Sampling at unknown locations, with an application in surface retrieval

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***Abstract*—We consider the problem of sampling at unknown locations. We prove that, in this setting, if we take arbitrarily many samples of a polynomial or real bandlimited signal, it is possible to find another function in the same class, arbitrarily far away from the original, that could have generated the same samples. In other words, the error can be arbitrarily large. Motivated by this, we prove that, for polynomials, if the sample positions are constrained such that they can be described by an unknown rational function, uniqueness can be achieved.**

In addition to our theoretical results, we show that, in 1-D, the problem of recovering a painted surface from a single image exactly fits this framework. Furthermore, we propose a simple iterative algorithm for recovering both the surface and the texture and test it with simple simulations.

I. Introduction

In today's information age, we are measuring the world around us like never before and sampling theory provides the tools to connect these digital measurements to their underlying continuous signals. The famous Whittaker-Kotelnikov-Shannon theorem [1], [2] states the sufficient condition on the sampling rate needed to recover bandlimited functions. It assumes that the samples are taken uniformly at known positions; however, in many applications the sample positions are not uniform and sometimes they are not even known.

Irregular sampling is well studied with fast iterative algorithms available for reconstructing functions from shift invariant spaces [3], [4], [5]. It has many applications such as in astronomy [6], where it is only possible to take measurements in certain conditions, and in magnetic resonance imaging, where it is more efficient to use a spiral sampling grid.

In contrast, there is very little work considering unknown sample positions. Such schemes arise, among others, when measurements are regular but some data points are lost during transmission, or when measurements are taken with a jittered clock or moving sensor [7]. Another interesting example, which we will investigate in this paper, is surface geometry estimation from images.

Unsurprisingly, sampling at unknown location may not have a unique solution: if we take m samples of a function which belongs to an n-dimensional space, there are $n + m$ degrees of freedom in total, but only m measurements. Despite this some recovery results have been achieved for some restricted cases. In particular, Marziliano et al. [8] studied the recovery of discrete-time bandlimited signals from samples at unknown locations. In the continuous case, Browning proposed an alternating least squares algorithm that converges to a local minimum [9]. In addition, Kumar studied the recovery of periodic bandlimited signals assuming that the sample positions follow some stochastic model. He showed that, if the sampling rate is sufficiently high, we can assume that the sampling locations are uniformly distributed and the reconstruction error is approximately inversely proportional to the number of samples [10] [11].

In this work, we investigate continuous-time signals with deterministic sample positions. We prove that, if we take arbitrarily many samples of a polynomial or real bandlimited signal, it is possible to find another function in the same class, arbitrarily far away from the original, that could have generated the same samples. Due to this non-uniqueness, we then investigate constraints on the sample positions. More precisely, we prove that, for polynomials, if the sample positions are constrained such that they can be described by an unknown rational function, uniqueness can be achieved.

Finally, we consider the problem of retrieving a 1-D linear surface, painted with a polynomial, from a single image. We show that the locations of the samples on the camera's image plane are governed by the geometry of the surface and, furthermore, they can be described by a rational function; i.e., this setup fits the conditions of our uniqueness result. In order to estimate the surface orientation and polynomial parameters, we use an adaptation of the alternating least squares recovery algorithm that was proposed in [9]. Our surface recovery technique is fundamentally different to traditional computer vision techniques, such as triangulation, photometric stereo and shape from shading/texture [12], [13]; however, it has many similarities to shape from bandwidth, which we have recently proposed [14].

In order to make the results of this paper reproducible, the code for all simulations is available at the following link: https://github.com/micha7a/surface-reconstruction.

II. Sampling at unknown locations

A. Setup

The problem is as follows: let $\mathcal{F}$ be a linear space of functions defined over the interval $[0, T]$. Assume we observe a function $f \in \mathcal{F}$ at m unknown and distinct locations over the interval; that is, we measure $f(t_1) \dots f(t_m)$. The only knowledge about the sampling instants is their linear order, that is $t_1 < t_2 < \cdots < t_m$. The question is whether we can recover the original f from the set of observations. Since $\mathcal{F}$ is a linear space, recovering functions is understood as finding the coefficients of the function f in the space $\mathcal{F}$.

