Gaussian versus Sparse Stochastic Processes: Construction, Regularity, Compressibility

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Abstract

Although this thesis contributes to the theory of random processes, it is motivated by signal processing applications, mainly the stochastic modeling of sparse signals. Specifically, we provide an in depth investigation of the innovation model, under which a signal is described as a random process s that can be linearly and deterministically transformed into a white noise. The noise represents the unpredictable part of the signal—called its innovation—and is a member of the family of Lévy white noises, which includes both Gaussian and Poisson noises. In mathematical terms, s satisfies the equation

$$\mathrm{L}s = w, \tag{1}$$

where L is a differential operator and w a Lévy noise. The problem is therefore to study the solutions of stochastic differential equations driven by Lévy noises. Gaussian models usually fail to reproduce the empirical sparsity observed in real-world signals. By contrast, Lévy models offer a wide range of random processes going from typically non-sparse (Gaussian) to very sparse ones (Poisson), and with many sparse signals standing between these two extremes.

Our contributions can be divided in four parts. First, the cornerstone of our work is the theory of generalized random processes. Within this framework, all the considered random processes are seen as random tempered generalized functions and can be observed through smooth and rapidly decaying windows. This allows us to define the solutions of (1), called generalized Lévy processes, in the most general setting. Then, we identify two limit phenomenons: the approximation of generalized Lévy processes by their Poisson counterparts, and the asymptotic behavior of generalized Lévy processes at coarse and fine scales. In the third part, we study the localization of Lévy noise in notorious function spaces (Hölder, Sobolev, Besov). As an application, we characterize the local smoothness and the asymptotic growth rate of the Lévy noise. Finally, we quantify the local compressibility of the generalized Lévy processes, understood as a measure of the decreasing rate of their approximation error in an appropriate basis. From this last result, we provide a theoretical justification of the ability of the innovation model (1) to represent sparse signals.

The guiding principle of our research is the duality between the local and asymptotic properties of generalized Lévy processes. In particular, we highlight the relevant quantities, called the local and asymptotic indices, that allow quantifying the local regularity, the asymptotic growth rate, the limit behavior at coarse and fine scales, and the level of compressibility of the solutions of generalized Lévy processes.

Keywords: Lévy white noise, sparse stochastic processes, stochastic differential equations, generalized random processes, infinite divisibility, convergence in law, Besov regularity, N-term approximation, wavelets bases.

Résumé

Si notre travail prend place dans le domaine des processus stochastiques, cette thèse a été motivée par des problématiques issues du traitement du signal, en particulier pour la modélisation stochastique des signaux parcimonieux. Il s'est agit d'étudier mathématiquement le modèle d'innovation. Celui-ci fait l'hypothèse qu'un signal, décrit par un processus stochastique s, peut être transformé en un bruit blanc par une opération linéaire et déterministe. Le bruit blanc représente la partie imprédictible—ou innovation—du signal et appartient à la famille des bruits de Lévy, contenant notamment le bruit gaussien et les bruits de Poisson. En quatre symboles :

$$\mathrm{L}s = w, \tag{1}$$

avec L un opérateur différentiel et w un bruit blanc de Lévy. Pour un mathématicien, il s'agit donc d'étudier les solutions d'équations différentielles stochastiques dirigées par un bruit blanc de Lévy. Si les modèles gaussiens échouent d'ordinaire à rendre compte de la forte compressibilité empirique observée chez les signaux réels, les modèles de Lévy offrent une gamme de processus allant du non parcimonieux (Gauss) au très parcimonieux (Poisson), de nombreux signaux réels se situant entre ces deux extrêmes.

Nous détaillons nos contributions, organisées en quatre parties. Tout d'abord, nous situons notre travail dans le cadre de la théorie des processus généralisés. Ainsi, nous voyons les processus en jeu comme des fonctions généralisées tempérées, qui s'observent donc *a priori* via des fonctions test infiniment régulières et à décroissance rapide. Ceci nous permet de définir les solutions de (1), appelées des processus de Lévy généralisés, dans le sens le plus large possible. Nous étudions ensuite deux phénomènes limites, que sont l'approximation des processus de Lévy généralisés par leurs contreparties poissonniennes et le comportement asymptotique des processus de Lévy généralisés observés à fines et larges échelles. Dans la troisième partie, nous étudions la localisation des bruits de Lévy dans des espaces de fonctions (Hölder, Sobolev, Besov). Cela nous permet de caractériser leur régularité locale et leur croissance asymptotique. Enfin, nous quantifions la compressibilité locale d'un processus de Lévy généralisé, comprise comme une mesure de la vitesse de décroissance de son erreur d'approximation dans une base adaptée. Fort de ce résultat, nous sommes à même d'expliquer théoriquement la pertinence de l'utilisation du modèle d'innovation (1) pour la modélisation de signaux parcimonieux.

Le fil conducteur de nos travaux se situe dans l'étude duale des propriétés locales et asymptotiques des processus stochastiques considérés. Nous nous sommes efforcés de mettre en évidence les quantités pertinentes, appelées respectivement les indices locaux et asymptotiques du processus, qui permettent de quantifier la régularité locale, le taux de croissance asymptotique, les comportements limites à fines et larges échelles, ainsi que le niveau de compressibilité des processsus stochastiques.

Mots clefs : Bruit blanc de Lévy, processus stochastiques parcimonieux, équations différentielle stochastiques, processus stochastiques généralisés, infinie divisibilité, convergence en loi, régularité de Besov, approximation N-term, bases ondelettes.

Contents

Notation

Sets

$\mathbb{N}$	Non-negative integers, including 0
$\mathbb{Z}$	Integers
$\mathbb{R}$	Real numbers
$\mathbb{C}$	Complex numbers
$\mathbb{R}^d$	d-dimensional Euclidian space
$\mathbb{T}^d = [0,1]^d$	d-dimensional torus

Miscellaneous

i	Imaginary unit such that $\mathrm{i}^2 = -1$
e	$\sum_{k\geq 0} 1/k!$
π	$\tau/2$
$\Re\{z\}$	Real part of $z \in \mathbb{C}$
$\Im\{z\}$	Imaginary part of $z \in \mathbb{C}$
z^*	Complex conjugate of $z \in \mathbb{C}$
$f^\vee$	Function $f^\vee(\boldsymbol{x}) = f(-\boldsymbol{x})$
Leb	Lebesgue measure
$\mathrm{Supp} f$	Support of the (generalized) function f
$\mathrm{Card} A$	Cardinal of the set A

Function spaces

$0 < p < \infty$	Integratibility rate
$\tau \in \mathbb{R}$	Smoothness parameter
$\rho \in \mathbb{R}$	Decay parameter
$\mathscr{X}$	Generic topological vector space
$\mathscr{D}(\mathbb{R}^d)$	Compactly supported smooth functions
$\mathscr{D}'(\mathbb{R}^d)$	Generalized functions
$\mathscr{S}(\mathbb{R}^d)$	Rapidly decaying smooth functions
$\mathscr{S}'(\mathbb{R}^d)$	Tempered generalized functions
$\mathscr{R}(\mathbb{R}^d)$	Rapidly decaying measurable functions
$L_p(\mathbb{R}^d)$	Functions such that $\int_{\mathbb{R}^d} \lvert f(\boldsymbol{x})\rvert^p \,\mathrm{d}\boldsymbol{x} < \infty$

$L_\infty(\mathbb{R}^d)$	Functions such that $\operatorname{ess\,sup}_{\boldsymbol{x}\in\mathbb{R}^d} \left\lvert f(\boldsymbol{x})\right\rvert < \infty$
$L_{p_0,p_\infty}(\mathbb{R}^d)$	Functions such that $f\mathbb{1}_{\lvert f\rvert>1} \in L_{p_0}(\mathbb{R}^d)$ and $f\mathbb{1}_{\lvert f\rvert\leq 1} \in L_{p_\infty}(\mathbb{R}^d)$
$L_\rho(\mathbb{R}^d)$	Functions such that $\int_{\mathbb{R}^d} \rho(f(\boldsymbol{x}))\mathrm{d}\boldsymbol{x}$ with $\rho : \mathbb{R} \to \mathbb{R}^+$
$L_\Theta(\mathbb{R}^d)$	Domain of definition of the Lévy noise
$L_{\Theta_p}(\mathbb{R}^d)$	Domain of finite pth-moments of the Lévy noise
$W_2^\tau(\mathbb{R}^d)$	Sobolev spaces
$W_2^\tau(\mathbb{R}^d;\rho)$	Weighted Sobolev spaces
$B_p^\tau(\mathbb{R}^d;\rho)$	Weighted Besov spaces
$\mathcal{S}(\mathbb{T}^d)$	Periodic smooth functions
$\mathcal{S}'(\mathbb{T}^d)$	Periodic generalized functions
$W_2^\tau(\mathbb{T}^d)$	Periodic Sobolev spaces
$B_p^\tau(\mathbb{T}^d)$	Periodic Besov spaces
$\dot{\mathcal{S}}(\mathbb{T}^d)$	Periodic smooth functions with 0-mean
$\dot{\mathcal{X}}$	Periodic functions in $\mathcal{X} \subseteq \mathcal{S}'(\mathbb{T}^d)$ with 0-mean

Functions and generalized functions

φ, ψ	Generic test functions in $\mathcal{S}(\mathbb{R}^d)$
u	Generic tempered generalized function in $\mathcal{S}'(\mathbb{R}^d)$
$\langle u, \varphi\rangle$	Duality product between $u \in \mathcal{S}'(\mathbb{R}^d)$ and $\varphi \in \mathcal{S}(\mathbb{R}^d)$
f	Generic measurable function from $\mathbb{R}^d$ to $\mathbb{R}$
$\mathbb{1}_B$	Indicator function of the set $B \subset \mathbb{R}^d$
δ	Dirac impulse
ρ_{L}	Green's function of the operator L
j	Scale parameter
G	Gender
$\boldsymbol{k}$	Shift parameter
ψ_F, ψ_M	Father and mother Daubechies wavelets
$\psi_{j,G.\boldsymbol{k}}$	Daubechies wavelets
$\psi_{j,G.\boldsymbol{k}}^{\mathrm{per}}$	Periodic Daubechies wavelets
$\Sigma_{N,p,\tau}(f)$	Best N-term approximation of f
$\sigma_{N,p,\tau}(f)$	N-term approximation error

Operators

L	Generic operator from $\mathcal{S}(\mathbb{R}^d)$ to $\mathcal{S}'(\mathbb{R}^d)$
L^*	Adjoint operator of L
T	Generic left-inverse of the adjoint L^*
Id	Identity operator
$\mathrm{T}_{\boldsymbol{x}_0}$	Translation operator with shift $\boldsymbol{x}_0 \in \mathbb{R}^d$
S_a	Scaling operator with scale $a > 0$
$\mathrm{R}_{\boldsymbol{\theta}_0}$	Rotation operator with rotation $\boldsymbol{\theta}_0 \in SO(d)$
$\mathcal{F}$	Fourier transform

D	Derivative
D_i	Partial derivative along direction x_i
$\mathrm{D}^{\boldsymbol{m}}$	Partial derivative of order $\boldsymbol{m} \in \mathbb{N}^d$
Δ	Laplacian operator
Λ	Partial derivative $\mathrm{D}^{(1,\dots,1)}$
D^γ	Fractional derivative of order $\gamma \geq 0$
$(-\Delta)^{\gamma/2}$	Fractional Laplacian of order $\gamma \geq 0$
J_γ	Bessel operator of order $\gamma \in \mathbb{R}$

Probability

$(\Omega, \mathscr{F}, \mathscr{P})$	Probability space
$\mathscr{B}(\mathscr{X})$	Borelian σ-field on the topological vector space $\mathscr{X}$
X, Y	Generic real random variables
$L_0(\Omega)$	Space of real random variables
$L_p(\Omega)$	Space of real random variables with finite pth-moment
$\mathscr{P}_X$	Probability law of X
$\widehat{\mathscr{P}}_X : \xi \mapsto \mathbb{E}[\mathrm{e}^{\mathrm{i}\xi X}]$	Characteristic function of X
$\boldsymbol{X} = (X_1, \dots, X_N)$	Generic random vector
$\Psi : \xi \mapsto \log \widehat{\mathscr{P}}_X(\xi)$	Lévy exponent of the infinitely divisible random variable X
$\nu(\mathrm{d}t)$	Lévy measure
(μ, σ^2, ν)	Lévy triplet
$\Theta : \xi \mapsto \Theta(\xi)$	Rajput-Rosinski exponent
s	Generic generalized random process in $\mathscr{S}'(\mathbb{R}^d)$
$\mathscr{P}_s$	Probability law of s (probability measure in $\mathscr{S}'(\mathbb{R}^d)$)
$\widehat{\mathscr{P}}_s : \varphi \mapsto \mathbb{E}[\mathrm{e}^{\mathrm{i}\langle s, \varphi\rangle}]$	Characteristic functional of s
$\langle s, \varphi \rangle$	Observation of the process s through the test function φ
w	Lévy white noise

Indices, exponents, parameters: local vs. asymptotic toolbox

α	Parameter of SαS random variables and noises
α_{loc}	Local index of an infinitely divisible law
α_{asymp}	Asymptotic index of an infinitely divisible law
H	Hurst exponent of a self-similar process
H_{loc}	Local Hurst exponent of a locally self-similar process
H_{asymp}	Asymptotic Hurst exponent of an asymptotic self-similar process
$\tau_p(f)$	Local smoothness of f for the rate p
$\rho_p(f)$	Asymptotic decay rate of f for the rate p
$\kappa_{p_0,\tau_0}(f)$	Local compressibility of f for the rate p_0 and the smoothness τ_0
$\kappa(f) = \kappa_{2,0}(f)$	Local compressibility of f in L_2

1 From Sparse Signals to Sparse Processes

The topic of this thesis is the mathematical study of stochastic differential and pseudo-differential equations driven by multivariate Lévy white noise. Three main aspects are developed: the construction of the solutions, the study of their regularity, and the quantification of their compressibility.

The original motivation of our work was the development of the theory of *sparse stochastic processes*, which represents the first systematic attempt for a stochastic and continuous-domain modeling of real world signals in line with the sparsity paradigm of signal processing [UTS14, UTAK14, UT14]. This work should therefore be seen as a mathematical continuation of the monograph of M. Unser and P.D. Tafti [UT14], in the sense that it deepens some mathematical questions (construction of sparse processes), and investigates new directions of research (scaling limits, Besov regularity, compressibility, etc.).

This introduction provides the opportunity to connect our work with signal processing, in particular with the framework of sparse stochastic processes. In Section 1.1, we introduce the innovation model, which is the signal processing formulation of the stochastic model we study. In Section 1.2, we review the current state of the theory of sparse stochastic processes. Then, we propose an overview of our own mathematical contributions in Section 1.3.

1.1 The Innovation Model

A signal is modeled as a continuous-domain random process that can be deterministically and linearly transformed into its innovation, understood as the unpredictable part of the signal, and itself captured by the concept of white noise. This is the spirit of the *innovation model*, of which we detail the assumptions.

A continuous-domain model. A signal is defined over the d-dimensional *continuum*. We only consider scalar-valued signals, seen as functions from $\mathbb{R}^d$ to $\mathbb{R}$. Most of the concepts are readily extended to vector-valued signals[1]. Nowadays, many popular signal processing formulations are inherently discrete, starting with the compressed sensing [Don06, CRT06] and the deep learning framework [GBC16]. This is driven by the constraint that practical algorithms are applied to discrete data and should produce discrete outputs. Nevertheless, we like to define the complete signal model in the continuous-domain as many physical phenomenon are inherently continuous and result in analog signals (such as images, sounds, etc.). The continuous framework also lends itself naturally to the specification of mathematical operations, such as geometric transformations (scaling, rotation) and differentiation, that are not well-defined in the discrete setting. It is then required to discretize the model—which corresponds to an approximation of the continuous-domain model—for the design of signal processing algorithms, which was largely investigated in [UT14].
The integer $d \geq 1$ specifies the dimension of the definition domain of the signal s. For instance, an acoustic signal—for which $d = 1$—is a function of time that measures the acoustic pressure $s(t)$ at each time t. A greyscale image is seen as a function that specifies the grey level $s(x, y)$ at each location $(x, y) \in \mathbb{R}^2$. More generally, one can consider *e.g.* $3D$ spatial signals $s(x, y, z)$, or $(2+1)D$ time-evolving two-dimensional signals $s(x, y; t)$.

A stochastic model. Real-world signals can be described deterministically using our knowledge of physical laws. Nevertheless, there are good reasons to introduce a stochastic approach in the modeling. First, physical phenomena are always affected with random fluctuations, that are studied by statistical physics. In signal processing, this leads to noisy observations. Moreover, the patterns observed in real-world signals appear to strongly depend on many variables which are often impossible to observe directly and possibly irrelevant to the question of interest [MD10]. This results in an irreducible uncertainty on real-world signals that has to be both diminished (by reducing the impact of the noise) and resolved (by inferring the hidden variables). Probability theory offers a powerful modeling of this uncertainty [VKG14, Section 3.8]. A signal is thus described as a continuous-domain random function, or *stochastic process*.

The innovation of a signal. The innovation approach can be traced back to H.W. Bode and C.E. Shannon [BS50], with important contributions by T. Kailath [Kai68, KF68, Kai70]. Following the definition of P. Tafti, *"innovation is that which cannot be predicted"* [Taf11], and is itself

[1] When they are not, there is a good chance that the question has been addressed in the doctoral dissertation of P.D. Tafti [Taf11].

modeled as a random process, the properties of which we now specify. We assume that the source of randomness of the signal is restricted to its innovation, as depicted in Figure 1.1, and that the signal is the deterministic recombination, or mixing, of its innovation. This implies that the signal is deterministic, conditionally on its innovation.

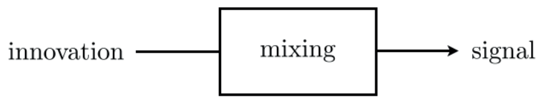


Figure 1.1 – Generative model.

In our model, the innovation is captured by the concept of *white noise*. This implies two assumptions. The innovation is a collection of *independent* atoms of randomness that have *identical statistics*. In a discrete setting, an innovation is therefore a collection of independent and identically distributed random variables. The adaptation of this concept in the continuous-domain requires more advanced mathematics that will be further introduced: It yields to the definition of a random process that is stationary and independent at every point.

The whitening operator. We assume that the signal is linearly linked to its innovation. Moreover, a small variation in the innovation should only produce a small variation in the signal. Mathematically, we ask that the deterministic mixing transformation that generate the signal from the innovation is *linear* and *continuous*. The inverse operation, which corresponds to extracting its innovation from the signal, is called the *whitening*, and shares the same properties. We summarize this in Figure 1.2.

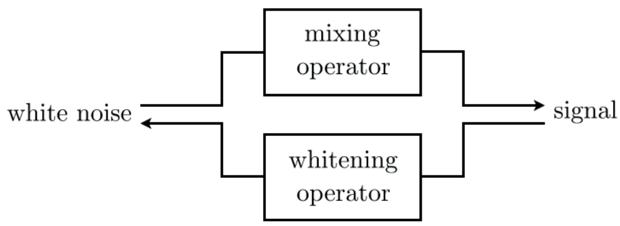


Figure 1.2 – Innovation model.

Differential and pseudo-differential operators are used as whitening operators because of their ability to reproduce both interesting dependency structures and statistical invariances (mainly stationarity and self-similarity). At that stage, we specify the innovation model as follows. A signal is modeled as a random process s such that

$$\mathrm{L}s = w \tag{1.1}$$

where L is a (pseudo-)differential operator, and w is a d-dimensional continuous-domain white noise.

Remarks. The innovation model as presented above is an idealisation. It goes beyond the

Gaussian paradigm, and is the richest possible framework under the linearity and stationarity assumptions. This simplified vision has a virtue. It allows to investigate in depth the sparsity of the random signals generated according to (1.1). Nevertheless, the statistical properties of real-world signals are rarely perfectly captured by linear and stationary models.

The choice of a stationary innovation leads us to the construction of random processes that are stationary or have stationary increments. For instance, Lévy processes, that correspond to (1.1) with w a 1-dimensional white noise and $\mathrm{L} = \mathrm{D}$ the derivative operator, have stationary and independent increments. One promising way to relax the stationarity is to replace Lévy processes by their generalization as Lévy-type processes [BSW14]. Essentially, one preserves the independence of the increments but allows them to vary with time. In the same spirit, one can define Lévy-type noises that are independent at every point but not stationary. One can also consider non-linear stochastic differential equations, which are a very active domain of research in probability theory. These two possible generalizations will not be discussed further.

1.2 Sparse Stochastic Processes

1.2.1 What is sparsity?

In the following, we do not provide a formal definition of the sparsity of a function, but outline what is required to understand the concepts of sparse signals and sparse processes. Roughly speaking, a signal is considered as sparse when most its energy is concentrated in a few coefficients in some transformation domain. Formally, given a basis $\boldsymbol{\psi} = (\psi_n)_{n\in\mathbb{N}}$ of $L_2(\mathbb{R}^d)$, the sparsity of a function $f \in L_2(\mathbb{R}^d)$ in $\boldsymbol{\psi}$ is measured by the speed of decay of the approximation error for the best N-term approximation of f, as N increases. We say that g *is sparser than* f *in the basis* $\boldsymbol{\psi}$ if

$$\frac{\|g - g_N\|_2}{\|g\|_2} \ll \frac{\|f - f_N\|_2}{\|f\|_2}$$

as N goes to infinity, where f_N (g_N, respectively) is the best N-term approximation of f (g, respectively) in the basis $\boldsymbol{\psi}$. The relation "being sparser in the basis $\boldsymbol{\psi}$" is a strict partial order on signals of $L_2(\mathbb{R}^d)$. Moreover, sparsity depends on the basis one selects. For instance, for any signal that is a finite linear combination of the ψ_n, the approximation error is zero for big enough N. This implies that the concept of sparsity is not absolute. We now specify how one usually proceeds to quantify the sparsity concretely.

- One considers only bases $\boldsymbol{\psi}$ with pleasing properties for signal processing purposes. Any function in $L_2(\mathbb{R}^d)$ should have a stable representation in the basis $\boldsymbol{\psi}$. This is typically the case for orthonormal bases or, more generally, for Riesz bases [UT14, Section 6.2.3]. Moreover, the coefficients of the basis decomposition should be computable using fast algorithms. This is typically the case for Fourier-based transforms or wavelet transforms [Mal99].

- One studies the sparsity of classes of functions rather than of isolated functions. Classes of functions, usually called function spaces, are characterized *e.g.* by their regularity or their decay rate. The analysis of the approximation properties of function spaces into interesting bases belongs to the field of approximation theory [Dev98].

- One analyses the properties of the signals of interest via their inclusions in appropriate function spaces (such as Besov spaces), for which we have quantified the sparsity level.

Gaussian models and sparsity. If we generate a Gaussian process s_{Gauss} that fits the second-order statistics of a real-world signal s_{real}, we will frequently observe that the s_{real} is sparser than s_{Gauss}; that is, for N big,

$$\frac{\|s_{\mathrm{real}} - s_{\mathrm{real},N}\|_2}{\|s_{\mathrm{real}}\|_2} \ll \frac{\|s_{\mathrm{Gauss}} - s_{\mathrm{Gauss},N}\|_2}{\|s_{\mathrm{Gauss}}\|_2},$$

with $s_{\mathrm{real},N}$ and $s_{\mathrm{Gauss},N}$ the corresponding best N-term approximations. Gaussian models are known to be unable to capture the kind of sparsity behaviors concretely observed for many signals. This limitation is well-documented [SLSZ03, HM99, MD10] and needs to be overcome.

Gaussian distributions are characterized by rare deviations from the average behavior. This lack of extreme values is inherited in any reasonable transform domain for a Gaussian process. The fact that real signals are much more compressible than Gaussian signals is actually very positive. It implies in particular that images, music, or movies are very efficiently compressed, allowing for the storage and the exchange of information to extents that would be unachievable in a Gaussian world.
In line with the sparsity paradigm in signal processing, this calls for stochastic models that should at the very least produce random processes sparser than their Gaussian counterparts. The theory of sparse stochastic processes provides such models.

1.2.2 Innovation Model and Sparsity

We have seen that Gaussian models fail to share an essential property of many real-world signals: the sparsity. This is true as well with the innovation model (1.1) when the innovation is Gaussian. It is possible, however to select non-Gaussian innovations to completely reverse this trend and to induce a behavior that is compatible with what is observed in real-world signals. The mathematics of the innovation model stands on two pillars: generalized random processes [GV64] and infinitely divisible laws [Sat13], the latter being required to understand why non-Gaussian innovations are sparse.

Generalized random processes. A continuous-domain white noise is too erratic to be defined as a pointwise random function. In (deterministic) functional analysis, one way to deal with “functions" that do not have a pointwise representation is to define them as generalized functions, or distributions[2], in the sense of L. Schwartz [Sch66]. For instance, the Dirac impulse is a generalized function such that $\langle \delta, \varphi \rangle = \varphi(0)$ for any smooth and compactly supported function. One defines the derivatives of any order of the Dirac impulse in the same way, by their effects on test functions. The theory of generalized random processes is the probabilistic counterpart of Schwartz theory of generalized functions, and is systematically exposed in [GV64]. This is the point of view that we are adopting in this thesis.

Infinite divisibility. A random variable is infinitely divisible if it can be decomposed as the sum of N i.i.d. random variables for all N. Consider a 1-dimensional white noise w, observed through the indicator function $\mathbb{1}_{[0,1)}$ and set $X = \langle w, \mathbb{1}_{[0,1)} \rangle$. If we define $X_{n,N} = \langle w, \mathbb{1}_{[(n-1)/N, n/N)} \rangle$, we have the decomposition, valid for every N,

$$X = X_{1,N} + \cdots + X_{n,N}.$$

For N fixed, the random variables $X_{n,N}$ are independent (as observations of the noise through windows with disjoint supports) and identically distributed (because the windows are shifted versions of each other). Thus, the observation of a white noise through an indicator function is infinitely divisible. This simple example highlights the connection between infinitely divisible

[2]We will not use the more usual term “distribution" thereafter, to avoid confusion with the probability distributions arising in probability theory.

random variables and continuous-domain white noise. More generally, the observation of any random process s solution of (1.1) through a test function φ produces an infinitely divisible random variable $\langle s, \varphi \rangle$. The infinitely divisibility of the observations of the processes satisfying the innovation model has several crucial consequences.

- The infinitely divisible random variable $\langle w, \mathbb{1}_{[0,1)} \rangle$ fully characterizes the law of the white noise w. There is actually a one-to-one correspondence between infinitely divisible laws and white noises. A noise is called a *Lévy white noise*, or simply a Lévy noise, in honour of P. Lévy for his role in the study of infinitely divisible random variables and their connection with continuous-domain random processes with stationarity and independence properties.

- The law of a random variable X is fully specified by its characteristic function $\widehat{\mathscr{P}}_X(\xi) = \mathbb{E}[e^{i\xi X}]$. The characteristic function of an infinitely divisible random variable X admits a *Lévy-Khintchine representation*. In the symmetric case, this means that we can write, for $\xi \in \mathbb{R}$, that

$$\log \widehat{\mathscr{P}}_X(\xi) = -\frac{\sigma^2 \xi^2}{2} + \int_{\mathbb{R}} (1 - \cos(\xi t)) \nu(\mathrm{d}t), \tag{1.2}$$

  where $\sigma^2 \geq 0$, and ν is a Lévy measure on $\mathbb{R}$, satisfying $\int_{\mathbb{R}} \min(1, t^2) \nu(\mathrm{d}t) < \infty$ and $\nu\{0\} = 0$. The log-characteristic function is denoted by $\Psi = \log \widehat{\mathscr{P}}_X$, and called the *Lévy exponent*. When $\nu = 0$, X is a Gaussian random variable. We say that X has no Gaussian part if $\sigma^2 = 0$. The Lévy-Khintchine representation is at the heart of the proofs of the fundamental results on Lévy noise and sparse stochastic processes.

- Many important properties (scaling limit, regularity, compressibility) of the Lévy noise are captured by its *indices*, that are related to the moments of the Lévy measure. They are defined as

$$\alpha_{\text{loc}} = \inf \left\{ p \geq 0 \,\middle|\, \int_{|t| \leq 1} |t|^p \, \nu(\mathrm{d}t) < \infty \right\},$$
$$\alpha_{\text{asymp}} = \sup \left\{ p \geq 0 \,\middle|\, \int_{|t| > 1} |t|^p \, \nu(\mathrm{d}t) < \infty \right\}.$$

Gaussian versus sparse stochastic processes. A Lévy noise with no Gaussian part is said to be *sparse*. We therefore reinterpret (1.2) as

$$\Psi = \Psi_{\text{Gauss}} + \Psi_{\text{sparse}},$$

with $\Psi_{\text{Gauss}}(\xi) = -\frac{\sigma^2 \xi^2}{2}$ and $\Psi_{\text{sparse}}(\xi) = \int_{\mathbb{R}} (1 - \cos(\xi t)) \nu(\mathrm{d}t)$. Equivalently, a Lévy noise is the sum of two independent white noises, one being sparse and the other Gaussian. Here, in accordance with the discussion of Section 1.2.1, *sparse* means sparser than Gaussian. We give several justifications for this terminology.

- In the discrete setting, random variables with heavy-tailed laws are known to produce i.i.d. sequences (or discrete white noise) that are more compressible than Gaussian ones [Cev09, AUM11, SP12, GCD12]. More generally, the asymptotic decay of the probability density appears to be critical for the compressibility of i.i.d. sequences. For infinitely divisible random variables, it is known that the Gaussian has the fastest decay. Moreover, the other non-Gaussian members of the theory cannot decay faster that $\exp\left(-\mathcal{O}(|x|\log|x|)\right)$ [AU14, Theorem 7]. This gap in the decay makes non-Gaussian infinitely divisible random variables good candidates for sparse *discrete* models.

- The compound Poisson processes, which correspond to the innovation model with an impulsive Poisson noise and the derivative operator, are piecewise constant, and are therefore easily shown to be sparser than the Brownian motion, in a suitable wavelet bases. This remark can be extended to the other innovation models for multivariate Poisson noise and general whitening operator [UT11].

- The symmetric-α-stable (SαS) noise are also part of the Lévy family [ST94]. They are parameterized by $0 < \alpha \leq 2$, where $\alpha = 2$ corresponds to the Gaussian case. The non-Gaussian SαS have infinite variance and are hence known to produce compressible sequences [AU14]. The sparsity is due to the presence of extreme values. The parameter α is a measure of the sparsity of the process: the smaller the α, the sparser the corresponding sparse process.

- More generally, there is empirical evidence that non-Gaussian processes are sparser than Gaussian ones in terms of approximation error. This is particularly visible in wavelet bases [Uns15, PU15, UT14]. In spite of this, a mathematical justification that a sparse stochastic process is *locally* sparser than its Gaussian counterpart is missing. This question will be addressed in this thesis.

Sparse processes in signal processing. Sparse stochastic processes and fields have been used to design algorithms for different signal processing tasks. The reconstruction of continuous-domain signals from their samples under the innovation model is analyzed in [AKBU13, ATWU13]. Different classes of sparse processes were used for the denoising of signals [KPAU13, KKBU13, BFKU13] and for inverse problems [BKNU13, Hos16]. In these works, the proposed algorithms are shown to outperform traditional Gaussian-based algorithms in many imaging science modalities. Some of them are state-of-the-art for the underlying class of stochastic models.

1.3 Contributions

In the following, we give an overview of the results presented in this thesis. All the mathematical concepts are introduced in more details in Chapter 2. The exposition is parallel to the thesis outline. For simplicity, we only consider *symmetric* random processes when presenting our contributions. Most of the results are taken from our published works [FAU14, FBU15, FUW17b], works in press [FFU], submitted works [FUU17, FU16, FUW17a], and works in preparation [AFU, DFHU].

We call a solution of (1.1) a *generalized Lévy process*. As we explained in Section 1.2.2, it includes both Gaussian processes (driven by the Gaussian white noise) and sparse stochastic processes (when the Lévy noise has no Gaussian part). Throughout the thesis, a special effort was done to particularise our results for interesting classes of noise, including Gaussian, SαS, compound Poisson, and Laplace noises.

1.3.1 Construction

All the random processes we shall encounter are defined as random elements of the space $\mathscr{S}'(\mathbb{R}^d)$ of tempered generalized functions. They are called *tempered generalized random processes*. Given a tempered generalized random process s, its characteristic functional is defined over $\mathscr{S}(\mathbb{R}^d)$ as

$$\widehat{\mathscr{P}}_s(\varphi) = \mathbb{E}[\mathrm{e}^{\mathrm{i}\langle s,\varphi\rangle}].$$

It is the infinite dimensional generalization of the characteristic function. The construction of tempered generalized random processes is achieved through their characteristic functional. It is based on the Bochner-Minlos theorem: A functional from $\mathscr{S}(\mathbb{R}^d)$ to $\mathbb{C}$ that is continuous, positive-definite, and which takes value 1 at $\varphi = 0$, is the characteristic functional of a generalized random process in $\mathscr{S}'(\mathbb{R}^d)$. Identifying valid characteristic functionals is therefore a powerful way to *construct* generalized random processes. We apply this principle for two classes of random processes: Lévy noise in $\mathscr{S}'(\mathbb{R}^d)$ and generalized Lévy processes.

Tempered Lévy noise. Gelfand and Vilenkin have introduced the complete family of Lévy white noise in the space $\mathscr{D}'(\mathbb{R}^d)$ of (not necessarily tempered) generalized functions [GV64]. There is actually a one-to-one correspondence between d-dimensional Lévy noises and infinitely divisible random variables, via the relation

$$w \mapsto X := \langle w, \mathbb{1}_{[0,1]^d}\rangle.$$

The random variable X is defined here as the limit in probability of random variables $\langle w,\varphi_k\rangle$ where the φ_k are smooth, compactly supported, and converge to $\mathbb{1}_{[0,1]^d}$ in an appropriate sense.

The adaptation of the theory to $\mathscr{S}'(\mathbb{R}^d)$ is motivated by mathematical purposes. In particular, we consider pseudo-differential operators and consider Besov spaces that are embedded in $\mathscr{S}'(\mathbb{R}^d)$. Thus, we have to identify the Lévy noise that are valid *tempered* generalized random processes. We show the following result (see Section 3.1.1 and [FAU14]).

If $\mathbb{E}[|\langle w, \mathbb{1}_{[0,1]^d}\rangle|^\epsilon] < \infty$ for some $\epsilon > 0$ arbitrarily small, then the Lévy noise w is tempered.

By following up of our investigation, R. Dalang and T. Humeau have recently proved that the converse result is true [DH15]. This provides a one-to-one correspondence between tempered Lévy noise and infinitely divisible random variables having a finite absolute moment. We also remark that the requirement for being tempered is extremely mild, and satisfied by the Lévy noises encountered in practice.

The domain of definition of the Lévy noise. As a preparatory result for the construction of generalized Lévy processes, we identify the broadest set of test functions such that the random variable $\langle w, f\rangle$ is well-defined, with w a tempered Lévy noise. We define this new random variable as the limit in probability of random variables $\langle w, \varphi_k\rangle$, where the compactly supported and smooth functions φ_k converge to f in an adequate sense. Our contribution is to connect the construction of Lévy noise as random elements in $\mathcal{S}'(\mathbb{R}^d)$ with the theory of independent scattered random measures of Rajput and Rosinski [RR89]. By doing so, we deduce the following result (see Section 3.2.2 and [DFHU]).

For f a measurable function, the random variable $\langle w, f\rangle$ is well-defined if and only if

$$\Theta(f) = \int_{\mathbb{R}^d} \Theta(f(\boldsymbol{x}))\mathrm{d}\boldsymbol{x} < \infty,$$

with $\Theta(\xi) = (\sigma\xi)^2 + \int_{\mathbb{R}} \min((t\xi)^2, 1)\nu(\mathrm{d}t)$ and (σ^2, ν) the variance and Lévy measure of the symmetric Lévy noise w (see (1.2)).

We call Θ the Rajput-Rosinski exponent of w. One easily remarks that $\Theta(f)$ is finite if f is compactly supported and bounded. In particular, $\Theta(\mathbb{1}_{[0,1]^d}) = \sigma^2 + \int_{\mathbb{R}} \min(t^2, 1)\nu(\mathrm{d}t) < \infty$, and the random variable $\langle w, \mathbb{1}_{[0,1]^d}\rangle$ is well-defined for any Lévy noise, as already announced.
We denote by $L_0(\Omega)$ the space of real random variables and by $L_\Theta(\mathbb{R}^d) = \{f \mid \Theta(f) < \infty\}$ the domain of definition of w. These spaces are both endowed with a topology of generalized Orlicz spaces. Then, we have two fundamental consequences that extend respectively the domain of definition of the noise, and the domain of continuity of its characteristic functional. While the two results below are *a priori* valid for test functions in $\mathcal{S}(\mathbb{R}^d)$ by definition, our contribution here is to delineate the maximal domain of definition of w (see Section 3.2.2 and [DFHU]).

The mapping that associates $\langle w, f\rangle$ to f is linear and continuous from $L_\Theta(\mathbb{R}^d)$ to $L_0(\Omega)$. Moreover, the characteristic functional $\widehat{\mathscr{P}}_w$ is continuous and positive-definite over $L_\Theta(\mathbb{R}^d)$.

In addition to these results, we provide simple criteria on Θ and ν to ensure a proper definition over L_p-type spaces. We give here our two main results (see Section 3.2.4 and [DFHU]). For $p_0, p_\infty \geq 0$, we set

$$L_{p_0,p_\infty}(\mathbb{R}^d) := \left\{ f \;\middle|\; \int_{\mathbb{R}^d} \left(\left|f(\boldsymbol{x})\right|^{p_0} \mathbb{1}_{|f(\boldsymbol{x})|>1} + \left|f(\boldsymbol{x})\right|^{p_\infty} \mathbb{1}_{|f(\boldsymbol{x})|\leq 1} \right) \mathrm{d}\boldsymbol{x} < \infty \right\}.$$

If $\Theta(\xi) \underset{0}{\sim} A|\xi|^{p_\infty}$ and $\Theta(\xi) \underset{\infty}{\sim} B|\xi|^{p_0}$, then

$$L_\Theta(\mathbb{R}^d) = L_{p_0,p_\infty}(\mathbb{R}^d). \tag{1.3}$$

If $\int_{|t|\leq 1} |t|^{p_0} \nu(\mathrm{d}t) + \int_{|t|>1} |t|^{p_\infty} \nu(\mathrm{d}t) < \infty$, then

$$L_{p_0,p_\infty}(\mathbb{R}^d) \subseteq L_\Theta(\mathbb{R}^d). \tag{1.4}$$

The criterion (1.3) allows identifying the domain of definition of Gaussian, SαS, Laplace, and compound Poisson noises. The embedding (1.4) that connects the moments of the Lévy measure to the domain of definition is used to specify general existence criteria for generalized Lévy processes.

Existence criterion for generalized Lévy processes. A (possibly fractional) differential operator L and a tempered Lévy noise w being given, can we construct a generalized random process s such that Ls and w have the same law? If yes, we say that L and w are compatible and we call s a *generalized Lévy process*. This question was addressed in [UT14]. Subfamilies for specific operators and/or noise are studied in [HL07, Taf11, SU12, UTS14].

The general principle is as follows. We want to specify s from its characteristic functional. To do so, assume that there exists a linear and continuous operator T, left-inverse[3] of the adjoint L* of L, such that the functional $\varphi \mapsto \widehat{\mathscr{P}}_w(\mathrm{T}\varphi)$ is the valid characteristic functional of a generalized random process s; that is,

$$\widehat{\mathscr{P}}_s(\varphi) = \widehat{\mathscr{P}}_w(\mathrm{T}\varphi). \tag{1.5}$$

We then have, by duality and using the left-inverse property, that

$$\widehat{\mathscr{P}}_{\mathrm{L}s}(\varphi) = \mathbb{E}[\mathrm{e}^{\mathrm{i}\langle \mathrm{L}s,\varphi\rangle}] = \mathbb{E}[\mathrm{e}^{\mathrm{i}\langle s,\mathrm{L}^*\varphi\rangle}] = \widehat{\mathscr{P}}_s(\mathrm{L}^*\varphi) = \widehat{\mathscr{P}}_w(\mathrm{TL}^*\varphi) = \widehat{\mathscr{P}}_w(\varphi). \tag{1.6}$$

In other terms, Ls and w have the same law and s is a generalized Lévy process. Our contribution is to identify the most general conditions (the key ingredient being the identification of the domain of definition of the Lévy noise) such that (1.5) is a valid characteristic functional (see Section 3.3.1 and [DFHU]).

Assume that there exists a linear operator T such that

- $\mathrm{TL}^*\{\varphi\} = \varphi$ for every $\varphi \in \mathscr{S}(\mathbb{R}^d)$; and
- T maps continuously $\mathscr{S}(\mathbb{R}^d)$ into $L_\Theta(\mathbb{R}^d)$.

Then, there exists a generalized random process s with characteristic functional $\widehat{\mathscr{P}}_s(\varphi) = \widehat{\mathscr{P}}_w(\mathrm{T}\{\varphi\})$, and s satisfies L$s = w$ in law.

[3] It is sufficient to know that T is a *left*-inverse, as seen in (1.6). This is important because it allows for a correction of the usual and unstable inverses related to differential or pseudo-differential operators.

A sufficient condition for the well-definiteness of s is the existence of T and $0 < p_0, p_\infty \leq 2$ such that T maps continuously $\mathscr{S}(\mathbb{R}^d)$ to $L_{p_0,p_\infty}(\mathbb{R}^d)$ and

$$\int_{|t|\leq 1} |t|^{p_0}\,\nu(\mathrm{d}t) + \int_{|t|>1} |t|^{p_\infty}\,\nu(\mathrm{d}t) < \infty.$$

This last criterion allows us to improve the known existence results, which involve L_p-stable operators $\mathrm{T}: \mathscr{S}(\mathbb{R}^d) \to L_p(\mathbb{R}^d)$.

1.3.2 Convergence Theorems

In the framework of tempered generalized random processes, the convergence in law of random processes is characterized by the pointwise convergence of their characteristic functionals. We exploit this characterization to deduce two convergence theorems for generalized Lévy processes.

Generalized Lévy processes as limits of generalized Poisson processes. It is known that any infinitely divisible random variable is the limit in law of compound Poisson random variables. We extend this result in the infinite-dimensional setting of generalized random processes (see Section 4.2 and [FUU17]).

Any generalized Lévy process s is the limit in law of a family of generalized Poisson processes with the same whitening operator.

The key idea is to consider compound Poisson noise with an increasing average number of impulses per unit of volume and a decreasing intensity of jumps. By combining these two effects adequately, one reconstructs the generalized Lévy process s at the limit.

A Generalized Poisson process is piecewise-smooth. Applying the whitening operator transforms it into a sum of weighted Dirac impulses with random weights and jumps locations. This allows us to interpret a generalized Poisson process as a random L-spline. This connection with splines gives a new interpretation of generalized Lévy processes. They are limits in law of random splines with more and more jumps per unit of volume, and whose weights of jumps are more and more concentrated towards the origin.

Scaling limits of generalized Lévy processes. We address the questions of the limit in law of a generalized Lévy process when we zoom into it (local behavior) and when we zoom out of it (asymptotic behavior). These questions are understood up to possible renormalization. More precisely, we aim at identifying H_{loc} (H_{asymp}, respectively) such that $a^{H_{\mathrm{loc}}} s(\cdot/a)$ ($a^{H_{\mathrm{asymp}}} s(\cdot/a)$, respectively) has a limit in law as $a \to \infty$ (as $a \to 0$, respectively).

For self-similar Lévy processes, the answer is straightforward since $a^H s(\cdot/a) = s$ in law, for any $a > 0$, where the exponent H is the self-similarity index of s. With the adequate renormalization, a reascaling of the process does not affect its law and $H_{\mathrm{loc}} = H_{\mathrm{asymp}} = H$. The only self-similar Lévy processes are driven by SαS white noise and whitened by homogeneous operators. For other members of the family, the previous argument is no longer valid. However, it is easy to see that if the limit of the rescaling exists (as $a \to 0$ or ∞), then the limiting process is self-similar. We therefore introduce the class of locally and asymptotically self-similar processes. We also give sufficient conditions on the generalized Lévy process such that it admits a local or asymptotic self-similar limits. We summarize our main results as follows (see Section 4.3 and [FU16]).

Let L be a γ-homogeneous operator ($\mathrm{L}\{\varphi(\cdot/a)\} = a^{-\gamma}\mathrm{L}\{\varphi\}(\cdot/a)$) and w be a Lévy noise with indices $\alpha_{\mathrm{loc}}, \alpha_{\mathrm{asymp}} > 0$. Under reinforced compatibility conditions between the whitening operator and the Lévy noise, we have the following convergences in law.

- *Coarse scale behavior:* The rescaled processes $a^{\gamma+d/\min(\alpha_{\mathrm{asymp}},2)-d} s(\cdot/a)$ converge in law to a SαS process with $\alpha = \min(\alpha_{\mathrm{asymp}}, 2)$ as $a \to 0$.
- *Fine scale behavior:* The rescaled processes $a^{\gamma+d/\alpha_{\mathrm{loc}}-d} s(\cdot/a)$ converge in law to a SαS process with $\alpha = \alpha_{\mathrm{loc}}$ as $a \to \infty$.

1.3.3 Regularity

We first focus on the Lévy noise, which is *a priori* a random element in $\mathcal{S}'(\mathbb{R}^d)$. We want to understand the smoothness and the growth rate of the noise. To do so, we consider the family of weighted Besov spaces[4] $B_p^\tau(\mathbb{R}^d;\rho)$, with $p \in (0,\infty]$ the integrability rate, $\tau \in \mathbb{R}$ the smoothness parameter, and $\rho \in \mathbb{R}$ the decay rate. The parameters τ and ρ are possibly fractional and possibly negative.

Our goal is to identify in which Besov spaces the Lévy noise is located, but also in which Besov spaces it is *not*. Assuming that we have a full answer to these questions, we are able, for any integrability rate $p > 0$, to identify the local smoothness $\tau_p(w)$ and the asymptotic decay rate $\rho_p(w)$ such that

- $w \in B_p^\tau(\mathbb{R}^d;\rho)$ almost surely as soon as $\tau < \tau_p(w)$ and $\rho < \rho_p(w)$, and
- $w \notin B_p^\tau(\mathbb{R}^d;\rho)$ almost surely as soon as $\tau > \tau_p(w)$ or $\rho > \rho_p(w)$.

Our contribution is to identify the quantities $\tau_p(w)$ and $\rho_p(w)$ for any $p > 0$ when w is Gaussian or compound Poisson, and for any $0 < p \leq 2$ or $p = 2k \geq 2$ an even integer for non-Gaussian and non-Poisson noise (see Section 5.2 and [FUW17b, FFU, AFU]).

[4] The Besov spaces are usually defined with an additional tuning parameter q. We consider here that $q = p$.

Consider a nontrivial Lévy noise w with indices $\alpha_{\mathrm{loc}} \in [0,2]$ and $\alpha_{\mathrm{asymp}} \in (0,\infty]$.

- If w is Gaussian, then, for $0 < p \leq \infty$,

$$\tau_p(w) = -d/2 \text{ and } \rho_p(w) = -d/p.$$

- If w is compound Poisson, then for every $0 < p \leq \infty$,

$$\tau_p(w) = d/p - d \text{ and } \rho_p(w) = -d/\min(\alpha_{\mathrm{asymp}}, p).$$

- If w is non-Gaussian and $\alpha_{\mathrm{loc}} = 0$ or $\alpha_{\mathrm{loc}} \neq 0$ and the Lévy exponent of w behaves like $-|\xi|^{\alpha_{\mathrm{loc}}}$ at infinity, then, for $0 < p \leq 2$ or $p = 2k \geq 2$ an even integer,

$$\tau_p(w) = d/\max(\alpha_{\mathrm{loc}}, p) - d \text{ and } \rho_p(w) = -d/\min(\alpha_{\mathrm{asymp}}, p).$$

From our results, we deduce in particular the Sobolev regularity ($p = 2$) and the Hölder regularity ($p = \infty$) of a Lévy noise.

Consider a nontrivial Lévy noise w with indices $\alpha_{\mathrm{loc}} \in [0,2]$ and $\alpha_{\mathrm{asymp}} \in (0,\infty]$.
The Sobolev smoothness and decay rate of a Lévy noise are

$$\tau_2(w) = -d/2 \text{ and } \rho_2(w) = -d/\min(\alpha_{\mathrm{asymp}}, 2).$$

If w and w_{Gauss} are respectively a non-Gaussian and a Gaussian noise, their Hölder smoothness is

$$\tau_\infty(w) = -d < \tau_\infty(w_{\mathrm{Gauss}}) = -d/2$$

and their the Hölder decay rate is

$$\rho_\infty(w) = -d/\alpha_{\mathrm{asymp}} \leq \rho_\infty(w_{\mathrm{Gauss}}) = 0.$$

We then extend our result to generalized Lévy processes s driven by the Lévy noise w by considering the smoothness only. More precisely, we identify conditions on the whitening operator L such that the local smoothnesses of s and w satisfy $\tau_p(s) = \tau_p(w) + \gamma$ for any $p > 0$ and a fixed $\gamma \geq 0$. Under these conditions, we directly deduce the local smoothness of a generalized Lévy process from the one of its innovation.

1.3.4 Compressibility

We have seen in Section 1.2.2 that non-Gaussian generalized Lévy processes are good candidates for the modeling of sparse signals. We have referred both to empirical evidence and theoretical arguments, the latter being focused on discrete results. We provide a mathematical justification—the first one, to the best of our knowledge—that innovations with no Gaussian parts are actually *sparse*, in the sense that they are *locally* more compressible than their Gaussian counterpart in wavelet bases. We also propose a way to quantify the compressibility, and therefore to sort generalized Lévy processes by their level of compressibility. We are interested in the local behavior, hence we consider the random processes over $\mathbb{T}^d = [0,1]^d$.

We approximate a generalized Lévy process s into a Daubechies wavelet basis. We denote by s_N its best N-term approximation. The speed of convergence of $\|s - s_N\|_2$ measures the sparsity of s. For generalized Lévy processes, this quantity has a polynomial, or faster-than-polynomial, decay. Roughly speaking, we can therefore define the *compressibility* of s as the quantity $\kappa(s)$ such that

$$\|s - s_N\|_2 \approx CN^{-\kappa(s)}$$

for some (random) constant $C > 0$, with the convention that $\kappa(s) = \infty$ if $N^\kappa \|s - s_N\|_2$ vanishes for any κ.

It is well-known that the speed of decay of $\|s - s_N\|_2$—which essentially measures the speed of decay of the wavelet coefficients of s—is strongly related to the smoothness of s: the more regular the process s, the faster the decay of its approximation error. More generally, the compressibility of a function is fully characterized by its Besov smoothness: knowing the local smoothness $\tau_p(f)$ for $p \leq 2$ completely determines the compressibility $\kappa(f)$. We apply the tools of approximation theory and our results on the Besov regularity of generalized Lévy processes to deduce the following results (see Section 6.2 and [FUW17a]).

Let s (s_{Gauss}, respectively) be a generalized Lévy process with whitening operator L and Lévy noise w (and Gaussian noise w_{Gauss}, respectively). We assume that L reduces the smoothness of any generalized function of an order $\gamma > d/2$ and denote by $\alpha_{\text{loc}} \in [0,2]$ the local index of w. Then, we have that

$$\kappa(s_{\text{Gauss}}) = \frac{\gamma}{d} - \frac{1}{2} \leq \frac{\gamma}{d} + \frac{1}{\alpha_{\text{loc}}} - 1 \leq \kappa(s).$$

Moreover, for the cases when $\tau_p(w)$ is completely determined ($\alpha_{\text{loc}} = 0$ or $\alpha_{\text{loc}} > 0$ and the Lévy exponent of w behaves like $-|\xi|^{\alpha_{\text{loc}}}$ at infinity), we have $\kappa(s) = \frac{\gamma}{d} + \frac{1}{\alpha_{\text{loc}}} - 1$.

As soon as $\alpha_{\text{loc}} < 2$, a generalized Lévy process is strictly more compressible than its Gaussian counterpart. Moreover, the compressibility of the process increases when α_{loc} diminishes. In the extreme case of $\alpha_{\text{loc}} = 0$ (for instance for compound Poisson noise), the compressibility is infinite: the approximation error decays faster than polynomial, which corresponds to the sparsest scenario.

Most of the results discussed above can be revisited from the duality between the local and asymptotic behavior of the Lévy noise or the generalized Lévy process. This will be further discussed in the conclusion (Chapter 7).

2 When Probability Meets Generalized Functions

A random process is a random function; that is, a random variable taking value in a function space. The probability law of the process is a probability measure on this function space. For instance, Brownian motion is the random process whose probability law is the Wiener measure on the space of continuous functions [KS12, Section 2.4]. This approach is admittedly quite abstract: The theory of random processes is built upon measure theory on infinite-dimensional Banach spaces [VTC87, LT13], nuclear spaces [GV64, Itô84], or more generally on topological vector spaces [Bog07, Mus96, Sch73b]. This is not the most standard construction, but it has the advantage of being very general.

We focus our attention on the theory of generalized random processes, initially introduced independently by K. Itō [Itô54] and I.M. Gelfand [Gel55] in the 50's, and brought to light by the latter, together with N.Y. Vilenkin, in [GV64, Chapter III]. A generalized random process is a random element in the space of generalized functions (or distribution, but we shall not use this terminology to avoid confusion with the concept of probability distribution). Generalized random processes are therefore the stochastic counterpart of the deterministic theory of generalized functions of Schwartz [Sch66].

The chapter is organized as follows. The mathematical backgrounds of probability theory and functional analysis are respectively covered in Sections 2.1 and 2.2, which are also useful to fix some notations and conventions. In Section 2.3, we introduce generalized random processes. A special emphasis is laid on the characteristic functional—the Fourier transform of the probability law of a generalized random process—as it will be one of our main tool for both the construction and the study of generalized Lévy processes.

Our personal contributions in Section 2.3 are twofolds. First, we present a systematic exposition of the framework in the space of *tempered* generalized functions $\mathscr{S}'(\mathbb{R}^d)$ that appears to be more convenient for signal processing applications, while the historical approach of Gelfand and Vilenkin was developed on $\mathscr{D}'(\mathbb{R}^d)$. Second, we extend some results on the measurability of function spaces into $\mathscr{S}'(\mathbb{R}^d)$ in order to include the complete family of Besov spaces.

2.1 Probability Theory in Finite Dimension

We review the basic notions of probability theory in finite dimension. The results of Section 2.1.1 are very classical; see for instance [Kal06]. Sections 2.1.2 and 2.1.3 focus on infinitely divisible random variables, which will play a crucial role when considering continuous-domain random processes [Sat13].

Once and for all, we fix a complete probability space $(\Omega, \mathscr{F}, \mathscr{P})$, where Ω is the *sample space* of all possible *outcomes* $\omega \in \Omega$, $\mathscr{F}$ is a set of *events*, assumed to be a σ-algebra on Ω, and $\mathscr{P} : \mathscr{F} \to [0,1]$ is a probability measure on Ω. The space Ω is the source of randomness that allows us to define the concepts of real random variables, random vectors, and (generalized) random processes. We assume that our probability space is rich enough so that all the stochastic objects encountered in our work are well-defined[1].

2.1.1 Real Random Variables and Vectors

The Borel σ-field $\mathscr{B}(\mathbb{R}^N)$ on $\mathbb{R}^N$ is the σ-field generated by the open balls of $\mathbb{R}^N$.

Definition 2.1. *A* real random variable X *is a measurable function from* $(\Omega, \mathscr{F})$ *to* $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$. *The* probability law *(or simply the* law*) of* X *is then the probability measure on* $\mathbb{R}$ *defined for* $B \in \mathscr{B}(\mathbb{R})$ *by*

$$\mathscr{P}_X(B) = \mathscr{P}(X \in B) = \mathscr{P}\{\omega \in \Omega \mid X(\omega) \in B\}.$$

Let $L_0(\Omega)$ be the space of real random variables. For $p > 0$, we also introduce $L_p(\Omega)$ as the space of real random variables $X \in L_0(\Omega)$ such that $\mathbb{E}[|X|^p] < \infty$.

Proposition 2.1. *The space* $L_0(\Omega)$ *is a complete linear metric space for the translation invariant metric*

$$\|X\|_0 := \mathbb{E}[\min(|X|, 1)].$$

The space $L_p(\Omega)$ *is a quasi-Banach space for* $0 < p < 1$*, and a Banach space for* $1 \leq p$*, for the following (quasi-)norm*

$$\|X\|_p := \left(\mathbb{E}[|X|^p]\right)^{1/p}.$$

For the spaces $L_p(\Omega)$ with $p \geq 1$, the result is well-known. The case $0 < p < 1$ is less classical; see for instance [Gra04, Section 1.1] for more details. The convergence in $L_0(\Omega)$ is equivalent to the convergence in probability.

Definition 2.2. *The* characteristic function *of a real random variable* X *is the function* $\widehat{\mathscr{P}}_X$:

[1] Even if it is at the heart of the axiomatisation of probability theory [Kol50], the construction of such a probability space will not be discussed here. It is sufficient to know that we can consider $\Omega = \mathscr{D}'(\mathbb{R}^d)$, the space of generalized functions (see Section 2.2.1), with the adequate σ-field, for the definition of a generalized random process (or $\Omega = \mathscr{S}'(\mathbb{R}^d)$ when this process is tempered).

$\mathbb{R} \to \mathbb{C}$ *such that*

$$\widehat{\mathscr{P}}_X(\xi) = \mathbb{E}\left[e^{i\xi X}\right] = \int_{\mathbb{R}} e^{i\xi x} d\mathscr{P}_X(x)$$

for every $\xi \in \mathbb{R}$.

The characteristic function is nothing more than the Fourier transform of the probability law of X (up to sign convention). Any characteristic function is continuous, normalized such that $\widehat{\mathscr{P}}_X(0) = 1$, and positive-definite over $\mathbb{R}$, meaning that

$$\sum_{m=1}^{N} \sum_{n=1}^{N} a_n a_m^* \widehat{\mathscr{P}}_X(\xi_n - \xi_m) \geq 0$$

for any $N \geq 1$, $a_n \in \mathbb{C}$, $\xi_n \in \mathbb{R}$. The converse of this result is true and is a characterization of the Fourier transforms of probability measures on $\mathbb{R}$: This is the Bochner theorem [Kat04, Section VI.2.8].

Proposition 2.2. *A function $\widehat{\mathscr{P}}$ that is continuous and positive-definite from $\mathbb{R}$ to $\mathbb{C}$ and such that $\widehat{\mathscr{P}}(0) = 1$ is the characteristic function of a real random variable $X \in L_0(\Omega)$.*

It is easy to check that $\xi \mapsto e^{-\xi^2/2}$ satisfies the conditions of Proposition 2.2 (see for instance [UT14, Appendix B.1]), and is therefore the characteristic function of a random variable. Of course, one recognizes the Gaussian law, more traditionally introduced via its probability density function $p_{\mathrm{Gauss}}(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} = \mathscr{F}^{-1}\{e^{-\cdot^2/2}\}(x)$. The Bochner theorem is an alternative to *construct* the Gaussian random variable without specifying its probability density function. Therefore, it does not require the existence of the Lebesgue measure. This will become crucial in infinite dimensional spaces, where the Lebesgue measure does not exist in general [Eld16, Theorem 1.1].

Definition 2.3. *We say that a sequence of random variables $(X_k)_{k\geq 0}$* converges in law *to the random variable X if*

$$\mathbb{E}\left[f(X_k)\right] \underset{k\to\infty}{\longrightarrow} \mathbb{E}\left[f(X)\right]$$

for any continuous and bounded function $f : \mathbb{R} \to \mathbb{R}$. We denote this situation by $X_k \xrightarrow[k\to\infty]{(\mathscr{L})} X$.

Theorem 2.1. *The sequence of random variables $(X_k)_{k\geq 0}$ converges in law to the random variable X if and only if*

$$\widehat{\mathscr{P}}_{X_k}(\xi) \underset{k\to\infty}{\longrightarrow} \widehat{\mathscr{P}}_X(\xi)$$

for any $\xi \in \mathbb{R}$.

This is the Lévy continuity theorem [Kal06, Theorem 5.3]. In other terms, the convergence in law of real random variables is equivalent to the pointwise convergence of the underlying characteristic functions to a characteristic function.

Definition 2.4. *Two random variables* X_1 *and* X_2 *are* independent *if the events* $\{X_1 \in B_1\}$ *and* $\{X_2 \in B_2\}$ *are independent for any* $B_1, B_2 \in \mathscr{B}(\mathbb{R})$*; that is, if*

$$\mathscr{P}((X_1, X_2) \in B_1 \times B_2) = \mathscr{P}(X_1 \in B_1)\mathscr{P}(X_2 \in B_2).$$

The independence of X_1 and X_2 is equivalent to the relation $\widehat{\mathscr{P}}_{X_1+X_2}(\xi) = \widehat{\mathscr{P}}_{X_1}(\xi)\widehat{\mathscr{P}}_{X_2}(\xi)$ for any $\xi \in \mathbb{R}$. Then, the law of $X_1 + X_2$ is the convolution of the laws of X_1 and X_2.
We now consider random variables with values in $\mathbb{R}^N$ for $N \geq 1$. Vectors will be denoted by $\boldsymbol{x} = (x_1, \dots, x_N) \in \mathbb{R}^N$.

Definition 2.5. *A* random vector $\boldsymbol{X} = (X_1, \dots, X_N)$ *of dimension* N *is a measurable function from* $(\Omega, \mathscr{F})$ *to* $(\mathbb{R}^N, \mathscr{B}(\mathbb{R}^N))$.

We define the law of a random vector as we did for real random variables. The characteristic function of $\boldsymbol{X}$ is the function $\widehat{\mathscr{P}}_{\boldsymbol{X}} : \mathbb{R}^N \to \mathbb{C}$ such that

$$\widehat{\mathscr{P}}_{\boldsymbol{X}}(\xi) = \mathbb{E}\left[e^{i\langle \boldsymbol{\xi}, \boldsymbol{X}\rangle}\right]$$

for any $\boldsymbol{\xi} \in \mathbb{R}^N$, where $\langle\cdot,\cdot\rangle$ is the usual scalar product on $\mathbb{R}^N$. Bochner's theorem, the convergence in law, Lévy's continuity theorem, and the notion of independence are easily extended to random vectors. We observe that the mutual independence of the random variables $X_1, \dots, X_N$ is equivalent to

$$\widehat{\mathscr{P}}_{(X_1,\dots,X_N)}(\xi_1, \dots, \xi_N) = \widehat{\mathscr{P}}_{X_1}(\xi_1)\cdots\widehat{\mathscr{P}}_{X_N}(\xi_N).$$

2.1.2 Infinitely Divisible Random Variables and their Indices

We briefly introduce the family of infinitely divisible random variables. They will play a crucial role when defining continuous-domain random processes in Section 3. We refer the reader to [Sat13] for an in-depth exposition on the subject and to [MR08] for a discussion on the origin of the concept.

Definition 2.6. *A random variable* X *is* infinitely divisible *if, for any* $N \geq 1$*, it can be decomposed as*

$$X = X_{1,N} + \cdots + X_{N,N}$$

where $X_{1,N}, \dots, X_{N,N}$ *are i.i.d. random variables.*

The characteristic function $\widehat{\mathscr{P}}_X$ of the infinitely divisible random variable X can therefore be written as $\widehat{\mathscr{P}}_X(\xi) = \widehat{\mathscr{P}}_{X_1^N}(\xi) \times \cdots \times \widehat{\mathscr{P}}_{X_N^N}(\xi) = (\widehat{\mathscr{P}}_{X_1^N}(\xi))^N$ for every N. An infinitely divisible random variable is therefore a random variable such that its characteristic function admits an Nth root that is itself a characteristic function for every $N \geq 1$.

Lévy exponent. If X is infinitely divisible, then $\widehat{\mathscr{P}}_X(\xi) \neq 0$ for every $\xi \in \mathbb{R}$ [Sat13, Lemma 7.5]. Then, one can show that there exists a continuous function Ψ such that

$$\widehat{\mathscr{P}}_X(\xi) = \exp(\Psi(\xi)).$$

We would like to emphasis that the existence of a *continuous* Ψ is not obvious, as explained in [Sat13, Lemma 7.6].
In general, the function $\xi \mapsto \exp(\Psi(\xi))$ is the characteristic function of an infinitely divisible law if and only if $\xi \mapsto \exp(\tau\Psi(\xi))$ is a characteristic function for any $\tau \in \mathbb{R}$.

Definition 2.7. *The continuous log-characteristic function of an infinitely divisible random variable is its* Lévy exponent.

In the literature, Ψ is often called the characteristic exponent of X.

Theorem 2.2. *Let Ψ be a continuous function with $\Psi(0) = 0$. The following statements are equivalent*

1. *The function Ψ is a Lévy exponent.*

2. *For every $\lambda \geq 0$, the function $\xi \mapsto e^{\lambda\Psi(\xi)}$ is positive-definite.*

3. *The function Ψ is* conditionally positive-definite on $\mathbb{R}$, *meaning that*

$$\sum_{m,n=1}^{N} a_m a_n^* \Psi(\xi_m - \xi_n) \geq 0$$

for any $N \geq 1$, $a_n \in \mathbb{C}$, and $\xi_n \in \mathbb{R}$ such that $\sum_{n=1}^N \xi_n = 0$.

4. *The function Ψ can be decomposed as*

$$\Psi(\xi) = i\mu\xi - \frac{\sigma^2\xi^2}{2} + \int_{\mathbb{R}} (e^{i\xi t} - 1 - i\xi t \mathbb{1}_{|t|\leq 1})\nu(dt), \tag{2.1}$$

where $\mu \in \mathbb{R}$, $\sigma^2 \geq 0$, and ν a Lévy measure*; that is, a measure on $\mathbb{R}$ such that*

$$\int_{\mathbb{R}} \min(1, t^2)\nu(dt) < \infty \tag{2.2}$$

and $\nu\{0\} = 0$.

