma 3.2. □

We now prove analogously the following two Lemmas.

Lemma 3.4. *For any* $|\alpha| \le \frac{1}{(\log L)^3}$*, and any multiplicative function,* g*, such that for some* $m > 0$*,* $0 \le g(p) \le m$ *for all* p*, we have*

$$\log\left(\prod_p \frac{1 + r(p)\lambda_f(p)^2 g(p) p^{\alpha-1/2}}{1 + r(p)\lambda_f(p)^2 g(p) p^{-1/2}}\right) \ll_m \alpha L \log\log L.$$

Proof. We may write

$$\begin{aligned}
&\sum_{L^2 \le p \le \exp(\log^2 L)} \log\left(1 + \frac{r(p)\lambda_f(p)^2 g(p) p^{-1/2}(p^\alpha - 1)}{1 + r(p)g(p)\lambda_f(p)^2 p^{-1/2}}\right) \\
&\quad = \sum_{L^2 \le p \le \exp(\log^2 L)} \frac{r(p)\lambda_f(p)^2 g(p) p^{-1/2}(p^\alpha - 1)}{1 + r(p)g(p)\lambda_f(p)^2 p^{-1/2}} \left(1 + O\left(\frac{r(p)\lambda_f(p)^2 g(p) p^{1/2}(p^\alpha - 1)}{1 + r(p)\lambda_f(p)^2 g(p) p^{-1/2}}\right)\right).
\end{aligned}$$

Since

$$(p^\alpha - 1)\frac{r(p)\lambda_f(p)^2 g(p) p^{-1/2}}{1 + r(p)g(p)\lambda_f(p)^2 p^{-1/2}} \le p^\alpha - 1 \ll \alpha \log p \ll \frac{1}{\log L},$$

we bound

$$\begin{aligned}
&\log\left(\prod_p \frac{1 + r(p)\lambda_f(p)^2 g(p) p^{\alpha-1/2}}{1 + r(p)\lambda_f(p)^2 g(p) p^{-1/2}}\right) \\
&\quad \le \sum_{L^2 \le p \le \exp(\log^2 L)} r(p)\lambda_f(p)^2 g(p) p^{-1/2}(p^\alpha - 1)\left(1 + O\left(\frac{1}{\log L}\right)\right) \\
&\quad \ll_m \sum_{L^2 \le p \le \exp(\log^2 L)} \alpha \frac{L}{p}\lambda_f(p)^2 \left(1 + O\left(\frac{1}{\log L}\right)\right) \\
&\quad = \alpha L(\log(\log^2 L) - \log\log L^2 + o(1))\left(1 + O\left(\frac{1}{\log L}\right)\right) \\
&\quad \ll \alpha L \log\log L.
\end{aligned}$$

□

As a corollary we deduce the following Lemma.

Lemma 3.5. *For* $Z > \exp(L(\log L)^5)$*, and* g *multiplicative such that for some* m*,* $0 \le g(p) \le m$ *for all* p*, we have*

$$\sum_{n \ge Z} \frac{r(n)}{\sqrt{n}}\lambda_f(n)^2 g(n) \le \exp\left(-(1 + o_m(1))\frac{\log Z}{(\log L)^3}\right),$$

and

$$\sum_{n < Z} \frac{r(n)}{\sqrt{n}}\lambda_f(n)^2 g(n) = \left(1 + O_m\left(\exp(-cL(\log L)^2)\right)\right)\prod_p \left(1 + \frac{r(p)}{\sqrt{p}}\lambda_f(p)^2 g(p)\right).$$

Proof. We use Rankin's trick to write

$$\begin{aligned}\sum_{n<Z}\frac{r(n)}{\sqrt{n}}\lambda_f(n)^2g(n) &= \sum_{n=1}^{\infty}\frac{r(n)}{\sqrt{n}}\lambda_f(n)^2g(n)-\sum_{n\geq Z}\frac{r(n)}{\sqrt{n}}\lambda_f(n)^2g(n)\\ &=\prod_p\left(1+\frac{r(p)}{\sqrt{p}}\lambda_f(p)^2g(p)\right)\\ &\quad+O\left(Z^{-\alpha}\prod_p(1+r(p)\lambda_f(p)^2g(p)p^{\alpha-1/2})\right).\end{aligned}$$

The result now follows from Lemma 3.4. □

Throughout the paper, we will also require a result of Tenenbaum [Ten15, Theorem 5.2, p. 281], inspired by previous work of Delange [Del54, Del71], that we give in the following lemma. We first need to set up some notation. Let $z\in\mathbb{C}$, and fix $c_0>0, 0<\delta\leq 1, M>0$, positive constants. Writing $s=\sigma+i\tau$, we say that a Dirichlet series $F(s)$ has the property $\mathscr{P}(z;c_0,\delta,M)$ if the Dirichlet series

$$G(s;z):=F(s)\zeta(s)^{-z}$$

may be continued as a holomorphic function for $\sigma\geq 1-c_0/(1+\log(2+|\tau|))$, and, in this domain, satisfies the bound

$$|G(s;z)|\leq M(1+|\tau|)^{1-\delta}.$$

If $F(s)=\sum a_n/n^s$ has the property $\mathscr{P}(z;c_0,\delta,M)$, and if there exists a sequence of non-negative real numbers $\{b_n\}_{n=1}^{\infty}$ such that $|a_n|\leq b_n$, $(n=1,2,\cdots)$, and the series

$$\sum_{n\geq 1}\frac{b_n}{n^s}$$

satisfies $\mathscr{P}(w;c_0,\delta,M)$ for some complex number w, we shall say that $F(s)$ has type $\mathscr{T}(z,w;c_0,\delta,M)$.

Lemma 3.6. *Let $F(s):=\sum a_n/n^s$ be a Dirichlet series of type $\mathscr{T}(z,w;c_0,\delta,M)$. For $x\geq 3, A>0, |z|\leq A, |w|\leq A$, there exist $d>0$ such that*

$$\sum_{n\leq x}a_n=x(\log x)^{z-1}\left\{\frac{G(1;z)}{\Gamma(z)}+O(M(e^{-d\sqrt{\log x}}+\log x^{-1}))\right\}.$$

The constant d and the implicit constant in the Landau symbol depend at most on c_0,δ, and A.

3.3 Computing the normalizing weight

In this section we compute the normalizing weight, NW, given by (3.2). We will require the following estimates on the coefficients a_n.

Lemma 3.7. *We have*

$$\sum_{n\leq T^{1-2\xi}}a_n^2\asymp(\log T)^{1/4}\prod_p(1+r(p)^2\lambda_f(p)^2)\prod_p\left(1+\frac{r(p)\lambda_f(p)^2}{\sqrt{p}}\right). \tag{3.8}$$

Proof. We have

$$a_n = \sum_{\substack{l \le T^{1-3\xi} \\ m \le T^{\xi} \\ lm=n}} r(l)\lambda_f(l) \frac{d_{1/2}(m)\lambda_f(m)}{m^{1/2}},$$

so that

$$\begin{aligned}
\sum_{n \le T^{1-2\xi}} a_n^2 &= \sum_{l_1,l_2 \le T^{1-3\xi}} r(l_1)r(l_2)\lambda_f(l_1)\lambda_f(l_2) \sum_{\substack{n_1,n_2 \le T^{\xi} \\ l_1 n_1 = l_2 n_2}} \frac{d_{1/2}(n_1)d_{1/2}(n_2)\lambda_f(n_1)\lambda_f(n_2)}{(n_1 n_2)^{1/2}} \\
&= \sum_{g \le T^{1-3\xi}} r(g)^2 \lambda_f(g)^2 \sum_{\substack{l_1,l_2 \le T^{1-3\xi}/g \\ (l_1,l_2)=(l_1 l_2, g)=1}} r(l_1 l_2)\lambda_f(l_1 l_2) \\
&\quad \times \sum_{\substack{n_1,n_2 \le T^{\xi} \\ l_1 n_1 = l_2 n_2}} \frac{d_{1/2}(n_1)d_{1/2}(n_2)\lambda_f(n_1)\lambda_f(n_2)}{(n_1 n_2)^{1/2}}.
\end{aligned}$$

We now let $n_2 := n_2/l_1$ and $n_1 := n_1/l_2$, so that we may rewrite this as

$$\begin{aligned}
&\sum_{g \le T^{1-3\xi}} r(g)^2 \lambda_f(g)^2 \sum_{\substack{l_1,l_2 \le T^{1-3\xi}/g \\ (l_1,l_2)=(l_1 l_2, g)=1}} \frac{r(l_1 l_2)}{(l_1 l_2)^{1/2}} \lambda_f(l_1 l_2) \\
&\sum_{n \le T^{\xi}/\max(l_1,l_2)} \frac{d_{1/2}(l_1 n)d_{1/2}(l_2 n)\lambda_f(l_1 n)\lambda_f(l_2 n)}{n}.
\end{aligned}$$

3.3.1 The n-sum

The idea is to treat the innermost sum by relating it to the fourth root of the Rankin-Selberg L-function,

$$\begin{aligned}
L(f \times f, s) &:= \prod_p (1-p^{-s})^{-1}(1-\alpha_p^2 p^{-s})^{-1}(1-\beta_p^2 p^{-s})^{-1}(1-\alpha_p\beta_p p^{-s})^{-1} \\
&= \zeta(s) L(\mathrm{sym}^2 f, s),
\end{aligned}$$

where

$$L(\mathrm{sym}^2 f, s) := \prod_p (1-p^{-s})^{-2}(1-\alpha_p^2 p^{-s})^{-1}(1-\beta_p^2 p^{-s})^{-1},$$

denotes the symmetric square L-function as studied by Gelbart and Jacquet [GJ78]. Following [Ten15, Chapter II.5], we define the generalized binomial coefficient by

$$\binom{\omega}{\nu} := \frac{1}{\nu!} \prod_{0 \le j < \nu} (\omega - j) \qquad (\omega \in \mathbb{C}, \nu \in \mathbb{N}),$$

so that

$$
\begin{aligned}
L^{1/4}(f\times f,s) &= \prod_p (1-p^{-s})^{-1/2}(1-\alpha_p^2 p^{-s})^{-1/4}(1-\beta_p^2 p^{-s})^{-1/4} \\
&= \prod_p \left(\sum_{k=0}^{\infty}\binom{k-\frac12}{k} p^{-ks}\right)\left(\sum_{k=0}^{\infty}\binom{k-\frac34}{k}\alpha_p^{2k}p^{-ks}\right) \\
&\quad\times\left(\sum_{k=0}^{\infty}\binom{k-\frac34}{k}\beta_p^{2k}p^{-ks}\right) \\
&= \prod_p\left(\sum_{k=0}^{\infty} a(p^k)p^{-ks}\right),
\end{aligned}
$$

where a is a multiplicative function such that

$$
a(p) = \frac{\lambda_f(p)^2}{4} = d_{1/2}^2(p)\lambda_f^2(p).
$$

Given that $L(\mathrm{sym}^2 f,s)$ is a cuspidal automorphic L-function (see [GJ78]), writing $s=\sigma+i\tau$, there exists a constant $c>0$, depending on f, such that $\zeta(s)L(\mathrm{sym}^2 f,s)$ is non-zero in the region $\sigma>1-c/\log(2+|\tau|)$ (see [Mic07]). We note that $L(\mathrm{sym}^2 f,s)$ is entire in that region, so that

$$
\sum_{n=1}^{\infty}\frac{d_{1/2}^2(n)\lambda_f^2(n)}{n^s} = \zeta^{1/4}(s)L^{1/4}(\mathrm{sym}^2 f,s)F(s),
$$

where $F(s)$ is a non-zero, bounded and holomorphic function in the region $\sigma>1-c/\log(2+|\tau|)$. It follows that

$$
\begin{aligned}
&\sum_{n=1}^{\infty}\frac{d_{1/2}(l_1 n)d_{1/2}(l_2 n)\lambda_f(l_1 n)\lambda_f(l_2 n)}{n^s} \\
&= \prod_{p\mid l_1 l_2}\left(\sum_{k=0}^{\infty}\frac{d_{1/2}(p^{k+1})d_{1/2}(p^k)\lambda_f(p^{k+1})\lambda_f(p^k)}{p^{ks}}\right)\times\prod_{p\nmid l_1 l_2}\left(\sum_{k=0}^{\infty}\frac{d_{1/2}^2(p^k)\lambda_f^2(p^k)}{p^{ks}}\right) \\
&= G(s;l_1 l_2)\prod_p\left(\sum_{k=0}^{\infty}\frac{d_{1/2}^2(p^k)\lambda_f^2(p^k)}{p^{ks}}\right) \\
&= G(s;l_1 l_2)F(s)L^{1/4}(\mathrm{sym}^2 f,s)\zeta^{1/4}(s),
\end{aligned}
$$

where

$$
G(s;l) = \prod_{p\mid l}\frac{\sum_{k=0}^{\infty}\frac{d_{1/2}(p^{k+1})d_{1/2}(p^k)\lambda_f(p^{k+1})\lambda_f(p^k)}{p^{ks}}}{\sum_{k=0}^{\infty}\frac{d_{1/2}^2(p^k)\lambda_f^2(p^k)}{p^{ks}}}.
$$

Observe that the denominator is non-zero because the coefficients are positive. We let

$$
G(s;1/4,l_1,l_2) := L^{1/4}(\mathrm{sym}^2 f,s)F(s)G(s,l_1,l_2),
$$

and wish to bound $|G(s;1/4,l_1,l_2)|$ in the aforementioned domain. Noting that k and $k+1$ have distinct parity, we estimate

$$\begin{aligned}|G(s,l_1l_2)| &= \left|\prod_{p|l_1l_2}\frac{\sum_{k=0}^{\infty}\frac{d_{1/2}(p^{k+1})d_{1/2}(p^k)\lambda_f(p^{k+1})\lambda_f(p^k)}{p^{ks}}}{\sum_{k=0}^{\infty}\frac{d_{1/2}^2(p^k)\lambda_f^2(p^k)}{p^{ks}}}\right|\\ &\leq d_{1/2}(l_1l_2)|\lambda_f(l_1l_2)\prod_{p|l_1l_2}M_G(l_1l_2)|,\end{aligned}$$

where M_G is a multiplicative function supported on squarefree integers satisfying at primes

$$M_G(p)^{\pm 1} \leq (1+C_1p^{-\delta_1}), \tag{3.9}$$

for some $\delta_1>0$ and an absolute constant C_1 (one may use bounds towards the Ramanujan-Petersson conjecture as given in [Kim03]). Since $L^{1/4}(s,\mathrm{sym}^2 f) \ll \tau^{\delta}$ for any arbitrarily small $\delta>0$, and letting $M>0$ be such that $F(s)\leq M$ in that region, we conclude that

$$|G(s;1/4,l_1,l_2)| \leq M|\lambda_f(l_1l_2)|d_{1/2}(l_1l_2)M_G(l_1l_2)(1+|\tau|^{\delta}).$$

By Lemma 3.6, we conclude that for $\max(l_1,l_2)\leq T^{\xi-\epsilon}$,

$$\begin{aligned}&\max(l_1,l_2)\sum_{n\leq T^{\xi}/\max(l_1,l_2)} d_{1/2}(l_1n)d_{1/2}(l_2n)\lambda_f(l_1n)\lambda_f(l_2n)\\ &=\frac{T^{\xi}}{\log(\frac{T^{\xi}}{\max(l_1,l_2)})^{3/4}}G(1;1/4,l_1,l_2)\frac{1}{\Gamma(\frac14)}+O\left(\frac{T^{\xi}|\lambda_f(l_1l_2)|d_{1/2}(l_1l_2)M_G(l_1l_2)}{(\log T^{\xi}/\max(l_1,l_2))^{7/4}}\right).\end{aligned}$$

It then follows by summation by parts, that whenever $\max(l_1,l_2)\leq T^{\xi-\epsilon}$, we can estimate

$$\begin{aligned}&\sum_{n\leq T^{\xi}/\max(l_1,l_2)}\frac{d_{1/2}(l_1n)d_{1/2}(l_2n)\lambda_f(l_1n)\lambda_f(l_2n)}{n} \\ &=4\frac{1}{\Gamma\left(\frac14\right)}G(1;\frac14,l_1,l_2)\log\left(\frac{T^{\xi}}{\max(l_1,l_2)}\right)^{1/4}+O\left(\frac{|\lambda_f(l_1l_2)|d_{1/2}(l_1l_2)M_G(l_1l_2)}{(\log\frac{T^{\xi}}{\max(l_1,l_2)})^{3/4}}\right).\end{aligned} \tag{3.10}$$