We call *a solution* any function $f_s \in \mathcal{F}$ which could have been a source for the observed samples; that is, a function for which there exist a sequence $\hat{t}_1 < \hat{t}_2 < \cdots < \hat{t}_m$ such that $t_i \in [0,T]$ and $f_s(\hat{t}_i) = f(t_i)$ for all $i = 1 \ldots m$. Of course, f is a solution.

B. Non-uniqueness of solutions/a trajectory of solutions

Without further constraints on the sample locations, it is easy to show that many solutions can exist. In [9], it is argued that a sufficiently small perturbation of a solution may also be a solution. We show that for some classes of functions a function may be arbitrary far (in L_2 norm) from the original solution and still fit the samples. For example, if the measured function is a polynomial of degree up to n, then for any affine transformation of the domain (that keeps sample positions inside the interval) one can easily find another polynomial of degree at most n which matches the samples exactly. Indeed, a scaled (or shifted) polynomial is also a polynomial. To avoid this issue one can fix the positions of first and last samples, and recover not one solution but a class of them.

It turns out that even with those assumptions the solution can be arbitrary far from the original function, not only in the class of polynomials of a finite degree, but also in the class of bandlimited functions:

Lemma 1. *Let $f \in \mathcal{F}[0,T]$ be a sampled function, and let f_i, $i = 1 \ldots m$ be samples taken from f at positions $0 = t_0 < t_1 < \cdots < t_m = T$. If*

1) *$\mathcal{F}$ is the class of polynomials of degree at most n, or*
2) *$\mathcal{F}$ is the class of real-valued, $2n$-bandlimited functions,*

then for any $M > 0$ there exist a function f_M such that $\|f - f_M\| \geq M$ and points $0 = s_1 < s_2 < \cdots < s_m = T$ such that $f_M(s_i) = f_i$.

Proof: We divide the proof of this lemma into two parts. In the first part, we prove that, if a function $g \in \mathcal{F}$ has the right sign at the extrema of f, then $f + \alpha g$ is also a solution for every $\alpha > 0$. In the second part we show that, for both classes, it is always possible to construct such a function g.

Part 1. Let $t_{max,1}, t_{max,2} \ldots$ and $t_{min,1}, t_{min,2}, \ldots$ be the arguments at which f attains its local maxima and minima, respectively, and let g be a non constant function, $g \in \mathcal{F}$, such that $g(0) = 0$, $g(t_n) = 0$, $g(t_{min,i}) \leq 0$ and $g(t_{max,i}) \geq 0$ for every $i = 1, \ldots, m$.

Consider a function $f_s = f + \alpha g$, where $\alpha \geq 0$. For any value of f we can a find point at which f_s has the same value by moving away from the closest local extremum, see Figure 1. Let us consider the interval $[a,b]$ between a maximum and a minimum. Since at point a the function f has its maximum, $g(a) \geq 0$, so $f(a) + \alpha g(a) \geq f(a)$. Similarly, $f(b) + \alpha g(b) \leq f(b)$. Therefore, since these functions are continuous, for any sample taken in the interval $[a,b]$, we can find a new point $t_s \in [a,b]$ such that $f(t_s) + \alpha g(t_s)$ is equal to this sample value. Continuity of the functions also allows us to preserve the order of the samples on this interval. A similar argument may be used on intervals between minima and maxima and between an extremum and the boundary. Therefore, we can find new sample positions that fit the data while preserving the order of the samples.