Note that $\left|e^{i\xi t} - 1 - i\xi t \mathbb{1}_{|t|\leq 1}\right| \leq 2\min(1, \xi^2 t^2)$ so that the integral in (2.1) is well-defined under the condition (2.2). These equivalences are proved in [GV64, Section III.4]. See also [UT14, Appendix B] for a discussion on positive-definite and conditionally positive-definite functions. The decomposition (2.1) is the famous Lévy-Khintchine representation of the Lévy exponent. The triplet (μ, σ^2, ν) is unique [Sat13, Theorem 8.1] and called the *Lévy triplet* of Ψ (or, equivalently, of the underlying infinitely divisible random variable).

Moments of infinitely divisible laws. The absolute moments of an infinitely divisible random variable are related to the absolute moments of the Lévy measure.

Proposition 2.3. *For the infinitely divisible law X with Lévy measure ν, we have the equivalence, for any $p > 0$,*

$$\mathbb{E}[|X|^p] < \infty \Longleftrightarrow \int_{|t|>1} |t|^p \,\nu(\mathrm{d}t) < \infty.$$

This is a particular case of [Sat13, Theorem 25.3]. In general, the Lévy exponent Ψ of the infinitely divisible law X can be bounded as

$$|\Psi(\xi)| \le C(1+|\xi|^2)$$

for some $C > 0$ and every $\xi \in \mathbb{R}$. When X has some finite moments, we have a better bound.

Proposition 2.4. *If the Lévy measure ν of the Lévy exponent Ψ satisfies the condition*

$$\int_{|t|>1} |t|^p \,\nu(\mathrm{d}t) < \infty$$

for some $0 < p \le 1$, then there exists a constant $C > 0$ such that, for every $\xi \in \mathbb{R}$,

$$|\Psi(\xi)| \le C\left(|\xi|^p + |\xi|^2\right). \tag{2.3}$$

The crucial point in (2.3) is that Ψ is dominated at the origin by a power law.

Proof. We recall the Lévy-Khintchine representation (2.1) of Ψ as

$$\Psi(\xi) = \mathrm{i}\mu\xi - \frac{\sigma^2\xi^2}{2} + \int_{\mathbb{R}} (\mathrm{e}^{\mathrm{i}\xi t} - 1 - \mathrm{i}\xi t \mathbb{1}_{|t|\le 1})\nu(\mathrm{d}t).$$

Since $\xi \mapsto \mathrm{i}\mu\xi - \frac{\sigma^2\xi^2}{2}$ is clearly dominated by $\xi \mapsto |\xi|^p + |\xi|^2$ (since $p \le 1$), we assume without lost of generality that $\mu = \sigma^2 = 0$. Then, we split the integral in two terms. First, we have that $\left|\mathrm{e}^{\mathrm{i}x} - 1 - \mathrm{i}x\right| \le x^2$ for any $x \in \mathbb{R}$. Applying this inequality to $x = \xi t$, we deduce that

$$\int_{|t|\le 1} \left|\mathrm{e}^{\mathrm{i}\xi t} - 1 - \mathrm{i}\xi t\right| \nu(\mathrm{d}t) \le \left(\int_{|t|\le 1} |t|^2 \,\nu(\mathrm{d}t)\right) |\xi|^2. \tag{2.4}$$

Moreover, we have that $\left|\mathrm{e}^{\mathrm{i}x} - 1\right|^2 = 2 - 2\cos x \le 2\min(2, x^2) \le 4\,|x|^{2p}$ (since $2p \le 2$), from which we deduce that

$$\int_{|t|>1} \left|\mathrm{e}^{\mathrm{i}\xi t} - 1\right| \nu(\mathrm{d}t) \le 2\left(\int_{|t|>1} |t|^p \,\nu(\mathrm{d}t)\right) |\xi|^p. \tag{2.5}$$

Combining (2.4) and (2.5), we easily obtain (2.3). □

The Lévy exponent is the cumulant generating function, in the sense that its Taylor expansion at the origin gives access to the cumulants [UT14, Section 9.6].

Proposition 2.5. *Let X be an infinitely divisible random variable with Lévy exponent Ψ. The Lévy exponent Ψ is N times continuously differentiable for $N \geq 1$ if and only if the Nth moment of X is finite. In that case, the Nth-cumulant $\kappa_N(X)$ of X is well-defined and is given by*

$$\kappa_N(X) = (-\mathrm{i})^N \Psi^{(N)}(0).$$

Indices of Infinitely divisible random variables. In this thesis, we assume that all the infinitely divisible laws satisfy the so-called *sector condition*: The imaginary part of the associated Lévy exponent is controlled by the real part in the sense that

$$|\Im\{\Psi(\xi)\}| \leq C|\Re\{\Psi(\xi)\}| \tag{2.6}$$

for some $C > 0$ and every $\xi \in \mathbb{R}$. Essentially, this conditions implies that the underlying infinitely divisible random variable is not dominated by a drift. For instance, the pure drift $X = \mu$, where $\mu \neq 0$ is a deterministic constant, is such that $\Psi(\xi) = \mathrm{i}\mu\xi$. It is therefore purely imaginary and does not satisfy the sector condition. The sector condition is automatically satisfied when X is symmetric, since Ψ is purely real in that case.

Definition 2.8. *Let X be a infinitely divisible random variable satisfying the sector condition and ν its Lévy measure. Then, we set*

$$\alpha_{\mathrm{loc}} := \inf\left\{ p \geq 0 \,\middle|\, \int_{|t|\leq 1} |t|^p \,\nu(\mathrm{d}t) < \infty \right\}, \tag{2.7}$$

$$\alpha_{\mathrm{asymp}} := \sup\left\{ p \geq 0 \,\middle|\, \int_{|t|> 1} |t|^p \,\nu(\mathrm{d}t) < \infty \right\}, \tag{2.8}$$

We call α_{loc} and α_{asymp} the local index *and the* asymptotic index *respectively.*

Remarks.

- Necessarily, $0 \leq \alpha_{\mathrm{loc}} \leq 2$, since $\int_{|t|\leq 1} t^2 \nu(\mathrm{d}t) < \infty$ for any Lévy measure. The asymptotic index, on the other hand, can take any value including 0 and ∞. Proposition 2.3 has two implications: The case $\alpha_{\mathrm{asymp}} = 0$ implies that X has no absolute positive moments, while $\alpha_{\mathrm{asymp}} = \infty$ when all the moments of X are finite. In particular, the latter is satisfied when $\nu = 0$, corresponding to the Gaussian law.

- The index α_{loc} is often referred to as the *Blumenthal-Getoor index* in the literature. It was introduced in [BG61], in order to measure the intensity of the small jumps of Lévy processes. This index is related to the asymptotic behavior of the Lévy exponent by the relation [BSW14, Chapter 5]

$$\alpha_{\mathrm{loc}} := \inf\left\{ p \geq 0 \,\middle|\, \limsup_{|\xi|\to\infty} \frac{|\Psi(\xi)|}{|\xi|^p} < \infty \right\}.$$

- In [Pru81], Pruitt measured the intensity of the large jumps of Lévy processes, by applying his results on the asymptotic behavior of series of i.i.d. random variables. To do so, he introduced several indices related to infinitely divisible laws. We should focus on the following one, that we call the *Pruitt index*, defined as

$$\beta_0 := \sup \left\{ p \geq 0 \;\middle|\; \limsup_{|\xi| \to 0} \frac{|\Psi(\xi)|}{|\xi|^p} < \infty \right\}.$$

It is actually known that $\beta_0 = \sup\left\{0 \leq p \leq 2 \mid \int_{|t|>1} |t|^p \, \nu(\mathrm{d}t) < \infty\right\}$ [Sat13, Proposition 48.10]. By comparing with (2.8), we deduce that the Pruitt index and the asymptotic index are linked by the relation

$$\beta_0 = \inf(\alpha_{\mathrm{asymp}}, 2). \tag{2.9}$$

Importantly, the Pruitt index can be deduced from the asymptotic index, while the converse is false.

- The Blumenthal-Getoor and Pruitt indices are respectively denoted by β_∞ and β_0 in the literature. This reminds us that they are respectively linked with the asymptotic behavior and the behavior at the origin of the Lévy exponent. We prefer to rename the Blumenthal-Getoor α_{loc}, and to introduce the new index α_{asymp} from which we can easily recover β_0 due to (2.9). We have several motivations for these new notations. First, for symmetric-α-stable random variables (see Section 2.1.3), one has $\alpha_{\mathrm{loc}} = \alpha_{\mathrm{asymp}} = \alpha$, so that the indices generalize the parameter α for non-stable infinitely divisible laws. Second, the local index characterizes the local smoothness of the Lévy noise, while the asymptotic index is linked to its asymptotic decay rate (see Theorem 5.3 in Section 5.2). The Pruitt index will be shown to play a crucial role on the behavior of Lévy noise and generalized Lévy processes at coarse scale (see Section 4.3). We prefer to use notations inspired by these fundamental properties.

2.1.3 Examples of Infinitely Divisible Laws

We present some classical families of infinitely divisible random variables: Gaussian, SαS, compound Poisson, and generalized Laplace. They will be our running examples, illustrating our results throughout the thesis. For each case, when closed forms are known, we provide the probability law, the characteristic function, the Lévy exponent, the Lévy triplet, as well as the local and asymptotic indices.

Gaussian random variables. A random variable X is called a *Gaussian random variable* of variance σ^2, which is denoted by $X \sim \mathcal{N}(0, \sigma^2)$, if its probability density is given by

$$p_X(x) = \frac{1}{\sqrt{2\pi}\sigma} \mathrm{e}^{-\frac{x^2}{2\sigma^2}}.$$

The characteristic function of X is

$$\widehat{\mathscr{P}}_X(\xi) = \mathrm{e}^{-\frac{\sigma^2 \xi^2}{2}}.$$

The random variable X is infinitely divisible, since it can be written as N independent Gaussian random variables with variance σ^2/N. Its Lévy exponent is $\Psi(\xi) = -\frac{\sigma^2\xi^2}{2}$ and Lévy triplet $(0, \sigma^2, 0)$. We easily see that $\alpha_{\text{loc}} = 2$ and $\alpha_{\text{asymp}} = \infty$.

Symmetric-α-stable random variables. We refer the reader to [ST94] for a complete exposition on stable laws, including the proofs of the results stated thereafter. A random variable X is stable if the sum $X_1 + X_2$ of two independent copies of X has the same law as $aX + b$ for some real numbers a and b. It is of course the case for Gaussian random variables. The other members of the family have an infinite variance. Stable laws are infinitely divisible.

We restrict our descriptions to symmetric stable random variables. In that case, we have necessarily $X_1 + X_2 \overset{(\mathscr{L})}{=} 2^\alpha X_1$ for some parameter $\alpha \in (0, 2]$. Symmetric stable random variables are therefore called SαS (for symmetric-α-stable). The characteristic function of X is of the form

$$\Phi_X(\xi) = \mathrm{e}^{-c^\alpha |\xi|^\alpha},$$

with $c > 0$ the *scaling parameter* and $\alpha \in (0, 2]$. We write in this case that $X \sim \mathscr{S}(\alpha, c)$. Observe that $X \sim \mathscr{S}(\alpha, 1)$ if and only if $cX \sim \mathscr{S}(\alpha, c)$ and that $\mathscr{S}(2, c) = \mathscr{N}(0, 2c)$. The Lévy exponent of X is $\Psi(\xi) = -c^\alpha |\xi|^\alpha$, and the Lévy triplet $(0, 0, c^\alpha \nu_\alpha)$, where the Lévy measure ν_α is given by

$$\nu_\alpha(\mathrm{d}t) = \frac{C_\alpha}{|t|^{\alpha+1}} \mathrm{d}t,$$

with $C_\alpha = (\int_{\mathbb{R}} (1 - \cos u) \frac{\mathrm{d}u}{|u|^{\alpha+1}})^{-1}$. The indices, which are easily computed from the Lévy measure, are $\alpha_{\text{loc}} = \alpha_{\text{asymp}} = \alpha$.

Compound-Poisson random variables. We say that X is a *compound-Poisson random variable* X if it can be written as

$$X = \sum_{n=1}^{N} X_n,$$

with N a Poisson random variable of parameter $\lambda > 0$—meaning that $\mathscr{P}(N = n_0) = \mathrm{e}^{-\lambda} \lambda^{n_0} / n_0!$—, and the X_n are i.i.d. with common probability law P such that $P\{0\} = 0$. The parameters of the compound Poisson random variables are therefore λ and P, respectively called the *sparsity parameter* and the *law of jumps* due to their role on compound Poisson noise (see Section 3.1.2). We denote this situation by $X \sim \mathscr{P}(\lambda, P)$. By conditioning the value of N, we see that the probability law of X is

$$\mathscr{P}_X = \mathrm{e}^{-\lambda} \sum_{n \geq 0} \frac{\lambda^n}{n!} P^{*n},$$

where $P^{*0} = \delta$ and $P^{*(n+1)} = (P * P^{*n})$. In Fourier domain, one has $\widehat{P^{*n}}(\xi) = \widehat{P}^n(\xi)$. We deduce the characteristic function of X, given for every $\xi \in \mathbb{R}$ by

$$\widehat{\mathscr{P}}_X(\xi) = \exp\left(\lambda(\widehat{P}(\xi) - 1)\right),$$

with $\widehat{P}$ the characteristic function associated to P. The Lévy exponent is $\Psi(\xi) = \lambda(\widehat{P}(\xi) - 1)$ and the Lévy triplet of X is $(\lambda\mu_P, 0, \lambda P)$, where $\mu_P := \int_{|t|\leq 1} tP(\mathrm{d}t)$. We have $|\widehat{P}(\xi)| \leq 1$; hence, Ψ is bounded and $\alpha_{\mathrm{loc}} = 0$. While the other index can take any value a priori, we remark that

$$\alpha_{\mathrm{asymp}} = \sup\left\{p > 0 \,\middle|\, \mathbb{E}[|Y|^p] < \infty\right\},$$

with Y a random variable with probability law P. Indeed, $\mathbb{E}[|Y|^p] = \int_{\mathbb{R}} |t|^p P(\mathrm{d}t) = \int_{|t|\leq 1} |t|^p P(\mathrm{d}t) + \int_{|t|>1} |t|^p P(\mathrm{d}t)$, the first term being always finite since P is a probability measure, and the second being finite for $p < \alpha_{\mathrm{asymp}}$ and infinite for $p > \alpha_{\mathrm{asymp}}$ (Proposition 2.3 applied to the Lévy measure λP).

Generalized Laplace random variables. Another interesting infinitely divisible family is given by the generalized-Laplace laws. We follow here the notations of [KKP01]. A *generalized-Laplace random variable* X has a characteristic function of the form

$$\widehat{\mathscr{P}}_X(\xi) = \frac{1}{(1 + \frac{1}{2}\sigma^2\xi^2)^\tau} = \exp\left(-\tau \log(1 + \frac{1}{2}\sigma^2\xi^2)\right),$$

with $\tau > 0$ the *shape parameter* and σ^2 the *scaling parameter*. We denote this situation by $X \sim \mathscr{GL}(\tau, \sigma^2)$. Generalized Laplace laws are infinitely divisible [KKP01, Section 2.4.1] with Lévy triplet $(0, 0, \nu_{\tau,\sigma^2})$ where [KKP01, Proposition 2.4.2]

$$\nu_{\tau,\sigma^2}(\mathrm{d}t) = \frac{\tau}{|t|} \mathrm{e}^{-2|t|/\sigma^2} \mathrm{d}t.$$

The Lévy exponent is $\Psi(\xi) = -\tau \log(1 + \frac{1}{2}\sigma^2\xi^2)$. The variance of X is then $\tau\sigma^2$. We easily see that $\alpha_{\mathrm{loc}} = 0$, since Ψ growths logarithmically at infinity. Moreover, all the moments of X are finite so that $\alpha_{\mathrm{asymp}} = \infty$.

2.2 Elements of Functional Analysis

A signal is modeled as a function from $\mathbb{R}^d$ to $\mathbb{R}$, with $d \geq 1$. A function space is a topological vector space whose elements are functions. Most of the spaces encountered in this section are included in the space $\mathscr{D}'(\mathbb{R}^d)$ of generalized functions, with a special emphasis on the space of tempered generalized functions $\mathscr{S}'(\mathbb{R}^d)$, introduced in Section 2.2.1. We recall important results on operators in Section 2.2.2, including the whitening operators that are considered in stochastic differential equations. The family of weighted Besov spaces, which allows to quantify the regularity of smoothness and the rate of decay of generalized functions, is presented in Section 2.2.3.

2.2.1 The Spaces $\mathscr{D}'(\mathbb{R}^d)$, $\mathscr{S}'(\mathbb{R}^d)$, and $\mathscr{S}'(\mathbb{T}^d)$

Most of the function spaces we will encounter are complete, Hausdorff, and locally convex. This means that their topology is associated to a separate family of semi-norms [Rud91], possibly infinite, possibly uncountable, for which they are complete. Among these spaces, Hilbert and Banach spaces have the simplest structure, since their topology is associated to a unique norm. As such, they will be our building blocks for the specification non-normable spaces.

We say that a semi-norm p on a topological vector space $\mathscr{X}$ is *separable* if the semi-normed space $(\mathscr{X}, N)$ has a countable dense subset. We also say that N is *Hilbertian* if it satisfies the parallelogram law. When N is a norm, this means that N is associated to a scalar product. A family of semi-norms $(N_i)_{i\in I}$ on $\mathscr{X}$ is said to be *separating* if $N_i(x) = 0$ for all $i \in I$ if and only if $x = 0$.

Definition 2.9. *A topological vector space $\mathscr{X}$ is a* multi-Hilbertian space *if there exists a separating family of Hilbertian semi-norms $(N_i)_{i\in I}$ such that the collection of sets*

$$V_{J,\epsilon,x_0} := \left\{x \in \mathscr{X} \;\middle|\; \forall j \in J,\ N_j(x - x_0) \leq \epsilon_j\right\}$$

form a complete system of neighbourhoods for the topology of $\mathscr{X}$, for J finite, $J \subset I$, $\epsilon_j > 0$, and $x_0 \in \mathscr{X}$. If the family $(N_i)_{i\in I}$ can be chosen countable, then $\mathscr{X}$ is a countably multi-Hilbertian space.

Notations. We consider functions from $\mathbb{R}^d$ to $\mathbb{R}$. A multi-index is written as $\boldsymbol{m} = (m_1, \dots, m_d) \in \mathbb{N}^d$. The partial derivative with respect to the ith-coordinate is denoted by D_i. For $\boldsymbol{m} \in \mathbb{N}^d$, we set $\mathrm{D}^{\boldsymbol{m}} = \mathrm{D}_1^{m_1} \cdots \mathrm{D}_d^{m_d}$ and $|\boldsymbol{m}| = m_1 + \cdots + m_d$.

For $0 < p \leq \infty$, the Lebesgue space $L_p(\mathbb{R}^d)$ of measurable functions with finite p-(quasi-)norm given by

$$\|f\|_p := \left(\int_{\mathbb{R}^d} \left|f(\boldsymbol{x})\right|^p \mathrm{d}\boldsymbol{x}\right)^{1/p} \quad \text{(if } p < \infty\text{)},$$

$$\|f\|_\infty := \operatorname*{ess\,sup}_{\boldsymbol{x}\in\mathbb{R}^d} \left|f(\boldsymbol{x})\right|$$

is a Banach space for $1 \leq p \leq \infty$, and a quasi-Banach space for $0 < p < 1$ [Gra04, Section 1.1]. It is a Hilbert space if and only if $p = 2$.

The space $\mathscr{D}(\mathbb{R}^d)$ and its dual. The space of compactly supported and smooth functions is denoted by $\mathscr{D}(\mathbb{R}^d)$. It is the union of the spaces $\mathscr{D}(K)$ of smooth functions whose support is included in K, where K is some compact subset of $\mathbb{R}^d$. For K fixed, the space $\mathscr{D}(K)$ is a countable multi-Hilbertian space for the Hilbertian semi-norms $(\|\mathrm{D}^{\boldsymbol{m}}\{\cdot\}\|_2)_{\boldsymbol{m}\in\mathbb{N}^d}$ [Itô84, Section 1.4]. Then, the space $\mathscr{D}(\mathbb{R}^d)$ is a complete topological vector space as the inductive limit of the spaces $\mathscr{D}([-n,n]^d)$, for $n \in \mathbb{N}$. A sequence $(\varphi_k)_{k\in\mathbb{N}}$ converges to 0 in $\mathscr{D}(\mathbb{R}^d)$ if the φ_k are in a common $\mathscr{D}(K)$ with K compact and converge to 0 in $\mathscr{D}(K)$. One can show that $\mathscr{D}(\mathbb{R}^d)$ is a multi-Hilbertian space, but not a countably multi-Hilbertian space [Itô84, Section 1.5].

The space of generalized functions $\mathscr{D}'(\mathbb{R}^d)$ is the topological dual of $\mathscr{D}(\mathbb{R}^d)$; that is, the space of continuous and linear function on $\mathscr{D}(\mathbb{R}^d)$. For $u \in \mathscr{D}'(\mathbb{R}^d)$, $\varphi, \psi \in \mathscr{D}(\mathbb{R}^d)$, and $\lambda \in \mathbb{R}$, we have $u(\varphi + \lambda\psi) = u(\varphi) + \lambda u(\psi)$. We denote $u(\varphi) = \langle u, \varphi\rangle$. Moreover, the continuity of the linear functional u is equivalent to the following condition: For every compact K, there exists $M \in \mathbb{N}$ and $C > 0$ such that for every $\varphi \in \mathscr{D}(K)$,

$$\left|\langle u, \varphi\rangle\right| \leq C \sum_{|\boldsymbol{m}|\leq M} \|\mathrm{D}^{\boldsymbol{m}}\{\varphi\}\|_2.$$

We endow the space $\mathscr{D}'(\mathbb{R}^d)$ with the weak topology. In particular, a sequence (u_k) converges to 0 in $\mathscr{D}'(\mathbb{R}^d)$ if and only if $\langle u_k, \varphi\rangle$ converges to 0 for every $\varphi \in \mathscr{D}(\mathbb{R}^d)$. A measurable function f that is locally integrable is identified with the generalized function $\varphi \mapsto \int_{\mathbb{R}^d} f(\boldsymbol{x})\varphi(\boldsymbol{x})\mathrm{d}\boldsymbol{x}$. With this identification, all the spaces $L_p(\mathbb{R}^d)$ are included in $\mathscr{D}'(\mathbb{R}^d)$ for $p \geq 1$.

The space $\mathscr{S}(\mathbb{R}^d)$ and its dual. We denote by $\mathscr{S}(\mathbb{R}^d)$ the space of smooth and rapidly decaying functions. Its topology is the one associated with the separable family of semi-norms,

$$\|\varphi\|_{2,\boldsymbol{m},\boldsymbol{n}} := \left\| \cdot^{\boldsymbol{n}}\, \mathrm{D}^{\boldsymbol{m}}\{\varphi\}\right\|_2 \tag{2.10}$$

where $\boldsymbol{m}, \boldsymbol{n} \in \mathbb{N}^d$, where $\cdot^{\boldsymbol{n}}$ is the function $\boldsymbol{x} \in \mathbb{R}^d \mapsto \boldsymbol{x}^{\boldsymbol{n}} = x_1^{n_1} \cdots x_d^{n_d}$. A sequence of functions (φ_k) converges to 0 in $\mathscr{S}(\mathbb{R}^d)$ if $\|\varphi_k\|_{2,\boldsymbol{m},\boldsymbol{n}} \to 0$ for every $\boldsymbol{m}, \boldsymbol{n} \in \mathbb{N}^d$ as $k \to \infty$. The semi-norms (2.10) are Hilbertian, so that $\mathscr{S}(\mathbb{R}^d)$ is a countably multi-Hilbertian space.

The topological dual of $\mathscr{S}(\mathbb{R}^d)$ is the space $\mathscr{S}'(\mathbb{R}^d)$ of tempered generalized functions. It is the space of continuous and linear functionals on $\mathscr{S}(\mathbb{R}^d)$. The duality product between a tempered generalized function $u \in \mathscr{S}'(\mathbb{R}^d)$ and a test function $\varphi \in \mathscr{S}(\mathbb{R}^d)$ is still denoted by $\langle u, \varphi\rangle$. For u a linear functional on $\mathscr{S}(\mathbb{R}^d)$, the continuity is equivalent to the following condition: There exists $M \in \mathbb{N}$ and $C > 0$ such that, for every $\varphi \in \mathscr{S}(\mathbb{R}^d)$,

$$\left|\langle u, \varphi\rangle\right| \leq C \sum_{|\boldsymbol{m}|\leq M} \sum_{|\boldsymbol{n}|\leq M} \|\varphi\|_{2,\boldsymbol{m},\boldsymbol{n}}. \tag{2.11}$$

As for $\mathscr{D}'(\mathbb{R}^d)$, we endow $\mathscr{S}'(\mathbb{R}^d)$ with the weak topology: a sequence (u_k) converges to 0 in

$\mathscr{S}'(\mathbb{R}^d)$ if and only if $\langle u_k, \varphi \rangle$ converge to 0 for every $\varphi \in \mathscr{S}(\mathbb{R}^d)$. The space $\mathscr{S}'(\mathbb{R}^d)$ is embedded in $\mathscr{D}'(\mathbb{R}^d)$, and consist of the generalized functions $u \in \mathscr{D}'(\mathbb{R}^d)$ such that (2.11) is valid for some $M, C > 0$ and any $\varphi \in \mathscr{D}(\mathbb{R}^d)$.
We define the space $\mathscr{R}(\mathbb{R}^d)$ of rapidly decaying measurable functions φ such that $\|\varphi\|_{2,\boldsymbol{n},0} < \infty$ for every $\boldsymbol{n} \in \mathbb{N}^d$. Again, $\mathscr{R}(\mathbb{R}^d)$ is a countably multi-Hilbertian space.

The space $\mathscr{S}(\mathbb{T}^d)$ and its dual. The d-dimensional torus is denoted by $\mathbb{T}^d = [-1/2, 1/2)^d$. Let $\mathscr{S}(\mathbb{T}^d)$ be the space of smooth functions on $\mathbb{T}^d$. It is isomorphic to the space of 1-periodic smooth functions from $\mathbb{R}^d$ to $\mathbb{R}$. Its topological dual $\mathscr{S}'(\mathbb{T}^d)$ is isomorphic to the space of periodic generalized functions; that is, generalized functions u such that $\langle u, \varphi(\cdot - 1)\rangle = \langle u, \varphi \rangle$ for every $\varphi \in \mathscr{D}(\mathbb{R}^d)$.
When $u \in \mathscr{S}'(\mathbb{T}^d)$, we define its Fourier coefficients as $c_{\boldsymbol{n}}(u) := \langle u, \mathrm{e}^{2\mathrm{i}\pi\langle \boldsymbol{n}, \cdot\rangle}\rangle$, where the duality product is defined over $\mathscr{S}'(\mathbb{T}^d) \times \mathscr{S}(\mathbb{T}^d)$. This quantity is always well-defined since $e_{\boldsymbol{n}} := \mathrm{e}^{2\mathrm{i}\pi\langle \boldsymbol{n}, \cdot\rangle}$ is in $\mathscr{S}(\mathbb{T}^d)$. In general, the sequence $c(u) := (c_{\boldsymbol{n}}(u))_{\boldsymbol{n} \in \mathbb{Z}^d}$ is of slow growth (bounded by a polynomial). A periodic generalized function is in $\mathscr{S}(\mathbb{T}^d)$ if and only if $c(u)$ is rapidly decaying.
For $\tau \in \mathbb{R}$, we define the periodic Sobolev space as

$$W_2^\tau(\mathbb{T}^d) := \left\{ u \in \mathscr{S}'(\mathbb{T}^d) \;\middle|\; \|u\|_{W_2^\tau(\mathbb{T}^d)} := \left(\sum_{\boldsymbol{n} \in \mathbb{Z}^d} \langle \boldsymbol{n} \rangle^{2\tau} \left| \langle u, \mathrm{e}^{2\mathrm{i}\pi\langle \boldsymbol{n}, \cdot\rangle}\rangle \right|^2 \right)^{1/2} < \infty \right\}.$$

Then, the dual of $W_2^\tau(\mathbb{T}^d)$ is isomorphic to $W_2^{-\tau}(\mathbb{T}^d)$ for all τ. Moreover, we have that

$$\mathscr{S}(\mathbb{T}^d) = \cap_{k \in \mathbb{Z}} W_2^k(\mathbb{T}^d) \quad \text{and} \quad \mathscr{S}'(\mathbb{T}^d) = \cup_{k \in \mathbb{Z}} W_2^k(\mathbb{T}^d).$$

Remarks. The spaces $\mathscr{D}(\mathbb{R}^d)$, $\mathscr{D}'(\mathbb{R}^d)$, $\mathscr{S}(\mathbb{R}^d)$, $\mathscr{S}'(\mathbb{R}^d)$, $\mathscr{S}(\mathbb{T}^d)$, and $\mathscr{S}'(\mathbb{T}^d)$ are not normable. They can be classified depending on the complexity of their topology.

- The space $\mathscr{S}(\mathbb{R}^d)$ is a countably Hilbertian space and is therefore metrizable. But it is not normable. The same remark holds for $\mathscr{R}(\mathbb{R}^d)$, $\mathscr{S}(\mathbb{T}^d)$, and $\mathscr{D}(K)$ for $K \subset \mathbb{R}^d$ compact. They are Fréchet spaces, which are systematically studied for instance in [MV97, Part IV].

- The space $\mathscr{S}'(\mathbb{R}^d)$ is the dual of a non-normable countably multi-Hilbertian space. It is therefore a non-metrizable (DF) space (for dual of Fréchet) [MV97, Part IV]. The same holds for $\mathscr{D}'(K)$ and $\mathscr{S}'(\mathbb{T}^d)$.

- The space $\mathscr{D}(\mathbb{R}^d)$ is a multi-Hilbertian space, but not a countable multi-Hilbertian space. It is therefore not metrizable. As the inductive limit of a family of countable multi-Hibertian (Fréchet) spaces, it is sometimes referred to as (LF)-spaces (for “limit of Fréchet") [Trè67, Section 13].

- The space $\mathscr{D}'(\mathbb{R}^d)$ is not metrizable. As the dual of a countable inductive limit of (non-Banach) countably multi-Hilbertian spaces, it is the more evolved structure based on

Hilbert spaces that we shall encounter.

Embeddings. We say that the topological vector space $\mathscr{X}$ is *embedded in* the topological vector space $\mathscr{Y}$, what we denote by $\mathscr{X} \subseteq \mathscr{Y}$, if it is included (as a set), and if the canonical injection is continuous. We have the following classical embeddings, valid for any $p \in [1,\infty]$:

$$\mathscr{D}(\mathbb{R}^d) \subseteq \mathscr{S}(\mathbb{R}^d) \subseteq \mathscr{R}(\mathbb{R}^d) \subseteq L_p(\mathbb{R}^d) \subseteq \mathscr{S}'(\mathbb{R}^d) \subseteq \mathscr{D}'(\mathbb{R}^d),$$

$$\mathscr{S}(\mathbb{R}^d) \subseteq \mathscr{S}(\mathbb{T}^d) \subseteq L_p(\mathbb{T}^d) \subseteq \mathscr{S}'(\mathbb{T}^d) \subseteq \mathscr{S}('\mathbb{R}^d).$$

Nuclear Spaces In functional analysis, the nuclear structure was introduced by A. Grothendieck [Gro95] to remedy the absence of normed topologies for many fundamental function spaces in the theory of generalized functions. To quote A. Pietsch in [Pie72]: *"The locally convex spaces encountered in analysis can be divided into two classes. First, there are the normed spaces (...). The second class consists of the so-called nuclear locally convex spaces."* When considering measure theory on multi-Hilbertian spaces in Section 2.3, the nuclearity of the considered topologies will appear to be crucial. The reason is that, contrary to Banach spaces, many finite dimensional results of probability theory have direct generalizations on nuclear spaces while this is typically not feasible for Banach spaces. Note that normed spaces and nuclear spaces are mutually exclusive in infinite dimension: The only complete topological vector spaces that are nuclear and normable are finite-dimensional [Trè67, Corollary 2, pp. 520].

Definition 2.10. *A linear operator* L *between two separable Hilbert spaces* $\mathscr{H}_1$ *and* $\mathscr{H}_2$ *is* Hilbert-Schmidt *if for any orthonormal basis* $(e_n)_{n\in\mathbb{N}}$ *of* $\mathscr{H}_1$*, one has*

$$\sum_{n\geq 0} \|\mathrm{L}\{e_n\}\|^2_{\mathscr{H}_2} < \infty,$$

with $\|\cdot\|_{\mathscr{H}_2}$ *the Hilbertian norm of* $\mathscr{H}_2$*.*

For instance, the identity is *not* Hilbert-Schmidt on an infinite dimensional separable Hilbert space.

Definition 2.11. *Consider a multi-Hilbertian space* $\mathscr{X}$ *whose topology is associated to the family of Hilbertian semi-norms* $\mathscr{N}$*. We denote by* $\mathscr{X}_N$ *the Hilbert space obtained as the completion of* $\mathscr{X}$ *for the semi-norm* $N \in \mathscr{N}$*. We say that* $\mathscr{X}$ *is* nuclear *if for any* $M \in \mathscr{N}$*, there exists* $N \in \mathscr{N}$ *such that* $\mathscr{X}_M \subseteq \mathscr{X}_N$ *and the identity is Hilbert-Schmidt from* $\mathscr{X}_M$ *to* $\mathscr{X}_N$*.*

There exists more general definitions of the nuclearity (not only for multi-Hilbertian spaces); see for instance [Trè67]. One can show that $\mathscr{S}(\mathbb{R}^d)$, $\mathscr{D}(K)$ for K compact, $\mathscr{S}(\mathbb{T}^d)$, and $\mathscr{D}(\mathbb{R}^d)$, together with their duals, are nuclear spaces [Itô84, Chapter 1].

2.2.2 Linear Operators

We chose to work with tempered generalized functions rather than with generalized functions. Among our motivations, we aim at considering whitening operators associated with Fourier multipliers in $\mathcal{S}'(\mathbb{R}^d)$ in our stochastic model developed in Section 3. With this constraint in mind, we focus on linear and continuous operators L from $\mathcal{S}(\mathbb{R}^d)$ to $\mathcal{S}'(\mathbb{R}^d)$. This is the most general form of operators we shall consider.

Definition 2.12. *Let* L *be a linear and continuous operator from* $\mathcal{S}(\mathbb{R}^d)$ *to* $\mathcal{S}'(\mathbb{R}^d)$. *The* adjoint *of* L *is the unique operator* L^* *linear and continuous from* $\mathcal{S}(\mathbb{R}^d)$ *to* $\mathcal{S}'(\mathbb{R}^d)$ *such that*

$$\langle \mathrm{L}\{\varphi\}, \psi\rangle = \langle \mathrm{L}^*\{\psi\}, \varphi\rangle \tag{2.12}$$

for every $\varphi, \psi \in \mathcal{S}(\mathbb{R}^d)$.

In (2.12), the two duality products are between a tempered generalized function and a rapidly decaying smooth function, so that all the quantities are well-defined.

The Schwartz kernel theorem.

Theorem 2.3. *For any linear and continuous operator* L *from* $\mathcal{S}(\mathbb{R}^d)$ *to* $\mathcal{S}'(\mathbb{R}^d)$, *there exists a unique generalized function* $h \in \mathcal{S}'(\mathbb{R}^d \times \mathbb{R}^d)$ *such that*

$$\langle \mathrm{L}\{\varphi\}, \psi\rangle = \langle h, \varphi \otimes \psi\rangle \tag{2.13}$$

for any $\varphi, \psi \in \mathcal{S}(\mathbb{R}^d)$, *where* $(\varphi \otimes \psi)(\boldsymbol{x}, \boldsymbol{y}) = \varphi(\boldsymbol{x})\psi(\boldsymbol{y})$ *is the tensor product between* φ *and* ψ.

The generalized function h is called the *kernel* of L. With a slight abuse of notation (valid when both h and $\mathrm{L}\{\varphi\}$ are locally integrable functions), we rewrite (2.13) as

$$\mathrm{L}\{\varphi\}(\boldsymbol{x}) = \int_{\mathbb{R}^d} h(\boldsymbol{x}, \boldsymbol{y})\varphi(\boldsymbol{y})\mathrm{d}\boldsymbol{y}.$$

Theorem 2.3 is known as the Schwartz kernel theorem. It tells us that a linear and continuous operator can be represented by a kernel. It is intimately linked to the nuclearity of $\mathcal{S}(\mathbb{R}^d)$ [Trè67, Sections 50, 51]. The corresponding result is also valid for linear and continuous operators from $\mathcal{D}(\mathbb{R}^d)$ to $\mathcal{D}'(\mathbb{R}^d)$, with kernels in $\mathcal{D}'(\mathbb{R}^d \times \mathbb{R}^d)$ [Trè67, Theorem 51.7], and more generally on locally convex nuclear spaces [Gro95]. The general result requires advanced functional analysis material. For the case of $\mathcal{S}(\mathbb{R}^d)$, an equivalent result on continuous bilinear forms on $\mathcal{S}(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d)$ is demonstrated with relatively elementary tools in [Sim03, Theorem 5].

Extension by duality. Assume that the adjoint L^* of L is continuous from $\mathcal{S}(\mathbb{R}^d)$ to itself. In that case, we can extend L as a linear and continuous operator from $\mathcal{S}'(\mathbb{R}^d)$ to itself. For $u \in \mathcal{S}'(\mathbb{R}^d)$, we define $\mathrm{L}\{u\}$ as the tempered generalized function such that

$$\langle \mathrm{L}\{u\}, \varphi\rangle = \langle u, \mathrm{L}^*\{\varphi\}\rangle$$

for $\varphi \in \mathcal{S}(\mathbb{R}^d)$. For instance, the derivative operator D is continuous from $\mathcal{S}(\mathbb{R})$ to itself, so is its adjoint $(-\mathrm{D})$. Therefore, the derivative is extended to any generalized function in $\mathcal{S}'(\mathbb{R}^d)$

More generally, if L^* is continuous from $\mathcal{S}(\mathbb{R}^d)$ to $\mathcal{X}$ where $\mathcal{X}$ is a locally convex topological vector space such that $\mathcal{S}(\mathbb{R}^d) \subseteq \mathcal{X} \subseteq \mathcal{S}'(\mathbb{R}^d)$, then L can be extended to the dual $\mathcal{X}'$ of $\mathcal{X}$ following the same principle.

Geometric transformation and invariances. For $\boldsymbol{x}_0 \in \mathbb{R}^d$, the *translation operator* $\mathrm{T}_{\mathbf{r}_0}$ is $\mathrm{T}_{\boldsymbol{x}_0}\{\varphi\} = \varphi(\cdot - \boldsymbol{x}_0)$ with $\varphi \in \mathcal{S}(\mathbb{R}^d)$. For $a > 0$, the *scaling operator* S_a is $\mathrm{S}_a\{\varphi\} = a^{-d/2}\varphi(\cdot/a)$ with $\varphi \in \mathcal{S}(\mathbb{R}^d)$. For $\boldsymbol{\theta}_0 \in SO(d)$, the special orthogonal group (or group of d-dimensional rotations), the *rotation operator* $\mathrm{R}_{\boldsymbol{\theta}_0}$ is $\mathrm{R}_{\boldsymbol{\theta}_0}\{\varphi\} = \varphi(\boldsymbol{\theta}_0^T.)$. We have the relations $\mathrm{T}^*_{\boldsymbol{x}_0} = \mathrm{T}^{-1}_{\boldsymbol{x}_0} = \mathrm{T}_{-\boldsymbol{x}_0}$ and $\mathrm{S}^*_a = \mathrm{S}^{-1}_a = \mathrm{S}_{a^{-1}}$, and $\mathrm{R}^*_{\boldsymbol{\theta}_0} = \mathrm{R}^{-1}_{\boldsymbol{\theta}_0} = \mathrm{R}^T_{\boldsymbol{\theta}_0}$. Translation, scaling, and rotation operators are extended to $\mathcal{S}'(\mathbb{R}^d)$ by duality.

Definition 2.13. *A linear operator* L *continuous from* $\mathcal{S}(\mathbb{R}^d)$ *to* $\mathcal{S}'(\mathbb{R}^d)$ *is said to be*

- shift-invariant *if* $\mathrm{LT}_{\boldsymbol{x}_0} = \mathrm{T}_{\boldsymbol{x}_0}\mathrm{L}$ *for all* $\boldsymbol{x}_0 \in \mathbb{R}^d$,
- homogeneous of order γ *(or* γ-homogeneous*) with* $\gamma \in \mathbb{R}$ *if* $\mathrm{LS}_a = a^{-\gamma}\mathrm{S}_a\mathrm{L}$ *for all* $a > 0$, *and*
- rotation-invariant *if* $\mathrm{LR}_{\boldsymbol{\theta}_0} = \mathrm{R}_{\boldsymbol{\theta}_0}\mathrm{L}$ *for all* $\boldsymbol{\theta}_0 \in SO(d)$.

When the operator L is shift-invariant, its kernel h satisfies $h(\boldsymbol{x},\boldsymbol{y}) = h(\boldsymbol{x}-\boldsymbol{y})$. Then, L is a convolution operator of the form $\mathrm{L}\{\varphi\} = h * \varphi$ with $h \in \mathcal{S}'(\mathbb{R}^d)$. The adjoint of L is itself a convolution and we have $\mathrm{L}^*\{\varphi\} = h^\vee * \varphi$ with $h^\vee(\boldsymbol{x}) = h(-\boldsymbol{x})$. In that case, for any $\varphi \in \mathcal{S}(\mathbb{R}^d)$, $\mathrm{L}^*\{\varphi\}$ is a smooth function. It means in particular that L^* is a continuous and linear operator from $\mathcal{S}(\mathbb{R}^d)$ to $\mathcal{E}(\mathbb{R}^d)$, the space of smooth functions (on which a nuclear topology can be defined as for $\mathcal{D}'(\mathbb{R}^d)$; see [Trè67, Corollary p.530]). Therefore, we can extend L by duality to any generalized function in the dual $\mathcal{E}'(\mathbb{R}^d)$ of $\mathcal{E}(\mathbb{R}^d)$, which is the space of compactly supported generalized functions [Bon01]. In particular, $\mathrm{L}\{u\}$ is well-defined as soon as $u \in \mathcal{S}'(\mathbb{R}^d)$ is compactly supported. This is the case for the Dirac impulse δ and all its (partial) derivatives. Then, we remark that $\mathrm{L}\{\delta\} = h * \delta = h$: the generalized function h is called the *impulse response* of L.

Differential and pseudo-differential operators. In view of studying stochastic differential equations, we specify here the class of operators that we will consider. For $\mathrm{L} : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ linear, continuous, and shift-invariant, the Fourier transform of the impulse response h is called the *Fourier multiplier* of L, denoted by $\widehat{L}$. We have then

$$\widehat{\mathrm{L}\{\varphi\}} = \widehat{L}\widehat{\varphi}.$$

Examples of whitening operators. We introduce some classical families of differential or pseudo-differential operators that we shall use as whitening operators. For all of them, we specify their adjoint, their Fourier multiplier, and recap their invariance properties.

- *Differential operators:* In the 1-D setting, a differential operator has the form $\mathrm{L} = P(\mathrm{D})$ with $N \geq 1$, $P(X) = a_0 + a_1 X + \cdots + a_N X^N$ a polynomial, and $a_N \neq 0$. We call N the *order* of L. It is shift-invariant with Fourier multiplier

$$\widehat{L}(\omega) = P(\mathrm{i}\omega) = a_0 + a_1(\mathrm{i}\omega) + \cdots + a_N(\mathrm{i}\omega)^N.$$

for any $\omega \in \mathbb{R}$. The adjoint of $P(\mathrm{D})$ is $P(-\mathrm{D})$. When $P(X) = X^N$, the operator $\mathrm{L} = \mathrm{D}^N$ is N-homogeneous.

- *Fractional derivatives:* The fractional derivative of order $\gamma \geq 0$ is the shift-invariant operator with Fourier multiplier given for $\omega \in \mathbb{R}$ by

  $$\widehat{L}(\omega) = (\mathrm{i}\omega)^\gamma,$$

  denoted by D^γ. The fractional derivative and its adjoint are shift-invariant and γ-homogeneous. For $\gamma \in \mathbb{N}$, it is consistent with the usual derivative.

  More generally, M. Unser and T. Blu have identified the complete class of one-dimensional shift-invariant and homogeneous operators. They shown that this family is parameterized by γ and τ with γ the order of homogeneity and τ a phase parameter [UB07, Proposition 2]. The adjoint of D^γ, with Fourier multiplier $(-\mathrm{i}\omega)^\gamma$, lies in this family.

- *Separable operators:* If L is a 1-dimensional whitening operator, one defines its separable extension, denoted by L_d, with Fourier multiplier

  $$\widehat{L}_d(\boldsymbol{\omega}) = \prod_{i=1}^d \widehat{L}(\omega_i)$$

  for any $\boldsymbol{\omega} = (\omega_1, \dots, \omega_d) \in \mathbb{R}^d$.

  When $\mathrm{L} = \mathrm{D}$, we denote by Λ its separable extension, given by $\Lambda = \prod_{i=1}^d \mathrm{D}_i$. The operator Λ is shift-invariant, d-homogeneous, and not rotation-invariant. Its adjoint is $\Lambda^* = (-1)^d \Lambda$.

  When $\mathrm{L} = \mathrm{D}^\gamma$ is the fractional derivative of order $\gamma > 0$, we denote by Λ^γ its separable version. This operator is shift-invariant, $(d\gamma)$-homogeneous, and not rotation-invariant.

- *Laplacian:* The Laplacian operator in dimension d is defined as $\mathrm{L} = \Delta = \mathrm{D}_1^2 + \cdots + \mathrm{D}_d^2$. It is continuous from $\mathscr{S}(\mathbb{R}^d)$ to itself and self-adjoint. As such, we extend Δ as a continuous operator from $\mathscr{S}'(\mathbb{R}^d)$ to $\mathscr{S}'(\mathbb{R}^d)$. The Laplacian is shift-invariant, 2-homogeneous, rotation-invariant, and its Fourier multiplier is $\widehat{\Delta}(\boldsymbol{\omega}) = -\|\boldsymbol{\omega}\|^2$.

- *Fractional Laplacian:* The fractional Laplacian of order $\gamma \geq 0$ is associated to the Fourier multiplier $\widehat{L}(\boldsymbol{\omega}) = \|\boldsymbol{\omega}\|^\gamma$. It is self-adjoint, γ-homogeneous, and rotation-invariant. When $\gamma = 2$, we recognize the opposite of the Laplacian operator, and we denote the fractional Laplacian by $(-\Delta)^{\gamma/2}$. See [Gra04, Section 6.1] for more details.

- *Bessel operator:* We recall that $\langle \boldsymbol{x} \rangle := \sqrt{1 + \|\boldsymbol{x}\|^2}$. For $\gamma \in \mathbb{R}$, the *Bessel operator of order* γ is the operator $\mathrm{J}_\gamma = (\mathrm{Id} - \Delta)^{\gamma/2}$ with Fourier multiplier $\widehat{J}_\gamma(\boldsymbol{\omega}) = \langle \boldsymbol{\omega} \rangle^\gamma$. It is a self-adjoint operator, with inverse $\mathrm{J}_{-\gamma}$. Since $\widehat{J}_\gamma$ and $(\widehat{J}_\gamma)^{-1} = \widehat{J}_{-\gamma}$ are infinitely differentiable functions of slow growth, the Bessel operator is a continuous bijection from $\mathcal{S}(\mathbb{R}^d)$ to itself, and by extension a continuous bijection from $\mathcal{S}'(\mathbb{R}^d)$ to itself. See [Gra04, Section 6.1.2] for more details.

Operators on periodic function spaces. We recall that $e_{\boldsymbol{n}}$ is the trigonometric function $e_{\boldsymbol{n}}(\boldsymbol{x}) = \mathrm{e}^{2\mathrm{i}\pi\langle \boldsymbol{x},\boldsymbol{n}\rangle}$, with $\boldsymbol{n} \in \mathbb{Z}^d$. We assume that L is now a continuous, linear, and shift-invariant operator from $\mathcal{S}(\mathbb{T}^d)$ to $\mathcal{S}(\mathbb{T}^d)$. Then, L can be extended by duality from $\mathcal{S}'(\mathbb{T}^d)$ to $\mathcal{S}'(\mathbb{T}^d)$. The $e_{\boldsymbol{n}}$ are the eigenfunctions of L, and we write $\mathrm{L}\{e_{\boldsymbol{n}}\} = \lambda_{\boldsymbol{n}} e_{\boldsymbol{n}}$.
The sequence $(\lambda_{\boldsymbol{n}})_{\boldsymbol{n}\in\mathbb{Z}^d}$ is slowly growing (bounded by a polynomial). Reciprocally, any slowly growing sequence specify a linear, continuous, and shift-invariant operator L from $\mathcal{S}(\mathbb{T}^d)$ to $\mathcal{S}(\mathbb{T}^d)$ by the relation

$$\mathrm{L}\varphi := \sum_{\boldsymbol{n}\in\mathbb{Z}^d} \lambda_{\boldsymbol{n}} c_{\boldsymbol{n}}(\varphi) e_{\boldsymbol{n}},$$

where the convergence holds in $\mathcal{S}(\mathbb{T}^d)$.
If L is also a linear, continuous, and shift-invariant operator from $\mathcal{S}(\mathbb{R}^d)$ to $\mathcal{S}'(\mathbb{R}^d)$ and its Fourier multiplier $\widehat{L}$ is a continuous function, then we have that $\lambda_{\boldsymbol{n}} = \widehat{L}(2\pi\boldsymbol{n})$. All the differential and pseudo-differential operators defined above satisfy this property, and can therefore be seen as operators from $\mathcal{S}(\mathbb{T}^d)$ to $\mathcal{S}(\mathbb{T}^d)$.

2.2.3 Weighted Besov Spaces

Besov spaces are parameterized by three values: the regularity parameter τ, the integrability order p, and an additional tuning parameter q. Concretely, the parameter q plays only a secondary role. Moreover, it appears to be unnecessary for our results on the compressibility of generalized Lévy processes (Chapter 6). We therefore restrict the presentation to the subfamily of the so-called Sobolev-Slobodeckij spaces, that corresponds to the case $p = q$. However, we prefer to keep the denomination of Besov spaces for two reasons. First, all the definitions and results presented in this section are developed for the complete family of Besov spaces in [Tri06, Tri08]. Second, interesting considerations on the parameter q can be done once the results are known for the case $p = q$. This calls for some possible refinements of our results that we shall discuss later.
Random processes do not decay at infinity in general, so that there is no hope to characterize their Besov regularity over the complete space $\mathbb{R}^d$ with classical Besov spaces. We will therefore consider weighted Besov spaces related to polynomial weights to overcome this issue. The parameter for the decay rate is ρ.
In what follows, we first consider the weighted Sobolev spaces, that corresponds to $p = q = 2$, based on the Fourier transform and the Bessel operators. Then, we consider the weighted Besov spaces (with $p = q$). We chose to use the wavelet characterization of Triebel as our definition. This section is essentially based on our publications [FFU, AFU].

Weighted Sobolev spaces. We recall that J_τ is the Bessel operator of order τ (see Section 2.2.2).

Definition 2.14. *Let* $\tau, \rho \in \mathbb{R}$. *The* Sobolev space of smoothness τ *is defined by*

$$W_2^\tau(\mathbb{R}^d) := \left\{ f \in \mathscr{S}'(\mathbb{R}^d) \,\middle|\, \mathrm{J}_\tau\{f\} \in L_2(\mathbb{R}^d) \right\}$$

and the Sobolev space of smoothness τ and decay ρ *is*

$$W_2^\tau(\mathbb{R}^d;\rho) := \left\{ f \in \mathscr{S}'(\mathbb{R}^d) \,\middle|\, \langle \cdot \rangle^\rho f \in W_2^\tau(\mathbb{R}^d) \right\}.$$

We also set $L_2(\mathbb{R}^d;\rho) := W_2^0(\mathbb{R}^d;\rho)$.

We summarize now the basic properties on weighted Sobolev spaces that are useful for our work, with short proofs for the sake of completeness. More details can be found in [Tri06]; in particular, in Chapter 6, a broader class of weighted spaces with their embedding relations is considered.

Proposition 2.6. *The following properties hold for weighted Sobolev spaces.*

- *For* $\rho, \tau \in \mathbb{R}$, $W_2^\tau(\mathbb{R}^d;\rho)$ *is a Hilbert space for the scalar product*

  $$\langle f, g \rangle_{W_2^\tau(\mathbb{R}^d;\rho)} := \left\langle \mathrm{J}_\tau\{\langle \cdot \rangle^\rho f\}, \mathrm{J}_\tau\{\langle \cdot \rangle^\rho g\} \right\rangle_{L_2(\mathbb{R}^d)}.$$

  We denote by $\|f\|_{W_2^\tau(\mathbb{R}^d;\rho)} = \langle f, f \rangle^{1/2}_{W_2^\tau(\mathbb{R}^d;\rho)}$ *the corresponding Hilbertian norm.*

- *For* $\rho \in \mathbb{R}$ *fixed and for every* $\tau_1 \le \tau_2$, *we have the continuous embedding*

  $$W_2^{\tau_2}(\mathbb{R}^d;\rho) \subseteq W_2^{\tau_1}(\mathbb{R}^d;\rho). \tag{2.14}$$

- *For* $\tau \in \mathbb{R}$ *fixed and for every* $\rho_1 \le \rho_2$, *we have the continuous embedding*

  $$W_2^{\tau}(\mathbb{R}^d;\rho_2) \subseteq W_2^{\tau}(\mathbb{R}^d;\rho_1). \tag{2.15}$$

- *For* $\rho, \tau \in \mathbb{R}$, *the operator* $\mathrm{J}_{\tau,\rho} : f \mapsto \langle \cdot \rangle^\rho \mathrm{J}_\tau\{f\}$ *is an isometry from* $L_2(\mathbb{R}^d)$ *to* $W_2^{-\tau}(\mathbb{R}^d;-\rho)$.

- *The dual space of* $W_2^\tau(\mathbb{R}^d;\rho)$ *is* $W_2^{-\tau}(\mathbb{R}^d;-\rho)$ *for every* $\tau, \rho \in \mathbb{R}$.

- *We have the countable projective limit*

  $$\mathscr{S}(\mathbb{R}^d) = \bigcap_{\tau,\rho \in \mathbb{R}} W_2^\tau(\mathbb{R}^d;\rho) = \bigcap_{n \in \mathbb{N}} W_2^n(\mathbb{R}^d;n). \tag{2.16}$$

- *We have the countable inductive limit*

  $$\mathscr{S}'(\mathbb{R}^d) = \bigcup_{\tau,\rho \in \mathbb{R}} W_2^\tau(\mathbb{R}^d;\rho) = \bigcup_{n \in \mathbb{N}} W_2^{-n}(\mathbb{R}^d;-n). \tag{2.17}$$

Proof. The space $W_2^\tau(\mathbb{R}^d;\rho)$ inherits the Hilbertian structure of $L_2(\mathbb{R}^d)$. For $\tau_1 \le \tau_2$ and $\rho_1 \le \rho_2$, we have moreover the inequalities,

$$\|f\|_{W_2^{\tau_1}(\mathbb{R}^d;\rho)} \leq \|f\|_{W_2^{\tau_2}(\mathbb{R}^d;\rho)},$$
$$\|f\|_{W_2^{\tau}(\mathbb{R}^d;\rho_1)} \leq \|f\|_{W_2^{\tau}(\mathbb{R}^d;\rho_2)},$$

from which we deduce (2.14) and (2.15). The relation

$$\|\mathrm{J}_{\tau,\rho} f\|_{W_2^{-\tau}(\mathbb{R}^d;-\rho)} = \|\mathrm{J}_{-\tau}\{\langle\cdot\rangle^{-\rho}\mathrm{J}_{\tau,\rho} f\}\|_{L_2(\mathbb{R}^d)} = \|f\|_{L_2(\mathbb{R}^d)}$$

proves that $\mathrm{J}_{\tau,\rho}$ is an isometry. For every $f, g \in L_2(\mathbb{R}^d)$, we have that

$$\langle \mathrm{J}_\tau\{\langle\cdot\rangle^{\rho} f\}, \mathrm{J}_{-\tau}\{\langle\cdot\rangle^{-\rho} g\}\rangle_{L_2(\mathbb{R}^d)} = \langle f, g\rangle_{L_2(\mathbb{R}^d)}. \tag{2.18}$$

Since $W_2^\tau(\mathbb{R}^d;\rho) = \{\mathrm{J}_\tau\{\langle\cdot\rangle^\rho f\},\ f \in L_2(\mathbb{R}^d)\}$, we easily deduce the dual of $W_2^\tau(\mathbb{R}^d;\rho)$ from (2.18). Finally, we can reformulate the topology on $\mathcal{S}(\mathbb{R}^d)$ as (2.16). This implies directly (2.17). □

Weighted Sobolev-Slobodeckij (Besov) spaces. We use a wavelet-based approach, as exposed in [Tri08]. Essentially, Besov spaces are subspaces of $\mathcal{S}'(\mathbb{R}^d)$ that are characterized by weighted sequence norms of the wavelet coefficients.

The scale and shift parameters of the wavelets are respectively denoted by $j \geq 0$ and $\boldsymbol{k} \in \mathbb{Z}^d$. The letters M and F refer to the *gender* of the wavelet (F for the father wavelets and G for the mother wavelet). Consider two functions ψ_M and $\psi_F \in L_2(\mathbb{R})$. We set $\mathrm{G}^0 = \{M, F\}^d$ and, for $j \geq 1$, $\mathrm{G}^j = \mathrm{G}^0 \backslash \{F^d\}$. For $G = (G_1, \dots, G_d) \in \mathrm{G}^0$, called a gender, we set, for every $\boldsymbol{x} = (x_1, \dots, x_d) \in \mathbb{R}^d$, $\psi_G(\boldsymbol{x}) = \prod_{i=1}^d \psi_{G_i}(x_i)$. For $j \geq 0$, $G \in \mathrm{G}^j$, and $\boldsymbol{k} \in \mathbb{Z}^d$, we define

$$\psi_{j,G,\boldsymbol{k}}(\boldsymbol{x}) := 2^{jd/2}\psi_G(2^j\boldsymbol{x} - \boldsymbol{k})$$

for any $\boldsymbol{x} \in \mathbb{R}^d$.

For any regularity parameter $r_0 \geq 1$, there exists two functions $\psi_M, \psi_F \in L_2(\mathbb{R})$ that are compactly supported, with at least r_0 continuous derivatives such that the family

$$\{\psi_{j,G,\boldsymbol{k}}\}_{(j,G,\boldsymbol{k}) \in \mathbb{N} \times \mathrm{G}^j \times \mathbb{Z}^d}$$

is an orthonormal basis of $L_2(\mathbb{R}^d)$ [Tri08]. Concretely, one consider the family of Daubechies wavelets [Dau88].

The following definition of weighted Besov spaces is equivalent to the more usual Fourier-based definitions. This equivalence is proved in [Tri08].

Definition 2.15. *Let $\tau, \rho \in \mathbb{R}$ and $0 < p \leq \infty$. Fix $r_0 > \max(\tau, d(1/p - 1)_+ - \tau)$ and consider a family of compactly supported wavelets $\{\psi_{j,G,\boldsymbol{k}}\}_{(j,G,\boldsymbol{k}) \in \mathbb{N} \times \mathrm{G}^j \times \mathbb{Z}^d}$ with at least r_0 continuous derivatives.*

The weighted Besov space $B_p^\tau(\mathbb{R}^d;\rho)$ *is the collection of tempered generalized functions $f \in$*

$\mathscr{S}'(\mathbb{R}^d)$ that can be written as

$$f = \sum_{(j,G,\boldsymbol{k})\in\mathbb{N}\times\mathrm{G}^j\times\mathbb{Z}^d} c_{j,G,\boldsymbol{k}}\psi_{j,G,\boldsymbol{k}}, \tag{2.19}$$

where the $c_{j,G,\boldsymbol{k}}$ satisfy

$$\sum_{j\geq 0} 2^{j(\tau p - d + \frac{dp}{2})} \sum_{G\in\mathrm{G}^j}\sum_{\boldsymbol{k}\in\mathbb{Z}^d} \langle 2^{-j}\boldsymbol{k}\rangle^{\rho p}|c_{j,G,\boldsymbol{k}}|^p < \infty,$$

where we recall that $\langle \boldsymbol{x}\rangle = (1+\|\boldsymbol{x}\|)^{1/2}$ and and where the convergence (2.19) *holds unconditionally on $\mathscr{S}'(\mathbb{R}^d)$. The usual modification should be done when $p=\infty$.*

The parameter r_0 in Definition 2.15 is chosen such that the wavelet is regular enough to be applied to a function of $B_p^\tau(\mathbb{R}^d;\rho)$. When the convergence (2.19) occurs, the duality product $\langle f, \psi_{j,G,\boldsymbol{k}}\rangle$ is well defined and we have $c_{j,G,\boldsymbol{k}} = \langle f, \psi_{j,G,\boldsymbol{k}}\rangle$. Moreover, the quantity

$$\|f\|_{B_p^\tau(\mathbb{R}^d;\rho)} := \left(\sum_{j\geq 0} 2^{j(\tau p - d + \frac{dp}{2})} \sum_{G\in\mathrm{G}^j}\sum_{\boldsymbol{k}\in\mathbb{Z}^d} \langle 2^{-j}\boldsymbol{k}\rangle^{\rho p}|\langle f, \psi_{j,G,\boldsymbol{k}}\rangle|^p\right)^{1/p} \tag{2.20}$$

is finite for $f\in B_p^\tau(\mathbb{R}^d;\rho)$ and specifies a norm (a quasi-norm, respectively) on $B_p^\tau(\mathbb{R}^d;\rho)$ for $p\geq 1$ ($p<1$, respectively). The space $B_p^\tau(\mathbb{R}^d;\rho)$ is a Banach (a quasi-Banach, respectively) for this norm (quasi-norm, respectively). When $p=2$, weighted Sobolev spaces and Besov spaces coincide; that is, $W_2^\tau(\mathbb{R}^d;\rho) = B_2^\tau(\mathbb{R}^d;\rho)$, the two norms—the one of Proposition 2.6 and (2.20)—being equivalent.

As a simple example, we obtain the Besov localization of the Dirac impulse. Of course, this result is known, and an alternative proof can be found for instance in [ST87]. We believe that it is interesting to give our own proof here. First, it illustrates how to use the wavelet-based characterization of Besov spaces, and second, it will be used to obtain sharp results for compound Poisson processes.

Proposition 2.7. *The Dirac impulse δ is in $B_p^\tau(\mathbb{R}^d;\rho)$ if and only if $\tau < \frac{d}{p} - d$.*

Proof. The definition of the Besov (quasi-)norm readily gives

$$\|\delta\|^p_{B_p^\tau(\mathbb{R}^d;\rho)} = \sum_{j\geq 0} 2^{j(\tau p - d + dp)} \sum_{G\in\mathrm{G}^j}\sum_{\boldsymbol{k}\in\mathbb{Z}^d} \langle 2^{-j}\boldsymbol{k}\rangle^{\rho p}|\psi_G(\boldsymbol{k})|^p.$$

The common support K of the ψ_G is compact. Therefore, only finitely many $\psi_G(\boldsymbol{k})$ are non zero, and for such $\boldsymbol{k}$ and every j we have

$$0 < \min_{\boldsymbol{x}\in K}\langle \boldsymbol{x}\rangle^{\rho p} = \langle 2^{-j}\boldsymbol{k}\rangle^{\rho p} \leq \max_{\boldsymbol{x}\in K}\langle \boldsymbol{x}\rangle^{\rho p} < \infty.$$

Since $2^d - 1 \leq \mathrm{Card}(\mathrm{G}^j) \leq 2^d$ and all the ψ_G are bounded, it is then easy to find $0 < A \leq B < \infty$

such that

$$A \sum_{j\geq 0} 2^{j(\tau-d+dp)} \leq \|\delta\|^p_{B^\tau_p(\mathbb{R}^d;\rho)} \leq B \sum_{j\geq 0} 2^{j(\tau-d+dp)}.$$

The sum converges for $\tau - d + dp < 0$ and diverges otherwise, implying the result. □

Embeddings between weighted Besov spaces.

Proposition 2.8. *Let* $0 < p_0 \leq p_1 \leq \infty$ *and* $\tau_0, \tau_1, \rho_0, \rho_1 \in \mathbb{R}$.

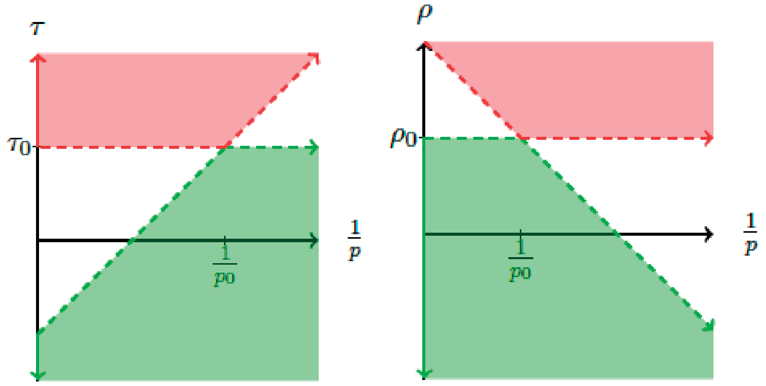
- *We have the embedding* $B^{\tau_0}_{p_0}(\mathbb{R}^d;\rho_0) \subseteq B^{\tau_1}_{p_1}(\mathbb{R}^d;\rho_1)$ *as soon as*

$$\tau_0 - \tau_1 > \frac{d}{p_0} - \frac{d}{p_1} \quad \textit{and} \quad \rho_0 > \rho_1. \tag{2.21}$$

- *We have the embedding* $B^{\tau_1}_{p_1}(\mathbb{R}^d;\rho_1) \subseteq B^{\tau_0}_{p_0}(\mathbb{R}^d;\rho_0)$ *as soon as*

$$\rho_1 - \rho_0 > \frac{d}{p_0} - \frac{d}{p_1} \quad \textit{and} \quad \tau_1 > \tau_0. \tag{2.22}$$

A proof of the sufficiency of (2.21) can be found in [ET08, Section 4.2.3] for unweighted Besov spaces. The extension to the weighted case is obvious. For the embedding (2.22), see [FFU, Proposition 3]. Proposition 2.8 is summarized in the two diagrams of Figure 2.1.



