ESAIM: COCV 25 (2019) 29
https://doi.org/10.1051/cocv/2017078

**ESAIM: Control, Optimisation and Calculus of Variations**
www.esaim-cocv.org

# CLOAKING USING COMPLEMENTARY MEDIA FOR ELECTROMAGNETIC WAVES

Hoai-Minh Nguyen*

**Abstract.** Negative index materials are artificial structures whose refractive index has negative value over some frequency range. The study of these materials has attracted a lot of attention in the scientific community not only because of their many potential interesting applications but also because of challenges in understanding their intriguing properties due to the sign-changing coefficients in equations describing their properties. In this paper, we establish cloaking using complementary media for electromagnetic waves. This confirms and extends the suggestions of Lai *et al.* [*Phys. Rev. Lett.* **102** (2009) 093901] for the full Maxwell equations. The analysis is based on the reflecting and removing localized singularity techniques, three-sphere inequalities, and the fact that the Maxwell equations can be reduced to a weakly coupled second order elliptic equations.

**Mathematics Subject Classification.** 35B34, 35B35, 35B40, 35J05, 78A25.

Received September 8, 2017. Accepted December 3, 2017.

## 1. Introduction

Negative index materials (NIMs) are artificial structures whose refractive index has negative value over some frequency range. These materials were investigated theoretically by Veselago in [33]. The existence of such materials was confirmed by Shelby *et al.* in [32]. The study of NIMs has attracted a lot of attention in the scientific community not only because of their many potential interesting applications but also because of challenges in understanding intriguing properties of these materials.

One of the interesting applications of NIMs is cloaking using complementary media, which was inspired by the concept of complementary media, see [14, 16, 23, 31]. Cloaking using complementary media was proposed and studied numerically by Lai *et al.* in [14] in two dimensions. The idea of this cloaking technique is to cancel the light effect of an object using its complementary media. Cloaking using complementary media was mathematically established in [20] for the quasistatic regime. The method used in [20] also works for the Helmholtz equation. Nevertheless, it requires small size of the cloaked region for large frequency due to the use of the (standard) three-sphere inequality. In [26], we gave a proof of cloaking using complementary media in the finite frequency regime for acoustic waves without imposing any condition on the size of the cloaked region. To successfully apply the approach in [20], we established a new three-sphere inequality for the Helmholtz equations which holds for arbitrary radii.

*Keywords and phrases:* Negative index materials, cloaking, complementary media, localized resonance, electromagnetic waves.

EPFL SB MATHAA CAMA, Station 8, CH-1015 Lausanne, Switzerland.

* Corresponding author: hoai-minh.nguyen@epfl.ch

Article published by EDP Sciences 

Another cloaking object technique using NIMs is cloaking an object *via* anomalous localized resonance technique. This was suggested and studied in [22]. Concerning this cloaking technique, an object is cloaked by the complementary property (or more precisely by the doubly complementary property) of the medium; hence the cloaking device is independent of cloaked objects. This cloaking technique is inspired by the work of Milton and Nicorovici in [15]. In their work, they discovered cloaking a source *via* anomalous localized resonance for constant radial plasmonic structures in the two-dimensional quasistatic regime (see [4, 8, 13, 17, 18, 24] for recent results in this direction). Another interesting application of NIMs is superlensing, *i.e.*, the possibility to beat the Rayleigh diffraction limit: no constraint between the size of the object and the wavelength is imposed, see [19, 23] and references therein.

Two difficulties in the study of cloaking using complementary media are as follows. Firstly, the problem is unstable. This can be explained by the fact that the equations describing the phenomena have sign-changing coefficients; hence the ellipticity and the compactness are lost in general. Secondly, the localized resonance might appear, *i.e.*, the field explodes in some regions and remains bounded in some others as the loss goes to 0. It is worthy noting that the character of resonance associated with NIMs is quite complex; localized resonance and complete resonance can occur in very similar settings, see [25].

In this paper, we study cloaking using complementary media for electromagnetic waves (Thm. 1.1). Let us now describe in details a scheme to cloak an arbitrary object using complementary media for the Maxwell equations. A more general class of schemes is considered in Section 4. Let $B_r$ denote the ball centered at the origin and of radius $r$ in $\mathbb{R}^3$ unless specified otherwise and let $\langle \cdot, \cdot, \rangle$ denote the Euclidean scalar product in $\mathbb{R}^3$. Assume that the cloaked region is the annulus $B_{2r_2} \setminus B_{r_2}$ in $\mathbb{R}^3$ for some $r_2 > 0$ in which the medium is characterized by a pair of two matrix-valued functions $(\varepsilon_O, \mu_O)$ of the permittivity $\varepsilon_O$ and the permeability $\mu_O$ of the region. The assumption on the cloaked region by all means imposes no restriction since any bounded set is a subset of such a region provided that the radius and the origin are appropriately chosen. We assume that $\varepsilon_O$ and $\mu_O$ are uniformly elliptic, *i.e.*,

$$\frac{1}{\Lambda}|\xi|^2 \le \langle \varepsilon_O(x)\xi, \xi \rangle \le \Lambda |\xi|^2 \text{ and } \frac{1}{\Lambda}|\xi|^2 \le \langle \mu_O(x)\xi, \xi \rangle \le \Lambda |\xi|^2 \quad \forall \xi \in \mathbb{R}^3, \text{ a.e. } x \in B_{r_2} \setminus B_{r_1}. \tag{1.1}$$

In this paper, we use schemes in the spirit of [20] with roots in the work of Lai *et al.* [14]. The cloak then contains two parts. The first one, in $B_{r_2} \setminus B_{r_1}$, makes use of complementary media to cancel the effect of the cloaked region and the second one, in $B_{r_1}$, is to fill the space which "disappears" from the cancellation by the homogeneous medium. Concerning the first part, instead of $B_{2r_2} \setminus B_{r_2}$, we consider $B_{r_3} \setminus B_{r_2}$ for some $r_3 > 0$ as the cloaked region in which the medium is given by

$$(\widetilde{\varepsilon}_O, \widetilde{\mu}_O) = \begin{cases} (\varepsilon_O, \mu_O) & \text{in } B_{2r_2} \setminus B_{r_2}, \\ (I, I) & \text{in } B_{r_3} \setminus B_{2r_2}. \end{cases} \tag{1.2}$$

The (reflecting) complementary medium in $B_{r_2} \setminus B_{r_1}$ is then given by

$$\big(F_*^{-1}\widetilde{\varepsilon}_O, F_*^{-1}\widetilde{\mu}_O\big), \tag{1.3}$$

where $F : B_{r_2} \setminus \bar{B}_{r_1} \to B_{r_3} \setminus \bar{B}_{r_2}$ is the Kelvin transform with respect to $\partial B_{r_2}$, *i.e.*,

$$F(x) = \frac{r_2^2}{|x|^2} x. \tag{1.4}$$

Here

$$\mathcal{T}_* a(y) = \frac{\nabla \mathcal{T}(x) a(x) \nabla \mathcal{T}^T(x)}{J(x)}, \tag{1.5}$$

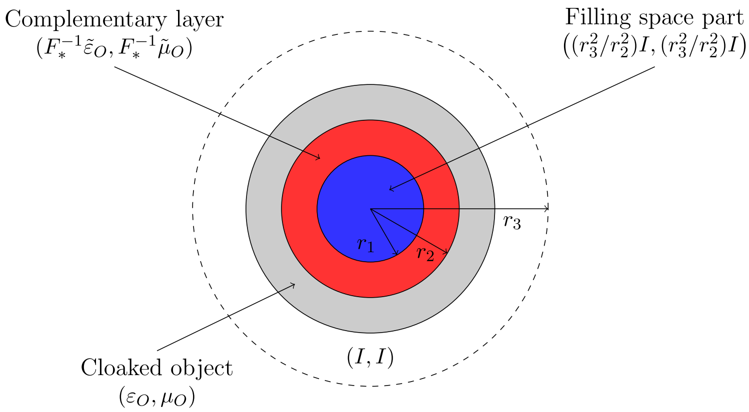


FIGURE 1. Cloaking scheme for an object $(\varepsilon_O, \mu_O)$ in $B_{2r_2} \setminus B_{r_2}$. Two parts are used: the complementary one in $B_{r_2} \setminus B_{r_1}$ (the red region) which is the complementary medium of the medium $(\tilde{\varepsilon}_O, \tilde{\mu}_O)$ in $B_{r_3} \setminus B_{r_2}$ and the filling space part in $B_{r_1}$ (the blue region). (Color online.)

where $x = \mathcal{T}^{-1}(y)$ and $J(x) = \det \nabla \mathcal{T}(x)$ for a diffeomorphism $\mathcal{T}$. It follows that

$$r_1 = r_2^2/r_3. \tag{1.6}$$

Note that in the definition of $T_*$ given in (1.5), $J(x) := \det \nabla T(x)$ **not** $|\det \nabla T(x)|$ as often used in the acoustic setting.[1] With this convention, one can easily verify that $F_*^{-1}\varepsilon$ and $F_*^{-1}\mu$ are negative symmetric matrices since $\det \nabla F(x) < 0$. This clarifies the point that one uses NIMs to construct a complementary medium for the cloaked object.

