Existence and Uniqueness of Load-Flow Solutions in Three-Phase Distribution Networks

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Abstract—We present sufficient conditions for the existence and uniqueness of load-flow solutions in three-phase distribution networks. The conditions can be efficiently verified for real distribution systems.

Index Terms—load flow solution, fixed point method, existence and uniqueness, distribution networks.

I. INTRODUCTION

IN distribution networks, many control procedures rely on the existence and uniqueness of load-flow solutions. However, due to the non-linearity of the load-flow equations, these properties are difficult to study. In this letter, we give efficiently verifiable conditions that guarantee the existence and uniqueness of load-flow solutions in three-phase distribution networks.

II. PROBLEM FORMULATION

We consider a three-phase network that has one slack bus indexed by 0, N PQ buses indexed by $1, \dots, N$, and a generic topology (i.e., radial or meshed). Let $\mathbf{v}_j \triangleq (v_j^a, v_j^b, v_j^c)^T$, $\mathbf{s}_j \triangleq (s_j^a, s_j^b, s_j^c)^T$ be the complex vectors representing three-phase nodal voltage and power injection at bus $j \in \{0, \dots, N\}$, and define $\mathbf{v} \triangleq (\mathbf{v}_1^T, \dots, \mathbf{v}_N^T)^T$, $\mathbf{s} \triangleq (\mathbf{s}_1^T, \dots, \mathbf{s}_N^T)^T$. Complex conjugates are denoted by adding an overline $\bar{\cdot}$. Then, the load-flow problem consists in solving for $\mathbf{v}$ and $\mathbf{s}_0$ in the following equations, where $\mathbf{s}$ and $\mathbf{v}_0$ are given and $\mathbf{Y}$ is the three-phase compound admittance matrix [1]:

$$\begin{bmatrix} \mathbf{s}_0 \\ \mathbf{s} \end{bmatrix} = \begin{bmatrix} \text{diag}(\mathbf{v}_0) & \\ & \text{diag}(\mathbf{v}) \end{bmatrix} \bar{\mathbf{Y}} \begin{bmatrix} \bar{\mathbf{v}}_0 \\ \bar{\mathbf{v}} \end{bmatrix}$$

Notice that matrix $\mathbf{Y}$ can be partitioned as

$$\mathbf{Y} = \begin{bmatrix} \mathbf{Y}_{00} & \mathbf{Y}_{0L} \\ \mathbf{Y}_{L0} & \mathbf{Y}_{LL} \end{bmatrix}$$

with 3×3 matrix $\mathbf{Y}_{00}$, $3 \times 3N$ matrix $\mathbf{Y}_{0L}$, $3N \times 3$ matrix $\mathbf{Y}_{L0}$, and $3N \times 3N$ matrix $\mathbf{Y}_{LL}$ (see Section IV-C for its invertibility). Then by defining the zero-load voltage $\mathbf{w} \triangleq -\mathbf{Y}_{LL}^{-1} \mathbf{Y}_{L0} \mathbf{v}_0 = (w_1^a, w_1^b, w_1^c, \dots, w_N^a, w_N^b, w_N^c)^T$, the load-flow problem can be reduced to Eq.(1), where $\mathbf{v}$ is unknown and $\mathbf{s}$ is given :

$$\mathbf{v} = \mathbf{w} + \mathbf{Y}_{LL}^{-1} \text{diag}(\bar{\mathbf{v}})^{-1} \bar{\mathbf{s}} \quad (1)$$

Eq.(1) is called the implicit Z_{bus} formulation [2]; it is a fixed point equation in $\mathbf{v}$ and can be solved by the iterative scheme in Eq.(2) :

$$\mathbf{v}^{(k+1)} = \mathbf{w} + \mathbf{Y}_{LL}^{-1} \text{diag}(\bar{\mathbf{v}}^{(k)})^{-1} \bar{\mathbf{s}} \quad (2)$$

In the rest of this letter, we give sufficient conditions under which there exists a unique load-flow solution that can be found by the iteration in Eq.(2).

III. RESULTS

In Theorem 1 we give a result assuming that we have prior knowledge of one pair $(\hat{\mathbf{v}}, \hat{\mathbf{s}})$ that satisfies Eq.(1). This is true in the cases where the load-flow problem is repeatedly solved for varying operational conditions (namely, $\hat{\mathbf{v}}$ is a solution computed in a previous instance of the problem when the power injection was $\hat{\mathbf{s}}$), or where a pair $(\hat{\mathbf{v}}, \hat{\mathbf{s}})$ is obtained by other methods. In Corollary 1, we give a result that does not depend on such prior knowledge.