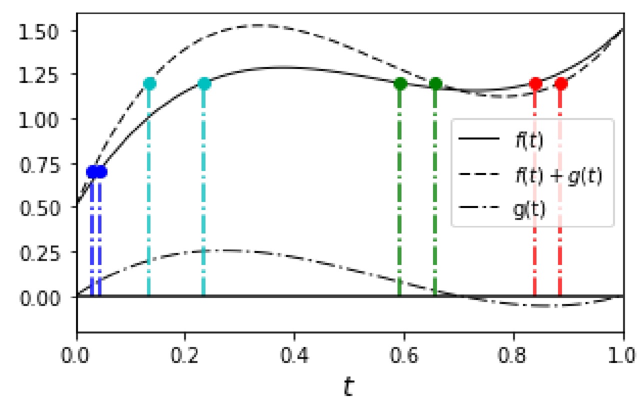

Fig. 1. Movement of samples: a polynomial of degree 3 (solid line), with extrema around 0.4 and 0.7, the corresponding polynomial g from Lemma 1 (dashed-dotted line) and $f + g$ (dashed line). With full circles are the samples at their old and new positions.

This means that f_s is also a solution, which preserve the values at the endpoints. Since g is not zero, the norm of the difference $\|f - f_s\| = \|\alpha g\| = \alpha\|g\|$ can be made larger than any constant M, by taking any $\alpha > \frac{M}{\|g\|}$.

Part 2. We have just shown a way to construct an additive perturbation to our original function that fits the samples. What is left is showing that such a function g exists and lies in $\mathcal{F}$.

Polynomial case: Let $\mathcal{F}$ be the space of polynomials of degree at most n. A polynomial $f \in \mathcal{F}$ has at most $n-1$ extrema inside the interval $[0,T]$. For every pair of consecutive extrema, we choose a number s_i between them, which gives $l \leq n-2$ numbers $s_1, s_2, \ldots, s_l$. Define

$$g(t) := \pm t(t-T)(t-s_1)\ldots(t-s_l). \tag{1}$$

This function is a polynomial of degree $2 + l \leq n$, so $g \in \mathcal{F}$. Moreover, $g(0) = 0$, $g(T) = 0$ and g has only single zeros so it has a different sign when f attains its maxima and its minima, so we can set the sign in such a way that when f attains its maxima, $g(t)$ is positive.

Band-limited case: The above reasoning cannot be repeated for bandlimited periodic functions because the nature of zeros of those functions is more subtle. Without loss of generality we assume that $T = 2\pi$. Note that, since $f(t)$ is bandlimited and defined on an interval, it can be extended to be periodic bandlimited and therefore $f(0) = f(2\pi)$. The function f has at most $2n-2$ extrema: let us call them $s_1, s_2, \ldots, s_l$, $l \leq 2n-2$.

We construct a function $g \in \mathcal{F}$, which is not zero everywhere, has zeros at $s_1, s_2, \ldots, s_{l-1}$ and 0, and has the proper sign at $t_{min,1}$. Consider the following function:

$$G(t) := e^{i(n-1)t}(1-e^{it})(e^{is_1}-e^{it})\ldots(e^{is_{l-1}}-e^{it}).$$

This function has zeros in the desired places, and it is a sum of powers of e^{it} from $e^{-i(n-1)t}$ to $e^{i(n-1)t}$, so it is periodic and bandlimited. However, it is not necessary real, so it is not necessarily in $\mathcal{F}$. If the function G has a non-zero real part, we can define the desired function g as the real part of G:

$$g(t) := \pm\frac{1}{2}\big(G(t) + \overline{G}(t)\big), \tag{2}$$

such that if $f(s_l)$ has a maximum, then $g(s_l)$ is positive and otherwise $g(s_l)$ is negative. In the case that G would be purely imaginary, we can define g as $\pm$ the imaginary part of G:

$$g(t) := \mp iG(t), \tag{3}$$

again setting the sign such that g is positive if and only if $f(s_l)$ is a maximum. In both cases, we get $g \in \mathcal{F}$, so g is the desired function. ■

Note that Lemma 1 implies not only the existence of one solution f_s, but a whole family of solutions, $f_\alpha = f + \alpha g$, for $\alpha \geq 0$.

