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Logarithmic Sobolev inequality revisited

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ABSTRACT

We provide a new characterization of the logarithmic Sobolev inequality.

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R É S U M É

Nous donnons une nouvelle caractérisation de l'inégalité de Sobolev logarithmique.

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1. Introduction

The classical Sobolev inequality translates information about the derivatives of a function into information about the size of the function itself. Precisely, for a function u with square summable gradient in dimension N , one obtains that u is $2N/(N-2)$ -summable, that is a gain in summability that depends on N and tends to deteriorate as $N \rightarrow \infty$. On the other hand, since the middle 1950s, people have started looking at possible replacements of the Sobolev inequality in order to provide an improvement in the summability *independent* of the dimension N , which can be done in terms of the integrability properties of $u^2 \log u^2$. This was firstly done by Stam [23], who proved the logarithmic Sobolev inequality with Gauss measure $d\mathcal{G}$

$$\int_{\mathbb{R}^N} u^2 \log \frac{u^2}{\|u\|_{2, d\mathcal{G}}^2} d\mathcal{G} \leq \frac{1}{\pi} \int_{\mathbb{R}^N} |\nabla u|^2 d\mathcal{G}, \quad d\mathcal{G} = e^{-\pi|x|^2} dx.$$

The formula was originally discovered in quantum field theory in order to handle estimates that are uniform in the space dimension, for systems with a large number of variables. A different proof and further insight was obtained by Gross in [17]. See also the work of Adams and Clarke [1] for an elementary proof of the previous inequality. These properties are

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widely used in statistical mechanics, quantum field theory and differential geometry. A variant of the logarithmic Sobolev inequality with Gauss measure is given by the following one-parameter family of Euclidean inequalities [18, Theorem 8.14]

$$\int_{\mathbb{R}^N} u^2 \log \frac{u^2}{\|u\|_2^2} dx + N(1 + \log a) \|u\|_2^2 \leq \frac{a^2}{\pi} \int_{\mathbb{R}^N} |\nabla u|^2 dx,$$

for any $u \in H^1(\mathbb{R}^N)$ and $a > 0$. A version of this inequality for fractional Sobolev spaces $H^s(\mathbb{R}^N)$ can be found in [13]. Recently, some new characterization of the Sobolev spaces were provided in [2,19,21] (see also [3–9,20]) in terms of the following family of nonlocal functionals

$$I_\delta(u) := \int \int_{\{|u(y)-u(x)|>\delta\}} \frac{\delta^2}{|x-y|^{N+2}} dx dy, \quad \delta > 0,$$

where u is a measurable function on \mathbb{R}^N . In particular, if $N \geq 3$ and $I_\delta(u) < \infty$ for some $\delta > 0$, then in [21] it was proved that

$$\int_{\{|u|>\lambda_N \delta\}} |u|^{2N/(N-2)} dx \leq C_N I_\delta(u)^{N/(N-2)}, \tag{1.1}$$

for some positive constants C_N and λ_N . This is a sort of nonlocal improvement of the classical Sobolev inequality, and it is also possible to show that in the singular limit $\delta \searrow 0$ one recovers the classical Sobolev result, since I_δ converges to the Dirichlet energy up to a normalization constant. The aim of this note is to remark that in this context also a logarithmic type estimate holds. Thus we have that the summability gain independent of N can be controlled in terms of $I_\delta(u)$.

More precisely, we have the following theorem.

Theorem 1.1. *Let $u \in L^2(\mathbb{R}^N)$ ($N \geq 3$). There is a positive constant C_N such that*

$$\int_{\mathbb{R}^N} \frac{u^2}{\|u\|_2^2} \log \frac{u^2}{\|u\|_2^2} dx + \frac{N}{2} \log \|u\|_2^2 \leq \frac{N}{2} \log \left(C_N \delta^{\frac{4}{N}} \|u\|_2^{\frac{2N-4}{N}} + C_N I_\delta(u) \right),$$

for all $\delta > 0$. In particular, if $u \in L^2(\mathbb{R}^N)$ is such that $I_\delta(u) < \infty$ for some $\delta > 0$, then