The result makes use of the following notation :

$$\xi(\mathbf{s}) \triangleq \|\mathbf{W}^{-1} \mathbf{Y}_{LL}^{-1} \bar{\mathbf{W}}^{-1} \text{diag}(\bar{\mathbf{s}})\|_\infty$$

$$u_{\min}(\mathbf{v}) \triangleq \min_{j \in \{1, \dots, N\}, \gamma \in \{a, b, c\}} |v_j^\gamma / w_j^\gamma|$$

$$\mathcal{D}(\rho, \hat{\mathbf{v}}) \triangleq \{\mathbf{v} : |v_j^\gamma - \hat{v}_j^\gamma| \leq \rho |w_j^\gamma|, j \in \{1, \dots, N\}, \gamma \in \{a, b, c\}\}.$$

In the above, $\mathbf{W} \triangleq \text{diag}(\mathbf{w})$, and $\|\bullet\|_\infty$ is the matrix norm induced by the ℓ^∞ norm (i.e., $\|A\|_\infty = \max_{1 \leq j \leq 3N} \sum_{k=1}^{3N} |A_{j,k}|$).

Theorem 1. *Let $\hat{\mathbf{v}}$ be a solution to Eq.(1) with power injection $\hat{\mathbf{s}}$ and assume that $(u_{\min}(\hat{\mathbf{v}}))^2 > \xi(\hat{\mathbf{s}})$.*

For any other power injection vector $\mathbf{s}$ that satisfies

$$\xi(\mathbf{s} - \hat{\mathbf{s}}) < \frac{1}{4} (u_{\min}(\hat{\mathbf{v}}) - \xi(\hat{\mathbf{s}})/u_{\min}(\hat{\mathbf{v}}))^2$$

there is a unique solution $\mathbf{v}$ in $\mathcal{D}(\rho^\dagger, \hat{\mathbf{v}})$ to Eq.(1) with power injection $\mathbf{s}$, where $\rho^\dagger \triangleq \frac{1}{2} (u_{\min}(\hat{\mathbf{v}}) - \xi(\hat{\mathbf{s}})/u_{\min}(\hat{\mathbf{v}}))$.

This unique solution can be reached by applying the iteration in Eq.(2) initialized with any $\mathbf{v}^{(0)} \in \mathcal{D}(\rho^\dagger, \hat{\mathbf{v}})$.

Moreover, this unique solution is located in the smaller domain $\mathcal{D}(\rho^\dagger, \hat{\mathbf{v}})$ with

$$\rho^\dagger \triangleq \rho^\dagger - \frac{1}{2} \sqrt{(u_{\min}(\hat{\mathbf{v}}) - \xi(\hat{\mathbf{s}})/u_{\min}(\hat{\mathbf{v}}))^2 - 4\xi(\mathbf{s} - \hat{\mathbf{s}})}.$$

Observe that the localization in the smaller domain $\mathcal{D}(\rho^\dagger, \hat{\mathbf{v}})$ is more accurate but depends (via $\rho^\dagger$) on the specific $\mathbf{s}$, unlike the larger domain $\mathcal{D}(\rho^\dagger, \hat{\mathbf{v}})$. Also note that the theorem implies that there is no solution $\mathbf{v}$ in $\mathcal{D}(\rho^\dagger, \hat{\mathbf{v}}) \setminus \mathcal{D}(\rho^\dagger, \hat{\mathbf{v}})$.

If no $(\hat{\mathbf{v}}, \hat{\mathbf{s}})$ is obtained, we can use the following corollary.

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Corollary 1. Suppose that the power injection $\mathbf{s}$ satisfies $\xi(\mathbf{s}) < 0.25$. There exists a unique solution $\mathbf{v}$ in $\mathcal{D}(0.5, \mathbf{w})$ to Eq.(1) with power injection $\mathbf{s}$. This solution can be reached by applying the iteration in Eq.(2) initialized with any $\mathbf{v}^{(0)} \in \mathcal{D}(0.5, \mathbf{w})$. Moreover, it is located in the smaller domain $\mathcal{D}(\rho, \mathbf{w})$ with $\rho = (1 - \sqrt{1 - 4\xi(\mathbf{s})})/2$.

Note that all the proposed conditions on $\hat{\mathbf{v}}, \hat{\mathbf{s}}, \mathbf{s}$ can be verified at low computational complexity prior to solving a load-flow problem. Thus, they can be used for network control applications like Distribution Management Systems that need to solve multiple instances of load-flow problems in real time.

Remark 1. Theorem 1 and Corollary 1 are essentially extension to general three-phase distribution networks of the main results for single-phase networks in [3]. The key steps of this extension are (i) formulating the three-phase load-flow problem in the same algebraic form as its single-phase counterpart, and (ii) proving the invertibility of matrix $\mathbf{Y}_{LL}$ in three-phase networks.