III. Sampling with constraints

In this section, we consider only the space $\mathcal{F}$ of polynomials of degree $\leq n$. From the previous section, we know that the reconstruction of polynomials is not unique in general. Now we present an intuition why constraining the relative sample positions can result in a uniquely-solvable problem, and then we analyse rational functions as a class of constraints.

Example 2 (Relative movement of sample positions)**.** Let $f \in \mathcal{F}$ be the original function and $g \in \mathcal{F}$ be a polynomial from Lemma 1, such that $f + \alpha g$ is a solution for every $\alpha \geq 0$.

Consider the movement of samples as α changes, see Figure 2. Let f be the sampled function and let $0 = t_1 < t_2 < \cdots < t_m = T$ be the original sample positions. We can construct a set of functions $s_i(\alpha)$, $i = 2 \ldots m-1$ such that

1) $s_i(\alpha)$ is the new sample position: $(f + \alpha g)(s_i(\alpha)) = f(t_i)$,
2) s_i are continuous,
3) $s_i(\alpha) < s_j(\alpha)$ for $i < j$ and every α.

We can do this because g was constructed in such a way that $(f + \alpha g)(t) - f(t_i)$ has at least one solution. There might be cases when, as α changes, a new solution appears, but then we can just pick any branch for which s_i is continuous. The resulting family of functions, indexed by i, is parameterized by a scalar α, which means that the set of possible sample positions contains a path in the m-dimensional space of all sample positions. △

If the possible movement along that path was the only source of ambiguity, it would be enough to restrict the sample positions to a subspace intersecting the path, but not following it. Of course, there may be more than just one function g such that $f + \alpha g$ is a solution (for every $\alpha \geq 0$), and there may be solutions producing the correct samples, but not connected to the original one by such a path. However, as s_i are the roots of polynomials of degree at most n, the sequence (s_i) has n degrees of freedom. If m is much bigger than n, it is very likely that, for any constraints, the intersection between the set of possible sample positions and the set of sample positions defined by (s_i) will be small.

Consider now the situation where we do not know the sample positions, but we restrict the way in which they can move. In particular, assume that sample positions are only allowed to move along trajectories defined by a rational function (we will see later an application where rational functions arise naturally as a constraint for sample trajectories). Then, we have the following result:

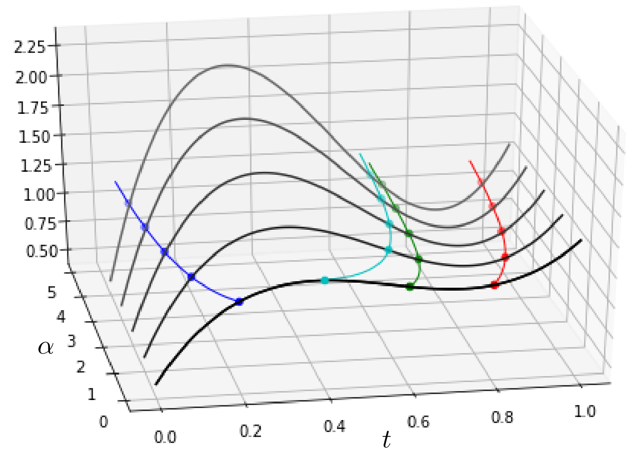

Fig. 2. Samples cannot move arbitrary. The polynomial $f(t) + \alpha g(t)$ describes a surface. Five slices of this surface are plotted corresponding to five different values of α. The first slice, when $\alpha = 0$, is the original polynomial f, which is sampled in four positions. We see that, as α increases, the position of the samples follow a trajectory in the α-t plane corresponding to a contour of the surface. Restricting the sample positions so that they cannot move along these contours allows us to distinguish between f and $f + \alpha g$

Lemma 3. *Let $\mathcal{F}$ be a space of polynomials of degree at most n. Let P be the sampled polynomial and let $0 \leq t_0 < t_1 < \cdots < t_m \leq T$ be the original sample positions. Let $\hat{t}_i = \hat{t}(t_i)$ be other possible sample positions,*

$$\hat{t}(t) = \frac{W(t)}{V(t)}, \tag{4}$$

where W and V are irreducible polynomials of degrees:

$$deg(W) = n_w \leq n_v = deg(V).$$

If the number of samples $m > n(n_v + 1)$, then there is no polynomial $Q \in \mathcal{F}$, $Q \not\equiv P$ such that $P(t_i) = Q(\hat{t}_i)$ for all i.