(a) $(1/p,\tau)$-diagram for fixed ρ_0 (b) $(1/p,\rho)$-diagram for fixed τ_0

Figure 2.1 – Representation of the embeddings between Besov spaces: If $f \in B^{\tau_0}_{p_0}(\mathbb{R}^d;\rho_0)$, then f is in any Besov space that is in the shaded green regions. Conversely, if $f \notin B^{\tau_0}_{p_0}(\mathbb{R}^d;\rho_0)$, then f is not in any of the Besov spaces of the shaded red regions.

If the only knowledge provided to us is that the generalized function f is in $\mathcal{S}'(\mathbb{R}^d)$, then this is not enough to set the regularity r_0 of the wavelet used to characterize the Besov smoothness of f. However, if we have additional information on f, for instance its inclusion in a weighted Sobolev space, then the situation is different. Proposition 2.9 gives a wavelet-domain criterion to determine if a generalized function f, known to be in $W^{\tau_0}_2(\mathbb{R}^d;\rho_0)$, is actually in $B^\tau_p(\mathbb{R}^d;\rho)$.

Note that $f \in \mathscr{S}'(\mathbb{R}^d)$ is in some Sobolev space $W_2^{\tau_0}(\mathbb{R}^d;\rho_0)$ because of (2.17). This result is taken from our work [FFU], where it is proved for general Besov spaces.

Proposition 2.9. *Let $\tau, \tau_0, \rho, \rho_0 \in \mathbb{R}$ and $0 < p \le \infty$. We set*

$$r_0 > \max(|\tau_0|, \left|\tau - d(1/p - 1/2)_+\right|). \tag{2.23}$$

Then, the generalized function $f \in W_2^{\tau_0}(\mathbb{R};\rho_0)$ is in $B_p^\tau(\mathbb{R}^d;\rho)$ if and only if

$$\sum_{j\ge 0} 2^{j(\tau p - d + dp/2)} \sum_{G\in G^j} \sum_{\boldsymbol{k}\in\mathbb{Z}^d} \langle 2^{-j}\boldsymbol{k}\rangle^{\rho p} |\langle f, \psi_{j,G,\boldsymbol{k}}\rangle|^p < \infty,$$

with $(\psi_{j,G,\boldsymbol{k}})$ a Daubechies wavelet basis of $L_2(\mathbb{R}^d)$ of regularity at least r_0, with the usual modifications when $p = \infty$.

Proof. Let $\tau_1 < \min(\tau_0, \tau - d(1/p - 1/2)_+)$ and $\rho_1 \le \min(\rho_0, \rho - d(1/p - 1/2)_+)$. Then, according to Proposition 2.8, we have the embeddings

$$B_{p,q}^\tau(\mathbb{R}^d;\rho) \subseteq W_2^{\tau_1}(\mathbb{R}^d;\rho_1) \text{ and } W_2^{\tau_0}(\mathbb{R}^d;\rho_0) \subseteq W_2^{\tau_1}(\mathbb{R}^d;\rho_1).$$

Condition (2.23) implies that we can apply Definition 2.15 to the Besov space $W_2^{\tau_1}(\mathbb{R}^d;\rho_1)$. In particular, if $(\psi_{j,G,\boldsymbol{k}})$ is a Dabauchies wavelet basis with regularity at least r_0, and for every function $f \in W_2^{\tau_1}(\mathbb{R}^d;\rho_1)$, then the wavelet coefficients $\langle f, \psi_{j,G,\boldsymbol{k}}\rangle$ are well-defined. Moreover, we have the characterization

$$f \in B_p^\tau(\mathbb{R}^d;\rho) \Leftrightarrow \|f\|_{B_p^\tau(\mathbb{R}^d;\rho)} < \infty$$

for $f \in W_2^{\tau_1}(\mathbb{R}^d;\rho_1)$ and, therefore, for $f \in W_2^{\tau_0}(\mathbb{R}^d;\rho_0)$. □

2.3 Generalized Random Processes and Fields

Generalized random processes are random elements in a space of generalized functions. In their seminal works [Gel55, GV64], Gelfand and Vilenkin examine generalized random processes in $\mathscr{D}'(\mathbb{R}^d)$. We prefer to develop the theory over the space $\mathscr{S}'(\mathbb{R}^d)$ of tempered generalized functions. This amounts to a slightly restriction on the class of processes, since $\mathscr{S}'(\mathbb{R}^d)$ is a strict subset of $\mathscr{D}'(\mathbb{R}^d)$. We are motivated by the fact that tempered generalized random processes are more adapted to the construction of solutions of stochastic differential equations. This is in line with the specification of whitening operators from $\mathscr{S}(\mathbb{R}^d)$ to $\mathscr{S}'(\mathbb{R}^d)$ via their Fourier multiplier (see Section 2.2.2). Moreover, adopting $\mathscr{S}'(\mathbb{R}^d)$ allows us to extend the space of test functions to the case of non-compactly supported functions, which are crucial in signal-processing applications.

2.3.1 Definition and Main Concepts

We fix a probability space $(\Omega, \mathscr{F}, \mathscr{P})$. We recall that the space $\mathscr{S}'(\mathbb{R}^d)$ is endowed with the weak*-topology. The associated Borel σ-field is denoted by $\mathscr{B}(\mathscr{S}'(\mathbb{R}^d))$. It is the σ-field generated by the open sets of $\mathscr{S}'(\mathbb{R}^d)$. Equivalently, $\mathscr{B}(\mathscr{S}'(\mathbb{R}^d))$ is generated by the *cylinders* of the form

$$\left\{u \in \mathscr{S}'(\mathbb{R}^d) \,\middle|\, \langle u, \boldsymbol{\varphi}\rangle \in B\right\} \tag{2.24}$$

with $N \geq 1$, $\boldsymbol{\varphi} \in (\mathscr{S}(\mathbb{R}^d))^N$, and $B \in \mathscr{B}(\mathbb{R}^N)$ Borel set[2].

Definition 2.16. *A* tempered generalized random process *is a measurable mapping from* $(\Omega, \mathscr{F})$ *to* $(\mathscr{S}'(\mathbb{R}^d), \mathscr{B}(\mathscr{S}'(\mathbb{R}^d)))$; *that is, a* $(\mathscr{S}'(\mathbb{R}^d))$*-valued random variable.*

When the context is clear, we will omit to specify that a generalized random process is tempered.

Definition 2.17. *The* law *of a generalized random process s is the probability measure on* $\mathscr{S}'(\mathbb{R}^d)$ *defined by*

$$\mathscr{P}_s(B) := \mathscr{P}(s \in B) = \mathscr{P}\{\omega \in \Omega \mid s(\omega) \in B\}$$

for any $B \in \mathscr{B}(\mathscr{S}'(\mathbb{R}^d))$. *Two generalized random processes s_1 and s_2 are* equal in law *if* $\mathscr{P}_{s_1} = \mathscr{P}_{s_2}$. *This is denoted by* $s_1 \overset{(\mathscr{L})}{=} s_2$.

For every tempered generalized random process s and $\varphi \in \mathscr{S}(\mathbb{R}^d)$, the mapping $\langle s, \varphi\rangle : (\Omega, \mathscr{F}) \to (\mathbb{R}, \mathscr{B}(\mathbb{R}))$ defined by $\langle s, \varphi\rangle(\omega) = \langle s(\omega), \varphi\rangle$ is measurable; that is, $\langle s, \varphi\rangle \in L_0(\Omega)$. Moreover, the map

$$\begin{aligned} s :\ & \mathscr{S}(\mathbb{R}^d) \to L_0(\Omega) \\ & \varphi \mapsto \langle s, \varphi\rangle \end{aligned}$$

[2]The cylinders (2.24) defines the *cylindrical σ-field* of $\mathscr{S}'(\mathbb{R}^d)$, that coincides with the Borel σ-field for the weak*-topology, as for any countably multi-Hilbertian spaces [Itô84].

is linear and continuous. The converse is also valid: Any linear and continuous map from $\mathscr{S}(\mathbb{R}^d)$ to $L_0(\Omega)$ specifies a tempered generalized random process. This is intimately related to the structure of nuclear and *countable* multi-Hilbertian space of $\mathscr{S}'(\mathbb{R}^d)$ [Itô84, Wal86]. See the introduction of [Sel07] for additional references on these questions.
More generally, for $\boldsymbol{\varphi} = (\varphi_1, \dots, \varphi_N) \in (\mathscr{S}(\mathbb{R}^d))^N$, we consider the N-dimensional random vector $\langle s, \boldsymbol{\varphi} \rangle := (\langle s, \varphi_1 \rangle, \dots, \langle s, \varphi_N \rangle)$. The random vectors

$$\big(\langle s, \boldsymbol{\varphi} \rangle\big)_{N \geq 1, \boldsymbol{\varphi} \in (\mathscr{S}(\mathbb{R}^d))^N}$$

are the *finite-dimensional marginals* of s. Two generalized random processes are equal in law if and only if their finite-dimensional marginals are equal in law.
If L is a continuous operator from $\mathscr{S}(\mathbb{R}^d)$ to $\mathscr{S}(\mathbb{R}^d)$, then L^*, when restricted to $\mathscr{S}(\mathbb{R}^d)$, shares this property and L can be extended by duality to $\mathscr{S}'(\mathbb{R}^d)$. Exploiting this principle, if s is a generalized random process, then we can define the process $\mathrm{L}\{s\}$ as

$$\langle \mathrm{L}\{s\}, \varphi \rangle = \langle s, \mathrm{L}^*\{\varphi\} \rangle$$

for $\varphi \in \mathscr{S}(\mathbb{R}^d)$. In particular, for any multi-integer $\boldsymbol{m} \in \mathbb{N}^d$, the process $\mathrm{D}^{\boldsymbol{m}}\{s\}$ is defined as

$$\langle \mathrm{D}^{\boldsymbol{m}}\{s\}, \varphi \rangle = (-1)^{|\boldsymbol{m}|} \langle s, \mathrm{D}^{\boldsymbol{m}}\{\varphi\} \rangle.$$

We remark that, contrary to classical random processes, the (partial) derivative of a generalized random process is always well-defined and is itself a generalized random process.

Definition 2.18. *We say that the two generalized random processes s_1 and s_2 are* independent *if for any B_1, B_2 in the Borel σ-field $\mathscr{B}(\mathscr{S}'(\mathbb{R}^d))$ of $\mathscr{S}'(\mathbb{R}^d)$, the events $\{s_1 \in B_1\}$ and $\{s_1 \in B_2\}$ are independent.*

Equivalently, two generalized random processes s_1 and s_2 are independent if their finite dimensional marginals are independent; that is, if the random vectors $\langle s_1, \boldsymbol{\varphi} \rangle$ and $\langle s_2, \boldsymbol{\varphi} \rangle$ are independent for every $N \geq 1$ and $\boldsymbol{\varphi} \in (\mathscr{S}'(\mathbb{R}^d))^N$.
Random processes are often classified by two characteristics: their statistical invariance properties and their dependency structure. We recall that the geometric transformations are introduced in Section 2.2.2.

Definition 2.19. *A generalized random process s is said to be*

- stationary *if for all $\boldsymbol{x}_0 \in \mathbb{R}^d$, $\mathrm{T}_{\boldsymbol{x}_0} s \overset{(\mathscr{L})}{=} s$;*
- symmetric *if $s^\vee \overset{(\mathscr{L})}{=} s$, where $s^\vee(\boldsymbol{x}) = s(-\boldsymbol{x})$;*
- self-similar of order $H \in \mathbb{R}$ *if for all $a > 0$, $a^H s(\cdot / a) \overset{(\mathscr{L})}{=} s$;*
- isotropic *if for all $\boldsymbol{\theta}_0 \in SO(d)$, $\mathrm{R}_{\boldsymbol{\theta}_0} s \overset{(\mathscr{L})}{=} s$.*

Definition 2.20. *A generalized random process is* independent at every point *if $\langle s, \varphi \rangle$ and $\langle s, \Psi \rangle$ are independent whenever φ and $\Psi \in \mathscr{S}(\mathbb{R}^d)$ have disjoint supports.*

We say that the generalized random process has *finite pth moments* for $p > 0$ if for any $\varphi \in \mathcal{S}(\mathbb{R}^d)$, $\mathbb{E}[|\langle s, \varphi\rangle|^p] < \infty$.
As we did for $\mathcal{S}'(\mathbb{R}^d)$, we define the Borel σ-field $\mathcal{B}(\mathcal{D}'(\mathbb{R}^d))$ of $\mathcal{D}'(\mathbb{R}^d)$ for the weak*-topology. A generalized random process is then a $(\mathcal{D}'(\mathbb{R}^d))$-valued random variable. All the concepts introduced above can be extended to random processes in $\mathcal{D}'(\mathbb{R}^d)$.

2.3.2 The Characteristic Functional

The characteristic functional of a random process was defined for the first time by A. Kolmogorov in the short paper [Kol35]. We shall see that most of the concepts introduced in Section 2.3.1 can be reformulated in terms of the characteristic functional. This is in line with the finite-dimensional case exposed in Section 2.1: The characteristic functional is the infinite-dimensional generalization of the characteristic function.

Definition 2.21. *The* characteristic functional *of the tempered generalized random process s is the functional from* $\mathcal{S}(\mathbb{R}^d)$ *to* $\mathbb{C}$ *defined by*

$$\widehat{\mathscr{P}}_s(\varphi) = \int_{\mathcal{S}'(\mathbb{R}^d)} \mathrm{e}^{\mathrm{i}\langle u,\varphi\rangle} \mathrm{d}\mathscr{P}_s(u) = \mathbb{E}[\mathrm{e}^{\mathrm{i}\langle s,\varphi\rangle}].$$

As for the characteristic function for random variables, the characteristic functional characterizes the law of the generalized random process: Two generalized random processes are equal in law if and only if $\widehat{\mathscr{P}}_{s_1} = \widehat{\mathscr{P}}_{s_2}$. The characteristic functional shares the defining properties of the characteristic function.

Proposition 2.10. *A characteristic functional* $\widehat{\mathscr{P}}_s$ *is*

- positive-definite *on* $\mathcal{S}(\mathbb{R}^d)$, *in the sense that*

  $$\sum_{n,m=1}^{N} a_n a_m^* \widehat{\mathscr{P}}_s(\varphi_n - \varphi_m) \geq 0$$

  for every $N \geq 1$, $\varphi_n \in \mathcal{S}(\mathbb{R}^d)$, *and* $a_n \in \mathbb{C}$.
- *continuous on* $\mathcal{S}(\mathbb{R}^d)$*;*
- *normalized as* $\widehat{\mathscr{P}}_s(0) = 1$.

The conditions of Proposition 2.10 are not only necessary, but also sufficient: This is the well-known generalization of Proposition 2.2 in $\mathcal{S}(\mathbb{R}^d)$, known as the Bochner-Minlos theorem.

Theorem 2.4. *A functional* $\widehat{\mathscr{P}}$ *from* $\mathcal{S}(\mathbb{R}^d)$ *to* $\mathbb{C}$ *is the characteristic functional of a generalized random process if and only if it is positive-definite, continuous, and satisfies* $\widehat{\mathscr{P}}(0) = 1$.

The Bochner-Minlos theorem was conjectured by Gelfand and demonstrated by Minlos in [Min59]. This theorem is also valid for processes in $\mathcal{D}'(\mathbb{R}^d)$ [GV64, Section III.2.6, Theorem 3]. As we did for the Gaussian random variable in Section 2.1, we can use the Bochner-Minlos theorem to *construct* generalized random processes. As a first example, consider the functional

$\widehat{\mathscr{P}}: \varphi \mapsto \mathrm{e}^{-\|\varphi\|_2^2/2}$. It is easy to check that it is continuous and positive-definite over $\mathscr{S}(\mathbb{R}^d)$ (for the latter, the proof is identical to the finite-dimensional case) and that $\widehat{\mathscr{P}}(0) = 1$. Therefore, $\widehat{\mathscr{P}}$ is the characteristic functional of a generalized random process, called the *Gaussian white noise.*

Proposition 2.11. *A characteristic functional $\widehat{\mathscr{P}}_s$ satisfies the relations*

$$\left|\widehat{\mathscr{P}}_s(\varphi)\right| \leq 1,$$
$$\left|\widehat{\mathscr{P}}_s(\varphi_2) - \widehat{\mathscr{P}}_s(\varphi_1)\right| \leq 2\left(1 - \Re\{\widehat{\mathscr{P}}_s(\varphi_2 - \varphi_1)\}\right) \tag{2.25}$$

for every $\varphi, \varphi_1, \varphi_2 \in \mathscr{S}(\mathbb{R}^d)$.

This result is actually valid for any positive-definite functional on a topological vector space. The relation (2.25) shows in particular that a positive-definite functional that is continuous around 0 is uniformly continuous; see for instance [Fer67, Section II.5.1] or [VTC87, Section IV.1.2, Proposition 1.1] for a proof.
We give now a collection of results on the characteristic functional on the finite-dimensional marginals, statistical invariances, independence properties, and moments. We sketch the simple proofs and give adequate references for the more evolved ones.

Proposition 2.12. *Let s be a generalized random process on $\mathscr{S}'(\mathbb{R}^d)$ and $\boldsymbol{\varphi} \in (\mathscr{S}(\mathbb{R}^d))^N$. Then, the characteristic function of the real random vector $\langle s, \boldsymbol{\varphi}\rangle$ is given by*

$$\widehat{\mathscr{P}}_{\langle s,\boldsymbol{\varphi}\rangle}(\boldsymbol{\xi}) = \widehat{\mathscr{P}}_s(\xi_1\varphi_1 + \cdots + \xi_N\varphi_N) = \widehat{\mathscr{P}}_s(\langle \boldsymbol{\xi}, \boldsymbol{\varphi}\rangle) \tag{2.26}$$

for every $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_N) \in \mathbb{R}^N$.

Note that the notation $\langle\cdot,\cdot\rangle$ is a duality product in the left term of (2.26), and the scalar product over $\mathbb{R}^N$ in the right term.

Proof. This is easily deduced from the computation

$$\langle\langle s, \boldsymbol{\varphi}\rangle, \boldsymbol{\xi}\rangle = \sum_{n=1}^N \langle s, \varphi_n\rangle \xi_n = \langle s, \sum_{n=1}^N \xi_n\varphi_n\rangle = \langle s, \langle \boldsymbol{\xi}, \boldsymbol{\varphi}\rangle\rangle.$$

□

Proposition 2.13. *Two random processes s_1 and s_2 are independent if and only if*

$$\widehat{\mathscr{P}}_{s_1+s_2}(\varphi) = \widehat{\mathscr{P}}_{s_1}(\varphi)\widehat{\mathscr{P}}_{s_2}(\varphi) \tag{2.27}$$

for every $\varphi \in \mathscr{S}(\mathbb{R}^d)$.

Proof. The processes s_1 and s_2 are independent if and only if $\langle s_1, \boldsymbol{\varphi}\rangle$ and $\langle s_1, \boldsymbol{\varphi}\rangle$ are independent for every $\boldsymbol{\varphi} \in (\mathscr{S}'(\mathbb{R}^d))^N$, $N \geq 1$. This is equivalent to $\widehat{\mathscr{P}}_{\langle s_1+s_2,\boldsymbol{\varphi}\rangle}(\boldsymbol{\xi}) = \widehat{\mathscr{P}}_{\langle s_1,\boldsymbol{\varphi}\rangle}(\boldsymbol{\xi})\widehat{\mathscr{P}}_{\langle s_1,\boldsymbol{\varphi}\rangle}(\boldsymbol{\xi})$ for any $\boldsymbol{\xi}, \boldsymbol{\varphi}$, that we can rewrite thanks to Proposition 2.12 as

$$\widehat{\mathscr{P}}_{s_1+s_2}(\langle \boldsymbol{\xi}, \boldsymbol{\varphi}\rangle) = \widehat{\mathscr{P}}_{s_1}(\langle \boldsymbol{\xi}, \boldsymbol{\varphi}\rangle)\widehat{\mathscr{P}}_{s_1}(\langle \boldsymbol{\xi}, \boldsymbol{\varphi}\rangle). \tag{2.28}$$

If s_1 and s_2 are independent, then we deduce (2.27) from (2.28) with $N = 1$ and $\xi = 1$. If now (2.27) is valid for any test function, we apply it with $\langle \boldsymbol{\varphi}, \boldsymbol{\xi} \rangle$, proving the equivalence. □

It is possible to read the independence at every point of a generalized random process on its characteristic functional. The following result on processes that are independent at every point is taken from [GV64, Section III.4.1, Theorem 1].

Proposition 2.14. *The generalized random process s is independent at every point if and only if*

$$\widehat{\mathscr{P}}_s(\varphi + \psi) = \widehat{\mathscr{P}}_s(\varphi)\widehat{\mathscr{P}}_s(\psi)$$

for every φ, ψ with disjoint supports.

The statistical invariances of s are related to the impact of the geometric transformations on the characteristic functional.

Proposition 2.15. *A generalized random process s is*

- *stationary if and only if for every $\varphi \in \mathscr{S}(\mathbb{R}^d)$ and $\boldsymbol{x}_0 \in \mathbb{R}^d$*

$$\widehat{\mathscr{P}}_s(\varphi) = \widehat{\mathscr{P}}_s(\mathrm{T}_{\boldsymbol{x}_0}\varphi).$$

- *symmetric if and only if for any $\varphi \in \mathscr{S}(\mathbb{R}^d)$, $\widehat{\mathscr{P}}_s(\varphi) = \widehat{\mathscr{P}}_s(\varphi^\vee)$.*
- *self-similar of order H if and only if for every $\varphi \in \mathscr{S}(\mathbb{R}^d)$ and $a > 0$.*

$$\widehat{\mathscr{P}}_s(\varphi) = \widehat{\mathscr{P}}_s(a^{H+d}\varphi(a\cdot)). \tag{2.29}$$

- *isotropic if and only if for every $\varphi \in \mathscr{S}(\mathbb{R}^d)$ and $\Omega \in SO(d)$*

$$\widehat{\mathscr{P}}_s(\varphi) = \widehat{\mathscr{P}}_s(\mathrm{R}_\Omega\varphi).$$

Proof. We prove the result for (2.29), the other proofs being very similar. We focus on the generalized random process $a^H s(\cdot/a)$. We readily see that

$$\widehat{\mathscr{P}}_{a^H s(\cdot/a)}(\varphi) = \mathbb{E}[\mathrm{e}^{\mathrm{i}\langle a^H s(\cdot/a), \varphi\rangle}] = \mathbb{E}[\mathrm{e}^{\mathrm{i}\langle s, a^{H+d}\varphi(a\cdot)\rangle}] = \widehat{\mathscr{P}}_s(a^{H+d}\varphi(a\cdot)).$$

Then, $s \overset{(\mathscr{L})}{=} a^H s(\cdot/a)$ if and only if $\widehat{\mathscr{P}}_s(a^{H+d}\varphi(a\cdot)) = \widehat{\mathscr{P}}_s(\varphi)$, as expected. □

2.3.3 Stochastic Functional Analysis

As we have seen, the theory of generalized random processes allows one to consider very general random processes, including the ones that do not admit a pointwise representation. It has another advantage: The Borel σ-field of $\mathscr{S}'(\mathbb{R}^d)$ appears to be very rich, and we will see, in particular, that the usual function spaces are measurable. This proves us with a strategy to probe the smoothness, the integrability, the decay rate, etc., of a generalized random process. In [Car63], P. Cartier compares the approach of Gelfand with more traditional ones in probability theory, in particular the theory developed by J.L. Doob [Doo90].

General principle. Consider a topological vector space $\mathscr{X}$ included in $\mathscr{S}'(\mathbb{R}^d)$, and endowed with the Borel σ-field $\mathscr{B}(\mathscr{X})$. Two questions arise:

1. Is the space $\mathscr{X}$ measurable in $\mathscr{S}'(\mathbb{R}^d)$; that is, $\mathscr{X} \in \mathscr{B}(\mathscr{S}'(\mathbb{R}^d))$?
2. Are the Borel σ-fields compatible in the sense that

$$\mathscr{B}(\mathscr{X}) = \mathscr{X} \cap \mathscr{B}(\mathscr{S}'(\mathbb{R}^d)) := \left\{ \mathscr{X} \cap B \;\middle|\; B \in \mathscr{B}(\mathscr{S}'(\mathbb{R}^d)) \right\}?$$

If these two assumptions hold, we know in particular that the probability $\mathscr{P}(s \in \mathscr{X})$ is well-defined for any tempered generalized random process s. The compatibility of the σ-field has also two consequences. First, an $\mathscr{X}$-valued random variable can be seen as an $\mathscr{S}'(\mathbb{R}^d)$-valued random variable such that $\mathscr{P}(s \in \mathscr{X}) = 1$. Second, an $\mathscr{S}'(\mathbb{R}^d)$-valued random variable for which $\mathscr{P}(s \in \mathscr{X}) = 1$ admits a version (identical up to a space of measure 0) that is an $\mathscr{X}$-valued random variable. In other terms, under 1. and 2., $\mathscr{X}$-valued random variables form a subspace of $\mathscr{S}'(\mathbb{R}^d)$-valued random variables, characterized by the relation $\mathscr{P}(s \in \mathscr{X}) = 1$ (up to modification on a space of measure 0 in $\mathscr{S}'(\mathbb{R}^d)$).
These questions were studied by X. Fernique [Fer67] and K. Itō [Itô84]. Fernique considers a very large class of function spaces, called *standard spaces*, for which the measurability structure is essentially compatible with the topological structure. This means in particular that the two questions above receive positive answers in this case. In [Fer67, Section III.3], Fernique applies his general principle, that we shall not detail here, to identify measurable spaces of $\mathscr{D}'(\mathbb{R}^d)$. The same ideas apply to $\mathscr{S}'(\mathbb{R}^d)$ and can be summarized as follows.

Proposition 2.16. *Fix $p \in [1,\infty)$ and $\tau \in \mathbb{R}$. Assume that $\mathscr{X}$ is one of the following function spaces: $\mathscr{D}(\mathbb{R}^d), \mathscr{S}(\mathbb{R}^d), W_2^\tau(\mathbb{R}^d), L_p(\mathbb{R}^d), \mathscr{E}'(\mathbb{R}^d)$. Then,*

$$\mathscr{X} \in \mathscr{B}(\mathscr{S}'(\mathbb{R}^d)) \;\; \text{and} \;\; \mathscr{B}(\mathscr{X}) = \mathscr{X} \cap \mathscr{B}(\mathscr{S}'(\mathbb{R}^d)).$$

Measurability of Besov Spaces in $\mathscr{S}'(\mathbb{R}^d)$. In this thesis, we shall investigate in which Besov space (local or weighted) is a given Lévy noise. Here, we first show that this question is meaningful in the sense that any Besov space $B^\tau_{p,q}(\mathbb{R}^d;\rho)$ is measurable in $\mathscr{S}'(\mathbb{R}^d)$. The principle developed by Fernique can be easily applied to Besov spaces that are Banach spaces (that is, when $p \geq 1$). In general, however, Besov spaces are quasi-Banach spaces and the results of Fernique cannot be directly applied.
Here, we give our own proof of the measurability of Besov spaces, taken from our works [FUW17b, FFU]. Our approach is different from the one of Fernique and does not rely on any topological argument. We essentially show that Besov spaces are included in the cylindrical σ-field of $\mathscr{S}'(\mathbb{R}^d)$. We say that C is a *cylinder* of $\mathscr{S}'(\mathbb{R}^d)$ if it can be written as

$$C = \{u \in \mathscr{S}'(\mathbb{R}^d),\; \langle u, \boldsymbol{\varphi}\rangle \in B\}$$

where $N \geq 1$, $\boldsymbol{\varphi} = (\varphi_1,\dots,\varphi_N) \in (\mathscr{S}(\mathbb{R}^d))^N$, and $B \in \mathscr{B}(\mathbb{R}^N)$. The *cylindrical σ-field* is the σ-field generated by the cylinders. In the case of $\mathscr{S}'(\mathbb{R}^d)$, it coincides with the topological σ-field [Itô84].

Proposition 2.17. *For every* $0 < p \leq \infty$ *and* $\tau, \rho \in \mathbb{R}$*, we have that*

$$B_p^\tau(\mathbb{R}^d;\rho) \in \mathscr{B}\left(\mathscr{S}'(\mathbb{R}^d)\right). \tag{2.30}$$

The proof of a similar measurability result is detailed for periodic generalized functions in [FUW17b, Theorem 4]. The difference here is that we deal with functions over $\mathbb{R}^d$ and with weighting functions. The adaptation to this case was exposed in [FUW17b, Lemma 1] for the complete family of Besov spaces (with, possibly, $q \neq p$). We reproduce here the proof for $p = q$. The proof uses Lemma 2.1.

Lemma 2.1. *Let* $\mathscr{X}$ *be a topological vector space,* $\mathscr{X}'$ *its topological dual and* $\mathscr{B}_c(\mathscr{X}')$ *the cylindrical* σ*-field on* $\mathscr{X}'$*, generated by the cylinders of the form*

$$C := \{f \in \mathscr{X}' \mid \langle f, \boldsymbol{\varphi}\rangle \in B\},$$

where $N \geq 1$*,* $\boldsymbol{\varphi} = (\varphi_1, \cdots, \varphi_N) \in \mathscr{X}^N$*, and* $B \in \mathscr{B}(\mathbb{R}^N)$*. Then, for every countable set* S*, every* $\varphi_n \in \mathscr{X}$*, and every* $p > 0$*, we have*

$$\left\{ f \in \mathscr{X}' \,\middle|\, \sum_{n\in S} \left|\langle f, \varphi_n\rangle\right|^p < \infty \right\} \in \mathscr{B}_c(\mathscr{X}').$$

Proof. We first remark that

$$\left\{ f \in \mathscr{X}' \,\middle|\, \sum_{n\in S} \left|\langle f, \varphi_n\rangle\right|^p < \infty \right\} = \bigcup_{N\geq 0} \left\{ f \in \mathscr{X}' \,\middle|\, \sum_{n\in S} \left|\langle f, \varphi_n\rangle\right|^p \leq N \right\}. \tag{2.31}$$

It therefore suffices to show that $\{f \in \mathscr{X}' \mid \sum_{n\in S} |\langle f, \varphi_n\rangle|^p \leq N\}$ is measurable. We denote by $\mathbb{R}^S$ the space of real sequences indexed by S, endowed with the product σ-field. By definition of the cylindrical σ-field, for fixed $\boldsymbol{\varphi} = (\varphi_n)_{n\in S}$, the projection

$$\pi_{\boldsymbol{\varphi}}(f) := \left(\langle f, \varphi_n\rangle\right)_{n\in S}$$

is measurable from $\mathscr{X}'$ to $\mathbb{R}^S$. Moreover, the function F_p from $\mathbb{R}^S$ to $\mathbb{R}^+$ that associates to a sequence $(a_n)_{n\in S}$ the quantity $\sum_{n\in S} |a_n|^p$ is measurable. Finally, since $[0, N]$ is measurable in $\mathbb{R}^+$,

$$\left\{ f \in \mathscr{X}' \,\middle|\, \sum_{n\in S} \left|\langle f, \varphi_n\rangle\right|^p \leq N \right\} = \pi_{\boldsymbol{\varphi}}^{-1}\left(F_p^{-1}([0,N])\right)$$

is measurable in $\mathscr{X}'$, as expected. □

Proof of Proposition 2.17. We obtain the desired result in three steps. We treat the case $p < \infty$ and let the reader adapt the proof for $p = \infty$.

- First, we show that $W_2^\tau(\mathbb{R}^d;\rho) \in \mathscr{B}\left(\mathscr{S}'(\mathbb{R}^d)\right)$ for every $\tau, \rho \in \mathbb{R}$. This corresponds to the case $p = 2$. Let $(h_n)_{n\in\mathbb{N}}$ be an orthonormal basis of $L_2(\mathbb{R}^d)$, with $h_n \in \mathscr{S}(\mathbb{R}^d)$ for all $n \geq 0$.

(We can for instance consider the Hermite functions, based on Hermite polynomials, see [Sim03, Section 2] or [Itô84, Section 1.3] for the definitions.) The interest of having basis functions in $\mathscr{S}(\mathbb{R}^d)$ is that we have the characterization

$$L_2(\mathbb{R}^d) = \left\{ f \in \mathscr{S}'(\mathbb{R}^d) \;\middle|\; \sum_{n\in\mathbb{N}} |\langle f, h_n\rangle|^2 < \infty \right\}.$$

More generally, with the notations of Section 2.2.3, $f \in W_2^\tau(\mathbb{R}^d;\rho)$ if and only if $\mathrm{J}_\tau\{\langle\cdot\rangle^\rho f\} \in L_2(\mathbb{R}^d)$, from which we deduce that

$$W_2^\tau(\mathbb{R}^d;\rho) = \left\{ f \in \mathscr{S}'(\mathbb{R}^d) \;\middle|\; \sum_{n\in\mathbb{N}} |\langle f, \langle\cdot\rangle^\rho \mathrm{J}_\tau\{h_n\}\rangle|^2 < \infty \right\}.$$

We can therefore apply Lemma 2.1 with $p = 2$, $S = \mathbb{N}$, and $\varphi_n = \langle\cdot\rangle^\rho \mathrm{J}_\tau\{h_n\}$, to deduce that $W_2^\tau(\mathbb{R}^d;\rho) \in \mathscr{B}(\mathscr{S}'(\mathbb{R}^d))$.

- For any $\tau,\rho \in \mathbb{R}$, the cylindrical σ-field of $W_2^\tau(\mathbb{R}^d;\rho)$ is the σ-field $\mathscr{B}_c(W_2^\tau(\mathbb{R}^d;\rho))$ generated by the sets

  $$\left\{ u \in W_2^\tau(\mathbb{R}^d;\rho),\ \langle u, \boldsymbol{\varphi}\rangle \in B \right\},$$

  where $N \geq 1$, $\boldsymbol{\varphi} = (\varphi_1,\dots,\varphi_N) \in (W_2^{-\tau}(\mathbb{R}^d;-\rho))^N$, and $B \in \mathscr{B}(\mathbb{R}^N)$. Then, knowing already that $W_2^\tau(\mathbb{R}^d;\rho) \in \mathscr{B}(\mathscr{S}'(\mathbb{R}^d))$ implies readily that

  $$\mathscr{B}_c(W_2^\tau(\mathbb{R}^d;\rho)) \subset \mathscr{B}(\mathscr{S}'(\mathbb{R}^d)). \tag{2.32}$$

- Finally, we show that $B_p^\tau(\mathbb{R}^d;\rho) \in \mathscr{B}_c(W_2^{\tau_1}(\mathbb{R}^d;\rho_1))$ for some adequately chosen $\tau_1,\rho_1 \in \mathbb{R}$. Coupled with (2.32), this suffices to show (2.30).

  Fix $\tau_1 \leq \tau + d\left(1/2 - 1/p\right)$ and $\rho_1 < \rho + d\left(1/p - 1/2\right)$. According to Proposition 2.8, we have the embedding $B_{p,q}^\tau(\mathbb{R}^d;\rho) \subseteq W_2^{\tau_1}(\mathbb{R}^d;\rho_1)$. Now, thanks to Proposition 2.9, we identify $B_p^\tau(\mathbb{R}^d;\rho)$ as the space of generalized functions $f \in W_2^{\tau_1}(\mathbb{R}^d;\rho_1)$ such that

  $$\sum_{j,G,\boldsymbol{m}} |\langle f, 2^{j(\tau-d/p+d/2)}\langle 2^{-j}\boldsymbol{m}\rangle^\rho \psi_{j,G,\boldsymbol{m}}\rangle|^p < \infty.$$

  Again, we apply Lemma 2.1 with $S = \left\{(j,G,\boldsymbol{m}) \;\middle|\; j \in \mathbb{Z}, G \in \mathrm{G}^j, \boldsymbol{m} \in \mathbb{Z}^d\right\}$, p, and $\varphi_{j,G,\boldsymbol{m}} = 2^{j(\tau-d/p+d/2)}\langle 2^{-j}\boldsymbol{m}\rangle^\rho \psi_{j,G,\boldsymbol{m}}$ to deduce that $B_p^\tau(\mathbb{R}^d;\rho) \in \mathscr{B}_c(W_2^{\tau_1}(\mathbb{R}^d;\rho_1))$. The inclusion (2.32) allows to conclude.

□

3 Construction of Generalized Lévy Processes

We aim at constructing generalized random processes solution of a stochastic differential equation of the form

$$\mathrm{L}s = w, \tag{3.1}$$

with L a linear (pseudo-)differential operator and w a Lévy white noise in $\mathcal{S}'(\mathbb{R}^d)$. Our main tool is the Bochner-Minlos theorem presented in Section 2.3.2. Two questions need to be addressed in order to define the broadest possible class of random processes: (i) the specification of the class of Lévy white noises on $\mathcal{S}'(\mathbb{R}^d)$, and (ii) the identification of compatibility conditions between a Lévy noise and a pseudo-differential operator. This is done respectively in Sections 3.1 and 3.3. In order to prepare the construction of general Lévy processes, we extend the domain of definition of the Lévy noise to test functions not necessarily smooth nor rapidly decaying in Section 3.2.

3.1 Lévy White Noise

The class of Lévy white noise on $\mathscr{D}'(\mathbb{R}^d)$ was introduced in [GV64, Chapter III]. Those processes are specified via their characteristic functional. Here, we will essentially follow the same line, except that we will consider Lévy noise on $\mathscr{S}'(\mathbb{R}^d)$. The question of whether or not a Lévy white noise is tempered has been recently resolved. In [FAU14], we gave a sufficient condition ensuring that a Lévy noise is actually located in $\mathscr{S}'(\mathbb{R}^d)$ in terms of moment conditions on the Lévy measure. This is the main contribution presented in Section 3.1.1. More recently, R. Dalang and T. Humeau have shown that our condition is actually sufficient [DH15]. This gives a complete characterization of tempered Lévy noises.

3.1.1 Construction: From $\mathscr{D}'(\mathbb{R}^d)$ to $\mathscr{S}'(\mathbb{R}^d)$

The construction of continuous-domain white noises and processes, including Lévy processes, is intimately linked with the infinite divisibility of the finite-dimensional marginals of those processes. The main idea is the following. If $(s(t))_{t\geq 0}$ is a valid pointwise process with stationary and independent increments and $s(0) = 0$ (in other terms, if s is a Lévy process), then we set, for all $N \geq 1$,

$$s(t) = \sum_{n=1}^{N} s\left(\frac{nt}{N}\right) - s\left(\frac{(n-1)t}{N}\right) := \sum_{n=1}^{N} X_{n,N}.$$

The $X_{n,N}$, $n = 1 \cdots N$, are independent (since the increments are independent) and identically distributed (since the increments are stationary). This is precisely the definition of an infinitely divisible random variable (Section 2.1.2).

Consider a vector of N i.i.d. infinitely divisible random variables $\boldsymbol{X}$ with common Lévy exponent Ψ. Then, the characteristic function of $\boldsymbol{X}$ is, for every $\boldsymbol{\xi} = (\xi_1, \dots, \xi_N)$,

$$\widehat{\mathscr{P}}_{\boldsymbol{X}}(\boldsymbol{\xi}) = \exp\left(\sum_{n=1}^{N} \Psi(\xi_n)\right). \tag{3.2}$$

Inspired by (3.2) and following Gelfand and Vilenkin, we consider infinite-dimensional functionals of the form

$$\widehat{\mathscr{P}}(\varphi) = \exp\left(\int_{\mathbb{R}^d} \Psi(\varphi(\boldsymbol{x}))\mathrm{d}\boldsymbol{x}\right). \tag{3.3}$$

The functional is, for instance, well-defined when $\Psi : \mathbb{R} \to \mathbb{C}$ is a continuous function that vanishes at 0 and φ is smooth and compactly supported. The idea is to replace the sum in (3.2) by an integral, and to use test functions as the running variable.

Lévy noise in $\mathscr{D}'(\mathbb{R}^d)$. The functional (3.3) is a valid characteristic functional over $\mathscr{D}\mathbb{R}^d)$ if and only if the function $\xi \mapsto \mathrm{e}^{\Psi(\xi)}$ is the characteristic function of an infinitely divisible law [GV64, Section 4.4, Theorem 6]; that is, if and only if Ψ is a Lévy exponent (according to Theorem 2.2). The Bochner-Minlos theorem then ensures that there exists a generalized random process whose characteristic functional is given by (3.3).

Definition 3.1. *Let Ψ be a Lévy exponent. Then, the generalized random process w with characteristic functional* (3.3) *is called a* Lévy white noise*, or simply a Lévy noise. By extension, we say that Ψ is the Lévy exponent of w.*

Lévy noise in $\mathscr{S}'(\mathbb{R}^d)$.

Definition 3.2. *We say that the Lévy exponent Ψ with Lévy triplet (μ, σ^2, ν) satisfies the ϵ-*condition *if there exists $\epsilon > 0$ such that*

$$\int_{t\geq 1} |t|^{\epsilon}\, \nu(\mathrm{d}t) < \infty$$

Since the moments of ν are related to the moment of the underlying infinitely divisible random variable X (Proposition 2.3), the ϵ-condition is equivalent to the existence of $\epsilon > 0$ such that $\mathbb{E}[|X|^{\epsilon}] < \infty$. It is also equivalent to $\alpha_{\mathrm{asymp}} > 0$, where α_{asymp} is the asymptotic index of Definition 2.8.
Here is a pedagogical example of an infinitely divisible law that does not satisfy the ϵ-condition. Consider the measure ν defined as

$$\nu(\mathrm{d}t) = \frac{\mathrm{d}t}{|t|\log^2(1+|t|)}.$$

Then, it is easy to see that $\int_{\mathbb{R}} \inf(1, t^2)\nu(\mathrm{d}t) < \infty$ since $(|t|\log^2(1+|t|))^{-1}$ is integrable at infinity (Bertrand integral), while $\int_{|t|\geq 1} |t|^{\epsilon}\, \nu(\mathrm{d}t) = \infty$ for any $\epsilon > 0$. Therefore, the Lévy exponent with Lévy triplet $(0, 0, \nu)$ does not satisfy the ϵ-condition. However, all the examples of Lévy noise that we will encountered (Gaussian, SαS, Laplace, compound Poisson with Gaussian jumps, etc.) easily satisfies the ϵ-condition. We characterize the tempered Lévy noise in Theorem 3.1.

Theorem 3.1. *A Lévy white noise on $\mathscr{D}'(\mathbb{R}^d)$ is almost surely tempered if and only if its Lévy exponent satisfies the ϵ-condition. This is equivalent to having finite ϵth-moment for some $\epsilon > 0$.*

We prove that the ϵ-condition is sufficient. For the necessity, see [DH15, Theorem 3.13]. The sufficiency was first proved in [FAU14, Theorem 3]. The proof that we propose here differs from the original one. We base our argument on the following proposition.

Proposition 3.1. *If Ψ is a continuous function from $\mathbb{R}$ to $\mathbb{C}$ such that*

- *the function $\xi \mapsto \exp(\lambda\Psi(\xi))$ is positive-definite for any $\lambda \geq 0$, and*
- *there exist $\epsilon > 0$ and $C > 0$ such that $|\Psi(\xi)| \leq C\left(|\xi|^{\epsilon} + |\xi|^2\right)$ for any $\xi \in \mathbb{R}$,*

then the functional $\widehat{\mathscr{P}} : \varphi \mapsto \exp(\int_{\mathbb{R}^d} \Psi(\varphi(\boldsymbol{x}))\mathrm{d}\boldsymbol{x})$ is well-defined and positive-definite over $\mathscr{S}(\mathbb{R}^d)$.

Proof. In [GV64, Section 4.2.2, Theorem 2], Gelfand and Vilenkin prove that $\widehat{\mathscr{P}}$ is positive-definite over $\mathscr{D}(\mathbb{R}^d)$ if and only if $\xi \mapsto \exp(\lambda\Psi(\xi))$ is positive-definite for any $\lambda > 0$. We essentially adapt their proof from $\mathscr{D}(\mathbb{R}^d)$ to $\mathscr{S}(\mathbb{R}^d)$. The positive-definiteness of $\widehat{\mathscr{P}}$ is equivalent to the following condition: For any $\varphi_1,\dots,\varphi_N \in \mathscr{S}(\mathbb{R}^d)$, the matrix A of size $N\times N$, defined as

$$\mathrm{A}[m,n] = \widehat{\mathscr{P}}(\varphi_n - \varphi_m),$$

is positive-definite. For $k \geq 1$ an integer, we set

$$\mathrm{A}_k[m,n] = \frac{1}{k}\sum_{\boldsymbol{u}\in\mathbb{Z}^d} \Psi\left(\varphi_n\left(\frac{\boldsymbol{u}}{k}\right) - \varphi_m\left(\frac{\boldsymbol{u}}{k}\right)\right).$$

For any $\varphi \in \mathscr{S}(\mathbb{R}^d)$, our bound on $\Psi(\xi)$ easily implies that

$$\int_{\mathbb{R}^d} \left|\Psi(\varphi(\boldsymbol{x}))\right| \mathrm{d}\boldsymbol{x} \leq C\left(\|\varphi\|_\epsilon^\epsilon + \|\varphi\|_2^2\right) < \infty. \tag{3.4}$$

It means in particular that $\boldsymbol{x} \mapsto \Psi(\varphi(\boldsymbol{x}))$ is integrable, and that $\widehat{\mathscr{P}}$ is well-defined over $\mathscr{S}(\mathbb{R}^d)$. Hence, $\boldsymbol{x} \mapsto \Psi\big(\varphi_n(\boldsymbol{x}) - \varphi_m(\boldsymbol{x})\big)$ is integrable, and we recognize a Riemann sum in (**??**), from which we deduce that $\mathrm{A}_k \to \mathrm{A}$ as $k\to\infty$. We also set, and for $k \geq 1$, $\boldsymbol{u}\in\mathbb{Z}^d$, and $M \geq 1$, the matrices with entries given by

$$\mathrm{A}_k^{\boldsymbol{u}}[m,n] = \frac{1}{k}\Psi(\varphi_n(\boldsymbol{u}/k) - \varphi_m(\boldsymbol{u}/k)) \text{ and } \mathrm{A}_{k,M}[m,n] = \prod_{|\boldsymbol{u}|\leq M} \mathrm{A}_k^{\boldsymbol{u}}[m,n].$$

Then, $\mathrm{A}_{k,M} \to \mathrm{A}_k$ when $M\to\infty$.
To conclude the proof, we remark that the matrix $\mathrm{A}_k^{\boldsymbol{u}}$ is positive-definite, using the positive-definiteness of $\xi \mapsto \exp(\frac{1}{k}\Psi(\xi))$ (chose $\xi_n = \varphi_n(\boldsymbol{u}/k)$ in the definition of the positive-definiteness of the function). The Schur product theorem ensures that the Hadamard product of positive-definite matrices is positive-definite. Therefore, $\mathrm{A}_{k,M}$ is positive-definite, a property that the A_k, and then A, inherit as $M,k\to\infty$. □

Proof of Theorem 3.1: The sufficiency. Let Ψ be the Lévy exponent of w. We need to prove that $\widehat{\mathscr{P}}$ is a valid characteristic functional on $\mathscr{S}(\mathbb{R}^d)$, knowing that it is a characteristic functional on $\mathscr{D}(\mathbb{R}^d)$. Of course, the functional vanishes at 0. We show that it is well-defined, positive-definite, and continuous over $\mathscr{S}(\mathbb{R}^d)$.
Positive-definiteness: The mapping $\xi \mapsto \exp(\lambda\Psi(\xi))$ is posititive-definite for any λ according to Theorem 2.2. Since we already know that Ψ satisfies (2.3) for some $p=\epsilon\in(0,1]$, we apply Proposition 3.1 to deduce that $\widehat{\mathscr{P}} : \varphi \mapsto \exp(\int_{\mathbb{R}^d}\Psi(\varphi(\boldsymbol{x}))\mathrm{d}\boldsymbol{x})$ is well-defined and positive-definite.
Continuity: The functional being positive-definite, it is enough to show its continuity at the origin (Proposition 2.11). For this, we simply remark that we have, using (3.4),

$$\left|\log\widehat{\mathscr{P}}_w(\varphi)\right| \leq C\left(\|\varphi\|_\epsilon^\epsilon + \|\varphi\|_2^2\right).$$

Hence, $\log\widehat{\mathscr{P}}_w(\varphi) \to 0 = \log\widehat{\mathscr{P}}_w(0)$ when $\varphi \to 0$ in $\mathscr{S}(\mathbb{R}^d)$. □

Remark. In order to apply the Bochner-Minlos theorem on $\mathscr{S}'(\mathbb{R}^d)$, it is required to prove the continuity and the positive-definiteness of the functional (3.3) over $\mathscr{S}(\mathbb{R}^d)$. In [FAU14], we proved the sufficiency in Theorem 3.1 using a different approach. We first showed the continuity of the characteristic functional over $\mathscr{S}(\mathbb{R}^d)$, and deduce the positive-definiteness by density (knowing *a priori* that the characteristic functional is positive-definite over the space $\mathscr{D}(\mathbb{R}^d)$, dense in $\mathscr{S}(\mathbb{R}^d)$). By contrast, we gave here a proof of the positive-definiteness before investigating the continuity. It is then sufficient to establish the continuity at the origin, which happens to be much less technical. Based on Theorem 3.1, we define the class of tempered Lévy noises.

Definition 3.3. *Let Ψ be a Lévy exponent satisfying the ϵ-condition. Then, the generalized random process w with characteristic functional* (3.3) *is called a* tempered Lévy white noise.

When the context is clear, we omit to specify that the noise is tempered.

Tempered Lévy noise in $\mathscr{D}'(\mathbb{R}^d)$. This discussion is highly linked with the results of Section 2.3.3. If Ψ is a Lévy exponent satisfying the ϵ-condition, we apply the Bochner-Minlos theorem to specify two probability measures as follows.

- We denote by $\mathscr{P}_{\mathscr{D}'}$ the probability measure on $\mathscr{D}'(\mathbb{R}^d)$ with characteristic functional $\widehat{\mathscr{P}}_{\mathscr{D}'}(\varphi) = \exp(\int_{\mathbb{R}^d} \Psi(\varphi(\boldsymbol{x}))\mathrm{d}\boldsymbol{x})$ for any $\varphi \in \mathscr{D}(\mathbb{R}^d)$.

- We denote by $\mathscr{P}_{\mathscr{S}'}$ the probability measure on $\mathscr{S}'(\mathbb{R}^d)$ with characteristic functional $\widehat{\mathscr{P}}_{\mathscr{S}'}(\varphi) = \exp(\int_{\mathbb{R}^d} \Psi(\varphi(\boldsymbol{x}))\mathrm{d}\boldsymbol{x})$ for any $\varphi \in \mathscr{S}(\mathbb{R}^d)$.

We recall that the spaces $\mathscr{D}'(\mathbb{R}^d)$ and $\mathscr{S}'(\mathbb{R}^d)$ are endowed with the Borel σ-fields $\mathscr{B}(\mathscr{D}'(\mathbb{R}^d))$ and $\mathscr{B}(\mathscr{S}'(\mathbb{R}^d))$ associated to their respective weak*-topology. The connection between the two probability measure is deduced from the work of X. Fernique, and summarized here. The following result is included in [Fer67, Section III.3].

Proposition 3.2. *The space $\mathscr{S}'(\mathbb{R}^d)$ is measurable in $\mathscr{D}'(\mathbb{R}^d)$*, i.e., $\mathscr{S}'(\mathbb{R}^d) \in \mathscr{B}(\mathscr{D}'(\mathbb{R}^d))$. *Moreover, we have that*

$$\mathscr{B}(\mathscr{S}'(\mathbb{R}^d)) = \mathscr{B}(\mathscr{D}'(\mathbb{R}^d)) \cap \mathscr{S}'(\mathbb{R}^d).$$

Proposition 3.2 implies that, for any $B \in \mathscr{B}(\mathscr{S}'(\mathbb{R}^d)) \subset \mathscr{B}(\mathscr{D}'(\mathbb{R}^d))$, we have $\mathscr{P}_{\mathscr{D}'}(B) = \mathscr{P}_{\mathscr{S}'}(B)$. In particular $\mathscr{P}_{\mathscr{D}'}(\mathscr{S}'(\mathbb{R}^d)) = 1$. This has two direct consequences. First, the generalized random process $s_{\mathscr{D}'}$ in $\mathscr{D}'(\mathbb{R}^d)$ with law $\mathscr{P}_{\mathscr{D}'}$ is almost surely tempered, so it admits a version in $\mathscr{S}'(\mathbb{R}^d)$. Second, the tempered generalized random process $s_{\mathscr{S}'}$ in $\mathscr{S}'(\mathbb{R}^d)$ with law $\mathscr{P}_{\mathscr{S}'}$ can be extended into a generalized random process in $\mathscr{D}'(\mathbb{R}^d)$ with law $\mathscr{P}_{\mathscr{D}'}(B) = \mathscr{P}_{\mathscr{S}'}(B \cap \mathscr{S}'(\mathbb{R}^d))$ for any $B \in \mathscr{B}(\mathscr{D}'(\mathbb{R}^d))$. This new process is almost surely in $\mathscr{S}'(\mathbb{R}^d)$.
Finally, the Lévy noises on $\mathscr{D}'(\mathbb{R}^d)$ whose Lévy exponent satisfy the ϵ-conditions admit a tempered version that is the associated tempered Lévy noise. In the next chapters, we will only consider tempered Lévy white noises with this connection with the original construction of Gelfand and Vilenkin in mind.

3.1.2 Independence, Invariance, and Examples of Lévy noises

We recall the independence and invariances properties of the Lévy noise, as studied in [GV64, Chapter III] The proofs are simple when relying on the characteristic functional.

Proposition 3.3. *A tempered Lévy noise is independent at every point.*

This is deduced from the form of the characteristic functional of the Lévy noise and Proposition 2.14.

Proposition 3.4. *A tempered Lévy noise is stationary and isotropic. It is symmetric if and only if the underlying infinitely divisible random variable is.*

Again, the form of the characteristic functional coupled with Proposition 2.15 directly gives the result. Propositions 3.3 and 3.4 are really reasonable in the sense that a white noise should clearly satisfy them. We note however that they do not characterize the class of Lévy noises. For instance, in dimension 1, the derivative of a Lévy noise is also independent at every point, stationary, and isotropic. This remark is extended in dimension d when considering partial derivatives of the Lévy noise.

Nomenclature of Lévy noise. Consider a Lévy exponent Ψ satisfying the ϵ-condition. Let X and w be the underlying infinitely divisible random variable and Lévy noise, respectively. The law of w is fully characterized by the one of X. By convention, the terminology for the random variable X is inherited by the Lévy noise w. It means in particular that we define Gaussian, SαS, compound Poisson, and generalized Laplace noise from their corresponding Gaussian, SαS, compound Poisson, and generalized Laplace random variables introduced in Section 2.1.3.

The compound Poisson case. Consider a compound Poisson random variable with parameter $\lambda > 0$ and law of jump P and w the corresponding compound Poisson noise. Then, we have that

$$w \overset{(\mathscr{L})}{=} \sum_{k\geq 0} a_k \delta(\cdot - \boldsymbol{x}_k) \tag{3.5}$$

where the a_k are i.i.d with common law P, and the $\boldsymbol{x}_k$, independent of the a_k, are such that $\mathrm{Card}\{k,\ \boldsymbol{x}_k \in B\}$ is a Poisson random variable with parameter $\lambda \mathrm{Leb}(B)$ for any bounded Borel set $B \subset \mathbb{R}^d$. This is a standard result in the theory of scattered random measure [RR89]: Poisson random measures are characterized by their jump locations (the $\boldsymbol{x}_k$) and the intensity of the jumps (the a_k). In fact, (3.5) can be shown almost surely; that is, the random variables a_k and the random vectors $\boldsymbol{x}_k$ can be specified from w, but this will not be exploited in the sequel. For a proof of (3.5) based on the computation of the characteristic functional of the right term of the relation, see [UT11, Appendix II]. The representation (3.5) of a compound Poisson noise will be exploited many times in the sequel.

3.2 The Domain of Definition of Lévy Noise

This section is based on our work done in collaboration with T. Humeau [DFHU]. In Section 3.1.1, a tempered Lévy white noise is a random element in $\mathscr{S}'(\mathbb{R}^d)$. This means that we can *a priori* apply the noise against a smooth and rapidly decaying test function. As shall be illustrated throughout this thesis, this very conservative restriction can to be relaxed. We give here some motivations in that direction.

- *From Lévy noises to Lévy processes:* A Lévy process s is solution of the stochastic differential equation $\mathrm{D}s = w$ with boundary condition $s(0) = 0$. It is well known that, contrary to the Lévy noise, the Lévy process is a pointwise process, with càdlàg[1] trajectories [Ber98]. Formally, a Lévy process satisfies the relation $s(t) = \langle w, \mathbb{1}_{[0,t]}\rangle$, where $\mathbb{1}_A$ denotes the indicator function of the set A. In particular, we aim to define rigorously $\langle w, f\rangle$ for test functions of the form $f = \mathbb{1}_{[0,t]}$. This question was already addressed, for instance in [LS06]. Our construction will also provide a full answer.

- *Expansion of the Lévy noise into orthonormal bases:* Consider an orthonormal basis (f_n) of $L_2(\mathbb{R}^d)$. We want to know when it is reasonable to consider the family of the coefficients $\langle w, f_n\rangle$ of the Lévy noise w. This will for instance be exploited in Section 5.2 where we use the Daubechies wavelets coefficients of a Lévy noise to estimate its regularity. Daubechies wavelets are compactly supported but have a limited smoothness [Dau88]. We will see that the expansion on any Daubechies wavelet basis is possible for every Lévy noise. More generally, we may be interested in bases whose elements are not compactly supported and/or not smooth.

- *Support localization of the Lévy white noise:* The domain of definition of Lévy noise is also the domain of continuity of its characteristic functional. There are strong connections between the continuity properties of the characteristic functional and the localization of the process, for instance in Sobolev spaces. The more we can extend the domain of definition, the more we learn about the regularity of the Lévy noise. This idea has been exploited in [FFU, Section 5].

- *Construction of solutions of SDEs driven by Lévy noise:* By extending the domain of definition of the Lévy noise, one weakens the conditions on the compatibility between whitening operator L and the noise w. Indeed, we have formally that

  $$\langle s, \varphi\rangle = \langle \mathrm{L}^{-1} w, \varphi\rangle = \langle w, (\mathrm{L}^{-1})^* \{\varphi\}\rangle, \tag{3.6}$$

  where $(\mathrm{L}^{-1})^*$ is the adjoint of L^{-1}. We therefore see that we essentially need that $(\mathrm{L}^{-1})^*\{\varphi\}$ belongs to the domain of definition of w for any $\varphi \in \mathscr{S}(\mathbb{R}^d)$ to give a meaning to (3.6). This principle will be used extensively in Section 3.3 to construct generalized Lévy processes and in Chapters 4 and 5 when studying generalized Lévy processes.

The previous examples show the interest of extending the domain of definition of the Lévy noise. We also want to go further, and to identify the broadest possible set of test functions

[1] *Càdlàg* is the French acronym for right continuous functions with left limit at each point.

such that the random variable $\langle w, f\rangle$ is well-defined. To do so, we connect the concept of Lévy noise as generalized random process with the independently scattered random measures studied by B. Rajput and J. Rosinski in [RR89].

3.2.1 Lévy Noises As Independently Scattered Random Measures

A random measure is as a random process whose test functions are indicator functions: To each measurable set, we associate a random variable. It is very popular for stochastic integration, the integral being defined for simple functions (*i.e.*, linear combinations of indicator functions), and extended by a limit argument. Essentially, a random measure is independently scattered when two indicator functions with disjoint supports define independent random variables. For a proper definition, see [RR89, Section 1].

We show in this section that a Lévy noise is an example of an independently scattered random measure. In [DFHU], we treat the general case of a Lévy noise in $\mathcal{D}'(\mathbb{R}^d)$. In accordance with the rest of the thesis, we restrict ourselves to tempered Lévy noise. A consequence is that the Lévy exponent is easier to control, which simplifies the proofs. We first extend the domain of definition of the noise to test functions of the form $\mathbb{1}_B$ where $B \in \mathcal{B}(\mathbb{R})^d$ a Borel set with finite Lebesgue measure. A *mollifier* is a function $\theta \in \mathcal{D}(\mathbb{R}^d)$ that is positive and such that $\int_{\mathbb{R}^d} \theta(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = 1$. We set $\theta_k(\boldsymbol{x}) = k^d \theta(k\boldsymbol{x})$.

Proposition 3.5. *We consider a Lévy noise w in $\mathcal{S}'(\mathbb{R}^d)$. Let $B \in \mathcal{B}(\mathbb{R}^d)$ be a Borel set and θ be a mollifier.*

- *If $\varphi \in \mathcal{D}(\mathbb{R}^d)$, then the random variables $\langle w, \varphi \cdot (\theta_k * \mathbb{1}_B)\rangle$ converge in $L_0(\Omega)$. The limit does not depend on θ and is denoted by $\langle w, \varphi \cdot \mathbb{1}_B\rangle$.*

- *In particular, if B is bounded, then the random variables $\langle w, \varphi \cdot \mathbb{1}_B\rangle$ do not depend on φ as soon as φ equals 1 on B. We denote by $\langle w, \mathbb{1}_B\rangle$ the common random variable.*

- *If $\mathrm{Leb} B < \infty$, then the random variables $\langle w, \mathbb{1}_{B \cap [-k,k]^d}\rangle$ converge in $L_0(\Omega)$ to a random variable denoted by $\langle w, \mathbb{1}_B\rangle$.*

Proof. The function $\theta_k * \mathbb{1}_B$ is smooth, therefore $\varphi \cdot (\theta_k * \mathbb{1}_B) \in \mathcal{D}(\mathbb{R}^d)$ and the random variable $X_k := \langle w, \varphi \cdot (\theta_k * \mathbb{1}_B)\rangle$ is well-defined in $L_0(\Omega)$. The space $L_0(\Omega)$ being complete, we need to show that the X_k are Cauchy in probability. Because the convergence in law to 0 implies the convergence in probability, it suffices to show that X_k is Cauchy in law. We have, for $k, \ell \geq 0$, that

$$\mathbb{E}[\mathrm{e}^{\mathrm{i}\xi(X_k - X_\ell)}] = \exp\left(\int_{\mathbb{R}^d} \Psi(\varphi(\boldsymbol{x})((\theta_k - \theta_\ell) * \mathbb{1}_B)(\boldsymbol{x}))\mathrm{d}\boldsymbol{x}\right).$$

According to Proposition 2.4, there exists $0 < \epsilon \leq 1$ and $C > 0$ such that $|\Psi(\xi)| \leq C(|\xi|^\epsilon + |\xi|^2)$.

Let K be the support of φ. We readily see that

$$\begin{aligned}\left|\int_{\mathbb{R}^d}\Psi(\varphi(\boldsymbol{x})((\theta_k-\theta_\ell)*\mathbb{1}_B)(\boldsymbol{x}))\mathrm{d}\boldsymbol{x}\right| &\leq C\Big(\int_K\left|\varphi(\boldsymbol{x})\right|^\epsilon|((\theta_k-\theta_\ell)*\mathbb{1}_B)(\boldsymbol{x})|^\epsilon\,\mathrm{d}\boldsymbol{x}\\ &\quad+\int_K\left|\varphi(\boldsymbol{x})\right|^2|((\theta_k-\theta_\ell)*\mathbb{1}_B)(\boldsymbol{x})|^2\,\mathrm{d}\boldsymbol{x}\Big)\\ &\leq C\big(\|\varphi\|_\infty^\epsilon\int_K|((\theta_k-\theta_\ell)*\mathbb{1}_B)(\boldsymbol{x})|^\epsilon\,\mathrm{d}\boldsymbol{x}\\ &\quad+\|\varphi\|_\infty^2\int_K|((\theta_k-\theta_\ell)*\mathbb{1}_B)(\boldsymbol{x})|^2\,\mathrm{d}\boldsymbol{x}\big).\end{aligned}\tag{3.7}$$

The two terms in (3.7) go to 0. This is well-known for the second term, because the regularization of a function in $L_2(K)$ converges to the function in $L_2(K)$. It is still valid for the first term since the integral is over the compact set K. Indeed, the Hölder inequality implies that

$$\left(\int_K|((\theta_k-\theta_\ell)*\mathbb{1}_B)(\boldsymbol{x})|^\epsilon\,\mathrm{d}\boldsymbol{x}\right)^{\frac{1}{\epsilon}}\leq\mathrm{Leb}(K)^{\frac{1}{\epsilon}-\frac{1}{2}}\left(\int_K|((\theta_k-\theta_\ell)*\mathbb{1}_B)(\boldsymbol{x})|^2\,\mathrm{d}\boldsymbol{x}\right)^{\frac{1}{2}},$$

and we are back to the L_2 case. Thus, $\mathbb{E}[\mathrm{e}^{\mathrm{i}\xi(X_k-X_\ell)}]$ vanishes as $k,\ell\to\infty$ and (X_k) is a Cauchy sequence converging to a limit X in the complete space $L_0(\Omega)$.
If $\tilde{\theta}$ is another mollifier and Y is the limit of the Cauchy sequence $Y_k:=\langle w,\varphi(\tilde{\theta}_k*\mathbb{1}_B)\rangle$, then we readily see that $X_k-Y_k=\langle w,\varphi\cdot(\theta_k-\tilde{\theta}_k)*\mathbb{1}_B\rangle$ vanishes. This implies that $X=Y$ in probability and the limit does not depend on the choice of the mollifier θ.
For the second point, we simply remark that $\varphi\mathbb{1}_B=\tilde{\varphi}\mathbb{1}_B$ if φ and $\tilde{\varphi}$ are equal to 1 over B, therefore $\langle w,\varphi\mathbb{1}_B\rangle=\langle w,\tilde{\varphi}\mathbb{1}_B\rangle$. For the last point, we show as we did for the first point that $(\langle w,\mathbb{1}_{B\cap[-k,k]^d}\rangle)$ is a Cauchy sequence in $L_0(\Omega)$. □

Proposition 3.6. *Let w be a Lévy noise and B a Borel set of $\mathbb{R}^d$ with finite Lebesgue measure. The characteristic function of the random variable $\langle w,\mathbb{1}_B\rangle$ is given for $\xi\in\mathbb{R}$ by*

$$\widehat{\mathscr{P}}_{\langle w,\mathbb{1}_B\rangle}(\xi)=\exp\left(\mathrm{Leb}(B)\Psi(\xi)\right)\tag{3.8}$$

where Ψ is the Lévy exponent of w.
For any disjoint sets $A,B\in\mathscr{B}(\mathbb{R}^d)$ with finite Lebesgue measure, the random variables $\langle w,\mathbb{1}_A\rangle$ and $\langle w,\mathbb{1}_B\rangle$ are independent and

$$\langle w,\mathbb{1}_{A\cup B}\rangle=\langle w,\mathbb{1}_A\rangle+\langle w,\mathbb{1}_B\rangle\tag{3.9}$$

almost surely.