Concerning the second part, the medium in $B_{r_1}$ is given by

$$\big((r_3^2/r_2^2)I, (r_3^2/r_2^2)I\big). \tag{1.7}$$

Taking into account the loss, the medium in the whole space $\mathbb{R}^3$ is thus characterized by $(\varepsilon_\delta, \mu_\delta)$ defined as follows (see Fig. 1 for the case $\delta = 0$)

$$(\varepsilon_\delta, \mu_\delta) = \begin{cases} \big(\widetilde{\varepsilon}_O, \widetilde{\mu}_O\big) & \text{in } B_{r_3} \setminus B_{r_2}, \\ \big(F_*^{-1}\widetilde{\varepsilon}_O + i\delta I, F_*^{-1}\widetilde{\mu}_O + i\delta I\big) & \text{in } B_{r_2} \setminus B_{r_1}, \\ \big((r_3^2/r_2^2)I, (r_3^2/r_2^2)I\big) & \text{in } B_{r_1}, \\ \big(I, I\big) & \text{in } \mathbb{R}^3 \setminus B_{r_3}. \end{cases} \tag{1.8}$$

Physically, $\varepsilon_\delta$ and $\mu_\delta$ are the permittivity and permeability of the medium, $k$ denotes the frequency, and the imaginary parts of $\varepsilon_\delta$ and $\mu_\delta$ in $B_{r_2} \setminus B_{r_1}$ describe the dissipative property (the loss) of this (negative index) region.

[1] This convention is very suitable for the electromagnetic setting when a change of variables is used (see (2.43) of Lem. 2.8).

Given (a current) $j \in \big[L^2(\mathbb{R}^3)\big]^3$ with compact support, let $(E_\delta, H_\delta), (E, H) \in [H_{loc}(\operatorname{curl}, \mathbb{R}^3)]^2$ be respectively the unique outgoing solutions to the Maxwell systems

$$\begin{cases} \nabla \times E_\delta = ik\mu_\delta H_\delta & \text{in } \mathbb{R}^3 \\ \nabla \times H_\delta = -ik\varepsilon_\delta E_\delta + j & \text{in } \mathbb{R}^3, \end{cases} \tag{1.9}$$

and

$$\begin{cases} \nabla \times E = ikH & \text{in } \mathbb{R}^3 \\ \nabla \times H = -ikE + j & \text{in } \mathbb{R}^3. \end{cases} \tag{1.10}$$

For an open subset $\Omega$ of $\mathbb{R}^3$, the following standard notations are used:

$$H(\operatorname{curl}, \Omega) := \Big\{ u \in [L^2(\Omega)]^3;\ \nabla \times u \in [L^2(\Omega)]^3 \Big\},$$

$$\|u\|_{H(\operatorname{curl}, \Omega)} := \|u\|_{L^2(\Omega)} + \|\nabla \times u\|_{L^2(\Omega)},$$

and

$$H_{loc}(\operatorname{curl}, \Omega) := \Big\{ u \in [L^2_{loc}(\Omega)]^3;\ \nabla \times u \in [L^2_{loc}(\Omega)]^3 \Big\}.$$

Recall that a solution $(\mathcal{E}, \mathcal{H}) \in [H_{loc}(\operatorname{curl}, \mathbb{R}^3 \setminus B_R)]^2$ (for some $R > 0$) of the system

$$\begin{cases} \nabla \times \mathcal{E} = ik\mathcal{H} & \text{in } \mathbb{R}^3 \setminus B_R, \\ \nabla \times \mathcal{H} = -ik\mathcal{E} & \text{in } \mathbb{R}^3 \setminus B_R, \end{cases}$$

is said to satisfy the outgoing condition (or the Silver-Müller radiation condition) if

$$\mathcal{E} \times x + r\mathcal{H} = O(1/r), \tag{1.11}$$

as $r = |x| \to +\infty$.

We shall extend $(\widetilde{\varepsilon}_O, \widetilde{\mu}_O)$ by $(I, I)$ in $B_{r_2}$ and still denote this extension by $(\widetilde{\varepsilon}_O, \widetilde{\mu}_O)$. We also assume that

$$(\widetilde{\varepsilon}_O, \widetilde{\mu}_O) \text{ is } C^2 \text{ in } B_{r_3}. \tag{1.12}$$

Condition (1.12) is required for the use of the unique continuation principle and three-sphere inequalities for Maxwell equations.

Cloaking effect of scheme (1.8) (see Fig. 1) is mathematically confirmed in the following main result of this paper.

**Theorem 1.1.** *Let $R_0 > r_3$, $j \in \big[L^2(\mathbb{R}^3)\big]^3$ with $\operatorname{supp} j \subset\subset B_{R_0} \setminus B_{r_3}$ and let $(E_\delta, H_\delta), (E, H) \in \big[H_{loc}(\operatorname{curl}, \mathbb{R}^3)\big]^2$ be the unique outgoing solution to (1.9) and (1.10) respectively. Given $0 < \gamma < 1/2$, there* *exists a positive constant* $\ell = \ell(\gamma) > 0$*, depending* **only** *on the elliptic constant of* $\widetilde{\varepsilon}_O$ *and* $\widetilde{\mu}_O$ *in* $B_{2r_2} \setminus B_{r_2}$ *and* $\|(\widetilde{\varepsilon}_O, \widetilde{\mu}_O)\|_{W^{2,\infty}(B_{4r_2})}$ *such that if* $r_3 > \ell r_2$ *then*

$$\|(E_\delta, H_\delta) - (E, H)\|_{H(\mathrm{curl}, B_R \setminus B_{r_3})} \le C_R \delta^\gamma \|j\|_{L^2}, \tag{1.13}$$

*for some positive constant* $C_R$ *independent of* $j$ *and* $\delta$*.*

For an observer outside $B_{r_3}$, the medium in $B_{r_3}$ looks like the homogeneous one by (1.10): one has cloaking. The starting point of the proof of Theorem 1.1 is to use reflections (see (3.1) and (3.2)) to obtain Cauchy problems. We then explore the construction of the cloaking device (its complementary property), use various three-sphere inequalities (Lems. 2.5 and 2.7), and the removing localized singularity technique to deal with the localized resonance. Using reflections is also the starting point in the study of stability of Helmholtz equations with sign changing coefficients in [21] (see also [6, 12, 30] for different approaches) and also plays a role in the study of superlensing applications of hyperbolic metamaterials in [7]. A numercial algorithm used for NIMs in the spirit [21] is considered in [1]. Various techniques developed to study NIMs were explored in the context of interior transmission eigenvalues in [27]. The study of NIMs in time domain is recently investigated in [10, 28] and references therein.

The paper is organized as follows. The proof of Theorem 1.1 is given in Section 3 after presenting several useful results in Section 2. In Section 4, we present a class of cloaking schemes *via* the concept of reflecting complementary media.

## 2. Preliminaries

In this section, we present several results which are used in the proof of Theorem 1.1. We first recall a known result on the trace of $H(\mathrm{curl}, D)$ (see [3, 9]).

**Lemma 2.1.** *Let* $D$ *be a smooth open bounded subset of* $\mathbb{R}^3$ *and set* $\Gamma = \partial D$*. The tangential trace operator*

$$\begin{array}{rccc} \gamma_0 & : H(\mathrm{curl}, D) & \to & H^{-1/2}(\mathrm{div}_\Gamma, \Gamma) \\ & u & \mapsto & u \times \nu \end{array}$$

*is continuous. Moreover, for all* $\phi \in H^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$*, there exists* $u \in H(\mathrm{curl}, D)$ *such that*

$$\gamma_0(u) = \phi \quad \text{and} \quad \|u\|_{H(\mathrm{curl}, D)} \le C \|\phi\|_{H^{-1/2}(\mathrm{div}_\Gamma, \Gamma)},$$

*for some positive constant* $C$ *independent of* $\phi$*.*

Here

$$H^{-1/2}(\mathrm{div}_\Gamma, \Gamma) := \Big\{ \phi \in [H^{-1/2}(\Gamma)]^3;\ \phi \cdot \nu = 0 \text{ and } \mathrm{div}_\Gamma \phi \in H^{-1/2}(\Gamma) \Big\}$$

$$\|\phi\|_{H^{-1/2}(\mathrm{div}_\Gamma, \Gamma)} := \|\phi\|_{H^{-1/2}(\Gamma)} + \|\mathrm{div}_\Gamma \phi\|_{H^{-1/2}(\Gamma)}.$$

The next result implies the well-posedness and *a priori* estimates of $(E_\delta, H_\delta)$ defined in (1.9).