The above result states the minimum number of samples required for uniqueness of the underlying function $f \in \mathcal{F}$. Before we prove Lemma 3, let us make a few remarks on the transformation $\hat{t}$. Rational functions are not defined everywhere, so we have to assume that the coefficients of V are such that $\hat{t}$ is well defined on the interval $[0, T]$. This of course will be true for every realistic transformation. In the general problem of sampling at unknown locations, we assumed that we knew the order of the samples, so for consistency we will assume that if $t_i < t_j$ then $\hat{t}_i < \hat{t}_j$, which means that $\hat{t}(t)$ is strictly increasing.

Proof: Let $Q \in \mathcal{F}$ be a polynomial such that

$$Q(\hat{t}_i) = Q\left(\frac{W(t_i)}{V(t_i)}\right) = P(t_i)$$

for every $i = 1, \ldots, m$. All t_i satisfy the following equation:

$$\sum_{k=0}^{n} p_k t^k = \sum_{k=0}^{n} q_k \left(\frac{W(t)}{V(t)}\right)^k, \tag{5}$$

where p_k and q_k, $k = 1, \ldots, n$ are the coefficients of the

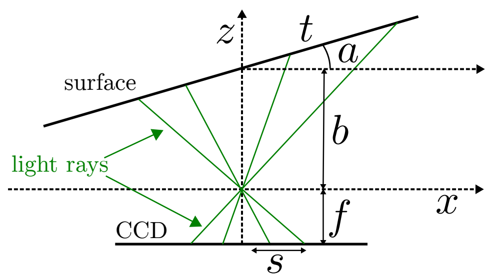

Fig. 3. Setting of painted line with pinhole camera model. Light rays reflected from the surface travel through the origin and are recorded on the image plane located at the distance f from the origin. The surface is parametrised by the z-intercept $b > 0$ and slope a. The line is parametrised by variable t, with 0 at the intersection with z-axis.

polynomials P and Q, respectively. We can rewrite this as

$$(V(t))^n \sum_{k=0}^{n} p_k t^k = \sum_{k=0}^{n} q_k (W(t))^k (V(t))^{n-k}. \tag{6}$$

This equation defines a polynomial with degree at most $\nu = \max(n_v n + n, n_w n)$. But, since $n_v \geq n_w$, $\nu = (n_v + 1)n$.

If the degree of P is greater than 1, the left hand side of Equation (6) cannot be equal to the right hand side everywhere. Therefore, Equation (5) has at most ν solutions and hence the polynomial P is unique provided that $m > (n_v + 1)n$.

If P is a constant it is possible that both sides of Equation (6) are equal everywhere but this can only occur if $P \equiv Q$. ■

We considered here an arbitrary rational function. However, if we fix some parameters of this rational function, we could exploit the structure of the polynomials obtained and lower the number of samples needed.

IV. Applications

Rational mapping functions appear naturally when reconstructing the geometry and texture of a painted surface. To see this, consider a pinhole camera viewing a 1-D linear surface painted with some texture, as depicted in Figure 3. The samples are taken uniformly on the image sensor but, due to pinhole projection and the surface's geometry, the texture is sampled irregularly. Furthermore, since we do not know the surface's orientation, we do not know the sample positions. However, as we will show, the sample positions are governed by a rational function and, if the texture is a polynomial, Lemma 3 states the conditions for its uniqueness.

To see that the mapping is rational, consider a sample which was recorded at a distance s from the optical axis. In the notation of Figure 3, the surface has planar coordinates $x(t) = \cos(a)t$ and $z(t) = \sin(a)t + b$, so the sample had to come from a point t such that

$$\frac{s}{f} = \frac{x(t)}{z(t)} = \frac{\cos(a)t}{\sin(a)t + b}, \tag{7}$$

where f is the focal length.