Proof. We have the convergence $\Psi(\varphi(\boldsymbol{x})(\theta_k*\mathbb{1}_B)(\boldsymbol{x}))\to\Psi(\varphi(\boldsymbol{x})\mathbb{1}_B(\boldsymbol{x}))$ for every $\boldsymbol{x}$ as k increases. Moreover, with Proposition 2.4, we have that

$$\begin{aligned}\left|\Psi(\varphi(\boldsymbol{x})(\theta_k*\mathbb{1}_B)(\boldsymbol{x}))\right| &\leq C(|\varphi(\boldsymbol{x})|^\epsilon|(\theta_k*\mathbb{1}_B)(\boldsymbol{x})|^\epsilon+|\varphi(\boldsymbol{x})|^2|(\theta_k*\mathbb{1}_B)(\boldsymbol{x})|^2)\\ &\leq C(\left|\varphi(\boldsymbol{x})\right|^\epsilon+\left|\varphi(\boldsymbol{x})\right|^2),\end{aligned}\tag{3.10}$$

that is an integrable function. In the second inequality of (3.10), we used that $0\leq\theta_k*\mathbb{1}_B(\boldsymbol{x})=$

$\int_B \theta_k(\boldsymbol{x}-\boldsymbol{y})\mathrm{d}\boldsymbol{y} \leq \int_{\mathbb{R}^d} \theta_k(\boldsymbol{x}-\boldsymbol{y})\mathrm{d}\boldsymbol{y} = 1$. The Lebesgue dominated convergence theorem then implies that, for any $\xi \in \mathbb{R}$,

$$\widehat{\mathscr{P}}_{\langle w, \varphi(\theta_k * \mathbb{1}_B)\rangle}(\xi) \underset{k\to\infty}{\longrightarrow} \exp\left(\int_B \Psi(\xi\varphi(\boldsymbol{x}))\mathrm{d}\boldsymbol{x}\right).$$

If B is included in a compact set, we deduce, by selecting $\varphi \in \mathscr{D}(\mathbb{R}^d)$ such that $\varphi = 1$ on B, that

$$\widehat{\mathscr{P}}_{\langle w, \mathbb{1}_B\rangle}(\xi) = \exp\left(\int_B \Psi(\xi)\mathrm{d}\boldsymbol{x}\right) = \exp\left(\mathrm{Leb}(B)\Psi(\xi)\right).$$

The third point of Proposition 3.5 ensures that this property is extended to B with finite Lebesgue measure, but not necessarily bounded.

If A and B are disjoint, we directly deduce from the form of the characteristic function (3.8) that $\widehat{\mathscr{P}}_{\langle w, \mathbb{1}_{A\cup B}\rangle}(\xi) = \widehat{\mathscr{P}}_{\langle w, \mathbb{1}_A\rangle + \langle w, \mathbb{1}_B\rangle}(\xi) = \widehat{\mathscr{P}}_{\langle w, \mathbb{1}_A\rangle}(\xi)\cdot\widehat{\mathscr{P}}_{\langle w, \mathbb{1}_B\rangle}(\xi)$, implying the independence property. The almost sure equality (3.9) is due to the linearity of w, easily extended to indicator functions, and to the fact that $\mathbb{1}_{A\cup B} = \mathbb{1}_A + \mathbb{1}_B$. □

We denote by $\mathscr{A}(\mathbb{R}^d)$ the δ-ring[2] of Borel subsets of $\mathbb{R}^d$ with finite Lebesgue measure.

Theorem 3.2. *Let w be a Lévy noise on $\mathscr{S}'(\mathbb{R}^d)$. We consider the extension of w to indicator functions on Borel sets with finite Lebesgue measure. The mapping $B \mapsto \langle w, \mathbb{1}_B\rangle$ from $\mathscr{A}(\mathbb{R}^d)$ to $L_0(\Omega)$ defines an independently scattered random measure in the sense of [RR89, Section 1].*

Proof. Consider a sequence $(B_k)_{k\in\mathbb{N}}$ of disjoint elements of $\mathscr{A}(\mathbb{R}^d)$. We have to show that: (i) the B_k are independent, and (ii) the series $\sum_{k\in\mathbb{N}}\langle w, \mathbb{1}_{B_k}\rangle$ converges to $\langle w, \mathbb{1}_{\cap_k B_k}\rangle$ as soon as $\cap_k B_k \in \mathscr{A}(\mathbb{R}^d)$. For the first point, we simply adapt the proof given in Proposition 3.6 for two random variables to the case of any finite collection of B_k. For the second point, we know that $\sum_{k=0}^K \langle w, \mathbb{1}_{B_k}\rangle = \langle w, \mathbb{1}_{\cup_{k=0}^K B_k}\rangle$ almost surely for any $K \in \mathbb{N}$. If, in addition, $\sum_{k\in\mathbb{N}}\mathrm{Leb}(B_k) < \infty$, then $\cap_{k\in\mathbb{N}} B_k \in \mathscr{A}(\mathbb{R}^d)$. With the expression of the characteristic function (3.8), we easily show that

$$\widehat{\mathscr{P}}_{\sum_{k=0}^K\langle w, \mathbb{1}_{B_k}\rangle}(\xi) = \widehat{\mathscr{P}}_{\langle w, \mathbb{1}_{\cup_{k=0}^K B_k}\rangle}(\xi) \underset{K\to\infty}{\longrightarrow} \widehat{\mathscr{P}}_{\langle w, \mathbb{1}_{\cup_{k\in\mathbb{N}} B_k}\rangle}(\xi)$$

for any $\xi \in \mathbb{R}$. Therefore, the series of the independent random variables $\langle w, \mathbb{1}_{B_k}\rangle$ converges in probability to $\langle w, \mathbb{1}_{\cup_{k\in\mathbb{N}} B_k}\rangle$. By [Chu01, Theorem 5.3.4], the sum converges almost surely, which concludes the proof. □

3.2.2 Extension of the Domain of Definition

Having connected Lévy white noises with independently scattered random measures, it is then possible to extend the domain to other test functions. This was done by Rajput and Rosinski in [RR89]. We restate here the main definitions and theorems of their work.

[2] A δ-ring is a collection of sets that is closed under finite union, countable intersection, and relative complementation [Bog07, Definition 1.2.13]. It appears in measure theory, especially when one want to avoid sets with infinite measure.

We say that f is a *simple function* if it can be written as $f = \sum_{n=1}^{N} a_n \mathbb{1}_{B_n}$, where $a_n \in \mathbb{R}$ and the $B_n \in \mathscr{A}(\mathbb{R}^d)$ are Borel subsets of $\mathbb{R}^d$ with finite Lebesgue measure. For any Borel set B and simple function f, we use Proposition 3.5 to define the random variable

$$\langle w, f \cdot \mathbb{1}_B \rangle := \sum_{n=1}^{N} a_n \langle w, \mathbb{1}_{B_n \cap B} \rangle.$$

Definition 3.4. *Consider a Lévy noise w. We say that a measurable function $f : \mathbb{R}^d \to \mathbb{R}$ is* w-integrable *if there exists a sequence of simple functions $(f_k)_{k \in \mathbb{N}}$ such that*

- *the f_k converge almost everywhere (for the Lebesgue measure) to f, and*
- *for any Borel set B in $\mathbb{R}^d$, the random variables $\langle w, f_k \cdot \mathbb{1}_B \rangle$ converge in probability.*

Then, we define the random variable

$$\langle w, f \cdot \mathbb{1}_B \rangle := \lim_{k \to \infty} \langle w, f_k \cdot \mathbb{1}_B \rangle.$$

Definition 3.4 identifies the class of measurable test functions such that $\langle w, f \rangle$ is well-defined. We have the following characterization of w-integrable functions, proved in [RR89, Theorem 2.7].

Theorem 3.3. *Let w be a Lévy noise with characteristic triplet (μ, σ^2, ν), and $f : \mathbb{R}^d \to \mathbb{R}$ be a measurable function. Then, the measurable function f is w-integrable if and only if the following conditions are satisfied:*

1. $\int_{\mathbb{R}^d} \left| \mu f(\boldsymbol{x}) + \int_{\mathbb{R}} t f(\boldsymbol{x}) \left(\mathbb{1}_{|t f(\boldsymbol{x})| \le 1} - \mathbb{1}_{|t| \le 1} \right) \nu(\mathrm{d}t) \right| \mathrm{d}\boldsymbol{x} < \infty,$
2. $\int_{\mathbb{R}^d} \sigma^2 \left| f(\boldsymbol{x}) \right|^2 \mathrm{d}\boldsymbol{x} < \infty,$
3. $\int_{\mathbb{R}^d \times \mathbb{R}} \min(1, |t f(\boldsymbol{x})|^2) \nu(\mathrm{d}t) \mathrm{d}\boldsymbol{x} < \infty.$

Then, if we set

$$\Theta(\xi) = \left| \mu\xi + \int_{\mathbb{R}} t\xi \left(\mathbb{1}_{|t\xi| \le 1} - \mathbb{1}_{|t| \le 1} \right) \nu(\mathrm{d}t) \right| + \sigma^2 \xi^2 + \int_{\mathbb{R}} \min(1, |t\xi|^2) \nu(\mathrm{d}t), \tag{3.11}$$

the measurable function f is w-integrable if and only if $\int_{\mathbb{R}^d} \Theta(f(\boldsymbol{x})) \mathrm{d}\boldsymbol{x} < \infty$.

We propose to call the function Θ the *Rajput-Rosinski exponent* of the Lévy noise w. We denote by $L_\Theta(\mathbb{R}^d)$ the space of w-integrable functions of the d-dimensional Lévy noise w with Rajput-Rosinski exponent Θ. The space $L_\Theta(\mathbb{R}^d)$ is called the *domain of definition* of w.

Moments of $\langle w, f \rangle$. When we restrict ourselves to $\varphi \in \mathscr{S}(\mathbb{R}^d)$, the random variables $\langle w, \varphi \rangle$ have finite pth moments if and only if the underlying infinitely divisible random variable has a finite pth moment itself [UT14]. The situation is different once we have extended the domain. The following characterization arises [RR89, Theorem 3.3].

Proposition 3.7. *Consider a Lévy noise w with finite pth-moments for $p>0$. For $f \in L_\Theta(\mathbb{R}^d)$, we have the equivalence*

$$\mathbb{E}[|\langle w,f\rangle|^p]<\infty \Longleftrightarrow \int_{\mathbb{R}^d}\int_{\mathbb{R}}\left(|tf(\boldsymbol{x})|^p \mathbb{1}_{|tf(\boldsymbol{x})|>1}+|tf(\boldsymbol{x})|^2\mathbb{1}_{|tf(\boldsymbol{x})|\le 1}\right)\nu(\mathrm{d}t)\mathrm{d}\boldsymbol{x}<\infty.$$

Therefore, if we set

$$\begin{aligned}\Theta_p(\xi) :=& \left|\mu\xi+\int_{\mathbb{R}} t\xi\left(\mathbb{1}_{|t\xi|\le 1}-\mathbb{1}_{|t|\le 1}\right)\nu(\mathrm{d}t)\right|+\sigma^2\xi^2\\ &+\int_{\mathbb{R}}\left(|t\xi|^p\mathbb{1}_{|t\xi|>1}+|t\xi|^2\mathbb{1}_{|t\xi|\le 1}\right)\nu(\mathrm{d}t),\end{aligned} \tag{3.12}$$

then, $\mathbb{E}[|\langle w,f\rangle|^p]<\infty$ *if and only if* $\int_{\mathbb{R}^d}\Theta_p(f(\boldsymbol{x}))\mathrm{d}\boldsymbol{x}<\infty$.

The function Θ_p is called the *pth-order Rajput-Rosinski exponent* and the *domain of finite pth-moments* is denoted by $L_{\Theta_p}(\mathbb{R}^d)$. From now on, we also denote $\Theta_0=\Theta$ and $L_{\Theta_0}(\mathbb{R}^d)=L_\Theta(\mathbb{R}^d)$. If the pth moments of w are infinite (that is, if the underlying infinite divisible random variable X satisfies $\mathbb{E}[|X|^p]=\infty$), then the exponent Θ_p defined in (3.12) is infinite for every $\xi\neq 0$. Therefore, we can extend the definition of the domain of finite pth-moments by setting $L_{\Theta_p}(\mathbb{R}^d)=\{0\}$.

Structure of $L_{\Theta_p}(\mathbb{R}^d)$.

Definition 3.5. *We say that $\rho:\mathbb{R}\to\mathbb{R}^+$ is a φ-*function *if $\rho(0)=0$ and ρ is symmetric, continuous, and nondecreasing on $\mathbb{R}^+$. The φ-function ρ is* Δ_2-regular *if*

$$\rho(2\xi)\le M\rho(\xi)$$

for some $M,\xi_0>0$, and every $\xi\ge\xi_0$.
Let ρ be a φ-function. For $f:\mathbb{R}^d\to\mathbb{R}$, we set $\rho(f):=\int_{\mathbb{R}^d}\rho(f(\boldsymbol{x}))\mathrm{d}\boldsymbol{x}$. The generalized Orlicz space *associated to ρ is*

$$L_\rho(\mathbb{R}^d):=\left\{f \text{ measurable} \,\middle|\, \exists\lambda>0,\ \rho(f/\lambda)<\infty\right\}.$$

Remark. Orlicz spaces were introduced in [BO31] as natural generalizations of L_p-spaces for $p\ge 1$. A systematic study with important extensions was done by J. Musielak [Mus83]. The initial theory deals with Banach spaces, excluding for instance the L_p-spaces with $0<p<1$. Definition 3.5 generalizes the Orlicz spaces in two ways: One does not require that ρ is convex, neither that $\rho(\xi)\to\infty$ as $\xi\to\infty$. The need for a non-locally convex framework (related to non-convex φ-function) is notable in stochastic integration. It was initiated by K. Urbanik and W.A. Woyczyns [UW67]. It is at the heart of the study of the structure developed by Rajput and Rosinski. We follow here the exposition of M.M. Rao and Z.D. Ren in [RR91, Chapter X]. Proposition 3.8 summarizes the results on generalized Orlicz spaces.

Proposition 3.8. *If ρ is a Δ_2-regular φ-function, then we have*

$$L_\rho(\mathbb{R}^d) = \left\{f \text{ measurable} \,\middle|\, \forall \lambda > 0,\ \rho(f/\lambda) < \infty\right\} = \left\{f \text{ measurable} \,\middle|\, \rho(f) < \infty\right\}.$$

The space $L_\rho(\mathbb{R}^d)$ is a complete linear metric space for the F-norm

$$\|f\|_\rho := \inf\left\{\lambda > 0 \,\middle|\, \rho(f/\lambda) \leq \lambda\right\}$$

on which simple functions are dense. Moreover, we have the equivalence, for any sequence of elements $f_k \in L_\rho(\mathbb{R}^d)$,

$$\|f_k\|_\rho \underset{k\to\infty}{\longrightarrow} 0 \Leftrightarrow \rho(f_k) \underset{k\to\infty}{\longrightarrow} 0.$$

For $p > 0$, the exponent Θ_p of a white noise with finite pth moment is a Δ_2-regular φ-function [RR89, Lemma 3.1]. We set $\Theta_p(f) = \int_{\mathbb{R}^d} \Theta_p(f(\boldsymbol{x}))\mathrm{d}\boldsymbol{x}$. Proposition 3.8 then directly implies the following result.

Proposition 3.9. *Fix $p > 0$ and w a Lévy noise with pth-order Rajput-Rosinski exponent Θ_p. Then, $L_{\Theta_p}(\mathbb{R}^d)$ is a generalized Orlicz space. In particular, it is a complete linear metric space. A sequence $(f_k)_{k\in\mathbb{N}}$ converges to 0 in $L_{\Theta_p}(\mathbb{R}^d)$ if and only if*

$$\Theta_p(f_k) = \int_{\mathbb{R}^d} \Theta_p(f_k(\boldsymbol{x}))\mathrm{d}\boldsymbol{x} \underset{k\to\infty}{\longrightarrow} 0.$$

Lévy noise as a random linear function on its domain. We are now ready to extend the domain of definition of w, according to [RR89, Theorem 3.3].

Theorem 3.4. *Let w be a Lévy white noise with finite pth-moments for $p \geq 0$. Then, the functional*

$$w : L_{\Theta_p}(\mathbb{R}^d) \to L_p(\Omega), \quad f \mapsto \langle w, f\rangle$$

is linear and continuous.

Theorem 3.4 with $p = 0$ identifies the domain of definition of w; that is, the broadest class of test functions on which w is a random linear functional. Once the random variable $\langle w, f\rangle$ is well-defined, it is important to identify its characteristic function. The following result is the last part of [RR89, Theorem 2.7].

Proposition 3.10. *For $f \in L_\Theta(\mathbb{R}^d)$, the characteristic function of $\langle w, f\rangle$ is given by*

$$\widehat{\mathscr{P}}_{\langle w,f\rangle}(\xi) = \exp\left(\int_{\mathbb{R}^d} \Psi(\xi f(\boldsymbol{x}))\mathrm{d}\boldsymbol{x}\right).$$

3.2.3 The spaces $L_{p_0,p_\infty}(\mathbb{R}^d)$

We introduce the family of function spaces that generalize the L_p-spaces for $0 < p < \infty$. They will be identified in the sequel as the domains of definition of important classes of Lévy white noises. We first give some notations. For $0 \le p_0, p_\infty < \infty$, we set

$$\rho_{p_0,p_\infty}(\xi) := |\xi|^{p_0}\mathbb{1}_{|\xi|>1} + |\xi|^{p_\infty}\mathbb{1}_{|\xi|\le 1},$$
$$\rho_{\log,p_\infty}(\xi) := (1+\log|\xi|)\mathbb{1}_{|\xi|>1} + |\xi|^{p_\infty}\mathbb{1}_{|\xi|\le 1},$$

with the convention that $0^0 = 1$.

Definition 3.6. *For $0 \le p_0, p_\infty < \infty$, we set*

$$L_{p_0,p_\infty}(\mathbb{R}^d) = \left\{ f \text{ measurable} \,\middle|\, \rho_{p_0,p_\infty}(f) := \int_{\mathbb{R}^d} \rho_{p_0,p_\infty}(f(\boldsymbol{x}))\mathrm{d}\boldsymbol{x} < \infty \right\},$$
$$L_{\log,p_\infty}(\mathbb{R}^d) = \left\{ f \text{ measurable} \,\middle|\, \rho_{p_0,p_\infty}(f) := \int_{\mathbb{R}^d} \rho_{\log,p_\infty}(f(\boldsymbol{x}))\mathrm{d}\boldsymbol{x} < \infty \right\}.$$

For $p > 0$, we have $L_{p,p}(\mathbb{R}^d) = L_p(\mathbb{R}^d)$. Roughly speaking, p_0 measures the local integrability of a function, while p_∞ indicates the asymptotic one. This is illustrated by the following example. For $\alpha, \beta > 0$, the function $f(\boldsymbol{x}) = \|\boldsymbol{x}\|^{-\alpha}\mathbb{1}_{\|\boldsymbol{x}\|<1} + \|\boldsymbol{x}\|^{-\beta}\mathbb{1}_{\|\boldsymbol{x}\|\ge 1}$ is such that

$$\rho_{p_0,p_\infty}(f) = \int_{\mathbb{R}^d} \left(|f(\boldsymbol{x})|^{p_0}\mathbb{1}_{|f(\boldsymbol{x})|>1} + |f(\boldsymbol{x})|^{p_\infty}\mathbb{1}_{|f(\boldsymbol{x})|\le 1}\right)\mathrm{d}\boldsymbol{x}$$
$$= \int_{\|\boldsymbol{x}\|<1} \|\boldsymbol{x}\|^{-p_0\alpha}\mathrm{d}\boldsymbol{x} + \int_{\|\boldsymbol{x}\|\ge 1} \|\boldsymbol{x}\|^{-p_\infty\beta}\mathrm{d}\boldsymbol{x}.$$

Therefore, f is in $L_{p_0,p_\infty}(\mathbb{R}^d)$ if and only if

$$\alpha < \frac{d}{p_0} \text{ and } \beta > \frac{d}{p_\infty}.$$

The first inequality effectively refers to the integrability of f at the origin (or local integrability), while the second covers its asymptotic integrability.

Structure of $L_{p_0,p_\infty}(\mathbb{R}^d)$. As we did in Section 3.2.2 with the spaces $L_{\Theta_p}(\mathbb{R}^d)$, we rely on generalized Orlicz spaces [RR91, Chapter X] to identify the structure of the spaces $L_{p_0,p_\infty}(\mathbb{R}^d)$.

Proposition 3.11. *We fix $p_0 \ge 0$ and $p_\infty > 0$. The function $\rho_{p_0,p_\infty} : \mathbb{R} \to \mathbb{R}$ is a Δ_2-regular φ-function. Therefore, $L_{p_0,p_\infty}(\mathbb{R}^d)$ is a complete linear metric space on which the convergence of f_k to 0 is equivalent to*

$$\rho_{p_0,p_\infty}(f_k) \underset{k\to\infty}{\longrightarrow} 0.$$

The same conclusions occur for the function $\rho_{\log,p_\infty}$ and the space $L_{\log,p_\infty}(\mathbb{R}^d)$.

Proof. To simplify the notation, we write $\rho = \rho_{p_0,p_\infty}$ in this proof. The function ρ is continuous, non-decreasing, symmetric, and vanishes at the origin (since $p_\infty \ne 0$). It is therefore a φ-

function. Moreover, we have that, for any $\xi \in \mathbb{R}$ and $\lambda > 0$,

$$m\rho(\xi) \leq \rho(\xi/\lambda) \leq M\rho(\xi), \tag{3.13}$$

where we set $I = [\min(1,\lambda), \max(1,\lambda)]$ and

$$m = \min_I \frac{x^{p_0 - p_\infty}}{\lambda^{p_0}} = \min(\lambda^{-p_0}, \lambda^{-p_\infty}),$$
$$M = \max_I \frac{x^{p_0 - p_\infty}}{\lambda^{p_0}} = \max(\lambda^{-p_0}, \lambda^{-p_\infty}).$$

To show these inequalities, we first remark that, for any $\min(1,\lambda) \leq |\xi| \leq \max(1,\lambda)$,

$$m|\xi|^{p_\infty} \leq \lambda^{-p_0}|\xi|^{p_0} \leq M|\xi|^{p_\infty}. \tag{3.14}$$

Then, we have the following decomposition

$$\rho(\xi/\lambda) = \lambda^{-p_0}|\xi|^{-p_0}\mathbb{1}_{|\xi|>1} + \lambda^{-p_0}|\xi|^{p_0}\mathbb{1}_{\lambda<|\xi|\leq 1} + \lambda^{-p_\infty}|\xi|^{p_\infty}\mathbb{1}_{|\xi|\leq\lambda}. \tag{3.15}$$

Using (3.14) to bound $\lambda^{-p_0}|\xi|^{p_0}\mathbb{1}_{\lambda<|\xi|\leq 1}$ in (3.15), we easily obtain (3.13). Taking $\lambda = 1/2$, this shows that ρ is Δ_2-regular. The structure of $L_{p_0,p_\infty}(\mathbb{R}^d)$ then follows from Proposition 3.8. The proof for $\rho_{\log,p_\infty}$ and $L_{\log,p_\infty}(\mathbb{R}^d)$ is very similar. □

Remark. In Proposition 3.11, we restricted ourselves to the case when $p_\infty \neq 0$. The reason is that $\rho_{p_0,0}(0) \neq 0$, so that $\rho_{p_0,0}$ is not a φ-function. Therefore, we do not define a generalized Orlicz space in the sense of Rao and Ren [RR91]. The space $L_{p_0,0}(\mathbb{R}^d)$ can be described as follows. It is the space of functions in $L_{p_0}(\mathbb{R}^d)$ whose support has a finite Lebesgue measure. We do not specify any topological structure on those vector spaces, since they will not appear as the domain of definition of any Lévy noise. However, the space $L_{2,0}(\mathbb{R}^d)$ will play a role as a common subspace to all the domains of definition of the Lévy noises (see Proposition 3.17).

3.2.4 Practical Determination of the Domain

We provide here several criteria for the practical identification of the domain of definition of a Lévy noise. We apply our result to Gaussian, SαS, compound Poisson, and generalized Laplace noises. To the best of our knowledge, the results presented here are new for the two latter classes of noise. Similar considerations are given for the domain of finite pth moments for $0 < p \leq 2$.

Proposition 3.12. *Let w be a Lévy noise with finite pth-moments for $p \geq 0$.*

- Linearity: *for $f, g \in L_{\Theta_p}(\mathbb{R}^d)$ and $\lambda \in \mathbb{R}$, $f + \lambda g \in L_{\Theta_p}(\mathbb{R}^d)$.*
- Invariances: *for $f \in L_{\Theta_p}(\mathbb{R}^d)$ and $\mathrm{H} : \mathbb{R}^d \to \mathbb{R}^d$ a C_1-diffeomorphism, we have*

$$\boldsymbol{x} \mapsto f(\mathrm{H}(\boldsymbol{x})) \in L_{\Theta_p}(\mathbb{R}^d).$$

In particular, the translations $\mathrm{T}_{\boldsymbol{x}_0} f$, rescalings $\mathrm{S}_a f$, and rotations $\mathrm{R}_{\boldsymbol{\theta}_0} f$ of f are in $L_{\Theta_p}(\mathbb{R}^d)$.

Proof. We already know that $L_{\Theta_p}(\mathbb{R}^d)$ is a linear space (Proposition 3.9). For the invariance, we simply remark that, by the substitution $\boldsymbol{y} = \mathrm{H}(\boldsymbol{x})$, we have

$$\int_{\mathbb{R}^d} \Theta_p(f(\mathrm{H}(\boldsymbol{x})))\mathrm{d}\boldsymbol{x} = \frac{1}{|\det J_{\mathrm{H}}|}\int_{\mathbb{R}^d} \Theta_p(f(\boldsymbol{y}))\mathrm{d}\boldsymbol{y}$$

with J_H the invertible Jacobian matrix of H. □

If w is a Lévy noise, so are aw and the rescaling $w(\cdot/a)$ for $a \neq 0$. If w_1 and w_2 are two independent Lévy noises, then $w_1 + w_2$ is also a Lévy noise. In Proposition 3.13, we denote by $\Theta_p(w)$ the pth-order Rajput-Rosinski exponent of w, in order to distinguish the exponents of the different noises.

Proposition 3.13. *Let w be a Lévy noise with finite pth-moments for $p \geq 0$. Then we have, for $a \neq 0$,*

$$L_{\Theta_p(w)}(\mathbb{R}^d) = L_{\Theta_p(aw)}(\mathbb{R}^d) = L_{\Theta_p(w(\cdot/a))}(\mathbb{R}^d).$$

If w_1 and w_2 are two independent Lévy noises, then

$$L_{\Theta_p(w_1)}(\mathbb{R}^d) \cap L_{\Theta_p(w_2)}(\mathbb{R}^d) \subseteq L_{\Theta_p(w_1+w_2)}(\mathbb{R}^d), \tag{3.16}$$

with equality when at least one of the two Lévy noises is symmetric.

Proof. We have $\langle w(\cdot/a), f\rangle = \langle w, a^d f(a\cdot)\rangle$. Thus, $f \in L_{\Theta_p(w(\cdot/a))}(\mathbb{R}^d)$ if and only if $a^d f(a\cdot) \in L_{\Theta_p}(\mathbb{R}^d)$. Since $L_{\Theta_p}(\mathbb{R}^d)$ is a linear space invariant by rescaling (Proposition 3.12), the latter condition is equivalent to $f \in L_{\Theta_p}(\mathbb{R}^d)$. This shows that $L_{\Theta_p(w(\cdot/a))}(\mathbb{R}^d) = L_{\Theta_p}(\mathbb{R}^d)$. We proceed similarly for $L_{\Theta_p(aw)}(\mathbb{R}^d)$.
For $i = 1,2$, the Lévy triplet of w_i (w, respectively) is denoted by $(\mu_i, \sigma_i^2, \nu_i)$ ((μ, σ^2, ν), respectively), and the corresponding Rajput-Rosinski exponent is $\Theta_{p,i}$ (Θ_p, respectively). If w_1 and w_2 are independent, we have the relations

$$\mu = \mu_1 + \mu_2,\ \sigma^2 = \sigma_1^2 + \sigma_2^2,\ \nu = \nu_1 + \nu_2.$$

Therefore, we have, by the triangular inequality,

$$\begin{aligned}\Theta_p(\xi) &= \left|(\mu_1+\mu_2)\xi + \int_{\mathbb{R}} t\xi\left(\mathbb{1}_{|t\xi|\leq 1} - \mathbb{1}_{|t|\leq 1}\right)(\nu_1+\nu_2)(\mathrm{d}t)\right| \\ &\quad + (\sigma_1^2+\sigma_2^2)\xi^2 + \int_{\mathbb{R}} \min(|\xi t|^p, |\xi t|^2)(\nu_1+\nu_2)(\mathrm{d}t) \\ &\leq \Theta_{p,1}(\xi) + \Theta_{p,2}(\xi),\end{aligned}$$

which proves (3.16). If for instance w_1 is symmetric, the latter inequality is an equality since $\mu_1\xi + \int_{\mathbb{R}} t\xi\left(\mathbb{1}_{|t\xi|\leq 1} - \mathbb{1}_{|t|\leq 1}\right)\nu_1(\mathrm{d}t) = 0$ and (3.16) is an equality. □

In general, (3.16) is only an inclusion. Consider for instance the case where w_1 and w_2 have Lévy triplet $(1,1,0)$ and $(-1,0,0)$ respectively, meaning that w_1 is a Gaussian noise with drift $\mu = 1$ and w_2 a pure drift $\mu = -1$. Then, w_1 and w_2 are clearly independent, and $w_1 + w_2$ is a Gaussian noise without drift. Therefore, $L_{\Theta_p(w_1+w_2)}(\mathbb{R}^d) = L_2(\mathbb{R}^d)$ but $L_{\Theta_p(w_1)}(\mathbb{R}^d) \cap L_{\Theta_p(w_2)}(\mathbb{R}^d) = L_1(\mathbb{R}^d) \cap L_2(\mathbb{R}^d)$.

Reduction to the symmetric case without Gaussian part. For $\mu \in \mathbb{R}$ and ν a Lévy measure, we set

$$m_{\mu,\nu}(\xi) = \left| \mu\xi + \int_{\mathbb{R}} t\xi(\mathbb{1}_{|t\xi|\leq 1} - \mathbb{1}_{|t|\leq 1})\nu(\mathrm{d}t) \right|.$$

Proposition 3.14. *Let (μ, σ^2, μ) be a Lévy triplet and Θ the corresponding Rajput-Rosinski exponent. We also denote by ν_{sym} the symmetrization of ν. We consider the following Lévy noises:*

- *w with Lévy triplet (μ, σ^2, ν),*
- *w_2 with Lévy triplet $(\mu, 0, \nu)$ and Rajput-Rosinski exponent $\Theta_{p,2}$,*
- *w_{sym} with Lévy triplet $(0, \sigma^2, \nu_{\mathrm{sym}})$ and Rajput-Rosinski exponent $\Theta_{p,\mathrm{sym}}$.*

Then, we have the following relations for $p \geq 0$:

- *If $\sigma^2 \neq 0$, then*

$$L_{\Theta_p}(\mathbb{R}^d) = L_2(\mathbb{R}^d) \cap L_{\Theta_{p,2}}(\mathbb{R}^d). \tag{3.17}$$

- *In any case,*

$$L_{\Theta_p}(\mathbb{R}^d) = L_{\Theta_{p,\mathrm{sym}}}(\mathbb{R}^d) \cap \left\{ f \in L_{\Theta}(\mathbb{R}^d) \;\middle|\; \int_{\mathbb{R}^d} m_{\mu,\nu}(f(\boldsymbol{x}))\mathrm{d}\boldsymbol{x} < \infty \right\}. \tag{3.18}$$

Proof. We can decompose $w = w_2 + w_{\mathrm{G}}$, where w_2 and w_{G} are independent with respective Lévy triplets $(\mu, 0, \nu)$ and $(0, \sigma^2, 0)$. Then, w_{G} is a Gaussian noise, for which $L_{\Theta_{p,\mathrm{G}}}(\mathbb{R}^d) = L_2(\mathbb{R}^d)$. We apply (3.16) with equality (w_{G} being symmetric) to obtain (3.17). Finally, (3.18) is a reformulation of [RR89, Proposition 2.9]. □

Based on Proposition 3.14, we restrict our attention to symmetric Lévy noises without Gaussian parts. We first reduce to the case when $\sigma^2 = 0$, thanks to (3.17). The only remaining part to deduce the general case from the symmetric one is the identification of functions f satisfying

$$\int_{\mathbb{R}^d} m_{\mu,\nu}(f(\boldsymbol{x}))\mathrm{d}\boldsymbol{x} < \infty.$$

Primarily, for non-symmetric noise, this usually relies on L_1-type conditions, but we shall not investigate this question in details here.

Practical criteria. We consider a symmetric Lévy noise w without Gaussian part and with symmetric Lévy measure ν. The function Θ_p defined in (3.12) simply becomes, for $p \geq 0$,

$$\begin{aligned}\Theta_p(\xi) &= \int_{\mathbb{R}} \min(|t\xi|^p, |t\xi|^2)\nu(\mathrm{d}t) \\ &= |\xi|^2 \int_{|t|\leq 1/|\xi|} |t|^2 \nu(\mathrm{d}t) + |\xi|^p \int_{|t|>1/|\xi|} |t|^p \nu(\mathrm{d}t) \\ &= \int_{\mathbb{R}} \rho_{p,2}(t\xi)\nu(\mathrm{d}t).\end{aligned} \tag{3.19}$$

We recall that Θ_p is finite as soon as $\int_{|t|>1} |t|^p \nu(\mathrm{d}t) = \infty$. Otherwise, we have that $L_{\Theta_p}(\mathbb{R}^d) = \{0\}$ and no nontrivial test function has a finite pth moment.
We provide powerful results that will be used in practice to determine the domain of definition of specific Lévy noise (SαS, compound Poisson, generalized Laplace). The first criterion is useful as soon as we are able to estimate the behavior of the Rajput-Rosinski exponent at the origin or at infinity.

Proposition 3.15. *Let w be a symmetric Lévy noise without Gaussian part and $p \geq 0$. The pth-order Rajput-Rosinski exponent of w is denoted by Θ_p.*

1. *Assume that $\Theta_p(\xi) \leq C\rho_{p_0,p_\infty}(\xi)$ for some constant $C > 0$ and every ξ, then we have the embedding*
$$L_{p_0,p_\infty}(\mathbb{R}^d) \subseteq L_{\Theta_p}(\mathbb{R}^d). \tag{3.20}$$
2. *Assume that $\rho_{p_0,p_\infty}(\xi) \leq C\Theta_p(\xi)$ for some constant $C > 0$ and every ξ, then we have the embedding*
$$L_{\Theta_p}(\mathbb{R}^d) \subseteq L_{p_0,p_\infty}(\mathbb{R}^d). \tag{3.21}$$
3. *Assume that $\Theta_p(\xi) \underset{0}{\sim} A|\xi|^{p_\infty}$ and $\Theta_p(\xi) \underset{\infty}{\sim} B|\xi|^{p_0}$, then*
$$L_{\Theta_p}(\mathbb{R}^d) = L_{p_0,p_\infty}(\mathbb{R}^d). \tag{3.22}$$
4. *The same holds with $L_{\log,p_\infty}(\mathbb{R}^d)$ instead of $L_{p_0,p_\infty}(\mathbb{R}^d)$ if we replace $|\xi|^{p_0}$ by $\log|\xi|$.*

Proof. The condition $\Theta_p(\xi) \leq C\rho_{p_0,p_\infty}(\xi)$ implies that, for any function $f \in L_{p_0,p_\infty}(\mathbb{R}^d)$, we have

$$\Theta_p(f) = \int_{\mathbb{R}^d} \Theta_p(f(\boldsymbol{x}))\mathrm{d}\boldsymbol{x} \leq C \int_{\mathbb{R}^d} \rho_{p_0,p_\infty}(f(\boldsymbol{x}))\mathrm{d}\boldsymbol{x} = C\|f\|_{p_0,p_\infty}.$$

Therefore, the identity is continuous from $L_{p_0,p_\infty}(\mathbb{R}^d)$ to $L_{\Theta_p}(\mathbb{R}^d)$ proving (3.20). The proof of (3.21) is similar. For the last point, we remark that the two functions Θ_p and ρ_{p_0,p_∞} do not vanish for $\xi \neq 0$. Moreover, they are continuous (for Θ_p, this comes from [RR89, Lemma 3.1]) and are equivalent at 0 and ∞. Thus, there exists two constants such that

$$C_1\rho_{p_0,p_\infty}(\xi) \le \Theta_p(\xi) \le C_2\rho_{p_0,p_\infty}(\xi).$$

We then apply (3.20) and (3.21) to obtain (3.22). □

Note that the local integrability of test functions (parameter p_0) is linked with the asymptotic behavior of Θ_p, while the asymptotic integrability (parameter p_∞) is linked to the behavior of Θ_p at 0.

If we know that the Lévy measure has some finite moments, then we obtain new information on the domain of definition of the Lévy noise. For $p, q \ge 0$, we set

$$m_{p,q}(\nu) := \int_{\mathbb{R}} \rho_{p,q}(t)\nu(\mathrm{d}t),$$

called the *generalized moments* of ν. Then, ν being a Lévy measure, we have that $m_{0,2}(\nu) < \infty$. If in addition the underlying infinitely divisible random variable has a finite pth moment, we can reformulate Proposition 2.3 as $m_{p,2}(\nu) < \infty$.

Proposition 3.16. *Let w be a symmetric Lévy noise without Gaussian part and with Lévy measure ν.*

- *We assume that $m_{p,2}(\nu) < \infty$ for some $0 \le p \le 2$. Then, we have, for any $\xi \in \mathbb{R}$, that*

$$m_{p,2}(\nu)\rho_{p,2}(\xi) \le \Theta_p(\xi) \le m_{p,2}(\nu)\rho_{2,p}(\xi). \tag{3.23}$$

- *We assume that $m_{p,2}(\nu) < \infty$ for some $p \ge 2$. Then, we have, for any $\xi \in \mathbb{R}$, that*

$$m_{p,2}(\nu)\rho_{2,p}(\xi) \le \Theta_p(\xi) \le m_{p,2}(\nu)\rho_{p,2}(\xi). \tag{3.24}$$

- *For $p > 0$, we condense* (3.23) *and* (3.24) *as*

$$m_{p,2}(\nu)\rho_{\min(p,2),\max(p,2)}(\xi) \le \Theta_p(\xi) \le m_{p,2}(\nu)\rho_{\max(p,2),\min(p,2)}(\xi).$$

- *If $m_{p_\infty,p_0}(\nu) < \infty$ for some $0 \le p_0 \le 2$, $0 < p_\infty < \infty$ and if $p \le p_0, p_\infty$, then*

$$\Theta_p(\xi) \le m_{\min(p_\infty,2),p_0}(\nu)\rho_{p_0,\min(p_\infty,2)}(\xi). \tag{3.25}$$

Proof. All the inequalities will be obtained by exploiting the position of $|t|$, $|\xi|$, or $|t\xi|$ with respect to 1. We first show (3.23), the proof for (3.24) being very similar. We start proving the upper bound of (3.23). We first assume that $|\xi| \le 1$. Then, using (3.19), we decompose Θ_p as

$$\Theta_p(\xi) = \int_{|t|\le 1} |t\xi|^2\nu(\mathrm{d}t) + \int_{1<|t|\le\frac{1}{|\xi|}} |t\xi|^2\nu(\mathrm{d}t) + \int_{|t|>\frac{1}{|\xi|}} |t\xi|^p\nu(\mathrm{d}t). \tag{3.26}$$

Since $p \le 2$, we have that

$$\begin{aligned}\Theta_p(\xi) &\le \int_{|t|\le 1} |t|^2|\xi|^p \nu(\mathrm{d}t) + \int_{1<|t|\le\frac{1}{|\xi|}} |t\xi|^p \nu(\mathrm{d}t) + \int_{|t|>\frac{1}{|\xi|}} |t\xi|^p \nu(\mathrm{d}t) \\ &= \left(\int_{|t|\le 1} |t|^2\, \nu(\mathrm{d}t) + \int_{1<|t|} |t|^p\, \nu(\mathrm{d}t)\right)|\xi|^p \\ &= m_{p,2}(\nu)\,|\xi|^p. \end{aligned} \tag{3.27}$$

Assume now that $|\xi| > 1$. Then, we use the decomposition

$$\Theta_p(\xi) = \int_{|t|\le\frac{1}{|\xi|}} |t\xi|^2 \nu(\mathrm{d}t) + \int_{\frac{1}{|\xi|}<|t|\le 1} |t\xi|^p \nu(\mathrm{d}t) + \int_{|t|>1} |t\xi|^p \nu(\mathrm{d}t). \tag{3.28}$$

Again, due to $p \le 2$, we have that

$$\begin{aligned}\Theta_p(\xi) &\le \int_{|t|\le\frac{1}{|\xi|}} |t\xi|^2 \nu(\mathrm{d}t) + \int_{\frac{1}{|\xi|}<|t|\le 1} |t\xi|^2 \nu(\mathrm{d}t) + \int_{|t|>1} |t|^p\,|\xi|^2\, \nu(\mathrm{d}t) \\ &= \left(\int_{|t|\le 1} |t|^2\, \nu(\mathrm{d}t) + \int_{1<|t|} |t|^p\, \nu(\mathrm{d}t)\right)|\xi|^2 \\ &= m_{p,2}(\nu)\,|\xi|^2. \end{aligned} \tag{3.29}$$

Combining (3.27) and (3.29), we deduce that $\Theta_p(\xi) \le m_{p,2}(\nu)\rho_{2,p}(\xi)$.

For the lower bound in (3.23), we first assume that $|\xi| \le 1$. Then, starting from (3.26), we have that

$$\begin{aligned}\Theta_p(\xi) &\ge \int_{|t|\le 1} |t|^2|\xi|^2 \nu(\mathrm{d}t) + \int_{1<|t|\le\frac{1}{|\xi|}} |t|^p\,|\xi|^2\, \nu(\mathrm{d}t) + \int_{|t|>\frac{1}{|\xi|}} |t|^p|\xi|^2 \nu(\mathrm{d}t) \\ &= m_{p,2}(\nu)\,|\xi|^2. \end{aligned} \tag{3.30}$$

And finally, when $|\xi| > 1$, we have, using (3.28), that

$$\begin{aligned}\Theta_p(\xi) &\ge \int_{|t|\le\frac{1}{|\xi|}} |t|^2\,|\xi|^p\, \nu(\mathrm{d}t) + \int_{\frac{1}{|\xi|}<|t|\le 1} |t|^2\,|\xi|^p\, \nu(\mathrm{d}t) + \int_{|t|>1} |t\xi|^p\, \nu(\mathrm{d}t) \\ &= m_{p,2}(\nu)\,|\xi|^p. \end{aligned} \tag{3.31}$$

With (3.30) and (3.32), we deduce that $\Theta_p(\xi) \ge m_{p,2}(\nu)\rho_{p,2}(\xi)$ and (3.23) is proved.

Finally, (3.25) is proved on the same principle. Assume that $|\xi| \le 1$ and $p \le p_\infty \le 2$. Then, using (3.26), we deduce that

$$\begin{aligned}\Theta_p(\xi) &\le \int_{|t|\le 1} |t|^2|\xi|^{p_\infty} \nu(\mathrm{d}t) + \int_{1<|t|\le\frac{1}{|\xi|}} |t\xi|^{p_\infty} \nu(\mathrm{d}t) + \int_{|t|>\frac{1}{|\xi|}} |t\xi|^{p_\infty} \nu(\mathrm{d}t) \\ &= m_{p_\infty,2}(\nu)\,|\xi|^p. \end{aligned}$$

If now $p_\infty > 2$, we have, still for $|\xi| \le 1$, that

$$\begin{aligned}\Theta_p(\xi) &\le \int_{|t|\le 1} |t|^2|\xi|^2 \nu(\mathrm{d}t) + \int_{1<|t|\le \frac{1}{|\xi|}} |t|^{p_\infty} |\xi|^2 \nu(\mathrm{d}t) + \int_{|t|>\frac{1}{|\xi|}} |\xi|^2 \nu(\mathrm{d}t) \\ &= m_{2,2}(\nu)\,|\xi|^2 .\end{aligned}$$

We deduce that $\Theta_p(\xi) \le m_{\min(p_\infty,2),2}(\nu)\,|\xi|^{\min(p_\infty,2)}$.
When $|\xi| > 1$, $p \le p_0 \le 2$, and $p < p_\infty$, we have using (3.28) that

$$\begin{aligned}\Theta_p(\xi) &\ge \int_{|t|\le \frac{1}{|\xi|}} |t\xi|^{p_0} \nu(\mathrm{d}t) + \int_{\frac{1}{|\xi|}<|t|\le 1} |t\xi|^{p_0} \nu(\mathrm{d}t) + \int_{|t|>1} |t|^{\min(p_\infty,2)} |\xi|^{p_0} \nu(\mathrm{d}t) \\ &= m_{\min(p_\infty,2),p_0}(\nu)\,|\xi|^{p_0} .\end{aligned} \tag{3.32}$$

Remarking that $m_{\min(p_\infty,2),2}(\nu) \le m_{\min(p_\infty,2),p_0}(\nu)$ and combining the bounds for $|\xi| \le 1$ and $|\xi| > 1$, we deduce (3.25). □

Proposition 3.17. *For any Lévy noise, we have*

$$L_{2,0}(\mathbb{R}^d) \subseteq L_\Theta(\mathbb{R}^d) \subseteq L_{0,2}(\mathbb{R}^d), \tag{3.33}$$

Let $0 < p \le 2$*. For any symmetric Lévy noise such that* $m_{p,2}(\nu) < \infty$*, we have*

$$L_{2,p}(\mathbb{R}^d) \subseteq L_{\Theta_p}(\mathbb{R}^d) \subseteq L_{p,2}(\mathbb{R}^d). \tag{3.34}$$

Let $p \ge 2$*. For any symmetric Lévy noise such that* $m_{p,2}(\nu) < \infty$*, we have*

$$L_{p,2}(\mathbb{R}^d) \subseteq L_{\Theta_p}(\mathbb{R}^d) \subseteq L_{2,p}(\mathbb{R}^d). \tag{3.35}$$

For $p > 0$*, assuming that* $m_{p,2}(\nu) < \infty$*, we condense* (3.34) *and* (3.35) *as*

$$L_{\max(p,2),\min(p,2)}(\mathbb{R}^d) \subseteq L_{\Theta_p}(\mathbb{R}^d) \subseteq L_{\min(p,2),\max(p,2)}(\mathbb{R}^d). \tag{3.36}$$

In particular, for any symmetric finite-variance Lévy noise

$$L_{\Theta_2}(\mathbb{R}^d) = L_2(\mathbb{R}^d). \tag{3.37}$$

For any symmetric Lévy noise without Gaussian part such that $m_{p_\infty,p_0}(\nu) < \infty$*, with* $0 \le p \le p_0, p_\infty \le 2$*, we have*

$$L_{p_0,p_\infty}(\mathbb{R}^d) \subseteq L_{\Theta_p}(\mathbb{R}^d). \tag{3.38}$$

Proof. When w is symmetric without Gaussian part, (3.33), (3.34), and (3.37) are directly deduced from (3.23) by taking $p = 0$, p general, and $p = 2$, respectively. Adding a Gaussian part with Rajput-Rosinski exponent Θ_{G} does not change the conclusions since $L_{2,p}(\mathbb{R}^d) \subseteq L_{\Theta_{p,\mathrm{G}}}(\mathbb{R}^d) = L_2(\mathbb{R}^d) \subseteq L_{p,2}(\mathbb{R}^d)$ for all $0 \le p \le 2$ and thanks to (3.17).
We now consider a general Lévy noise w with Lévy triplet (μ, σ^2, ν) and w_{sym} its symmetric ver-

sion with triplet $(0,\sigma^2,\nu_{\mathrm{sym}})$. We already now that $L_{2,0}(\mathbb{R}^d) \subseteq L_{\Theta_{\mathrm{sym}}}(\mathbb{R}^d) \subseteq L_{0,2}(\mathbb{R}^d)$. Moreover, from (3.18), we know that

$$L_{\Theta}(\mathbb{R}^d) = L_{\Theta_{\mathrm{sym}}}(\mathbb{R}^d) \cap \left\{ f \in L_{\Theta}(\mathbb{R}^d) \,\middle|\, \int_{\mathbb{R}^d} m_{\mu,\nu}(f(\boldsymbol{x}))\mathrm{d}\boldsymbol{x} < \infty \right\}. \tag{3.39}$$

First, we have that $L_{\Theta}(\mathbb{R}^d) \subseteq L_{\Theta_{\mathrm{sym}}}(\mathbb{R}^d) \subseteq L_{0,2}(\mathbb{R}^d)$. Second, due to (3.39), it is sufficient to prove that

$$L_{2,0}(\mathbb{R}^d) \subseteq \left\{ f \in L_{\Theta}(\mathbb{R}^d) \,\middle|\, \int_{\mathbb{R}^d} m_{\mu,\nu}(f(\boldsymbol{x}))\mathrm{d}\boldsymbol{x} < \infty \right\}$$

to deduce that $L_{2,0}(\mathbb{R}^d) \subseteq L_{\Theta}(\mathbb{R}^d)$. We remark that, for $|\xi| \leq 1$,

$$\begin{aligned} m_{\mu,\nu}(\xi) &= \left| \mu\xi + \int_{1\leq|t|\leq\frac{1}{|\xi|}} \xi t \nu(\mathrm{d}t) \right| \leq |\mu\xi| + \int_{1\leq|t|\leq\frac{1}{|\xi|}} \nu(\mathrm{d}t) \\ &\leq |\mu| + \int_{1\leq|t|} \nu(\mathrm{d}t), \end{aligned}$$

and that, for $|\xi| > 1$,

$$\begin{aligned} m_{\mu,\nu}(\xi) &= \left| \mu\xi + \int_{\frac{1}{|\xi|}\leq|t|\leq 1} \xi t \nu(\mathrm{d}t) \right| \leq |\mu\xi| + \int_{\frac{1}{|\xi|}\leq|t|\leq 1} |\xi t|^2 \nu(\mathrm{d}t) \\ &\leq \left(|\mu| + \int_{|t|\leq 1} t^2 \nu(\mathrm{d}t) \right) \xi^2. \end{aligned}$$

Therefore, we have $m_{\mu,\nu}(\xi) \leq C\rho_{2,0}(\xi)$ for some constant C, which implies that $L_{2,0}(\mathbb{R}^d)$ is included into $\left\{ f \in L_{\Theta}(\mathbb{R}^d) \mid \int_{\mathbb{R}^d} m_{\mu,\nu}(f(\boldsymbol{x}))\mathrm{d}\boldsymbol{x} < \infty \right\}$, as expected.
Finally, (3.38) is a direct consequence of (3.25). □

Remarks.

- The embeddings (3.33) inform on the extreme cases. In particular, a function in $L_{2,0}(\mathbb{R}^d)$—the space of functions in $L_2(\mathbb{R}^d)$ whose support has a finite Lebesgue measure—can be applied to any Lévy noise. This includes in particular all the indicator functions $\mathbb{1}_B$ with B a Borel set with finite Lebesgue measure, or the Daubechies wavelets that are compactly supported and in $L_2(\mathbb{R}^d)$. Finite-variance compound Poisson noises reach the largest possible domain of definition $L_{0,2}(\mathbb{R}^d)$ (see Proposition 3.19 below).

- Moreover, (3.38) is particularly important as it gives the implication of having finite moments of the form $\int_{|t|>1} |t|^{p_\infty} \nu(\mathrm{d}t) < \infty$ and $\int_{|t|\leq 1} |t|^{p_0} \nu(\mathrm{d}t) < \infty$. This result will play a crucial role when identifying compatibility conditions between a whitening operator and a Lévy noise in Section 3.3.1.

- The embeddings (3.36) are useful to understand the finiteness of the moments of $\langle w, f\rangle$ for a Lévy noise with finite pth-moments. In particular, a test function f that is bounded with compact support is in the domain of definition of any noise and $\langle w, f\rangle$ has a finite pth-moment as soon as w has.

- Finally, we point out that the behavior of the Rajput-Rosinski exponent Θ at the origin (at the infinity, respectively) is related to the moments of ν at the infinity (at the origin, respectively): The local and asymptotic behaviors of ν and Θ are inverted. This reminds us of the Fourier transform. The local regularity of a function is directly connected to the decay properties of its Fourier transform, and *vice versa*. This is not surprising. For instance, for compound Poisson processes, the Lévy exponent is the Fourier transform of the Lévy measure up to the addition of a constant term, and the Rajput-Rosinski exponent is highly related to the Lévy exponent.

We see how the indices α_{loc} and α_{asymp} influence the domain of definition and the domain of pth-moments of the Lévy noise.

Proposition 3.18. *Assume that w is a symmetric Lévy noise with local and asymptotic indices $\alpha_{\text{loc}} \in [0,2]$, $\alpha_{\text{asymp}} \in (0,\infty]$. For $p \leq \alpha_{\text{loc}}, 2$ and $p < \alpha_{\text{asymp}}$, if $\epsilon > 0$ is small enough, we have the embedding*

$$L_{\alpha_{\text{loc}}+\epsilon,\alpha_{\text{asymp}}-\epsilon}(\mathbb{R}^d) \subseteq L_{\Theta_p}(\mathbb{R}^d)$$

if $\alpha_{\text{asymp}} \leq 2$, and

$$L_{\alpha_{\text{loc}}+\epsilon,2}(\mathbb{R}^d) \subseteq L_{\Theta_p}(\mathbb{R}^d)$$

if $\alpha_{\text{asymp}} > 2$, with Θ_p the pth-order Rajput-Rosinski exponent of w.

Proof. Let ϵ be small enough such that $p \leq \alpha_{\text{asymp}} - \epsilon$. Then, we have that

$$\int_{\mathbb{R}} \rho_{\min(\alpha_{\text{asymp}}-\epsilon,2),\alpha_{\text{loc}}}(t)\nu(\mathrm{d}t) < \infty,$$

by definition of the indices (see Definition 2.8). We can therefore apply (3.38) with the adequate conditions on p to deduce Proposition 3.18. The distinction between $\alpha_{\text{asymp}} \leq 2$ and $\alpha_{\text{asymp}} > 2$ comes from the fact that $p_\infty \leq 2$ in (3.38). □

Examples. We shall see how our results apply to specific Lévy noises. For these different classes, introduced in Section 2.1.3, we specify the domain of definition $L_\Theta(\mathbb{R}^d)$ and the domains of finite pth moments $L_{\Theta_p}(\mathbb{R}^d)$.

The Gaussian noise of variance σ^2 is characterized by the Lévy triplet $(0,\sigma^2,0)$. With Theorem 3.3, we obtain that, for every $0 \leq p \leq 2$,

$$L_{\Theta_{p,\text{Gauss}}}(\mathbb{R}^d) = L_2(\mathbb{R}^d).$$

Note that Theorem 3.3 is on Θ_0, but $\Theta_p = \Theta_0$ in the Gaussian case. Based on these considerations and on Proposition 3.14, we shall consider Lévy triplets with $\sigma^2 = 0$ from now on.

Proposition 3.19. *The domains of definition of the following Lévy noises are completely characterized.*

- *If w_α is a SαS noise with $0 < \alpha < 2$, then, for every $0 \leq p < \alpha$, we have*

$$L_{\Theta_{p,\alpha}}(\mathbb{R}^d) = L_\alpha(\mathbb{R}^d).$$

  For $p \geq \alpha$, we have $L_{\Theta_{p,\alpha}}(\mathbb{R}^d) = \{0\}$.

- *If w_{Poisson} is a symmetric compound Poisson noise with finite variance, then*

$$L_{\Theta_{p,\text{Poisson}}}(\mathbb{R}^d) = L_{p,2}(\mathbb{R}^d).$$

  for every $0 \leq p \leq 2$.

- *If w_{Laplace} is a generalized Laplace noise, then we have*

$$L_{\Theta_{\text{Laplace}}}(\mathbb{R}^d) = L_{\log,2}(\mathbb{R}^d). \tag{3.40}$$

  Moreover, for $0 < p \leq 2$, we have

$$L_{\Theta_{p,\text{Laplace}}}(\mathbb{R}^d) = L_{p,2}(\mathbb{R}^d). \tag{3.41}$$

Proof. We study each case separately.

- *SαS:* Without loss of generality, one can assume that $\gamma = 1$. The Lévy measure of w_α is $\nu(\mathrm{d}t) = \frac{C_\alpha}{|t|^{\alpha+1}}\mathrm{d}t$ with C_α a constant (see Section 2.1.3). A non-trivial SαS random variable has an infinite pth-moment for $p \geq \alpha$, and for every $f \in L(w_\alpha)$, $\langle w, f\rangle$ is a SαS random variable. Hence $L_p(w) = \{0\}$ for $p \geq \alpha$. The case of interest is therefore $0 \leq p < \alpha$. Then, from (3.19),

$$\begin{aligned}\Theta_p(\xi) &= 2C_\alpha \int_0^{1/|\xi|} \frac{\xi^2}{t^{\alpha+1}}\mathrm{d}t + 2C_\alpha \int_{1/|\xi|} \frac{|\xi|^p}{x^{\alpha+1-p}}\mathrm{d}t \\ &= 2C_\alpha |\xi|^\alpha \left(\int_0^1 \frac{\mathrm{d}y}{y^{\alpha-1}} + \int_1^\infty \frac{\mathrm{d}y}{y^{\alpha+1-p}} \right) \\ &= \left(\frac{2(2-p)C_\alpha}{(2-\alpha)(\alpha-p)} \right) |\xi|^\alpha.\end{aligned}$$

  where we perfomed the change of variable $y = \xi x$. The result eventually follows from Proposition 3.15.

- *Compound Poisson:* We denote by λ and P the sparsity parameter and the law of jumps of w_{Poisson}, respectively. The Lévy measure is then λP. First, $L_{\Theta_{p,\text{Poisson}}}(\mathbb{R}^d) \subseteq L_{p,2}(\mathbb{R}^d)$ as for any symmetric Lévy noise, according to (3.34). Moreover, for a compound Poisson noise with finite variance, we have for every $q \in [0,2]$ that $\int_{\mathbb{R}} |t|^q P(\mathrm{d}t) < \infty$. Therefore, we have

$$\begin{aligned}\Theta_p(\xi) &= \lambda \int_{\mathbb{R}} \min(|t\xi|^p, |t\xi|^2) P(\mathrm{d}t) \\ &\leq \lambda \min\left(|\xi|^p \int_{\mathbb{R}} |t|^p P(\mathrm{d}t), |\xi|^2 \int_{\mathbb{R}} |t|^2 P(\mathrm{d}t) \right) \\ &\leq C \min(|\xi|^p, |\xi|^2) = \rho_{p,2}(\xi),\end{aligned}$$

so that $\|f\|_\Theta \leq C \|f\|_{p,2}$. This means that $L_{p,2}(\mathbb{R}^d) \subseteq L_{\Theta_{p,\mathrm{Poisson}}}(\mathbb{R}^d)$, concluding the proof.

- *Laplace:* Let $0 \leq p \leq 2$. Without loss of generality, we fix the parameters of the generalized Laplace noise as $\sigma^2 = 2$ and $\tau = 1$. Then, the Lévy measure is $\nu(\mathrm{d}t) = \frac{\mathrm{e}^{-|t|}}{|t|} \mathrm{d}t$. We start from (3.19) and write

$$\Theta_p(\xi) = \xi^2 \int_{|t| \leq 1/|\xi|} x^2 \nu(\mathrm{d}t) + |\xi|^p \int_{|x| > 1/|\xi|} |t|^p \nu(\mathrm{d}t) := \Theta_{p,1}(\xi) + \Theta_{p,2}(\xi).$$

Then, by integration by parts, we have

$$\begin{aligned}\Theta_{p,1}(\xi) &= 2|\xi|^2 \int_0^{1/|\xi|} t\mathrm{e}^{-t} \mathrm{d}t \\ &= 2|\xi|^2 \left(1 - \mathrm{e}^{-1/|\xi|} \left(1 + \frac{1}{|\xi|} \right) \right)\end{aligned}$$

Hence, we have $\Theta_{p,1}(\xi) \underset{\xi \to \infty}{\longrightarrow} 2$ and $\Theta_{p,1}(\xi) \underset{\xi \to 0}{\sim} 2|\xi|^2$.

For $\Theta_{p,2}(\xi) = |\xi|^p \int_{|t| > 1/|\xi|} |t|^p \nu(\mathrm{d}t)$, we shall distinguish between $p = 0$ and $p > 0$. For $p > 0$, the function $t^{p-1}\mathrm{e}^{-t}$ is integrable over $\mathbb{R}$, so that $\Theta_{p,2}(\xi) \underset{\xi \to \infty}{\sim} \left(\int_{\mathbb{R}} t^{p-1} \mathrm{e}^{-t} \mathrm{d}t \right) |\xi|^p$. For $p = 0$, the function $t^{-1}\mathrm{e}^{-t}$ is not anymore integrable around 0. Using the equivalence $t^{-1}\mathrm{e}^{-t} \underset{t \to 0}{\sim} t^{-1}$, we deduce that

$$\Theta_{0,2}(\xi) = 2 \int_{\frac{1}{|\xi|}}^{\infty} t^{-1} \mathrm{e}^{-t} \mathrm{d}t \underset{\xi \to \infty}{\sim} 2 \int_{\frac{1}{|\xi|}}^{1} t^{-1} \mathrm{e}^{-t} \mathrm{d}t \underset{\xi \to \infty}{\sim} 2 \int_{\frac{1}{|\xi|}}^{1} t^{-1} \mathrm{d}t = 2 \log |\xi|.$$

Moreover, since $p \leq 2$, we have by integration by parts,

$$\Theta_{p,2}(\xi) = 2 \int_{t|\xi| > 1} (t|\xi|)^p \mathrm{e}^{-t} \frac{\mathrm{d}t}{t} \leq 2 \int_{t|\xi| > 1} (t|\xi|)^2 \mathrm{e}^{-t} \frac{\mathrm{d}t}{t} = 2|\xi|(1 + |\xi|)\mathrm{e}^{-1/|\xi|},$$

implying that $\Theta_{p,2}(\xi) \underset{\xi \to 0}{=} o(|\xi|^2)$. By combining the results on $\Theta_{p,1}$ and $\Theta_{p,2}$, we obtain that

  - for $0 \leq p \leq 2$, $\Theta_p(\xi) \underset{\xi \to 0}{\sim} 2|\xi|^2$;
  - for $0 < p \leq 2$, $\Theta_p(\xi) \underset{\xi \to \infty}{\sim} \left(\int_{\mathbb{R}} x^{p-1} \mathrm{e}^{-x} \mathrm{d}t \right) |\xi|^p$;
  - for $p = 0$, $\Theta_0(\xi) = \Theta(\xi) \underset{\xi \to \infty}{\sim} 2 \log |\xi|$.

Table 3.1 – Domain of Definition of Lévy Noise

Lévy noise	$\Psi(\xi)$	$L_\Theta(\mathbb{R}^d)$	$L_{\Theta_p}(\mathbb{R}^d)$ $0 < p \leq 2$
Gaussian	$-\frac{1}{2}\sigma^2\xi^2$	$L_2(\mathbb{R}^d)$	$L_2(\mathbb{R}^d)$
SαS	$-c^\alpha\lvert\xi\rvert^\alpha$	$L_\alpha(\mathbb{R}^d)$	$\begin{cases} L_\alpha(\mathbb{R}^d) & \text{if } p < \alpha \\ \{0\} & \text{if } p \geq \alpha \end{cases}$
symmetric finite-variance compound Poisson	$\lambda(\widehat{P}(\xi) - 1)$	$L_{0,2}(\mathbb{R}^d)$	$L_{p,2}(\mathbb{R}^d)$
generalized Laplace	$-\tau\log(1 + \sigma^2\xi/2)$	$L_{\log,2}(\mathbb{R}^d)$	$L_{p,2}(\mathbb{R}^d)$

We finally apply Proposition 3.15 to deduce (3.40) and (3.41).

□

We summarize the results of this section in Table 3.1. The Lévy noises are characterized by their Lévy exponent. We refer to Section 2.1.3 for the complete definition of the corresponding infinite divisible laws.

3.3 Generalized Lévy Processes

We constructed and studied the Lévy noise on $\mathscr{S}'(\mathbb{R}^d)$ in Sections 3.1 and 3.2. We now investigate the existence of generalized Lévy processes that are solutions of a stochastic differential equations driven by tempered Lévy noise. Section 3.3.1 is dedicated to the specification of a general criterion for the construction of generalized Lévy processes. It is based on [DFHU, Section 6] and extends previous results of [FAU14, UT14, UTS14, FU16]. Section 3.3.2 presents classes of generalized Lévy processes associated with specific differential or pseudo-differential whitening operators.

3.3.1 Existence Criterion

Our goal is to give general conditions of compatibility between the operator L and the Lévy noise w such that the process s in (3.1) exists. By exploiting the results of Section 3.2, we first show that the domain of definition of the Lévy noise is also the domain of continuity of its characteristic functional.

Proposition 3.20. *The characteristic functional of the Lévy noise w is well-defined, continuous, positive-definite over $L_\Theta(\mathbb{R}^d)$, and normalized such that $\widehat{\mathscr{P}}_w(0) = 1$.*

Proof. The characteristic functional $\varphi \mapsto \widehat{\mathscr{P}}_w(\varphi) = \mathbb{E}[\mathrm{e}^{\mathrm{i}\langle w,\varphi\rangle}]$ is *a priori* continuous over $\mathscr{S}(\mathbb{R}^d)$. For $f \in L_\Theta(\mathbb{R}^d)$, the random variable $\langle w, f\rangle$ is well-defined and its characteristic function is $\xi \mapsto \mathbb{E}[\mathrm{e}^{\mathrm{i}\xi\langle w,f\rangle}] = \exp\left(\int_{\mathbb{R}^d} \Psi(\xi f(\boldsymbol{x}))\mathrm{d}\boldsymbol{x}\right)$ (Proposition 3.10). We can therefore extend $\widehat{\mathscr{P}}_w$ to $L_\Theta(\mathbb{R}^d)$ by setting

$$\widehat{\mathscr{P}}_w(f) = \mathbb{E}[\mathrm{e}^{\mathrm{i}\langle w,f\rangle}] = \exp\left(\int_{\mathbb{R}^d} \Psi(f(\boldsymbol{x}))\mathrm{d}\boldsymbol{x}\right)$$

for $f \in L_\Theta(\mathbb{R}^d)$.