$$\int_{\mathbb{R}^N} u^2 \log u^2 dx < +\infty. \tag{1.2}$$

Proof. By a simple normalization argument, we may reduce the assertion to proving that

$$\int_{\mathbb{R}^N} u^2 \log u^2 dx \leq \frac{N}{2} \log \left(C_N \delta^{\frac{4}{N}} + C_N I_\delta(u) \right), \quad \text{for all } \delta > 0, \tag{1.3}$$

for any $u \in L^2(\mathbb{R}^N)$ such that $\|u\|_2 = 1$. Considering the normalized outer measure

$$\mu(E) := \int_E u^2(x) dx, \quad \mu(\mathbb{R}^N) = 1,$$

and using Jensen’s inequality for concave nonlinearities and with measure μ , we have

$$\log \left(\int_{\mathbb{R}^N} |u|^{\frac{2N}{N-2}} dx \right) = \log \left(\int_{\mathbb{R}^N} |u|^{\frac{4}{N-2}} d\mu \right) \geq \int_{\mathbb{R}^N} \log |u|^{\frac{4}{N-2}} d\mu = \frac{2}{N-2} \int_{\mathbb{R}^N} u^2 \log u^2 dx. \tag{1.4}$$

On the other hand, applying (1.1), we derive that, for all $\delta > 0$,

$$\frac{2}{N-2} \int_{\mathbb{R}^N} u^2 \log u^2 dx \leq \log \left(D_N \delta^{\frac{4}{N-2}} + C_N I_\delta(u)^{\frac{N}{N-2}} \right),$$

for some positive constant D_N , which implies (1.3). Here we used the fact that

$$\int_{\{|u| \leq \lambda_N \delta\}} |u|^{\frac{2N}{N-2}} dx \leq \lambda_N^{\frac{4}{N-2}} \delta^{\frac{4}{N-2}},$$

since $\int_{\mathbb{R}^N} u^2 dx = 1$. \square

Defining a notion of *entropy* as typical in statistical mechanics:

$$\text{Ent}_\mu(f) := \int_{\mathbb{R}^N} \frac{f}{\|f\|_{1,\mu}} \log \frac{f}{\|f\|_{1,\mu}} d\mu + \frac{N}{2} \log \|f\|_{1,\mu}, \quad f \geq 0, \quad \|f\|_{1,\mu} := \int f d\mu,$$

the conclusion of the previous results reads as

$$u \in L^2(\mathbb{R}^N), \exists \delta > 0 : I_\delta(u) < +\infty \implies \text{Ent}_{\mathcal{L}^N}(u^2) < +\infty.$$

Remark 1.2 (*Logarithmic NLS*). If $u \in H^1(\mathbb{R}^N)$, then the results of [19] show that

$$\lim_{\delta \searrow 0} I_\delta(u) = Q_N \int_{\mathbb{R}^N} |\nabla u|^2 dx, \tag{1.5}$$

for some constant $Q_N > 0$. Hence, passing to the limit as $\delta \searrow 0$ in the inequality of Theorem 1.1, one recovers classical forms of the logarithmic inequality. The logarithmic Schrödinger equation

$$i\partial_t \phi + \Delta \phi + \phi \log |\phi|^2 = 0, \quad \phi : [0, \infty) \times \mathbb{R}^N \rightarrow \mathbb{C}, \quad N \geq 3, \tag{1.6}$$

admits applications to quantum mechanics, quantum optics, transport and diffusion phenomena, theory of superfluidity and Bose–Einstein condensation (see [25] and [10–12]). The *standing waves* solutions to (1.6) solve the following semi-linear elliptic problem

$$-\Delta u + \omega u = u \log u^2, \quad u \in H^1(\mathbb{R}^N). \tag{1.7}$$

These equations were recently investigated in [15,24]. From a variational point of view, the search for solutions to (1.7) can be associated with the study of critical points (in a nonsmooth sense) of the lower semi-continuous functional $J : H^1(\mathbb{R}^N) \rightarrow \mathbb{R} \cup \{+\infty\}$ defined by