Let $\hat{a}$ and $\hat{b}$ be the parameters of another planar surface, and $\hat{t}$ be a position on this line which could produce this sample. Applying the above equation to both t and $\hat{t}$ yields

$$\hat{t}(t, a, b) = \frac{\hat{b}\cos(a)t}{b\cos(\hat{a}) + \sin(a - \hat{a})t}. \tag{8}$$

The transformation $\hat{t}$ is a rational function in terms of t. If $\hat{a} \neq a$, then the degree of the numerator is equal to the degree of the denominator. From Lemma 3, we know that the set of at least $2n + 1$ samples (where n is the degree of P) could be generated by only one of those lines, and with the assumption that the tilt is a, no polynomial can be fitted to those $2n + 1$ samples perfectly. However, if $\hat{a} = a$ the transformation $\hat{t}$ becomes just scaling. Therefore parameter a can be reconstructed but parameter b cannot, and the sample positions can be reconstructed up to a scaling.

To actually find the parameter a, we can use an adaptation of the alternating least squares algorithm [9]. This algorithm minimises the mean squared error, $\|\boldsymbol{M}\boldsymbol{p} - \boldsymbol{f}\|^2$, where $\boldsymbol{M}$ is a Vandermonde matrix consisting of the powers of $\hat{t}_i$, $\boldsymbol{p}$ is the vector of coefficients and $\boldsymbol{f}$ is the vector of sample values. It works by alternating between the following two steps:

1) fix the matrix $\boldsymbol{M}$ and solve for the coefficients $\boldsymbol{p}$ using ordinary least squares,
2) fix the vector $\boldsymbol{p}$ and make one step of gradient descent with respect to $\hat{t}_i$.

In our case, the $\hat{t}_i$ are functions of the parameters a and b and the position on the CCD and therefore the gradient can be calculated in terms of a and b. As expected, if the algorithm starts with the right distance parameter b, the sample positions are recovered exactly, see Figure 4.

To analyse the performance with respect to the degree of the polynomial, we performed the following numerical simulation. For polynomial degrees ranging from 2 to 8, we ran our reconstruction algorithm on 50 randomly generated polynomials painted onto a linear surface with 13 different angles (equally spaced within $[-30^\circ, +30^\circ]$). As can be seen from Figure 5, the performance is better for lower-degree polynomials. This might be due to the non-convexity of the constrained cost function, since the algorithm is more likely to converge to a local minimum with the higher degrees. Furthermore, higher order polynomials performed especially poorly when the leading coefficient was small compared to the others.

To analyse the noise robustness, we randomly generated 50 polynomials of degree 3 and painted them onto the same linear surfaces. We tested the algorithm for different amounts of noise and sampling rates. The results are depicted in Figure 5. Although the error grows quite quickly as the noise amplitude is increased, the results are still reasonable for a relative noise amplitude of $\sigma = 0.1$. However, oversampling improves the results only slightly and brings with it a large increase in the computational cost, since the complexity is of the order of $n^3 \times$ (the number of iterations).

V. Conclusion and Future Work

We investigated the problem of sampling at unknown locations. We showed that, for polynomial and real bandlimited signals, the error can be arbitrarily large; however, by constraining the path along which samples can move, uniqueness