**Lemma 2.2.** *Let* $k > 0$*,* $0 < \delta < 1$*,* $R_0 > 0$*,* $D \subset B_{R_0}$ *be a smooth bounded open subset of* $\mathbb{R}^3$*. Let* $\varepsilon, \mu$ *be two* **real** *measurable matrix-valued functions defined in* $\mathbb{R}^3$ *such that* $\varepsilon, \mu$ *are uniformly elliptic and piecewise* $C^1$ *in* $\mathbb{R}^3$*, and*

$$\varepsilon = \mu = I \text{ in } \mathbb{R}^3 \setminus B_{R_0}. \tag{2.1}$$

*Set, for $\delta > 0$,*

$$(\varepsilon_\delta, \mu_\delta) = \begin{cases} (-\varepsilon + i\delta I, -\mu + i\delta I) & \textit{if } x \in D, \\ (\varepsilon, \mu) & \textit{otherwise.} \end{cases} \tag{2.2}$$

*Let $j \in L^2(\mathbb{R}^3)$ with $\operatorname{supp} j \subset B_{R_0}$. There exists a unique outgoing solution $(E_\delta, H_\delta) \in [H_{_{loc}}(\operatorname{curl}, \mathbb{R}^3)]^2$ to the Maxwell system*

$$\begin{cases} \nabla \times \mathcal{E}_\delta = ik\mu_\delta \mathcal{H}_\delta & \textit{in } \mathbb{R}^3, \\ \nabla \times \mathcal{H}_\delta = -ik\varepsilon_\delta \mathcal{E}_\delta + j & \textit{in } \mathbb{R}^3. \end{cases} \tag{2.3}$$

*Moreover,*

$$\|(\mathcal{E}_\delta, \mathcal{H}_\delta)\|^2_{H(\operatorname{curl}, B_R)} \le C_R \left( \frac{1}{\delta} \|j\|_{L^2} \|(\mathcal{E}_\delta, \mathcal{H}_\delta)\|_{L^2(\operatorname{supp} j)} + \|j\|^2_{L^2} \right). \tag{2.4}$$

*Here $C_R$ denotes a positive constant depending on $R$, $R_0$, $\varepsilon$, $\mu$ but independent of $j$ and $\delta$. Consequently, we have*

$$\|(\mathcal{E}_\delta, \mathcal{H}_\delta)\|_{H(\operatorname{curl}, B_R)} \le \frac{C_R}{\delta} \|j\|_{L^2}. \tag{2.5}$$

*Proof.* The existence of $(\mathcal{E}_\delta, \mathcal{H}_\delta)$ can be derived from the uniqueness of $(\mathcal{E}_\delta, \mathcal{H}_\delta)$ as usual. The uniqueness of $(\mathcal{E}_\delta, \mathcal{H}_\delta)$ can be deduced from the estimates of $(\mathcal{E}_\delta, \mathcal{H}_\delta)$. Estimate (2.5) is a direct consequence of (2.4). We hence only give the proof of (2.4). We have, by (2.3),

$$\nabla \times (\mu_\delta^{-1} \nabla \times \mathcal{E}_\delta) - k^2 \varepsilon_\delta \mathcal{E}_\delta = ikj \text{ in } \mathbb{R}^3.$$

Set

$$M_\delta = \frac{1}{\delta} \|j\|_{L^2} \|(\mathcal{E}_\delta, \mathcal{H}_\delta)\|_{L^2(\operatorname{supp} j)} + \|j\|^2_{L^2}.$$

We have

$$\nabla \times (\mu_\delta^{-1} \nabla \times \mathcal{E}_\delta) - k^2 \varepsilon_\delta \mathcal{E}_\delta = ikj \text{ in } \mathbb{R}^3. \tag{2.6}$$

Multiplying this equation by $\bar{\mathcal{E}}_\delta$ (the conjugate of $\mathcal{E}_\delta$), integrating in $B_R$, and using the fact that $\operatorname{supp} j \subset B_{R_0}$, we have, for $R > R_0$,

$$\int_{B_R} \langle \mu_\delta^{-1} \nabla \times \mathcal{E}_\delta, \nabla \times \mathcal{E}_\delta \rangle - \int_{\partial B_R} \big\langle (\mu_\delta^{-1} \nabla \times \mathcal{E}_\delta) \times \nu, \mathcal{E}_\delta \big\rangle - k^2 \int_{B_R} \langle \varepsilon_\delta \mathcal{E}_\delta, \mathcal{E}_\delta \rangle = \int_{B_R} \langle ikj, \mathcal{E}_\delta \rangle.$$

Since $\mu_\delta = I$ and so $\nabla \times \mathcal{E}_\delta = ik\mathcal{H}_\delta$ in $\mathbb{R}^3 \setminus B_{R_0}$, we derive that, for $R > R_0$,

$$\int_{B_R} \langle \mu_\delta^{-1} \nabla \times \mathcal{E}_\delta, \nabla \times \mathcal{E}_\delta \rangle + \int_{\partial B_R} \langle ik\mathcal{H}_\delta, \mathcal{E}_\delta \times \nu \rangle - k^2 \int_{B_R} \langle \varepsilon_\delta \mathcal{E}_\delta, \mathcal{E}_\delta \rangle = \int_{B_R} \langle ikj, \mathcal{E}_\delta \rangle.$$

Letting $R \to +\infty$, using the outgoing condition ($\mathcal{E}_\delta(x) \times \nu(x) = -\mathcal{H}_\delta(x) + O(1/R^2)$ for $x \in \partial B_R$), and considering the imaginary part, we obtain

$$\|\mathcal{E}_\delta\|^2_{H(\mathrm{curl}, D)} \le C M_\delta. \tag{2.7}$$

This implies, by Lemma 2.1, with the notation $\Gamma = \partial D$,

$$\|\mathcal{E}_\delta \times \nu\|^2_{H^{-1/2}(\mathrm{div}_\Gamma, \Gamma)} \le C M_\delta. \tag{2.8}$$

Using the equations of $(\mathcal{E}_\delta, \mathcal{H}_\delta)$ in $D$, we derive from (2.7) that

$$\|\mathcal{H}_\delta\|^2_{H(\mathrm{curl}, D)} \le C M_\delta; \tag{2.9}$$

which yields, by Lemma 2.1 again,

$$\|\mathcal{H}_\delta \times \nu\|^2_{H^{-1/2}(\mathrm{div}_\Gamma, \Gamma)} \le C M_\delta. \tag{2.10}$$

Let $D_1^c$ be the unbounded connected component of $\mathbb{R}^3 \setminus \bar{D}$ and let $D_2^c$ be the complement of $D_1^c$ in $\mathbb{R}^3 \setminus \bar{D}$, *i.e.*, $D_2^c = (\mathbb{R}^3 \setminus \bar{D}) \setminus D_1^c$.[2] We have

$$\begin{cases} \nabla \times \mathcal{E}_\delta = ik\mu \mathcal{H}_\delta & \text{in } D_1^c, \\ \nabla \times \mathcal{H}_\delta = -ik\varepsilon \mathcal{E}_\delta + j & \text{in } D_1^c. \end{cases}$$

It follows that

$$\|(\mathcal{E}_\delta, \mathcal{H}_\delta)\|_{H(\mathrm{curl}, B_R \cap D_1^c)} \le C_R \big( \|j\|_{L^2} + \|\mathcal{E}_\delta \times \nu\|_{H^{-1/2}(\mathrm{div}_{\partial D_1^c}, \partial D_1^c)} \big).$$

We deduce from (2.8) that

$$\|(\mathcal{E}_\delta, \mathcal{H}_\delta)\|^2_{H(\mathrm{curl}, B_R \cap D_1^c)} \le C_R M_\delta, \tag{2.11}$$

and, by Lemma 2.3 below, we derive from (2.8) and (2.10) that

$$\|(\mathcal{E}_\delta, \mathcal{H}_\delta)\|^2_{H(\mathrm{curl}, D_2^c)} \le C M_\delta. \tag{2.12}$$

A combination of (2.7), (2.9), (2.11), and (2.12) yields

$$\|(\mathcal{E}_\delta, \mathcal{H}_\delta)\|_{H(\mathrm{curl}, B_R)} \le C_R M_\delta; \tag{2.13}$$

which is (2.4). □

In the proof of Lemma 2.2 we use the following result whose proof follows directly from the unique continuation principle for the Maxwell equations (see *e.g.*, [2, 5, 29]) *via* a contradiction argument.

[2] We will apply Lemma 2.2 with $D = B_{r_2} \setminus B_{r_1}$; in this case $D_1^c = \mathbb{R}^3 \setminus \bar{B}_{r_2}$ and $D_2^c = B_{r_1}$.

**Lemma 2.3.** *Let $k > 0$, $D$ be a smooth bounded open subset of $\mathbb{R}^3$, $f, g \in [L^2(D)]^3$, and $h_1, h_2 \in H^{-1/2}(\operatorname{div}_{\partial D}, \partial D)$, and let $\varepsilon$ and $\mu$ be two piecewise $C^1$, symmetric uniformly elliptic matrix-valued functions defined in $D$. Assume that $(\mathcal{E}, \mathcal{H}) \in [H(\operatorname{curl}, D)]^2$ is a solution to*

$$\begin{cases} \nabla \times \mathcal{E} = ik\mu\mathcal{H} + f & \text{in } D, \\ \nabla \times \mathcal{H} = -ik\varepsilon\mathcal{E} + g & \text{in } D, \\ \mathcal{H} \times \nu = h_1;\ \mathcal{E} \times \nu = h_2 & \text{on } \partial D. \end{cases}$$

*Then*

$$\|(\mathcal{E}, \mathcal{H})\|_{H(\operatorname{curl}, D)} \le C\Big(\|(f, g)\|_{L^2(D)} + \|(h_1, h_2)\|_{H^{-1/2}(\operatorname{div}_\Gamma, \partial D)}\Big), \tag{2.14}$$

*for some positive constant $C$ depending on $D$, $\varepsilon$, $\mu$, and $k$ but independent of $f$, $g$, $h_1$, and $h_2$.*

We next present a known result which reveals a connection between Maxwell equations with weakly coupled elliptic systems.