$$J(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{\omega + 1}{2} \int_{\mathbb{R}^N} u^2 dx - \frac{1}{2} \int_{\mathbb{R}^N} u^2 \log u^2 dx,$$

which is well defined by the logarithmic Sobolev inequality. Due to Theorem 1.1 and (1.5), one could handle a kind of *nonlocal approximations* of (1.7), formally defined for $\delta > 0$ by

$$I'_\delta(u) + \omega u = u \log u^2,$$

which are associated with the energy functional $J_\delta : H^1(\mathbb{R}^N) \rightarrow \mathbb{R} \cup \{+\infty\}$ defined by

$$J_\delta(u) = I_\delta(u) + \frac{\omega + 1}{2} \int_{\mathbb{R}^N} u^2 dx - \frac{1}{2} \int_{\mathbb{R}^N} u^2 \log u^2 dx.$$

Since there holds $I_\delta(u) \leq C_N \int_{\mathbb{R}^N} |\nabla u|^2 dx$ for all $\delta > 0$ and $u \in H^1(\mathbb{R}^N)$ (cf. [19, Theorem 2]), the energy functional J_δ is well defined, for every $\delta > 0$.

Remark 1.3 (*Magnetic case*). If $A : \mathbb{R}^N \rightarrow \mathbb{R}^N$ is locally bounded and $u : \mathbb{R}^N \rightarrow \mathbb{C}$, we set

$$\Psi_u(x, y) := e^{i(x-y) \cdot A\left(\frac{x+y}{2}\right)} u(y), \quad x, y \in \mathbb{R}^N.$$

It was observed in [14] that the following *diamagnetic inequality* holds

$$\| |u(x)| - |u(y)| \| \leq | \Psi_u(x, x) - \Psi_u(x, y) |, \quad \text{for a.e. } x, y \in \mathbb{R}^N.$$

In turn, by defining

$$I_\delta^A(u) := \int_{\{|\Psi_u(x,y) - \Psi_u(x,x)| > \delta\}} \frac{\delta^2}{|x-y|^{N+2}} dx dy,$$

we have

$$I_\delta(|u|) \leq I_\delta^A(u), \quad \text{for all } \delta > 0 \text{ and all measurable } u : \mathbb{R}^N \rightarrow \mathbb{C}. \tag{1.8}$$

Then, [Theorem 1.1](#) yields the following *magnetic logarithmic Sobolev inequality*. For $u \in L^2(\mathbb{R}^N)$, there is a positive constant C_N such that

$$\int_{\mathbb{R}^N} \frac{|u|^2}{\|u\|_2^2} \log \frac{|u|^2}{\|u\|_2^2} dx + \frac{N}{2} \log \|u\|_2^2 \leq \frac{N}{2} \log \left(C_N \delta^{\frac{4}{N}} \|u\|_2^{\frac{2N-4}{N}} + C_N I_\delta^A(u) \right).$$

Notice that, since $I_\delta(|u|) \approx \|\nabla|u|\|_2^2$ as $\delta \searrow 0$ [[19](#)] and $I_\delta^A(u) \approx \|\nabla u - iAu\|_2^2$ as $\delta \searrow 0$ [[22](#)], from inequality (1.8) one recovers $\|\nabla|u|\|_2 \leq \|\nabla u - iAu\|_2$, which follows from the well-know diamagnetic inequality for the gradients $|\nabla|u|| \leq |\nabla u - iAu|$, see [[18](#)].

As a companion to [Theorem 1.1](#), we also have the following theorem.