3.3.2 The l_i and g sums

We let $Z=\exp((\log N)^{2/3})$ and consider first the contribution from the main term above when $\max(l_1,l_2)<Z$. Namely, we estimate

$$\sum_{g\leq T^{1-3\xi}} r(g)^2\lambda_f(g)^2 \sum_{\substack{l_1,l_2\leq T^{1-3\xi}/g\\ \max(l_1,l_2)<Z\\ (l_1,l_2)=(l_1l_2,g)=1}}\frac{r(l_1l_2)}{(l_1l_2)^{1/2}}\lambda_f(l_1l_2)^2d_{1/2}(l_1l_2)H(l_1l_2)\left(\log T\right)^{1/4},$$

where $H(l)$ is a non-negative multiplicative function supported on squarefree integers, satisfying (3.9) on primes, possibly with a different constant. By Lemma 3.3 we thus estimate

$$\begin{aligned}
&(\log T)^{1/4} \sum_{\substack{l_1,l_2<Z\\(l_1,l_2)=1}} \frac{r(l_1l_2)}{\sqrt{l_1l_2}}\lambda_f(l_1l_2)^2 d_{1/2}(l_1l_2)H(l_1l_2) \sum_{\substack{g\le\frac{T^{1-2\xi}}{\max(l_1,l_2)}\\(g,l_1l_2)=1}} r(g)^2\lambda_f(g)^2\\
&\sim (\log T)^{1/4} \sum_{\substack{l_1,l_2<Z\\(l_1,l_2)=1}} \frac{r(l_1l_2)}{\sqrt{l_1l_2}}\lambda_f(l_1l_2)^2 d_{1/2}(l_1l_2)H(l_1l_2) \prod_{p\nmid l_1l_2}(1+r^2(p)\lambda_f(p)^2)\\
&= (\log T)^{1/4}\prod_p(1+r(p)^2\lambda_f(p)^2) \sum_{\substack{l_1,l_2<Z\\(l_1,l_2)=1}} \frac{r(l_1l_2)}{\sqrt{l_1l_2}}\lambda_f(l_1l_2)^2 d_{1/2}(l_1l_2)\tilde{H}(l_1l_2),
\end{aligned}$$

where $\tilde{H}(l)$ is a non-negative, multiplicative function, absolutely bounded on primes and satisfying

$$\tilde{H}(p) = \frac{H(p)}{(1+r(p)^2\lambda_f(p)^2)}.$$

We make the change of variables $l = l_1l_2$ to reduce our estimation to that of

$$\sum_{l<Z}\frac{r(l)}{\sqrt{l}}\lambda_f(l)^2\tilde{H}(l) \sim \prod_p\left(1+\frac{r(p)}{\sqrt{p}}\lambda_f(p)^2\tilde{H}(p)\right) \sim \prod_p\left(1+\frac{r(p)}{\sqrt{p}}\lambda_f(p)^2\right),$$

by Lemma 3.5. The contribution from the tail $\max(l_1,l_2)\ge Z$ is bounded by

$$\begin{aligned}
&\log T\sum_{g\le T^{1-3\xi}} r(g)^2\lambda_f(g)^2 \sum_{l_1\le T^{1-3\xi}/g}\frac{r(l_1)\lambda_f(l_1)^2}{\sqrt{l_1}} \sum_{Z<l_2\le T^{1-3\xi}/g}\frac{r(l_2)\lambda_f(l_2)^2}{\sqrt{l_2}}\\
&\ll \log T\exp(-(\log N)^{2/3-\epsilon})\prod_p(1+r(p)^2\lambda_f^2(p))\prod_p\left(1+\frac{r(p)\lambda_f(p)^2}{\sqrt{p}}\right),
\end{aligned}$$

by Lemma 3.5, which is negligible. We are only left with estimating the contribution coming from the error term in (3.10), with $\max(l_1,l_2)<Z$. We thus care to bound

$$\begin{aligned}
&(\log T)^{-3/4} \sum_{\substack{l_1,l_2<Z\\(l_1,l_2)=1}} \frac{r(l_1l_2)}{\sqrt{l_1l_2}}\lambda_f(l_1l_2)^2 d_{1/2}(l_1l_2)M_G(l_1l_2) \sum_{\substack{g\le\frac{T^{1-3\xi}}{\max(l_1,l_2)}\\(g,l_1l_2)=1}} r(g)^2\lambda_f(g)^2\\
&\sim (\log T)^{-3/4} \sum_{\substack{l_1,l_2<Z\\(l_1,l_2)=1}} \frac{r(l_1l_2)}{\sqrt{l_1l_2}}\lambda_f(l_1l_2)^2 d_{1/2}(l_1l_2)M_G(l_1l_2) \prod_{p\nmid l_1l_2}(1+r(p)^2\lambda_f(p)^2)\\
&\sim (\log T)^{-3/4}\prod_p(1+r(p)^2\lambda_f(p)^2)\prod_p\left(1+\frac{r(p)M_G(p)\lambda_f^2(p)}{\sqrt{p}(1+r(p)^2\lambda_f(p)^2)}\right)
\end{aligned}$$

which is negligible. Putting all of the estimates together, we obtain (3.8).

□

We conclude this section by computing the normalizing weight.

Proposition 3.2. *We may estimate the normalizing weight,*

$$NW \asymp (\log T)^{1/4} \prod_p (1 + r(p)^2 \lambda_f(p)^2) \prod_p \left(1 + \frac{r(p)}{\sqrt{p}} \lambda_f(p)^2\right) I_T.$$

Proof. We denote by Γ_ϵ the contour defined by the line $\Re(s) = 1/2 + \epsilon$ clockwards and $\Re(s) = -1/2 - \epsilon$ anticlockwards, so that up to negligible error, we have

$$NW \sim \frac{1}{2\pi i} \int_{\Gamma_\epsilon} R(s) R(-s) \frac{\Delta'(s)}{\Delta(s) - e^{2i\theta}} \frac{\mathrm{d}s}{\cos\left(\frac{iT-s}{H}\right)}.$$

The integral on $\Re(s) = \frac{1}{2} + \epsilon$ is negligible by (3.5). On $\Re(s) = -\frac{1}{2} - \epsilon$, we substitute $s \mapsto -s$ and thus need to estimate

$$\begin{aligned}
&\int_{\Re(s)=\frac{1}{2}+\epsilon} R(s)R(-s) \left(-\frac{\Delta'(s)}{\Delta(s)} \frac{1}{1 - e^{2i\theta}\Delta(s)}\right) \frac{\mathrm{d}s}{\cos\left(\frac{iT+s}{H}\right)} \\
&= \sum_{k=0}^{\infty} \int_{\Re(s)=\frac{1}{2}+\epsilon} R(s)R(-s) \left(-\frac{\Delta'(s)}{\Delta(s)} \Delta^k(s) e^{2ik\theta}\right) \frac{\mathrm{d}s}{\cos\left(\frac{iT+s}{H}\right)} \\
&= -\int_{\Re(s)=\frac{1}{2}+\epsilon} R(s)R(-s) \frac{\Delta'(s)}{\Delta(s)} \frac{\mathrm{d}s}{\cos\left(\frac{iT+s}{H}\right)} \\
&\quad - \int_{\Re(s)=\frac{1}{2}+\epsilon} R(s)R(-s)\Delta'(s) \frac{e^{2i\theta}}{1 - e^{2i\theta}\Delta(s)} \frac{\mathrm{d}s}{\cos\left(\frac{iT+s}{H}\right)} \\
&= -\sum_{m,n \le T^{1-2\xi}} a_m a_n \int_{\Re(s)=\frac{1}{2}+\epsilon} \left(\frac{m}{n}\right)^s \frac{\Delta'(s)}{\Delta(s)} \frac{\mathrm{d}s}{\cos\left(\frac{iT+s}{H}\right)} \\
&\quad - e^{2i\theta} \sum_{m,n \le T^{1-2\xi}} a_m a_n \int_{\Re(s)=\frac{1}{2}+\epsilon} \left(\frac{m}{n}\right)^s \frac{\Delta'(s)}{\Delta(s)} \frac{\Delta(s)}{1 - e^{2i\theta}\Delta(s)} \frac{\mathrm{d}s}{\cos\left(\frac{iT+s}{H}\right)}.
\end{aligned}$$

Using Lemma 3.1 and Lemma 3.7, we conclude that

$$NW \asymp (\log T)^{1/4} \prod_p (1 + r(p)^2 \lambda_f(p)^2) \prod_p \left(1 + \frac{r(p)}{\sqrt{p}} \lambda_f(p)^2\right) I_T.$$

□

3.4 The unsigned moment

We denote by $\mathbf{E}_{w_{T,\theta}}$ the expectation over T_θ with respect to the measure $w_{T,\theta}$ and wish to give a lower bound to

$$\begin{aligned}
&\mathrm{NW}.\mathbf{E}_{w_{T,\theta}} \left[\left|L\left(f, \frac{1}{2} + it\right)\right|\right] \ge \mathrm{NW}. \left|\mathbf{E}_{w_{T,\theta}} \left[L\left(f, \frac{1}{2} - it\right) \frac{A_{1/2}\left(\frac{1}{2} + it\right)^2}{\left|A_{1/2}\left(\frac{1}{2} + it\right)\right|^2}\right]\right| \\
&\sim \frac{1}{2\pi} \left|\int_{\Gamma_\epsilon} L\left(f, \frac{1}{2} - s\right) A_{1/2}\left(\frac{1}{2} + s\right)^2 R^*(s)R^*(-s) \frac{\Delta'(s)}{\Delta(s) - e^{2i\theta}} \frac{\mathrm{d}s}{\cos\left(\frac{iT-s}{H}\right)}\right| \qquad (3.11) \\
&= \frac{1}{2\pi} \left|\int_{\Re(s)=1/2+\epsilon} \cdots + \int_{\Re(s)=-1/2-\epsilon} \cdots\right|.
\end{aligned}$$

3.4.1 Contribution from the integral along the line $\Re(s) = \frac{1}{2} + \epsilon$

We show that the contribution from this term is negligible. We first note that by Mellin inversion, for a smooth $\phi : \mathbb{R} \to [0,1]$ compactly supported in $[-1,1]$ such that $\phi \equiv 1$ in a neighborhood of 0, we have uniformly in $\{s = \sigma + it : T/2 \le t \le 2T, 0 \le \sigma \le 2\}$, and for all $\epsilon > 0, A > 0$,

$$L(f,s) = \sum_{n\ge 1} \frac{\lambda_f(n)}{n^s}\phi\Big(\frac{n}{T^{2+\epsilon}}\Big) + O(T^{-A}).$$

By the definition of $\Delta(s)$, the integral becomes

$$\begin{aligned}
&\int_{\Re(s)=\frac{1}{2}+\epsilon} L\Big(f,\frac{1}{2}+s\Big) A_{1/2}\Big(\frac{1}{2}+s\Big)^2 R^*(s)R^*(-s)\frac{\Delta'}{\Delta}(s)\frac{1}{\Delta(s)-e^{2i\theta}}\frac{\mathrm{d}s}{\cos\big(\frac{iT-s}{H}\big)}\\
&= \sum_{n\ge 1}\frac{\lambda_f(n)}{n^{1/2}}\phi\Big(\frac{n}{T^{2+\epsilon}}\Big)\sum_{l_1,l_2<T^{1-3\xi}}\sum_{m_1,m_2<T^{\xi}}\frac{d_{1/2}(m_1)d_{1/2}(m_2)\lambda_f(m_1)\lambda_f(m_2)}{(m_1m_2)^{1/2}}\\
&\times r(l_1)r(l_2)\lambda_f(l_1)\lambda_f(l_2)\int_{\Re(s)=\frac{1}{2}+\epsilon}\Big(\frac{l_2}{nm_1m_2l_1}\Big)^s\frac{\Delta'}{\Delta}(s)\frac{1}{\Delta(s)-e^{2i\theta}}\frac{\mathrm{d}s}{\cos\big(\frac{iT-s}{H}\big)} + O(T^{-A}).
\end{aligned}$$

We write

$$(\Delta(s)-e^{2i\theta})^{-1} = -e^{-2i\theta} - e^{-4i\theta}\frac{\Delta(s)}{1-e^{-2i\theta}\Delta(s)},$$

and the contribution of the second term to the s-integral is

$$\int_{\Re(s)=\frac{1}{2}+\epsilon}\Big(\frac{l_2}{nm_1m_2l_1}\Big)^s\frac{\Delta'}{\Delta}(s)\frac{\Delta(s)}{1-e^{-2i\theta}\Delta(s)}\frac{\mathrm{d}s}{\cos\big(\frac{iT-s}{H}\big)},$$

which by (3.5) is negligible. It remains to bound the contribution of the first term above, which is

$$\frac{1}{2\pi i}\int_{\Re(s)=\frac{1}{2}+\epsilon}\Big(\frac{l_2}{nm_1m_2l_1}\Big)^s\frac{\Delta'}{\Delta}(s)\frac{\mathrm{d}s}{\cos\big(\frac{iT-s}{H}\big)} = -\frac{1}{4\pi}\delta_{l_2=nm_1m_2l_1}I_T + O(T^{-A}),$$

by (3.6). We therefore just need to estimate

$$\begin{aligned}
&\sum_{l_1,l_2<T^{1-3\xi}}\frac{\sqrt{l_1}r(l_1)r(l_2)\lambda_f(l_1)\lambda_f(l_2)}{\sqrt{l_2}}\sum_{\substack{m_1,m_2<T^{\xi}\\ nm_1m_2=\frac{l_2}{l_1}}}\lambda_f(n)d_{1/2}(m_1)d_{1/2}(m_2)\lambda_f(m_1)\lambda_f(m_2)I_T\\
&= \sum_{l_1,l_2<T^{1-3\xi}}\frac{\lambda_f(l_2)^2\sqrt{l_1}r(l_1)r(l_2)}{\sqrt{l_2}}\sum_{\substack{m_1,m_2<T^{\xi}\\ nm_1m_2=\frac{l_2}{l_1}}}d_{1/2}(m_1)d_{1/2}(m_2)I_T,
\end{aligned}$$

After making a change of variables $l_2 = l_2/l_1$, we thus estimate

$$\begin{aligned}
&\sum_{l_1<T^{1-3\xi}} r(l_1)^2\lambda_f(l_1)^2 \sum_{\substack{l_2\le \frac{T^{1-3\xi}}{l_1}\\ (l_1,l_2)=1}} \frac{\lambda_f(l_2)^2 r(l_2)}{\sqrt{l_2}} \sum_{\substack{m_1,m_2<T^{\xi}\\ nm_1m_2=l_2}} d_{1/2}(m_1)d_{1/2}(m_2)I_T\\
&\le \sum_{l_1<T^{1-3\xi}} r(l_1)^2\lambda_f(l_1)^2 \sum_{\substack{l_2\le \frac{T^{1-3\xi}}{l_1}\\ (l_1,l_2)=1}} \frac{\lambda_f(l_2)^2 r(l_2)}{\sqrt{l_2}} \sum_{m|l_2<T^{2\xi}} d(m)d_{1/2}(m)I_T\\
&\le \sum_{l_1<T^{1-3\xi}} r(l_1)^2\lambda_f(l_1)^2 \sum_{\substack{l_2\le \frac{T^{1-3\xi}}{l_1}\\ (l_1,l_2)=1}} \frac{\lambda_f(l_2)^2 r(l_2)}{\sqrt{l_2}} d(l_2)I_T\\
&\ll \prod_p (1+r(p)^2\lambda_f(p)^2)\left(\prod_p\left(1+\frac{r(p)}{\sqrt{p}}\lambda_f(p)^2\right)\right)^2 I_T.
\end{aligned}$$

Dividing by the normalizing weight, using Proposition 3.2, we see that the contribution from $\Re(s) = 1/2+\epsilon$ in (3.11) is bounded by

$$\ll (\log T)^{-1/4}\prod_p\left(1+\frac{r(p)}{\sqrt{p}}\lambda_f(p)^2\right),$$

which is smaller than (3.3) by a factor of $\log T$.