**Lemma 2.4.** *Let $D$ be an open subset of $\mathbb{R}^3$, $\varepsilon, \mu$ be two matrix-valued functions defined in $D$, and let $(\mathcal{E}, \mathcal{H}) \in \big[H^1(D)\big]^2$ be a solution of the system*

$$\begin{cases} \nabla \times \mathcal{E} = ik\mu\mathcal{H} & \text{in } D, \\ \nabla \times \mathcal{H} = -ik\varepsilon\mathcal{E} & \text{in } D. \end{cases} \tag{2.15}$$

*Then, for $1 \le a \le 3$,*

$$\operatorname{div}(\mu\nabla\mathcal{H}_a) + \operatorname{div}(\partial_a\mu\mathcal{H} - ik\mu\epsilon^a\varepsilon\mathcal{E}) = 0 \text{ in } D, \tag{2.16}$$

$$\operatorname{div}(\varepsilon\nabla\mathcal{E}_a) + \operatorname{div}(\partial_a\varepsilon\mathcal{E} + ik\varepsilon\epsilon^a\mu\mathcal{H}) = 0 \text{ in } D. \tag{2.17}$$

*Here the $bc$ component $\epsilon^a_{bc}$ $(1 \le b, c \le 3)$ of $\epsilon^a$ $(1 \le a \le 3)$ denotes the usual Levi Civita permutation, i.e.,*

$$\epsilon^a_{bc} = \begin{cases} \textit{sign } (abc) & \text{if } abc \text{ is a permuation}, \\ 0 & \text{otherwise}. \end{cases} \tag{2.18}$$

*Proof.* The proof is quite simple as follows. Using the fact, for $1 \le a \le 3$,

$$\partial_a\mathcal{H} = \nabla\mathcal{H}_a + \epsilon^a(\nabla \times \mathcal{H}) \quad \text{and} \quad \partial_a\mathcal{E} = \nabla\mathcal{E}_a + \epsilon^a(\nabla \times \mathcal{E}),$$

we derive from (2.15) that, for $1 \le a \le 3$,

$$\partial_a\mathcal{H} = \nabla\mathcal{H}_a - ik\epsilon^a\varepsilon\mathcal{E} \quad \text{and} \quad \partial_a\mathcal{E} = \nabla\mathcal{E}_a + ik\epsilon^a\mu\mathcal{H} \text{ in } D. \tag{2.19}$$

Since

$$\operatorname{div}(\mu\mathcal{H}) = 0 \text{ in } D,$$

it follows that, for $1 \le a \le 3$,

$$0 = \partial_a \operatorname{div}(\mu \mathcal{H}) = \operatorname{div}(\mu \partial_a \mathcal{H}) + \operatorname{div}(\partial_a \mu \mathcal{H}) \text{ in } D.$$

This implies, by the first identity of (2.19),

$$\operatorname{div}(\mu \nabla \mathcal{H}_a) + \operatorname{div}(\partial_a \mu \mathcal{H} - ik\mu \epsilon^a \varepsilon \mathcal{E}) = 0 \text{ in } D;$$

which is (2.16). Similarly, we obtain (2.17). □

Hadamard proved the following three-circle inequality: Assume that $\Delta v = 0$ in $B_{R^*} \setminus B_{R_*} \subset \mathbb{R}^2$ and $0 < R_* < R_1 < R_2 < R_3 < R^*$. Then

$$\|v\|_{L^\infty(\partial B_{R_2})} \le \|v\|^{\alpha}_{L^\infty(\partial B_{R_1})} \|v\|^{1-\alpha}_{L^\infty(\partial B_{R_3})},$$

with $\alpha = \ln(R_3/R_2)/\ln(R_3/R_1)$. Here is its variant which is used in the proof of Theorem 1.1.

**Lemma 2.5.** *Let $d = 2, 3$, $k$, $R_*$, $R^* > 0$, and let $v \in H^1(B_{R_*} \setminus B_{R^*})$ be a solution to the equation $\Delta v + k^2 v = 0$ in $B_{R_3} \setminus B_{R_1} \subset \mathbb{R}^d$. We have, for $R_* \le R_1 < R_2 < R_3 \le R^*$,*

$$\|v\|_{\mathbf{H}(\partial B_{R_2})} \le C \|v\|^{\alpha}_{\mathbf{H}(B_{R_1})} \|v\|^{1-\alpha}_{\mathbf{H}(B_{R_3})}, \tag{2.20}$$

*where $\alpha = \ln(R_3/R_2)/\ln(R_3/R_1)$ and $C$ is a positive constant depending only on $k$, $R_*$, and $R^*$. Here*

$$\|v\|_{\mathbf{H}(\partial B_r)} := \|v\|_{H^{1/2}(\partial B_r)} + \|\partial_r v\|_{H^{-1/2}(\partial B_r)}. \tag{2.21}$$

**Remark 2.6.** Note that in the case $k \neq 0$, one must use both the information of $v$ and its normal derivative in (2.21); otherwise the conclusion does not hold in general, see [26] for a discussion on this matter.

Before giving the proof of Lemma 2.5, we recall some properties of the spherical Bessel and Neumann functions and the Bessel and Neumann functions of large order. We first introduce, for $n \ge 1$,

$$\hat{j}_n(t) = 1 \cdot 3 \cdots (2n+1) j_n(t) \quad \text{and} \quad \hat{y}_n = -\frac{y_n(t)}{1 \cdot 3 \cdots (2n-1)}, \tag{2.22}$$

and for $n \ge 0$,

$$\hat{J}_n(r) = 2^n n! J_n(r) \quad \text{and} \quad \hat{Y}_n(r) = \frac{\pi i}{2^n (n-1)!} Y_n(r), \tag{2.23}$$

where $j_n$ and $y_n$ are the spherical Bessel and Neumann functions, and $J_n$ and $Y_n$ are the Bessel and Neumann functions of order $n$ respectively. Then, see, *e.g.*, [11], (2.37), (2.38), (3.57), and (3.58), as $n \to +\infty$,

$$\hat{j}_n(r) = r^n \big[1 + O(1/n)\big], \quad \hat{y}_n(r) = r^{-n-1} \big[1 + O(1/n)\big], \tag{2.24}$$

$$\hat{J}_n(t) = t^n \big[1 + O(1/n)\big], \quad \text{and} \quad \hat{Y}_n(t) = t^{-n} \big[1 + O(1/n)\big]. \tag{2.25}$$

One also has, see, *e.g.*, ([11], (2.36) and (3.56)),

$$j_n(r)y_n'(r) - j_n'(r)y_n(r) = \frac{1}{r^2} \tag{2.26}$$

and

$$J_n(r)Y_n'(r) - J_n'(r)Y_n(r) = \frac{2}{\pi r}. \tag{2.27}$$

We are ready to give:

*Proof of Lemma 2.5.* By rescaling, one can assume that $k = 1$. We consider the case $d = 2$ and $d = 3$ separately.

**Case 1: d=3.** Since $\Delta v + v = 0$ in $B_{R_3} \setminus B_{R_1}$, $v$ can be represented in the form

$$v = \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \Big(a_m^n \hat{j}_n(|x|) + b_m^n \hat{y}_n(|x|)\Big) Y_m^n(\hat{x}) \quad \text{ in } B_{R_3} \setminus B_{R_1},$$

for $a_m^n \in \mathbb{C}$ and $\hat{x} = x/|x|$ where $Y_m^n$ is the spherical harmonic function of degree $n$ and of order $m$. In what follows in this proof, $C$ denotes a positive constant depending only on $R_*$ and $R^*$ and can change from one place to another and $a \sim b$ means that $a \le Cb$ and $b \le Ca$. Using the fact $(Y_m^n)$ is an orthonormal basis of $L^2(\partial B_1)$ and

$$\mathit{\Delta}_{\partial B_1} Y_m^n + n(n+1) Y_m^n = 0 \text{ on } \partial B_1,$$

we derive that, for $R_1 \le r \le R_3$,

$$\|v\|^2_{\mathbf{H}(\partial B_r)} \sim \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \big(n|c_m^n(r)|^2 + n^{-1}|d_m^n(r)|^2\big), \tag{2.28}$$

where

$$c_m^n(r) = a_m^n \hat{j}_n(r) + b_m^n \hat{y}_n(r) \quad \text{ and } \quad d_m^n(r) = a_m^n \hat{j}_n'(r) + b_m^n \hat{y}_n'(r). \tag{2.29}$$

From (2.28) and (2.29), we have

$$\|v\|^2_{\mathbf{H}(\partial B_r)} \le C \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \Big( |a_m^n|^2 \big(n|\hat{j}_n(r)|^2 + n^{-1}|\hat{j}_n'(r)|^2\big) + |b_m^n|^2 \big(n|\hat{y}_n(r)|^2 + n^{-1}|\hat{y}_n'(r)|^2\big)\Big);$$

which yields, by (2.24),

$$\|v\|^2_{\mathbf{H}(\partial B_r)} \le C \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \big(n r^{2n} |a_m^n|^2 + n r^{-2n} |b_m^n|^2\big). \tag{2.30}$$