Positive-definiteness. Let $N \geq 1$, $a_n \in \mathbb{C}$, $f_n \in L_\Theta(\mathbb{R}^d)$, $n = 1,\dots,N$. Simple functions are dense in the generalized Orlicz space $L_\Theta(\mathbb{R}^d)$. Moreover, any simple function can be approximated by functions of $\mathscr{S}(\mathbb{R}^d)$ in $L_\Theta(\mathbb{R}^d)$, so that, $\mathscr{S}(\mathbb{R}^d)$ is dense in $L_\Theta(\mathbb{R}^d)$. Let us fix N sequences $(\varphi_{n,k})_{k\in\mathbb{N}}$ such that the $\varphi_{k,n}$ converge to f_n in $L_\Theta(\mathbb{R}^d)$ for $n = 1,\dots,N$ as k goes to infinity. From Theorem 3.4, we know that $f \mapsto \langle w, f\rangle$ is continuous from $L_\Theta(\mathbb{R}^d)$ to $L_0(\Omega)$. In particular, we have $\mathbb{E}[\mathrm{e}^{\mathrm{i}\langle w,\varphi_k^i-\varphi_k^j\rangle}] \underset{k\to\infty}{\longrightarrow} \mathbb{E}[\mathrm{e}^{\mathrm{i}\langle w,f_i-f_j\rangle}]$ for every $1 \leq i,j \leq N$. This implies that

$$\begin{aligned}
\sum_{1\leq i,j\leq N} a_i a_j^* \widehat{\mathscr{P}}_w(f_i - f_j) &= \sum_{1\leq i,j\leq N} a_i a_j^* \mathbb{E}[\mathrm{e}^{\mathrm{i}\langle w,f_i-f_j\rangle}] \\
&= \lim_{k\to\infty} \sum_{1\leq i,j\leq N} a_i a_j^* \mathbb{E}[\mathrm{e}^{\mathrm{i}\langle w,\varphi_k^i-\varphi_k^j\rangle}] \\
&= \lim_{k\to\infty} \sum_{1\leq i,j\leq N} a_i a_j^* \widehat{\mathscr{P}}_w(\varphi_k^i - \varphi_k^j) \\
&\geq 0,
\end{aligned}$$

where we used the positive-definiteness of $\widehat{\mathscr{P}}_w$ over $\mathscr{S}(\mathbb{R}^d)$.

Continuity. Using the Lévy-Khintchine representation (2.1) of Ψ with Lévy triplet (μ, σ^2, ν), we have

$$\begin{aligned}|\Psi(\xi)| &= \left|\mathrm{i}\mu\xi + \mathrm{i}\int_{\mathbb{R}} t\xi\left(\mathbb{1}_{|t\xi|\le 1} - \mathbb{1}_{|t|\le 1}\right)\nu(\mathrm{d}t) + \sigma^2\xi^2 + \int_{\mathbb{R}}(\mathrm{e}^{\mathrm{i}t\xi} - 1 - \mathrm{i}t\xi\mathbb{1}_{|t\xi|\le 1})\nu(\mathrm{d}t)\right| \\ &\le \left|\mu\xi + \int_{\mathbb{R}} t\xi\left(\mathbb{1}_{|t\xi|\le 1} - \mathbb{1}_{|t|\le 1}\right)\nu(\mathrm{d}t)\right| + \sigma^2\xi^2 + 2\int_{\mathbb{R}}\min(1, |t\xi|^2)\nu(\mathrm{d}t) \\ &\le 2\Theta(\xi), \end{aligned} \tag{3.42}$$

where we used the triangular inequality and the relation $|\mathrm{e}^{\mathrm{i}y} - 1 - \mathrm{i}y\mathbb{1}_{|y|\le 1}| \le 2\min(1, y^2)$ applied to $y = t\xi$. Applying (3.42) to $\xi = f(\boldsymbol{x})$ and integrating over $\mathbb{R}^d$, we have for every $f \in L_\Theta(\mathbb{R}^d)$,

$$|\log \widehat{\mathscr{P}}_w(f)| \le \int_{\mathbb{R}^d} |\Psi(f(\boldsymbol{x}))| \mathrm{d}\boldsymbol{x} \le 2\Theta(f).$$

This shows that $\widehat{\mathscr{P}}_w$ is continuous at 0. The functional $\widehat{\mathscr{P}}_w$ is positive-definite and continuous at 0, and therefore continuous (Proposition 2.11). □

Combining Proposition 3.20 with the Bochner-Minlos theorem, we obtain the following general criterion for the existence of solution of stochastic differential equations driven by Lévy noise.

Theorem 3.5. *Consider a Lévy noise w in $\mathscr{S}(\mathbb{R}^d)$. For any linear operator* T *continuous from $\mathscr{S}(\mathbb{R}^d)$ to $L_\Theta(\mathbb{R}^d)$, there exists a generalized random process s such that*

$$\widehat{\mathscr{P}}_s(\varphi) = \widehat{\mathscr{P}}_w(\mathrm{T}\{\varphi\}). \tag{3.43}$$

In particular, if T *is a left-inverse of the adjoint* L^* *of a linear, continuous, and shift-invariant operator* L *from $\mathscr{S}(\mathbb{R}^d)$ to $\mathscr{S}'(\mathbb{R}^d)$, then*

$$\mathrm{L}s \overset{(\mathscr{L})}{=} w. \tag{3.44}$$

If moreover the operator T *continuously maps $\mathscr{S}(\mathbb{R}^d)$ to $L_{\Theta_p}(\mathbb{R}^d)$ for some $0 < p \le 2$, then the process s has finite pth-moments.*

Proof. The operator T is continuous from $\mathscr{S}(\mathbb{R}^d)$ to $L_\Theta(\mathbb{R}^d)$ and $\widehat{\mathscr{P}}_w$ is continuous over $L_\Theta(\mathbb{R}^d)$ according to Proposition 3.20. Hence, the functional $\widehat{\mathscr{P}} = \widehat{\mathscr{P}}_w(\mathrm{T}\{\cdot\})$ is continuous over $\mathscr{S}(\mathbb{R}^d)$. The positive-definiteness of $\widehat{\mathscr{P}}$ over $\mathscr{S}(\mathbb{R}^d)$ is a direct consequence of the positive-definiteness of $\widehat{\mathscr{P}}_w$ over $L_\Theta(\mathbb{R}^d)$ (again thanks to Proposition 3.20), and the fact that $\mathrm{T}\{\varphi\} \in L_\Theta(\mathbb{R}^d)$ for $\varphi \in \mathscr{D}(\mathbb{R}^d)$. Finally, $\widehat{\mathscr{P}}(0) = \widehat{\mathscr{P}}_w(\mathrm{T}\{0\}) = \widehat{\mathscr{P}}_w(0) = 1$. We are therefore in the conditions of the Bochner-Minlos theorem: The process s with characteristic functional (3.43) exists.
For the second part, we remark that, for $\varphi \in \mathscr{S}(\mathbb{R}^d)$,

$$\widehat{\mathscr{P}}_{\mathrm{L}s}(\varphi) = \widehat{\mathscr{P}}_s(\mathrm{L}^*\{\varphi\}) = \widehat{\mathscr{P}}_w(\mathrm{TL}^*\{\varphi\}) = \widehat{\mathscr{P}}_w(\varphi),$$

due to the left-inverse property. Then, the processes Ls and w, having the same characteristic functional, are equal in law.

For the last part, we simply remark that, for any $\varphi \in \mathscr{S}(\mathbb{R}^d)$, $\mathbb{E}[\left|\langle s, \varphi\rangle\right|^p] = \mathbb{E}[\left|\langle w, \mathrm{T}\{\varphi\}\rangle\right|^p] < \infty$ since $\mathrm{T}\{\varphi\} \in L_{\Theta_p}(\mathbb{R}^d)$. □

Definition 3.7. *Consider a tempered Lévy noise w and a continuous, linear, and shift-invariant operator* L *from $\mathscr{S}(\mathbb{R}^d)$ to $\mathscr{S}'(\mathbb{R}^d)$. We say that the generalized random process s is a* generalized Lévy process *driven by w and whitened by* L *if there exists a left-inverse operator* T *of* L^*, *continuous from $\mathscr{S}(\mathbb{R}^d)$ to $L_\Theta(\mathbb{R}^d)$, such that*

$$\widehat{\mathscr{P}}_s(\varphi) = \widehat{\mathscr{P}}_w(\mathrm{T}\{\varphi\}). \tag{3.45}$$

The operator L *is the* whitening operator *of s.*

Under the conditions of Definition 3.7, s satisfies (3.44). The following result links the stability property of the corrected left-inverse (operator T) with the finiteness of generalized moments of the Lévy measure of w.

Proposition 3.21. *We consider a symmetric Lévy noise without Gaussian part w and a linear, continuous, and shift-invariant operator* L. *We assume that, for $0 \leq p_0, p_\infty \leq 2$, we have*

- $\int_{\mathbb{R}} \rho_{p_\infty, p_0}(t) \nu(\mathrm{d}t) < \infty$, *and*
- *the adjoint operator* L^* *admits a left-inverse* T *that maps continuously* $\mathscr{S}(\mathbb{R}^d)$ *to* $L_{p_0, p_\infty}(\mathbb{R}^d)$.

Then, there exists a generalized Lévy process s with characteristic functional (3.43) *that satisfies* $\mathrm{L}s \overset{(\mathscr{L})}{=} w$.

Proof. Applying (3.38) with $p = 0$, the condition $\int_{\mathbb{R}} \rho_{p_\infty, p_0}(t) \nu(\mathrm{d}t) < \infty$ ensures that $L_{p_0, p_\infty}(\mathbb{R}^d) \subset L_\Theta(\mathbb{R}^d)$. This embedding and the assumption on T imply that T maps continuously $\mathscr{S}(\mathbb{R}^d)$ to $L_\Theta(\mathbb{R}^d)$, and Theorem 3.5 applies. □

Comparison with previous works. Proposition 3.21 can be compared with other conditions of compatibility between the whitening operator L and the Lévy noise w. The results are reformulated with our notation.

- For $1 \leq p \leq 2$, Ψ is *p-admissibile* if $|\Psi(\xi)| + |\xi| \left|\Psi'(\xi)\right| \leq C|\xi|^p$. Note that the derivative $\Psi'(\xi)$ is well-defined as soon as the first moment of the underlying infinitely divisible random variable is finite, what we assume now. This notion was introduced in [UT14] together with the following compatibility condition: if Ψ is p-admissible and T continuously map $\mathscr{S}(\mathbb{R}^d)$ to $L_p(\mathbb{R}^d)$, then (3.43) specifies a valid characteristic functional. A sufficient condition for the p-admissible is that $\int_{\mathbb{R}} |t|^p \nu(\mathrm{d}t) < \infty$. Therefore, (3.43) is a valid characteristic functional as soon as $\int_{\mathbb{R}} |t|^p < \infty$ and T maps continuously $\mathscr{S}(\mathbb{R}^d)$ to $L_p(\mathbb{R}^d)$ for some $1 \leq p \leq 2$. We recover this by selecting $p_0 = p_\infty = p$ in Proposition 3.21. Our result extends this criterion in two ways. First, we can distinguish between the behavior of ν around 0 and at ∞. Second, we do not restrict to the case $p \geq 1$ (this second improvement was already achieved in our work [FU16] thanks to a relaxation of the p-admissibility).

- In our work with A. Amini and M. Unser, we have shown that the characteristic functional (3.45) specifies a generalized Lévy process if $\int_{\mathbb{R}} \rho_{p_\infty, p_0}(t) \nu(\mathrm{d}t)$ and T maps continuously $\mathscr{S}(\mathbb{R}^d)$ to $L_{p_0,p_\infty}(\mathbb{R}^d)$ for $0 < p_\infty \leq p_0 \leq 2$ [FAU14, Theorem 5]. When $p_\infty \leq p_0$, we have that

$$\max(|\xi|^{p_0}, |\xi|^{p_\infty}) \leq \rho_{p_0,p_\infty}(\xi) \leq |\xi|^{p_0} + |\xi|^{p_\infty}.$$

  Therefore, $L_{p_0,p_\infty}(\mathbb{R}^d) = L_{p_0}(\mathbb{R}^d) \cap L_{p_\infty}(\mathbb{R}^d)$ and we recover our previous result (at least for symmetric Lévy noise without Gaussian part). Moreover, Proposition 3.21 is a improvement, since one can consider $p_\infty > p_0$. In that case, $L_{p_0,p_\infty}(\mathbb{R}^d)$ contains but is strictly bigger than $L_{p_0}(\mathbb{R}^d) \cap L_{p_\infty}(\mathbb{R}^d)$ and the requirement on T is less strong: our new criterion is applicable to a more general class of operators.

- Combining (3.38) and Proposition 3.20, we generalize [AU14, Theorem 2] again by considering the case $p_\infty > p_0$: we are able to specify a larger domain of definition and of continuity than $L_{p_0}(\mathbb{R}^d) \cap L_{p_\infty}(\mathbb{R}^d)$ in that case.

3.3.2 Specific Classes of Generalized Lévy Processes

We introduce the generalized Lévy processes associated with the classes of differential and pseudo-differential operators presented in Section 2.2.2. The model (3.44) appears to contain many classical families of random processes related to Lévy noise, both in the univariate and multivariate settings. The main aspect here is to understand on concrete examples when the operator L and the Lévy noise w are compatible so to generate a generalized Lévy process.

The whitening operators that we shall consider share the following properties. They are linear, shift-invariant, continuous from $\mathscr{S}(\mathbb{R}^d)$ to $\mathscr{S}'(\mathbb{R}^d)$, and admits a measurable Green's function of slow growth ρ_{L}; that is, a measurable function, bounded by a polynomial, such that such that $\mathrm{L}\{\rho_{\mathrm{L}}\} = \delta$. Then, the function $\rho_{\mathrm{L}^*}(\boldsymbol{x}) = \rho_{\mathrm{L}}(-\boldsymbol{x})$ is a Green's function of the adjoint operator L^*.

We have seen in Theorem 3.5 that a natural way to define a solution s to (3.44) is to identify a (left-)inverse to L^*. The natural candidate is the shift-invariant operator $(\mathrm{L}^*)^{-1}$, inverse of L^*, defined for $\varphi \in \mathscr{S}(\mathbb{R}^d)$ as

$$(\mathrm{L}^*)^{-1}\{\varphi\} = \rho_{\mathrm{L}^*} * \varphi. \tag{3.46}$$

Note that the convolution is well-defined since ρ_{L^*} is of slow growth. Two different scenarios occur in practice.

- If $(\mathrm{L}^*)^{-1}$ continuously maps $\mathscr{S}(\mathbb{R}^d)$ to $L_\Theta(\mathbb{R}^d)$, then one selects $\mathrm{T} = (\mathrm{L}^*)^{-1}$ in (3.45) to define s solution of (3.44). We then have $s = \rho_{\mathrm{L}} * w$, and the process s is stationary.

- For many operators L, the Green's function ρ_{L} does not decay at infinity, so that one easily finds $\varphi \in \mathscr{S}(\mathbb{R}^d)$ with $(\mathrm{L}^*)^{-1}\varphi \notin L_\Theta(\mathbb{R}^d)$. In that case, we need to correct the operator $(\mathrm{L}^*)^{-1}$. In doing so, we do not look for a standard two-sided inverse since we know from Theorem 3.5 that we only need to specify a *left*-inverse. The construction of valid left-inverses will be crucial in the examples below, for which we rely on existing works on operators.

When the operator $(\mathrm{L}^*)^{-1}$ is unstable, we are looking for left-inverses T of L^* satisfying one of the two following properties for the construction of generalized Lévy processes:

- **Condition (C1):** T is continuous from $\mathscr{S}(\mathbb{R}^d)$ to $\mathscr{R}(\mathbb{R}^d)$. In this scenario, $\mathrm{T}\varphi$ is possibly non-smooth but has nice decay properties.
- **Condition (C2):** T is continuous from $\mathscr{S}(\mathbb{R}^d)$ to $L_p(\mathbb{R}^d)$ for some $0 < p \leq 2$. Again, T should preserve some stability, but this is much less restrictive.

Condition (C1) will concern ordinary differential operators. This situation is particularly pleasant: Due to the embeddings $\mathscr{R}(\mathbb{R}^d) \subseteq L_\Theta(\mathbb{R}^d)$, valid for any noise, one can construct generalized Lévy processes whitened by L for any noise w. Condition (C2) is of interest for pseudo-differential operators that are fractional versions of the differential operators. In that case, the generalized Lévy process is well-defined provided that $L^p(\mathbb{R}^d) \subseteq L_\Theta(\mathbb{R}^d)$. Consequently, under (C2), there are restrictions on the class of Lévy noises that are compatible with L.

Lévy processes. Most traditionally, Lévy processes are introduced as the unique random processes $(s(t))_{t\in\mathbb{R}^+}$ that have stationary and independent increments, are continuous in probability, and vanishes at 0 [App09, Ber98]. They are unseparable from the infinitely divisible laws [Sat13].

In the framework of generalized random processes, Lévy processes are solutions of the equation $\mathrm{D}s = w$ where w is a one-dimensional Lévy noise and the whitening operator $\mathrm{L} = \mathrm{D}$ is the derivative. This construction is developed more extensively, with generalizations to Nth-order Lévy processes, in [UTS14]. The function $-\mathbb{1}_{\mathbb{R}^+}$ is a Green's function of $\mathrm{D}^* = -\mathrm{D}$. The inverse (3.46) is therefore $(\mathrm{D}^*)^{-1}\varphi = -(\mathbb{1}_{\mathbb{R}^+} * \varphi)(x)$. Since $-\mathbb{1}_{\mathbb{R}^+}$ does not decay at infinity, the operator $(\mathrm{D}^*)^{-1}$ is unstable. We introduce the corrected version of $(\mathrm{D}^*)^{-1}$ as the operator I_0 defined by

$$\mathrm{I}_0\{\varphi\}(x) = -(\mathbb{1}_{\mathbb{R}^+} * \varphi)(x) + \widehat{\varphi}(0)\mathbb{1}_{\mathbb{R}^+}(x). \tag{3.47}$$

The operator I_0 is a left-inverse of $-\mathrm{D}$ (since $\widehat{\mathrm{D}\varphi}(0) = 0$ for any $\varphi \in \mathscr{S}(\mathbb{R}^d)$), that continuously maps $\mathscr{S}(\mathbb{R}^d)$ to $\mathscr{R}(\mathbb{R}^d)$ [UTS14, Proposition 2].

For any tempered Lévy noise, I_0 is therefore continuous from $\mathscr{S}(\mathbb{R}^d)$ to $L_\Theta(\mathbb{R}^d)$. Applying Theorem 3.5, there exists a generalized random process s such that

$$\widehat{\mathscr{P}}_s(\varphi) = \widehat{\mathscr{P}}_w(\mathrm{I}_0\{\varphi\}) = \exp\left(\int_{\mathbb{R}} \Psi(\mathrm{I}_0\{\varphi\}(x))\mathrm{d}x\right) \tag{3.48}$$

for any $\varphi \in \mathscr{S}(\mathbb{R}^d)$, with Ψ the Lévy exponent of w. Such a process is called a *Lévy process.* It is solution of the differential equation

$$\mathrm{D}s \overset{(\mathscr{L})}{=} w.$$

We recover the well-known fact that the Lévy noise can be thought as the derivative of the Lévy process in 1D. Finally, $\mathrm{I}_0\{\delta\}(x) = \mathbb{1}_{\mathbb{R}^+}(x) - \mathbb{1}_{\mathbb{R}^+}(x) = 0$, so that

$$s(0) = \langle s, \delta \rangle \overset{(\mathscr{L})}{=} \langle w, \mathrm{I}_0\{\delta\} \rangle = 0.$$

By exploiting the properties of the characteristic functional (3.48) and the criteria in Propositions 2.14 and 2.15, we easily show that s has first-order independent and stationary increments. Again thanks to Proposition 2.15, we show that if the Lévy noise is H-self-similar, then the corresponding Lévy process is $(H+1)$-self-similar.

An in depth discussion on the two constructions—Lévy processes as generalized random processes and Lévy processes as pointwise stochastically continuous random processes with stationary and independent increments—can be found in [DH15].

Remark. The class of processes we study in this thesis are named generalized Lévy processes. They generalize Lévy processes in two ways: they are built using more for general differential or pseudo-differential whitening operators L, and they can be defined on $\mathbb{R}^d$ with $d \geq 2$—in which case we talk about generalized Lévy fields.

CARMA Lévy processes. A *Nth-order CARMA* (continuous auto-regressive moving average) *Lévy process* is a stationary solution of the stochastic differential equation

$$P(\mathrm{D})s = w \tag{3.49}$$

with w a 1-dimensional Lévy noise and $P(X)$ a polynomial of degree N. Requiring the stationarity of s put constraints on the roots of the polynomial P. Essentially, the condition is that P has no purely imaginary roots [UTS14]. For instance, when $P(X) = X$, we recover the stochastic differential equation (3.48) that does not admit any stationary solution.

We construct the CARMA Lévy process solution of (3.49) by decomposing P as

$$P = X^N + a_{N-1}X^{N-1} + \cdots + a_0 = \prod_{n=1}^{N}(X - \alpha_n)$$

with $\alpha_n \in \mathbb{C}$, $\Re\{\alpha_n\} \neq 0$ for all n. We assume here that the coefficient $a_N = 1$ without loss of generality.

For $\Re\{\alpha\} \neq 0$, the operator $(\mathrm{D} - \alpha\mathrm{Id})^* = -(\mathrm{D} + \alpha\mathrm{Id})$ is a continuous bijection from $\mathscr{S}(\mathbb{R}^d)$ to $\mathscr{S}(\mathbb{R}^d)$ with a continuous and shift-invariant inverse $-(\mathrm{D} + \alpha\mathrm{Id})^{-1}$. The Fourier multiplier of the inverse is $\omega \mapsto -(\mathrm{i}\omega + \alpha)^{-1}$ (the denominator does not vanish by assumption on α). By selecting $\mathrm{T} = -(\mathrm{D} + \alpha\mathrm{Id})^{-1}$ in (3.45), we construct s from its characteristic functional

$$\widehat{\mathscr{P}}_s(\varphi) = \exp\left(\int_{\mathbb{R}} \Psi(-(\mathrm{D} + \alpha\mathrm{Id})^{-1}\{\varphi\}(x))\mathrm{d}x\right)$$

by applying Theorem 3.5. Then, s satisfies $(\mathrm{D} - \alpha\mathrm{Id})s \overset{(\mathscr{L})}{=} w$ and is stationary. It is therefore a first-order CARMA process, often called an Ornstein-Uhlenbeck process driven by a Lévy noise. The general solution of (3.49) is constructed following the same principle by composing

the operators $-(\mathrm{D}+\alpha_n \mathrm{Id})^{-1}$ for $1 \leq n \leq N$.
Several authors are more generally considering *CARMA* (p,q) *processes*, that are solutions of $P(\mathrm{D})s = Q(\mathrm{D})w$ (the integers p and q are the degrees of P and Q, respectively), as classical processes [MS07] or generalized random processes [Bro01, BL09, BH10]. The construction is easily deduced from the one we exposed by applying $Q(\mathrm{D})$ to the solution of (3.49).
By combining the construction of Lévy processes and of CARMA processes, one can also construct the random solution of any differential equation of the form $P(\mathrm{D})s = w$, where P is any polynomial. See [UTS14] for more details.

Until now, we defined univariate random processes. The next ones are defined over $\mathbb{R}^d$ with $d \geq 1$. Note that there is not a unique way to extend Lévy processes to higher dimensions. One approach, different from ours, is proposed in [DJ12], with a discussion on the definition of multivariate Lévy processes. The same remark applies for multivariate generalized Lévy processes, for which we propose two types of random fields. The *random sheets*, are based on separable whitening operators, as direct transposition of the 1-dimensional case. The *isotropic random fields* involve rotation-invariant operators.

Lévy sheets. The d-dimensional *Lévy sheet* is a generalized Lévy process whitened by the operator $\Lambda = \mathrm{D}_1 \cdots \mathrm{D}_d$. Its markovian properties have been studied in the Gaussian case in [DW92b] and in the general case in [DW92a]. In the theory of sparse stochastic processes, it is presented as the *Mondrian process* for its ability to reproduce Mondrian-like patterns [UT11, UT14].
In the framework of generalized Lévy processes, the construction of the Lévy sheet is very similar to the one of the Lévy process. As for the derivative, the operator Λ has no stable inverse and we need to specify a corrected left-inverse. For $i = 1 \dots d$, we set

$$\mathrm{I}_{0,i}\{\varphi\}(\boldsymbol{x}) = -\int_0^{x_i} \varphi(\boldsymbol{x} + (y - x_i)\boldsymbol{e}_i)\mathrm{d}y + \mathbb{1}_{\mathbb{R}^+}(x_i) \int_{\mathbb{R}} \varphi(\boldsymbol{x} + (y - x_i)\boldsymbol{e}_i)\mathrm{d}y,$$

where the $\boldsymbol{e}_i$ are the canonical basis of $\mathbb{R}^d$. When $d = 1$, we recover the operator I_0 defined in (3.47). As for I_0, we show that $\prod_{i=1}^d \mathrm{I}_{0,i}$ is a left-inverse of $\Lambda^* = (-1)^d \Lambda$ that continuously maps $\mathscr{S}(\mathbb{R}^d)$ to $\mathscr{R}(\mathbb{R}^d)$. This is developed for a general class of multivariate operators in our work [FAU14, Section 4.2]. The stability of the left-inverse ensures that the Lévy sheet with characteristic functional

$$\exp\left(\int_{\mathbb{R}^d} \Psi\left(\prod_{i=1}^d \mathrm{I}_{0,i}\{\varphi\}(\boldsymbol{x}) \right) \mathrm{d}\boldsymbol{x} \right)$$

is well-defined for any tempered Lévy noise w with Lévy exponent Ψ.

CARMA Lévy sheets. We define CARMA Lévy sheets that generalizes CARMA Lévy processes. For $\alpha \in \mathbb{C}$ with $\Re\{\alpha\} \neq 0$, we set $\Lambda_\alpha = \prod_{i=1}^d (\mathrm{D}_i - \alpha \mathrm{Id})$. As we did in dimension 1, one defines a stable (right and left) inverse $(\Lambda_\alpha^*)^{-1}$ to $\Lambda_\alpha^* = (-1)^d \prod_{i=1}^d (\mathrm{D}_i + \alpha \mathrm{Id})$. Then, the generalized Lévy process with characteristic functional

$$\exp\left(\int_{\mathbb{R}^d} \Psi\left((\Lambda_\alpha^*)^{-1}\{\varphi\}(\boldsymbol{x})\right) \mathrm{d}\boldsymbol{x}\right)$$

is well-defined and stationary for every Lévy noise w with Lévy exponent Ψ. More details can be found in [FAU14, Section 4.2] that also includes the specification of a more general class of directional Lévy fields.

Fractional Lévy processes and fields. *Fractional Lévy processes* are solution of the equation

$$(-\Delta)^{\gamma/2} s = w$$

with $(-\Delta)^{\gamma/2}$ the fractional Laplacian of order $\gamma > 0$ and w a d-dimensional Lévy noise. In the 1-dimensional setting, one can consider similarly the stochastic pseudo-differential equation $\mathrm{D}^\gamma s = w$ with D^γ the fractional derivative. As soon as $d \geq 2$, we talk about *fractional Lévy fields*. As for Lévy processes, fractional Lévy processes are classically defined as pointwise processes. In dimension 1, the fractional Brownian motion is Gaussian with $0 < \gamma < 1$ and was introduced by B.B. Mandelbrot and J.W. Van Ness in [MN68]. Higher-order extensions ($\gamma \geq 1$) are studied in [PHBJ$^+$01]. Fractional SαS processes are studied as pointwise process, for instance, in [ST94, EM00], while the general Lévy case is considered in [Mar06, EW13]. In the framework of generalized random processes, the fractional Lévy process was constructed for Gaussian noise in [BU07, LSSW16], for Poisson noise in [UT11, SU12], and for SαS noise in [HL07] (for $\alpha > 1$). The multivariate case is studied for instance in [LSSW16, UT11, UT14].
The construction of fractional Lévy processes and fields in the framework of generalized random processes was considered in [SU12]. This work was dedicated to the construction of stable left-inverses of $(-\Delta)^{\gamma/2}$ with the adequate invariances that we exploited to extend the construction of fractional Lévy processes as generalized random processes in [FAU14, Section 4.1]. The operator $(-\Delta)^{\gamma/2}$ admits a unique shift- and scale- invariant left-inverse operator as soon as $(\gamma - d) \notin \mathbb{N}$. It is the operator I^γ with frequency response $\|\boldsymbol{\omega}\|^\gamma$ [SU12, Theorem 1.1]. In general, this operator is not stable from $\mathcal{S}(\mathbb{R}^d)$ to $L_p(\mathbb{R}^d)$. By giving up the shift-invariance property, it is possible to specify stable left-inverses. For $p \geq 1$, we define the operator $\mathrm{I}_{\gamma,p}$ for any $\varphi \in \mathcal{S}(\mathbb{R}^d)$ by

$$\mathcal{F}\{\mathrm{I}_{\gamma,p}\varphi\}(\boldsymbol{\omega}) := \left(\mathcal{F}\{\varphi\}(\boldsymbol{\omega}) - \sum_{|\boldsymbol{m}| \leq \gamma - d(1-1/p)} \frac{1}{\boldsymbol{m}!} (\mathrm{D}^{\boldsymbol{m}}\varphi)(\boldsymbol{0})\boldsymbol{\omega}^{\boldsymbol{m}}\right) \|\boldsymbol{\omega}\|^{-\gamma},$$

with the usual multi-index notation. Then, according to [SU12, Theorem 1.2], as soon as $\gamma \notin \mathbb{N}$ and $\gamma - d + d/p \notin \mathbb{N}$, the operator $\mathrm{I}_{\gamma,p}$ is the unique $(-\gamma)$-homogeneous left inverse of the fractional Laplacian $(-\Delta)^{\gamma/2}$ that continuously maps $\mathcal{S}(\mathbb{R}^d)$ to $L_p(\mathbb{R}^d)$.
Therefore, for any Lévy noise w with exponents Ψ and Θ such that $L_p(\mathbb{R}^d) \subseteq L_\Theta(\mathbb{R}^d)$, the generalized random process s with characteristic functional

$$\exp\left(\int_{\mathbb{R}^d} \Psi\left(\mathrm{I}_{\gamma,p}\{\varphi\}(\boldsymbol{x})\right) \mathrm{d}\boldsymbol{x}\right)$$

is well-defined and satisfies $(-\Delta)^{\gamma/2} s \stackrel{(\mathscr{L})}{=} w$ according to Theorem 3.5. From Proposition 3.15, we know, for instance, that $L_p(\mathbb{R}^d) \subseteq L_\Theta(\mathbb{R}^d)$ as soon as $\Theta(\xi) \leq C|\xi|^p$ over $\mathbb{R}$ for some constant $C > 0$. Sufficient conditions on the Lévy measures for the well-definiteness of s are given in [FAU14, Proposition 6].

To summarize, we have characterized classes of generalized Lévy processes based on their whitening operator. By focusing on specific types of Lévy noises, we may also define the class of *CARMA compound Poisson processes* (for L a differential operator with no purely imaginary characteristic roots and w a 1-dimensional compound Poisson noise), *fractional SαS processes* (for $\mathrm{L} = (-\Delta)^{\gamma/2}$ and w a SαS noise, assumed to be compatible), etc.

4 Limit Theorems for Generalized Lévy Processes

In this chapter, we establish two different asymptotic results for generalized Lévy processes. Both of them highlight important properties of the considered processes. In Section 4.1, we review the fundamental theorem on the convergence in law of generalized random processes, which is the source of our contributions: the Fernique-Lévy theorem. We then prove that any generalized Lévy process is the limit in law of a family of generalized Poisson processes in Section 4.2. In Section 4.3, we investigate the coarse and fine scale behavior of generalized Lévy Processes.

4.1 The Lévy-Fernique Theorem

In finite dimensions, the convergence in law of a sequence of random variables is equivalent to the pointwise convergence of the underlying characteristic function: This is the Lévy continuity theorem presented in Section 2.1.

The generalization of this result to infinite-dimensional Hilbert spaces is not straightforward. For instance, the domain of definition of the characteristic functional of a random variable in $\mathscr{H}'$ is not $\mathscr{H}$, in contrast to the finite-dimensional case. Nevertheless, there is one class of function spaces on which the Lévy theorem is directly generalizable: the nuclear multi-Hilbertian spaces introduced in Section 2.2.1. Here, we restrict ourselves to tempered generalized random processes in $\mathscr{S}'(\mathbb{R}^d)$, but the result remains valid for generalized random processes in $\mathscr{D}'(\mathbb{R}^d)$.

Definition 4.1. *We say that the sequence of tempered generalized random processes (s_k) converges in law to the tempered generalized random process s if the underlying probability laws $\mathscr{P}_{s_k}$ converge weakly to the probability law $\mathscr{P}_s$; that is, if*

$$\int_{\mathscr{S}'(\mathbb{R}^d)} f(u)\mathrm{d}\mathscr{P}_{s_k}(u) \underset{k\to\infty}{\longrightarrow} \int_{\mathscr{S}'(\mathbb{R}^d)} f(u)\mathrm{d}\mathscr{P}_s(u)$$

for every bounded and continuous functional f from $\mathscr{S}'(\mathbb{R}^d)$ to $\mathbb{R}$.

Theorem 4.1 (Lévy-Fernique continuity theorem)**.** *A sequence of tempered generalized random processes (s_k) converges in law to the tempered generalized random process s if and only if*

$$\widehat{\mathscr{P}}_{s_k}(\varphi) \underset{k\to\infty}{\longrightarrow} \widehat{\mathscr{P}}_s(\varphi)$$

for every $\varphi \in \mathscr{S}(\mathbb{R}^d)$.

Theorem 4.1 is a powerful tool to deduce the limit in law of generalized random processes. We shall exploit it extensively in this chapter. It was proved by X. Fernique on $\mathscr{D}'(\mathbb{R}^d)$ [Fer67, Theorem III.6.5]. Along the same line as the Bochner-Minlos theorem, we call Theorem 4.1 the Lévy-Fernique theorem, the result on random vectors of the former mathematician being generalized for generalized random processes by the latter. P. Boulicaut has shown that the result is valid on any nuclear space [Bou74, Theorem 4.5]. He also obtained a converse result applicable to countably multi-Hilbertian spaces. If $\mathscr{X}$ is a Fréchet space, or the dual of a Fréchet space ((DF)-space), the convergence in law of $\mathscr{X}'$-valued random variables is equivalent to the pointwise convergence of the characteristic functionals on $\mathscr{X}$ if and only if the space $\mathscr{X}$ is nuclear [Bou74, Theorems 5.3 and 5.4]. This demonstrates that the nuclearity is essential for a direct generalization of the finite-dimensional concepts of probability theory to function spaces. Other infinite-dimensional generalizations (not only for nuclear spaces) of the Lévy theorem are extensively developed in [Mus96].

4.2 Generalized Poisson Processes Generate Generalized Lévy Processes

In their landmark paper on linear prediction [BS50], H.W. Bode and C.E. Shannon proposed that *"a (...) noise can be thought of as made up of a large number of closely spaced and very short impulses."* In this section, we formulate this intuitive interpretation of a white noise in a mathematically rigorous way. This allows us to extend this intuition beyond noise and to draw additional properties for the class of generalized Lévy processes. More precisely, we show that these processes can be statistically approximated as closely as desired by generalized Poisson processes that can also be viewed as random L-splines. This section is mostly based on our publication [FUU17]. A preliminary version of this work was presented to the SampTA conference [FWU15].

4.2.1 Generalized Poisson Processes are L-Splines

Splines are continuous-domain functions characterized by a sequence of knots and sample values. They provide a powerful framework to build discrete descriptions of continuous objets in sampling theory [Uns99]. Initially defined as piecewise-polynomial functions [Sch73a], they were further generalized by exploiting their connection with differential operators [SV67, MN90, UB00]. Recently, in the one-dimensional setting, a very general formulation has been proposed to specify under which condition a linear operator can be associated to a spline [AMU].

A linear and continuous operator L from $\mathcal{S}(\mathbb{R}^d)$ to $\mathcal{S}'(\mathbb{R}^d)$ is *spline-admissible* if it is shift-invariant and if there exists a measurable function of slow growth ρ_{L} such that

$$\mathrm{L}\{\rho_{\mathrm{L}}\} = \delta.$$

The function ρ_{L} is a *Green's function* of L. The differential and pseudo-differential operators of Section 2.2.2 are spline-admissible. The corresponding Green's functions and adequate references are given in Table 4.1.

Definition 4.2. *Let* L *be a spline-admissible operator with measurable Green's function* ρ_{L}. *A* nonuniform L-spline *is a function s such that*

$$\mathrm{L}s = \sum_{k\geq 0} a_k \delta(\cdot - \boldsymbol{x}_k) := w_\delta. \tag{4.1}$$

The a_k *are the* weights, *the* $\boldsymbol{x}_k$ *the* knots, *and* w_δ *is the* innovation *of the spline.*

The generic expression for a nonuniform L-spline is

$$s = p_0 + \sum_{k\geq 0} a_k \rho_{\mathrm{L}}(\cdot - \boldsymbol{x}_k)$$

with p_0 in the null space of L (*i.e.*, $\mathrm{L}p_0 = 0$). Indeed, we have, by linearity and shift-invariance

Table 4.1 – Some families of spline-admeissible operators

Operator	Parameter	$\rho_{\mathrm{L}}(\boldsymbol{x})$	Spline type
D^N	$N \in \mathbb{N}\backslash\{0\}$	$\frac{1}{(N-1)!} x^{N-1} \mathbb{1}_{x\geq 0}$	B-splines [Uns99, Sch73a]
$(\mathrm{D}+\alpha\mathrm{Id})$	$\alpha \in \mathbb{C}, \Re(\alpha) > 0$	$\mathrm{e}^{-\alpha x} \mathbb{1}_{x\geq 0}$	E-splines [UB05]
D^γ	$\gamma > 0$	$\frac{1}{\Gamma(\gamma)} x^{\gamma-1} \mathbb{1}_{x\geq 0}$	fractional splines [UB00, UB07]
$\mathrm{D}_1 \cdots \mathrm{D}_d$	-	$\mathbb{1}_{\boldsymbol{x}\geq \boldsymbol{0}} = \prod_{i=1}^d \mathbb{1}_{x_i \geq 0}$	separable splines [UT14]
$(-\Delta)^{m/2}$	$m-d \in 2\mathbb{N}$	$c_{m,d} \lVert\boldsymbol{x}\rVert^{m-d} \log \lVert\boldsymbol{x}\rVert$	cardinal polyharmonic splines [MN90]
$(-\Delta)^{\gamma/2}$	$\gamma - d \in \mathbb{R}^+ \backslash 2\mathbb{N}$	$c_{\gamma,d} \lVert\boldsymbol{x}\rVert^{\gamma-d}$	fractional polyharmonic splines [VBU05]

of L, that

$$\mathrm{L}\Big\{s - \sum_{k\geq 0} a_k \rho_{\mathrm{L}}(\cdot - \boldsymbol{x}_k)\Big\} = \mathrm{L}s - \sum_{k\geq 0} a_k \delta(\cdot - \boldsymbol{x}_k).$$

Therefore, $\big(s - \sum_{k\geq 0} a_k \rho_{\mathrm{L}}(\cdot - \boldsymbol{x}_k)\big)$ is in the null space of L.
By comparing (3.1) and (4.1), one easily realizes that the operator L connects the random and deterministic frameworks. The link is even stronger when one notices that compound Poisson white noises can be written as $w_{\text{Poisson}} = w_\delta$ according to (3.5). This means that generalized Poisson processes are (random) L-splines.

4.2.2 The Convergence Theorem

Our main result uncovers the link between L-splines and generalized Lévy processes through the use of generalized Poisson processes. A compound Poisson noise is made of a sparse sequence of weighted impulses whose jumps follow a common law. The average density of impulses λ is the primary parameter of such a Poisson white noise: Upon increasing λ, one increases the average number of impulses by unit of time. Meanwhile, the intensity of the impulses is governed by the common law of the jumps of the noise: Upon decreasing this intensity, one makes the weights of the impulses smaller. By combining these two effects, one can recover the intuitive conceptualization of a white noise proposed by Bode and Shannon in [BS50].

We start by fixing some notation. For simplicity, we shall consider symmetric Lévy noise without Gaussian part. The extension to the general case can be made thanks to Proposition 3.14 and is done in [FUU17]. If Ψ is a Lévy exponent, we set P_Ψ the compound Poisson law with sparsity parameter $\lambda = 1$ and law of jumps the infinitely divisible law with exponent Ψ. Then, for $\lambda, \tau > 0$, the Lévy exponent associated to the compound Poisson law with sparsity parameter λ and law of jumps $P_{\tau\Psi}$ is

$$\Psi_{\lambda,\tau}(\xi) = \lambda(\mathrm{e}^{\tau\Psi(\xi)} - 1). \tag{4.2}$$

Proposition 4.1. *We consider a Lévy exponent* Ψ *with Lévy measure* ν*, associated to a symmetric Lévy noise without Gaussian part. For* $\lambda, \tau > 0$*, we denote by* $\nu_{\lambda,\tau}$ *the Lévy measure associated to the Lévy exponent* $\Psi_{\lambda,\tau}$ *defined in* (4.2)*. If*

$$\int_{\mathbb{R}} \rho_{p_\infty,p_0}(t)\nu(\mathrm{d}t) < \infty \tag{4.3}$$

for some $0 \leq p_\infty, p_0 \leq 2$*, then,*

$$\int_{\mathbb{R}} \rho_{p_\infty,p_0}(t)\nu_{\lambda,\tau}(\mathrm{d}t) < \infty \tag{4.4}$$

for any $\lambda, \tau > 0$.
Therefore, under (4.3)*, if* T *maps continuously* $\mathscr{S}(\mathbb{R}^d)$ *to* $L_{p_0,p_\infty}(\mathbb{R}^d)$*, then* T *maps continuously* $\mathscr{S}(\mathbb{R}^d)$ *to* $L_{\Theta_{\lambda,\tau}}(\mathbb{R}^d)$ *for any* $\lambda, \tau > 0$*, where* $\Theta_{\lambda,\tau}$ *is the Rajput-Rosinski exponent associated to the Lévy exponent* $\Psi_{\lambda,\tau}$*.*

Proof. The Lévy measure of the compound Poisson noise $w_{\lambda,\tau}$ is $\nu_{\lambda,\tau} = \lambda P_{\tau\Psi}$. Without loss of generality, one can assume that $\lambda = 1$. First, $\int_{|t|\leq 1} |t|^{p_0} P_{\tau\Psi}(\mathrm{d}t)$ is finite because $P_{\tau\Psi}$ is a probability measure. To show (4.4), it suffices to show that $\int_{|t|>1} |t|^{p_\infty} P_{\tau\Psi}(\mathrm{d}t) < \infty$. This is equivalent to $\mathbb{E}[|Y|^{p_\infty}] < \infty$, where

$$Y = \sum_{i=1}^{N} X_i$$

is a compound Poisson random variable, with $N \sim \mathscr{P}(1)$ and the X_k are i.i.d., infinitely divisible with common Lévy exponent $\tau\Psi$. Let us fix $x, y \in \mathbb{R}$. If $0 < p < 1$, then we have that

$$|x+y|^p \leq |x|^p + |y|^p.$$

On the contrary, if $1 \leq p \leq 2$, then the inequality

$$\left|\frac{x+y}{2}\right|^p \leq \frac{|x|^p + |y|^p}{2}$$

follows from the convexity of $x \mapsto x^p$ on $\mathbb{R}^+$. From these two inequalities, we readily see that for any $0 < p \leq 2$ and $(x_i)_{1\leq i\leq N}$,

$$\left|\sum_{i=1}^{N} x_i\right|^p \leq N^{\max(p-1,0)} \sum_{i=1}^{N} |x_i|^p \leq N \sum_{i=1}^{N} |x_i|^p. \tag{4.5}$$

Therefore, we have that

$$\begin{aligned} \mathbb{E}[|Y|^{p_\infty}] = \mathbb{E}\Big[\Big|\sum_{i=1}^{N} X_i\Big|^{p_\infty}\Big] &\leq \mathbb{E}\Big[N\sum_{i=1}^{N} |X_i|^{p_\infty}\Big] = \sum_{n\geq 0} \mathbb{P}(N=n)\mathbb{E}\Big[n\sum_{i=1}^{n} |X_i|^{p_\infty}\Big] \\ &= \Big(\sum_{n\geq 0} n^2 \mathbb{P}(N=n)\Big) \times \mathbb{E}\left[|X_1|^{p_\infty}\right]. \end{aligned} \tag{4.6}$$

Finally, using that $\sum_{n\geq 0} n^2 \mathbb{P}(N=n) = \mathbb{E}[N^2] = 2\lambda = 2$, we deduce that $\mathbb{E}[|Y|^{p_\infty}] = 2\mathbb{E}[|X_1|^{p_\infty}]$. We conclude by remarking that $\mathbb{E}[|X_1|^{p_\infty}] < \infty$, what is equivalent to $\int_{|t|>1} |t|^{p_\infty} \nu(\mathrm{d}t) < \infty$ according to Proposition 2.3.
From (4.4), we deduce with Proposition 3.17 that $L_{p_0,p_\infty}(\mathbb{R}^d) \subseteq L_{\Theta(w_{\lambda,\tau})}(\mathbb{R}^d)$, implying directly the second part of Proposition 4.1. □

Proposition 4.1 has an important consequence. If a whitening operator L and a Lévy noise w with Lévy exponent Ψ satisfy together the conditions of Proposition 3.21 (and are therefore compatible), then L is also compatible with any compound Poisson noise whose law of jumps has Lévy exponent $\tau\Psi$.

Proposition 4.2. *Let w be a symmetric Lévy noise without Gaussian part, whose Lévy measure ν satisfies*

$$\int_{\mathbb{R}} \rho_{p_\infty,p_0}(t)\nu(\mathrm{d}t) < \infty$$

for some $0 \leq p_0, p_\infty \leq 2$. *We define, for* $k \geq 1$,

$$\Psi_k(\xi) = k\left(\mathrm{e}^{\Psi(\xi)/k} - 1\right). \tag{4.7}$$

Then, the followings hold.

- *The function Ψ_k is the Lévy exponent of the compound Poisson noise w_k with sparsity parameter $\lambda = k$ and infinitely divisible law of jumps with Lévy exponent* $\mathrm{e}^{\Psi/k}$.
- *The characteristic functionals $\widehat{\mathscr{P}}_{w_k}$ are well-defined, continuous, and positive definite over* $L_{p_0,p_\infty}(\mathbb{R}^d)$.
- *For any $\varphi \in L_{p_0,p_\infty}(\mathbb{R}^d)$, we have that*

$$\widehat{\mathscr{P}}_{w_k}(\varphi) \underset{k\to\infty}{\longrightarrow} \widehat{\mathscr{P}}_w(\varphi). \tag{4.8}$$

Proof. The first point is obvious. Remark that $\mathrm{e}^{\Psi/k}$ is a valid characteristic function because Ψ is a Lévy exponent (see Theorem 2.2). We set Θ (Θ_k, respectively) the Rajput-Rosinski exponent of w (of w_k, respectively). The second point of Proposition 4.2 is a consequence of the embeddings $L_{p_0,p_\infty}(\mathbb{R}^d) \subseteq L_\Theta(\mathbb{R}^d)$ and $L_{p_0,p_\infty}(\mathbb{R}^d) \subseteq L_{\Theta_k}(\mathbb{R}^d)$, deduced from Proposition 4.1, and of the extension of the domain of continuity of the characteristic functional with Proposition 3.20. We can now prove the convergence (4.8). For every fixed $\boldsymbol{x} \in \mathbb{R}^d$, we have that

$$\Psi_k(\varphi(\boldsymbol{x})) = k\left(\mathrm{e}^{\Psi(\varphi(\boldsymbol{x}))/k} - 1\right) \underset{k\to\infty}{\longrightarrow} \Psi(\varphi(\boldsymbol{x})).$$

Due to the convexity of the exponential, we have that $|\mathrm{e}^x - 1| \leq |x|$ for any $x \leq 0$. The symmetry of w implies the one of the w_k. Hence, both Ψ and the Ψ_k are real and negative. Therefore, we have that

$$|\Psi_k(\varphi(\boldsymbol{x}))| = k|\mathrm{e}^{\Psi(\varphi(\boldsymbol{x}))/k} - 1| \leq |\Psi(\varphi(\boldsymbol{x}))|,$$

which is integrable for $\varphi \in L_{p_0,p_\infty}(\mathbb{R}^d)$. The Lebesgue dominated convergence theorem implies then that

$$\int_{\mathbb{R}^d} \Psi_k(\varphi(\boldsymbol{x}))\mathrm{d}\boldsymbol{x} \underset{k\to\infty}{\longrightarrow} \int_{\mathbb{R}^d} \Psi(\varphi(\boldsymbol{x}))\mathrm{d}\boldsymbol{x}$$

and, therefore, (4.8) holds. □

We are now ready to state the main result of this section.

Theorem 4.2. *Let s be a generalized Lévy process with characteristic functional $\varphi \mapsto \widehat{\mathscr{P}}_w(\mathrm{T}\varphi)$ as in Theorem 3.5, with w a symmetric Lévy noise without Gaussian part. We assume that there exist* $0 \le p_0, p_\infty \le 2$ *such that*

- *the Lévy measure ν of s satisfies $\int_{\mathbb{R}} \rho_{p_\infty,p_0}(t)\nu(\mathrm{d}t) < \infty$, and*
- *the operator* T *continuously maps $\mathscr{S}(\mathbb{R}^d)$ to $L_{p_0,p_\infty}(\mathbb{R}^d)$.*

For $k \ge 1$, let w_k be the compound Poisson noise with Lévy exponent (4.7)*, then*

- *the characteristic functional $\varphi \mapsto \widehat{\mathscr{P}}_{w_k}(\mathrm{T}\varphi)$ specifies a generalized Poisson process s_k, and*
- *we have the convergence in law*

$$s_k \underset{k\to\infty}{\overset{(\mathscr{L})}{\longrightarrow}} s. \tag{4.9}$$

Proof. The conditions on w and T ensures with Proposition 4.1 that T continuously maps $\mathscr{S}(\mathbb{R}^d)$ to the domain of definition of all the w_k. Therefore, the generalized Poisson process s_k is well-defined for all $k \ge 1$.

For the second point, we fix $\varphi \in \mathscr{S}(\mathbb{R}^d)$. Then, $\mathrm{T}\varphi \in L_{p_0,p_\infty}(\mathbb{R}^d)$, and we have with Proposition 4.2 that

$$\widehat{\mathscr{P}}_{s_k}(\varphi) = \widehat{\mathscr{P}}_{w_k}(\mathrm{T}\{\varphi\}) \underset{k\to\infty}{\longrightarrow} \widehat{\mathscr{P}}_w(\mathrm{T}\{\varphi\}) = \widehat{\mathscr{P}}_s(\varphi).$$

The Lévy-Fernique theorem then implies (4.9). □

4.2.3 Examples and Simulations

We illustrate the convergence result of Theorem 4.2 on generalized Lévy processes of three types, namely

- Gaussian processes based on Gaussian noise, which are non-sparse;
- Laplace processes based on Laplace noise, which are sparse and have finite variance;
- Cauchy processes based on Cauchy noise, our prototypical example of infinite-variance sparse model.

Table 4.2 – Lévy noises used in Section 4.2.3.

Lévy Noise	Parameters	Lévy Exponent
Gaussian	$\sigma^2 > 0$	$-\frac{\sigma^2\xi^2}{2}$
Laplace	$\sigma^2 > 0$	$-\log\left(1+\frac{\sigma^2\xi^2}{2}\right)$
Cauchy	$c > 0$	$-c\lvert\xi\rvert$
Gauss-Poisson	$\lambda, \sigma^2 > 0$	$\lambda\left(\mathrm{e}^{-\frac{\sigma^2\xi^2}{2\lambda}}-1\right)$
Laplace-Poisson	$\lambda, \sigma^2 > 0$	$\lambda\left(\frac{1}{(1+\sigma^2\xi^2/2)^{1/\lambda}}-1\right)$
Cauchy-Poisson	$\lambda, c > 0$	$\lambda\left(\mathrm{e}^{-\frac{c\lvert\xi\rvert}{\lambda}}-1\right)$

For a given Lévy noise w with Lévy exponent Ψ, we consider compound Poisson processes that follow the principle of Proposition 4.2. Therefore, we consider compound Poisson noise with parameter λ and law of jumps with Lévy exponent $\Psi(\xi)/\lambda$, for increasing values of λ. In Table 4.2, we specify the parameters and Lévy exponents of six types of noise: Gaussian, Laplace, Cauchy, and their corresponding compound Poisson noises. We name a compound Poisson noise in relation to the law of its jumps (*e.g.*, the compound Poisson noise with Gaussian jumps is called a Gauss-Poisson noise). As λ increases, the associated compound Poisson noise features more and more jumps on average (λ per unit of volume) and is more and more concentrated towards 0. For instance, in the Gaussian case, the Gauss-Poisson noise has jumps with variance $\frac{\sigma^2}{\lambda} \underset{\lambda\to\infty}{\longrightarrow} 0$. To illustrate our results, we provide simulations for the 1-D and 2-D settings.

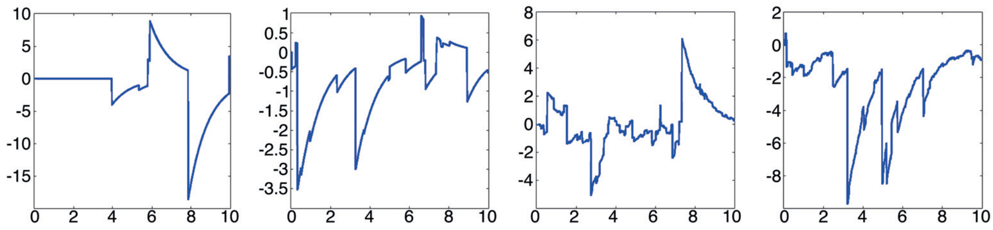
Simulations in 1-D. We consider two families of 1-D processes:

- $(\mathrm{D}+\alpha\mathrm{I})s = w$, with parameter $\alpha > 0$;
- $\mathrm{D}s = w$.

All the processes are plotted the interval $[0,10]$. We show in Figure 4.1 a Cauchy process generated by $\mathrm{D}+\alpha\mathrm{I}$. In Figure 4.2 and 4.3, we show a Gaussian and a Laplace process, respectively. Both of them are whitened by D. In all cases, we first plot the processes generated with compound Poisson noises with increasing values of λ. Then, we show the processes obtained from the corresponding Lévy noise.
Interestingly, we observe that the processes obtained with Poisson noises of small λ in Figures 4.2 and 4.3 are very similar. However, their asymptotic processes (large λ) differ, as expected from the fact that they converge to processes obtained from different Lévy noises.

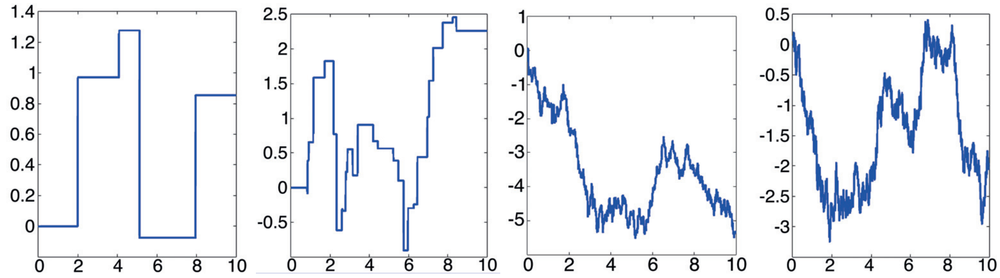
Simulations in 2-D. We consider three families of 2-D fields s:

- $\mathrm{D}_x\mathrm{D}_y s = w$;
- $(\mathrm{D}_x+\alpha\mathrm{I})(\mathrm{D}_y+\alpha\mathrm{I})s = w$, with parameter $\alpha > 0$;



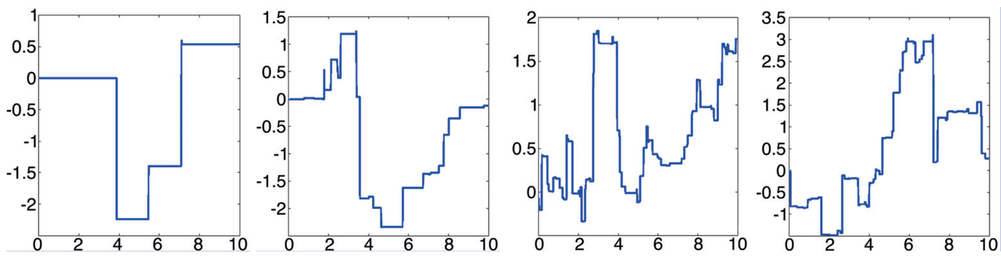
(a) $\lambda = 0.5$ (b) $\lambda = 3$ (c) $\lambda = 100$ (d) $\lambda \to \infty$

Figure 4.1 – Processes whitened by $\mathrm{D} + \alpha\mathrm{I}$, $\alpha = 0.1$. In (a)-(c), Cauchy-Poisson noises. In (d), Cauchy noise.



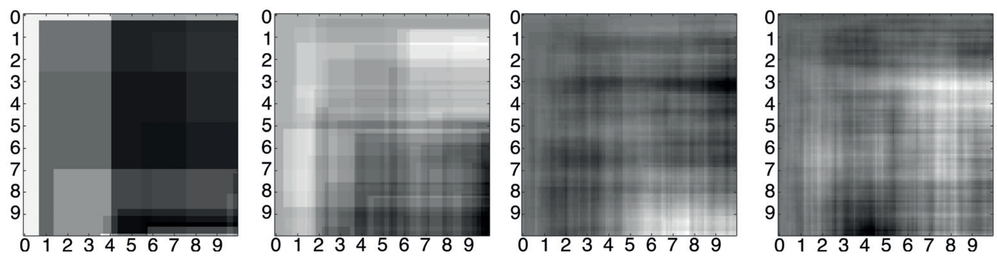
(a) $\lambda = 0.5$ (b) $\lambda = 3$ (c) $\lambda = 100$ (d) $\lambda \to \infty$

Figure 4.2 – Processes whitened by D. In (a)-(c) Gauss-Poisson noises. In (d), Gaussian noise.



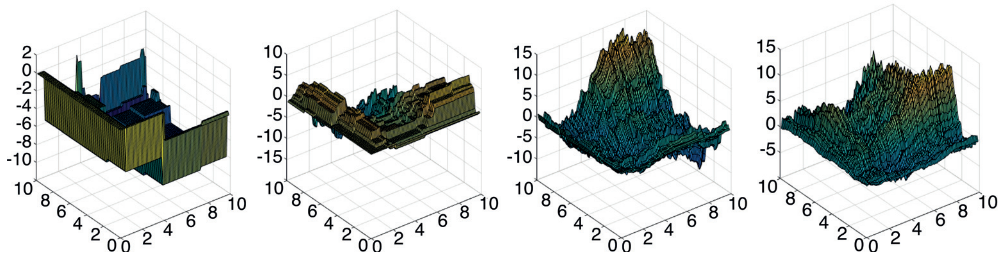
(a) $\lambda = 0.5$ (b) $\lambda = 3$ (c) $\lambda = 100$ (d) $\lambda \to \infty$

Figure 4.3 – Processes whitened by D. In (a)-(c), Laplace-Poisson noises. In (d), Laplace noise.



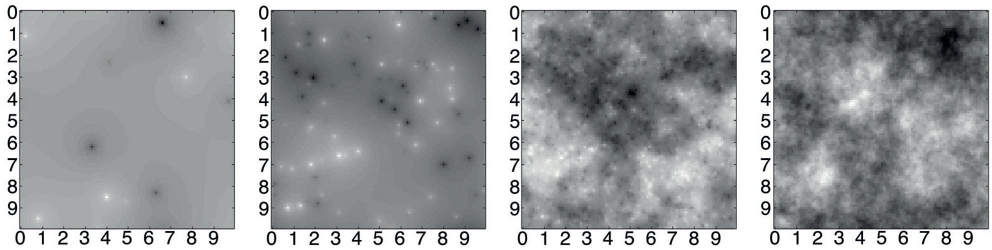
(a) $\lambda = 0.1$ (b) $\lambda = 1$ (c) $\lambda = 50$ (d) $\lambda \to \infty$

Figure 4.4 – Processes whitened by $\mathrm{D}_x\mathrm{D}_y$. In (a)-(c), Gauss-Poisson noise. In (d), Gaussian noise.



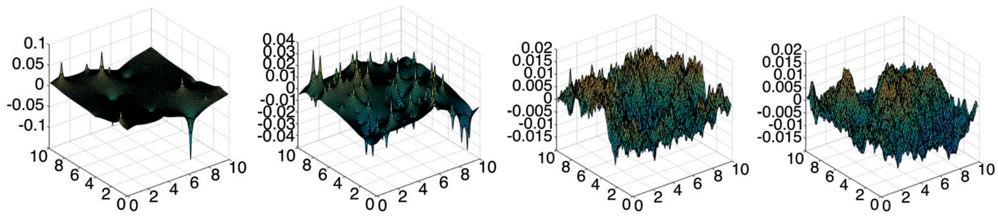
(a) $\lambda = 0.1$ (b) $\lambda = 1$ (c) $\lambda = 50$ (d) $\lambda \to \infty$

Figure 4.5 – 3-D representation of the processes in Figure 4.4.



(a) $\lambda = 0.1$ (b) $\lambda = 1$ (c) $\lambda = 50$ (d) $\lambda \to \infty$

Figure 4.6 – Processes whitened by $(-\Delta)^{\gamma/2}$, $\gamma = 1.5$. In (a)-(c), Gauss-Poisson noises. In (d), Gaussian noise.

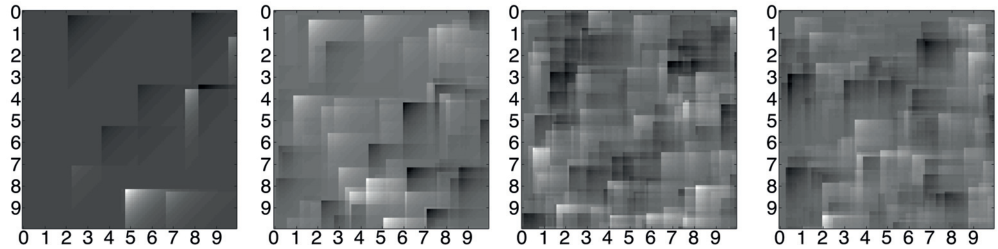


(a) $\lambda = 0.1$ (b) $\lambda = 1$ (c) $\lambda = 50$ (d) $\lambda \to \infty$

Figure 4.7 – 3-D representation of the processes in Figure 4.6

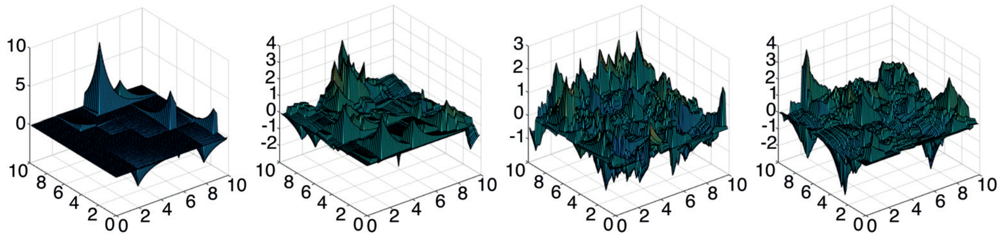
- $(-\Delta)^{\gamma/2} s = w$, with parameter $\gamma > 0$.

We represent our 2-D examples in two ways: first as an image, with gray levels that correspond to the amplitude of the process (lowest value is dark, highest value is white); second as a 3-D plot. All processes are plotted on $[0,10]^2$. In Figures 4.4 and 4.5, we show a Gaussian process with D as whitening operator. A Gaussian process generated by the fractional Laplacian $(-\Delta)^{\gamma/2}$ is illustrated in Figures 4.6 and 4.7. Finally, we plot in Figures 4.8 and 4.9 a Laplace process generated by $\mathrm{D} + \alpha\mathrm{I}$. We always first show the process generated with an appropriate Poisson noise with increasing λ and then plot the processes obtained from the corresponding Lévy noise.



(a) $\lambda = 0.1$ (b) $\lambda = 1$ (c) $\lambda = 50$ (d) $\lambda \to \infty$

Figure 4.8 – Processes whitened by $(\mathrm{D}_x + \alpha \mathrm{I})(\mathrm{D}_y + \alpha \mathrm{I})$, $\alpha = 0.1$. In (a)-(c), Laplace-Poisson noises. In (d), Laplace noise.



(a) $\lambda = 0.1$ (b) $\lambda = 1$ (c) $\lambda = 50$ (d) $\lambda \to \infty$

Figure 4.9 – 3-D representation of the processes in Figure 4.8.

4.3 Scaling Limits of Generalized Lévy Processes

In this section, we focus on the impact of rescaling operations for a broad class of generalized Lévy processes that are asymptotically self-similar. Consider a solution s of the stochastic (pseudo-)differential equation $\mathrm{L}s = w$, with w a d-dimensional Lévy noise and L a linear, continuous, and shift invariant operator. Our aim is to study the statistical behavior of the rescaling $\boldsymbol{x} \mapsto s(\boldsymbol{x}/a)$ of s when $a > 0$ is varying. The two questions we focus on are:

- What is the asymptotic behavior of $s(\cdot/a)$ when we zoom out the process (*i.e.*, when $a \to 0$)?
- What is the asymptotic behavior when we zoom in (*i.e.*, when $a \to \infty$)?