3.4.2 The main term

The integral along the line $\Re(s) = -1/2-\epsilon$ contributes to (3.11) as a main term. We make the change of variables $s \to -s$, and estimate

$$\begin{aligned}
&\int_{\Re(s)=\frac12+\epsilon} L\left(f,\frac12+s\right)A_{1/2}\left(\frac12-s\right)^2 R^*(s)R^*(-s)\frac{\Delta'(-s)}{\Delta(-s)-e^{2i\theta}}\frac{\mathrm{d}s}{\cos\left(\frac{iT+s}{H}\right)}\\
&=\int_{\Re(s)=\frac12+\epsilon} L\left(f,\frac12+s\right)A_{1/2}\left(\frac12-s\right)^2 R^*(s)R^*(-s)\frac{\Delta'}{\Delta}(s)\frac{1}{1-e^{2i\theta}\Delta(s)}\frac{\mathrm{d}s}{\cos\left(\frac{iT+s}{H}\right)}\\
&=\sum_{n\ge1}\frac{\lambda_f(n)}{n^{1/2}}\phi\left(\frac{n}{T^{2+\epsilon}}\right)\sum_{l_1,l_2<T^{1-3\xi}}\sum_{m_1,m_2<T^{\xi}}\frac{d_{1/2}(m_1)d_{1/2}(m_2)\lambda_f(m_1)\lambda_f(m_2)}{(m_1m_2)^{1/2}}\\
&\times r(l_1)r(l_2)\lambda_f(l_1)\lambda_f(l_2)\int_{\Re(s)=\frac12+\epsilon}\left(\frac{m_1m_2l_2}{nl_1}\right)^s\frac{\Delta'}{\Delta}(s)\frac{1}{1-e^{2i\theta}\Delta(s)}\frac{\mathrm{d}s}{\cos\left(\frac{iT+s}{H}\right)}.
\end{aligned}$$

We write

$$(1-e^{2i\theta}\Delta(s))^{-1} = 1+\frac{e^{2i\theta}\Delta(s)}{1-e^{2i\theta}\Delta(s)},$$

and by the same observation as before, only the contribution from the first term above is non-negligible. By (3.6), this term yields, up to negligible error term,

$$\begin{aligned}
&\sum_{l_1,l_2<T^{1-3\xi}} r(l_1)r(l_2)\lambda_f(l_1)\lambda_f(l_2)\sum_{\substack{m_1,m_2<T^{\xi}\\ m_1m_2l_2=nl_1}}\frac{\lambda_f(n)d_{1/2}(m_1)d_{1/2}(m_2)\lambda_f(m_1)\lambda_f(m_2)}{(nm_1m_2)^{1/2}}I_T\\
&=\sum_{g\le T^{1-3\xi}} S(g)I_T, \qquad (3.12)
\end{aligned}$$

where $S(g)$ is defined as

$$\sum_{\substack{l_1,l_2\le \frac{T^{1-3\xi}}{g}\\ (l_1,l_2)=(l_1l_2,g)=1}} r(l_1)r(l_2)\lambda_f(l_1)\lambda_f(l_2)\sum_{\substack{m_1,m_2<T^{\xi}\\ m_1m_2l_2=nl_1}}\frac{\lambda_f(n)d_{1/2}(m_1)d_{1/2}(m_2)\lambda_f(m_1)\lambda_f(m_2)}{\sqrt{nm_1m_2}}.$$

We let $l_{11}=(l_1,m_1)$, $l_{12}=l_1/l_{11}$ and $m_1:=m_1/l_{11}$, $m_2:=m_2/l_{12}$, so that

$$\begin{aligned} S(g)=&\sum_{\substack{l_1,l_2\le \frac{T^{1-3\xi}}{g}\\ (l_1,l_2)=(l_1l_2,g)=1}}\frac{r(l_1)r(l_2)\lambda_f(l_1)\lambda_f(l_2)}{\sqrt{l_1l_2}}\sum_{l_{11}l_{12}=l_1}\\ &\times\sum_{l_{11}m_1,l_{12}m_2<T^{\xi}}\frac{\lambda_f(m_1m_2l_2)d_{1/2}(l_{11}m_1)d_{1/2}(l_{12}m_2)\lambda_f(l_{11}m_1)\lambda_f(l_{12}m_2)}{m_1m_2}. \end{aligned}$$

We will estimate the outer sum by repeated use of Lemma 3.6. We first evaluate the m_1-sum and then the m_2-sum.

The m_1-sum

Writing l for l_{11} and m for m_1, we study the series

$$\sum_{m=1}^{\infty}\frac{\lambda_f(m_2l_2m)d_{1/2}(lm)\lambda_f(lm)}{m^s}=G_1(s;m_2,l_2,l)\prod_p\left(\sum_{k=0}^{\infty}\lambda_f(p^k)^2d_{1/2}(p^k)p^{-ks}\right),$$

where

$$G_1(s;m_2,l_2,l):=\prod_{p\mid m_2l_2l}\frac{\sum_{k=0}^{\infty}\lambda_f(p^{\nu_p(m_2l_2)+k})d_{1/2}(p^{\nu_p(l)+k})\lambda_f(p^{\nu_p(l)+k})p^{-ks}}{\sum_{k=0}^{\infty}\lambda_f(p^k)^2d_{1/2}(p^k)p^{-ks}}.$$

We wish to relate our Euler product to $L^{1/2}(\mathrm{sym}^2 f,s)$. We have

$$\begin{aligned} L^{1/2}(f\times f,s)=&\prod_p(1-p^{-s})^{-1}(1-\alpha_p^2p^{-s})^{-1/2}(1-\beta_p^2p^{-s})^{-1/2}\\ =&\prod_p\left(\sum_{k=0}^{\infty}p^{-ks}\right)\left(\sum_{k=0}^{\infty}\binom{k-\frac12}{k}\alpha_p^{2k}p^{-ks}\right)\\ &\times\left(\sum_{k=0}^{\infty}\binom{k-\frac12}{k}\beta_p^{2k}p^{-ks}\right)\\ =&\prod_p\left(\sum_{k=0}^{\infty}b(p^k)p^{-ks}\right), \end{aligned}$$

where b is a non-negative multiplicative function such that

$$b(p)=\frac{\lambda_f(p)^2}{2}=d_{1/2}(p)\lambda_f^2(p).$$

Writing $s=\sigma+i\tau$, we thus have

$$\prod_p\left(\sum_{k=0}^{\infty}\lambda_f(p^k)^2 d_{1/2}(p^k)p^{-ks}\right)=\zeta(s)^{1/2}L^{1/2}(\mathrm{sym}^2 f,s)B(s), \tag{3.13}$$

where $B(s)$ is a bounded holomorphic function in the region $\sigma>1-c/\log(2+|\tau|)$. We write

$$\sum_{m=1}^{\infty}\frac{\lambda_f(m_2l_2m)d_{1/2}(lm)\lambda_f(lm)}{m^s}=G_1\left(s;\frac{1}{2},m_2,l_2,l\right)\zeta(s)^{1/2},$$

where

$$G_1\left(s;\frac{1}{2},m_2,l_2,l\right):=L^{1/2}(\mathrm{sym}^2 f,s)B(s)G_1(s;m_2,l_2,l).$$

We define $M_1(m_2,l_2,l)$ to be

$$\prod_{p\mid m_2l_2l}\sup_{\sigma>1-\frac{c}{\log(2+|\tau|)}}\left|\frac{\sum_{k=0}^{\infty}|\lambda_f(p^{\nu_p(m_2l_2)+k})d_{1/2}(p^{\nu_p(l)+k})\lambda_f(p^{\nu_p(l)+k})|p^{-ks}}{\sum_{k=0}^{\infty}\lambda_f(p^k)^2 d_{1/2}(p^k)p^{-ks}}\right|,$$

and deduce by Lemma 3.6 the following lemma.

Lemma 3.8. *For $l\le T^{\xi-\epsilon}$, we have*

$$\sum_{m<\frac{T^{\xi}}{l}}\frac{\lambda_f(m_2l_2m)d_{1/2}(lm)\lambda_f(lm)}{m}=\frac{2+o(1)}{\Gamma\left(\frac{1}{2}\right)}\left(\log\frac{T^{\xi}}{l}\right)^{1/2}G_1\left(1;\frac{1}{2},m_2,l_2,l\right)$$
$$+O\left(\frac{M_1}{\log^{1/2}T}\right).$$

The m_2-sum

We now evaluate the contribution of the main term of Lemma 3.8 and study the associated Dirichlet series

$$\sum_{m_2}\frac{d_{1/2}(l_{12}m_2)\lambda_f(l_{12}m_2)G_1(1;m_2,l_2,l_{11})}{m_2^s}$$
$$=G_2(s;l_{11},l_{12},l_2)\prod_p\frac{\sum_{k,k'=0}^{\infty}\dfrac{d_{1/2}(p^k)\lambda_f(p^k)\lambda_f(p^{k+k'})d_{1/2}(p^{k'})\lambda_f(p^{k'})}{p^{ks+k'}}}{\sum_{k=0}^{\infty}\lambda_f(p^k)^2 d_{1/2}(p^k)p^{-k}},$$

where $G_2(s;l_{11},l_{12},l_2)$ is defined as

$$\prod_{p\mid l_2l_{11}l_{12}}G_{2,p}(s;l_{11},l_{12},l_2),$$

and $G_{2,p}$ is given by

$$\frac{\sum_{k,k'=0}^{\infty}d_{\frac{1}{2}}(p^{\nu_p(l_{12})+k})\lambda_f(p^{\nu_p(l_{12})+k})\lambda_f(p^{\nu_p(l_2)+k+k'})d_{\frac{1}{2}}(p^{\nu_p(l_{11})+k'})\lambda_f(p^{\nu_p(l_{11})+k'})p^{-ks-k'}}{\sum_{k,k'=0}^{\infty}d_{\frac{1}{2}}(p^k)\lambda_f(p^k)\lambda_f(p^{k+k'})d_{\frac{1}{2}}(p^{k'})\lambda_f(p^{k'})p^{-ks-k'}}.$$

We note that the prime factors of l_1,l_2 are, by the support of r, large enough so that the denominator above does not vanish.

Claim 3.1. *Let $s = \sigma + i\tau$; there exists a function, $C(s)$, bounded and holomorphic in the region $\sigma > 1 - c/\log(2 + |\tau|)$ such that*

$$\prod_p \frac{\sum_{k,k'=0}^{\infty} \frac{d_{\frac{1}{2}}(p^k)\lambda_f(p^k)\lambda_f(p^{k+k'})d_{\frac{1}{2}}(p^{k'})\lambda_f(p^{k'})}{p^{ks+k'}}}{\sum_{k=0}^{\infty} \lambda_f(p^k)^2 d_{\frac{1}{2}}(p^k)p^{-k}} = \zeta^{1/2}(s)L^{1/2}(sym^2 f, s)C(s).$$

Proof. We have

$$\prod_p \frac{\sum_{k,k'=0}^{\infty} d_{1/2}(p^k)\lambda_f(p^k)\lambda_f(p^{k+k'})d_{1/2}(p^{k'})\lambda_f(p^{k'})p^{-ks-k'}}{\sum_{k=0}^{\infty} \lambda_f(p^k)^2 d_{1/2}(p^k)p^{-k}}$$
$$= \prod_p \left(1 + d_{1/2}(p)\lambda_f(p)^2 p^{-s} + O(p^{-s-\frac{1}{2}})\right),$$

and the claim follows immediately. □

By the above claim, we have

$$\sum_{m_2} \frac{d_{1/2}(l_{12}m_2)\lambda_f(l_{12}m_2)G_1(1; m_2, l_2, l_{11})}{m_2^s} = \zeta^{1/2}(s)G_2\left(s; \frac{1}{2}, l_{11}, l_{12}, l_2\right),$$

where

$$G_2\left(s; \frac{1}{2}, l_{11}, l_{12}, l_2\right) = G_2(s; l_{11}, l_{12}, l_2)L^{1/2}(\mathrm{sym}^2 f, s)C(s).$$

We let

$$M_2(l_{11}, l_{12}, l_2) = \prod_{p|l_{11}l_{12}l_2} \sup_{\sigma > 1 - \frac{c}{\log(2+|\tau|)}} |M_{2,p}(s)|$$

where $M_{2,p}(s)$ is given by

$$\frac{\sum_{k,k'=0}^{\infty} \left| d_{\frac{1}{2}}(p^{\nu_p(l_{12})+k})\lambda_f(p^{\nu_p(l_{12})+k})\lambda_f(p^{\nu_p(l_2)+k+k'})d_{\frac{1}{2}}(p^{\nu_p(l_{11})+k'})\lambda_f(p^{\nu_p(l_{11})+k'})\right| p^{-ks-k'}}{\sum_{k,k'=0}^{\infty} |d_{\frac{1}{2}}(p^k)\lambda_f(p^k)\lambda_f(p^{k'})\lambda_f(p^{k+k'})d_{\frac{1}{2}}(p^{k'})|p^{-ks-k'}}.$$

We note that by the parity of $\nu_p(l_{12}) + k, \nu_p(l_2) + k + k'$, and $\nu_p(l_{11}) + k'$, we have

$$M_2(l_{11}, l_{12}, l_2) \le d_{1/2}(l_1)|\lambda_f(l_1 l_2)|M_2(l_1 l_2),$$

where $M_2(l)$ is a positive multiplicative function supported on squarefree integers and satisfying

$$M_2(p)^{\pm 1} \le (1 + C_2 p^{-\delta_2}), \tag{3.14}$$

for some absolute constant C_2 and some $\delta_2 > 0$. By Lemma 3.6, we obtain for $l_{12} < T^{\xi-\epsilon}$,

$$\sum_{m_2 < \frac{T^\xi}{l_{12}}} \frac{d_{1/2}(l_{12}m_2)\lambda_f(l_{12}m_2)G(1; m_2, l_2, l_{11})}{m_2} = \frac{2 + o(1)}{\Gamma\left(\frac{1}{2}\right)} G_2\left(1; \frac{1}{2}, l_{11}, l_{12}, l_2\right) \log^{1/2}\left(\frac{T^\xi}{l_{12}}\right)$$
$$+ O\left(\frac{d_{1/2}(l_1)|\lambda_f(l_1 l_2)|M_2(l_1 l_2)}{\log^{1/2} T}\right).$$

We may control the contribution from the error term in Lemma 3.8 similarly. Namely, with $s=\sigma+i\tau$ and $z=\delta+i\gamma$, we let

$$M_3(l_{11},l_{12},l_2):=\prod_{p|l_{11}l_{12}l_2}\sup_{\sigma>1-\frac{c}{\log(2+|\tau|)}}\frac{\sum_{k=0}^{\infty}\sup_{\delta>1-\frac{c}{\log(2+|\gamma|)}}|G_{3,p}(k;s,z,l_{11},l_{12},l_2)|}{\sum_{k=0}^{\infty}\sup_{\delta>1-\frac{c}{\log(2+|\gamma|)}}|G^{\dagger}_{3,p}(k;s,z,l_{11},l_{12},l_2)|},$$

where $G_{3,p}(k;s,z,l_{11},l_{12},l_2)$ is given by

$$\frac{\sum_{k'=0}^{\infty}|\lambda_f(p^{\nu_p(l_2)+k+k'})d_{1/2}(p^{\nu_p(l_{11})+k'})\lambda_f(p^{\nu_p(l_{11})+k'})d_{1/2}(p^{\nu_p(l_{12})+k})\lambda_f(p^{\nu_p(l_{12})+k})|p^{-ks-k'z}}{\sum_{k'=0}^{\infty}\lambda_f(p^{k'})^2d_{1/2}(p^{k'})p^{-k'z}},$$

and $G^{\dagger}_{3,p}(k;s,z,l_{11},l_{12},l_2)$ is given by

$$\frac{\sum_{k'=0}^{\infty}|\lambda_f(p^{k+k'})d_{1/2}(p^{k'})\lambda_f(p^{k'})d_{1/2}(p^{k})\lambda_f(p^{k})|p^{-ks-k'z}}{\sum_{k'=0}^{\infty}\lambda_f(p^{k'})^2d_{1/2}(p^{k'})p^{-k'z}}.$$

We note that we also have

$$M_3(l_{11},l_{12},l_2)\le d_{1/2}(l_1)|\lambda_f(l_1l_2)|M_3(l_1l_2),$$

where $M_3\ge M_2$ is a function satisfying (3.14) possibly with a different constant. Using these to bound the contribution from the error term, we conclude the following lemma.