From (2.29), we have

$$a_m^n = \frac{c_m^n(r)\hat{y}_n'(r) - d_m^n(r)\hat{y}_n(r)}{\hat{j}_n(r)\hat{y}_n'(r) - \hat{j}_n'(r)\hat{y}_n(r)} \quad \text{ and } \quad b_m^n = \frac{c_m^n(r)\hat{j}_n'(r) - d_m^n(r)\hat{j}_n(r)}{\hat{y}_n(r)\hat{j}_n'(r) - \hat{y}_n'(r)\hat{j}_n(r)}. \tag{2.31}$$

From (2.26), we obtain, for some $c_n \neq 0$,

$$\hat{y}_n(r)\hat{j}'_n(r) - \hat{y}'_n(r)\hat{j}_n(r) = \frac{c_n}{t^2}. \tag{2.32}$$

Combining (2.24), (2.31), and (2.32) yields

$$|a^n_m| \le C\big(|c^n_m| r^{-n} + n^{-1}|d^n_m| r^{-n}\big) \quad \text{and} \quad |b^n_m| \le C\big(|c^n_m| r^{n} + n^{-1}|d^n_m| r^{n}\big). \tag{2.33}$$

We derive from (2.28) and (2.33) that

$$\sum_{n=1}^{\infty}\sum_{m=-n}^{n}\big(n r^{2n}|a^n_m|^2 + n r^{-2n}|b^n_m|^2\big) \le C\|v\|^2_{\mathbf{H}(\partial B_r)}. \tag{2.34}$$

A combination of (2.30) and (2.34) yields

$$\|v\|^2_{\mathbf{H}(\partial B_r)} \sim \sum_{n=1}^{\infty}\sum_{m=-n}^{n}\big(n r^{2n}|a^n_m|^2 + n r^{-2n}|b^n_m|^2\big). \tag{2.35}$$

Inequality (2.20) is now a consequence of (2.35) after applying Hölder's inequality and noting that $R_2 = R_1^{\alpha} R_3^{1-\alpha}$.

**Case 2: d=2.** Since $\Delta v + v = 0$ in $B_{R^*} \setminus B_{R_*}$, one can represent $v$ of the form

$$v = \sum_{n=0}^{\infty}\sum_{\pm}\big(a_{n,\pm}\hat{J}_n(|x|) + b_{n,\pm}\hat{Y}_n(|x|)\big)e^{\pm in\theta} \text{ in } B_{R^*} \setminus B_{R_*},$$

with the convention $a_{0,-} = a_{0,+}$ and $b_{0,-} = b_{0,+}$. Using (2.25) and (2.27), as in the previous case, one can prove that

$$\|v\|^2_{\mathbf{H}(\partial B_r)} \sim \sum_{n=0}^{\infty}\sum_{\pm}\big(n r^{2n}|a_{n,\pm}|^2 + n^{-1} r^{-2n}|b_{n,\pm}|^2\big). \tag{2.36}$$

Inequality (2.20) is now a consequence of (2.35) after applying Hölder's inequality and noting that $R_2 = R_1^{\alpha} R_3^{1-\alpha}$. □

We next state a three-sphere inequality for an "elliptic system".

**Lemma 2.7.** *Let $m, n \in \mathbb{N}$ ($m \ge 2$, $n \ge 1$), $0 < R_* < R_1 < R_2 < R_3 < R^*$, $c_1, c_2 > 0$ and let $M^1, \cdots, M^n$ be such that $M^k$ is an $(m \times m)$ matrix defined in $B_{R^*} \subset \mathbb{R}^m$ for $1 \le k \le n$.*[3] *Assume that $M^k$ is Lipschitz and uniformly elliptic in $B_{R^*}$ for $1 \le k \le n$ and $V \in \big[H^1(B_{R_3} \setminus \bar{B}_{R_1})\big]^n$ satisfies*

$$|\operatorname{div}(M^k \nabla V_k)| \le c_1|\nabla V| + c_2|V| \text{ a.e. in } B_{R_3} \setminus \bar{B}_{R_1} \text{ for } 1 \le k \le n. \tag{2.37}$$

*There exists a constant $q \ge 1$, depending only on $m$, $n$, and the elliptic and the Lipschitz constants of $M^k$ for $1 \le k \le n$ such that, for any $\lambda_0 > 1$ and $R_2 \in (\lambda_0 R_1, R_3/\lambda_0)$, we have*

$$\|V\|_{\mathbf{H}(\partial B_{R_2})} \le C\|V\|^{\alpha}_{\mathbf{H}(\partial B_{R_1})}\|V\|^{1-\alpha}_{\mathbf{H}(\partial B_{R_3})} \quad \text{where} \quad \alpha := \frac{R_2^{-q} - R_3^{-q}}{R_1^{-q} - R_3^{-q}}, \tag{2.38}$$

[3] In this lemma, $B_r$ denotes the ball centered at the origin with radius $r$ in $\mathbb{R}^m$.

*and*

$$\|V_k\|_{\mathbf{H}(\partial B_r)} = \|V_k\|_{H^{1/2}(\partial B_r)} + \|M_k \nabla V_k \cdot e_r\|_{H^{-1/2}(\partial B_r)}, \quad \|V\|_{\mathbf{H}(\partial B_r)} = \sum_{k=1}^{n} \|V_k\|_{\mathbf{H}(\partial B_r)}. \tag{2.39}$$

*Here $C$ is a positive constant depends on the elliptic and the Lipschitz constants of $M^k$ $(1 \le k \le n)$, $c_1$, $c_2$, $R_*, R^*, m, n$, and $\lambda_0$ but independent of $v$.*

In inequality (2.38), the constant $q$ does **not** depend on $c_1$, $c_2, R_*, R^*$ but the constant $C$ **does**. No upper bound on $R^*$ is imposed as often required in a three-sphere inequality for Helmholtz equations (see *e.g.*, Thm. 4.1 from [2]). Nevertheless, both information of $V$ and $M\nabla V \cdot e_r$ are used (2.39); this is the key point to ensure that (2.38) holds without imposing any condition on $R^*$. Lemma 2.7 is proved in Theorem 2 from [26], for the case $n = 1$. The proof for the case $n \ge 1$ follows similar and is omitted.

We finally state a change of variables formula

**Lemma 2.8.** *Let $D, D'$ be two bounded connected open subsets of $\mathbb{R}^3$ and $\mathcal{T} : D \to D'$ be bijective such that $\mathcal{T} \in C^1(\bar{D})$ and $\mathcal{T}^{-1} \in C^1(\bar{D}')$. Assume that $\varepsilon, \mu \in [L^\infty(D)]^{3\times 3}$, $j \in [L^2(D)]^3$ and $(E, H) \in [H(\operatorname{curl}, D)]^2$ is a solution to*

$$\begin{cases} \nabla \times E = ik\mu H & \text{in } D, \\ \nabla \times H = -ik\varepsilon E + j & \text{in } D. \end{cases}$$

*Define $(E', H')$ in $D'$ as follows*

$$E'(x') = \mathcal{T} * E(x') := \nabla \mathcal{T}^{-T}(x) E(x) \text{ and } H'(x') = \mathcal{T} * H(x') := \nabla \mathcal{T}^{-T}(x) H(x), \tag{2.40}$$

*with $x' = \mathcal{T}(x)$. Then $(E', H')$ is a solution to*

$$\begin{cases} \nabla' \times E' = ik\mu' H' & \text{in } D', \\ \nabla' \times H' = -ik\varepsilon' E' + j' & \text{in } D', \end{cases} \tag{2.41}$$

*where*

$$\varepsilon' = \mathcal{T}_* \varepsilon, \quad \mu' = \mathcal{T}_* \mu, \quad j' = T_* j,$$

*and*

$$\mathcal{T}_*\varepsilon(x') = \frac{\nabla \mathcal{T}(x) \varepsilon(x) \nabla \mathcal{T}^T(x)}{J(x)}, \quad \mathcal{T}_*\mu(x') = \frac{\nabla \mathcal{T}(x) \mu(x) \nabla \mathcal{T}^T(x)}{J(x)}, \quad \text{and} \quad \mathcal{T}_* j(x') = \frac{j(x)}{J(x)}, \tag{2.42}$$

*with $x = \mathcal{T}^{-1}(x')$ and $J(x) = \det \nabla \mathcal{T}(x)$. Assume in addition that $D$ is of class $C^1$ and $\mathbf{T} = \mathcal{T}\big|_{\partial D} : \partial D \to \partial D'$ is a diffeomorphism. We have*[4]

$$\text{if } E \times \nu = g \text{ and } H \times \nu = h \text{ on } \partial D \text{ then } E' \times \nu' = \mathbf{T}_* g \text{ and } H' \times \nu' = \mathbf{T}_* h \text{ on } \partial D', \tag{2.43}$$

*where $\mathbf{T}_*$ is given in (2.44).*

[4] Here $\nu$ and $\nu'$ denote the outward unit normal vector on $\partial D$ and $\partial D'$.

For a tangential vector field $g$ defined in $\partial D$, we denote

$$\mathbf{T}_* g(x') = \text{sign } \cdot \frac{\nabla_{\partial D}\mathbf{T}(x) g(x)}{|\det \nabla_{\partial D}\mathbf{T}(x)|} \quad \text{with} \quad x' = \mathbf{T}(x), \tag{2.44}$$

where sign $:= \det \nabla \mathcal{T}(x)/|\det \nabla \mathcal{T}(x)|$ for some $x \in D$.