IV. PROOFS

A. Proof of Theorem 1

Proof. The three-phase implicit Z_{bus} formulation in Eq.(1) has exactly the same algebraic form as its single-phase counterpart in [3], thus we can directly follow and apply the proof of Lemma 1 and Lemma 2 in [3]. Let $\mathbf{G}(\mathbf{v})$ express the right-handside of Eq.(1). It follows that, whenever the conditions in Theorem 1 are true, $\mathbf{G}()$ is a self-mapping and contraction mapping on $\mathcal{D}(\rho, \hat{\mathbf{v}})$ for any $\rho \in [\rho^\dagger, \rho^\ddagger]$. Therefore, according to Banach's fixed point theorem [4], there is a unique solution in $\mathcal{D}(\rho^\dagger, \hat{\mathbf{v}})$ to Eq.(1), and it can be reached by the iteration in Eq.(2) for any $\mathbf{v}^{(0)} \in \mathcal{D}(\rho^\dagger, \hat{\mathbf{v}})$; the same also holds if we replace $\rho^\dagger$ by $\rho^\ddagger$. Since $\mathcal{D}(\rho^\dagger, \hat{\mathbf{v}}) \subseteq \mathcal{D}(\rho^\ddagger, \hat{\mathbf{v}})$, we conclude that (i) there is a unique solution in $\mathcal{D}(\rho^\ddagger, \hat{\mathbf{v}})$; (ii) the solution is located in $\mathcal{D}(\rho^\ddagger, \hat{\mathbf{v}})$; (iii) it can be reached by iteration in Eq.(2) with any $\mathbf{v}^{(0)} \in \mathcal{D}(\rho^\ddagger, \hat{\mathbf{v}})$. $\square$

B. Proof of Corollary 1

Proof. Apply Theorem 1 with $\hat{\mathbf{v}} = \mathbf{w}$ and $\hat{\mathbf{s}} = \mathbf{0}$. $\square$

C. Invertibility of $\mathbf{Y}_{LL}$

The admittance matrix depends on specific device modeling. Here, we show that $\mathbf{Y}_{LL}$ is invertible when the following system assumptions hold, which covers most practical cases.

- The longitudinal component and shunt elements of transmission line between any pair of buses are described by circulant matrices;
- Transformer between any pair of buses is depicted by models in [5], but equipped with complex ratio;
- The connection from the slack bus to a PQ bus can be realized via (i) a transmission line, or (ii) a transformer of either Delta – Wye_G or Wye_G – Wye_G configuration;
- The connection between two PQ buses can be established through either a transmission line or a transformer of Wye_G – Wye_G configuration;
- Transformers may have additional core losses in the form of transverse components attached to related buses;

- Transmission lines and transformers do not generate active power, and their longitudinal components have positive resistance in zero-, positive-, negative sequences.

Proof. To prove that $\mathbf{Y}_{LL}$ is invertible we show that if $\mathbf{x} \in \mathbb{C}^{3N}$ is a vector such that $\mathbf{Y}_{LL}\mathbf{x} = \mathbf{0}$ then we must have $\mathbf{x} = \mathbf{0}$. We can view $\mathbf{Y}_{LL}$ as the admittance matrix of a fictitious N -bus network (i.e., the original network with the slack bus grounded) and $\mathbf{x}$ as its nodal voltage vector; then the sum of all nodal power injections of this N -bus network is $s^{\text{total}} = \mathbf{x}^T \bar{\mathbf{Y}}_{LL} \bar{\mathbf{x}} = 0$.

Let $\mathcal{V}^{\text{slack}}$ be the set of buses that are connected with the slack bus in the original $N + 1$ -bus network, and write $\mathbf{x}$ as $\mathbf{x} = (\mathbf{x}_1^T, \dots, \mathbf{x}_N^T)^T$ with $\mathbf{x}_j, j \in \{1, \dots, N\}$ interpreted as the three-phase nodal voltage at bus j . Note that s^{total} equals the system power loss and can be further decomposed as

$$s^{\text{total}} = s^{\text{slack}} + s^{\text{line}} + s^{\text{shunt}} + s^{\text{leakage}} + s^{\text{core-loss}}$$

where

- s^{slack} is the power consumption in the transverse components that result from connections to the slack bus in the original $N + 1$ -bus network of all buses $j \in \mathcal{V}^{\text{slack}}$;
- s^{line} is the power consumption in the longitudinal components of all transmission lines;
- s^{shunt} is the power consumption in all shunt elements;
- s^{leakage} is the power consumption that results from all transformer leakage impedances;
- $s^{\text{core-loss}}$ is the power consumption caused by core losses in all transformers.

By the last item of assumption, $s^{\text{total}} = 0$ implies that all the five terms contain zero real parts. Therefore, according to the system assumptions, we have

- 1) $\Re(s^{\text{slack}}) = 0$ implies that $\mathbf{x}_j = \mathbf{0}$ for all $j \in \mathcal{V}^{\text{slack}}$;
- 2) From $\Re(s^{\text{line}}) = 0$, it can be obtained that $\mathbf{x}_j = \mathbf{x}_l$ for transmission line between any buses $j, l \in \{1, \dots, N\}$;
- 3) By $\Re(s^{\text{leakage}}) = 0$ and the Wye_G – Wye_G configuration, we have $\mathbf{x}_j = K_{jl}\mathbf{x}_l$ for transformer with ratio K_{jl} between any buses $j, l \in \{1, \dots, N\}$.

From the first item, there is at least one bus that has zero voltage. From the second and the third items, the zero voltage propagates throughout the N -bus network. Thus, we have $\mathbf{x} = \mathbf{0}$. $\square$

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