Lemma 3.9. *For $l_{11},l_{12}<T^{\xi-\epsilon}$, we have*

$$\begin{aligned}\sum_{l_{11}m_1,l_{12}m_2<T^{\xi}}&\frac{\lambda_f(m_1m_2l_2)d_{1/2}(l_{11}m_1)d_{1/2}(l_{12}m_2)\lambda_f(l_{11}m_1)\lambda_f(l_{12}m_2)}{m_1m_2}\\&=\left(\frac{4+o(1)}{\Gamma\left(\frac{1}{2}\right)}\right)^2\left(\log\frac{T^{\xi}}{l_{11}}\right)^{1/2}\left(\log\frac{T^{\xi}}{l_{12}}\right)^{1/2}G_2\left(1;\frac{1}{2},l_{11},l_{12},l_2\right)\\&\qquad+O(d_{1/2}(l_1)|\lambda_f(l_1l_2)|M_3(l_1l_2)).\end{aligned}$$

The l_1 and l_2 sums

We let $Z=\exp((\log N)^{2/3})$ and note that by Lemma 3.5 the contribution from $l_1,l_2\ge Z$ to $S(g)$ is negligible. We first consider the contribution from the main term in Lemma 3.9 to (3.12), yielding

$$\begin{aligned}
&\log T \sum_{\substack{l_1,l_2<Z\\(l_1,l_2)=1}} \frac{r(l_1l_2)\lambda_f(l_1l_2)}{\sqrt{l_1l_2}} \sum_{l_{11}l_{12}=l_1} G_2\left(1;\frac{1}{2},l_{11},l_{12},l_2\right) \sum_{\substack{g\le \frac{T^{1-3\xi}}{\max(l_1,l_2)}\\(g,l_1l_2)=1}} r(g)^2\lambda_f(g)^2 I_T\\
&\sim \log T \sum_{\substack{l_1,l_2<Z\\(l_1,l_2)=1}} \frac{r(l_1l_2)\lambda_f(l_1l_2)}{\sqrt{l_1l_2}} \sum_{l_{11}l_{12}=l_1} G_2\left(1;\frac{1}{2},l_{11},l_{12},l_2\right) \prod_{p\nmid l_1l_2}(1+r(p)^2\lambda_f(p)^2)I_T\\
&\sim \log T \sum_{\substack{l_1,l_2<Z\\(l_1,l_2)=1}} \frac{r(l_1l_2)\lambda_f(l_1l_2)}{\sqrt{l_1l_2}} d(l_1)G_2(l_1,l_2)\prod_p(1+r(p)^2\lambda_f(p)^2)I_T,
\end{aligned}$$

where up to a constant $G_2(l_1,l_2)$ is given by

$$\prod_{p|l_1l_2} \frac{\sum_{k,k'=0}^{\infty} d_{\frac12}(p^{\nu_p(l_1)+k})\lambda_f(p^{\nu_p(l_1)+k})\lambda_f(p^{\nu_p(l_2)+k+k'})d_{\frac12}(p^{k'})\lambda_f(p^{k'})p^{-k-k'}}{(1+r(p)^2\lambda_f(p)^2)\sum_{k,k'=0}^{\infty} d_{\frac12}(p^{k})\lambda_f(p^{k})\lambda_f(p^{k+k'})d_{\frac12}(p^{k'})\lambda_f(p^{k'})p^{-k-k'}}.$$

Since for any l_1,l_2,k,k', one of $\nu_p(l_1)+k,\nu_p(l_2)+k+k'$ and k' must be odd, we may factorize $\lambda_f(l_1l_2)$ and obtain

$$\lambda_f(l_1l_2)G_2(l_1,l_2)\ge d_{1/2}(l_1)\lambda_f(l_1l_2)^2G(l_1l_2),$$

where G is some multiplicative function supported on squarefree integers and satisfying

$$G(p)^{\pm1}\le(1+C_3p^{-\delta_3}),$$

for some absolute constant C_3 and some $\delta_3>0$. From Lemma 3.5 we have the following sequence of estimates:

$$\sum_{l<Z}\frac{d(l)r(l)\lambda_f(l)^2}{\sqrt{l}}G(l)\sim\prod_p\left(1+\frac{2r(p)\lambda_f(p)^2}{\sqrt{p}}G(p)\right)\sim\prod_p\left(1+\frac{2r(p)\lambda_f(p)^2}{\sqrt{p}}\right). \tag{3.15}$$

The lower bound (3.3) follows after dividing by the Normalizing Weight.

We now consider the contribution from the error terms in Lemma 3.9. We estimate

$$\begin{aligned}
&\sum_{\substack{l_1,l_2<Z\\(l_1,l_2)=1}} \frac{r(l_1l_2)|\lambda_f(l_1l_2)|}{\sqrt{l_1l_2}} \sum_{l_{11}l_{12}=l_1} d_{1/2}(l_1)|\lambda_f(l_1l_2)|M_3(l_1l_2)\prod_{p\nmid l_1l_2}(1+r(p)^2\lambda_f(p)^2) \qquad (3.16)\\
&=\sum_{\substack{l_1,l_2<Z\\(l_1,l_2)=1}} \frac{r(l_1l_2)|\lambda_f(l_1l_2)|^2}{\sqrt{l_1l_2}}\tilde{M}_3(l_1,l_2)\prod_p(1+r(p)^2\lambda_f(p)^2),
\end{aligned}$$

where $\tilde{M}_3$ is a multiplicative function supported on squarefree integers defined on primes by

$$\tilde{M}_3(p)=\frac{M_3(p)}{(1+r(p)^2\lambda_f(p)^2)}.$$

We then may evaluate (3.16) as in (3.15), however the contribution from this term is smaller as we save a factor of $\log T$ in the error term of Lemma 3.9.

3.5 The signed moment

In this section we prove (3.4), by studying

$$\mathrm{NW}.\mathbf{E}_{w_{T,\theta}}\left[L\left(f,\frac{1}{2}+it\right)\right]$$
$$\sim\int_{\Gamma_\epsilon}L\left(f,\frac{1}{2}+s\right)A_{\frac{1}{2}}\left(\frac{1}{2}+s\right)A_{\frac{1}{2}}\left(\frac{1}{2}-s\right)R^*(s)R^*(-s)\frac{\Delta'(s)}{\Delta(s)-e^{2i\theta}}\frac{\mathrm{d}s}{\cos\left(\frac{iT-s}{H}\right)}.$$

The contribution of the integral along the line $\Re(s)=1/2+\epsilon$ is

$$-e^{-2i\theta}\sum_{n\geq 1}\frac{\lambda_f(n)}{n^{1/2}}\phi\left(\frac{n}{T^{2+\epsilon}}\right)\sum_{l_1,l_2<T^{1-3\xi}}\sum_{m_1,m_2<T^{\xi}}\frac{d_{1/2}(m_1)d_{1/2}(m_2)\lambda_f(m_1)\lambda_f(m_2)}{(m_1m_2)^{1/2}}$$
$$\times r(l_1)r(l_2)\lambda_f(l_1)\lambda_f(l_2)\int_{\Re(s)=\frac{1}{2}+\epsilon}\left(\frac{m_2l_2}{nm_1l_1}\right)^s\frac{\Delta'}{\Delta}(s)\frac{\Delta(s)}{1-e^{-2i\theta}\Delta(s)}\frac{\mathrm{d}s}{\cos\left(\frac{iT-s}{H}\right)},$$

which by (3.5) is negligible. We thus only care to estimate the integral along the line $\Re(s)=-1/2-\epsilon$. We make a change of variables $s\to -s$ and use the definition of $\Delta(s)$ to find

$$\int_{\Re(s)=\frac{1}{2}+\epsilon}L\left(f,\frac{1}{2}+s\right)A_{1/2}\left(\frac{1}{2}+s\right)A_{1/2}\left(\frac{1}{2}-s\right)R^*(s)R^*(-s)\frac{\Delta'(-s)}{1-e^{2i\theta}\Delta(s)}\frac{\mathrm{d}s}{\cos\left(\frac{iT+s}{H}\right)}$$
$$=\sum_{n\geq 1}\frac{\lambda_f(n)}{n^{1/2}}\phi\left(\frac{n}{T^{2+\epsilon}}\right)\sum_{m_1,m_2<T^{\xi}}\frac{d_{1/2}(m_1)d_{1/2}(m_2)\lambda_f(m_1)\lambda_f(m_2)}{(m_1m_2)^{1/2}}$$
$$\times\sum_{l_1,l_2<T^{1-2\xi}}r(l_1)r(l_2)\lambda_f(l_1)\lambda_f(l_2)\int_{\Re(s)=\frac{1}{2}+\epsilon}\left(\frac{m_2l_2}{nm_1l_1}\right)^s\frac{\Delta'(-s)}{1-e^{2i\theta}\Delta(s)}\frac{\mathrm{d}s}{\cos\left(\frac{iT+s}{H}\right)}.$$

We write

$$(1-e^{2i\theta}\Delta(s))^{-1}=1+e^{2i\theta}\Delta(s)+\frac{e^{4i\theta}\Delta^2(s)}{1-e^{2i\theta}\Delta(s)}$$

to obtain the following three terms

$$\mathrm{I}:=\sum_{l_1,l_2<T^{1-3\xi}}r(l_1)r(l_2)\lambda_f(l_1)\lambda_f(l_2)\sum_{m_1,m_2<T^{\xi}}\frac{d_{1/2}(m_1)d_{1/2}(m_2)\lambda_f(m_1)\lambda_f(m_2)}{(m_1m_2)^{1/2}}$$
$$\times\int_{\Re(s)=\frac{1}{2}+\epsilon}L\left(f,\frac{1}{2}+s\right)\left(\frac{m_2l_2}{m_1l_1}\right)^s\Delta'(-s)\frac{\mathrm{d}s}{\cos\left(\frac{iT+s}{H}\right)},$$

$$\mathrm{II}:=e^{2i\theta}\sum_{n\geq 1}\frac{\lambda_f(n)}{n^{1/2}}\phi\left(\frac{n}{T^{2+\epsilon}}\right)\sum_{m_1,m_2<T^{\xi}}\frac{d_{1/2}(m_1)d_{1/2}(m_2)\lambda_f(m_1)\lambda_f(m_2)}{(m_1m_2)^{1/2}}$$
$$\times\sum_{l_1,l_2<T^{1-2\xi}}r(l_1)r(l_2)\lambda_f(l_1)\lambda_f(l_2)\int_{\Re(s)=\frac{1}{2}+\epsilon}\left(\frac{m_2l_2}{nm_1l_1}\right)^s\frac{\Delta'}{\Delta}(s)\frac{\mathrm{d}s}{\cos\left(\frac{iT+s}{H}\right)},$$

and

$$\mathrm{III}:=e^{4i\theta}\sum_{n\geq 1}\frac{\lambda_f(n)}{n^{1/2}}\phi\left(\frac{n}{T^{2+\epsilon}}\right)\sum_{m_1,m_2<T^{\xi}}\frac{d_{1/2}(m_1)d_{1/2}(m_2)\lambda_f(m_1)\lambda_f(m_2)}{(m_1m_2)^{1/2}}$$
$$\times\sum_{l_1,l_2<T^{1-2\xi}}r(l_1)r(l_2)\lambda_f(l_1)\lambda_f(l_2)\int_{\Re(s)=\frac{1}{2}+\epsilon}\left(\frac{m_2l_2}{nm_1l_1}\right)^s\frac{\Delta'(s)}{1-e^{2i\theta}\Delta(s)}\frac{\mathrm{d}s}{\cos\left(\frac{iT+s}{H}\right)}.$$

We can see from (3.5) that III is negligible, and we shall therefore focus solely on I and II.

3.5.1 Bounding II

Using (3.6), II is bounded up to negligible error term by

$$\sum_{g<T^{1-3\xi}} r(g)^2\lambda_f(g)^2 \sum_{\substack{l_1,l_2<T^{1-3\xi}/g\\(l_1,l_2)=(l_1l_2,g)=1}} \lambda_f(l_1l_2)r(l_1l_2)$$
$$\times \sum_{\substack{m_1,m_2<T^{\xi}\\nm_1l_1=m_2l_2}} \frac{\lambda_f(n)d_{1/2}(m_1)d_{1/2}(m_2)\lambda_f(m_1)\lambda_f(m_2)}{(nm_1m_2)^{1/2}} I_T.$$

We let $l_{21}=(l_2,m_1), l_{22}=(l_2,n)$ and replace $m_1 := \frac{m_1}{l_{21}}, n := \frac{n}{l_{22}}$ to reduce the problem to estimating

$$\sum_{g<T^{1-3\xi}} r(g)^2\lambda_f(g)^2 \sum_{\substack{l_1,l_2<T^{1-3\xi}/g\\(l_1,l_2)=(l_1l_2,g)=1}} \frac{\lambda_f(l_1l_2)r(l_1l_2)}{\sqrt{l_1l_2}}\times$$
$$\sum_{l_{21}l_{22}=l_2}\ \sum_{nm_1l_1,l_{21}m_1<T^{\xi}} \frac{\lambda_f(l_{22}n)d_{1/2}(l_{21}m_1)d_{1/2}(m_1nl_1)\lambda_f(l_{21}m_1)\lambda_f(m_1nl_1)}{nm_1}.$$

We note that the innermost sum is bounded by

$$\sum_{nl_1,l_{21}m_1<T^{\xi}} \frac{|\lambda_f(l_{22}n)d_{1/2}(l_{21}m_1)d_{1/2}(l_1m_1n)\lambda_f(l_{21}m_1)\lambda_f(m_1nl_1)|}{nm_1}. \tag{3.17}$$

Bounding (3.17)

We study

$$\sum_n \frac{|\lambda_f(l_{22}n)d_{1/2}(l_1m_1n)\lambda_f(m_1nl_1)|}{n^s} = \prod_p \left(\sum_{k=0}^{\infty} \lambda_f(p^k)^2 d_{1/2}(p^k)p^{-ks}\right) G_3(s;l_1,m_1,l_{22}),$$

where

$$G_3(s;l_1,m_1,l_{22}) := \prod_{p|l_1m_1l_{22}} \frac{\sum\limits_{k=0}^{\infty} |\lambda_f(p^{\nu_p(l_{22})+k})d_{\frac12}(p^{\nu_p(l_1m_1)+k})\lambda_f(p^{\nu_p(l_1m_1)+k})|p^{-ks}}{\sum\limits_{k=0}^{\infty} \lambda_f(p^k)^2 d_{\frac12}(p^k)p^{-ks}}.$$

By (3.13), we conclude that

$$\sum_n \frac{|\lambda_f(l_{22}n)d_{1/2}(l_1m_1n)\lambda_f(m_1nl_1)|}{n^s} = \zeta^{1/2}(s)G_3\left(s;\frac{1}{2},l_1,m_1,l_{22}\right),$$

where

$$G_3\left(s;\frac{1}{2},l_1,m_1,l_{22}\right) = G_3(s;l_1,m_1,l_{22})L^{1/2}(\mathrm{sym}^2 f,s)B(s),$$

where $B(s)$ is given in (3.13). Letting $M_3(l_1,m_1,l_{22})$ denote

$$\prod_{p|l_1m_1l_{22}} \sup_{\sigma>1-c/\log(2+|\tau|)} \left| \frac{\sum_{k=0}^{\infty} |\lambda_f(p^{\nu_p(l_{22})+k})d_{\frac12}(p^{\nu_p(l_1m_1)+k})\lambda_f(p^{\nu_p(l_1m_1)+k})|p^{-ks}}{\sum_{k=0}^{\infty} \lambda_f(p^k)^2 d_{\frac12}(p^k)p^{-ks}} \right|,$$

we use Lemma 3.6 to conclude that for $l_1 < T^{\xi-\epsilon}$, we have

$$\sum_{l_1 n < T^{\xi}} \frac{|\lambda_f(l_{22}n) d_{1/2}(l_1 m_1 n)\lambda_f(m_1 n l_1)|}{n} \ll (\log T)^{1/2} G_3\left(1; \frac{1}{2}, l_1, m_1, l_{22}\right) + O\left(\frac{M_3}{(\log T)^{1/2}}\right). \tag{3.18}$$