**Remark 2.9.** In the change of variables, the definition of $\mathcal{T}*$ in (2.40) is different from $\mathcal{T}_*$ in (2.42). It is worthy remembering that for electromagnetic fields (2.40) is used whereas for sources, (1.5) is involved. In the proof of Theorem 1.1, we use both (2.41) and (2.43). Assertion (2.41) is known and used in the cloaking *via* a change of variables technique, assertion (2.43) is less known – see *e.g.*, Lemma 7 from [23].

## 3. Proof of Theorem 1.1

Let $(E_\delta^{(1)}, H_\delta^{(1)}) \in \big[H^1_{loc}(\operatorname{curl}, \mathbb{R}^3 \setminus B_{r_2})\big]^2$ be the reflection of $(E_\delta, H_\delta)$ through $\partial B_{r_2}$ by the Kelvin transform $F$ with respect to $\partial B_{r_2}$, *i.e.*,

$$\big(E_\delta^{(1)}, H_\delta^{(1)}\big) = \big(F * E_\delta, F * H_\delta\big) \text{ in } \mathbb{R}^3 \setminus B_{r_2}, \tag{3.1}$$

where $F*$ is defined by (2.40). Let $\big(E_\delta^{(2)}, H_\delta^{(2)}\big) \in \big[H(\operatorname{curl}, B_{r_3})\big]^2$ be the reflection of $(E_\delta^{(1)}, H_\delta^{(1)})$ through $\partial B_{r_3}$ by the Kelvin transform $G : \mathbb{R}^3 \setminus B_{r_3} \mapsto B_{r_3}$ with respect to $\partial B_{r_3}$., *i.e.*, $G(x) = r_3^2 x/|x|^2$ and

$$\big(E_\delta^{(2)}, H_\delta^{(2)}\big) = \big(G * E_\delta^{(1)}, G * H_\delta^{(1)}\big) \text{ in } B_{r_3}. \tag{3.2}$$

Since $G \circ F(x) = \big(r_3^2/r_2^2\big)x$ and $G_* F_* = (G \circ F)_*$, it follows from (1.5) and (1.7) that

$$(G_* F_* \varepsilon_\delta, G_* F_* \mu_\delta) = (G_* F_* \varepsilon_O, G_* F_* \mu_O) = (I, I) \text{ in } B_{r_3}. \tag{3.3}$$

Set

$$Data(j, \delta) := \left(\frac{1}{\delta}\|(E_\delta, H_\delta)\|_{L^2(B_{R_0}\setminus B_{r_3})}\|j\|_{L^2} + \|j\|^2_{L^2}\right)^{1/2}. \tag{3.4}$$

Applying Lemma 2.2 to $D = B_{r_2} \setminus B_{r_1}$, we have

$$\|(E_\delta, H_\delta)\|^2_{[L^2(B_{R_0})]^2} \le C Data(j, \delta)^2. \tag{3.5}$$

Here and in what follows in the proof, $C$ denotes a positive constant independent of $\delta$ and $j$ and the fact $\ell > 10$ is assumed.

The proof now is divided into two steps.

- Step 1: We prove that if $\ell$ is large enough then

$$\|(E_\delta^{(1)} - E_\delta) \times \nu, (H_\delta^{(1)} - H_\delta) \times \nu\|_{H^{-1/2}(\operatorname{div}_\Gamma, \partial B_{2r_2})} \le C\delta^{\gamma + 1/2} Data(j, \delta). \tag{3.6}$$

- Step 2: Define

$$(\mathcal{E}_\delta, \mathcal{H}_\delta) = \begin{cases} (E_\delta, H_\delta) & \text{in } \mathbb{R}^3 \setminus B_{r_3}, \\ (E_\delta, H_\delta) - \big(E_\delta^{(1)} - E_\delta^{(2)}, H_\delta^{(1)} - H_\delta^{(2)}\big) & \text{in } B_{r_3} \setminus B_{2r_2}, \\ \big(E_\delta^{(2)}, H_\delta^{(2)}\big) & \text{in } B_{2r_2}. \end{cases}$$

We prove that if (3.6) holds then

$$\|(\mathcal{E}_\delta, \mathcal{H}_\delta) - (E, H)\|_{L^2(B_R \setminus B_{r_3})} \le C\delta^\gamma \|j\|_{L^2}. \tag{3.7}$$

It is clear that the conclusion follows after Steps 1 and 2.

Step 1: Using the fact that

$$ikF_*^{-1}\widetilde{\mu}_O + i\delta I = ik\big(F_*^{-1}\widetilde{\mu}_O + (\delta/k)F_*^{-1}F_*I\big) \text{ in } B_{r_2} \setminus B_{r_1}$$

and

$$-ikF_*^{-1}\widetilde{\varepsilon}_O + i\delta I = -ik\big(F_*^{-1}\widetilde{\varepsilon}_O - (\delta/k)F_*^{-1}F_*I\big) \text{ in } B_{r_2} \setminus B_{r_1},$$

and applying Lemma 2.8, we have

$$\begin{cases} \nabla \times E_\delta^{(1)} = ik\widetilde{\mu}_O H_\delta^{(1)} + i\delta F_* I H_\delta^{(1)} & \text{in } B_{r_3} \setminus B_{r_2}, \\ \nabla \times H_\delta^{(1)} = -ik\widetilde{\varepsilon}_O E_\delta^{(1)} + i\delta F_* I E_\delta^{(1)} & \text{in } B_{r_3} \setminus B_{r_2}, \end{cases} \tag{3.8}$$

and

$$\big(E_\delta^{(1)} \times \nu, H_\delta^{(1)} \times \nu\big) = \big(E_\delta \times \nu, H_\delta \times \nu\big)\Big|_{\text{ext}} \text{ on } \partial B_{r_2}. \tag{3.9}$$

In (3.9), we use the fact that $F(x) = x$ on $\partial B_{r_2}$. Set

$$(\varepsilon, \mu) = \begin{cases} (\widetilde{\varepsilon}_O, \widetilde{\mu}_O) & \text{in } B_{r_3} \setminus B_{r_2}, \\ (I, I) & \text{otherwise.} \end{cases}$$

Let $\big(\mathbf{E}_\delta^{(1)}, \mathbf{H}_\delta^{(1)}\big) \in [H_{_{loc}}(\mathbb{R}^3)]^2$ be the unique outgoing solution to

$$\begin{cases} \nabla \times \mathbf{E}_\delta^{(1)} = ik\mu \mathbf{H}_\delta^{(1)} + i\delta \mathbb{1}_{B_{r_3} \setminus B_{r_2}} F_* I H_\delta^{(1)} & \text{in } \mathbb{R}^3, \\ \nabla \times \mathbf{H}_\delta^{(1)} = -ik\varepsilon \mathbf{E}_\delta^{(1)} + i\delta \mathbb{1}_{B_{r_3} \setminus B_{r_2}} F_* I E_\delta^{(1)} & \text{in } \mathbb{R}^3. \end{cases} \tag{3.10}$$

Here $\mathbb{1}_D$ denotes the characteristic function of a subset $D$ of $\mathbb{R}^3$. Note that $\varepsilon, \mu$ are uniformly elliptic. Using (3.5), we can derive from (3.10) (see [23], Lem. 4) that

$$\|\big(\mathbf{E}_\delta^{(1)}, \mathbf{H}_\delta^{(1)}\big)\|_{H(\text{curl}, B_{r_3} \setminus B_{r_2})} \le C\delta Data(j, \delta). \tag{3.11}$$

Set

$$(\widetilde{E}_\delta, \widetilde{H}_\delta) = \begin{cases} \Big(E_\delta^{(1)} - E_\delta - \mathbf{E}_\delta^{(1)}, E_\delta^{(1)} - H_\delta - \mathbf{H}_\delta^{(1)}\Big) & \text{in } B_{r_3} \setminus B_{r_2}, \\ \Big(-\mathbf{E}_\delta^{(1)}, -\mathbf{H}_\delta^{(1)}\Big) & \text{in } B_{r_2}. \end{cases}$$

It follows from (3.8) and (3.10) that

$$\begin{cases} \nabla \times \widetilde{E}_\delta = ik\mu \widetilde{H}_\delta & \text{in } B_{r_3}, \\ \nabla \times \widetilde{H}_\delta = -ik\varepsilon \widetilde{E}_\delta & \text{in } B_{r_3}. \end{cases} \tag{3.12}$$