Our main contribution is to identify sufficient conditions such that the rescaling $a^H s(\cdot/a)$ has a self-similar asymptotic limit as a goes to 0 or ∞. When this limit exists, the parameter H is unique and depends essentially on the degree of homogeneity γ of L and on the indices α_{loc} and α_{asymp} of w introduced in Definition 2.8. These indices are used in the literature to characterize the local and asymptotic behaviors of Lévy processes that are not self-similar [BG61, Pru81, BSW14].

This section is based on two publications [FBU15, FBU14]. In [FBU15], we study the coarse scale behavior of finite-variance generalized Lévy processes and apply our results to the wavelet expansion of wide-sense self-similar sparse processes. This work also contains statistical experiments on real-world images and is an extension of an earlier conference paper [FBU14]. We address the general case in [FU16], both at coarse and fine scales, for possibly infinite-variance processes. The organization of this section is mainly based on this second publication.

4.3.1 Self-Similar Generalized Lévy Processes

The study of self-similar processes and self-similar fields is a branch of probability theory [EM00]. Self-similar processes and fields have been applied in areas such as signal and image processing [BU07, FBU15, PPV02] or traffic network [LTWW94, MRRS02], among others [Man82]. Many notorious random processes are self-similar, starting with fractional Brownian motions [MN68] and their higher-order extensions [PHBJ$^+$01]. It also allows for infinite-variance stable processes [ST94] and their fractional versions [HL07]. Self-similar random fields have also been investigated both in the Gaussian [BS81, Dob79, LSSW16, TU10] and the α-stable case [ARX07, BS81].

Self-similar processes are intimately linked with stable laws [ST94]. Stable laws are indeed known to be the only possible probabilistic limits of the renormalized sum of independent and identically distributed random variables: This is the well-known (generalized) central-limit theorem. From this result, self-similar processes are scaling limits of many discretization schemes and stochastic models [Sur81, BEK10, BD09, DGP09, KLNS07, Sin76].

We recall (Definition 2.19) that a generalized random process s is self-similar of order H if $a^H s(\cdot/a)$ and s have the same law for all $a > 0$. The parameter H is often referred to as the Hurst exponent of s. The coarse and fine scale behaviors of a self-similar process are obvious, since the law of the process is not changed by scaling, up to renormalization.

Here, we consider generalized Lévy processes solution of (pseudo-)differential equations of the form

$$\mathrm{L}_\gamma s = w_\alpha, \tag{4.10}$$

with L_γ a γ-homogeneous operator with adequate invertibility properties and w a SαS noise.

Proposition 4.3. *Assume that*

- $w = w_\alpha$ *is a SαS noise with* $\alpha \in (0,2]$, *and*
- $\mathrm{T} = \mathrm{T}_\gamma$ *is a linear, continuous, and* $(-\gamma)$*-homogeneous operator from* $\mathscr{S}(\mathbb{R}^d)$ *to* $L_\alpha(\mathbb{R}^d)$.

Then, the generalized random process s with characteristic functional

$$\widehat{\mathscr{P}}_s(\varphi) = \exp\left(-c^\alpha \|\mathrm{T}_\gamma\{\varphi\}\|_\alpha^\alpha\right),$$

where $c > 0$, *is well-defined, self-similar, with Hurst exponent*

$$H = \gamma + d\left(\frac{1}{\alpha} - 1\right). \tag{4.11}$$

In particular, when T_γ *is a left-inverse of the adjoint* L_γ^* *of a γ-homogeneous whitening operator* L_γ, *then s is a self-similar generalized Lévy process solution of* (4.10).

Proof. First, the domain of definition of the SαS noise w_α is $L_\alpha(\mathbb{R}^d)$ (Proposition 3.19). Therefore, the assumption on T_γ ensure that s is well-defined, according to Theorem 3.5.
Fix H according to (4.11). Then, $H + d = \gamma + d/\alpha$ and we have, for any $\varphi \in \mathscr{S}(\mathbb{R}^d)$,

$$\begin{aligned}
\widehat{\mathscr{P}}_s(a^{H+d}\varphi(a\cdot)) &= \widehat{\mathscr{P}}_s(a^{\gamma+d/\alpha}\varphi(a\cdot)) \\
&= \exp\left(-c^\alpha \|a^{\gamma+d/\alpha}\mathrm{T}_\gamma\{\varphi(a\cdot)\}\|_\alpha^\alpha\right) \\
&= \exp\left(-c^\alpha \|a^{d/\alpha}\{\mathrm{T}_\gamma\varphi\}(a\cdot)\|_\alpha^\alpha\right) && (4.12) \\
&= \exp\left(-c^\alpha \|\mathrm{T}_\gamma\varphi\|_\alpha^\alpha\right) && (4.13) \\
&= \widehat{\mathscr{P}}_s(\varphi),
\end{aligned}$$

where we used respectively the $(-\gamma)$-homogeneity of T_γ and the change of variable $\boldsymbol{y} = a\boldsymbol{x}$ in (4.12) and (4.13). According to Proposition 2.15, this implies that s is self-similar with Hurst exponent H. □

Remark. We should comment on the assumptions of Proposition 4.3. First, in order to have a well-defined generalized Lévy process s, we require T_γ to be at least continuous form $\mathscr{S}(\mathbb{R}^d)$ to $L(w_\alpha) = L_\alpha(\mathbb{R}^d)$ (see Theorem 3.5). The additional assumption is on the homogeneity of T_γ. If T_γ is a homogeneous left-inverse of a γ-homogeneous operator L_γ^*, then the order of homogeneity of T_γ is necessarily $(-\gamma)$. However, we do not know *a priori* whether one can select a left-inverse with this invariance. The construction of such a stable homogeneous left-inverse is not straightforward, as seen for instance the case of the fractional Laplacian studied in [SU12]. This assumption is nevertheless crucial to ensures the self-similarity of s.

Among the classes of generalized Lévy processes introduced in Section 3.3.2, the self-similarity is achieved under two conditions. First, the underlying Lévy noise must be stable. Second, the adjoint of L_γ must admit a $(-\gamma)$-homogeneous left-inverse with the adequate stability properties.

- *SαS noise:* Any d-dimensional stable noise is self-similar. Stable noises are actually the only self-similar Lévy noise [EM00, Theorem 4.2]. The complete family of stable laws is presented for instance in [ST94]. Here, we restrict ourselves to symmetric ones, called SαS (see Section 2.1.3 for more details). All the self-similar generalized Lévy processes that we consider are driven by SαS noise. The Hurst exponent of a SαS noise is $H = d\,(1/\alpha - 1)$.

- *SαS processes:* The operator $\mathrm{D}^* = -\mathrm{D}$ admits a (-1)-homogeneous left-inverse with adequate stability property (it is the operator I_0 introduced in 2.2.2). Therefore, the Lévy process driven by a SαS noise is self-similar. Its Lévy exponent is $H = 1 + (1/\alpha - 1) = 1/\alpha$.

- *Fractional SαS processes:* The fractional derivative D^γ is γ homogeneous. It admits a $(-\gamma)$-homogeneous that is continuous from $\mathscr{S}(\mathbb{R}^d)$ to $L_\alpha(\mathbb{R}^d)$ if $\alpha \geq 1$, $\gamma > 0$, and $\gamma - 1 + 1/\alpha \notin \mathbb{N}$ [UT14]. Under this condition, the fractional SαS process exists according to Theorem 3.5 and is self-similar with Hurst exponent $H = \gamma + (1/\alpha - 1)$. In particular, we recover the fractional Brownian motion when $\alpha = 2$, which gives $H = \gamma - 1/2$.

- *SαS sheets:* We now consider generalized random fields. The d-dimensional Lévy sheets driven by a SαS is also self-similar. It is based on the $(-d)$-homogeneous left-inverse of the adjoint of $\Lambda = \mathrm{D}^{(1,\dots,1)}$ introduced for instance in [FAU14, Section 4.2]. Its Hurst exponent is $H = d/\alpha$.

- *Fractional Lévy fields:* The fractional Laplacian $(-\Delta)^{\gamma/2}$ of order $\gamma > 0$ admits a continuous and $(-\gamma)$-homogeneous left-adjoint from $\mathscr{S}(\mathbb{R}^d)$ to $L_\alpha(\mathbb{R}^d)$ as soon as $\alpha \geq 1$, $\gamma \notin \mathbb{N}$, and $\gamma - d(1 - 1/\alpha) \notin \mathbb{N}$ [SU12]. Using Theorem 3.5, one can construct the generalized Lévy process such that $(-\Delta)^{\gamma/2} s = w_\alpha$. Then, the process s is self-similar with Hurst exponent $H = \gamma + d\,(1/\alpha - 1)$.

4.3.2 Generalized Lévy Processes at Coarse and Fine Scales

The self-similarity imposes a strong constraint on the law of the random process. In particular, it intimately links the behaviors at coarse and fine scales. Many phenomenon are adequately modeled by self-similar processes [Man97]. However, it can also appear to be too restrictive. An advantage of the general class of Lévy processes is to overcome this restriction. Poisson processes are dramatic examples that are piecewise constant and possibly self-similar at coarse scales as we shall see. As such, they can be used as stochastic models for piecewise constant signals [UT11]. In the study of many physical systems, the Cauchy process (also referred to as the Lévy flight) share many good properties with the observations, while the variance of the phenomenon is by essence finite. This motivated the construction of the so-called truncated Lévy flight, that allows for a tradeoff between this two *a priori* contradictory requirements [MS94]. More generally, Rosinski introduced tempered stable processes that are α-stable at fine scale (with $0 < \alpha < 2$) and Gaussian at coarse scales [Ros07]. Here we consider the general problem of characterizing the coarse and large scale behaviors of generalized Lévy processes.

Inspired by Theorem 3.5, we study random processes s with characteristic functional of the form

$$\widehat{\mathscr{P}}_s(\varphi) = \widehat{\mathscr{P}}_w(\mathrm{T}_\gamma\{\varphi\}) \tag{4.14}$$

with w a Lévy noise and T_γ a linear and continuous operator from $\mathscr{S}(\mathbb{R}^d)$ to $L(w)$.

We have seen in Section 4.3.1 that two ingredients are sufficient to make a generalized Lévy process self-similar: the self-similarity of the Lévy noise and the homogeneity of the left-inverse operator appearing in (4.14). This second point is the reason why we index the operator with γ, the order of homogeneity of the underlying whitening operator L_γ. Moreover, the self-similarity of a Lévy noise is equivalent to the stability of the underlying infinitely divisible random variable [ST94]. Even if generalized Lévy processes are not self-similar in general, one can recover some self-similarity by zooming the process in or out.

Definition 4.3. *We say that the generalized random process s is* asymptotically self-similar of order H_{asymp} *if the rescaled processes $a^{H_{\mathrm{asymp}}} s(\cdot/a)$ converge in law to a non-trivial H_{asymp}-self-similar process as $a \to 0$.*

We say that the generalized random process s is locally self-similar of order H_{loc} *if the rescaled processes $a^{H_{\mathrm{loc}}} s(\cdot/a)$ converge in law to a non-trivial H_{loc}-self-similar process as $a \to \infty$.*

The main issues that remain are the following: When is a generalized Lévy process asymptotically self-similar, when is it locally self-similar, and, if so, what are the asymptotic and local Hurst exponents?

One crucial question is the compatibility of the noise w and the operator L_γ, through the operator T_γ). We are used to this for well-defined processes s. Here, we reinforce the stability properties for the left-inverse T_γ so that L_γ is also compatible with the adequate SαS noise. In what follows, we consider two scenarios:

- **Condition (C1):** We assume that T_γ is continuous from $\mathscr{S}(\mathbb{R}^d)$ to $\mathscr{R}(\mathbb{R}^d)$, with no restriction on the noise w.

- **Condition (C2):** We assume that T_γ is continuous from $\mathscr{S}(\mathbb{R}^d)$ to $L_p(\mathbb{R}^d)$ and that $L_p(\mathbb{R}^d)$ is embedded into $L(w)$ for some adequate value of $p \in (0,2]$.

The whitening operators presented in Section 2.2.2 satisfy one of these two properties. Typically, differential operators satisfy (C1), while pseudo-differential (or fractional) ones meet (C2). These assumptions will be discussed later.

In order to emphasize the different assumptions, we analyse the coarse and fine scale behavior separately even if the methods of proof are similar. The relevant parameter of the underlying white noise is the index α_{asymp} at coarse scales and α_{loc} at fine scales.

Theorem 4.3. *Let* L_γ *be a homogeneous whitening operator of order* $\gamma \geq 0$ *and* w *be a Lévy noise with Lévy exponent* Ψ *and asymptotic index* $0 < \alpha_{\text{asymp}} \leq 2$*. We assume that there exists a* $(-\gamma)$*-homogeneous left-inverse* T_γ *of* L_γ^* *that satisfies one of the two following conditions.*

- ***Condition (C1):*** T_γ *is continuous from* $\mathcal{S}(\mathbb{R}^d)$ *to* $\mathcal{R}(\mathbb{R}^d)$*, or*
- ***Condition (C2):*** T_γ *is continuous from* $\mathcal{S}(\mathbb{R}^d)$ *to* $L_{\min(\alpha_{\text{asymp}},2)}(\mathbb{R}^d)$ *and the Lévy exponent is bounded as* $|\Psi(\xi)| \leq M|\xi|^{\min(\alpha_{\text{asymp}},2)}$ *for some constant* $M > 0$*.*

Let s *be the generalized Lévy process with characteristic function* $\widehat{\mathscr{P}}_s(\varphi) = \widehat{\mathscr{P}}_w(\mathrm{T}_\gamma\varphi)$*. Then, if the Lévy exponent* Ψ *satisfies* $\Psi(\xi) \underset{0}{\sim} -C|\xi|^{\min(\alpha_{\text{asymp}},2)}$ *for some constant* $C > 0$*, we have the convergence in law*

$$a^{\gamma+d\left(\frac{1}{\min(\alpha_{\text{asymp}},2)}-1\right)} s(\cdot/a) \xrightarrow[a\to 0]{(\mathscr{L})} s_{\mathrm{L}_\gamma,\min(\alpha_{\text{asymp}},2)}, \tag{4.15}$$

where $\mathrm{L}_\gamma s_{\mathrm{L}_\gamma,\min(\alpha_{\text{asymp}},2)} \overset{(\mathscr{L})}{=} w_{\min(\alpha_{\text{asymp}},2)}$ *is a* $S\alpha S$ *white noise with* $\alpha = \min(\alpha_{\text{asymp}},2)$*. In particular, the process* s *is asymptotically self-similar with asymptotic Hurst exponent*

$$H_{\text{asymp}} = \gamma + \frac{d}{\min(\alpha_{\text{asymp}},2)} - d.$$

Proof. For this proof, we set $\alpha = \min(\alpha_{\text{asymp}},2)$. Assume first that (C1) holds. Then, T_γ is continuous over $L(w)$ and $L(w_\alpha) = L_\alpha(\mathbb{R}^d)$ since $\mathcal{R}(\mathbb{R}^d)$ is embedded in the domain of definition of any Lévy noise. We can therefore apply Theorem 3.5 to deduce that both s and $s_{\mathrm{L}_\gamma,\alpha}$ are well-defined. Now, if (C2) holds, then the bound of Ψ implies that $L_\alpha(\mathbb{R}^d) \subseteq L(w)$, and T_γ is still continuous from $\mathcal{S}(\mathbb{R}^d)$ to $L(w)$. The processes are thus again well-defined with Theorem 3.5.

By the Lévy-Fernique theorem (Theorem 4.1), we know in addition that the convergence in law (4.15) is equivalent to the pointwise convergence of the characteristic functionals. Hence, we have to prove that, for every $\varphi \in \mathcal{S}(\mathbb{R}^d)$,

$$\log \widehat{\mathscr{P}}_{a^{\gamma+d(1/\alpha-1)}s(\cdot/a)}(\varphi) \underset{a\to 0}{\longrightarrow} \log \widehat{\mathscr{P}}_{w_\alpha}(\mathrm{T}_\gamma\varphi) = -C\|\mathrm{T}_\gamma\varphi\|_\alpha^\alpha. \tag{4.16}$$

Let $\varphi \in \mathcal{S}(\mathbb{R}^d)$. Then, we have

$$\begin{aligned}\langle a^{\gamma+d(1/\alpha-1)}s(\cdot/a),\varphi\rangle &= \langle w, a^{\gamma+d/\alpha}\varphi(a\cdot)\rangle \\ &= \langle w, \mathrm{T}_\gamma\{a^{\gamma+d/\alpha}\varphi(a\cdot)\}\rangle \qquad (4.17)\\ &= \langle w, a^{d/\alpha}\{\mathrm{T}_\gamma\varphi\}(a\cdot)\rangle, \qquad (4.18)\end{aligned}$$

where we have used that $\langle s,\varphi\rangle = \langle w,\mathrm{T}_\gamma\varphi\rangle$ and the $(-\gamma)$-homogeneity of T in (4.17) and (4.18), respectively. Therefore, we have

$$\begin{aligned}\log \widehat{\mathscr{P}}_{a^{\gamma+d(1/\alpha-1)} s(\cdot/a)}(\varphi) &= \log \widehat{\mathscr{P}}_w(a^{d/\alpha}\{\mathrm{T}_\gamma\varphi\}(a\cdot)) \\ &= \int_{\mathbb{R}^d} \Psi(a^{d/\alpha}\{\mathrm{T}_\gamma\varphi\}(a\boldsymbol{x}))\mathrm{d}\boldsymbol{x} \\ &= \int_{\mathbb{R}^d} \left(a^{-d}\Psi(a^{d/\alpha}\mathrm{T}_\gamma\varphi(\boldsymbol{y})\right)\mathrm{d}\boldsymbol{y}. \end{aligned} \tag{4.19}$$

By assumption on Ψ, we moreover have that, for every $\boldsymbol{y} \in \mathbb{R}^d$,

$$a^{-d}\Psi(a^{d/\alpha}\mathrm{T}_\gamma\varphi(\boldsymbol{y})) \underset{a\to 0}{\longrightarrow} -C\left|\mathrm{T}_\gamma\varphi(\boldsymbol{y})\right|^\alpha.$$

We split the proof in two parts depending on whether T_γ and Ψ satisfy (C1) or (C2).

- We start with (C2). The bound on Ψ implies that

$$\left|a^{-d}\Psi(a^{d/\alpha}\mathrm{T}_\gamma\varphi(\boldsymbol{y}))\right| \le M\left|\mathrm{T}_\gamma\varphi(\boldsymbol{y})\right|^\alpha \tag{4.20}$$

  for every $\boldsymbol{y} \in \mathbb{R}^d$. The right term of (4.20) is integrable by assumption on T_γ. The Lebesgue dominated convergence theorem therefore applies and (4.16) is proven.

- We assume now (C1). In that case, we do not have a full bound on Ψ. However, we know that $\mathrm{T}_\gamma\varphi$ is bounded, so that $\|\mathrm{T}_\gamma\varphi\|_\infty < \infty$. Since Ψ is continuous and behaves like $(-C|\xi|^\alpha)$ at 0, there exists $M > 0$ such that $|\Psi(\xi)| \le M|\xi|^\alpha$ for every $|\xi| \le \|\mathrm{T}_\gamma\varphi\|_\infty$. Hence, for all $a \le 1$, we have $\left|a^{d/\alpha}\mathrm{T}_\gamma\varphi(\boldsymbol{y})\right| \le 1$, and (4.20) is still valid. Again, we deduce (4.16) from the Lebesgue dominated convergence theorem.

□

Theorem 4.4. *Let* L_γ *be a homogeneous whitening operator of order* $\gamma \ge 0$ *and* w *be a Lévy noise with Lévy exponent* Ψ *and Blumenthal-Getoor index* $0 < \alpha_{\mathrm{loc}} \le 2$. *We assume that there exists a* $(-\gamma)$*-homogeneous left-inverse* T_γ *of* L_γ^* *that satisfies one of the two following conditions.*

- ***Condition (C1):*** T_γ *is continuous from* $\mathscr{S}(\mathbb{R}^d)$ *to* $\mathscr{R}(\mathbb{R}^d)$, *or*

- ***Condition (C2):*** T_γ *is continuous from* $\mathscr{S}(\mathbb{R}^d)$ *to* $L_{\alpha_{\mathrm{loc}}}(\mathbb{R}^d)$ *and the Lévy exponent is bounded as* $|\Psi(\xi)| \le M|\xi|^{\alpha_{\mathrm{loc}}}$ *for some constant* $M > 0$.

Let s *be the generalized Lévy process with characteristic function* $\widehat{\mathscr{P}}_s(\varphi) = \widehat{\mathscr{P}}_w(\mathrm{T}_\gamma\varphi)$. *Then, if the Lévy exponent* Ψ *satisfies* $\Psi(\xi) \underset{\infty}{\sim} -C|\xi|^{\alpha_{\mathrm{loc}}}$ *for some constant* $C > 0$, *we have the convergence in law*

$$a^{\gamma+d\left(\frac{1}{\alpha_{\mathrm{loc}}}-1\right)} s(\cdot/a) \underset{a\to\infty}{\overset{(\mathscr{L})}{\longrightarrow}} s_{\mathrm{L}_\gamma,\alpha_{\mathrm{loc}}},$$

where $\mathrm{L}s_{\mathrm{L}_\gamma,\alpha_{\mathrm{loc}}} \overset{(\mathscr{L})}{=} w_{\alpha_{\mathrm{loc}}}$ *is a* $S\alpha S$ *white noise with* $\alpha = \alpha_{\mathrm{loc}}$. *In particular, the process* s *is locally* *self-similar with local Hurst exponent*

$$H_{\text{loc}} = \gamma + \frac{d}{\alpha_{\text{loc}}} - d.$$

Proof. The proof is very similar to the one of Theorem 4.3, thus we only develop the parts that differ. If T_γ and Ψ satisfy (C2), the proof follows exactly the line of Theorem 4.3. We should therefore assume that T_γ continuously maps $\mathscr{S}(\mathbb{R}^d)$ to $\mathscr{R}(\mathbb{R}^d)$. Restarting from (4.19) with α_{loc} instead of $\min(\alpha_{\text{asymp}}, 2)$, we split the integral into two parts and get

$$\begin{aligned}\log \widehat{\mathscr{P}}_{a^{\gamma+d(1/\alpha_{\text{loc}}-1)}s(\cdot/a)}(\varphi) &= \int_{\mathbb{R}^d} \mathbb{1}_{|\mathrm{T}_\gamma\varphi(\boldsymbol{y})|a^{d/\alpha_{\text{loc}}}\geq 1}\, a^{-d}\Psi(a^{d/\alpha_{\text{loc}}}\mathrm{T}_\gamma\varphi(\boldsymbol{y}))\mathrm{d}\boldsymbol{y} \\ &\quad + \int_{\mathbb{R}^d} \mathbb{1}_{|\mathrm{T}_\gamma\varphi(\boldsymbol{y})|a^{d/\alpha_{\text{loc}}}< 1}\, a^{-d}\Psi(a^{d/\alpha_{\text{loc}}}\mathrm{T}_\gamma\varphi(\boldsymbol{y}))\mathrm{d}\boldsymbol{y} \\ &:= I(a) + J(a).\end{aligned}$$

Control of $I(a)$: By assumption on Ψ, we have that, for any $\boldsymbol{y} \in \mathbb{R}^d$,

$$\mathbb{1}_{|\mathrm{T}_\gamma\varphi(\boldsymbol{y})|a^{d/\alpha_{\text{loc}}}\geq 1}\, a^{-d}\Psi(a^{d/\alpha_{\text{loc}}}\mathrm{T}_\gamma\varphi(\boldsymbol{y})) \underset{a\to\infty}{\longrightarrow} -C\left|\mathrm{T}_\gamma\varphi(\boldsymbol{y})\right|^{\alpha_{\text{loc}}}.$$

Moreover, since the continuous function Ψ asymptotically behaves like $(-C|\xi|^{\alpha_{\text{loc}}})$, there exists a constant C' such that $|\Psi(\xi)| \leq C'|\xi|^{\alpha_{\text{loc}}}$ for every ξ with $|\xi| \geq 1$. The function $\mathrm{T}_\gamma\varphi$, which is in $\mathscr{R}(\mathbb{R}^d)$, is bounded. Hence, for any $a > 0$, we have that

$$\begin{aligned}\left|\mathbb{1}_{|\mathrm{T}_\gamma\varphi(\boldsymbol{y})|a^{d/\alpha_{\text{loc}}}\geq 1}\, a^{-d}\Psi(a^{d/\alpha_{\text{loc}}}\mathrm{T}_\gamma\varphi(\boldsymbol{y}))\right| &\leq C'\mathbb{1}_{|\mathrm{T}_\gamma\varphi(\boldsymbol{y})|a^{d/\alpha_{\text{loc}}}\geq 1}\left|\mathrm{T}_\gamma\varphi(\boldsymbol{y}))\right|^{\alpha_{\text{loc}}} \\ &\leq C'\left|\mathrm{T}_\gamma\varphi(\boldsymbol{y})\right|^{\alpha_{\text{loc}}}\end{aligned}$$

for all $\boldsymbol{y} \in \mathbb{R}^d$. The function on the right is integrable, and the Lebesgue dominated convergence thus applies We obtain finally that $I(a) \underset{a\to\infty}{\longrightarrow} -C\|\mathrm{T}_\gamma\varphi\|_{\alpha_{\text{loc}}}^{\alpha_{\text{loc}}}$.
Control of $J(a)$: The Lévy noise being tempered and, according to Proposition 2.4, there exists $C' > 0$ and $\epsilon > 0$ such that $|\Psi(\xi)| \leq C'(|\xi|^\epsilon + |\xi|^2)$. Without loss of generality, one can choose $\epsilon < \alpha_{\text{loc}}$. Then, for $|\xi| \leq 1$, we have $|\Psi(\xi)| \leq 2C'|\xi|^\epsilon$ and, therefore,

$$\left|\int_{\mathbb{R}^d} \mathbb{1}_{|\mathrm{T}_\gamma\varphi(\boldsymbol{y})|a^{d/\alpha_{\text{loc}}}< 1}\, a^{-d}\Psi(a^{d/\alpha_{\text{loc}}}\mathrm{T}_\gamma\varphi(\boldsymbol{y}))\mathrm{d}\boldsymbol{y}\right| \leq 2C' a^{d(\epsilon/\alpha_{\text{loc}}-1)}\|\mathrm{T}_\gamma\varphi\|_\epsilon^\epsilon.$$

Since $\mathscr{R}(\mathbb{R}^d) \subset L_\epsilon(\mathbb{R}^d)$ and $\epsilon < \alpha_{\text{loc}}$, we have $\|\mathrm{T}_\gamma\varphi\|_\epsilon^\epsilon < \infty$ and $a^{d(\epsilon/\alpha_{\text{loc}}-1)} \underset{a\to\infty}{\longrightarrow} 0$, which implies that $J(a) \underset{a\to\infty}{\longrightarrow} 0$. Finally, we have shown that

$$\log \widehat{\mathscr{P}}_{a^{\gamma+d(1/\alpha_{\text{loc}}-1)}s(\cdot/a)}(\varphi) = I(a) + J(a) \underset{a\to\infty}{\longrightarrow} -C\left|\mathrm{T}_\gamma\varphi(\boldsymbol{y})\right|^{\alpha_{\text{loc}}},$$

as expected. □

Remarks

- The renormalization procedures in Theorems 4.3 and 4.4 have to be compared with the index $H = \gamma + d(1/\alpha - 1)$ of a self-similar generalized Lévy process (see Proposition 4.3). One can say that the lack of self-similarity of s is asymptotically or locally removed.

- (C1) has to be understood as the sufficient assumption on the operator T_γ such that the process s with characteristic functional $\widehat{\mathscr{P}}_w(\mathrm{T}_\gamma\varphi)$ is well-defined without any additional assumption on the Lévy white noise w. Therefore, (C1) is restrictive for the operator but not for the noise.

- The previous remark is in contrast to (C2). Here, the restriction on T_γ is minimal since the process $s_{\mathrm{L}_\gamma,\alpha}$ should be well-defined for $\alpha = \min(\alpha_{\mathrm{asymp}}, 2)$ or $\alpha = \alpha_{\mathrm{loc}}$. Therefore, T_γ should at least map $\mathscr{S}(\mathbb{R}^d)$ into $L_\alpha(\mathbb{R}^d)$. It means that (C2) gives sufficient assumptions on the Lévy white noise (more precisely on the bound of the Lévy exponent) such that the minimal assumption on T_γ is also sufficient.

- When the variance of the noise is finite, we have in particular that $\alpha_{\mathrm{asymp}} \geq 2$, and therefore $\min(\alpha_{\mathrm{asymp}}, 2) = 2$. Under the assumptions of Theorem 4.3, the process $a^{\gamma - d/2} s(\cdot/a)$ converges to a Gaussian self-similar process. This can be seen as a central limit theorem for finite-variance generalized Lévy processes.

- For important classes of Lévy white noises, the parameter α_{loc} vanishes, and Theorem 4.4 does not apply. This includes generalized Laplace noises and compound Poisson noises (see Section 3.1.2). In that case, the underlying processes do not admit any scaling limit at fine scales, at least when T_γ satisfies (C1), as shown in Proposition 4.4.

Proposition 4.4. *Let L_γ be a homogeneous whitening operator of order $\gamma \geq 0$ and w be a Lévy noise with Lévy exponent Ψ and Blumenthal-Getoor index $\alpha_{\mathrm{loc}} = 0$. We assume that there exists a $(-\gamma)$-homogeneous left-inverse T_γ of L_γ^*, continuous from $\mathscr{S}(\mathbb{R}^d)$ to $\mathscr{R}(\mathbb{R}^d)$. Let s be the generalized Lévy process with characteristic function $\widehat{\mathscr{P}}_s(\varphi) = \widehat{\mathscr{P}}_w(\mathrm{T}_\gamma\varphi)$. Then, for every $H \in \mathbb{R}$,*

$$a^H s(\cdot/a) \xrightarrow[a\to\infty]{(\mathscr{L})} 0.$$

Proof. Due to the Lévy-Fernique theorem, we have to show that, for every $\varphi \in \mathscr{S}(\mathbb{R}^d)$,

$$\log \widehat{\mathscr{P}}_{a^H s(\cdot/a)}(\varphi) \underset{a\to\infty}{\longrightarrow} 0.$$

Proceeding as in Theorem 4.3, we easily show that

$$\log \widehat{\mathscr{P}}_{a^H s(\cdot/a)}(\varphi) = \int_{\mathbb{R}^d} a^{-d} \Psi(a^{d+H} \mathrm{T}_\gamma \varphi(\boldsymbol{y})) \mathrm{d}\boldsymbol{y}. \tag{4.21}$$

According to Proposition 2.4, there exists $\epsilon, C' > 0$ such that $|\Psi(\xi)| \leq C' |\xi|^\epsilon$ for $|\xi| \leq 1$. Without loss of generality, one can assume that $\epsilon < \frac{d}{d+|H|}$. This implies in particular that $\epsilon(d+H) - d < 0$. The knowledge that $\alpha_{\mathrm{loc}} = 0$ is enough to deduce that $\Psi(\xi)$ is also dominated by $|\xi|^\epsilon$ for $|\xi| \geq 1$, and that there exists $C > 0$ such that

$$|\Psi(\xi)| \leq C |\xi|^\epsilon$$

for every $\xi \in \mathbb{R}$. Resuming from (4.21), we obtain that

$$\left|\log \widehat{\mathscr{P}}_{a^H s(\cdot/a)}(\varphi)\right| \leq C \int_{\mathbb{R}^d} a^{\epsilon(d+H)-d} \left|\mathrm{T}_\gamma \varphi(\boldsymbol{y})\right|^\epsilon \mathrm{d}\boldsymbol{y} = C \|\mathrm{T}_\gamma \varphi\|_\epsilon^\epsilon a^{\epsilon(d+H)-d},$$

which vanishes when $a \to \infty$ due to our choice of ϵ. This concludes the proof. □

4.3.3 Examples and Simulations

The processes consider in this section have been introduced in Section 3.3.2.

Lévy processes and sheets. We recall the notation $\Lambda = \mathrm{D}_1 \cdots \mathrm{D}_d$. We consider Lévy sheets solutions of $\Lambda s = w$. The left-inverse of Λ introduced in Section 3.3.2 is $(-d)$-homogeneous and continuous from $\mathscr{S}(\mathbb{R}^d)$ to $\mathscr{R}(\mathbb{R}^d)$. We satisfy therefore the Condition (C1) Applying the results of Section 4.3.2, we directly deduce Proposition 4.5. We denote the SαS Lévy sheet for $\alpha \in (0,2]$ by $s_{\Lambda,\alpha}$.

Proposition 4.5. *Let w be a Lévy noise with indices $\alpha_{\mathrm{asymp}} \in (0,\infty]$ and $\alpha_{\mathrm{loc}} \in [0,2]$ and s the Lévy sheet driven by w.*

- *If $\Psi(\xi) \underset{0}{\sim} -C|\xi|^{\min(\alpha_{\mathrm{asymp}},2)}$ for some $C > 0$, then*

$$a^{d/\min(\alpha_{\mathrm{asymp}},2)} s(\cdot/a) \underset{a\to 0}{\longrightarrow} s_{\Lambda,\min(\alpha_{\mathrm{asymp}},2)}.$$

- *If $\alpha_{\mathrm{loc}} \neq 0$ and $\Psi(\xi) \underset{\infty}{\sim} -C|\xi|^{\alpha_{\mathrm{loc}}}$ for some $C > 0$, then*

$$a^{d/\alpha_{\mathrm{loc}}} s(\cdot/a) \underset{a\to\infty}{\longrightarrow} s_{\Lambda,\alpha_{\mathrm{loc}}}.$$

We illustrate our results in 1-dimension with simulations of Lévy processes. First, we consider three Lévy processes driven by the Laplace white noise, the Gaussian-Poisson white noise, and the Cauchy-Poisson white noise, respectively. We look at the processes at three different scales by representing them on $[0,1]$, $[0,10]$, and $[0,1000]$. We only generate one process of each type and represent it on the different intervals, which corresponds to zooming out of it. The theoretical prediction at large scale is as follows. The Laplace and Poisson-Gaussian process should be statistically indistinguishable from the Brownian motion, while the Poisson-Cauchy process should be statistically indistinguishable from the Cauchy process (also called Lévy flight). We see in Figure 4.10 that this can indeed be observed on simulations. For comparison purposes, we also represent one realization of the expected limit process.

We also depict the difference between fine-scale and coarse scale behaviors. To do so, we consider a Lévy noise w, which is the sum of two independent Gaussian and Cauchy noises. The prediction states that the Lévy process driven by w converges to the Brownian motion at fine scales and to the Cauchy process (or Lévy flight) at coarse scales. Again, this theoretical result is observed on simulations in Figure 4.11, where one realization of the process is represented on $[0,0.1]$ (fine-scale), $[0,10]$ (medium scale), and $[0,1000]$ (coarse scale).

Fractional Lévy Processes and Fields In dimension d, we consider the stochastic differential equation $(-\Delta)^{\gamma/2} s = w$, where $(-\Delta)^{\gamma/2}$ is the fractional Laplacian. The conditions of existence of s were discussed in Section 3.3.2 and we assume that they are satisfied. Again, the direct application of the results of Section 4.3.2 yields Proposition 4.6. We denote by $s_{(-\Delta)^{\gamma/2},\alpha}$ the fractional Lévy process driven by the SαS Lévy noise (assuming that it is well-defined).

Proposition 4.6. *Let w be a Lévy noise with indices $\alpha_{\text{asymp}} \in (0,\infty]$ and $\alpha_{\text{loc}} \in [0,2]$ and s be the fractional Lévy process driven by w (which is assumed to exist).*

- *If $\Psi(\xi) \underset{0}{\sim} -C|\xi|^{\min(\alpha_{\text{asymp}},2)}$ and $|\Psi(\xi)| \leq C'|\xi|^{\min(\alpha_{\text{asymp}},2)}$ for some $C, C' > 0$, then*

$$a^{\gamma+d(1/\min(\alpha_{\text{asymp}},2)-1)} s(\cdot/a) \underset{a\to 0}{\longrightarrow} s_{(-\Delta)^{\gamma/2},\min(\alpha_{\text{asymp}},2)}.$$

- *If $\alpha_{\text{loc}} \neq 0$, $\Psi(\xi) \underset{\infty}{\sim} -C|\xi|^{\alpha_{\text{loc}}}$, and $|\Psi(\xi)| \leq C'|\xi|^{\alpha_{\text{loc}}}$ for some $C, C' > 0$, then*

$$a^{\gamma+d(1/\alpha_{\text{loc}}-1)} s(\cdot/a) \underset{a\to\infty}{\longrightarrow} s_{(-\Delta)^{\gamma/2},\alpha_{\text{loc}}}.$$

In dimension $d = 1$, one can construct generalized Lévy field whitened by the fractional derivative $\mathrm{L} = \mathrm{D}^{\gamma}$. This includes the fractional Brownian motion [MN68] when considering

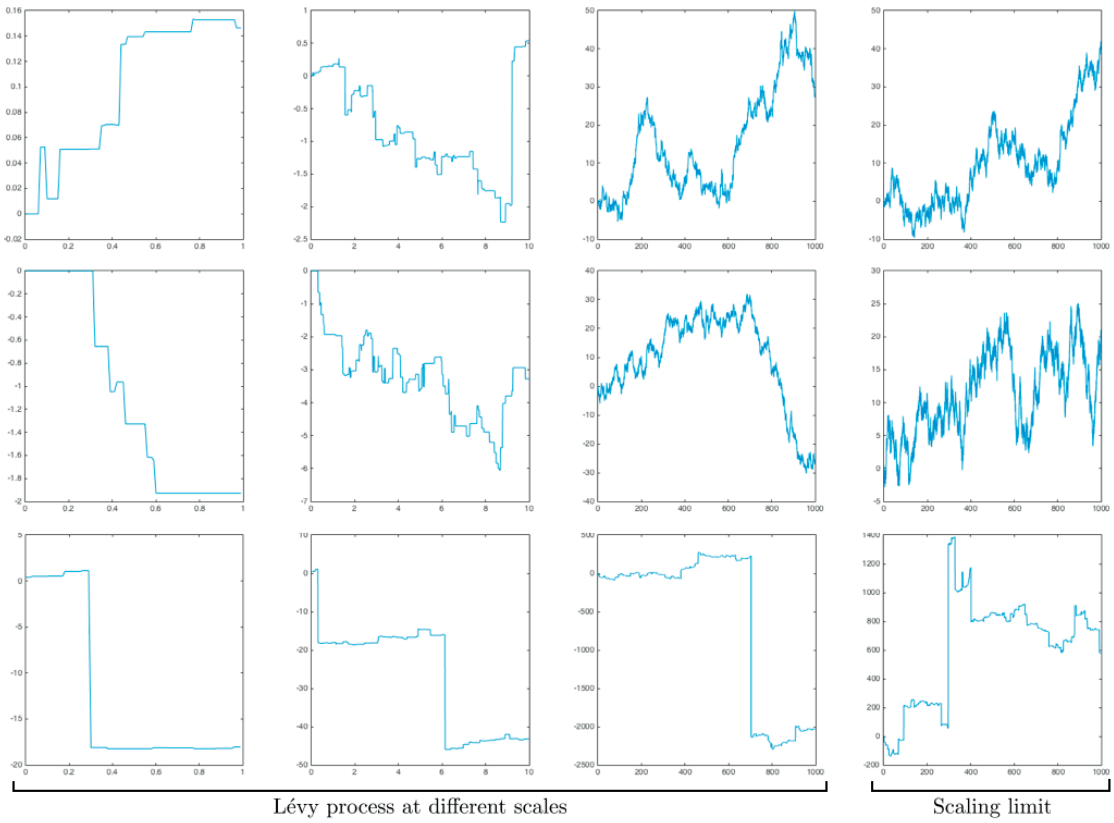


Figure 4.10 – Lévy processes at three different scale and comparison with the corresponding self-similar process at large scale according to Theorem 4.3.

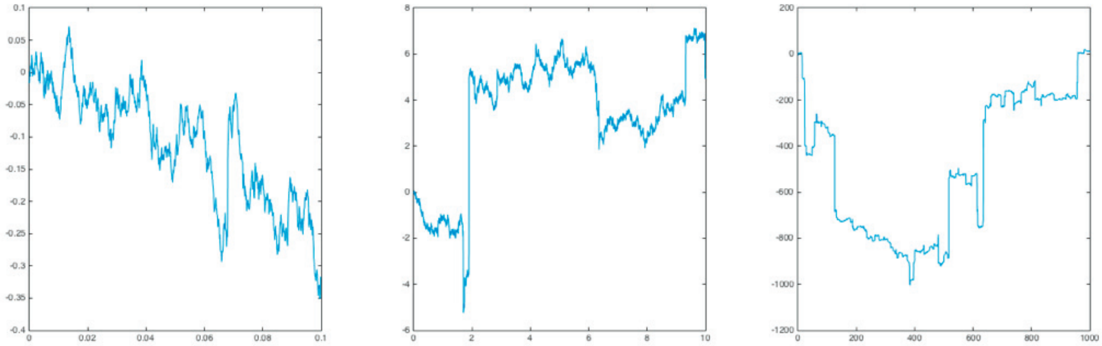


Figure 4.11 – Sum of a Lévy flight and a Brownian motion at three different scales.

the Gaussian noise and Lévy driven generalizations [EW13]. For a left-inverse-based approach in this case, see [UT14, Section 7.5].

5 Regularity of Generalized Lévy Processes

In this chapter, we aim at specifying in which function spaces, associated to different notions of regularity (Hölder, Sobolev, and more generally Besov), the generalized Lévy processes are localized. A special attention will be given to the Lévy noise, for which we identify the local smoothness and the asymptotic decay rate. We then deduce local smoothness of the generalized Lévy processes specified in the periodic framework. This chapter is based on our publications [FUW17b, FFU, AFU], in collaboration with S. Aziznejad, A. Fallah, M. Unser, and J.P. Ward.

5.1 Smoothness and Decay Rate in $\mathscr{S}'(\mathbb{R}^d)$

For us, random processes are constructed as random elements in the space of tempered generalized functions. We will therefore describe their smoothness and decay properties as we would do for a (deterministic) tempered generalized function.
For $p = 2$, we have seen that the space of tempered distribution is the union of the weighted Sobolev spaces (see (2.16)). More generally, if we fix the integrability rate $0 < p \leq \infty$, the space of tempered generalized functions satisfies [Kab08, Proposition 1]

$$\mathscr{S}'(\mathbb{R}^d) = \bigcup_{\tau,\rho\in\mathbb{R}} B_p^\tau(\mathbb{R}^d;\rho), \tag{5.1}$$

where the weighted Besov spaces $B_p^\tau(\mathbb{R}^d;\rho)$ are introduced in Section 2.2.3. Ideally, for $f \in \mathscr{S}'(\mathbb{R}^d)$, we want to identify the set

$$\mathscr{E}_p(f) = \left\{(\tau,\rho) \in \mathbb{R}^2 \;\middle|\; f \in B_p^\tau(\mathbb{R}^d;\rho)\right\}.$$

We remark that $\mathscr{E}_p(f)$ is non-empty due to (5.1). When $\tau_1 \leq \tau_2$ and $\rho_1 \leq \rho_2$, we have the embeddings

$$B_p^{\tau_2}(\mathbb{R}^d;\rho) \subseteq B_p^{\tau_1}(\mathbb{R}^d;\rho),$$
$$B_p^{\tau}(\mathbb{R}^d;\rho_2) \subseteq B_p^{\tau}(\mathbb{R}^d;\rho_1).$$

Thus, if $(\tau_0,\rho_0) \in \mathscr{E}_p(f)$, then

$$(-\infty,\tau_0] \times (-\infty,\rho_0] \subset \mathscr{E}_p(f).$$

Assume that we know two quantities $\tau_p(f) \in (-\infty,\infty]$ and $\rho_p(f) \in (-\infty,\infty]$ such that:

- if $\tau < \tau_p(f)$ and $\rho < \rho_p(f)$, then $f \in B_p^\tau(\mathbb{R}^d;\rho)$; while
- if $\tau > \tau_p(f)$ or $\rho > \rho_p(f)$, then $f \notin B_p^\tau(\mathbb{R}^d;\rho)$.

The case $\tau_p(f) = \infty$ corresponds to infinitely differentiable functions, and $\rho_p(f) = \infty$ means that f is rapidly decaying. When these two quantities are finite, we have that

$$(-\infty,\tau_p(f)) \times (-\infty,\rho_p(f)) \subset \mathscr{E}_p(f) \subset (-\infty,\tau_p(f)] \times (-\infty,\rho_p(f)]. \tag{5.2}$$

The value of $\tau_p(f)$ measures the *local smoothness* of f in the L_p-scale, while $\rho_p(f)$ quantifies its *asymptotic decay rate*. Knowing $\tau_p(f)$ and $\rho_p(f)$ allows to almost completely characterize in which weighted Besov spaces the generalized function f is. The only remaining part is precisely when $\tau = \tau_p(f)$ or $\rho = \rho_p(f)$.

Remark. We do not claim that $\tau_p(f)$ and $\rho_p(f)$ are well-defined for any $f \in \mathscr{S}'(\mathbb{R}^d)$ and $0 < p \leq \infty$. In particular, the space $\mathscr{E}_p(f)$ is not necessarily sandwiched between open and closed separable spaces in $\mathbb{R}^2$ as in (5.2). Nevertheless, the description of the Besov regularity of f is particularly simple when it occurs. This is true for the Lévy noise, as we shall see in

Section 5.2. It is also the case for the Dirac impulse, for which $\tau_p(\delta) = d/p - d$ and $\rho_p(\delta) = +\infty$, as easily deduced from Proposition 2.7.

5.2 Besov Regularity of the Lévy Noise

This section is dedicated to the identification of the local smoothness $\tau_p(w)$ and the asymptotic decay rate $\rho_p(w)$ of the Lévy noise for $0 < p \leq \infty$. We will see that these quantities are well-defined.

Our results are based on the wavelet characterization of Besov spaces exposed in Section 2.2.3, where we used Daubechies wavelet bases. In order to identify a certain Besov regularity, we therefore need to justify that one can analyse the Lévy noise with Daubechies wavelets. We have seen in Section 3.2 that any Lévy noise can be extended as a random linear and continuous functional on its domain of definition (Theorem 3.4). Moreover, the domain of definition always includes the space of compactly supported functions, which is a subspace of $L_{2,0}(\mathbb{R}^d)$ (see Proposition 3.17). This means that the family of random variables

$$(\langle w, \psi_{j,G,\boldsymbol{k}} \rangle)_{j\geq 0,\ G\in \mathrm{G}^j,\ \boldsymbol{k}\in\mathbb{Z}^d} \tag{5.3}$$

is always well-defined in a compatible way (with the notation of Section 2.2.3).

An alternative justification of the well-definiteness of (5.3) was exposed in our works [FFU, FUW17b]. There, we have shown, based on considerations on the characteristic functional, that a Lévy noise is almost surely located in the Sobolev space $W_2^\tau(\mathbb{R}^d;\rho)$ as soon as $\tau < -d/2$ and $\rho < -d/\min(\alpha_{\mathrm{asymp}}, 2)$ [FFU, Proposition 8]. Then, it suffices to take a Daubechies wavelet basis with a sufficient regularity in accordance with Proposition 2.9 to justify the wavelet analysis. The two approaches convey the same message: we can apply Daubechies wavelets to any Lévy noise.

We split the different cases as follows. In Sections 5.2.1 and 5.2.2, we fully characterize the Besov regularity of the Gaussian noise and the compound Poisson noise, respectively. The case of Lévy noise without Gaussian part (or sparse Lévy noise) is treated in Section 5.2.3. We combine and comment all the results in Section 5.2.4.

We briefly present the strategy of the proof, which is similar for the different classes of noise. Showing that the noise is almost surely (almost surely not, respectively) in a given Besov space is called a *positive result* (a *negative result*, respectively). Given a Lévy noise and a Besov space $B_p^\tau(\mathbb{R}^d;\rho)$, we study the random variable

$$\|w\|^p_{B_p^\tau(\mathbb{R}^d;\rho)} = \sum_{j\geq 0} 2^{j(\tau p - d + \frac{dp}{2})} \sum_{G\in\mathrm{G}^j} \sum_{\boldsymbol{k}\in\mathbb{Z}^d} \langle 2^{-j}\boldsymbol{k}\rangle^{\rho p} |\langle w, \psi_{j,G,\boldsymbol{k}}\rangle|^p. \tag{5.4}$$

We assume that we have identified (or guessed) the values $\tau_p(w)$ and $\rho_p(w)$.

- For $\tau < \tau_p(w)$ and $\rho < \rho_p(w)$, we show that $\|w\|_{B_p^\tau(\mathbb{R}^d;\rho)} < \infty$ almost surely. For $p < \alpha_{\mathrm{asymp}}$, we show the strongest result that $\mathbb{E}[\|w\|^p_{B_p^\tau(\mathbb{R}^d;\rho)}] < \infty$. This requires a precise estimation of the behavior of $\mathbb{E}[|\langle w, \psi_{j,G,\boldsymbol{k}}\rangle|^p]$ as j goes to infinity. When $p > \alpha_{\mathrm{asymp}}$, the random variables $\langle w, \psi_{j,G,\boldsymbol{k}}\rangle$ have an infinite pth moment and the method is not applicable. For $p \geq \alpha_{\mathrm{asymp}}$, we actually deduce the result using the embeddings between Besov spaces. It appears that this approach is sufficient to obtain sharp positive results.

- For $\tau > \tau_p(w)$, we show that $\|w\|_{B_p^\tau(\mathbb{R}^d;\rho)} = \infty$ almost surely. To do so, we consider only the mother wavelet (gender $G = M^d$) and restrain the shifts $\boldsymbol{k}$ to retain the lower bound

$$\|w\|^p_{B^\tau_p(\mathbb{R}^d;\rho)} \geq C \sum_{j\geq 0} 2^{j(\tau p - d + \frac{dp}{2})} \sum_{0\leq k_1,\ldots,k_d<2^j} |\langle w, \psi_{j,M^d,\boldsymbol{k}}\rangle|^p, \tag{5.5}$$

with C smaller than $\langle 2^{-j}\boldsymbol{k}\rangle^{\rho p}$ for the considered range of $\boldsymbol{k}$. We need to show then that $|\langle w, \psi_{j,G,\boldsymbol{k}}\rangle|$ cannot be too small, with Borel-Cantelli-type arguments.

- For $\rho > \rho_p(w)$, we show again that $\|w\|_{B^\tau_p(\mathbb{R}^d;\rho)} = \infty$ almost surely. It appears that the evolution among the scale j is not what makes the Besov norm explode. We only consider the father wavelet (gender $G = F^d$) in (5.4), and use the lower bound

$$\|w\|^p_{B^\tau_p(\mathbb{R}^d;\rho)} \geq \sum_{\boldsymbol{k}\in\mathbb{Z}^d} \langle \boldsymbol{k}\rangle^{\rho p} |\langle w, \psi_{0,F^d,\boldsymbol{k}}\rangle|^p. \tag{5.6}$$

Again, a Borel-Cantelli-type argument is used to show that the $|\langle w, \psi_{0,F^d,\boldsymbol{k}}\rangle|$ cannot be too small, and that the Besov norm is almost surely infinite.

5.2.1 Gaussian Noise

The Gaussian case is much simpler than the general one since the wavelet coefficients of the Gaussian noise are independent and identically distributed. We present it separately for three reasons: (i) it can be considered as an instructive toy problem that already contains some of the technicalities that will appear for the general case, (ii) it cannot be deduced from the other sections, where the results are based on a careful study of the Lévy measure, and (iii) the localization of the Gaussian noise in *weighted* Besov spaces has not been addressed in the literature, to the best of our knowledge (for the local Besov regularity, a complete answer was given in [Ver10]). We first state a lemma that will be useful throughout this section.

Lemma 5.1. *Assume that $(X_{\boldsymbol{k}})_{\boldsymbol{k}\in\mathbb{Z}^d}$, is a sequence of i.i.d. nonzero random variables. Then,*

$$\sum_{\boldsymbol{k}\in\mathbb{Z}^d} \frac{|X_{\boldsymbol{k}}|}{\langle \boldsymbol{k}\rangle^d} = \infty \text{ a.s.}$$

Proof. Lemma 5.1 can be easily proved using for instance Kolmogorov's three-series theorem. We propose here a short and self-contained proof. First of all, the result for any dimension d is easily deduced from the one-dimensional case. Moreover, $|k|$ and $\langle k\rangle$ are equivalent asymptotically, so that it is equivalent to show that $\sum_{k\geq 1} \frac{|X_k|}{k} = \infty$ for X_k i.i.d. For $k \geq 1$, we set $Z_k = \frac{1}{2^k}\sum_{l=2^{k-1}}^{2^k-1} |X_l|$, so that

$$\sum_{k\geq 1} \frac{|X_k|}{k} = \sum_{k\geq 1}\sum_{l=2^{k-1}}^{2^k-1} \frac{|X_l|}{l} \geq \sum_{k\geq 1} \frac{1}{2^k}\sum_{l=2^{k-1}}^{2^k-1} |X_l| = \sum_{k\geq 1} Z_k$$

The Z_k are independent because the X_k are. Moreover, we have $\mathbb{E}[Z_k] = \mathbb{E}[|X_1|]$ for all k. The weak law of large numbers ensures that $\mathscr{P}(Z_k > \mathbb{E}[|X_1|]/2)$ goes to 1, therefore $\sum_{k\geq 1}\mathscr{P}(Z_k > \mathbb{E}[|X_1|]/2) = \infty$. Since the events $\{Z_k > \mathbb{E}[|X_1|]/2\}$ are independent, we apply the Borel-Cantelli lemma to deduce that infinitely many Z_k are bigger than $\mathbb{E}[|X_1|]/2$ almost surely. Finally, this implies that $\sum_{k\geq 1} Z_k = \infty$ almost surely and the result is proved. □

Theorem 5.1. *Let* $0 < p < \infty$*. The Gaussian noise* w_{Gauss} *is*

- *almost surely in* $B_p^\tau(\mathbb{R}^d;\rho)$ *if* $\tau < -d/2$ *and* $\rho < -d/p$*, and*
- *almost surely not in* $B_p^\tau(\mathbb{R}^d;\rho)$ *if* $\tau \geq -d/2$ *or* $\rho \geq -d/p$*.*

Proof. Without loss of generality, we assume that the variance of the Gaussian noise is 1.
If $\tau < -d/2$ *and* $\rho < -d/p$. For $p > 0$, we denote by C_p the p-th moment of a Gaussian random variable with mean 0 and variance 1. For the Gaussian noise, $\langle w_{\text{Gauss}}, \varphi_1\rangle$ and $\langle w_{\text{Gauss}}, \varphi_2\rangle$ are independent if and only if $\langle \varphi_1, \varphi_2\rangle = 0$, and $\langle w_{\text{Gauss}}, \varphi\rangle$ is a Gaussian random variable with variance $\|\varphi\|_2^2$ [UT14]. The family of functions $(\psi_{j,G,\boldsymbol{k}})_{j,G,\boldsymbol{k}}$ being orthonormal, the random variables $\langle w_{\text{Gauss}}, \psi_{j,G,\boldsymbol{k}}\rangle$ are therefore i.i.d. with law $\mathcal{N}(0,1)$. We then have

$$\begin{aligned}\mathbb{E}[\|w_{\text{Gauss}}\|^p_{B_p^\tau(\mathbb{R}^d;\rho)}] &= \sum_{j\geq 0} 2^{j(\tau p - d + \frac{dp}{2})} \sum_{G\in \mathrm{G}^j} \sum_{\boldsymbol{k}\in\mathbb{Z}^d} \langle 2^{-j}\boldsymbol{k}\rangle^{\rho p} \mathbb{E}[|\langle w_{\text{Gauss}}, \psi_{j,G,\boldsymbol{k}}\rangle|^p] \\ &= C_p \sum_{j\geq 0} 2^{j(\tau p - d + \frac{dp}{2})} \mathrm{Card}(\mathrm{G}^j) \sum_{\boldsymbol{k}\in\mathbb{Z}^d} \langle 2^{-j}\boldsymbol{k}\rangle^{\rho p} \\ &\leq 2^d C_p \sum_{j\geq 0} 2^{j(\tau p - d + \frac{dp}{2})} \sum_{\boldsymbol{k}\in\mathbb{Z}^d} \langle 2^{-j}\boldsymbol{k}\rangle^{\rho p}.\end{aligned}$$

The last inequality is due to $\mathrm{Card}(\mathrm{G}^j) \leq 2^d$. Since $\rho p < -d$ and $\langle 2^{-j}\boldsymbol{k}\rangle \underset{\infty}{\sim} 2^{-j}\|\boldsymbol{k}\|$, we have that $\sum_{\boldsymbol{k}\in\mathbb{Z}^d} \langle 2^j \boldsymbol{k}\rangle^{\rho p} < \infty$. Moreover, we recognize a Riemann sum and have the convergence

$$2^{-jd} \sum_{\boldsymbol{k}\in\mathbb{Z}^d} \langle 2^j \boldsymbol{k}\rangle^{\rho p} \underset{j\to\infty}{\longrightarrow} \int_{\mathbb{R}^d} \langle \boldsymbol{x}\rangle^{\rho p} \mathrm{d}\boldsymbol{x} < \infty.$$

In particular, the series $\sum_j 2^{j(\tau p + \frac{dp}{2})} \left(2^{-jd} \sum_{\boldsymbol{k}} \langle 2^{-j}\boldsymbol{k}\rangle^{\rho p}\right)$ converges if and only if the series $\sum_j 2^{j(\tau p + \frac{dp}{2})}$ does; that is, if and only if $\tau < d/2$. Finally, if $\tau < d/2$ and $\rho < -d/p$, we have shown that $\mathbb{E}[\|w_{\text{Gauss}}\|^p_{B_p^\tau(\mathbb{R}^d;\rho)}] < \infty$, and therefore $w_{\text{Gauss}} \in B_p^\tau(\mathbb{R}^d;\rho)$ almost surely.

If $\tau \geq -d/2$. We then have $2^{j(\tau - d + dp/2)} \geq 2^{-jd}$. We aim at finding a lower bound for the Besov norm of w and we restrict ourselves to the wavelet with gender $G = M^d \in \mathrm{G}^j$ for any $j \geq 0$. Using (5.5), we have that

$$\|w_{\text{Gauss}}\|^p_{B_p^\tau(\mathbb{R}^d;\rho)} \geq C \sum_{j\geq 0} 2^{-jd} \sum_{0\leq k_1,\dots,k_d < 2^j} |\langle w_{\text{Gauss}}, \psi_{j,M^d,\boldsymbol{k}}\rangle|^p := C \sum_{j\geq 0} Z_j.$$

The random variables $Z_j = 2^{-jd} \sum_{0\leq k_1,\dots,k_d<2^j} |\langle w_{\text{Gauss}}, \psi_{j,M^d,\boldsymbol{k}}\rangle|^p$ are independent, non-negative, and have the same average $\mathbb{E}[Z_j] = C_p$ equals to the pth-moment of a Gaussian random variable with variance 1. The same argument as in Lemma 5.1 therefore implies that $\sum_{j\geq 0} Z_j = \infty$ almost surely, hence $\|w_{\text{Gauss}}\|^p_{B_p^\tau(\mathbb{R}^d;\rho)} = \infty$ almost surely.

If $\rho \geq -d/p$. Using (5.6), we have the lower bound

$$\|w_{\text{Gauss}}\|^p_{B^\tau_p(\mathbb{R}^d;\rho)} \geq \sum_{\boldsymbol{k}\in\mathbb{Z}^d} \langle \boldsymbol{k}\rangle^{\rho p} |\langle w_{\text{Gauss}}, \psi_{0,F^d,\boldsymbol{k}}\rangle|^p \geq \sum_{\boldsymbol{k}\in\mathbb{Z}^d} \frac{|\langle w_{\text{Gauss}}, \psi_{0,F^d,\boldsymbol{k}}\rangle|^p}{\langle \boldsymbol{k}\rangle^d}.$$

Finally, the random variables $\langle w_{\text{Gauss}}, \psi_{0,F^d,\boldsymbol{k}}\rangle$ being i.i.d., Lemma 5.1 applies, implying that $\|w_{\text{Gauss}}\|^p_{B^\tau_p(\mathbb{R}^d;\rho)} = \infty$ almost surely. □

Theorem 5.1 fully characterizes the Besov localization of the Gaussian noise. We reinterpret it as

$$\mathscr{E}_p(w_{\text{Gauss}}) = (-\infty, -d/2) \times (-\infty, -d/p)$$

for any $0 < p < \infty$.

The proof of Theorem 5.1 for the case $\rho \geq -d/p$ uses an argument that is valid for any Lévy noise. We state this result right now in full generality.

Proposition 5.1. *Fix $0 < p < \infty$ and $\tau, \rho \in \mathbb{R}$. If w is a nontrivial Lévy white noise, then, $w \notin B^\tau_p(\mathbb{R}^d;\rho)$ as soon as $\rho \geq -d/p$.*

Proof. By restricting to the scale $j \geq 0$, with only the father wavelet (gender $G = F^d$), and selecting k_0 such that the $\psi_{0,F^d,\boldsymbol{k}}$ have disjoint supports two by two for $\boldsymbol{k} \in k_0\mathbb{Z}^d$, we have the lower bound

$$\|w\|^p_{B^\tau_p(\mathbb{R}^d;\rho)} \geq \sum_{\boldsymbol{k}\in k_0\mathbb{Z}^d} \langle \boldsymbol{k}\rangle^{\rho p} |\langle w, \psi_{0,F^d,\boldsymbol{k}}\rangle|^p \geq \sum_{\boldsymbol{k}\in k_0\mathbb{Z}^d} \frac{|\langle w, \psi_{0,F^d,\boldsymbol{k}}\rangle|^p}{\langle \boldsymbol{k}\rangle^d}.$$

The $\psi_{0,F^d,\boldsymbol{k}}$ having disjoint supports, the random variables $\langle w, \psi_{0,F^d,\boldsymbol{k}}\rangle$ are i.i.d. when $\boldsymbol{k} \in k_0\mathbb{Z}^d$. Lemma 5.1 hence applies and $\|w\|^p_{B^\tau_p(\mathbb{R}^d;\rho)} = \infty$, as expected. □

In other terms, for any Lévy noise w, we have $\mathscr{E}_p(w) \subset \mathbb{R} \times (-\infty, -d/p)$.

5.2.2 Compound Poisson Noise

As for the Gaussian case, we treat the Besov regularity of the compound Poisson noise for every $0 < p < \infty$. Our positive results are based on a careful estimation of the moments for compound Poisson noise.

Lemma 5.2. *Let w_{Poisson} be a compound Poisson noise with asymptotic index α_{asymp} and $p < \alpha_{\text{asymp}} \leq \infty$. Then, there exists a constant C such that*

$$\mathbb{E}[\left|\langle w_{\text{Poisson}}, \psi_{j,G,\boldsymbol{k}}\rangle\right|^p] \leq C 2^{jpd/2 - jd}$$

for every $j \geq 0$, $G \in \mathrm{G}^j$, and $\boldsymbol{k} \in \mathbb{Z}^d$.

Proof. We denote by $\lambda > 0$ and P the sparsity parameter and the law of the jumps of w_{Poisson}, respectively. We know from (3.5) that we can write

$$w_{\text{Poisson}} \overset{(\mathcal{L})}{=} \sum_{k\geq 0} a_k \delta(\cdot - \boldsymbol{x}_k), \tag{5.7}$$

where the a_k are i.i.d. with law P, and the $\boldsymbol{x}_k$, independent from the a_k, are randomly located such that $\text{Card}\{\boldsymbol{x}_k \in B\}$ is a Poisson random variable with parameter $\lambda \text{Leb}(B)$ for any Borel set $B \subset \mathbb{R}^d$ with finite Lebesgue measure. For M big enough, the support of the Ψ_G is included in $[-M/2, M/2]^d$. Then, the support of $\Psi_{j,G,\boldsymbol{k}}$ is included in $I_{j,\boldsymbol{k}} := \prod_{i=1}^d [2^{-j}(k_i - M/2), 2^{-j}(k_i + M/2)]$. We set

$$A_{j,\boldsymbol{k}} = \text{Card}\left\{k \geq 0 \;\middle|\; \boldsymbol{x}_k \in I_{j,\boldsymbol{k}}\right\}.$$

It is a Poisson random variable with parameter $\lambda \text{Leb}(I_{j,\boldsymbol{k}}) = \frac{\lambda M^d}{2^{jd}}$. Then, we have the equality in law

$$\langle w_{\text{Poisson}}, \psi_{j,\boldsymbol{k}} \rangle \overset{(\mathcal{L})}{=} \sum_{n=1}^{A_{j,\boldsymbol{k}}} a'_n \psi_{j,\boldsymbol{k}}(\boldsymbol{x}'_n)$$

where the a'_n are i.i.d. with the same law than the a_k. The law of the $\boldsymbol{x}'_n$ can be specified explicitly but will play no role in the sequel.