We estimate the contribution from the first term of (3.18); the contribution of the second term is analogous. We thus study

$$\sum_{m_1} \frac{d_{1/2}(l_{21}m_1)|\lambda_f(l_{21}m_1)| G_3(1; l_1, m_1, l_{22})}{m_1^s}$$
$$= \prod_p \left(\frac{\sum_{k,k'=0}^{\infty} |d_{1/2}(p^k)\lambda_f(p^k)\lambda_f(p^{k'}) d_{1/2}(p^{k+k'})\lambda_f(p^{k+k'})| p^{-k'-ks}}{\sum_{k=0}^{\infty} \lambda_f(p^k)^2 d_{1/2}(p^k) p^{-k}} \right) G_4(s; l_1, l_2),$$

where

$$G_4(s; l_1, l_2) = \prod_{p | l_1 l_2} G_{4,p}(s; l_1, l_2),$$

and $G_{4,p}(s; l_1, l_2)$ is given by

$$\frac{\sum\limits_{k,k'=0}^{\infty} |d_{\frac{1}{2}}(p^{\nu_p(l_{21})+k})\lambda_f(p^{\nu_p(l_{21})+k})\lambda_f(p^{\nu_p(l_{22})+k'}) d_{\frac{1}{2}}(p^{\nu_p(l_1)+k+k'})\lambda_f(p^{\nu_p(l_1)+k+k'})| p^{-k'-ks}}{\sum\limits_{k,k'=0}^{\infty} |d_{\frac{1}{2}}(p^k)\lambda_f(p^k)\lambda_f(p^{k'}) d_{\frac{1}{2}}(p^{k+k'})\lambda_f(p^{k+k'})| p^{-k'-ks}}.$$

Claim 3.2. *Let $s = \sigma + i\tau$; there exists a function, D, bounded and holomorphic in the region $\sigma > 1 - c/\log|\tau|$ such that*

$$\prod_p \frac{\sum\limits_{k,k'=0}^{\infty} \dfrac{|d_{\frac{1}{2}}(p^k)\lambda_f(p^k)\lambda_f(p^{k'}) d_{\frac{1}{2}}(p^{k+k'})\lambda_f(p^{k+k'})|}{p^{k'+ks}}}{\sum\limits_{k=0}^{\infty} \lambda_f(p^k)^2 d_{\frac{1}{2}}(p^k) p^{-k}} = \zeta^{1/4}(s) L^{1/4}(sym^2 f, s) D(s).$$

Proof. We have

$$\prod_p \frac{\sum_{k,k'=0}^{\infty} d_{1/2}(p^k)\lambda_f(p^k)\lambda_f(p^{k+k'}) d_{1/2}(p^{k+k'})\lambda_f(p^{k'}) p^{-ks-k'}}{\sum_{k=0}^{\infty} \lambda_f(p^k)^2 d_{1/2}(p^k) p^{-k}}$$
$$= \prod_p \left(1 + d_{1/2}(p)^2 \lambda_f(p)^2 p^{-s} + O(p^{-s-\frac{1}{2}})\right),$$

and the claim follows immediately. □

We let $M_4(l_1 l_2)$ be a positive multiplicative function, supported on squarefree integers, such that

$$M_4(p)^{\pm 1} \leq (1 + C_4 p^{-\delta_4}),$$

for some constant $C_4 > 0$ and some $\delta_4 > 0$, chosen so that

$$\prod_{p|l_1 l_2} \sup_{\Re(s)>1-c/\log|\tau|} G_{4,p}(s; l_1, l_2) \leq |\lambda_f(l_1 l_2)| d_{1/2}(l_{21} l_1) M_4(l_1 l_2),$$

and use Lemma 3.6 to conclude the following lemma.

Lemma 3.10. *For $l_1, l_2 < T^{\xi-\epsilon}$, we have*

$$\begin{aligned}\sum_{nl_1, l_{21} m_1 < T^{\xi}} & \frac{|\lambda_f(l_{22} n) d_{1/2}(l_{21} m_1) d_{1/2}(l_1 m_1 n) \lambda_f(l_{21} m_1) \lambda_f(m_1 n l_1)|}{n m_1} \\ & \ll (\log T)^{3/4} G_4(1; l_1, l_2) + O\left(M_4 (\log T)^{-1/4}\right)\end{aligned}$$

Estimating the outer sums

We estimate the contribution from the first term of Lemma 3.10 to II, the second term being treated similarly. We notice that letting $Z = \exp(\log^{2/3} N)$ the contribution from $\max(l_1, l_2) > Z$ is negligible, and thus only care to estimate

$$\begin{aligned}&\sum_{g < T^{1-3\xi}} r(g)^2 \lambda_f(g)^2 \sum_{\substack{l_1, l_2 < T^{1-3\xi}/g, Z \\ (l_1, l_2) = (l_1 l_2, g) = 1}} \frac{\lambda_f(l_1 l_2) r(l_1 l_2)}{\sqrt{l_1 l_2}} \sum_{l_{21} l_{22} = l_2} G_4(1; l_1, l_2) \\ \ll & \sum_{g < T^{1-3\xi}} r(g)^2 \lambda_f(g)^2 \sum_{\substack{l_1, l_2 < T^{1-3\xi}/g, Z \\ (l_1, l_2) = (l_1 l_2, g) = 1}} \frac{\lambda_f(l_1 l_2) r(l_1 l_2)}{\sqrt{l_1 l_2}} d_{1/2}(l_1) d_{3/2}(l_2) \tilde{G}_4(l_1 l_2),\end{aligned}$$

where $\tilde{G}_4$ is a multiplicative function supported on squarefree integers such that

$$\tilde{G}_4(l) = \prod_{p|l} (|\lambda_f(p^{\nu_p(l)})| + O(p^{-\delta_5}),$$

for some $\delta_5 > 0$. By Lemma 3.3 and 3.5 we obtain that the contribution of II is bounded by

$$\begin{aligned}&(\log T)^{3/4} \prod_p (1 + r(p)^2 \lambda_f(p)^2) \prod_p \left(1 + \frac{\lambda_f(p)^2 r(p)}{2\sqrt{p}}\right) \prod_p \left(1 + \frac{3\lambda_f(p)^2 r(p)}{2\sqrt{p}}\right) I_T \\ \ll & (\log T)^{3/4} \prod_p (1 + r(p)^2 \lambda_f(p)^2) \prod_p \left(1 + \frac{2\lambda_f(p)^2 r(p)}{\sqrt{p}}\right) I_T,\end{aligned}$$

so that dividing by NW we obtain an acceptable upper bound towards (3.4).

3.5.2 Estimating I

We have reduced the proof of the upper bound of Proposition 3.1 to bounding I. We will do so by showing it is bounded by II. By Proposition 1.2, we have that

$$L\left(f, \frac{1}{2} + i\nu\right) = \sum_n \frac{\lambda_f(n)}{n^{1/2+i\nu}} V_\nu(n) + \Delta(i\nu) \sum_n \frac{\lambda_f(n)}{n^{1/2-i\nu}} V_{-\nu}(n),$$

where
$$V_\nu(y) := \frac{1}{2\pi i}\int_{(3)} y^{-u} e^{u^2} \frac{L_\infty\left(\frac{1}{2}+i\nu+u\right)}{L_\infty\left(\frac{1}{2}+i\nu\right)}\frac{\mathrm{d}u}{u}.$$

We may thus write
$$\int_{\mathbb{R}} L\left(f,\frac{1}{2}+i\nu\right)\left(\frac{m_2 l_2}{m_1 l_1}\right)^{i\nu}\Delta'(-i\nu)\frac{\mathrm{d}\nu}{\cosh\left(\frac{T+\nu}{H}\right)} = S_1+S_2,$$

where
$$S_1 := \sum_n \frac{\lambda_f(n)}{n^{1/2}}\int_{\mathbb{R}}\left(\frac{m_2 l_2}{m_1 l_1 n}\right)^{i\nu} V_\nu(n)\Delta'(-i\nu)\frac{\mathrm{d}\nu}{\cosh\left(\frac{T+\nu}{H}\right)},$$

and
$$S_2 := \sum_n \frac{\lambda_f(n)}{n^{1/2}}\int_{\mathbb{R}}\left(\frac{m_2 l_2 n}{m_1 l_1}\right)^{i\nu} V_{-\nu}(n)\Delta(i\nu)\Delta'(-i\nu)\frac{\mathrm{d}\nu}{\cosh\left(\frac{T+\nu}{H}\right)}.$$

We note that by the support of V_ν (see Proposition 1.3), we only need to consider the contribution from $|\nu| \asymp T$ and $n \ll T^{1+\epsilon}$, for both S_1 and S_2 . We also note that in the definition of V_ν only the contribution from $u \ll T^\epsilon$ is non-negligible. We thus estimate
$$S_1 = \sum_{n \ll T^{1+\epsilon}} \frac{\lambda_f(n)}{2\pi n^{1/2}}\int_{u \ll T^\epsilon} \frac{e^{(3+iu)^2}}{n^{3+iu}(3+iu)} K_T(n;m_1,m_2,l_1,l_2,u)\mathrm{d}u + O(T^{-A}),$$

where $K_T(n;m_1,m_2,l_1,l_2,u)$ is defined to be
$$\int_{\mathbb{R}}\left(\frac{m_2 l_2}{m_1 l_1 n}\right)^{i\nu}\frac{L_\infty\left(\frac{7}{2}+i(\nu+u)\right)}{L_\infty\left(\frac{1}{2}+i\nu\right)}\Delta'(-i\nu)W(\nu)\frac{\mathrm{d}\nu}{\cosh\left(\frac{T+\nu}{H}\right)},$$

and W is a smooth function supported on $[-2T,-T/2]$ such that $W^{(j)}(x) \ll x^{-j}$ for all $j \geq 0$.

We recall that by (3.7),
$$\Delta'(-i\nu) = -\frac{\Delta(-i\nu)}{2}\log\left(\frac{\frac{1}{16}+\frac{1}{4}((r+\nu)^2+(\nu-r)^2)+(\nu^2-r^2)^2}{16\pi^4}\right) + O(|\nu|^{-1+\epsilon}),$$

and
$$\frac{L_\infty\left(\frac{7}{2}+i(\nu+u)\right)}{L_\infty\left(\frac{1}{2}+i\nu\right)}\Delta(-i\nu) = c_1\pi^{-2i\nu}\left|\frac{r+u+\nu}{2e}\right|^{i\frac{r+u+\nu}{2}+\frac{5}{4}}\left|\frac{\nu+u-r}{2e}\right|^{i\frac{\nu+u-r}{2}+\frac{5}{4}}$$
$$\times\left|\frac{r+\nu}{2e}\right|^{i\frac{(r+\nu)}{2}+\frac{1}{4}}\left|\frac{\nu-r}{2e}\right|^{i\frac{(\nu-r)}{2}+\frac{1}{4}} e^{\frac{\pi u}{2}}(1+O(|\nu|^{-1})),$$

for some absolute constant c_1. We then write
$$K_T(n;m_1,m_2,l_1,l_2,u) = \int_{\mathbb{R}} g_T(\nu)e(f_T(\nu))\mathrm{d}\nu,$$

where
$$g_T(\nu) = c_2\frac{\log\left(\frac{\frac{1}{16}+\frac{1}{4}((r+\nu)^2+(\nu-r)^2)+(\nu^2-r^2)^2}{16\pi^4}\right)+O(|\nu|^{-1+\epsilon})}{\cosh\left(\frac{T+\nu}{H}\right)}W(\nu)$$
$$\times\left|\frac{r+u+\nu}{2}\right|^{5/4}\left|\frac{\nu+u-r}{2}\right|^{5/4}\left|\frac{r+\nu}{2}\right|^{1/4}\left|\frac{\nu-r}{2}\right|^{1/4}e^{\pi u},$$

fore some absolute constant c_2, and

$$2\pi f_T(v) = v\log\left(\frac{m_2 l_2}{nm_1 l_1 \pi^2}\right) + \frac{r+v}{2}\log\left|\frac{r+v}{2e}\right| + \frac{v-r}{2}\log\left|\frac{v-r}{2e}\right| + \frac{r+u+v}{2}\log\left|\frac{r+u+v}{2e}\right| + \frac{v+u-r}{2}\log\left|\frac{v+u-r}{2e}\right|.$$

We now wish to run a stationary phase analysis on K_T, and we therefore compute

$$2\pi f_T'(v) = \log\left(\frac{m_2 l_2}{nm_1 l_1 \pi^2}\right) + \frac{1}{2}\log\left|\frac{r+v}{2e}\right| + \frac{1}{2}\log\left|\frac{v-r}{2e}\right| + 2 + \frac{1}{2}\log\left|\frac{r+u+v}{2e}\right| + \frac{1}{2}\log\left|\frac{v+u-r}{2e}\right|.$$

We note that in the support of the integral, $|f_T'(v)| \geq 1$ as otherwise we would require to have $m_2 l_2 T^2 \asymp nm_1 l_1$, however the right hand side is always bounded by $T^{2-\epsilon}$. By repeated integration by parts, we find

$$K_T(n; m_1, m_2, l_1, l_2, u) \ll e^{\pi u} T^{-A},$$

so that the contribution from S_1 is negligible. Similarly, we now study

$$S_2 = \sum_{n \ll T^{1+\epsilon}} \frac{\lambda_f(n)}{2\pi n^{1/2}} \int_{u \ll T^\epsilon} \frac{e^{(3+iu)^2}}{(\pi n)^{3+iu}(3+iu)} \tilde{K}_T(n; m_1, m_2, l_1, l_2, u)\mathrm{d}u + O(T^{-A}),$$

where $\tilde{K}_T(n; m_1, m_2, l_1, l_2, u)$ is defined to be

$$\int_{\mathbb{R}} \left(\frac{m_2 l_2 n}{m_1 l_1}\right)^{iv} \frac{L_\infty\left(\frac{7}{2} + i(u-v)\right)}{L_\infty\left(\frac{1}{2} - iv\right)} \Delta(iv)\Delta'(-iv) \frac{W(v)\mathrm{d}v}{\cosh\left(\frac{T+v}{H}\right)}.$$

We write

$$\tilde{K}_T(n; m_1, m_2, l_1, l_2, u) = \int_{\mathbb{R}} \tilde{g}_T(v) e(\tilde{f}_T(v))\mathrm{d}v,$$

where up to a constant, $\tilde{g}_T(v)$ is given by

$$e^{-\frac{\pi u}{2}} \left|\frac{u+r-v}{2}\right|^{5/4} \left|\frac{u-v-r}{2}\right|^{5/4} \left|\frac{r-v}{2}\right|^{1/4} \left|\frac{r+v}{2}\right|^{1/4} \times \log\left(\frac{\frac{1}{16} + \frac{1}{4}((r+v)^2 + (v-r)^2) + (v^2-r^2)^2}{16\pi^4}\right) \frac{W(v)}{\cosh\left(\frac{T+v}{H}\right)}(1 + O(|v|^{-1+\epsilon})),$$

and

$$2\pi \tilde{f}_T(v) = v\log\left(\frac{m_2 l_2 n}{m_1 l_1}\right) + \frac{u+r-v}{2}\log\left|\frac{u+r-v}{2e}\right| + \frac{u-v-r}{2}\log\left|\frac{u-v-r}{2e}\right| + \frac{v-r}{2}\log\left|\frac{r-v}{2e}\right| + \frac{v+r}{2}\log\left|\frac{v+r}{2e}\right|.$$

We compute

$$2\pi \tilde{f}_T'(v) = \log\left(\frac{m_2 l_2 n}{m_1 l_1}\right) - \frac{1}{2}\left(\log\left(1 + \frac{u}{r-v}\right) + \log\left(1 - \frac{u}{v+r}\right)\right),$$

We thus see that if $m_2 l_2 n \neq m_1 l_1$, then $\tilde{f}'_T(v) \gg T^{\xi-1}$. Computing higher derivatives, one finds that $\tilde{f}_T^{(j)} \ll T^{\epsilon-j}$, so that by Lemma 1.5 one concludes that

$$\tilde{K}_T(n; m_1, m_2, l_1, l_2, u) \ll O(T^{-A}).$$

Using the bound $V_{-\nu}(n) \ll 1$, we consider thus have

$$S_2 \ll \delta_{m_2 l_2 n = m_1 l_1} \frac{|\lambda_f(n)|}{n^{1/2}} I_T + O(T^{-A}).$$

The contribution from the main term thereof to I is therefore bounded by

$$\sum_{l_1, l_2 < T^{1-3\xi}} r(l_1) r(l_2) |\lambda_f(l_1) \lambda_f(l_2)| \sum_{\substack{m_1, m_2 < T^{\xi} \\ m_2 l_2 n = m_1 l_1}} \frac{d_{\frac{1}{2}}(m_1) d_{\frac{1}{2}}(m_2) |\lambda_f(m_1) \lambda_f(m_2) \lambda_f(n)|}{(n m_1 m_2)^{1/2}} I_T,$$

which is the same sum as that appearing in II. This concludes the proof of the upper bound of Proposition 3.1.