Applying Lemma 2.4, we have, for $1 \le a \le 3$,

$$\operatorname{div}\big(\varepsilon \nabla \widetilde{E}_{\delta,a}\big) = -\operatorname{div}\big(\partial_a \varepsilon \widetilde{E}_\delta + ik\varepsilon \epsilon^a \mu \widetilde{H}_\delta\big) \quad \text{in } B_{r_3}$$

and

$$\operatorname{div}\big(\mu \nabla \widetilde{H}_{\delta,a}\big) = -\operatorname{div}\big(\partial_a \mu \widetilde{H}_\delta - ik\mu \epsilon^a \varepsilon_\delta \widetilde{E}_\delta\big) \quad \text{in } B_{r_3},$$

where $\epsilon^a_{bc}$ $(1 \le a, b, c \le 3)$ denote the usual Levi Civita permutation, see (2.18). Let $q$ be the constant in Lemma 2.7 with $m = 3$, $n = 6$, $M^1 = M^2 = M^3 = \varepsilon$, and $M^4 = M^5 = M^6 = \mu$. Define, for $0 < r \le r_3$,

$$\|\widetilde{E}\|_{\mathbf{H}(\partial B_r)} = \|\widetilde{E}\|_{H^{1/2}(\partial B_r)} + \|\varepsilon \nabla \widetilde{E} \cdot e_r\|_{H^{-1/2}(\partial B_r)},$$

and

$$\|\widetilde{H}\|_{\mathbf{H}(\partial B_r)} = \|\widetilde{H}\|_{H^{1/2}(\partial B_r)} + \|\mu \nabla \widetilde{H} \cdot e_r\|_{H^{-1/2}(\partial B_r)}.$$

By Lemma 2.7, there exists some positive constant $C$ independent of $\delta$ such that

$$\|\big(\widetilde{E}_\delta, \widetilde{H}_\delta\big)\|_{\mathbf{H}(\partial B_{2r_2})} \le C \|\big(\widetilde{E}_\delta, \widetilde{H}_\delta\big)\|^{\alpha}_{\mathbf{H}(\partial B_{r_2/2})} \|\big(\widetilde{E}_\delta, \widetilde{H}_\delta\big)\|^{1-\alpha}_{\mathbf{H}(\partial B_{4r_2})}, \tag{3.13}$$

with

$$\alpha = \frac{(2r_2)^{-q} - (4r_2)^{-q}}{(r_2/2)^{-q} - (4r_2)^{-q}} = \frac{2^{-q} - 4^{-q}}{2^{q} - 4^{-q}}. \tag{3.14}$$

Since $\varepsilon = \mu = I$ in $B_{r_2} \cup (B_{r_3} \setminus B_{r_3/4})$ (recall that $\ell > 10$), it follows from (3.12) that

$$\Delta \widetilde{E}_\delta + k^2 \widetilde{E}_\delta = \Delta \widetilde{H}_\delta + k^2 \widetilde{H}_\delta = 0 \text{ in } B_{r_3} \setminus B_{2r_2}.$$

Applying Lemma 2.5, we have

$$\|(\widetilde{E}, \widetilde{H})\|_{\mathbf{H}(\partial B_{4r_2})} \le C \|(\widetilde{E}, \widetilde{H})\|^{\beta}_{\mathbf{H}(\partial B_{2r_2})} \|(\widetilde{E}, \widetilde{H})\|^{1-\beta}_{\mathbf{H}(\partial B_{r_3/2})}, \tag{3.15}$$

where

$$\beta = \ln\Big(\frac{r_3}{4r_2}\Big) \Big/ \ln\Big(\frac{r_3}{2r_2}\Big). \tag{3.16}$$

Combining (3.13) and (3.15) yields

$$\|\big(\widetilde{E}_\delta, \widetilde{H}_\delta\big)\|_{\mathbf{H}(\partial B_{2r_2})} \le C\|\big(\widetilde{E}_\delta, \widetilde{H}_\delta\big)\|^{\rho}_{\mathbf{H}(\partial B_{r_2/2})}\|\big(\widetilde{E}_\delta, \widetilde{H}_\delta\big)\|^{1-\rho}_{\mathbf{H}(\partial B_{r_3/2})}, \tag{3.17}$$

with

$$\rho = \frac{\alpha}{1-(1-\alpha)\beta}. \tag{3.18}$$

On the other hand, since

$$\Delta \widetilde{E}_\delta + k^2 \widetilde{E}_\delta = \Delta \widetilde{H}_\delta + k^2 \widetilde{H}_\delta = 0 \text{ in } B_{r_2} \cup (B_{r_3} \setminus B_{2r_2}),$$

we derive that

$$\|(\widetilde{E}, \widetilde{H})\|_{\mathbf{H}(\partial B_{r_2/2})} \le C\|(\widetilde{E}, \widetilde{H})\|_{L^2(B_{r_2})} \text{ and } \|(\widetilde{E}, \widetilde{H})\|_{\mathbf{H}(\partial B_{r_3/2})} \le C\|(\widetilde{E}, \widetilde{H})\|_{L^2(B_{r_3}\setminus B_{r_3/4})}. \tag{3.19}$$

From (3.5), (3.11), (3.17), and (3.19), we obtain

$$\|\big(\widetilde{E}_\delta, \widetilde{H}_\delta\big)\|_{\mathbf{H}(\partial B_{2r_2})} \le C\delta^{\rho} Data(j,\delta). \tag{3.20}$$

By taking $l$ large enough, we derive from (3.14), (3.16), and (3.18) that $\rho > 1/2 + \gamma$ if $r_3 > l r_2$. The conclusion of Step 1 follows.

Step 2: We have, since $G(x) = x$ on $\partial B_{r_3}$,

$$[\mathcal{E}_\delta \times \nu] = (E_\delta^{(1)} - E_\delta^{(2)}) \times \nu = 0 \text{ on } \partial B_{r_3},$$

and

$$[\mathcal{H}_\delta \times \nu] = (H_\delta^{(1)} - H_\delta^{(2)}) \times \nu = 0 \text{ on } \partial B_{r_3}.$$

Applying Lemma 2.8 (see also (3.8)), we obtain

$$\begin{cases} \nabla \times \mathcal{E}_\delta = ik\mathcal{H}_\delta + i\delta F_* I H_\delta^{(1)} \mathbb{1}_{B_{r_3}\setminus B_{r_2}} & \text{in } \mathbb{R}^3 \setminus \partial B_{2r_2}, \\ \nabla \times \mathcal{H}_\delta = -ik\mathcal{E}_\delta + j + i\delta F_* I E_\delta^{(1)} \mathbb{1}_{B_{r_3}\setminus B_{r_2}} & \text{in } \mathbb{R}^3 \setminus \partial B_{2r_2}. \end{cases} \tag{3.21}$$

We derive from (3.5) and (3.6) (see ([23], Lem. 4)) that

$$\|(\mathcal{E}_\delta, \mathcal{H}_\delta)\|_{H(\mathrm{curl}, B_{R_0}\setminus \partial B_{r_3})} \le C\Big(\|j\|_{L^2} + \delta^{\gamma+1/2} Data(j,\delta)\Big). \tag{3.22}$$

Since $\gamma > 0$ and $(\mathcal{E}_\delta, \mathcal{H}_\delta) = (E_\delta, H_\delta)$ in $\mathbb{R}^3 \setminus B_{r_3}$, it follows from (3.4) and (3.22) that

$$\|(\mathcal{E}_\delta, \mathcal{H}_\delta)\|_{H(\mathrm{curl}, B_{R_0}\setminus B_{r_3})]^2} \le C\|j\|_{L^2}.$$

We obtain from (3.4) that

$$Data(j,\delta) \le C\delta^{-1/2}\|j\|_{L^2}; \tag{3.23}$$

which in turn implies, by (3.6),

$$\|(E_\delta^{(1)} - E_\delta) \times \nu, (H_\delta^{(1)} - H_\delta) \times \nu\|_{\left[H^{-1/2}(\mathrm{div}_\Gamma, \partial B_{2r_2})\right]^2} \le C\delta^\gamma \|j\|_{L^2}. \tag{3.24}$$

It is clear from (1.10) and (3.21) that

$$\begin{cases} \nabla \times (\mathcal{E}_\delta - E) = -ik(\mathcal{H}_\delta - H) & \text{in } \mathbb{R}^3 \setminus \partial B_{2r_2}, \\ \nabla \times (\mathcal{H}_\delta - H) = ik(\mathcal{E}_\delta - E) & \text{in } \mathbb{R}^3 \setminus \partial B_{2r_2}. \end{cases}$$

Using (3.24), one obtains the conclusion of Step 2.

The proof is complete. □

**Remark 3.1.** The definition of $(\mathcal{E}_\delta, \mathcal{H}_\delta)$ is one of the key points of the proof. The idea is to remove from $(E_\delta, H_\delta)$ the term $(E_\delta^{(1)} - E_\delta^{(2)}, H_\delta^{(1)} - H_\delta^{(2)})$ in $B_{r_3} \setminus B_{2r_2}$; which is singular in general. This is the spirit of the removing of localized singularity technique introduced in [19, 20].