Theorem 1.4. Let $u \in L^2(\mathbb{R}^N)$ ($N \geq 3$). Assume that there exists a non-decreasing function $F : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that $F(ts) \leq t^\beta F(s)$ for any $s, t \geq 0$ and some $\beta > 0$ and

$$\int_{\mathbb{R}^{2N}} \frac{F(|u(x) - u(y)|)}{|x - y|^{N+2}} dx dy < +\infty. \tag{1.9}$$

Then there exists a positive constant $C_{N,F}$ such that

$$\int_{\mathbb{R}^N} \frac{u^2}{\|u\|_2^2} \log \frac{u^2}{\|u\|_2^2} dx + \frac{N}{2} \log \|u\|_2^\beta \leq \frac{N}{2} \log \left(C_{N,F} \|u\|_2^\beta + C_{N,F} \int_{\mathbb{R}^{2N}} \frac{F(|u(x) - u(y)|)}{|x - y|^{N+2}} dx dy \right).$$

In particular, condition (1.2) holds.

Proof. Consider the statement when $\|u\|_2 = 1$. In light of inequality (1.4), since by [[21, Proposition 6](#)] there exists $C_N > 0$ and $\lambda_N > 0$ such that

$$\int_{\{|u| > \lambda_N F(1/2)\}} |u|^{2N/(N-2)} dx \leq C_N \left(\frac{1}{F(1/2)} \int_{\mathbb{R}^{2N}} \frac{F(|u(x) - u(y)|)}{|x - y|^{N+2}} dx dy \right)^{N/(N-2)}, \tag{1.10}$$

by arguing as in the previous proof, we get

$$\frac{2}{N-2} \int_{\mathbb{R}^N} u^2 \log u^2 \leq \log \left(D_{N,F} + D_{N,F} \left(\int_{\mathbb{R}^{2N}} \frac{F(|u(x) - u(y)|)}{|x - y|^{N+2}} dx dy \right)^{N/(N-2)} \right),$$

where we used the fact that

$$\int_{\{|u| \leq \lambda_N F(1/2)\}} |u|^{\frac{2N}{N-2}} dx \leq \lambda_N^{\frac{4}{N-2}} F(1/2)^{\frac{4}{N-2}},$$

since $\int_{\mathbb{R}^N} u^2 dx = 1$. Then, we get

$$\int_{\mathbb{R}^N} u^2 \log u^2 \leq \frac{N}{2} \log \left(C_{N,F} + C_{N,F} \int_{\mathbb{R}^{2N}} \frac{F(|u(x) - u(y)|)}{|x - y|^{N+2}} dx dy \right).$$

In the general case, using the sub-homogeneity condition on F yields

$$\int_{\mathbb{R}^N} \frac{u^2}{\|u\|_2^2} \log \frac{u^2}{\|u\|_2^2} \leq \frac{N}{2} \log \left(C_{N,F} + \frac{C_{N,F}}{\|u\|_2^\beta} \int_{\mathbb{R}^{2N}} \frac{F(|u(x) - u(y)|)}{|x - y|^{N+2}} dx dy \right),$$

which yields the desired conclusion. \square

Remark 1.5 ($L^p(\mathbb{R}^N)$ -version). If $p > 1$ and $N > p$, one has a variant of (1.4), namely

$$\log \left(\int_{\mathbb{R}^N} |u|^{\frac{Np}{N-p}} dx \right) \geq \frac{p}{N-p} \int_{\mathbb{R}^N} |u|^p \log |u|^p dx. \quad (1.11)$$

Then, by arguing as in the proofs of Theorems 1.1 and 1.4 with

$$u \mapsto \int \int_{\{|u(y)-u(x)|>\delta\}} \frac{\delta^p}{|x-y|^{N+p}} dx dy, \quad u \mapsto \int_{\mathbb{R}^{2N}} \frac{F(|u(x)-u(y)|)}{|x-y|^{N+p}} dx dy, \quad (1.12)$$

in place of $I_\delta(u)$ and (1.9) respectively, it is possible to get the corresponding log-Sobolev inequalities as for the case $p = 2$, via the results of [21]. In particular, if $u \in L^p(\mathbb{R}^N)$ and the functionals in (1.12) are finite at u for some $\delta > 0$, then

$$\int_{\mathbb{R}^N} |u|^p \log |u|^p dx < +\infty.$$

The Euclidean logarithmic Sobolev inequalities for the p -case have been intensively studied, see, e.g., the work of Del Pino and Dolbeault [16] and the references therein.

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