4 Analogies with Frobenius Trace Functions and further directions

We give the definition of Frobenius trace functions as studied in [FKM15a] and look at certain analytic properties for which we have analogs in the context of analytic trace functions as defined in Chapter 2.

The class of analytic trace functions we consider unfortunately don't have a clear geometric nature as that of Frobenius trace functions. However we give some heuristics as to how we may define similar concepts.

4.1 Frobenius Trace Functions

Let p be a prime, and let $l \neq p$ be an auxiliary prime, and fix

$$\iota : \overline{\mathbb{Q}_l} \to \mathbb{C},$$

an isomorphism of fields. Let $K := \mathbb{F}_p(X)$ be the function field of $\mathbb{P}^1(\mathbb{F}_p)$, and let K^{s} denote a separable closure of K in which an algebraic closure $\overline{\mathbb{F}_p}$ is contained. We have an exact sequence

$$1 \to G^{\mathrm{g}} := \mathrm{Gal}(K^{\mathrm{s}}/\overline{\mathbb{F}_p}(X)) \to G^{\mathrm{a}} := \mathrm{Gal}(K^{\mathrm{s}}/K) \to \mathrm{Gal}(\overline{\mathbb{F}_p}/\mathbb{F}_p) \to 1.$$

Definition 4.1. *Let $U \subset \mathbb{P}^1_{\mathbb{F}_p}$ be an open set. An l-*adic sheaf lisse *on U is a continuous representation*

$$\rho : G^{a} \to \mathrm{GL}(V),$$

for some finite-dimensional $\overline{\mathbb{Q}_l}-$vector space, V, such that for all closed point $x \in U$, the inertia group I_x acts trivially on V.

For any closed point $x \in \mathbb{A}^1_{\mathbb{F}_p}$, let k_x denote its residue field. We have an isomorphism

$$D_x/I_x \cong \mathrm{Gal}(\overline{\mathbb{F}_p}/k_x) = \langle \mathrm{Fr}_x \rangle,$$

where D_x is the decomposition group and Fr_x denotes the *geometric Frobenius element,* i.e. the inverse of the usual Frobenius element $x \mapsto x^{|k_x|}$.

Definition 4.2. *Let ρ be an l-adic sheaf. The trace function attached to ρ, $t_\rho : \mathbb{A}^1(\mathbb{F}_p) = \mathbb{F}_p \to \mathbb{C}$, is defined to be*

$$t_\rho(x) := \iota\left(Tr(\rho(Fr_x|V^{I_x}))\right),$$

where V^{I_x} denotes the subspace of V that is invariant under I_x.

We define the complexity of the trace function t_ρ by the conductor, a geometric invariant given by

$$c(t_\rho) := \dim(V) + p + 1 - |U| + \mathrm{Swan}(\rho),$$

where $\mathrm{Swan}(\rho)$ is a non-negative integer called the Swan conductor of ρ.

In order to finalise our definition of Frobenius trace functions, we require the notion l-adic sheafs of weight 0.

Definition 4.3. *Let ρ be an l-adic sheaf. We say that ρ is of weight 0 if for all closed point $x \in U$, we have that the eigenvalues of $\rho(Fr_x)$ are of absolute value 1*

We are now in a position to define formally Frobenius trace functions.

Definition 4.4. *A function $t : \mathbb{F}_p \to \mathbb{C}$ is said to be a Frobenius trace function if there exists ρ, some l-adic sheaf of weight 0, such that $t = t_\rho$. We define the conductor of t ,$c(t)$, to be*

$$c(t) := \min\{c(\rho) | t = t_\rho\}.$$

We now give some examples of trace functions.

Example 4.1. *Let $\psi : \mathbb{F}_p \to \mathbb{C}^*$ be a non-trivial additive character. Then ψ is a Frobenius trace function. In particular, there exists an l-adic sheaf $\mathscr{L}_\psi$, satisfying $t_{\mathscr{L}_\psi} = \psi$, called the* Artin-Schreier sheaf *attached to ψ.*

Example 4.2. *Let $\chi : \mathbb{F}_p^* \to \mathbb{C}$ be a non-trivial multiplicative character. Then χ is a Frobenius trace function. In particular, there exists an l-adic sheaf $\mathscr{L}_\chi$, satisfying $t_{\mathscr{L}_\chi} = \chi$, called the* Kummer sheaf *attached to χ.*

Example 4.3. *We define the Kloosterman sum by*

$$Kl_2(x) := p^{-1/2} \sum_{m \in \mathbb{F}_p^*} e\left(\frac{xm + m^{-1}}{p}\right).$$

The Kloosterman sum is a Frobenius trace function.

Example 4.4. *For any $n \geq 3$ we define the Hyper-Kloosterman sum by*

$$Kl_n(x) := p^{\frac{1-n}{2}} \sum_{m_1 \cdots m_n \equiv x} e\left(\frac{m_1 + \cdots + m_n}{p}\right).$$

All Hyper-Kloosterman sums are Frobenius trace functions.

Remark 4.1. *The complex conjugate of any Frobenius trace function is again a Frobenius trace function (Via the dual representation). The sum of any two Frobenius trace functions is also a Frobenius trace function (via the direct sum representation). The product of any two Frobenius trace functions is close to being a Frobenius trace function (via the tensor product representation). We won't make the last statement precise.*

We conclude this section by giving two deep theorems of Deligne. First we give the quasi-orthogonality relation by means of the Riemann hypothesis over finite fields [Del74]. We then define the Fourier transform and give a theorem explaining that generically the Fourier transform of a Frobenius trace function is also a Frobenius trace function [Lau87, Kat88, Del80].

We first require the notion of geometric irreducibility and isomorphism.

Definition 4.5. *Let $\rho : G^{a} \to \mathrm{GL}(V)$ be an l-adic sheaf. We say that ρ is geometrically irreducible if its restriction to G^{g} is irreducible.*

Let t be a trace function. We say that t is geometrically irreducible if there exist a geometrically irreducible l-adic sheaf, ρ, such that $t = t_\rho$.

We note here the following fact, for which one may imagine an analog when considering irreducibility in the context of analytic trace functions.

Fact 4.1. *Let t be a Frobenius trace function that decomposes as a sum of n geometrically irreducible components. We have*

$$\sum_{\mathbb{F}_p} |K(x)|^2 = (n + o(1))p.$$

Definition 4.6. *Let $\rho_1 : G^{a} \to \mathrm{GL}(V_1)$ and $\rho_2 : G^{a} \to \mathrm{GL}(V_2)$ be two l-adic sheaves. We say that ρ_1 and ρ_2 are geometrically isomorphic if their restrictions to G^{g} are isomorphic.*

Let t_1 and t_2 be two Frobenius trace functions. We say that t_1 and t_2 are geometrically isomorphic if there exist two geometrically isomorphic l-adic sheaves, ρ_1 and ρ_2, such that $t_1 = t_{\rho_1}$ and $t_2 = t_{\rho_2}$.

We may now state the Riemann hypothesis.

Theorem 4.1. *Let K_1 and K_2 be two geometrically irreducible Frobenius trace functions, of conductors c_1 and c_2 respectively.*

- *If K_1 and K_2 are not geometrically isomorphic, then*

$$\left| \sum_{x \in \mathbb{F}_p} K_1(x)\overline{K_2(x)} \right| \le 4c_1^2 c_2^2 p^{1/2}.$$

- *If K_1 and K_2 are geometrically isomorphic, then there exists $\alpha \in \mathbb{C}^1$ such that $K_2 = \alpha K_1$, and we have*

$$\left| \sum_{x \in \mathbb{F}_p} K_1(x)\overline{K_2(x)} - \overline{\alpha} p \right| \le 4c_1^2 c_2^2 p^{1/2}.$$

Note that Fact 4.1 follows from the second statement of the Theorem.

We conclude this section by describing the Fourier transform, that is crucial in the proof of Theorem 0.1.

Definition 4.7. *Let*

$$\psi : \mathbb{F}_p \to \mathbb{C}^*$$

be a fixed non-trivial additive character. Let $C(\mathbb{F}_p)$ *be the space of functions* $\mathbb{F}_p \to \mathbb{C}$*. The Fourier transform with respect to* ψ *is the linear operator*

$$FT_\psi : C(\mathbb{F}_p) \to C(\mathbb{F}_p)$$

defined by

$$FT_\psi(\varphi)(y) = -p^{-1/2} \sum_{x \in \mathbb{F}_p} \varphi(x)\psi(xy)$$

for any $\varphi \in C(\mathbb{F}_p)$ *and any* $y \in \mathbb{F}_p$*.*

We have the following deep theorem of Deligne.

Theorem 4.2. *Let* ρ *be a geometrically irreducible* l*-adic sheaf that is not geometrically isomorphic to an Artin-Schreier sheaf. Then there exists a geometrically irreducible* l*-adic sheaf, the* Fourier sheaf $FT_\psi(\rho)$*, such that*

$$t_{FT_\psi(\rho)} = FT_\psi(t_\rho).$$

Moreover, $c(t_{FT_\psi(\rho)}) \leq 10c(t_\rho)^2$*.*

This theorem therefore tells us that the Fourier transform of a Frobenius trace function is also a Frobenius trace function (generically). We note that the bound on the conductor of the Fourier sheaf is proved by means of a geometric analog of the stationary phase method.

4.2 Analytic trace functions

In this section, we compare certain properties of analytic trace functions, as defined in Chapter 2, to those of Frobenius trace functions. We recall their definition here.

Definition 4.8. *A family of smooth functions* $(K_t)_{t\in\mathbb{R}}, K_t : \mathbb{R}_{>0} \to \mathbb{C}$ *is called a family of analytic trace functions if there exist real numbers* $a < b, b > 0$ *and a family of analytic functions* $(M_t(s))_{t\in\mathbb{R}}$ *in the strip* $a < \Re(s) < b$*, such that the following conditions hold uniformly for* $a < \Re(s) < b$*.*

1. *The following integral converges for any* $a < \sigma < b$*,*

$$\frac{1}{2\pi i}\int_{(\sigma)} M_t(s)x^{-s}\mathrm{d}s, \tag{4.1}$$

and is equal to $K_t(x)$ *for all* $x \in \mathbb{R}_{>0}$*,* $t \in \mathbb{R}$*.*

2. *There exist constants c_1, c_2 depending on the family $(K_t)_{t\in\mathbb{R}}$, independent of t, such that we may write $M_t(\sigma+i\nu) = g_t(\sigma+i\nu)e(f_t(\sigma+i\nu))$, in such a way that for all $x \in [t,2t]$, the following*

$$g_t^{(j)}(\sigma+i\nu) \ll_j (1+|\nu|)^{\sigma-1/2-j} \qquad \forall j \geq 0, \tag{4.2}$$

holds, as well as the following conditions on f_t.

   (a) *Whenever $|\nu| \leq c_1 t$ or $|\nu| \geq c_2 t$, we have*

$$\left| f_t'(\sigma+i\nu) - \frac{1}{2\pi}\log(x) \right| \gg 1, \tag{4.3}$$

   where the implicit constant does not depend on t.

   (b) *When $c_1 t \leq |\nu| \leq c_2 t$, either (4.3) holds, or we have*

$$f_t''(\sigma+i\nu) \gg (1+|\nu|)^{-1}, \tag{4.4}$$

   while for all $\epsilon > 0$, $j \geq 0$,

$$f_t^{(j)}(\sigma+i\nu) \ll_{j,\epsilon} (1+|\nu|)^{1+\epsilon-j}, \tag{4.5}$$

   where all the implicit constants do not depend on t.

   (c) *Finally, we require that*

$$f_t''(\sigma+i\nu) - \frac{1}{2\pi\nu} \gg (1+|\nu|)^{-1}, \tag{4.6}$$

   whenever $c_1 t \leq |\nu| \leq c_2 t$, and where the implicit constant does not depend on t.

We first give a quasi-orthogonality relation for analytic trace functions, reminiscent of Theorem 4.1.

Proposition 4.1. *Let $K_{1,t}$ and $K_{2,t}$ be two analytic trace functions, with associated Mellin transforms given by $M_{k,t}(\sigma+i\nu) = g_{k,t}(\sigma+i\nu)e(f_{k,t}(\sigma+i\nu))$ for $k=1,2$. Let c_{i1}, c_{i2} be the constants attached to $K_{i,t}$ appearing in condition (4.3). Let further $C_1 = \min(c_{11}, c_{21})$ and $C_2 = \max(c_{12}, c_{22})$. Suppose that $f_{1,t}''(\sigma+i\nu) - f_{2,t}''(\sigma'+i(\nu+a)) \gg \nu^{-1}$ for $C_1 \leq |\nu| \leq C_2 t$ and $a \ll t^{\epsilon}$. Let V be a smooth compactly supported function in $[1,2]$ satisfying $x^j V^{(j)}(x) \ll 1$ for all $j \geq 0$, then*

$$\int_{\mathbb{R}} K_{1,t}(x)\overline{K_{2,t}(x)}V\left(\frac{x}{t}\right)\mathrm{d}x \ll t^{1/2+\epsilon}.$$

Proof. Following the notation in Chapter 2, we let $\mathscr{I} = \{0\} \cup_{j\geq 0} \{\pm\left(\frac{4}{3}\right)^j\}$, such that for each $l \in \mathscr{I}$, we take a smooth function $W_l(x)$ supported in $[\frac{3l}{4}, \frac{4l}{3}]$ for $l \neq 0$ and such that

$$x^k W_l^{(k)}(x) \ll_k 1,$$

for all $k \geq 0$. For $l = 0$, take $W_0(x)$ supported in $[-2,2]$ with $W_0^{(k)}(x) \ll_l 1$, and such that $1 = \sum_{l \in \mathscr{I}} W_l(x)$. We then let for any $i \in \mathscr{I}$ and $k = 1,2$,

$$I_{k,l,t}(x) := \frac{1}{2\pi} \int_{\mathbb{R}} g_{k,t}(\sigma_k + i\nu) e(f_{k,t}(\sigma_k + i\nu)) x^{-\sigma_k - i\nu} W_l(\nu).$$

We recall that by Lemma 2.1 only the contributions from $|l| \leq t^{\epsilon}$ and $|l| \in [t^{1-\epsilon}, t^{1+\epsilon}]$ are non-negligible.

If $|l| \leq t^{\epsilon}$, up to replacing $K_{k,t}$ by the functions $K_{k,t}V(x/t)$, we may take the Mellin transform for any $\sigma > 0$, and therefore $I_{k,l,t}$ is negligible for $x \in [t, 2t]$.