## 4. Further discussion

The requirement that $F$ is the Kelvin transform with respect to $\partial B_{r_2}$ in Theorem 1.1 can be relaxed. In fact, as seen in the proof of Theorem 1.1, one can replace the Kelvin transform by any transformation $F : B_{r_2} \setminus \bar{B}_{r_1} \to B_{r_3} \setminus \bar{B}_{r_2}$ such that *i)* $F(x) = x$ on $\partial B_{r_2}$; *ii)* There exists a diffeomorphism extension of $F$, which is still denoted by $F$, from $B_{r_2} \setminus \{0\}$ onto $\mathbb{R}^3 \setminus \bar{B}_{r_2}$; *iii)* There exists a diffeomorphism $G : \mathbb{R}^3 \setminus \bar{B}_{r_3} \to B_{r_3} \setminus \{0\}$ such that $G \in C^1(\mathbb{R}^3 \setminus B_{r_3})$, $G(x) = x$ on $\partial B_{r_3}$, and $G \circ F : B_{r_1} \to B_{r_3}$ is a diffeomorphism if one sets $G \circ F(0) = 0$. In this context, the first layer in $B_{r_2} \setminus B_{r_1}$ is also given by (1.3) and the second layer in $B_{r_1}$ is changed correspondingly by

$$\left(F_*^{-1} G_*^{-1} I, F_*^{-1} G_*^{-1} I\right). \tag{4.1}$$

Set

$$(\varepsilon_\delta, \mu_\delta) = \begin{cases} \left(\widetilde{\varepsilon}_O, \widetilde{\mu}_O\right) & \text{in } B_{r_3} \setminus B_{r_2}, \\ \left(F_*^{-1}\widetilde{\varepsilon}_O + i\delta I, F_*^{-1}\widetilde{\mu}_O + i\delta I\right) & \text{in } B_{r_2} \setminus B_{r_1}, \\ \left(F_*^{-1} G_*^{-1} I, F_*^{-1} G_*^{-1} I\right) & \text{in } B_{r_1}, \\ \left(I, I\right) & \text{in } \mathbb{R}^3 \setminus B_{r_3}. \end{cases} \tag{4.2}$$

We have

**Proposition 4.1.** *Let $R_0 > r_3$, $j \in \left[L^2(\mathbb{R}^3)\right]^3$ with* $\operatorname{supp} j \subset\subset B_{R_0} \setminus B_{r_3}$ *and let $(E_\delta, H_\delta) \in \left[H_{loc}(\mathrm{curl}, \mathbb{R}^3)\right]^2$ be the unique outgoing solution to (1.9) where $(\varepsilon_\delta, \mu_\delta)$ is given by (4.2) and let $(E, H) \in \left[H_{loc}(\mathrm{curl}, \mathbb{R}^3)\right]^2$ be the unique outgoing solution to (1.10). Given $0 < \gamma < 1/2$, there exists a positive constant $\ell = \ell(\gamma) > 0$, depending* **only** *on the elliptic constant of $\widetilde{\varepsilon}_O$ and $\widetilde{\mu}_O$ in $B_{2r_2} \setminus B_{r_2}$ and $\|(\widetilde{\varepsilon}_O, \widetilde{\mu}_O)\|_{W^{2,\infty}(B_{4r_2})}$ such that if $r_3 > \ell r_2$ then*

$$\|(E_\delta, H_\delta) - (E, H)\|_{H(\mathrm{curl}, B_R \setminus B_{r_3})} \le C_R \delta^\gamma \|j\|_{L^2}, \tag{4.3}$$

*for some positive constant $C_R$ independent of $j$ and $\delta$.*

The constants $\gamma$ and $\ell(\gamma)$ in Proposition 4.1 can be chosen as the ones in Theorem 1.1. The proof of Proposition 4.1 follows the same line as the one of Theorem 1.1 and is omitted.

## References


[1] A. Abdulle, M.E. Huber and S. Lemaire, An optimization-based numerical method for diffusion problems with sign-changing coefficients. *C. R. Math. Acad. Sci. Paris* **355** (2017) 472–478.

[2] G. Alessandrini, L. Rondi, E. Rosset and S. Vessella, The stability for the Cauchy problem for elliptic equations. *Inverse Probl.* **25** (2009) 123004.

[3] A. Alonso and A. Valli, Some remarks on the characterization of the space of tangential traces of $H(\mathrm{rot};\Omega)$ and the construction of an extension operator. *Manuscr. Math.* **89** (1996) 159–178.

[4] H. Ammari, G. Ciraolo, H. Kang, H. Lee and G.W. Milton, Anomalous localized resonance using a folded geometry in three dimensions. *Proc. R. Soc. Lond. Ser. A* **469** (2013) 20130048.

[5] J. Ball, Y. Capdeboscq and B. Tsering-Xiao, On uniqueness for time harmonic anisotropic Maxwell's equations with piecewise regular coefficients. *Math. Models Methods Appl. Sci.* **22** (2012) 1250036.

[6] A.S. Bonnet-Ben Dhia, L. Chesnel and P. Ciarlet, T-coercivity for scalar interface problems between dielectrics and metamaterials. *ESAIM: M2AN* **46** (2012) 1363–1387.

[7] E. Bonnetier and H.-M. Nguyen, Superlensing using hyperbolic metamaterials: the scalar case. Preprint arXiv: 1606.05516 (2016).

[8] G. Bouchitté and B. Schweizer, Cloaking of small objects by anomalous localized resonance. *Q. J. Mech. Appl. Math.* **63** (2010) 437–463.

[9] A. Buffa, M. Costabel and D. Sheen, On traces for $H(\mathrm{curl},\Omega)$ in Lipschitz domains, *J. Math. Anal. Appl.* **276** (2002) 845–867.

[10] M. Cassier, C. Hazard and P. Joly, Spectral theory for Maxwell's equations at the interface of a metamaterial. Part I: Generalized Fourier transform. Preprint arXiv: 1610.03021 (2016).

[11] D. Colton and R. Kress, in Inverse Acoustic and Electromagnetic Scattering Theory. 2nd edn. Vol. 98 of *Applied Mathematical Sciences*. Springer-Verlag, Berlin (1998).

[12] M. Costabel and E. Stephan, A direct boundary integral equation method for transmission problems. *J. Math. Anal. Appl.* **106** (1985) 367–413.

[13] R.V. Kohn, J. Lu, B. Schweizer and M.I. Weinstein, A variational perspective on cloaking by anomalous localized resonance. *Commun. Math. Phys.* **328** (2014) 1–27.

[14] Y. Lai, H. Chen, Z. Zhang and C.T. Chan, Complementary media invisibility cloak that cloaks objects at a distance outside the cloaking shell. *Phys. Rev. Lett.* **102** (2009) 093901.

[15] G.W. Milton and N.-A.P. Nicorovici, On the cloaking effects associated with anomalous localized resonance. *Proc. R. Soc. Lond. Ser. A* **462** (2006) 3027–3059.

[16] H.-M. Nguyen, Asymptotic behavior of solutions to the Helmholtz equations with sign changing coefficients. *Trans. Am. Math. Soc.* **367** (2015) 6581–6595.

[17] H.-M. Nguyen, Cloaking via anomalous localized resonance. A connection between the localized resonance and the blow up of the power for doubly complementary media. *C. R. Math. Acad. Sci. Paris* **353** (2015) 41–46.

[18] H.-M. Nguyen, Cloaking via anomalous localized resonance for doubly complementary media in the quasi static regime. *J. Eur. Math. Soc.* **17** (2015) 1327–1365.

[19] H.-M. Nguyen, Superlensing using complementary media. *Ann. Inst. Henri Poincaré Anal. Non Linéaire* **32** (2015) 471–484.

[20] H.-M. Nguyen, Cloaking using complementary media in the quasistatic regime. *Ann. Inst. Henri Poincaré Anal. Non Linéaire* **33** (2016) 1509–1518.

[21] H.-M. Nguyen, Limiting absorption principle and well-posedness for the Helmholtz equation with sign changing coefficients. *J. Math. Pures Appl.* **106** (2016) 797–836.

[22] H.-M. Nguyen, Cloaking an arbitrary object via anomalous localized resonance: the cloak is independent of the object. *SIAM J. Math. Anal.* **49** (2017) 3208–3232.

[23] H.-M. Nguyen, Superlensing using complementary media and reflecting complementary media for electromagnetic waves. *Adv. Nonlinear Anal.* (2018).

[24] H.-M. Nguyen, Cloaking via anomalous localized resonance for doubly complementary media in the finite frequency regime. *J. Anal. Math.* (2018).

[25] H.-M. Nguyen and H.L. Nguyen, Complete resonance and localized resonance in plasmonic structures. *ESAIM: M2AN* **49** (2015) 741–754.

[26] H.-M. Nguyen and H.L. Nguyen, Cloaking using complementary media for the Helmholtz equation and a three spheres inequality for second order elliptic equations. *Trans. Am. Math. Soc. B* **2** (2016) 93–112.

[27] H.-M. Nguyen and Q.H. Nguyen, Discreteness of interior transmission eigenvalues revisited. *Calc. Var. Partial Differ. Equ.* **56** (2017) 51.

[28] H.-M. Nguyen and V. Vinoles, Electromagnetic wave propagation in media consisting of dispersive metamaterials. Preprint arXiv: 1710.08648 (2017).

[29] T. Nguyen and J.-N. Wang, Quantitative uniqueness estimate for the Maxwell system with Lipschitz anisotropic media. *Proc. Am. Math. Soc.* **140** (2012) 595–605.

[30] P. Ola, Remarks on a transmission problem. *J. Math. Anal. Appl.* **196** (1995) 639–658.

[31] S. Ramakrishna and J.B. Pendry, Spherical perfect lens: solutions of Maxwell's equations for spherical geometry. *Phys. Rev. B* **69** (2004) 115115.

[32] R.A. Shelby, D.R. Smith and S. Schultz, Experimental verification of a negative index of refraction. *Science* **292** (2001) 77–79.

[33] V.G. Veselago, The electrodynamics of substances with simultaneously negative values of $\varepsilon$ and $\mu$. *Usp. Fiz. Nauk* **92** (1964) 517–526.