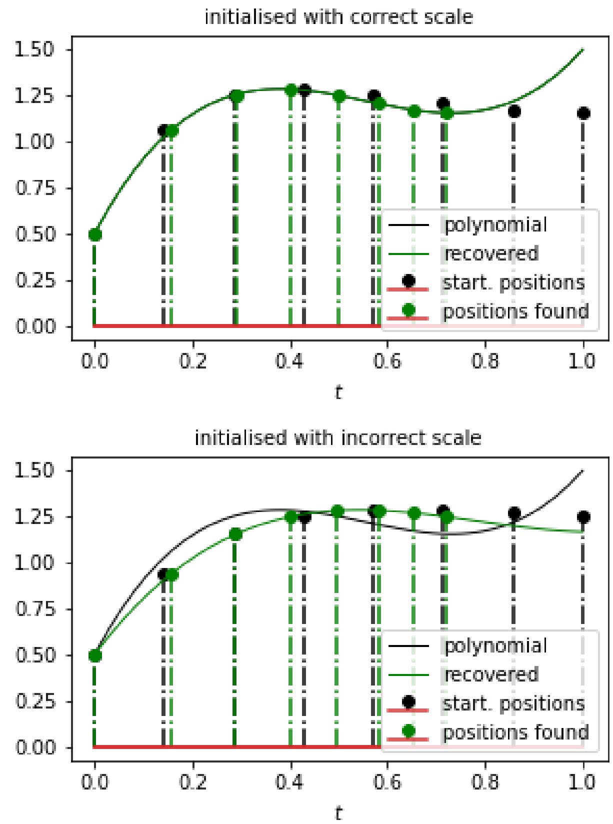

Fig. 4. ALS algorithm reconstructing a polynomial of degree 3. In the upper image, the algorithm was initialised with the correct scale b. Given this initialisation it calculates the correct orientation a and polynomial coefficients. In the lower image, the algorithm was initialised with an incorrect scale b. The algorithm still returns the correct orientation a but, as expected, the estimated polynomial is stretched in time compared to the original one.

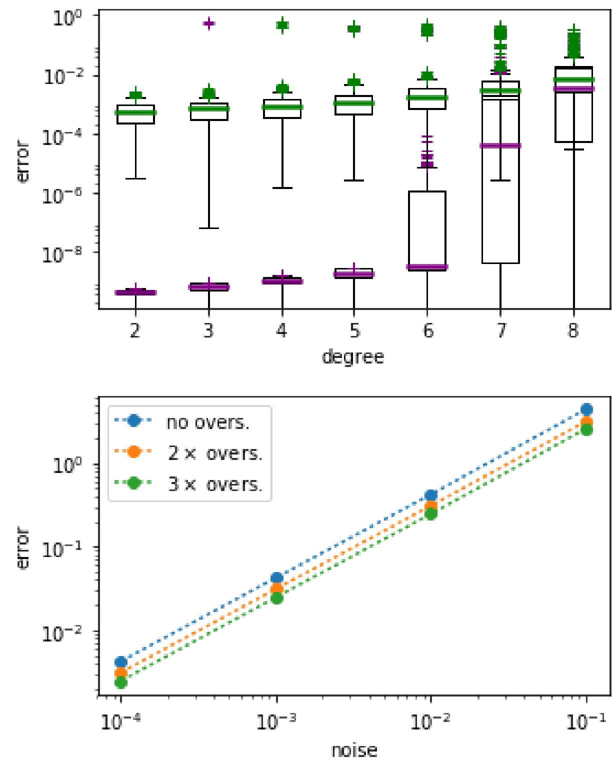

Fig. 5. Results of the ALS algorithm. The upper plot shows the dependence of the recovery error (in degrees) on the degree of the polynomial, for no noise (purple, lower error) and for noise with $\sigma = 10^{-3}$ (green). The points outside the boxes are in the 95th percentile. The lower plot shows the mean error for different amounts of noise. Here the degree of the polynomial is 3 and the following number of samples have been used: $8, 24$ and 64 samples.

can be achieved. We proved this for the case where the sample locations are governed by a rational function and we hope to extend our results to other transformations.

As shown in this work, rational mapping functions have applications in surface retrieval. However, to be practical, our framework must be extended to $2D$. We believe that this is possible and can envision higher dimensional theoretical results, similar to those developed in this paper, and iterative algorithms that jointly estimate a $2D$ surface and texture.

Furthermore, we believe that other mapping functions could be applicable in a wide range of applications including simultaneous localisation and mapping (SLAM). In the SLAM case, we expect the $1D$ case to be relatively simply but the extension to motion trajectories in higher dimensions could be far from straightforward.

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