We therefore assume that $|l| \in [t^{1-\epsilon}, t^{1+\epsilon}]$. By changing the order of summation, we have

$$\int_{\mathbb{R}} I_{1,l_1,t}(x) \overline{I_{2,l_2,t}(x)} V\left(\frac{x}{t}\right) \mathrm{d}x =$$
$$\frac{1}{4\pi^2} \int_{\mathbb{R}^2} M_{1,t}(\sigma_1 + i\nu_1) \overline{M_{2,t}(\sigma_2 + i\nu_2)} W_{l_1}(\nu_1) W_{l_2}(\nu_2) \int_{\mathbb{R}} x^{-\sigma_1 - \sigma_2 + i(\nu_2 - \nu_1)} V\left(\frac{x}{t}\right) \mathrm{d}x \mathrm{d}\nu_1 \mathrm{d}\nu_2.$$

Now by Lemma 1.5, the innermost integral is negligible unless $|\nu_1 - \nu_2| \ll t^{\epsilon}$. We therefore need to estimate for $x \in [t, 2t]$,

$$\int_{\mathbb{R}} \int_{|\nu_2 - \nu_1| \ll t^{\epsilon}} g_{1,t}(\sigma_1 + i\nu_1) \overline{g_{2,t}(\sigma_2 + i\nu_2)} W_{l_1}(\nu_1) W_{l_2}(\nu_2) e(f_{1,t}(\sigma_1 + i\nu_1) - f_{2,t}(\sigma_2 + i\nu_2)) x^{i(\nu_2 - \nu_1)} \mathrm{d}\boldsymbol{\nu}.$$

We will estimate this integral by the two variables second-derivative bound for oscillatory integrals as in [Mun15]. We therefore rewrite this integral as

$$\int_{a \ll t^{\epsilon}} \int_{\mathbb{R}} g(\nu, a) e(f(\nu, a)) \mathrm{d}\nu \mathrm{d}a,$$

where

$$g(\nu, a) = g_{1,t}(\sigma_1 + i\nu) \overline{g_{2,t}(\sigma_2 + i(\nu + a))} W_{l_1}(\nu) W_{l_2}(\nu + a) W(a),$$

where W is a smooth, non-oscillatory, compactly supported function on $[-t^{\epsilon}, t^{\epsilon}]$, and

$$f(\nu, a) = f_{1,t}(\sigma_1 + i\nu) - f_{2,t}(\sigma_2 + i(\nu + a)) + \frac{a}{2\pi} \log x.$$

We compute

$$\frac{\mathrm{d}^2}{\mathrm{d}\nu^2} f(\nu, a) = \frac{\mathrm{d}^2}{\mathrm{d}\nu^2} f_{1,t}(\sigma_1 + i\nu) - \frac{\mathrm{d}^2}{\mathrm{d}\nu^2} f_{2,t}(\sigma_2 + i(\nu + a)) \gg \nu^{-1},$$

and

$$\frac{\mathrm{d}^2}{\mathrm{d}a^2} f(\nu, a) = \frac{\log x}{2\pi} - \frac{\mathrm{d}^2}{\mathrm{d}a^2} f_{2,t}(\sigma_2 + i(\nu + a)) \gg 1.$$

We also need to compute the total variation of g. By the properties of $g_{k,t}$ for $k = 1,2$ and of W_l, we have

$$\frac{\mathrm{d}^2}{\mathrm{d}\nu \mathrm{a}} g(\nu, a) \ll \nu^{\sigma_1 + \sigma_2 - 2},$$

so that

$$\mathrm{var}(g) := \int_{t^{1-\epsilon}}^{t^{1+\epsilon}} \int_{-t^{\epsilon}}^{t^{\epsilon}} \left| \frac{\mathrm{d}^2}{\mathrm{d}\nu \mathrm{d}a} g(\nu, a) \right| \ll t^{\sigma_1 + \sigma_2 - 1 + \epsilon}.$$

By the second derivative bound for multivariable oscillatory integrals [Mun15], we thus obtain

$$\int_{a\ll t^{\epsilon}}\int_{\mathbb{R}} g(v,a)e(f(v,a))\mathrm{d}v\mathrm{d}a \ll t^{\sigma_1+\sigma_2-1/2+\epsilon}.$$

We therefore have

$$\int_{\mathbb{R}} I_{1,l_1,t}(x)\overline{I_{2,l_2,t}(x)}V\left(\frac{x}{t}\right)\mathrm{d}x \ll t^{-1/2+\epsilon},$$

concluding the proof of the proposition. □

Hyper-Kloosterman sums of different weights are not geometrically isomorphic. We now give an archimedean analog of this fact.

Proposition 4.2. *Let F_{it} denote the normalized Bessel function,*

$$F_{it}(x) = t^{1/2}\Gamma\left(\frac{1}{2}+it\right)J_{it}(x),$$

where J_ν denotes the usual J-Bessel function of order ν. Then for any $n \geq 3$, the higher rank Bessel function,

$$J_{n,t} = \frac{t^{\frac{n-1}{2}}}{2\pi i n}\int_{(\frac{1}{4})}\Gamma\left(\frac{s-int}{n}\right)\Gamma\left(\frac{s}{n}+\frac{it}{n-1}\right)^{n-1}e\left(\frac{s}{4}\right)x^{-s}\mathrm{d}s,$$

satisfies

$$\int_{\mathbb{R}} F_{it}(x)\overline{J_{n,t}(x)}V\left(\frac{x}{t}\right)\mathrm{d}x \ll t^{1/2+\epsilon},$$

for any smooth, compactly supported in $[1,2]$ function, satisfying $V^{(j)}(x) \ll_j 1$ for all $j \geq 0$.

Proof. It suffices to check the conditions of Proposition 4.1. We denote by $M_{F,t}(s) = g_{F,t}(s)e(f_{F,t}(s))$ and $M_{J_n,t}(s) = g_{J_n,t}(s)e(f_{J_n,t}(s))$ the Mellin transforms respectively associated to F_{it} and $J_{n,t}$. We recall (see Propositions 2.2 and 2.3) the regions of interest for $M_{F,t}(\sigma+i\nu)$ and $M_{J_n,t}(\sigma'+i\nu')$ are respectively contained in $-100t \leq \nu \leq -t$ and $-100nt \leq \nu' \leq -\frac{n}{n-1}t$, where

$$2\pi\frac{\mathrm{d}^2}{\mathrm{d}\nu^2}f_{F,t}(\sigma+i\nu) = \frac{\nu}{\nu^2-t^2},$$

and

$$2\pi\frac{\mathrm{d}^2}{\mathrm{d}\nu'^2}f_{J_n,t}(\sigma'+i\nu') = \frac{(n-1)\nu'+nt(2-n)}{(\nu'-nt)((n-1)\nu'+nt)}.$$

For any $a \ll t^{\epsilon}$ and $-100t \leq \nu \leq -\frac{n}{n-1}t$, we check that

$$\begin{aligned}\frac{(n-1)\nu+nt(2-n)}{(\nu-nt)((n-1)\nu+nt)} - \frac{\nu+a}{(\nu+a)^2-t^2} &= \frac{\nu t^2(n-n^2-1)+nt^3(2-n)+O(t^2)}{(\nu-nt)((n-1)\nu+nt)((\nu+a)^2-t^2)}\\ &\gg \nu^{-1},\end{aligned}$$

concluding the proof. □

We also prove an analogous result for two higher rank Bessel functions of different rank.

Proposition 4.3. *Let $3 \leq n < m$ and let $J_{n,t}, J_{m,t}$ denote higher rank Bessel functions,*

$$J_{k,t} = \frac{t^{\frac{k-1}{2}}}{2\pi ik}\int_{(\frac{1}{4})}\Gamma\left(\frac{s-ikt}{k}\right)\Gamma\left(\frac{s}{k}+\frac{it}{k-1}\right)^{k-1}e\left(\frac{s}{4}\right)x^{-s}\mathrm{d}s,$$

for $k \geq 3$. Then

$$\int_{\mathbb{R}} J_{m,t}(x)\overline{J_{n,t}(x)}V\left(\frac{x}{t}\right)\mathrm{d}x \ll t^{1/2+\epsilon},$$

for any smooth, compactly supported in $[1,2]$ function, satisfying $V^{(j)}(x) \ll_j 1$ for all $j \geq 0$.

Proof. It suffices to check the conditions of Proposition 4.1. We denote by $M_{J_m,t}(s) = g_{J_m,t}(s)e(f_{J_m,t}(s))$ and $M_{J_n,t}(s) = g_{J_n,t}(s)e(f_{J_n,t}(s))$ the Mellin transforms respectively associated to $J_{m,t}$ and $J_{n,t}$. As in the proof of Proposition 4.2, the region of interest here is given by $-100nt \leq v \leq -\frac{m}{m-1}t$, where

$$2\pi\frac{\mathrm{d}^2}{\mathrm{d}v^2}f_{J_m,t}(\sigma+iv) = \frac{(m-1)v+mt(2-m)}{(v-mt)((m-1)v+mt)},$$

and

$$2\pi\frac{\mathrm{d}^2}{\mathrm{d}v'^2}f_{J_n,t}(\sigma'+iv') = \frac{(n-1)v'+nt(2-n)}{(v'-nt)((n-1)v'+nt)}.$$

For any $a \ll t^\epsilon$ and $-100nt \leq v \leq -\frac{m}{m-1}t$, we check that

$$\begin{aligned}&\frac{(m-1)v+mt(2-m)}{(v-mt)((m-1)v+mt)} - \frac{(n-1)(v+a)+nt(2-n)}{(v+a-nt)((n-1)(v+a)+nt)}\\ &= t^2\frac{(n^2-n^2m+m^2n-m^2)v+2nm(m-n)t+O(1)}{(v-mt)((m-1)v+mt)(v+a-nt)((n-1)(v+a)+nt)} \gg v^{-1},\end{aligned}$$

concluding the proof. □

We now wish to investigate the analog of Theorem 4.2, in the context of analytic trace functions. This was encapsulated by Proposition 2.4, that we recall here.

Proposition 4.4. *Let K_t be an analytic trace function. Then there exists an analytic trace function, $\tilde{K}_t(x)$, such that the Fourier transform,*

$$\hat{K}_t(x) := t^{1/2}\int_1^2 K_t(tu)V(u)e(-xu)\mathrm{d}u,$$

satisfies

$$\hat{K}_t(x) = \tilde{K}_t(x) + O(t^{-1/2}).$$

Indeed Proposition 4.4 tells us that up to a small error, the Fourier transform of an analytic trace function is also an analytic trace function. We also note that in Theorem 4.2, it is important to avoid Artin-Schreier sheaves. However in our definition of analytic trace functions, we already noted that condition (4.6) was in some sense a way to impose not being "too close" to an additive character.

4.3 Connection between Chapters 2 and 3

We make a link between the work on analytic twists of modular forms and large values of L-functions with prescribed argument. We start by giving a very rough sketch of the work of Hough on large values of $L(\chi, 1/2)$, for χ a Dirichlet character $\pmod q$, a large prime going to infinity. In his work, Hough requires Theorem 0.1 to bound sums of divisor functions against Hyper-Kloosterman sums. We then describe the analogies with our work and look at the links with Theorem 0.2.

4.3.1 Large values of $L(\chi, 1/2)$

Let χ be a non-principal Dirichelt character $\pmod q$, a large prime. We assume that χ is even for simplicity. We then attach the Dirichlet L-function

$$L(\chi, s) := \sum_n \frac{\chi(n)}{n^s}, \qquad \Re(s) > 1,$$

and the completed L-function

$$\Lambda(\chi, s) := \left(\frac{q}{\pi}\right)^{s/2} \Gamma\left(\frac{s}{2}\right) L(\chi, s).$$

The completed L-function satisfies the functional equation

$$\Lambda(\chi, s) = \epsilon(\chi)\Lambda(\overline{\chi}, 1-s),$$

where

$$\epsilon(\chi) := \frac{\tau(\chi)}{\sqrt{q}} = \frac{1}{\sqrt{q}} \sum_{a \mod q} \chi(a) e\left(\frac{a}{q}\right)$$

is the normalized Gauss sum attached to χ.

Theorem 4.3 (Hough)**.** *Let $F(x)$ be a growth function, satisfying for all large x, $F(x) = o((\log x)^{1/2})$ and fix $\eta < 1/32$. For all primes $q > q_0(F)$, for all $\theta \in \mathbb{R}/\mathbb{Z}$, for all $\delta > \frac{1}{F(q)}$, there exists a non-principal $\chi \pmod q$ such that,*

$$\left\| \frac{1}{2\pi} \arg\left(L\left(\chi, \frac{1}{2}\right)\right) - \theta \right\|_{\mathbb{R}/\mathbb{Z}} \leq \delta, \qquad \Re\left(\log L\left(\chi, \frac{1}{2}\right)\right) \geq \sqrt{\eta \frac{\log q}{\log\log q}}.$$

We now explain some of the ideas involved in the proof of this Theorem. We let θ_χ denote the argument of $L(\chi, 1/2)$, and note that if $L\left(\chi, \frac{1}{2}\right) \neq 0$, since

$$\frac{L\left(\chi, \frac{1}{2}\right)}{L\left(\overline{\chi}, \frac{1}{2}\right)} = e^{i2\theta_\chi},$$

we have

$$e^{2i\theta_\chi} = \frac{\tau(\chi)}{\sqrt{q}}.$$

Unlike in the continuous case, we may not pin down exactly $2\theta_\chi = \theta$. Instead of using the residue theorem, Hough therefore uses a sort of equidistribution result for typical values of

$L(\chi, 1/2)$. We let $\mathscr{L} = \log q \prod_p \left(1 + \frac{r(p)}{\sqrt{p}(1+r(p)^2)}\right)$, where r is given by (3.1). The idea is to compute certain signed and unsigned moments. We let $N = q^\eta$ for some $0 < \eta < 1/32$ and

$$R(\chi) = \sum_{n \le N} r(n)\chi(n),$$

be a resonating polynomial. By a clever argument involving a quantitative equidistribution result, he captures large values by proving that

$$\sum_{\chi \,(\mathrm{mod}\, q)} \left[\left| L\left(\chi, \frac{1}{2}\right) \right|^2 \left|R(\chi)\right|^2 \right] \gg \sum_{\chi \,(\mathrm{mod}\, q)} \left[\left|R(\chi)\right|^2 \right] \mathscr{L}^4,$$

while

$$\sum_{\chi \,(\mathrm{mod}\, q)} \left[L\left(\chi, \frac{1}{2}\right)^k \left|R(\chi)\right|^2 \right] = o\left(\sum_{\chi \,(\mathrm{mod}\, q)} \left[\left|R(\chi)\right|^2 \right] \mathscr{L}^4 \right),$$

for $k = 1, 2$. He then encapsulates the equidistribution properties by proving that for all $m \ge 1$,

$$\sum_{\chi \,(\mathrm{mod}\, q)} \left[L\left(\chi, \frac{1}{2}\right)^k e(2m\theta_\chi) \left|R(\chi)\right|^2 \right] \ll_{m,\epsilon} q^{\eta - 1/8 + \epsilon} \sum_{\chi \,(\mathrm{mod}\, q)} \left[\left|R(\chi)\right|^2 \right] \mathscr{L}^4, \tag{4.7}$$

for $k = 1, 2$.

We now focus on the ideas behind the proof of (4.7), as that is where a non-correlation problem similar to that of Theorem 0.1 appears. We will focus on the case $k = 2$ which is the hardest. By the approximate functional equation for $L(\chi, 1/2)$, opening the square and writing $e(2m\theta_\chi) = \epsilon(\chi)^m$, the left hand side of (4.7) is essentially given by

$$\sum_{l_1, l_2 < q^\eta} r(l_1) r(l_2) \sum_{\chi \,(\mathrm{mod}\, q)} \chi(l_1)\overline{\chi(l_2)}\epsilon(\chi)^m \sum_n \frac{\chi(n) d(n)}{\sqrt{n}} V\left(\frac{n}{q}\right), \tag{4.8}$$

for a smooth, non-oscillatory, compactly supported function on $[1,2]$. By orthogonality of characters, we have the following proposition.

Proposition 4.5. *Let $a, b \not\equiv 0 \,(\mathrm{mod}\, q)$. For each $m \ge 0$, we have*

$$\frac{q^{1/2}}{q-1} \sum_{\chi \,(\mathrm{mod}\, q)} \chi(a)\overline{\chi(b)}\epsilon(\chi)^m = Kl_m(\overline{a}b).$$

From this Proposition, we conclude that (4.8) is given by

$$\frac{q-1}{q^{1/2}} \sum_{l_1, l_2 < q^\eta} r(l_1) r(l_2) \sum_n \frac{d(n)}{\sqrt{n}} \mathrm{Kl}_m(\overline{l_1 n} l_2) V\left(\frac{n}{q}\right),$$

which is small by Theorem 0.1.