By conditioning on $A_{j,\boldsymbol{k}}$ and using the inequality (4.5), we deduce that

$$\begin{aligned}
\mathbb{E}[\left|\langle w_{\text{Poisson}}, \psi_{j,G,\boldsymbol{k}} \rangle\right|^p] &= \sum_{N=1}^{\infty} \mathscr{P}(A_{j,\boldsymbol{k}} = N) \mathbb{E}\left[\left|\langle w_{\text{Poisson}}, \psi_{j,G,\boldsymbol{k}} \rangle\right|^p | A_{j,\boldsymbol{k}} = N\right] \\
&= \sum_{N=1}^{\infty} \mathscr{P}(A_{j,\boldsymbol{k}} = N) \mathbb{E}\left[\left|\sum_{n=1}^{N} a'_n \psi_{j,G,\boldsymbol{k}}(\boldsymbol{x}'_n)\right|^p\right] \\
&\leq \sum_{N=1}^{\infty} \mathscr{P}(A_{j,\boldsymbol{k}} = N) N^{\max(0,p-1)} \mathbb{E}\left[\sum_{n=1}^{N} \left|a'_n \psi_{j,G,\boldsymbol{k}}(\boldsymbol{x}'_n)\right|^p\right] \\
&\leq \|\psi_{j,G,\boldsymbol{k}}\|_\infty^p \sum_{N=1}^{\infty} \mathscr{P}(A_{j,\boldsymbol{k}} = N) N^{\max(1,p)} \mathbb{E}\left[|a_1|^p\right] \\
&= 2^{jdp/2} \|\psi_G\|_\infty^p \mathbb{E}\left[|a_1|^p\right] \sum_{N=1}^{\infty} N^{\max(1,p)} \mathscr{P}(A_{j,\boldsymbol{k}} = N).
\end{aligned} \tag{5.8}$$

We used at the end the relation $\|\psi_{j,G,\boldsymbol{k}}\|_\infty^p = 2^{jdp/2} \|\psi_G\|_\infty^p$. Knowing the law of $A_{j,\boldsymbol{k}}$, we then have

$$\sum_{N=1}^{\infty} N^{\max(1,p)} \mathscr{P}(A_{j,\boldsymbol{k}} = N) = \sum_{N=1}^{\infty} N^{\max(1,p)} \frac{1}{N!} (M^d \lambda)^N 2^{-jdN} \mathrm{e}^{-\lambda M^d 2^{-jd}}.$$

Then, $2^{-jdN} \leq 2^{-jd}$ for every $N \geq 1$ and $\mathrm{e}^{-\lambda M^d 2^{-jd}} \leq 1$, hence,

$$\sum_{N=1}^{\infty} N^{\max(1,p)} \mathscr{P}(A_{j,\boldsymbol{k}} = N) \leq \tilde{C} 2^{-jd} \tag{5.9}$$

where $C' = \sum_{N=1}^{\infty} N^{\max(1,p)} \frac{1}{N!}(M^d\lambda)^N < \infty$. Finally, including (5.9) into (5.8), we deduce the result with $C = C'\mathbb{E}[|a_1|^p]\|\psi_G\|_\infty^p$. □

Theorem 5.2. *Let w_{Poisson} be a compound Poisson noise with asymptotic index $\alpha_{\text{asymp}} \in (0,\infty]$ and $0 < p < \infty$. Then, w_{Poisson} is*

- *almost surely in $B_p^\tau(\mathbb{R}^d;\rho)$ if $\tau < d/p - d$ and $\rho < -d/\min(p,\alpha_{\text{asymp}})$,*
- *almost surely not in $B_p^\tau(\mathbb{R}^d;\rho)$ for $p \le \alpha_{\text{asymp}}$ if $\tau \ge d/p - d$ or $\rho \ge -d/p$, and*
- *almost surely not in $B_p^\tau(\mathbb{R}^d;\rho)$ for $p > \alpha_{\text{asymp}}$ if $\tau \ge d/p - d$ or $\rho > -d/\alpha_{\text{asymp}}$.*

Proof. *If $p < \alpha_{\text{asymp}}$, $\tau < d/p - d$, and $\rho < -d/p$.* Under these assumptions, we apply Lemma 5.2 to deduce that

$$\begin{aligned}\mathbb{E}[\|w_{\text{Poisson}}\|^p_{B_p^\tau(\mathbb{R}^d;\rho)}] &= \sum_{j\ge 0} 2^{j(\tau p - d + dp/2)} \sum_{G,\boldsymbol{k}} \langle 2^{-j}\boldsymbol{k}\rangle^{\rho p} \mathbb{E}[|\langle w_{\text{Poisson}}, \psi_{j,G,\boldsymbol{k}}\rangle|^p] \\ &\le C2^d \sum_{j\ge 0} 2^{j(\tau p - d + dp)} \frac{1}{2^{jd}} \sum_{\boldsymbol{k}\in\mathbb{Z}^d} \langle 2^{-j}\boldsymbol{k}\rangle^{\rho p}.\end{aligned}$$

The sum over the gender was removed using that $\text{Card}(\mathrm{G}^j) \le 2^d$. Then,

$$\frac{1}{2^{jd}} \sum_{\boldsymbol{k}\in\mathbb{Z}^d} \langle 2^{-j}\boldsymbol{k}\rangle^{\rho p} \underset{j\to\infty}{\longrightarrow} \int_{\mathbb{R}^d} \langle \boldsymbol{x}\rangle^{\rho p} \mathrm{d}\boldsymbol{x} < \infty$$

as soon as $\rho < -d/p$. Assuming this condition on ρ, the sum in is finite if and only if $\sum_j 2^{j(\tau p - d + dp)} < \infty$, that is, if and only if $\tau p - d + d/p < 0$, as expected.

If $p \ge \alpha_{\text{asymp}}$, $\tau < d/p - d$, and $\rho < -d/\alpha_{\text{asymp}}$. From the conditions on τ and ρ, one can find p_0, ρ_0, and τ_0 such that

$$\begin{aligned}&\rho < \rho_0 < -\frac{d}{p_0} < -\frac{d}{\alpha_{\text{asymp}}}, \\ &\tau + \frac{d}{p_0} - \frac{d}{p} < \tau_0 < \frac{d}{p_0} - d.\end{aligned} \tag{5.10}$$

Then, in particular, $p_0 < p$, $\tau_0 - \tau > d/p_0 - d/p$, and $\rho_0 > \rho$, so that $B_{p_0}^{\tau_0}(\mathbb{R}^d;\rho_0)$ is embedded in $B_p^\tau(\mathbb{R}^d;\rho)$ (according to (2.21)). Moreover, $p_0 < \alpha_{\text{asymp}}$, $\tau_0 < d/p_0 - d$, and $\rho_0 < -d/p_0$. We are therefore back to the first case, for which we have shown that $w_{\text{Poisson}} \in B_{p_0}^{\tau_0}(\mathbb{R}^d;\rho_0)$ almost surely. In conclusion, $w_{\text{Poisson}} \in B_p^\tau(\mathbb{R}^d;\rho)$ almost surely.

Combining these first two cases, we obtain that $w_{\text{Poisson}} \in B_p^\tau(\mathbb{R}^d;\rho)$ if $\tau < d/p - d$ and $\rho < -d/\min(p,\alpha_{\text{asymp}})$.

If $\tau \ge d/p - d$. We use again the representation (5.7) of the compound Poisson noise. Assume that w_{Poisson} is in $B_p^\tau(\mathbb{R}^d;\rho)$ for some $\rho \in \mathbb{R}$. Then, the product of w_{Poisson} by any compactly supported smooth test function $\varphi \in \mathscr{D}(\mathbb{R}^d)$ is also in $B_p^\tau(\mathbb{R}^d;\rho)$. Choosing φ such that $\varphi(\boldsymbol{x}_0) = 1$ and $\varphi(\boldsymbol{x}_k) = 0$ for $k \neq 0$, we get

$$\varphi \cdot w_{\text{Poisson}} = a_0 \delta(\cdot - \boldsymbol{x}_0) \in B_p^\tau(\mathbb{R}^d;\rho),$$

which is absurd due to Proposition 2.7. This proves that $w_{\text{Poisson}} \notin B_p^\tau(\mathbb{R}^d;\rho)$ for any $\rho \in \mathbb{R}$.

If $\rho \geq -d/p$. We already know that $w_{\text{Poisson}} \notin B_p^\tau(\mathbb{R}^d;\rho)$ for any $\tau \in \mathbb{R}$ according to Proposition 5.1.

If $p > \alpha_{\text{asymp}}$ *and* $\rho > -d/\alpha_{\text{asymp}}$. This means in particular that $\alpha_{\text{asymp}} < \infty$. We treat the case $\rho < 0$, the extension for $\rho \geq 0$ comes easily by embedding. We set $q = -d/\rho$. Using (5.6), we have that

$$\| w_{\text{Poisson}} \|^p_{B_p^\tau(\mathbb{R}^d;\rho)} \geq \sum_{\boldsymbol{k} \in k_0 \mathbb{Z}^d} \frac{|\langle w_{\text{Poisson}}, \psi_{0,F^d,\boldsymbol{k}} \rangle|^p}{\langle \boldsymbol{k} \rangle^{dp/q}}. \tag{5.11}$$

Consider the events $A_{\boldsymbol{k}} = \{ \left| \langle w_{\text{Poisson}}, \psi_{0,F^d,\boldsymbol{k}} \rangle \right| \geq \langle \boldsymbol{k} \rangle^{d/q} \}$ for $\boldsymbol{k} \in k_0 \mathbb{Z}^d$. The $A_{\boldsymbol{k}}$ are independent because the $X_{\boldsymbol{k}} = \langle w_{\text{Poisson}}, \psi_{0,F^d,\boldsymbol{k}} \rangle$ are. Moreover, the $X_{\boldsymbol{k}}$ have the same law since w_{Poisson} is stationary. Set $Y = |X_{\boldsymbol{0}}|^q$. Then,

$$\sum_{\boldsymbol{k} \in k_0 \mathbb{Z}^d} \mathscr{P}(A_{\boldsymbol{k}}) = \sum_{\boldsymbol{k} \in k_0 \mathbb{Z}^d} \mathscr{P}(Y \geq \langle \boldsymbol{k} \rangle^d) \geq \sum_{m \geq 1} \mathscr{P}(Y \geq m k_0). \tag{5.12}$$

Moreover, exploiting that $\mathscr{P}(Y \geq x)$ is decreasing in x, we have that

$$\mathbb{E}[Y] = \int_0^\infty \mathscr{P}(Y \geq x) \mathrm{d}x = \sum_{m \geq 1} \int_{m k_0}^{(m+1)k_0} \mathscr{P}(Y \geq x) \mathrm{d}x \leq \sum_{m \geq 1} \mathscr{P}(Y \geq m k_0). \tag{5.13}$$

The relation $q = \frac{d}{-\rho} > \alpha_{\text{asymp}}$ implies that $\mathbb{E}[Y] = \mathbb{E}[|X_{\boldsymbol{0}}|^q] = \infty$. Hence, from (5.12) and (5.13), we deduce that $\sum_{\boldsymbol{k} \in k_0 \mathbb{Z}^d} \mathscr{P}(A_{\boldsymbol{k}}) = \infty$. The Borel-Cantelli lemma implies that $|X_{\boldsymbol{k}}|^p \geq \langle \boldsymbol{k} \rangle^{pd/q}$ for infinitely many $\boldsymbol{k}$ almost surely. Due to (5.11), this implies that $\| w_{\text{Poisson}} \|^p_{B_p^\tau(\mathbb{R}^d;\rho)} = \infty$ almost surely and the result is proved. □

Theorem 5.2 can be reformulated in terms of the Besov localization of the compound Poisson noise as follows:

- If $p \leq \alpha_{\text{asymp}}$, then

$$\mathscr{E}_p(w_{\text{Poisson}}) = (-\infty, d/p - d) \times (-\infty, -d/p).$$

- If $p > \alpha_{\text{asymp}}$, then

$$(-\infty, d/p - d) \times (-\infty, -d/\alpha_{\text{asymp}}) \subset \mathscr{E}_p(w_{\text{Poisson}}),$$

$$\mathscr{E}_p(w_{\text{Poisson}}) \subset (-\infty, d/p - d) \times (-\infty, -d/\alpha_{\text{asymp}}].$$

The only remaining part for a complete characterization of the Besov regularity is when $p > \alpha_{\text{asymp}}$, $\tau < d/p - d$, and $\rho = -d/\alpha_{\text{asymp}}$. In particular, our results are complete when all the moments of w_{Poisson} are finite ($\alpha_{\text{asymp}} = \infty$). We conjecture that $w_{\text{Poisson}} \notin B_p^\tau(\mathbb{R}^d; -d/\alpha_{\text{asymp}})$ when $p > \alpha_{\text{asymp}}$ and $\tau \in \mathbb{R}$.

5.2.3 Non-Gaussian Lévy Noise

A *non-Gaussian Lévy noise* is a Lévy noise whose Lévy measure is nonzero. When it has no Gaussian part ($\sigma^2 = 0$ in the Lévy-Khintchine representation (2.1)), we say that the noise is *sparse*[1]. When the Gaussian part is nonzero, we say that the noise is *composed* in the sense that it has both a Gaussian and a sparse part. This section is at the heart of our contributions on the Besov regularity of Lévy noise.

Moment estimations. We start with preliminary results that will be used in the proof. We estimate the moments of a random variable by relaying the fractional moments to the characteristic function. Proposition 5.2 can be found for instance in [DS15, Lau80, MP13] with some variations. For the sake of completeness, we recall the proof, similar to the one of [DS15].

Proposition 5.2. *For a random variable X with characteristic function $\widehat{\mathscr{P}}_X$ and $0 < p < 2$, we have the relation*

$$\mathbb{E}[|X|^p] = c_p \int_{\mathbb{R}} \frac{1 - \Re(\widehat{\mathscr{P}_X})(\xi)}{|\xi|^{p+1}} \mathrm{d}\xi \in [0, \infty], \tag{5.14}$$

for some finite constant $c_p > 0$.

Proof. For $p \in (0,2)$, we have, for every $x \in \mathbb{R}$,

$$h(x) = \int_{\mathbb{R}} (1 - \cos(x\xi)) \frac{\mathrm{d}\xi}{|\xi|^{p+1}} = \left(\int_{\mathbb{R}} (1 - \cos(u)) \frac{\mathrm{d}u}{|u|^{p+1}} \right) |x|^p,$$

which is obtained by the change of variable $u = x\xi$. Applying this relation to $x = X$ and denoting $c_p = \left(\int_{\mathbb{R}} (1 - \cos(u)) \frac{\mathrm{d}u}{|u|^{p+1}} \right)^{-1}$, we have by Fubini's theorem that

$$\begin{aligned} \mathbb{E}[|X|^p] &= c_p \mathbb{E}\left[\int_{\mathbb{R}} (1 - \cos(\xi X)) \frac{\mathrm{d}\xi}{|\xi|^{p+1}} \right] \\ &= c_p \int_{\mathbb{R}} (1 - \Re(\mathbb{E}[\mathrm{e}^{\mathrm{i}\xi X}])) \frac{\mathrm{d}\xi}{|\xi|^{p+1}} \\ &= c_p \int_{\mathbb{R}} \frac{1 - \Re(\widehat{\mathscr{P}_X})(\xi)}{|\xi|^{p+1}} \mathrm{d}\xi. \end{aligned}$$

□

Proposition 5.3. *We consider a Lévy noise w with indices α_{loc} and α_{asymp}. Then, for* $0 < p <$

[1] This terminology will be justified in Chapter 6.

$\min(\alpha_{\mathrm{asymp}}, 2)$ *and* $\epsilon > 0$ *small enough, there exists* $C > 0$ *such that*

$$\mathbb{E}[\left|\langle w, f\rangle\right|^p] \leq C(\|f\|^p_{\alpha_{\mathrm{loc}}+\epsilon} + \|f\|^p_{\min(\alpha_{\mathrm{asymp}},2)-\epsilon}) \tag{5.15}$$

for any $f \in L_{\Theta_p}(\mathbb{R}^d)$*, the domain of finite* p*th moments of* w*.*

Proof. For simplicity, we write $\tilde{\alpha} = \min(\alpha_{\mathrm{asymp}}, 2)$ in this proof. We start with a preliminary property: There exists a constant $C > 0$ such that, for every $z \in \mathbb{C}$ with $\Re(z) \leq 0$, we have that

$$|1 - \mathrm{e}^z| \leq C\left(1 - \mathrm{e}^{-|z|}\right). \tag{5.16}$$

Indeed, the function $h(z) = \frac{|1-\mathrm{e}^z|}{1-\mathrm{e}^{-|z|}}$ is easily shown to be bounded for $\Re(z) \leq 0$ by a continuity argument.
Defining $X = \langle w, f\rangle$ with $f \in L_\Theta(\mathbb{R}^d)$, the characteristic function of X is (Proposition 3.10)

$$\widehat{\mathscr{P}}_X(\xi) = \exp\left(\int_{\mathbb{R}^d} \Psi(\xi f(\boldsymbol{x}))\mathrm{d}\boldsymbol{x}\right).$$

Moreover, using (3.42) and Proposition 3.18, we deduce that

$$|\Psi(\xi)| \leq 2\Theta(\xi) \leq C\rho_{\alpha_{\mathrm{loc}}+\epsilon, \tilde{\alpha}-\epsilon}(\xi) \leq C\left(|\xi|^{\tilde{\alpha}-\epsilon} + |\xi|^{\alpha_{\mathrm{loc}}+\epsilon}\right).$$

This implies that

$$\int_{\mathbb{R}^d} |\Psi(\xi f(\boldsymbol{x}))|\mathrm{d}\boldsymbol{x} \leq C\left(\|f\|^{\tilde{\alpha}-\epsilon}_{\tilde{\alpha}-\epsilon}|\xi|^{\tilde{\alpha}-\epsilon} + \|f\|^{\alpha_{\mathrm{loc}}+\epsilon}_{\alpha_{\mathrm{loc}}+\epsilon}|\xi|^{\alpha_{\mathrm{loc}}+\epsilon}\right). \tag{5.17}$$

We therefore have that

$$\begin{aligned}
1 - \Re(\widehat{\mathscr{P}}_X)(\xi) &\leq |1 - \widehat{\mathscr{P}}_X(\xi)| \\
&\overset{(i)}{\leq} C\left(1 - \exp\left(-\left|\int \Psi(\xi f(\boldsymbol{x}))\mathrm{d}\boldsymbol{x}\right|\right)\right) \\
&\overset{(ii)}{\leq} C\left(1 - \exp\left(-\int |\Psi(\xi f(\boldsymbol{x}))|\mathrm{d}\boldsymbol{x}\right)\right) \\
&\overset{(iii)}{\leq} C'\left(1 - \mathrm{e}^{-\|f\|^{\tilde{\alpha}-\epsilon}_{\tilde{\alpha}-\epsilon}|\xi|^{\tilde{\alpha}-\epsilon}}\mathrm{e}^{-\|f\|^{\alpha_{\mathrm{loc}}+\epsilon}_{\alpha_{\mathrm{loc}}+\epsilon}|\xi|^{\alpha_{\mathrm{loc}}+\epsilon}}\right) \\
&\overset{(iv)}{\leq} C'\left((1 - \mathrm{e}^{-\|f\|^{\tilde{\alpha}-\epsilon}_{\tilde{\alpha}-\epsilon}|\xi|^{\tilde{\alpha}-\epsilon}}) + (1 - \mathrm{e}^{-\|f\|^{\alpha_{\mathrm{loc}}+\epsilon}_{\alpha_{\mathrm{loc}}+\epsilon}|\xi|^{\alpha_{\mathrm{loc}}+\epsilon}})\right),
\end{aligned}$$

where (i) comes from (5.16), (ii) and (iii) from the fact that $x \mapsto 1 - \mathrm{e}^{-x}$ is increasing, (iii) from (5.17), and (iv) from the inequality $(1 - xy) \leq (1 - x) + (1 - y)$. By a change of variable, we remark that for $\alpha \in (0, 2)$ and $p < \alpha$, there exists a constant $c_{p,\alpha}$ such that

$$\int_{\mathbb{R}} \frac{1 - \mathrm{e}^{-|x\xi|^\alpha}}{|\xi|^{p+1}}\mathrm{d}\xi = c_{p,\alpha}|x|^p.$$

Applying this result with $(x = \|f\|_{\tilde{\alpha}-\epsilon}, \alpha = \tilde{\alpha} - \epsilon)$ and $(x = \|f\|_{\alpha_{\mathrm{loc}}+\epsilon}, \alpha = \alpha_{\mathrm{loc}} + \epsilon)$ respectively, we deduce using (5.14) that

$$\mathbb{E}[|X|^p] = c_p \int_{\mathbb{R}} \frac{1 - \Re(\widehat{\mathscr{P}_X})(\xi)}{|\xi|^{p+1}} \mathrm{d}\xi \leq C'' \left(\|f\|_{\tilde{\alpha}-\epsilon}^p + \|f\|_{\alpha_{\mathrm{loc}}+\epsilon}^p \right),$$

which completes the proof. □

Sparse Lévy noise. We first assume that the Lévy noise has no Gaussian part ($\sigma^2 = 0$ in the Lévy triplet). We split the main result in different subcases.

Proposition 5.4. *Let $0 < p < \infty$. Then, the Lévy noise w with indices α_{loc} and α_{asymp} is almost surely in $B_p^\tau(\mathbb{R}^d;\rho)$ if $\tau < d/\max(p,\alpha_{\mathrm{loc}}) - d$ and $\rho < -d/\min(p,2,\alpha_{\mathrm{asymp}})$.*
In particular, if $0 < p < 2$, then w is almost surely in $B_p^\tau(\mathbb{R}^d;\rho)$ if $\tau < d/\max(p,\alpha_{\mathrm{loc}}) - d$ and $\rho < -d/\min(p,\alpha_{\mathrm{asymp}})$.

We base the proof on the following estimation.

Lemma 5.3. *Let w be a Lévy noise whose indices satisfy $\alpha_{\mathrm{loc}} < \min(\alpha_{\mathrm{asymp}},2)$. We fix $\alpha_{\mathrm{loc}} < \alpha < \min(\alpha_{\mathrm{asymp}},2)$ and $p < \alpha$. Then , there exists $C > 0$ such that*

$$\mathbb{E}[\left|\langle w, \psi_{j,G,\boldsymbol{k}}\rangle\right|^p] \leq C 2^{jdp(1/\alpha - 1/2)} \tag{5.18}$$

for any $j \geq 0$, $G \in \mathrm{G}^j$, and $\boldsymbol{k} \in \mathbb{Z}^d$.

Proof. For $\epsilon > 0$ small enough such that $\alpha_{\mathrm{loc}} + \epsilon \leq \alpha \leq \min(\alpha_{\mathrm{asymp}},2) - \epsilon$, we have the embedding $L_\alpha(\mathbb{R}^d) \subseteq L_{\alpha_{\mathrm{loc}}+\epsilon}(\mathbb{R}^d) \cap L_{\min(\alpha_{\mathrm{asymp}},2)-\epsilon}(\mathbb{R}^d)$ and there exists $M > 0$ such that

$$\|f\|_{\alpha_{\mathrm{loc}}+\epsilon}^p + \|f\|_{\min(\alpha_{\mathrm{asymp}},2)-\epsilon}^p \leq M\|f\|_\alpha^p. \tag{5.19}$$

Applying (5.15) and (5.19) to $f = \psi_{j,G,\boldsymbol{k}}$, we get

$$\begin{aligned}\mathbb{E}[\left|\langle w, \psi_{j,G,\boldsymbol{k}}\rangle\right|^p] &\leq C(\|\psi_{j,G,\boldsymbol{k}}\|_{\alpha_{\mathrm{loc}}+\epsilon}^p + \|\psi_{j,G,\boldsymbol{k}}\|_{\min(\alpha_{\mathrm{asymp}},2)-\epsilon}^p) \\ &\leq CM\|\psi_{j,G,\boldsymbol{k}}\|_\alpha^p \\ &= CM\|\psi_G\|_\alpha^p 2^{jdp(1/\alpha-1/2)}.\end{aligned}$$

Finally, (5.18) is proved for the constant $CM\sup_G\|\psi_G\|_\alpha^p$. □

Proof of Proposition 5.4. The second part of Proposition 5.4 is directly deduced from the first part because $\min(p,\alpha_{\mathrm{asymp}}) = \min(p,2,\alpha_{\mathrm{asymp}})$ when $p < 2$. We now prove the first part.

If $\alpha_{\mathrm{loc}} < \min(\alpha_{\mathrm{asymp}},2)$ and $p < \min(\alpha_{\mathrm{asymp}},2)$. One select α close enough to $\max(p,\alpha_{\mathrm{loc}})$ such that

$$\max(p,\alpha_{\mathrm{loc}}) < \alpha < \min(\alpha_{\mathrm{asymp}},2) \text{ and } \tau < \frac{d}{\alpha} - d < \frac{d}{\max(p,\alpha_{\mathrm{loc}})} - d.$$

Since, in addition, $p < \alpha_{\mathrm{asymp}}$ and $p < 2$, we have $p < \min(\alpha_{\mathrm{asymp}}, 2)$. We are in the conditions of Lemma 5.3. Therefore, we know that there exists a constant $C > 0$ such that

$$\mathbb{E}[\left|\langle w, \psi_{j,G,\boldsymbol{k}}\rangle\right|^p] \leq C 2^{jdp(1/2-1/\alpha)}.$$

Then, we have that

$$\begin{aligned}\mathbb{E}[\|w\|^p_{B^\tau_p(\mathbb{R}^d;\rho)}] &= \sum_{j\geq 0} 2^{j(\tau p - d + dp/2)} \sum_{G,\boldsymbol{k}} \langle 2^{-j}\boldsymbol{k}\rangle^{\rho p} \mathbb{E}[\left|\langle w, \psi_{j,G,\boldsymbol{k}}\rangle\right|^p] \\ &\leq 2^d C \sum_{j\geq 0} 2^{j(\tau p - d + dp/2) + jdp(1/2-1/\alpha)} \left(\sum_{\boldsymbol{k}} \langle 2^{-j}\boldsymbol{k}\rangle^{\rho p}\right).\end{aligned}$$

By assumption, we have that $\rho < -d/\min(p, \alpha_{\mathrm{asymp}}) = -d/p$, and $\langle \boldsymbol{x}\rangle^{\rho p}$ is hence integrable over $\mathbb{R}^d$. We recognize a Riemman sum and deduce that

$$\sum_{\boldsymbol{k}} \langle 2^{-j}\boldsymbol{k}\rangle^{\rho p} \underset{j\to\infty}{\sim} 2^{jd} \int_{\mathbb{R}^d} \langle \boldsymbol{x}\rangle^{\rho p} \mathrm{d}\boldsymbol{x}. \tag{5.20}$$

Therefore, for C' big enough, we have that

$$\mathbb{E}[\|w\|^p_{B^\tau_p(\mathbb{R}^d;\rho)}] \leq C' \sum_{j\geq 0} (2^{\tau p + dp - dp/\alpha})^j.$$

The sum converges if and only if $\tau < d/\alpha - d$, which we have assumed. Finally, we have shown that w is almost surely in $B^\tau_p(\mathbb{R}^d;\rho)$.

If $\alpha_{\mathrm{loc}} < \min(\alpha_{\mathrm{asymp}}, 2)$ *and* $p \geq \min(\alpha_{\mathrm{asymp}}, 2)$. We deduce the result by embeddings (Proposition 2.8) from the case $p < \min(\alpha_{\mathrm{asymp}}, 2)$, as we did in (5.10).

General case. The Lévy noise w can be decomposed as $w = w_1 + w_2$ where w_1 a compound Poisson noise and w_2 a noise with all its moments finite. Then, we have that $\alpha_{\mathrm{asymp}} = \alpha_{\mathrm{asymp}}(w_1) \leq \alpha_{\mathrm{asymp}}(w_2) = \infty$ and $\alpha_{\mathrm{loc}} = \alpha_{\mathrm{loc}}(w_2) \geq \alpha_{\mathrm{loc}}(w_1) = 0$. From this, we easily see that $\tau < d/\max(p, \alpha_{\mathrm{loc}}(w_i)) - d$ and $\rho < -d/\min(p, 2, \alpha_{\mathrm{asymp}}(w_i))$ for $i = 1, 2$. Moreover, we have that $\alpha_{\mathrm{loc}}(w_1) = 0 < \alpha_{\mathrm{asymp}}(w_1)$ and $\alpha_{\mathrm{loc}}(w_2) \leq 2 = \min(\alpha_{\mathrm{asymp}}(w_2), 2)$. Thus, we can apply the first cases ($\alpha_{\mathrm{loc}} < \min(\alpha_{\mathrm{asymp}}, 2)$) to deduce that both w_1 and $w_2 \in B^\tau_p(\mathbb{R}^d;\rho)$ almost surely. Besov spaces being linear, w inherits this property. □

Remark. Proposition 5.4 gives sufficient conditions relying on the indices α_{loc} and α_{asymp}. For $p < 2$, we have seen that one can replace $\min(\alpha_{\mathrm{asymp}}, 2)$ by α_{asymp}. Actually, we shall see that the decay rate is captured by the asymptotic index α_{asymp} and not by the Pruitt index $\beta_0 = \min(\alpha_{\mathrm{asymp}}, 2)$. This means in particular that Proposition 5.4 is sharp only for $p \leq 2$. This is the reason why we have reformulated the result for $p < 2$, with α_{asymp} instead on $\min(\alpha_{\mathrm{asymp}}, 2)$.

Proposition 5.5. *Let* $p \geq 2$ *be an even integer. Then, the Lévy noise* w *with indices* α_{loc} *and* α_{asymp} *is almost surely in* $B^\tau_p(\mathbb{R}^d;\rho)$ *if* $\tau < d/p - d = d/\max(p, \alpha_{\mathrm{loc}}) - d$ *and* $\rho < -d/\min(p, \alpha_{\mathrm{asymp}})$.

The proof is based on the estimation of the moments of the wavelet decomposition of the noise, in particular with the evolution with the scale j.

Lemma 5.4. *Let w be a Lévy with finite $(2k)$-moments, with $k \geq 1$ an integer. Then, there exists a constant C such that*

$$\mathbb{E}[\left|\langle w, \psi_{j,G,\boldsymbol{k}}\rangle\right|^{2k}] \leq C 2^{jd(k-1)}$$

for every $j \geq 0$, $G \in \mathrm{G}^j$, and $\boldsymbol{k} \in \mathbb{Z}^d$.

Proof. Consider a test function $f \in L_\Theta(\mathbb{R}^d)$ and set $X = \langle w, f\rangle$. The characteristic function of X is (Proposition 3.10)

$$\widehat{\mathscr{P}}_X(\xi) = \exp\left(\int_{\mathbb{R}^d} \Psi(\xi f(\boldsymbol{x}))\mathrm{d}\boldsymbol{x}\right) := \exp(\Psi_f(\xi)).$$

The functions $\widehat{\mathscr{P}}_X$ and Ψ_f are $(2k)$-differentiable because the $(2k)$-moment of X is finite. Their Taylor expansions give the moments and the cumulants of X, respectively. In particular, we have that $\mathbb{E}[X^{2k}] = (-1)^k \widehat{\mathscr{P}}_X^{(2k)}(0)$. The $(2k)$th derivative of $\widehat{\mathscr{P}}_X$ is deduced from the Faà di Bruno formula [Fra78], and is

$$\widehat{\mathscr{P}}_X^{(2k)}(\xi) = \left(\sum_{n_1,\dots n_{2k}: \sum_u u n_u = 2k} \frac{(2k)!}{n_1! \dots n_{2k}!} \prod_{v=1}^{2k} \left(\frac{\Psi_f^{(v)}(\xi)}{v!}\right)^{n_v}\right) \widehat{\mathscr{P}}_X(\xi).$$

Exploiting that $\Psi_f^{(v)}(0) = \left(\int_{\mathbb{R}^d} (f(\boldsymbol{x}))^v \mathrm{d}\boldsymbol{x}\right) \Psi^{(v)}(0)$ we obtain the bound, for $\xi = 0$,

$$\left|\widehat{\mathscr{P}}_X^{(2k)}(0)\right| \leq C' \sum_{n_1,\dots n_{2k}: \sum_u u n_u = 2k} \prod_{v=1}^{2k} \left|\int_{\mathbb{R}^d} f(\boldsymbol{x})^v \mathrm{d}\boldsymbol{x}\right|^{n_v} \tag{5.21}$$

with $C > 0$ a constant.

We now apply (5.21) to $f = \psi_{j,G,\boldsymbol{k}}$. Since we have

$$\int_{\mathbb{R}^d} \psi_{j,G}(\boldsymbol{x})^v \mathrm{d}\boldsymbol{x} = 2^{jdv/2} \int_{\mathbb{R}^d} \psi_G(2^j \boldsymbol{x} - \boldsymbol{k})^v \mathrm{d}\boldsymbol{x} = 2^{jd(v/2-1)} \int_{\mathbb{R}^d} \psi_G(\boldsymbol{x})^v \mathrm{d}\boldsymbol{x},$$

we deduce from (5.21) the new bound

$$\begin{aligned}
\mathbb{E}[\langle w, \psi_{j,G,\boldsymbol{k}}\rangle^{2k}] &= \left|\widehat{\mathscr{P}}^{(2k)}_{\langle w, \psi_{j,G,\boldsymbol{k}}\rangle}(0)\right| \\
&\leq C'' \sum_{n_1,\dots n_{2k}: \sum_u u n_u = 2k} \prod_{v=1}^{2k} 2^{jd(v/2-1)n_v} \\
&= C'' \sum_{n_1,\dots n_{2k}: \sum_u u n_u = 2k} 2^{jd\sum_v (n_v(v/2-1))}.
\end{aligned}$$

Finally, since $\sum_v v n_v = 2k$ and $\sum_v n_v \geq 1$, we have $\sum_v (n_v(v/2-1)) \leq k-1$, and therefore

$$\mathbb{E}[\langle w, \psi_{j,G,\boldsymbol{k}}\rangle^{2k}] \leq C 2^{jd(k-1)}$$

for an adequate $C > 0$, as expected. □

Proof of Proposition 5.5. We set $p = 2k$ with $k \geq 1$, $k \in \mathbb{N}$. Then, we assume that $\tau < d/2k - d$ and $\rho < -d/\min(2k, \alpha_{\text{asymp}})$.

If $\alpha_{\text{asymp}} = \infty$. The assumption on ρ becomes $\rho < -d/2k$. According to Lemma 5.4, there exists a constant $C > 0$ such that

$$\mathbb{E}[|\langle w, \psi_{j,G,\boldsymbol{k}}\rangle|^p] \leq C 2^{jd(k-1)}. \tag{5.22}$$

Applying (5.22), we deduce that

$$\begin{aligned}
\mathbb{E}[\|w\|^{2k}_{B^{\tau}_{2k}(\mathbb{R}^d;\rho)}] &= \sum_{j\geq 0} 2^{j(2k\tau - d + dk)} \sum_{G,\boldsymbol{k}} \langle 2^{-j}\boldsymbol{k}\rangle^{2k\rho} \mathbb{E}[|\langle w, \psi_{j,G,\boldsymbol{k}}\rangle|^{2k}] \\
&\leq 2^d C \sum_{j\geq 0} 2^{j(2k\tau - d + dk + dk - d)} \left(\sum_{\boldsymbol{k}} \langle 2^{-j}\boldsymbol{k}\rangle^{2\rho k} \right) \\
&\leq C' \sum_{j \geq 0} (2^{2k\tau + 2kd - d})^j,
\end{aligned}$$

where we have finally used (5.20) for the last inequality, which holds since $\rho < -d/2k$. The final sum converges if and only if $\tau < d/2k - d$, which we have assumed. Finally, we have shown that $w \in B^{\tau}_{2k}(\mathbb{R}^d;\rho)$ almost surely.

General case. We decompose $w = w_1 + w_2$ with w_1 a compound Poisson noise and w_2 a Lévy noise with $\alpha_{\text{asymp}}(w_2) = \infty$, w_1 and w_2 being independent. Then, the conditions on τ and ρ easily imply that $\tau < d/\max(p, \alpha_{\text{loc}}(w_i)) - d$ and $\rho < -d/\min(p, 2, \alpha_{\text{asymp}}(w_i))$ for $i = 1, 2$. Therefore, $w_1 \in B^{\tau}_p(\mathbb{R}^d;\rho)$ according to Theorem 5.2, and $w_2 \in B^{\tau}_p(\mathbb{R}^d;\rho)$ as we have seen in the previous case. Finally, by linearity, $w = w_1 + w_2 \in B^{\tau}_p(\mathbb{R}^d;\rho)$ almost surely. □

Remark. The second part of Proposition 5.4 and Proposition 5.5 state the same result for different ranges of p. We conjecture that this result is actually valid for any $p \in (0, \infty]$. What is missing is an adequate estimation of the moments $\mathbb{E}[|\langle w, \psi_{j,G,\boldsymbol{k}}\rangle|^p]$ for general p, in the spirit of Lemmas 5.3 and 5.4.

We now prove negative results; that is, we identify the Besov spaces to which the Lévy noise does not belong almost surely. We split the results for the smoothness (for which we have the result for any $p > 0$) and for the decay rate (for which we do not consider the case $p > 2$, $p/2 \notin \mathbb{N}$).

Proposition 5.6. *Let $p > 0$. Then, the non-Gaussian Lévy noise w is not in $B^{\tau}_p(\mathbb{R}^d;\rho)$ almost surely if $\tau > d/p - d$.*

Proof. We adapt the proof of the compound Poisson case to the general case. The main idea is as follows. We decompose $w = w_1 + w_2$ with w_1 a compound Poisson noise and w_2 a Lévy noise with finite moments. We can always impose that w_1 is not zero, since w is non-Gaussian.

Then, we will see that the jumps of the compound Poisson part forces the Besov norm to explode, and cannot be compensated by w_2.

First, we remark that it is sufficient to show the existence of a test function $\varphi \in \mathscr{D}(\mathbb{R}^d)$ such that $\|w \cdot \varphi\|_{B_p^\tau(\mathbb{R}^d)} = \infty$ almost surely. This proves that $w \notin B_p^\tau(\mathbb{R}^d;\mathrm{loc})$, the local Besov space, and therefore $w \notin B_p^\tau(\mathbb{R}^d;\rho) \subseteq B_p^\tau(\mathbb{R}^d;\mathrm{loc})$.

According to (3.5), we can write $w_1 = \sum_{k\geq 0} a_k \delta(\cdot - \boldsymbol{x}_k)$. The random variables $|a_k|$ are i.i.d. and almost surely strictly positive. Let $c_0 > 0$ be such that $\mathscr{P}(|a_k| \geq c_0) > 0$. Then, almost surely, there exists $k \geq 0$ such that $|a_k| \geq c_0$. We fix such a random k_0 in the sequel. We therefore have $|a_{k_0}| \geq c_0 > 0$ almost surely. We chose $\varphi \in \mathscr{D}(\mathbb{R}^d)$ random such that $\varphi(\boldsymbol{x}_k) = 0$ for $k \neq k_0$, and $\varphi = 1$ on a neighbourhood $\{\boldsymbol{x} \in \mathbb{R}^d \mid \|\boldsymbol{x} - \boldsymbol{x}_{k_0}\|_\infty \leq \delta\}$ of $\boldsymbol{x}_{k_0}$.

We consider a Daubechies mother wavelet such that $|\psi_{M^d}(\boldsymbol{x})| \geq m_0 > 0$ for $\boldsymbol{x} \in [-1/2, 1/2]^d$. This is always possible because the Daubechies wavelets converge to the sinc function, which admits a strictly positive lower bound over $[-1/2, 1/2]$. Therefore, it is sufficient to take Daubechies wavelets of a large enough order.

Then, let $\boldsymbol{k}_j \in \mathbb{Z}^d$ be the closed multi-integer to $2^j \boldsymbol{x}_{k_0}$. In particular, $2^j \boldsymbol{x}_{k_0} - \boldsymbol{k}_j \in [-1/2, 1/2]^d$ and

$$\left|\psi_{M^d}(2^j \boldsymbol{x}_{k_0} - \boldsymbol{k}_j)\right| \geq m_0 > 0.$$

This relation is important since it provides a uniform and deterministic lower bound on the random quantities $|\psi_{M^d}(2^j \boldsymbol{x}_{k_0} - \boldsymbol{k}_j)|$. We fix $J \in \mathbb{N}$ such that

$$\mathrm{Leb}(\mathrm{Supp}\psi_{j,M^d,\boldsymbol{k}}) = 2^{-jd}\mathrm{Leb}(\mathrm{Supp}\psi_{M^d}) \leq \delta$$

for every $j \geq J$ and $\boldsymbol{k} \in \mathbb{Z}^d$. Then, $\mathrm{Supp}\Psi_{j,M^d,\boldsymbol{k}_j} \subset \{\boldsymbol{x} \in \mathbb{R}^d, \| \mid \boldsymbol{x} - \boldsymbol{x}_{k_0}\|_\infty \leq \delta\}$ due to the size of the support of $\psi_{j,M^d,\boldsymbol{k}_j}$. Therefore, for every $j \geq J$, we have that $\varphi(\boldsymbol{x}) \cdot \psi_{j,M^d,\boldsymbol{k}_j}(\boldsymbol{x}) = \psi_{j,M^d,\boldsymbol{k}_j}(\boldsymbol{x})$, since $\varphi(\boldsymbol{x}) = 1$ on the support of $\psi_{j,M^d,\boldsymbol{k}_j}$.

Then, we set a lower bound on the Besov norm of $\varphi \cdot w$ by restricting to the gender $G = M^d$, the scales $j \geq J$, and $\boldsymbol{k} = \boldsymbol{k}_j$. We then exploit that $\varphi \cdot \psi_{j,M^d,\boldsymbol{k}_j} = \psi_{j,M^d,\boldsymbol{k}_j}$ and that $\langle w_1 \cdot \varphi, \psi_{j,M^d,\boldsymbol{k}_j}\rangle = a_{k_0}\psi_{j,M^d,\boldsymbol{k}_j}(\boldsymbol{x}_{k_0})$ to deduce that

$$\begin{aligned}\|\varphi \cdot w\|_{B_p^\tau(\mathbb{R}^d)} &\geq \sum_{j\geq J} 2^{j(\tau p - d + dp)} 2^{-jdp/2} \left|\langle w, \varphi \cdot \psi_{j,M^d,\boldsymbol{k}_j}\rangle\right|^p \\ &\geq \max_{j\geq J} 2^{j(\tau p - d + dp)} \left|\langle w, \psi_{M^d}(2^j \cdot - \boldsymbol{k}_j)\rangle\right|^p \\ &= \max_{j\geq J} 2^{j(\tau p - d + dp)} \left|\langle w_2, \psi_{M^d}(2^j \cdot - \boldsymbol{k}_j)\rangle + a_{k_0}\psi_{M^d}(2^j \boldsymbol{x}_{k_0} - \boldsymbol{k}_j)\right|^p. \end{aligned} \tag{5.23}$$

We apply the Markov inequality $\mathscr{P}(|X| \geq x) \leq \mathbb{E}[|X|^2]/x^2$ to $x = c_0 m_0/2$ and $X = \langle w_2, \psi_{M^d}(2^j \cdot -\boldsymbol{k}_j)\rangle$ and get

$$\mathscr{P}\left(\left|\langle w_2, \psi_{M^d}(2^j \cdot - \boldsymbol{k}_j)\rangle\right| \geq \frac{1}{2} c_0 m_0\right) \leq \frac{4}{c_0^2 m_0^2} \mathbb{E}\left[\langle w_2, \psi_{M^d}(2^j \cdot - \boldsymbol{k}_j)\rangle^2\right].$$

The mean of $\langle w_2, \psi_{M^d}(2^j\cdot - \boldsymbol{k}_j)\rangle$ is 0 because the mother wavelet has a 0 mean. We denote by σ_0^2 the variance of the noise w_2. Then, we have that

$$\mathbb{E}\left[\langle w_2, \psi_{M^d}(2^j\cdot - \boldsymbol{k}_j)\rangle^2\right] = \sigma_0^2 \|\psi_{M^d}(2^j\cdot - \boldsymbol{k}_j)\|_2^2 = \sigma_0^2 2^{-jd},$$

using that the wavelet is normalized. Finally, we have shown that

$$\mathscr{P}\left(\left|\langle w_2, \psi_{M^d}(2^j\cdot - \boldsymbol{k}_j)\rangle\right| \geq \frac{1}{2}c_0 m_0\right) \leq \frac{4\sigma_0^2}{c_0^2 m_0^2} 2^{-jd}.$$

From this, and because $\left|\psi_{M^d}(2^j \boldsymbol{x}_{k_0} - \boldsymbol{k}_j)\right| \geq m_0$ and $\left|a_{k_0}\right| \geq c_0$ almost surely, we deduce that

$$\mathscr{P}\left(\left|\langle w_2, \psi_{M^d}(2^j\cdot - \boldsymbol{k}_j)\rangle\right| \geq \frac{1}{2}\left|a_{k_0}\right|\left|\psi_{M^d}(2^j \boldsymbol{x}_{k_0} - \boldsymbol{k}_j)\right|\right) \\ \leq \mathscr{P}\left(\left|\langle w_2, \psi_{M^d}(2^j\cdot - \boldsymbol{k}_j)\rangle\right| \geq \frac{1}{2}c_0 m_0\right) \underset{j\to\infty}{\longrightarrow} 0.$$

This implies that

$$\mathscr{P}\left(\exists j \geq J, \left|\langle w_2, \psi_{M^d}(2^j\cdot - \boldsymbol{k}_j)\rangle\right| < \frac{1}{2}\left|a_{k_0}\right|\left|\psi_{M^d}(2^j \boldsymbol{x}_{k_0} - \boldsymbol{k}_j)\right|\right) = 1.$$

We denote by Ω_0 this space of probability 1. On Ω_0, we have that

$$\left|\langle w_2, \psi_{M^d}(2^j\cdot - \boldsymbol{k}_j)\rangle + a_{k_0}\psi_{M^d}(2^j \boldsymbol{x}_{k_0} - \boldsymbol{k}_j)\right| \geq \frac{\left|a_{k_0}\psi_{M^d}(2^j \boldsymbol{x}_{k_0} - \boldsymbol{k}_j)\right|}{2} \geq \frac{c_0 m_0}{2}$$

for some $j \geq J$. Finally, using (**??**) in (5.23), we deduce that

$$\|\varphi \cdot w\|_{B_p^\tau(\mathbb{R}^d)} \geq \max_{j\geq J} 2^{j(\tau p - d + dp)} \frac{c_0 m_0}{2} = \infty$$

almost surely, since $\tau p - d + dp > 0$ by assumption. Finally, the noise is almost surely not in $B_p^\tau(\mathbb{R}^d;\rho)$.

□

Proposition 5.7. *Let w be a non-Gaussian Lévy noise with local index $\alpha_{\mathrm{loc}} > 0$ and Lévy exponent Ψ. We assume that*

$$\Psi(\xi) \underset{\infty}{\sim} -C|\xi|^{\alpha_{\mathrm{loc}}} \tag{5.24}$$

for some constant $C > 0$. We fix $p \in (0,\infty]$, $\rho, \tau \in \mathbb{R}$. Then, w is almost surely not in $B_p^\tau(\mathbb{R}^d;\rho)$ almost surely if $\tau > d/\alpha_{\mathrm{loc}} - d$.

We base the proof on the following estimation.

Lemma 5.5. *Let w be a non-Gaussian Lévy noise with indices $\alpha_{\mathrm{asymp}}, \alpha_{\mathrm{loc}} > 0$ and Lévy expo-*

nent Ψ *satisfying* (5.24). *Then, for every* $p < \alpha_{\text{loc}}, \alpha_{\text{asymp}}$, *we have, for every* $\boldsymbol{k}, G$,

$$\mathbb{E}[|\langle w, \Psi_{j,G,\boldsymbol{k}}\rangle|^p] \underset{j\to\infty}{\sim} C_{G,p,\alpha} 2^{jdp\left(\frac{1}{2}-\frac{1}{\alpha_{\text{loc}}}\right)} \tag{5.25}$$

with $C_{G,p,\alpha} > 0$ *a constant.*

Proof. We first remark that

$$\begin{aligned}\langle w, \Psi_{j,G,\boldsymbol{k}}\rangle &= 2^{-jd/2}\langle w(\cdot/2^j), \Psi_G(\cdot - \boldsymbol{k})\rangle \\ &= 2^{j(d/2-d/\alpha_{\text{loc}})}\langle 2^{jd(1/\alpha_{\text{loc}}-1)} w(\cdot/2^j), \Psi_G(\cdot - \boldsymbol{k})\rangle. \end{aligned} \tag{5.26}$$

Moreover, with Theorem 4.4, we know that $2^{jd(1/\alpha_{\text{loc}}-1)} w(\cdot/2^j)$ converges to a SαS noise $w_{\alpha_{\text{loc}}}$ with $\alpha = \alpha_{\text{loc}}$ and $\widehat{\mathscr{P}}_{w_{\alpha_{\text{loc}}}}(\varphi) = \mathrm{e}^{-C\|\varphi\|_{\alpha_{\text{loc}}}^{\alpha_{\text{loc}}}}$. In particular, for $p < \alpha_{\text{loc}}$, we have the convergence

$$\mathbb{E}\left[\left|\langle 2^{jd(1/\alpha_{\text{loc}}-1)} w(\cdot/2^j), \Psi_G(\cdot - \boldsymbol{k})\rangle\right|^p\right] \underset{j\to\infty}{\longrightarrow} \mathbb{E}\left[|\langle w_{\alpha_{\text{loc}}}, \Psi_G\rangle|^p\right]. \tag{5.27}$$

Finally, (5.25) is a consequence of (5.26) and (5.27). □

Proof of Proposition 5.7. By the embeddings $B_q^{\tau+\epsilon}(\mathbb{R}^d;\rho) \subseteq B_p^{\tau}(\mathbb{R}^d;\rho)$ valid for every $q > p$ and $\epsilon > 0$, it is sufficient to show the result for p arbitrarily small. We assume that $p < \alpha_{\text{asymp}}$.
Let $k_0 \geq 1$ be such that the families of random variables $(\langle w, \Psi_{j,G,\boldsymbol{k}}\rangle)_{\boldsymbol{k}\in k_0\mathbb{Z}^d}$ are independent at $j \geq 0$ and $G \in \mathrm{G}^j$ fixed. This is possible because the wavelets are compactly supported. It therefore suffices to take k_0 big enough such that the supports do not intersect at a given gender and scale. By restricting to $G = M^d$ and the range of $\boldsymbol{k}$, we have that

$$\|w\|^p_{B_p^\tau(\mathbb{R}^d;\rho)} \geq C\sum_{j\geq 0} 2^{j(\tau p-d+dp/2)} \sum_{\boldsymbol{k}\in k_0\mathbb{Z}^d, 0\leq k_i<k_0 2^j} \left|\langle w, \psi_{j,M^d,\boldsymbol{k}}\rangle\right|^p.$$

We set $X_{j,\boldsymbol{k}} = 2^{jd\left(\frac{1}{\alpha_{\text{loc}}}-\frac{1}{2}\right)}\langle w, \psi_{j,M^d,\boldsymbol{k}}\rangle$ and

$$M_{j,p} := 2^{-jd} \sum_{\boldsymbol{k}\in k_0\mathbb{Z}^d, 0\leq k_i<k_0 2^j} |X_{j,\boldsymbol{k}}|^p,$$

which is an average among 2^{jd} random variables. According to Lemma 5.5, the sequence $(2^{jdp\left(\frac{1}{\alpha_{\text{loc}}}-\frac{1}{2}\right)}\mathbb{E}[|\langle w, \Psi_{j,G,\boldsymbol{k}}\rangle|^p])_{j\geq 0}$ converges to a strictly positive constant, and is therefore bounded below and above by some constants $m_p, M_p > 0$, respectively. In particular, we have that $m_p \leq \mathbb{E}[M_{j,p}] \leq M_p$ for every $p < \alpha_{\text{loc}}$ and $j \geq 0$.
We now assume that $p < \alpha_{\text{loc}}/2$. Then, by exploiting the independence of the $X_{j,\boldsymbol{k}}$, we have

$$\begin{aligned}\mathbb{E}[M_{j,p}^2] &= 2^{-jd}\mathbb{E}[2^{-jd}\left(\sum |X_{j,\boldsymbol{k}}|^p\right)^2] \\ &= 2^{-jd}\mathbb{E}[2^{-jd}\sum |X_{j,\boldsymbol{k}}|^{2p}] \\ &= 2^{-jd}\mathbb{E}[M_{j,2p}] \\ &\leq 2^{-jd} M_{2p}.\end{aligned}$$

Moreover, due to the Markov inequality, we have that

$$\begin{aligned}\mathscr{P}\left(\left|M_{j,p}-\mathbb{E}[M_{j,p}]\right|\geq\delta\right)&\leq\delta^{-2}\mathbb{E}\left[\left|M_{j,p}-\mathbb{E}[M_{j,p}]\right|^2\right]\\&\leq\delta^{-2}\mathbb{E}[M_{j,p}^2]\\&\leq\delta^{-2}2^{-jd}M_{2p}.\end{aligned}$$

Taking $\delta=2^{-j\epsilon}$ with $0<\epsilon<d/2$, we have that

$$\mathscr{P}\left(\left|M_{j,p}-\mathbb{E}[M_{j,p}]\right|\geq 2^{-j\epsilon}\right)\leq 2^{j(2\epsilon-d)}M_{2p}\underset{j\to\infty}{\longrightarrow}0. \tag{5.28}$$

We fix now $N\geq 0$. Let $J\geq 0$ big enough such that $2^{Jp(\tau+d-d/\alpha_{\text{loc}})}\geq N$ (it exists because $\tau+d-d/\alpha_{\text{loc}}>0$). According to (5.28), if we denote by $\Omega_0=\{\exists j\geq J, M_{j,p}\geq m_p/2\}$, we have that $\mathscr{P}(\Omega_0)=1$. Then, on Ω_0, we have that

$$\|w\|^p_{B_p^\tau(\mathbb{R}^d;\rho)}\geq\sum_{j\geq J}2^{jp(\tau-d+d/\alpha_{\text{loc}})}M_{j,p}\geq N\sum_{j\geq J}M_{j,p}\geq\frac{Nm_p}{2}.$$

This is valid for every $N\geq 0$, hence $\|w\|^p_{B_p^\tau(\mathbb{R}^d;\rho)}=\infty$ almost surely. □

Proposition 5.8. *Let* $0<p<2$ *be real or* $p\geq 2$ *be an even integer. Then, the non-Gaussian Lévy noise* w *with asymptotic index* α_{asymp} *is almost surely not in* $B_p^\tau(\mathbb{R}^d;\rho)$ *if* $\rho>-d/\min(p,\alpha_{\text{asymp}})$.

Proof. If $\rho\geq -d/p$, we already know that $w\notin B_p^\tau(\mathbb{R}^d;\rho)$ almost surely with Proposition 5.1. One can therefore assume that $p>\alpha_{\text{asymp}}$ and that $\rho>-d/\alpha_{\text{asymp}}$. We make the additional assumption that $\rho<-d/p$ (possible since $\alpha_{\text{asymp}}<p$) and that $\tau<d/\max(p,\alpha_{\text{loc}})-d$. Then, we decompose $w=w_1+w_2$ with w_1 a nontrivial compound Poisson noise and w_2 a Lévy noise with finite moments. Since $\rho>-d/\alpha_{\text{asymp}}=-d/\alpha_{\text{asymp}}(w_1)$, we apply Theorem 5.2 to deduce that $w_1\notin B_p^\tau(\mathbb{R}^d;\rho)$ almost surely. Moreover, the upper bounds $\tau<d/\max(p,\alpha_{\text{loc}})-d=d/\max(p,\alpha_{\text{loc}}(w_2))-d$ and $\rho<-d/p=-d/\min(p,\alpha_{\text{asymp}}(w_2))$ imply that the Lévy noise $w_2\in B_p^\tau(\mathbb{R}^d;\rho)$. This come from Proposition 5.4 for $p<2$ and from Proposition 5.5 if $p\geq 2$ is an even integer. Thus, $w\notin B_p^\tau(\mathbb{R}^d;\rho)$ as the sum between an element of $B_p^\tau(\mathbb{R}^d;\rho)$ and an element that is not in $B_p^\tau(\mathbb{R}^d;\rho)$.
Finally, the assumptions $\tau<d/\max(p,\alpha_{\text{loc}})-d$ and $\rho<-d/p$ can be removed by embedding. □

Remarks. Proposition 5.6 does not assume any restriction on $p>0$. On the other hand, Proposition 5.8 has the same restriction than the one we had for the positive results. This is due to the fact that the proof uses these positive results for the Besov localization of the Lévy noise with finite moments w_2. Therefore, if one extends Proposition 5.5 to any $p\geq 2$, it automatically implies that Proposition 5.8 is also valid for any $p>0$.

Besov regularity of non-Gaussian Lévy noise:. Theorem 5.3 condenses the results of Section 5.2.3 and handles the case of composed Lévy noise, that is, Lévy noise with both nonzero sparse and Gaussian parts.

Theorem 5.3. *Consider a non-Gaussian Lévy noise with indices* α_{loc} *and* α_{asymp}*. We fix* $0 < p < 2$ *a real number or* $p \geq 2$ *an even integer. Then,* w *is*

- *almost surely in* $B_p^\tau(\mathbb{R}^d;\rho)$ *if*

$$\tau < \frac{d}{\max(p,\alpha_{\text{loc}})} - d \text{ and } \rho < -\frac{d}{\min(p,\alpha_{\text{asymp}})};$$

- *almost surely not in* $B_p^\tau(\mathbb{R}^d;\rho)$ *if*

$$\tau > \frac{d}{p} - d \text{ or } \rho > -\frac{d}{\min(p,\alpha_{\text{asymp}})}; \text{ and}$$

- *almost surely not in* $B_p^\tau(\mathbb{R}^d;\rho)$ *if*

$$\tau > \frac{d}{\max(p,\alpha_{\text{loc}})} - d \text{ or } \rho > -\frac{d}{\min(p,\alpha_{\text{asymp}})}$$

  and under the additional assumption that the Lévy exponent satisfies

$$\Psi(\xi) \underset{\infty}{\sim} -C|\xi|^{\alpha_{\text{loc}}} \tag{5.29}$$

  for some $C > 0$ *when* $\alpha_{\text{loc}} > 0$.

Proof. When the Lévy noise is sparse (without a Gaussian part), Theorem 5.3 is a reformulation of Propositions 5.4 to 5.8. We now assume that w is composed, that is, $w = w_{\text{Gauss}} + w_{\text{sparse}}$ with $w = w_{\text{Gauss}}$ and w_{sparse} two independent Gaussian and sparse noise, respectively. In that case, one has that $\alpha_{\text{loc}} = 2$. □

Theorem 5.3 implies that, for a non-Gaussian Lévy noise and if $0 < p < 2$ or $p \geq 2$ is an even integer, then

$$A_p(\alpha_{\text{loc}},\alpha_{\text{asymp}}) \subset \mathscr{E}_p(w) \subseteq \bar{A}_p(p,\alpha_{\text{asymp}}).$$

with $A_p(x,y) = \left(-\infty, \frac{d}{\max(p,x)} - d\right) \times \left(-\infty, -\frac{d}{\min(p,y)}\right)$ and $\bar{A}_p(x,y)$ its closure. If in addition the Lévy exponent behaves adequately at infinity, then we have the more precise estimate

$$A_p(\alpha_{\text{loc}},\alpha_{\text{asymp}}) \subset \mathscr{E}_p(w) \subseteq \bar{A}_p(\alpha_{\text{loc}},\alpha_{\text{asymp}}). \tag{5.30}$$

Several questions remain for a complete characterization of the Besov localization of Lévy noise.

- First, and most importantly, the negative result on the smoothness is not complete. In the general case, we only showed that the Lévy noise is not in the corresponding Besov space if $\tau > d/p - d$. Under an additional assumption on the Lévy exponent (see (5.29)) , we showed that this condition becomes $\tau > d/\max(p,\alpha_{\text{loc}}) - d$. This latter condition is sharp, as we see by comparing with the positive results. We conjecture that this results remain valid in general. In particular, this would imply that (5.30) is valid for any non-Gaussian Lévy noise.

- We did not treat the case $p \geq 2$ when p is not an even integer. We conjecture that our conclusions are also valid in this case.
- Finally, we did not consider in full generality the limit cases when $\tau = d/\max(p, \alpha_{\mathrm{loc}}) - d$ or $\rho = -d/\min(p, \alpha_{\mathrm{asymp}})$. For these smoothness or decay rate values, we conjecture that the Lévy noise is not in the corresponding Besov space, in analogy with the Gaussian case.

5.2.4 Smoothness and Decay Rate of Lévy Noise

In light of the above, we deduce the local smoothness and the asymptotic decay rate of Lévy noise in the following cases.

Theorem 5.4. *Let w be a Lévy noise with local and asymptotic indices $\alpha_{\mathrm{loc}} \in [0,2]$ and $\alpha_{\mathrm{asymp}} \in (0,\infty]$. All the following equalities are almost sure.*

- *If $w = w_{\mathrm{Gauss}}$ is Gaussian, then, for every $0 < p \leq \infty$,*

$$\tau_p(w_{\mathrm{Gauss}}) = -\frac{d}{2} \text{ and } \rho_p(w_{\mathrm{Gauss}}) = -\frac{d}{p}. \tag{5.31}$$

- *If $w = w_{\mathrm{Poisson}}$ is compound Poisson, then, for every $0 < p \leq \infty$,*

$$\tau_p(w_{\mathrm{Poisson}}) = \frac{d}{p} - d \text{ and } \rho_p(w_{\mathrm{Poisson}}) = -\frac{d}{\min(p, \alpha_{\mathrm{asymp}})}. \tag{5.32}$$

- *If w is non-Gaussian, $\alpha_{\mathrm{loc}} > 0$, and under the assumption (5.29), then, for every real $0 < p < 2$, even integer $p \geq 2$, or $p = \infty$,*

$$\tau_p(w) = \frac{d}{\max(p, \alpha_{\mathrm{loc}})} - d \text{ and } \rho_p(w) = -\frac{d}{\min(p, \alpha_{\mathrm{asymp}})}. \tag{5.33}$$

- *If w is non-Gaussian, then, for every real $0 < p < 2$, even integer $p \geq 2$, or $p = \infty$,*

$$\frac{d}{\max(p, \alpha_{\mathrm{loc}})} - d \leq \tau_p(w) \leq \frac{d}{p} - d \text{ and } \rho_p(w) = -\frac{d}{\min(p, \alpha_{\mathrm{asymp}})}. \tag{5.34}$$

Proof. We treat the case of the compound Poisson noise, the other being very similar. We fix $0 < p < \infty$. The positive results of Theorem 5.2 imply that

$$\tau_p(w_{\mathrm{Poisson}}) \geq d/p - d \text{ and } \rho_p(w_{\mathrm{Poisson}}) \geq -d/\min(p, \alpha_{\mathrm{asymp}}).$$

The negative results imply the other inequalities, therefore we deduce (5.48).

If now $p = \infty$, the results are deduced from $p < \infty$ by taking $p \to \infty$ and $\epsilon > 0$ in the embedding $B_p^{\tau+d/p-\epsilon}(\mathbb{R}^d; \rho+\epsilon) \subseteq B_\infty^\tau(\mathbb{R}^d; \rho)$ valid for all $p < \infty$ and $\epsilon > 0$ (Proposition 2.8). For Lévy noise that are non-Gaussian and non-Poisson, the same argument works with $p = 2k$ and $k \to \infty$. □

Remarks.

- For Gaussian and Poisson noises, the local smoothness $\tau_p(w)$ and asymptotic decay rate $\rho_p(w)$ are fully characterized for every $p > 0$.
- The local smoothness and the asymptotic decay rate are if $p < 0$ is real, $p \geq 2$ is an even integer, or $p = \infty$ when $\alpha_{\text{loc}} = 0$ or when $\alpha_{\text{loc}} > 0$ and under the condition (5.29).
- In the general case, the results are for $p < 0$ real, $p \geq 2$ an even integer, or $p = \infty$. Under this restriction, the asymptotic decay rate is characterized. Moreover, the local smoothness is known for $p \geq \alpha_{\text{loc}}$. It is in particular the case for $p = 2$ and $p = \infty$.
- What remains is to show that $\tau_p(w) = d/\alpha_{\text{loc}} - d$ when $p < \alpha_{\text{loc}}$, without the assumption (5.29).

Sobolev and Hölder regularity. By specifying the value of p, one deduces the Sobolev ($p = 2$) and the Hölder ($p = \infty$) regularity of the Lévy noise.

Corollary 5.1. *For any nontrivial Lévy noise, we have that*

$$\tau_2(w) = -d/2, \text{ and}$$
$$\rho_2(w) = -d/\min(\alpha_{\text{asymp}}, 2). \tag{5.35}$$

Proof. We simply remark that all the local smoothness of Theorem 5.4 are equal to $-d/2$ when $p = 2$ (since $\alpha_{\text{loc}} \leq 2$). When w is non-Gaussian, the value $\rho_2(w)$ is always $-d/\min(\alpha_{\text{asymp}}, 2)$. Moreover, $\min(\alpha_{\text{asymp}}, 2) = 2$ for the Gaussian noise and (5.35) is coherent with (5.47). □

Remarks. It is remarkable that the local Sobolev regularity of the Lévy noises is identical. The case $p = 2$ is not sufficient to distinguish between different noises when considering the local regularity. If the variance of the noise is finite ($\alpha_{\text{asymp}} \geq 2$), we have that $\rho_2(w) = -d/2$, independently of the Lévy noise. Otherwise, the smaller α_{asymp}, the bigger $\rho_2(w)$ (in absolute value). We need to compensate the asymptotic decay due to the heavy-tailedness of the noise. The Pruitt index $\beta_0 = \min(\alpha_{\text{asymp}}, 2)$ is therefore the relevant quantity to measure the Sobolev decay rate of a Lévy noise.

Corollary 5.2. *Let w_{Gauss} and w be a Gaussian noise and a non-Gaussian Lévy noise ($\nu \neq 0$), respectively. Then, we have almost surely that*

$$\tau_\infty(w_{\text{Gauss}}) = -d/2 \text{ and } \tau_\infty(w) = -d.$$

Moreover, for any nontrivial Lévy noise, we almost surely have that

$$\rho_\infty(w) = -d/\alpha_{\text{asymp}}. \tag{5.36}$$

Remarks.

- The non-Gaussian noises have an identical Hölder regularity $\tau_\infty(w) = -d$, that is also the one of the Dirac implies δ. The case of Gaussian noise is different. With the same idea, the Brownian motion is the unique Lévy process that has continuous sample paths, other Lévy processes being only càdlàg [Ber98]. The interest of Corollary 5.2 is to quantify the gap of Hölder regularity between the two types of noise. The fact that the Hölder regularities are all negative is coherent with the idea that Lévy noises have no pointwise interpretation and should be described by their effects on test functions.

- When all the moments of the noise are finite, we have that $\rho_\infty(w) = 0$. For heavy-tailed noises ($\alpha_{\text{asymp}} < \infty$), it is required to compensate with a weight of order $-d/\alpha_{\text{asymp}}$.

- Conversely to the Sobolev regularity, it is the asymptotic index α_{asymp} that is relevant to quantify the Hölder decay rate of a Lévy noise. Comparing (5.35) and (5.36), we have another justification for our choice of notation for the asymptotic indices of the Lévy noise. The Pruitt index $\min(\alpha_{\text{asymp}}, 2)$ is associated to the Sobolev rate of decay $\rho_2(w)$, while α_{asymp} is inversely proportional to the Hölder decay rate $\rho_\infty(w)$.