4.3.2 Large values of $L(f, 1/2+it)$

The proof of Theorem 0.4 follows analogous steps, however since we may use the Residue theorem to capture the angle of $L(f, 1/2+it)$, we only need to compute first signed and unsigned moments. We recall that in the computation of the signed moment, we have to estimate sums I, II and III (see p. 65). We recall that term III is negligible, while term II contains a sum of Fourier coefficients twisted by a non-oscillatory function. These terms should be seen as analogs to the moments

$$\sum_{\chi \,(\mathrm{mod}\, q)} \left[L\left(\chi, \frac{1}{2}\right) e(2m\theta_\chi) \left|R(\chi)\right|^2 \right]$$

for $m \geq 2$, and

$$\sum_{\chi \,(\mathrm{mod}\, q)} \left[L\left(\chi, \frac{1}{2}\right) \left|R(\chi)\right|^2 \right],$$

respectively. Term III is analogous to the case $m = 1$ above. As explained in the previous section, Hough bounds this moment by relating it to sums of the divisors function against Kloosterman sums.

In our context, we don't see this analogy exactly. That is due to the fact that the estimates for term III essentially boil down to bounding a sum of the shape

$$\sum_n \lambda_f(n) V\left(\frac{n}{t}\right) K_T(n; l_1, l_2)$$

for $l_1, l_2 < T^{1-\epsilon}$, and where

$$K_T(n; l_1, l_2) := \int_{\Re(s)=\frac{1}{2}+\epsilon} \left(\frac{l_2}{l_1 n}\right)^s \Delta'(-s) \frac{\mathrm{d}s}{\cos\left(\frac{s+iT}{H}\right)}. \tag{4.9}$$

In said region, we prove that $K_T(x; l_1, l_2) \ll T^{-A}$, by the stationary phase method, and therefore we don't require Theorem 0.2.

However, as soon as the range of l_1, l_2 is allowed to go beyond T, we can no longer estimate trivially $K_T(n; l_1, l_2)$. In fact, we may prove that $K_t(n; l_1, l_2)$ is an analytic trace function.

Proposition 4.6. *For any $l \asymp T$, there exists an analytic trace function, $K_{T,l}(x)$, such that the function $K_T(x; l, 1)$ satisfies*

$$K_T(x; l, 1) = T^{1/2} (\log T) K_{T,l}(x) + O(T^\epsilon).$$

Proof. We may, up to negligible error, shift the line of integration to $\Re(s) = 0$, and using the relation (3.7), we obtain

$$K_T(x; l, 1) = -\frac{1}{2} \int_{\mathbb{R}} \left(\frac{1}{lx}\right)^{i\nu} \Delta(-i\nu) \log\left(\frac{\frac{1}{16} + \frac{1}{4}((r+\nu)^2 + (\nu-r)^2) + (\nu^2 - r^2)^2}{16\pi^4}\right) \frac{\mathrm{d}\nu}{\cosh\left(\frac{T+\nu}{H}\right)} + O(T^\epsilon).$$

We now let

$$K_{T,l}(x) := T^{-1/2} - \frac{1}{2} \int_{\mathbb{R}} \left(\frac{1}{lx}\right)^{i\nu} \frac{\Delta(-i\nu)}{\log T} \log\left(\frac{\frac{1}{16} + \frac{1}{4}((r+\nu)^2 + (\nu-r)^2) + (\nu^2 - r^2)^2}{16\pi^4}\right) \frac{\mathrm{d}\nu}{\cosh\left(\frac{T+\nu}{H}\right)},$$

and we prove this is an analytic trace function. Let

$$M_T(iv) = -\pi T^{-1/2}(\log T)^{-1} l^{-iv} \Delta(-iv) \log\left(\frac{\frac{1}{16} + \frac{1}{4}((r+v)^2 + (v-r)^2) + (v^2-r^2)^2}{16\pi^4} \right) \frac{1}{\cosh\left(\frac{T+v}{H}\right)};$$

we check it satisfies condition (4.2- 4.6). By Stirling's formula, we have

$$\begin{aligned} \Delta(-iv) &= \pi^{-2iv} \frac{\Gamma\left(\frac{\frac{1}{2}+i(v+r)}{2}\right)\Gamma\left(\frac{\frac{1}{2}+i(v-r)}{2}\right)}{\Gamma\left(\frac{\frac{1}{2}+i(r-v)}{2}\right)\Gamma\left(\frac{\frac{1}{2}-i(v+r)}{2}\right)} \\ &= \pi^{-2iv} \left|\frac{v+r}{2e}\right|^{i(v+r)} \left|\frac{v-r}{2e}\right|^{i(v-r)} (1+O(v^{-1})). \end{aligned}$$

We thus write $M_T(iv) = g_T(iv)e(f_T(iv))$, where

$$g_T(iv) = -\pi T^{-1/2}(\log T)^{-1} \log\left(\frac{\frac{1}{16} + \frac{1}{4}((r+v)^2 + (v-r)^2) + (v^2-r^2)^2}{16\pi^4} \right) \frac{1+O(v^{-1})}{\cosh\left(\frac{T+v}{H}\right)},$$

and

$$2\pi f_T(iv) = v\log\frac{1}{l\pi^2} + (v+r)\log\left|\frac{v+r}{2e}\right| + (v-r)\log\left|\frac{v-r}{2e}\right|.$$

We easily check that

$$g_T^{(j)}(iv) \ll_j v^{-1/2-j}, \qquad \forall j \geq 0.$$

We now compute

$$2\pi f_T'(iv) = \log\left|\frac{(v^2-r^2)}{4\pi^2 l}\right|,$$

so that for $x \in [T, 2T]$,

$$\left| f_T'(iv) - \frac{1}{2\pi}\log x \right| \gg 1,$$

whenever $v \neq T$. We also compute

$$2\pi f_T''(iv) = \frac{2v}{v^2-r^2} \gg v^{-1},$$

and

$$f_T^{(j)}(iv) \ll_{j,\epsilon} v^{1+\epsilon-j},$$

for all $j \geq 0$. It remains to check condition (4.6):

$$f_T''(iv) - \frac{1}{2\pi v} = \frac{v^2+r^2}{2\pi v(v^2-\rho^2)} \gg v^{-1},$$

concluding the proof. □

4.4 Further directions

We conclude the thesis by giving some ideas for further research and developments.

4.4.1 Analytic trace functions

We have presented some analogies between Frobenius and analytic trace functions, however many notions and properties remain elusive. Most importantly, the analogy between the algebraic geometric considerations leading to the formalism of Frobenius trace functions and a geometric structure behind the definition of analytic trace functions is lacking. For instance one could imagine considering sheafs arising as D-modules, and building similar geometric invariants such as irreducibility, isomorphisms and others in an analogous way as for Frobenius trace functions.

Remark 4.2. *For irreducibility, one could imagine to have an analytic statement defining irreducibility by giving an analog of Fact 4.1. More precisely, by computing the L^2-norm of an analytic trace function, one might be able to detect what would be the number of irreducible components.*

Remark 4.3. *The property that analytic trace functions are closed under addition does not to follow from the definition. One could imagine to bypass this issue by defining the space of analytic trace functions as being functions generated as sums of analytic trace functions as defined in this thesis.*

We encapsulate in the following table the analogies we do have, and those for which work has yet to be done.

	Frobenius trace functions	Analytic trace functions
Algebraic geometric considerations	Yes: constructions arising as l-adic sheaves.	Unclear: constructions arising as D-modules?
Notion of irreducibility	Yes: via representation theory.	Unclear: one could expect an answer here if given a more algebraic interpretation. Also see Remark 4.2
Notion of isomorphism	Yes: via representation theory.	Unclear: one could expect an answer here if given a more algebraic interpretation. **Note**: The hypotheses in Proposition 4.1 could also hint to a more analytic description.
Quasi-orthogonality relations	Yes: via the Riemann hypothesis for varieties over finite fields.	Yes: this is Proposition 4.1
Stability under Fourier transform	Yes: this is Theorem 4.2	Yes: this is Proposition 4.4
Stability under addition	Yes: via representation theory.	Unclear: one could expect an answer here if given a more algebraic interpretation. Also see Remark 4.3.
Stability under multiplication	Essentially Yes: via representation theory, however with some subtleties.	Unclear: one could expect an answer here if given a more algebraic interpretation.

4.4.2 Large values of $L(f, 1/2 + it)$ with prescribed argument

Theorem 0.4 proves the existence of large values of $L(f, 1/2 + it)$, however it doesn't give any information on the density of such large values. As noted previously, our proof of Theorem 0.4 only relies on estimates for the first moments of $L(f, 1/2 + it)$.

Let $T_\theta = \{t \in \mathbb{R} | \arg\left(L\left(f, \frac{1}{2} + it\right)\right) = \theta \,(\mathrm{mod}\, \pi)\}$ and $H = T/\log^2 T$. In order to obtain information about the density of large values, one is led to the study of the second moment

$$\sum_{t \in T_\theta} \left| L\left(f, \frac{1}{2} + it\right) \right|^2 \frac{1}{\cosh\left(\frac{t-T}{H}\right)}.$$

One way to link these two problems may be given by means of the Cauchy-Schwartz inequality,

$$\sum_{\substack{t \in T_\theta \\ |L| \gg \exp\left(\sqrt{\frac{\log T}{\log\log T}}\right)}} \left| L\left(f, \frac{1}{2} + it\right) \right| \leq \left(\sum_{\substack{t \in T_\theta \\ |L| \gg \exp\left(\sqrt{\frac{\log T}{\log\log T}}\right)}} 1 \right)^{1/2} \left(\sum_{\substack{t \in T_\theta \\ |L| \gg \exp\left(\sqrt{\frac{\log T}{\log\log T}}\right)}} \left| L\left(f, \frac{1}{2} + it\right) \right|^2 \right)^{1/2},$$

so that given a lower bound for the first moment and an upper bound for the second moment, one obtains information about the number of elements with large value.

By the Residue theorem, analogously to the first moment, one is led to estimating type I sums, i.e. sums of the type

$$\sum_{n,m \ll T} \rho_f(n) \rho_f(m) K_T(n; m, 1), \tag{4.10}$$

where $K_t(n; m, 1)$ is given by (4.9). Now by Proposition 4.6, we see that in each variable $K_T(n; m, 1)$ is an analytic trace function. Motivated by this problem, a natural continuation of this work would be to study bilinear sums of type (4.10).

4.4.3 Graphs of certain analytic trace functions

We include graphs of the normalized Bessel function and of Bessel functions of higher rank, which were the main examples of analytic trace functions. The first two graphs correspond, respectively, to the plots of the Bessel function and the Bessel function of rank 3, against a cos function to illustrate that these example do not resemble the additive character. The last graph plots the Bessel function and the Bessel function of rank 3 which do not correlate as proven in Proposition 4.1.

Plot of Bessel function against additive character

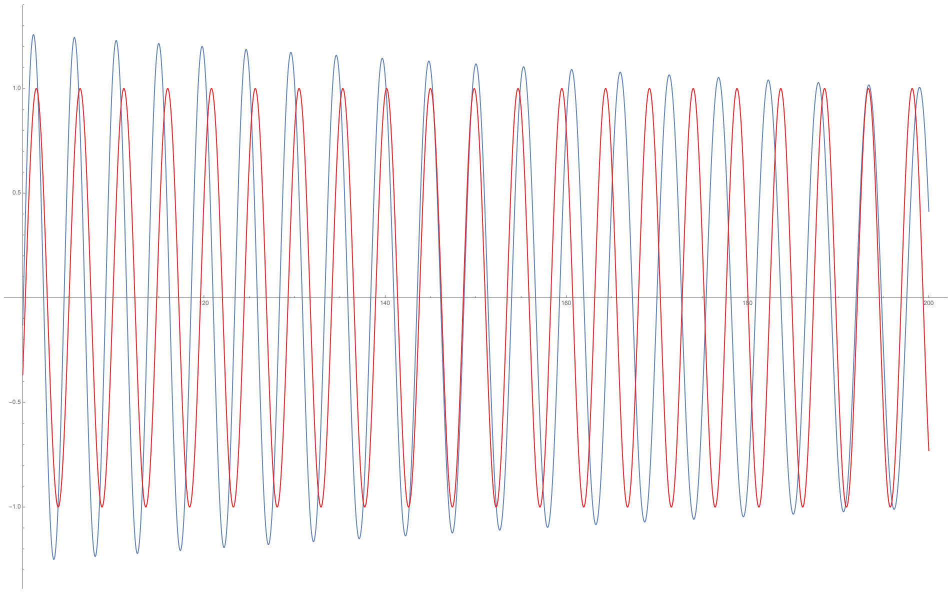

Figure 4.1 – Blue: J_{100i}, Red: $\cos(1.3x)$

Plot of Bessel function of rank 3 against additive character

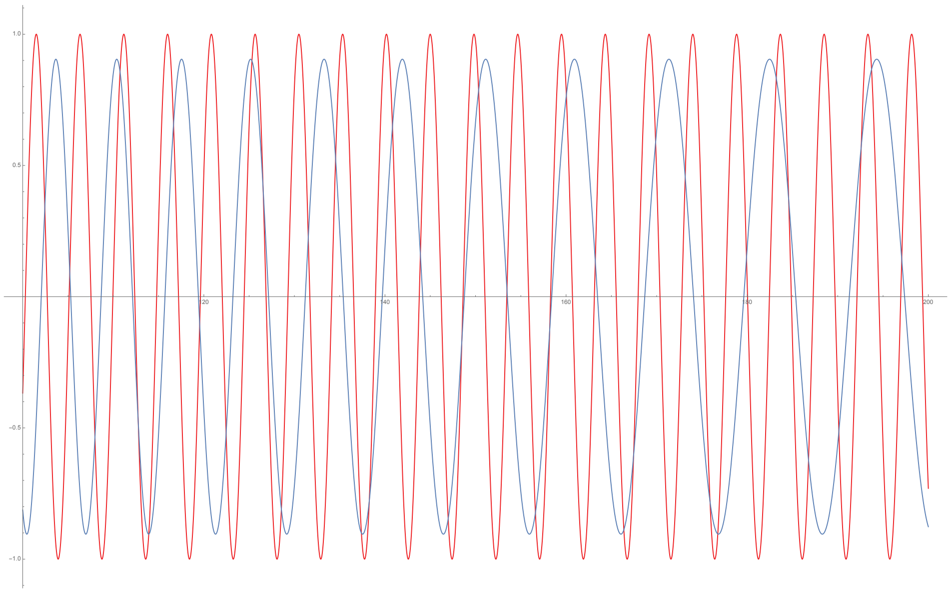

Figure 4.2 – Blue: $J_{3,100i}$, Red: $\cos(1.3x)$

Plot of Bessel function of rank 3 against Bessel function

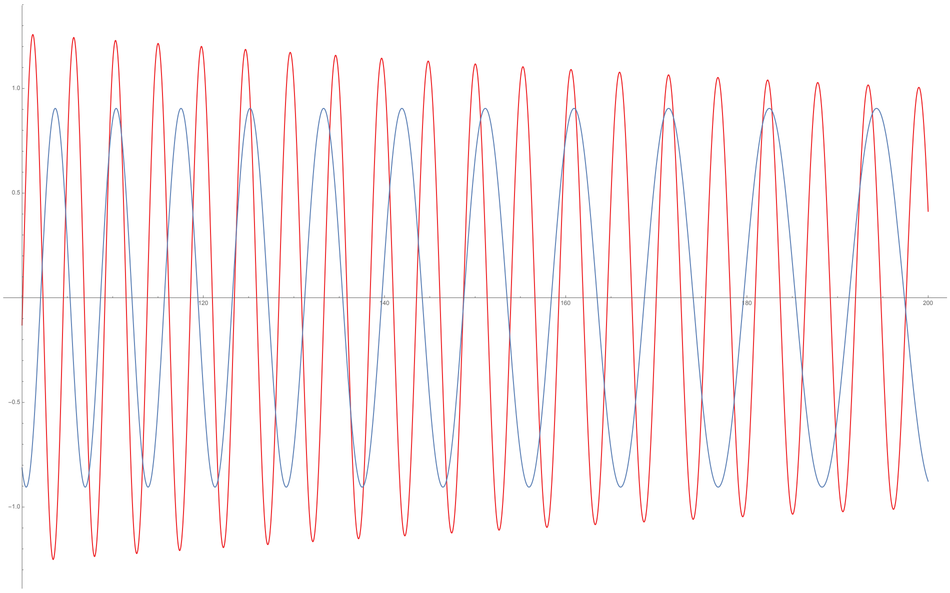

Figure 4.3 – Blue: $J_{3,100i}$, Red: J_{100i}

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Talks

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Université Blaise Pascal, France — June 2016
Sommes de coefficients de Fourier de formes modulaires tordues analytiquement et applications

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