Comparison with known results. Several authors have studied the Besov regularity of Lévy processes or Lévy white noises. For comparison purposes, we interpret their results in terms of the functions $\tau_p(s)$ and $\rho_p(s)$, with s the random process of interest. When the study is local, the only information is on $\tau_p(s)$. In the literature, most of the results are expressed with the index $\beta_0 = \min(\alpha_{\text{asymp}}, 2)$. Most of the authors work with Besov spaces $B^{\tau}_{p,q}$, where $q \in (0,\infty]$ is an additional parameter. In our case, we have only considered $p = q$. This is reasonable for our purpose because the parameter q plays a secondary role, due to the embeddings $B^{\tau+\epsilon}_{p,q}(\mathbb{R}^d;\rho) \subseteq B^{\tau}_{p,r}(\mathbb{R}^d;\rho)$, valid for any $\epsilon > 0$ and $0 < p, q, r \leq \infty$. Finally, we sometimes complete the results we refer to by using embeddings between Besov spaces without specifying it.

Lévy processes. In the past, the Besov regularity of Lévy processes has received more attention than the one of Lévy noises. A Lévy process X is solution of the stochastic differential equation $\mathrm{D}X = w$ with D the derivative and w a one-dimensional Lévy noise.
Ciesielski *et al.* have studied the Gaussian and SαS cases locally in [CKR93]. Their results imply that

$$\tau_p(X_{\text{Gauss}}) = 1/2, \tag{5.37}$$

$$\tau_p(X_\alpha) \begin{cases} = 1/\alpha & \text{if } p < \alpha \\ \geq 1/p & \text{if } p \geq \alpha, \end{cases} \tag{5.38}$$

for $1 \leq p \leq \infty$, with X_{Gauss} the Brownian motion and X_α the SαS process for $1 \leq \alpha < 2$. In a series of papers [Sch97, Sch98, Sch00], summarized in [BSW14], Schilling obtained the following results for Lévy processes:

$$\frac{1}{\max(p,\alpha_{\mathrm{loc}})} \leq \tau_p(X) \leq \frac{1}{p}, \tag{5.39}$$

$$-\frac{1}{p} - \frac{1}{\min(\alpha_{\mathrm{asymp}},2)} \leq \rho_p(X). \tag{5.40}$$

This yields several comments.

- The regularity of a Lévy process and the underlying noise are linked by the relation $\tau_p(X) = \tau_p(w)+1$. With that respect, (5.37), (5.38), and (5.39) are coherent with Theorem 5.4.
- Ciesielski *et al.* obtained an exact estimation for stable processes by exploiting the self-similarity. On the contrary, the general results of Schilling mostly deal with positive results that imply a lower bound on $\tau_p(X)$. The upper bound in (5.39) is not sharp and exploits the discontinuity of the trajectories of non-Gaussian Lévy processe; see [BSW14, Corollary 5.28]. The results (5.39) are equivalent with our smoothness result (5.34). Under the assumption (5.29), we improved the result by showing that the lower bound of (5.39) is sharp.
- Conversely to the smoothness, the decay rate $\rho_p(X)$ of the Lévy process and the one of $\rho_p(w)$ of the underlying Lévy noise seem not to be related by a constant (with respect to p). This needs to be confirmed by a precise estimation of $\tau_p(X)$ for which only a lower bound is known, together with a precise estimation of $\tau_p(w)$ when $p>2$ is not an even integer. Our conjecture is that the lower bound (5.40) is sharp for any $p>0$. If this is true, it means that $\min(\alpha_{\mathrm{asymp}},2)$ is the relevant quantity for the growth rate of the Lévy process, contrary to the Lévy noise for which it is α_{asymp}.

Lévy noise. Veraar studied the local Besov regularity of the Gaussian white noise. As a corollary of [Ver10, Theorem 3.4], we deduce that $\tau_p(w) = -d/2$. We gave a new proof of this result with alternative technics based on wavelets, while Veraar was considering Fourier series expansions. The localization of the Gaussian white noise in weighted Sobolev spaces was studied by Kusuoka [Kus82].

Application of the Results to Specific Lévy Noises. The Gaussian and Poisson cases have already been treated. Knowing their indices (*cf.* Section 2.1.3), the SαS and Laplace cases are easily deduced from Corollary 5.4. Note that the local smoothness is known for these two examples, because $\alpha_{\mathrm{loc}} = 0$ for the Laplace noise, while the Lévy exponent satisfies (5.29) for SαS.

Corollary 5.3. *Let $0<p<2$ be a real number or $p \geq 2$ be an even integer.*

- *The SαS noise w_α almost surely satisfies*

$$\tau_p(w_\alpha) = \frac{d}{\max(p,\alpha)} - d \ \text{ and } \ \rho_p(w_\alpha) = -\frac{d}{\min(p,\alpha)}.$$

- *The Laplace noise* w_{Laplace} *almost surely satisfies*

$$\tau_p(w_{\text{Laplace}}) = \frac{d}{p} - d \quad \textit{and} \quad \rho_p(w_{\text{Laplace}}) = -\frac{d}{p}.$$

5.3 Smoothness of Periodic Generalized Lévy Processes

In this section, we identify the local smoothness $\tau_p(s)$ of a large class of generalized Lévy processes s. To do so, we work on the d-dimensional torus, and therefore specify the considered processes as the periodized version of the generalized Lévy processes of Section 6. However, we will not address here the question of the asymptotic decay rate of s.

The section is mostly based on [FUW17a]. Since we are only interested in the local smoothness, we simplify the study of the stochastic differential equation $\mathrm{L}s = w$ by introducing spaces of homogeneous (or 0-mean) periodic functions in Section 5.3.1. On such spaces, the study of the whitening operators is particularly pleasant. It is exposed in Section 5.3.2. Finally, we collect the results in Section 5.3.3, where the local smoothness of periodic generalized Lévy processes is quantified.

5.3.1 Homogeneous Periodic Function Spaces

We work with periodic generalized functions in $\mathcal{S}'(\mathbb{T}^d)$. The set of *homogeneous smooth functions* is

$$\dot{\mathcal{S}}(\mathbb{T}^d) := \left\{ \varphi \in \mathcal{S}(\mathbb{T}^d) \,\middle|\, c_{\mathbf{0}}(\varphi) = \langle \varphi, 1 \rangle = 0 \right\}.$$

Its topological dual $\dot{\mathcal{S}}(\mathbb{T}^d)$ is the space

$$\dot{\mathcal{S}}'(\mathbb{T}^d) := \left\{ u \in \mathcal{S}'(\mathbb{T}^d) \,\middle|\, c_{\mathbf{0}}(u) = \langle u, 1 \rangle = 0 \right\}.$$

The space $\dot{\mathcal{S}}(\mathbb{T}^d)$ inherits the structure of nuclear countably multi-Hilbert (or nuclear Fréchet) space of $\mathcal{S}(\mathbb{T}^d)$ (see Section 2.2.1). Thus, the space $\dot{\mathcal{S}}'(\mathbb{T}^d)$ is a nuclear (DF)-space.

It is possible to specify periodic Besov spaces (homogeneous periodic Besov spaces, respectively) in $\mathcal{S}'(\mathbb{T}^d)$ (in $\dot{\mathcal{S}}'(\mathbb{T}^d)$, respectively) using wavelet methods, as we did for weighted Besov spaces in Section 2.2.3. Here, we follow a different but equivalent approach, based on Fourier transform, that is more adapted to the study of operators in periodic function spaces. The equivalence between the wavelet-based and the Fourier-based constructions is proven in [Tri08, Section 1.3.3].

The following definition of homogeneous periodic Besov spaces (Definition 5.1) is taken from [Tri08, Definition 1.27]. The idea is to decompose a function f by grouping dyadic frequency bands using a partition of unity in the Fourier domain. In what follows, we fix $\widehat{v} \in \mathcal{S}(\mathbb{R}^d)$ such that

- $\widehat{v}(\boldsymbol{\omega}) = 0$ if $\|\boldsymbol{\omega}\| \leq 1/2$ or $\|\boldsymbol{\omega}\| \geq 2$,
- $\widehat{v}(\boldsymbol{\omega}) > 0$ if $1/2 < \|\boldsymbol{\omega}\| < 2$,
- $\sum_{j \geq 0} \widehat{v}(2^{-j}\boldsymbol{\omega}) = 1$ if $1 \leq \|\boldsymbol{\omega}\|$.

We say that $\widehat{v}$ generates a hierarchical partition of unity outside the ball of radius $1/2$ centered at the origin.

Definition 5.1. *Suppose* $0 < p \leq \infty$ *and* $\tau \in \mathbb{R}$. *A generalized function* $f \in \dot{\mathcal{S}}'(\mathbb{T}^d)$ *with Fourier coefficients* $c_{\boldsymbol{n}}(f)$ *is in* $\dot{B}_p^\tau(\mathbb{T}^d)$ *if the quantity*

$$\|f\|_{\dot{B}_p^\tau(\mathbb{T}^d)} := \left(\sum_{j=0}^{\infty} 2^{j\tau p} \left\| \sum_{\boldsymbol{n} \in \mathbb{Z}^d \setminus \{\boldsymbol{0}\}} c_{\boldsymbol{n}}(f) \widehat{v}(2^{-j}\boldsymbol{n}) \mathrm{e}^{2\pi \mathrm{i} \langle \boldsymbol{n}, \cdot \rangle} \right\|_{L_p(\mathbb{T}^d)}^p \right)^{1/p} \tag{5.41}$$

is finite, with the usual modification when $p = \infty$.

The Besov spaces $\dot{B}_p^\tau(\mathbb{T}^d)$ are Banach spaces for the norm (5.41) when $p \geq 1$. For $p < 1$, (5.41) is a quasi-norm and Besov spaces are quasi-Banach spaces. The validity of the embeddings between homogeneous periodic Besov spaces is governed by Proposition 5.9 [Tri08], which is the periodic version of Proposition 2.8.

Proposition 5.9. *Let* $0 < p_0 \leq p_1 \leq \infty$ *and* $\tau_0, \tau_1, \rho_0, \rho_1 \in \mathbb{R}$.

- *We have the embedding* $\dot{B}_{p_0}^{\tau_0}(\mathbb{T}^d) \subseteq \dot{B}_{p_1}^{\tau_1}(\mathbb{T}^d)$ *as soon as*

$$\tau_0 - \tau_1 > \frac{d}{p_0} - \frac{d}{p_1}. \tag{5.42}$$

- *We have the embedding* $\dot{B}_{p_1}^{\tau_1}(\mathbb{T}^d) \subseteq \dot{B}_{p_0}^{\tau_0}(\mathbb{T}^d)$ *as soon as*

$$\tau_0 < \tau_1. \tag{5.43}$$

If we fix the integrability rate $p \in (0, \infty]$, we define the local smoothness of $f \in \dot{S}'(\mathbb{T}^d)$ as

$$\tau_p(f) := \sup \left\{ \tau \in \mathbb{R} \;\middle|\; f \in \dot{B}_p^\tau(\mathbb{R}^d) \right\}. \tag{5.44}$$

Proposition 5.10. *For every* $0 < p_0 \leq p_1 \leq \infty$, *and every* $f \in \dot{S}'(\mathbb{T}^d)$, *we have*

$$\tau_{p_0}(f) - \left(\frac{d}{p_0} - \frac{d}{p_1} \right) \leq \tau_{p_1}(f) \leq \tau_{p_0}(f). \tag{5.45}$$

In particular, $p \mapsto \tau_p(f)$ *is a decreasing continuous function.*

Proof. We prove the second inequality in (5.45), the first one being similar. Let $\tau < \tau_{p_1}(f)$ and $\epsilon > 0$. Then, $f \in \dot{B}_{p_1}^\tau(\mathbb{T}^d) \subseteq \dot{B}_{p_0}^{\tau-\epsilon}(\mathbb{T}^d)$ according to (5.43). Therefore, for every $\tau < \tau_{p_1}(f)$ and $\epsilon > 0$, we have $\tau_{p_0}(f) \geq \tau - \epsilon$. We deduce the result with $\tau \to \tau_{p_1}(f)$ and $\epsilon \to 0$. □

5.3.2 Operators on Homogeneous Periodic Functions

We shall consider the class of differential and pseudo-differential operators that reduce the Besov regularity of a function by some (possibly fractional) order $\gamma > 0$. Importantly, since we are interested in the regularity properties of the solutions of the differential equation $\mathrm{L}s = w$, we focus on those operators that are continuous bijections from $\dot{B}_p^{\tau+\gamma}(\mathbb{T}^d)$ to $\dot{B}_p^\tau(\mathbb{T}^d)$. For those operators, the smoothness of the generalized Lévy process is easily deduced from that of the underlying Lévy noise.

We consider linear and shift-invariant operators L that continuously maps $\mathscr{S}(\mathbb{R}^d)$ to $\mathscr{S}'(\mathbb{R}^d)$. We assume that L has a continuous Fourier multiplier $\widehat{L}$. We have seen in Section 2.2.2 that L specifies a continuous operator from $\mathscr{S}(\mathbb{T}^d)$ to itself (and by extension from $\mathscr{S}'(\mathbb{T}^d)$ to itself) if and only if the sequence $(\widehat{L}(2\pi\boldsymbol{n}))_{\boldsymbol{n}\in\mathbb{Z}^d}$ is slowly growing.
By working on homogeneous function space, we can also consider operators for which $\widehat{L}(\boldsymbol{\omega})$ has no limit when $\boldsymbol{\omega}$ vanishes. Therefore, L specifies a continuous operator from $\dot{\mathscr{S}}(\mathbb{T}^d)$ to itself (and by extension from $\dot{\mathscr{S}}'(\mathbb{T}^d)$ to itself) if and only if the sequence $(\widehat{L}(2\pi\boldsymbol{n}))_{\boldsymbol{n}\in\mathbb{Z}^d\setminus\{\boldsymbol{0}\}}$ is slowly growing. For instance, the integrator D^{-1} with impulse response $\mathbb{1}_{\mathbb{R}^+}$ does not specify a operator from $\mathscr{S}(\mathbb{T})$ to itself ($\mathrm{D}^{-1}\varphi\in\mathscr{S}(\mathbb{T})$ if and only if φ has zero mean) However, it is a valid operator on $\dot{\mathscr{S}}(\mathbb{T})$, and by extension on $\dot{\mathscr{S}}'(\mathbb{T})$, characterized by the relation

$$\mathrm{D}^{-1}\{u\} = \sum_{n\in\mathbb{Z}\setminus\{0\}} (\mathrm{i}n)^{-1} c_n(u) e_n$$

for any $u\in\dot{\mathscr{S}}'(\mathbb{T})$, where $e_n(x) = \mathrm{e}^{\mathrm{i}nx}$. This motivates the use of homogeneous function spaces: we do not have to pay attention to the mean of the function, which can always be considered as being equal to 0. This makes the operators such as D^{-1} stable in $\dot{S}'(\mathbb{T}^d)$. The operator D^{-1} is actually a continuous bijection from $\dot{\mathscr{S}}(\mathbb{T})$ to $\dot{\mathscr{S}}(\mathbb{T})$, which reduces the regularity of any function of one order ($\tau_{\mathrm{D}s}(p) = \tau_s(p) - 1$). The following definition generalizes this idea.

Definition 5.2. *An operator* L*, continuous from* $\dot{\mathscr{S}}(\mathbb{T}^d)$ *to itself, is said to be* γ-admissible *for* $\gamma\in\mathbb{R}$ *if* $\mathrm{L} : \dot{B}_p^{\tau+\gamma}(\mathbb{T}^d) \to \dot{B}_p^{\tau}(\mathbb{T}^d)$ *is a continuous bijection and* L^{-1} *is continuous for every* $0 < p \leq \infty$ *and* $\tau\in\mathbb{R}$.

In particular, a γ-admissible operator is a bijection from $\dot{\mathscr{S}}(\mathbb{T}^d)$ to itself. This imposes that $\widehat{L}(2\pi\boldsymbol{n}) \neq 0$ for any $\boldsymbol{n}\neq\boldsymbol{0}$, and that the sequence $(\widehat{L}(2\pi\boldsymbol{n}))_{\boldsymbol{n}\neq\boldsymbol{0}}$ and $(\widehat{L}(2\pi\boldsymbol{n})^{-1})_{\boldsymbol{n}\neq\boldsymbol{0}}$ are slowly increasing.
The fractional Laplacian $(-\Delta)^{\gamma/2}$ of order $\gamma>0$ is the canonical example of a γ-admissible operator. Moreover, perturbations of the fractional Laplacian are also γ-admissible. The next few results make this statement precise. The idea is the following: An operator L is γ-admissible if and only if $(-\Delta)^{\gamma/2}\mathrm{L}^{-1}$ and $(-\Delta)^{-\gamma/2}\mathrm{L}$ are automorphisms on Besov spaces.

Proposition 5.11. *The fractional Laplacian* $(-\Delta)^{\gamma/2}$ *is a* γ*-admissible operator.*

Proof. This follows from the homogeneity of the Fourier multiplier of the fractional Laplacian. Applying Theorem 3.3.4 of [ST87] to Definition 5.1 gives the result. □

Theorem 5.5. *Let* L *be an admissible operator with continuous Fourier multiplier* $\widehat{L}$*. For* $\gamma>0$*, we define* $m_{\mathrm{L},\gamma}(\boldsymbol{\omega}) = \|\boldsymbol{\omega}\|^{-\gamma}\widehat{L}(\boldsymbol{\omega})$*. Also, let* ζ *be any function in* $\mathscr{S}(\mathbb{R}^d)$ *satisfying*

$$0 \leq \zeta(\boldsymbol{x}) \leq 1, \quad \zeta(\boldsymbol{x}) = \begin{cases} 0 & \text{if } \|\boldsymbol{x}\| \leq 1/4 \text{ or } \|\boldsymbol{x}\| \geq 4, \\ 1 & \text{if } 1/2 \leq \|\boldsymbol{x}\| \leq 2. \end{cases}$$

If the function m satisfies

$$\sup_{j\in\mathbb{N}} \left(\left\| \zeta(\cdot) m_{\mathrm{L},\gamma}(2^j\cdot) \right\|_{W_2^\tau(\mathbb{R}^d)} + \left\| \zeta(\cdot) m_{\mathrm{L},\gamma}(2^j\cdot)^{-1} \right\|_{W_2^\tau(\mathbb{R}^d)} \right) < \infty$$

for all $\tau > 0$, *then* L *is* γ*-admissible.*

Proof. This follows from a sufficient condition for Fourier multipliers on Besov spaces [ST87, Theorem 3.6.3]. To summarize, if $0 < p < \infty$ and

$$\tau > d\left(\frac{1}{\min(1,p)} - \frac{1}{2}\right),$$

then there exists $C > 0$ such that

$$\left\| \sum_{\boldsymbol{n}\in\mathbb{Z}^d\setminus\{\boldsymbol{0}\}} m_{\mathrm{L},\gamma}(2\pi\boldsymbol{n})c_{\boldsymbol{n}}(f)\mathrm{e}^{2\mathrm{i}\pi\langle\boldsymbol{n},\cdot\rangle}\right\|_{\dot{B}_p^\tau(\mathbb{T}^d)} \leq C\left(\sup_{j\in\mathbb{N}}\left\|\zeta(\cdot)m_{\mathrm{L},\gamma}(2^j\cdot)\right\|_{W_2^\tau(\mathbb{R}^d)}\right)\|f\|_{\dot{B}_p^\tau(\mathbb{T}^d)}$$

holds for all functions $m \in L_\infty(\mathbb{R}^d)$ and all $f \in \dot{B}_p^\tau(\mathbb{T}^d)$. □

Examples. The following whitening operators are γ-admissible.

- The derivative D is 1-admissible.
- The differential operators $\mathrm{D}^N + a_{N-1}\mathrm{D}^{N-1} + \cdots + a_0\mathrm{Id}$ with non-vanishing Fourier multipliers (except possibly at 0) are N-admissible.
- The fractional derivative D^γ is γ-admissible for any $\gamma > 0$.
- The fractional Laplacian $(-\Delta)^{\gamma/2}$ is γ-admissible for any $\gamma > 0$.
- The Bessel operator $\mathrm{J}_\gamma = (\mathrm{Id} - \Delta)^{\gamma/2}$ is γ-admissible for any $\gamma > 0$.

5.3.3 From Lévy Noises to Generalized Lévy Processes

The definition of generalized random processes, characteristic functionals, and the corresponding results of Section 2.3 are still valid over the nuclear space $\dot{\mathcal{S}}'(\mathbb{T}^d)$. Let Ψ be a Lévy exponent. We define the *periodic Lévy noise* w as the periodic generalized random process with characteristic functional

$$\widehat{\mathscr{P}}_w(\varphi) = \exp\left(\int_{\mathbb{T}^d} \Psi(\varphi(\boldsymbol{x}))\mathrm{d}\boldsymbol{x}\right)$$

for every $\varphi \in \dot{\mathscr{S}}(\mathbb{T}^d)$. If L is a γ-admissible operator for some $\gamma \geq 0$, then the functional $\varphi \mapsto \widehat{\mathscr{P}}_w(\mathrm{L}^{-1}\varphi)$ is a valid characteristic functional over $\dot{S}(\mathbb{T}^d)$, because L^{-1} is an automorphism on $\dot{S}(\mathbb{T}^d)$. Thus, the generalized Lévy process $s = \mathrm{L}^{-1}w$ with characteristic functional $\widehat{\mathscr{P}}_s(\varphi) = \widehat{\mathscr{P}}_w(\mathrm{L}^{-1}\varphi)$ is well-defined. We call s a *periodic generalized Lévy process.*

For Ψ a Lévy exponent, we have two notions of Lévy noise: one over $\mathbb{R}^d$ and the other on $\mathbb{T}^d$, that we denote by w and w_{per}, respectively. In particular, $\tau_p(w)$ is characterized in Section 5.1, while $\tau_p(w_{\mathrm{per}})$ is defined by (5.44). One important difference between the periodic and the global settings is that $\tau_p(f_{\mathrm{per}})$ is effectively well-defined for any periodic function f_{per}. In the global setting, we have characterized $\tau_p(f)$ in a unique fashion, but we did not prove its existence in general, as commented at the end of Section 5.1. In particular, the following result holds for the periodic setting.

Proposition 5.12. *If f_{per} satisfies $\tau_p(f_{\mathrm{per}}) = d/p + \tau_0$ for every even integer $p \geq 2$, then $\tau_p(f_{\mathrm{per}}) = d/p + \tau_0$ for every real $p \geq 2$.*

Proof. We fix $2k < p < 2(k+1)$ with $k \geq 1$ an integer. According to Proposition 5.10, we have that

$$\tau_p(f) \geq \tau_{2k}(f) - \left(\frac{d}{2k} - \frac{d}{p}\right) = \frac{d}{p} + \tau_0. \tag{5.46}$$

If now $\tau > d/p + \tau_0$, then

$$\tau - \left(\frac{d}{p} - \frac{d}{2(k+1)}\right) > \frac{d}{2(k+1)} + \tau_0 = \tau_{2(k+1)}(f),$$

implying that $f \notin \dot{B}_{2(k+1)}^{\tau-(d/p-d/2(k+1)}(\mathbb{T}^d)$. According to the embedding (5.42), this implies that $f \notin \dot{B}_p^{\tau+\epsilon}(\mathbb{T}^d)$ for every $\epsilon > 0$. In particular, $\tau_p(f) \leq \tau + \epsilon$. By taking $\epsilon \to 0$ and $\tau \to d/p + \tau_0$, we deduce that $\tau_p(f) \leq d/p + \tau_0$, which, together with (5.46), gives the result. □

The proofs for the Besov regularity of the Lévy noise in the global setting can be adapted to the periodic setting, and we obtain that $\tau_p(w) = \tau_p(w_{\mathrm{per}})$. Based on this principle, we deduce the smoothness of periodic generalized Lévy processes.

Corollary 5.4. *We consider a periodic generalized Lévy process $s = \mathrm{L}^{-1} w$, where w is a Lévy white noise with local index $\alpha_{\mathrm{loc}} \in [0,2]$ and L is a γ-admissible operator with $\gamma \geq 0$.*

- *If $w = w_{\mathrm{Gauss}}$ is Gaussian, then, for every $0 < p \leq \infty$, we have almost surely*

$$\tau_p(s) = \gamma - \frac{d}{2}. \tag{5.47}$$

- *If $w = w_{\mathrm{Poisson}}$ is compound Poisson, then, conditionally to $w_{\mathrm{Poisson}} \neq 0$, for every $0 < p \leq \infty$, we have almost surely*

$$\tau_p(s) = \gamma + \frac{d}{p} - d. \tag{5.48}$$

- *If w is non-Gaussian and non-Poisson with $\alpha_{\mathrm{loc}} > 0$ and its Lévy exponent satisfies* (5.29)*, then, for every $0 < p \leq \infty$, we have almost surely*

$$\tau_p(s) = \gamma + \frac{d}{\max(p, \alpha_{\mathrm{loc}})} - d. \tag{5.49}$$

- *If w is non-Gaussian and non-Poisson, then, for every $0 < p \leq \infty$, we have almost surely*

$$\gamma + \frac{d}{\max(p, \alpha_{\mathrm{loc}})} - d \leq \tau_p(s) \leq \gamma + \frac{d}{p} - d. \tag{5.50}$$

Proof. For compound Poisson noise, w is zero with probability $e^{-\lambda}$ (that corresponds to a number of jump $N = 0$ over $\mathbb{T}^d$. In that case, of course, $\tau_p(w) = \infty$. We condition to the event $N \neq 0$ to avoid this case. The case of the Lévy noise (L is the identity and $\gamma = 0$) is treated by adapting the proof of Theorem 5.4 to the periodic setting (which is possible using the wavelet-domain characterization of periodic Besov spaces; see [FUW17b] for the case of SαS noise). With Proposition 5.12, we extend the result to any $p > 0$ for sparse and composed Lévy noise. Finally, the result is extended to s because L is γ-admissible, implying that $\tau_p(\mathrm{L}f) = \tau_p(f) - \gamma$ for any f. □

In the periodic framework, we have identified the local regularity of many generalized Lévy process whitened by a γ-admissible operator. As for the Lévy noises, what remains is to show that the lower bound of (5.50) is sharp, even when $\alpha_{\mathrm{loc}} > 0$ but the Lévy exponent does not satisfies (5.29).

6 Local Compressibility of Generalized Lévy Processes

In Chapter 1, we have argued that non-Gaussian generalized Lévy processes are good candidates for the stochastic modeling of sparse signals. In this section, we define and evaluate the *local compressibility* of generalized Lévy processes. The compressibility of a function is measured by the decay rate of the error of its best N-term approximation. Our results are based on the estimations of the Besov regularity of the Lévy white noises and generalized Lévy processes presented in Chapter 5. We show, in particular, that non-Gaussian generalized Lévy processes are more compressible in a wavelet basis than their Gaussian counterpart in the sense that the error of their best N-term approximation decays faster. We quantify the compressibility in terms of the local (or Blumenthal-Getoor) index α_{loc} of the Lévy noise and of the order γ of the whitening operator. This section is mostly based on our work from [FUW17a], with important extensions taking advantage of the results of [AFU].

6.1 N-Term Approximation and Besov Regularity

In this section, we highlight the link between the Besov regularity and the decay rate of the approximation error of a (deterministic) generalized function. The application of these results to random processes will be done in Section 6.2. We here focus on homogeneous periodic function spaces, in order to study the *local* properties of functions. We are mostly interested in the approximation error in the space $\dot{L}_2(\mathbb{T}^d)$ of homogeneous square-integrable functions in $\mathbb{T}^d$, but we shall consider the approximation error in a general homogeneous periodic Besov space.

Following Triebel [Tri08], we briefly introduce the Daubechies wavelets in the d-dimensional torus. We also give a wavelet-based characterization of homogeneous Besov spaces. Periodizing the compactly supported Daubechies wavelets [Dau92] results in the orthonormal basis of $L_2(\mathbb{T}^d)$. With the exception of the Haar wavelet, the support of classical Daubechies wavelets is larger than $\mathbb{T}^d = [0,1]^d$. Consequently, the coarsest scale is scaled by 2^L, where the parameter $L \in \mathbb{N}$ ensures that the support is included in $\mathbb{T}^d$. For the rest of this chapter, we set L (as a function of the Daubechies wavelet order) to be the smallest integer that guarantees this condition on the support. The wavelet translates are still indexed by $\boldsymbol{k}$, and the set of translations at scale j is given by

$$\mathbb{P}_j^d = \left\{ \boldsymbol{k} \in \mathbb{Z}^d \;\middle|\; 0 \leq k_i < 2^{j+L}, i = 1, \dots, d \right\}.$$

Using the notation of Section 2.2.3, we set $\mathscr{I} := \left\{ (j, G, \boldsymbol{k}) \;\middle|\; j \in \mathbb{N}, G \in \mathrm{G}^j, \boldsymbol{k} \in \mathbb{P}_j^d \right\}$. The Daubechies wavelet basis is denoted by $(\psi^{\mathrm{per}}_{j,G,\boldsymbol{k}})_{(j,G,\boldsymbol{k})\in\mathscr{I}}$, where

$$\psi^{\mathrm{per}}_{j,G,\boldsymbol{k}} = 2^{jd/2} \psi^{\mathrm{per}}_{0,G,0}(2^j \cdot - \boldsymbol{k}).$$

The wavelet decomposition of $f \in L_2(\mathbb{T}^d)$ is $f = \sum_{j,G,\boldsymbol{k}} \langle f, \psi^{\mathrm{per}}_{j,G,\boldsymbol{k}} \rangle \psi^{\mathrm{per}}_{j,G,\boldsymbol{k}}$, with $\langle \cdot, \cdot \rangle$ the canonical scalar product on $L_2(\mathbb{T}^d)$. More details on the periodization of wavelet bases can be found in [Tri08, Section 1.3].

The following characterization of the periodic Besov spaces can be found in [ST87, Theorem 1.36]. It is the periodic version of Proposition 2.9.

Proposition 6.1. *Let* $\tau, \tau_0 \in \mathbb{R}$ *and* $0 < p \leq \infty$. *We set*

$$r_0 > \max\left(|\tau_0|, (d(1/p - 1))_+ - \tau \right).$$

Then, the periodic generalized function $f \in \dot{B}_2^{\tau_0}(\mathbb{T}^d) = \dot{W}_2^{\tau_0}(\mathbb{T}^d)$ *is in* $\dot{B}_p^{\tau}(\mathbb{T}^d)$ *if and only if*

$$\sum_{j \geq 0} 2^{j(\tau p - d + dp/2)} \sum_{G \in \mathrm{G}^j} \sum_{\boldsymbol{k} \in \mathbb{P}_j^d} \left| \langle f, \psi^{\mathrm{per}}_{J,G,\boldsymbol{k}} \rangle \right| < \infty.$$

with $(\psi^{\mathrm{per}}_{j,G,\boldsymbol{k}})$ *a Daubechies wavelet basis of* $L_2(\mathbb{T}^d)$ *with a regularity of at least* r_0, *with the usual modification for* $p = \infty$.

N-term Approximation. We fix a generalized function $f \in \dot{\mathcal{S}}'(\mathbb{T}^d)$ and a Daubechies wavelet basis with enough regularity such that the duality products between f and the wavelets are well-defined. An *N-term approximation* to f is a finite sum of the form

$$\sum_{(j,G,\boldsymbol{k})\in\mathcal{J}} c_{j,G,\boldsymbol{k}} \psi^{\text{per}}_{j,G,\boldsymbol{k}},$$

with $c_{j,G,\boldsymbol{k}} \in \mathbb{R}$ and $\mathcal{J}$ a finite subset of $\mathcal{I}$ of size N. If moreover $f \in \dot{B}^\tau_p(\mathbb{T}^d)$ for $0 < p \leq \infty$ and $\tau \in \mathbb{R}$, we denote by $\Sigma_{N,p,\tau}(f)$ the *best N-term approximation of f in $\dot{B}^\tau_p(\mathbb{T}^d)$*, defined as the N-term approximation that minimizes the approximation error in $\dot{B}^\tau_p(\mathbb{T}^d)$. We also set

$$\sigma_{N,p,\tau}(f) = \left\| f - \Sigma_{N,p,\tau}(f) \right\|_{\dot{B}^\tau_p(\mathbb{T}^d)},$$

which is the approximation error of f in $\dot{B}^\tau_p(\mathbb{T}^d)$. When $p = 2$ and $\tau = 0$, *i.e.*, $\dot{B}^\tau_p(\mathbb{T}^d) = \dot{L}_2(\mathbb{T}^d)$, we simply write $\Sigma_{N,2,0}(f) = \Sigma_N(f)$ and $\sigma_{N,2,0}(f) = \sigma_N(f)$.

Control of the approximation error. The speed of decay of the Fourier series coefficients of a function is well-known to be tightly related to its smoothness. This is also valid in wavelet bases [Mal99]. As a consequence, it is possible to relate the decay rate of the approximation error of functions in $L^2(\mathbb{T}^d)$, and more generally in $B^\tau_p(\mathbb{T}^d)$, to their inclusion in periodic Besov spaces. This topic has been investigated extensively in (deterministic) approximation theory [CDH00, Dev98, GH04]. We give now some insight for the case of the approximation error in $\dot{L}_2(\mathbb{T}^d)$.

- If we know that $f \in \dot{L}_2(\mathbb{T}^d)$ is in the Sobolev space $\dot{W}^\tau_2(\mathbb{T}^d)$ for some $\tau > 0$, this implies that the approximation error $\sigma_N(f)$ is dominated by $N^{-\tau/d}$. The higher τ, the faster the decay of the upper bound. When f is infinitely smooth, we deduce that the approximation error vanishes faster than any polynomial.

- The previous result focuses on the integrality rate $p = 2$. It can be improved if we have additional information on the Besov localization for other integrability rates $p < 2$. The Besov regularity is indeed characterized by weighted ℓ_p-norms on the wavelet coefficients Correspondingly, the minimization of ℓ_p-norms for $p < 2$ induces sparser approximations. This is true in particular for $p = 1$ [UFW16, UFG16]. The limit case is when $p \to 0$, with strong connections to the notion of sparsity in the theory of compressed sensing [FR13]. The quantitative study of this fact is specified in Theorem 6.1 thereafter.

- It moreover appears that the complete characterization of the Besov localization of f fully determines the decay rate of its approximation error. Basically, the approximation error of a non-smooth function cannot have a fast rate of decay. This phenomenon can be captured sharply once one knows the Besov smoothness $\tau_p(f)$ of f for integrability rate $p \in (0,2]$. The simplest case $p = 2$ is usually not sufficient to obtain sharp results. Again, this is quantified in Theorem 6.1, in which we consider the decay of the approximation error in a general Besov space $\dot{B}^{\tau_0}_{p_0}(\mathbb{T}^d)$ and not only $\dot{L}_2(\mathbb{T}^d)$.

Theorem 6.1. *We fix $0 < p_0 < \infty$ and $\tau \in \mathbb{R}$. Assume that p and τ are such that*

$$0 < p < p_0 \text{ and } \Delta\tau := \tau - \tau_0 = \frac{d}{p} - \frac{d}{p_0}.$$

i) If $f \in \dot{B}_p^\tau(\mathbb{T}^d)$, then there is a constant $C > 0$ such that

$$\sigma_{N,p_0,\tau_0}(f) \leq C N^{-\Delta\tau/d} \|f\|_{\dot{B}_p^\tau(\mathbb{T}^d)}.$$

ii) If there are constants $C, \epsilon > 0$ such that

$$\sigma_{N,p_0,\tau_0}(f) \leq C N^{-\Delta\tau/d-\epsilon},$$

then $f \in \dot{B}_p^\tau(\mathbb{T}^d)$.

Proof. We define the Besov sequence spaces b_p^τ as the sequences λ such that

$$\|\lambda\|_{b_p^\tau} := \left(\sum_{(j,G,\boldsymbol{k}) \in \mathscr{I}} 2^{j(\tau p - d)} \left|\lambda_{j,G,\boldsymbol{k}}\right|^p \right)^{1/p} < \infty.$$

This proof uses Corollary 6.2 of [GH04], which characterizes N-term approximation spaces as Besov spaces. In particular,

$$b_p^{\tau_0+\Delta\tau} = A_p^{\Delta\tau/d}(b_{p_0}^{\tau_0}), \tag{6.1}$$

where $A_p^{\Delta\tau/d}(b_{p_0}^{\tau_0})$ is an approximation space with error measured in $b_{p_0}^{\tau_0}$. Essentially, $A_p^{\Delta\tau/d}(b_{p_0}^{\tau_0})$ is the collection of sequences f for which the sequence of error terms

$$N^{\Delta\tau/d} \sigma_{N,p_0,\tau_0}(f)$$

is in ℓ_p with respect to a Haar-type measure on $\mathbb{N}$.
This characterization along with standard embedding properties of approximation spaces [DL93, Chapter 7] allow us to derive our result. In particular, (6.1) together with the aforementioned embedding implies that

$$b_p^{\tau_0+\Delta\tau} \subset A_\infty^{\Delta\tau/d}(b_{p_0}^{\tau_0}).$$

Similarly, we have that

$$A_\infty^{\Delta\tau/d+\epsilon}(b_{p_0}^{\tau_0}) \subset b_p^{\tau_0+\Delta\tau}.$$

The fact that the continuous-domain Besov spaces are isomorphic to Besov sequence spaces [ST87, Theorem 1.36] completes the proof. □

Compressibility of a function. The compressibility of a (generalized) function quantifies the speed of convergence of its approximation error in a wavelet basis.

Definition 6.1. *For a generalized function $f \in \dot{B}^{\tau_0}_{p_0}(\mathbb{T}^d)$, we define its* (p_0,τ_0)-compressibility *as*

$$\kappa_{p_0,\tau_0}(f) := \sup\left\{\kappa \geq 0 \;\middle|\; \sup_{N\in\mathbb{N}} (N+1)^{\kappa} \left\| f - \Sigma_{N,p_0,\tau_0}(f)\right\|_{\dot{B}^{\tau_0}_{p_0}(\mathbb{T}^d)} < \infty\right\} \in [0,\infty]. \tag{6.2}$$

The quantity (6.2) is well-defined for $f \in \dot{B}^{\tau_0}_{p_0}(\mathbb{T}^d)$. If the approximation error has a faster-than-algebraic decay, then $\kappa_{p_0,\tau_0}(f) = \infty$. The value of $\kappa_{p_0,\tau_0}(f)$ quantifies the local compressibility of f in a wavelet basis: the higher the $\kappa_{p_0,\tau_0}(f)$, the more compressible the function f. In particular, we say that f is *strictly more compressible* than g in $\dot{B}^{\tau_0}_{p_0}(\mathbb{T}^d)$ if $\kappa_{p_0,\tau_0}(f) > \kappa_{p_0,\tau_0}(g)$. The (p_0,τ_0)-compressibility of $f \in \dot{B}^{\tau}_{p_0}(\mathbb{T}^d)$ is fully determined by the inclusion of f in the Besov spaces $\dot{B}^{d/p-d/p_0+\tau_0}_{p}(\mathbb{T}^d)$, where p describes $(0,p_0)$.

Proposition 6.2. *Let $f \in \dot{B}^{\tau}_{p_0}(\mathbb{T}^d)$, with $0 < p_0 < \infty$ and $\tau \in \mathbb{R}$. We set*

$$\begin{aligned} p_{p_0,\tau_0}(f) &:= \inf\left\{p \leq p_0 \;\middle|\; f \in \dot{B}^{d/p-d/p_0+\tau_0}_{p}(\mathbb{T}^d)\right\} \\ &= \inf\left\{p \leq p_0 \mid d/p - d/p_0 + \tau_0 < \tau_p(f)\right\} \\ &\in [0,p_0]. \end{aligned}$$

Then, we have

$$\kappa_{p_0,\tau_0}(f) = \frac{1}{p_{p_0,\tau_0}(f)} - \frac{1}{p_0}. \tag{6.3}$$

Proof. First, $f \in \dot{B}^{d/p_0-d/p_0+\tau_0}_{p_0}(\mathbb{T}^d) = \dot{B}^{\tau_0}_{p_0}(\mathbb{T}^d)$; hence $p_{p_0,\tau_0}(f)$ is well-defined. We set $\tau = d/p - d/p_0 - \tau_0$. If $p > p_{p_0,\tau_0}(f)$, then $f \in B^{\tau}_{p}(\mathbb{T}^d)$. Applying the first part of Theorem 6.1, we deduce that $\sigma_{N,p_0,\tau_0}(f) \leq CN^{-(1/p-1/p_0)}\|f\|_{\dot{B}^{\tau}_{p}(\mathbb{T}^d)}$, and therefore that $\kappa_{p_0,\tau_0} \geq 1/p - 1/p_0$. By taking $p \to p_{p_0,\tau_0}(f)$, we deduce that $\kappa_{p_0,\tau_0} \geq 1/p_{p_0,\tau_0}(f) - 1/p_0$.
If now $p < p_{p_0,\tau_0}(f)$, then $f \notin B^{\tau}_{p}(\mathbb{T}^d)$. From the second part of Theorem 6.1, we know that, for every $\epsilon > 0$, the quantity $\sigma_{N,p_0,\tau_0}(f)N^{1/p-1/p_0+\epsilon}$ is not bounded. This implies that $\kappa_{p_0,\tau_0} \leq 1/p - 1/p_0 - \epsilon$. With $\epsilon \to 0$ and $p \to p_{p_0,\tau_0}(f)$, we deduce that $\kappa_{p_0,\tau_0} \leq 1/p_{p_0,\tau_0}(f) - 1/p_0$. Finally, we have shown (6.3). □

Proposition 6.2 implies that the compressibility of f can easily be read using the graphical representation of $\tau_p(f)$ in the $(1/p,\tau)$-diagram.

6.2 The Compressibility of Generalized Lévy Processes

From what precedes, we know:

- The Besov localization of periodic generalized Lévy processes (Section 5.3.3);
- The characterization of the compressibility of a (deterministic) function via its Besov localization (Section 6.1).

We are therefore ready to deduce the compressibility of the generalized Lévy processes.

Theorem 6.2. *Let* $s = \mathrm{L}^{-1} w$ *be a generalized Lévy process in* $\dot{\mathcal{S}}'(\mathbb{T}^d)$*, with* L *a* γ*-admissible operator,* $\gamma \geq 0$*, and* w *a periodic Lévy noise. We fix* $0 < p_0 < \infty$ *and* $\tau \in \mathbb{R}$*.*

- *Assume that* $w = w_{\mathrm{Gauss}}$ *so that* $s = s_{\mathrm{Gauss}}$ *is Gaussian. If*

$$\gamma > \tau_0 + \frac{d}{2},$$

then, almost surely, $s_{\mathrm{Gauss}} \in \dot{B}^{\tau_0}_{p_0}(\mathbb{T}^d)$ *and*

$$\kappa_{p_0,\tau_0}(s_{\mathrm{Gauss}}) = \frac{\gamma - \tau_0}{d} - \frac{1}{2}.$$

- *Assume that* w *is non-Gaussian with local index* $\alpha_{\mathrm{loc}} = 0$*, or* $\alpha_{\mathrm{loc}} > 0$ *and the Lévy exponent of* w *satisfies* (5.29)*. If*

$$\gamma > \tau_0 + d - \frac{d}{\max(p_0, \alpha_{\mathrm{loc}})}, \tag{6.4}$$

then, almost surely, $s \in \dot{B}^{\tau_0}_{p_0}(\mathbb{T}^d)$ *and*

$$\kappa_{p_0,\tau_0}(s) = \frac{\gamma - \tau_0}{d} + \frac{1}{\alpha_{\mathrm{loc}}} - 1.$$

- *Assume that* w *is non-Gaussian with local index* $\alpha_{\mathrm{loc}} \in [0,2]$*. If*

$$\gamma > \tau_0 + d - \frac{d}{\max(p_0, \alpha_{\mathrm{loc}})},$$

then, almost surely, $s \in \dot{B}^{\tau_0}_{p_0}(\mathbb{T}^d)$ *and*

$$\kappa_{p_0,\tau_0}(s) \geq \frac{\gamma - \tau_0}{d} + \frac{1}{\alpha_{\mathrm{loc}}} - 1.$$

Proof. The proofs for the Gaussian and non-Gaussian cases are very similar. We shall therefore only develop the non-Gaussian case, with $\alpha_{\mathrm{loc}} = 0$ or $\alpha_{\mathrm{loc}} > 0$ and the Lévy exponent satisfies (5.29). In particular, $\tau_p(s) = \gamma + \frac{d}{\max(p, \alpha_{\mathrm{loc}})} - d$ (Corollary 5.4). Condition (6.4) ensures that the process s is almost surely in $\dot{B}^{\tau_0}_{p_0}(\mathbb{T}^d)$ according to Corollary 5.4. We identify $\kappa_{p_0,\tau_0}(s)$ thanks to

(6.3). Let us first remark that $p_{p_0,\tau_0}(s) \le \alpha_{\text{loc}}$. This is straightforward when $\alpha_{\text{loc}} \ge p_0$. If now $\alpha_{\text{loc}} < p_0$, we fix $p \in (\alpha_{\text{loc}}, p_0)$ and we easily check that

$$\frac{d}{p} - \frac{d}{p_0} < \tau_p(s) = \gamma - \tau_0 + \frac{d}{p} - d.$$

This condition is equivalent to $0 < \gamma - \tau_0 + \frac{d}{p_0} - d = \gamma - \tau_0 + \frac{d}{\max(\alpha_{\text{loc}}, p_0)} - d$, which is precisely (6.4).

Once we know that $p_{p_0,\tau_0}(s) \le \alpha_{\text{loc}}$, we can restrict to $p \le \alpha_{\text{loc}}$ and therefore have

$$p_{p_0,\tau_0}(s) = \inf\left\{p \le \alpha_{\text{loc}} \;\middle|\; d/p - d/p_0 + \tau_0 < \gamma + d/\alpha_{\text{loc}} - d\right\}.$$

Finally, this means that

$$\frac{d}{p_{p_0,\tau_0}} - \frac{d}{p_0} = \gamma - \tau_0 + \frac{d}{\alpha_{\text{loc}}} - d,$$

and, according to (6.3), that $\kappa_{p_0,\tau_0}(s) = 1/p_{p_0,\tau_0}(s) - 1/p_0 = (\gamma - \tau_0)/d + 1/\alpha_{\text{loc}} - 1$ as expected. In the general case, we only have a lower bound on $\tau_p(s)$, inducing a lower bound on the local compressibility $\kappa_{p_0,\tau_0}(s)$. □

Corollary 6.1. *Let $s = \mathrm{L}^{-1} w$ be a periodic generalized Lévy process. We assume that $s \in \dot{B}^{\tau_0}_{p_0}(\mathbb{T}^d)$ with $0 < p_0 < \infty$ and $\tau_0 \in \mathbb{R}$. Then, we have*

$$\kappa_{p_0,\tau_0}(s) = \frac{\tau_0(s) - \tau_0}{d}, \tag{6.5}$$

where

$$\tau_0(s) := \lim_{p \to 0} \tau_p(s) \in [\gamma - d/2, \infty].$$

Proof. First of all, the limit of $\tau_p(f)$ exists for every $f \in \dot{S}'(\mathbb{T}^d)$ when $p \to 0$, because the function $p \mapsto \tau_p(f)$ is decreasing (Proposition 5.10). With Corollary 5.4, we see that $\tau_0(s) = \gamma - d/2$ if w is Gaussian, and $\tau_0(s) = \gamma + d/\alpha_{\text{loc}} - d$ otherwise, where α_{loc} is the local index of w. Finally, it suffices to compare $\kappa_{p_0,\tau_0}(s)$ with the values of à $(\tau_0(s) - \tau_0)/d$ in each of the different cases to deduce (6.5). □

Remark: Corollary 6.1 connects the local compressibility with the weighted ℓ_p-quasi-norms of the wavelet coefficients when $p \to 0$. This reinforces the interpretation that the sparsity of a function—here a generalized Lévy process—is intimately linked with the "ℓ_0-norm" of its wavelet coefficients.

The following result is a direct consequence of Theorem 6.2.

Corollary 6.2. *We consider that s_{Gauss}, s_{Poisson}, and s are periodic generalized Gaussian, Poisson, and Lévy noises, respectively. Moreover, the three processes are assumed to be whitened by* *the same γ-admissible operator, for some $\gamma \geq 0$. We assume that*

$$\gamma > \tau_0 + d - d/\max(p_0, \alpha_{\mathrm{loc}}), \tag{6.6}$$

with α_{loc} the local index of s, $0 < p_0 < \infty$, and $\tau_0 \in \mathbb{R}$. Then, we have that

$$\frac{\gamma}{d} - \frac{1}{2} = \kappa_{p_0,\tau_0}(s_{\mathrm{Gauss}}) \leq \kappa_{p_0,\tau_0}(s) \leq \kappa_{p_0,\tau_0}(s_{\mathrm{Poisson}}) = \infty. \tag{6.7}$$

Moreover,

$$\kappa_{p_0,\tau_0}(s) = \kappa_{p_0,\tau_0}(s_{\mathrm{Gauss}}) \Longleftrightarrow \alpha_{\mathrm{loc}} = 2, \text{ and}$$
$$\kappa_{p_0,\tau_0}(s) = \kappa_{p_0,\tau_0}(s_{\mathrm{Poisson}}) \Longleftrightarrow \alpha_{\mathrm{loc}} = 0.$$

Proof. Condition (6.6) ensures that the three processes s_{Gauss}, s_{Poisson}, and s are in $\dot{B}^{\tau_0}_{p_0}(\mathbb{T}^d)$. Then, (6.7) is a direct consequence of Theorem 6.2, exploiting the fact that $\alpha_{\mathrm{loc}} \in [0,2]$. The extreme cases are easily deduced. □

Remarks.

- With Theorem 6.2, we see that the local compressibility is determined by the local index and the order γ of the whitening operator. For a fixed γ, the local compressibility of the generalized Lévy process s increases when α_{loc} decreases. Moreover, the compressibility also increases when γ increases: for a fixed Lévy noise, the more we smooth the process, the more compressible it becomes.

- Corollary 6.2 highlights the extreme cases. The Gaussian Lévy noise is the less compressible. This is in line with the empirical observations stated in Chapter 1. Simply stated, sparse processes are more compressible than Gaussian ones. Our characterization gives a new mathematical justification for the terminology of *sparse* processes introduced in [UT14].

  However, we point out that there exists non-Gaussian Lévy noises that induce the same local compressibility as the Gaussian ones. This corresponds to the case $\alpha_{\mathrm{loc}} = 2$. It is typically the case of any generalized Lévy process whose Lévy noise has a Gaussian part. It is also possible to construct Lévy noises without Gaussian part with a local index of $\alpha_{\mathrm{loc}} = 2$.

- The other extreme case is reached by compound Poisson processes. Here, the order γ of the operator is not relevant provided that $\gamma > \tau + d - d/p_0$ and the local compressibility is always infinite. This means that the approximation error has a faster-than-algebraic decay. Generalized Laplace processes are other examples of highly compressible random processes.

- As soon as $0 < \alpha_{\mathrm{loc}} < 2$, we are strictly located between the Gaussian and Poisson cases. The generalized Lévy process is then strictly sparser than its Gaussian counterpart and

Table 6.1 – Compressibility of Gaussian and sparse processes.

White noise w	Parameter	$\Psi(\xi)$	α_{loc}	$\kappa(s)$
Gaussian	$\sigma^2 > 0$	$-\sigma^2\xi^2/2$	2	$\gamma - \frac{d}{2}$
Cauchy [ST94]	—	$-\lvert\xi\rvert$	1	γ
SαS [ST94]	$\alpha \in (0,2)$	$-\lvert\xi\rvert^\alpha$	α	$\gamma + d/\alpha - d$
Compound Poisson [UT11]	$\lambda > 0, P$	$\lambda(\widehat{P}(\xi) - 1)$	0	∞
Laplace [KKP01]	—	$-\log(1+\xi^2)$	0	∞

has an approximation error that decays polynomially. This is the case with non-Gaussian SαS processes.

- In our initial work [FUW17a], we only obtained a lower bound on the compressibility of non-Gaussian and non-Poisson generalized Lévy processes. These earlier bounds provided in [FUW17a] are proved to be sharp in this chapter when $\alpha_{\text{loc}} = 0$ or when the Lévy exponent behaves asymptotically as a power law. This is possible thanks to the sharp estimation of $\tau_p(w)$ for a Lévy noise w developed in Chapter 5.

We summarize the results in Table 6.1 for different classes of Lévy noises. We express the compressibility for $p_0 = 2$ and $\tau_0 = 0$. We assume that γ is big enough such that s is in $\dot{L}_2(\mathbb{T}^d)$ almost surely. In that case, we denote its local compressibility by $\kappa := \kappa_{2,0}$.

7 Conclusion: Local versus Asymptotic

This thesis is dedicated to the mathematical study of the innovation model, specified by the stochastic differential equation $\mathrm{L}s = w$, with L a possibly fractional differential operator and w a Lévy white noise. Our contributions were organized in four chapters.

- In Chapter 3, we gave general conditions for the existence of generalized Lévy processes. This was achieved in three steps. We started with the characterization of Lévy noises that are in the space $\mathcal{S}'(\mathbb{R}^d)$ of tempered generalized functions. Then, we maximally extended the domain of definition of Lévy noises to non-smooth and non-rapidly decaying test functions. Finally, we applied these results to the construction of generalized Lévy processes.

- We obtained two limit theorems in Chapter 4. First, we have shown that any generalized Lévy process is the limit in law of generalized Poisson processes. The latter are particularly pleasant, since they can be described as random L-splines. Second, we gave general conditions on generalized Lévy processes such that they become self-similar at fine or coarse scales.

- In Chapter 5, we studied the Besov regularity of the Lévy noise in order to identify its local smoothness and its asymptotic decay rate. We then applied the local results to generalized Lévy processes.

- Finally, in Chapter 6, we used our smoothness results in order to quantify the local compressibility of generalized Lévy processes.

The principle underlying all of our research is the analyse of the local and asymptotic properties of generalized Lévy processes. When the Lévy noise is SαS, the two behaviors are intrinsically connected. We now propose to revisit our results for this particular case, and then recap the changes observed in the general case.

Local and asymptotic behaviors of generalized SαS processes. We consider the model $\mathrm{L}_\gamma s = w_\alpha$ with L_γ a γ-homogeneous differential operator and w_α a SαS stable noise. The model is Gaussian when $\alpha = 2$, and has infinite variance otherwise. The characteristic functional of w_α is

$$\widehat{\mathscr{P}}_{w_\alpha}(\varphi) = \exp(-\|\varphi\|_\alpha^\alpha).$$

We assume that the generalized SαS process is well-defined, in accordance with the construction of Section 3.3. We obtained the following results, where parameter α plays a crucial role.

- *Tempered Lévy noise:* For mathematical purposes, it is reasonable to ask for a noise model in $\mathscr{S}'(\mathbb{R}^d)$. The S$\alpha$S noise has finite pth moments (for every $p > 0$ when $\alpha = 2$ and for $0 < p < \alpha$ when $\alpha < 2$). From Theorem 3.1, it is therefore in $\mathscr{S}'(\mathbb{R}^d)$ for every $0 < \alpha \leq 2$.

- *Domain of definition:* The extension of the domain of the noise allows one to define the broadest possible class of generalized SαS processes in $\mathscr{S}'(\mathbb{R}^d)$. The Rajput-Rosinski exponent (see (3.11)) of the SαS noise is proportional to $\xi \mapsto |\xi|^\alpha$. This implies that the domain of definition of the SαS noise is $L_\alpha(\mathbb{R}^d)$ (Proposition 3.19).

- *Fine and coarse scales behaviors:* If L_γ admits a left-inverse with adequate stability and homogeneity properties, one can construct a self-similar process s solution of $\mathrm{L}_\gamma s = w_\alpha$. The self-similarity exponent is then $H = \gamma + d/\alpha - d$ and, for any $a > 0$, we have that

$$s \overset{(\mathscr{L})}{=} a^H s(\cdot/a). \tag{7.1}$$

  We zoom in the process when $a > 1$ and zoom out of it when $a < 1$. With (7.1), we see that the local ($a \to \infty$) and asymptotic ($a \to 0$) behaviors of s are identical. This property is not conserved for non-stable noises.

- *Besov Regularity:* When fixing the integrability rate $p \in (0, \infty]$, there exists a limit smoothness $\tau_s(w_\alpha)$ and a limit asymptotic decay rate $\rho_p(w_\alpha)$ such that w_α is in $B_p^\tau(\mathbb{R}^d;\rho)$ when τ and ρ are strictly smaller that these limits, and w_α is not in $B_p^\tau(\mathbb{R}^d;\rho)$ when one of them is strictly bigger that its corresponding limit. This is also valid for other tempered Lévy noises. The local smoothness and the asymptotic decay rate of w_α are given by

$$\tau_p(w_{\text{Gauss}}) = -\frac{d}{2} \text{ and } \rho_p(w_{\text{Gauss}}) = -\frac{d}{p}$$

  when $\alpha = 2$, in which case $w_2 = w_{\text{Gauss}}$ is therefore Gaussian, and by

$$\tau_p(w_\alpha) = \frac{d}{\max(p,\alpha)} - d \text{ and } \rho_p(w_\alpha) = -\frac{d}{\min(p,\alpha)}$$

  when $\alpha < 2$ (see Theorem 5.4 and Corollary 5.3). The local smoothness and the asymptotic decay rate are therefore both characterized by α.

- *Compressibility:* Consider a periodic generalized SαS process $s_{\gamma,\alpha}$ whitened by a γ-admissible operator (Definition 5.2). Its local compressibility in $L_2(\mathbb{T}^d)$ is given by (Theorem 6.2)

$$\kappa(s_{\gamma,\alpha}) = \frac{\gamma}{d} + \frac{1}{\alpha} - 1.$$

The local and asymptotic indices. We have seen that the parameter α is central for the quantification of the self-similarity exponent, the local regularity, the asymptotic decay rate, together with the local compressibility of a generalized SαS process. For non-stable noise, this parameter is not well-defined anymore. We recall that the characteristic functional of a Lévy noise has the general form

$$\widehat{\mathscr{P}}_w(\varphi) = \exp\left(\int_{\mathbb{R}^d} \Psi(\varphi(\boldsymbol{x}))\mathrm{d}\boldsymbol{x}\right),$$

with Ψ its Lévy exponent. Then, Ψ admits a Lévy-Khintchine representation (2.1) and is characterized by its Lévy triplet (μ,σ^2,ν), with ν the Lévy measure, as

$$\Psi(\xi) = \mathrm{i}\mu\xi - \frac{\sigma^2\xi^2}{2} + \int_{\mathbb{R}} (\mathrm{e}^{\mathrm{i}\xi t} - 1 - \mathrm{i}\xi t \mathbb{1}_{|t|\leq 1})\nu(\mathrm{d}t).$$

The relevant quantities that extend the parameter α for non-stable infinitely divisible laws are as follows. The local behavior of a generalized Lévy process is captured by the local index

$$\alpha_{\mathrm{loc}} = \inf\left\{p > 0 \;\middle|\; \int_{|t|\leq 1} |t|^p\,\nu(\mathrm{d}t) < \infty\right\} = \inf\left\{p > 0 \;\middle|\; \limsup_{|\xi|\to\infty} \frac{|\Psi(\xi)|}{|\xi|^p} < \infty\right\}.$$

The corresponding quantities for the asymptotic behaviors differ, depending if we consider the Lévy exponent or the Lévy measure. We define the asymptotic index as

$$\alpha_{\mathrm{asymp}} = \sup\left\{p > 0 \;\middle|\; \int_{|t|> 1} |t|^p\,\nu(\mathrm{d}t) < \infty\right\}.$$

Then, we have $\min(\alpha_{\mathrm{asymp}},2) = \sup\left\{p > 0 \;\middle|\; \limsup_{|\xi|\to 0} \frac{|\Psi(\xi)|}{|\xi|^p} < \infty\right\}$. Depending on the asymptotic question of interest, the relevant quantity is α_{asymp} or $\min(\alpha_{\mathrm{asymp}},2)$. Note that, for S$\alpha$S, one has

$$\alpha = \alpha_{\mathrm{loc}} = \alpha_{\mathrm{asymp}} = \min(\alpha_{\mathrm{asymp}},2).$$

Local versus asymptotic. The indices α_{loc} and α_{asymp} are not related. It is indeed possible to construct a Lévy noise with any possible pair $(\alpha_{\mathrm{loc}},\alpha_{\mathrm{asymp}}) \in [0,2] \times (0,\infty]$. Local and asymptotic indices have first been introduced for the local and asymptotic study of Lévy processes, by Blumenthal and Getoor [BG61] and Pruitt [Pru81], respectively. The role of the indices for the local and asymptotic behaviors of Lévy and Lévy-type processes is well-known

[Sat13, BSW14]. This thesis confirmed that fact by investigating new directions of research.

- *Tempered Lévy noise:* Theorem 3.1 can be reformulated as follow:

  $$w \in \mathcal{S}'(\mathbb{R}^d) \text{ a.s.} \Longleftrightarrow \alpha_{\text{asymp}} > 0.$$

  We recover that fact that the temperedness of a Lévy noise is an asymptotic property.

- *Domain of definition:* Roughly speaking, the Lévy exponent Ψ behaves like $-|\xi|^{\alpha_{\text{loc}}}$ at infinity, and like $-|\xi|^{\min(\alpha_{\text{asymp}},2)}$ around 0 and the Rajput-Rosinski exponent Θ inherits this property. The criteria for the domain of definition exposed in Section 3.2.4 can be summarized by the informal relation

  $$L_{\Theta}(\mathbb{R}^d) \approx L_{\alpha_{\text{loc}},\min(\alpha_{\text{asymp}},2)}(\mathbb{R}^d).$$

  This equality is in particular true when Θ behaves like a power law both at the origin and asymptotically. We recall that the functions in $L_{\alpha_{\text{loc}},\min(\alpha_{\text{asymp}},2)}(\mathbb{R}^d)$ are locally in $L_{\alpha_{\text{loc}}}$ and asymptotically in $L_{\min(\alpha_{\text{asymp}},2)}$ (see Section 3.2.3). The duality local/asymptotic can be read on the domain of definition.

- *Fine and coarse scales behaviors:* When the noise is not stable, the generalized Lévy process is not self-similar anymore. Therefore, a rescaling of the process impacts its probability law.

  Under some reasonable conditions (existence of a stable and homogeneous left-inverse and conditions on the Lévy exponent), a generalized Lévy process admits self-similar limits at coarse and fine scales. We assume that we are in the conditions of Theorems 4.3 and 4.4 respectively. In particular, $\Psi(\xi)$ behaves like $-A|\xi|^{\alpha_{\text{loc}}}$ at infinity, and like $-B|\xi|^{\min(\alpha_{\text{asymp}},2)}$ around 0 for some constant $A, B > 0$. Then, at coarse scales, the rescaled processes $a^{H_\infty} s(\cdot/a)$ converges in law to a H_∞-self-similar process as $a \to 0$ with

  $$H_\infty = \gamma + \frac{d}{\min(\alpha_{\text{asymp}},2)} - d. \tag{7.2}$$

  At fine scales, $a^{H_\infty} s(\cdot/a)$ converges in law to a H_{loc}-self-similar process as $a \to \infty$ with

  $$H_{\text{loc}} = \gamma + \frac{d}{\alpha_{\text{loc}}} - d. \tag{7.3}$$

  The asymptotic and local self-similarity exponents are characterized by the truncated asymptotic and local index, respectively.

- *Besov Regularity:* The integrability rate $0 < p \leq \infty$ being fixed, the local smoothness $\tau_p(w)$ and the asymptotic decay rate $\rho_p(w)$ of a non-Gaussian Lévy noise w are given by (see Theorem 5.4)

$$\tau_p(w) = \frac{d}{\max(p, \alpha_{\mathrm{loc}})} - d \text{ and } \rho_p(w) = -\frac{d}{\min(p, \alpha_{\mathrm{asymp}})}.$$

We have shown that this is valid for $\alpha_{\mathrm{loc}} = 0$, or under minor assumptions of the Lévy exponent when $\alpha_{\mathrm{loc}} > 0$. However, in future works, we hope to remove these assumptions. Contrarily to the SαS case, the local and asymptotic behaviors are dissociated. The parameter α_{loc} characterizes the local smoothness, while α_{asymp} determines the asymptotic decay rate.

Three integrability rate are especially interesting: $p = \infty$ (Hölder), $p = 2$ (Sobolev), and $p = 0$ (as the limit of $p \to 0$). If w_{Gauss} is Gaussian and w is a non-Gaussian Lévy noise, then we have:

$$\tau_\infty(w) = -d < -\frac{d}{2} = \tau_\infty(w_{\mathrm{Gauss}}) \text{ and } \rho_\infty(w) = -\frac{d}{\alpha_{\mathrm{asymp}}} \leq 0 = \rho_\infty(w_{\mathrm{Gauss}});$$

$$\tau_2(w) = -\frac{d}{2} = \tau_\infty(w_{\mathrm{Gauss}}) \text{ and } \rho_2(w) = -\frac{d}{\min(\alpha_{\mathrm{asymp}}, 2)} \leq -\frac{d}{2} = \rho_2(w_{\mathrm{Gauss}});$$

$$\tau_0(w) = \frac{d}{\alpha_{\mathrm{loc}}} - d \geq -\frac{d}{2} = \tau_0(w_{\mathrm{Gauss}}) \text{ and } \rho_0(w) = -\infty = \rho_0(w_{\mathrm{Gauss}}).$$

- *Compressibility:* We have studied the local compressibility of a generalized Lévy process s via its wavelet coefficients

$$\langle s, \Psi_{j,G,\boldsymbol{k}} \rangle, \quad j \geq 0, G \in \mathrm{G}^j, \|\boldsymbol{k}\|_\infty < 2^j.$$

The crucial point for the local study is to restrict the range of the shifts $\boldsymbol{k}$ (or, equivalently, to work on the d-dimensional torus). If $s = \mathrm{L}_\gamma^{-1} w$ with L_γ a γ-admissible operator, the local compressibility of s in $L_2(\mathbb{T}^d)$ is then given by (Theorem 6.2)

$$\kappa(s) = \frac{\gamma}{d} + \frac{1}{\alpha_{\mathrm{loc}}} - 1.$$

Our proof covers all the cases for which we have an exact estimation of the local smoothness. Again, the local compressibility is captured by the local index α_{loc}. When α_{loc} increases, the local compressibility of the process decreases. The local compressibility therefore implies the following local hierarchy, from non-sparse to sparse:

Gauss $\ll$ non-Gaussian SαS $\ll$ Laplace = Poisson.

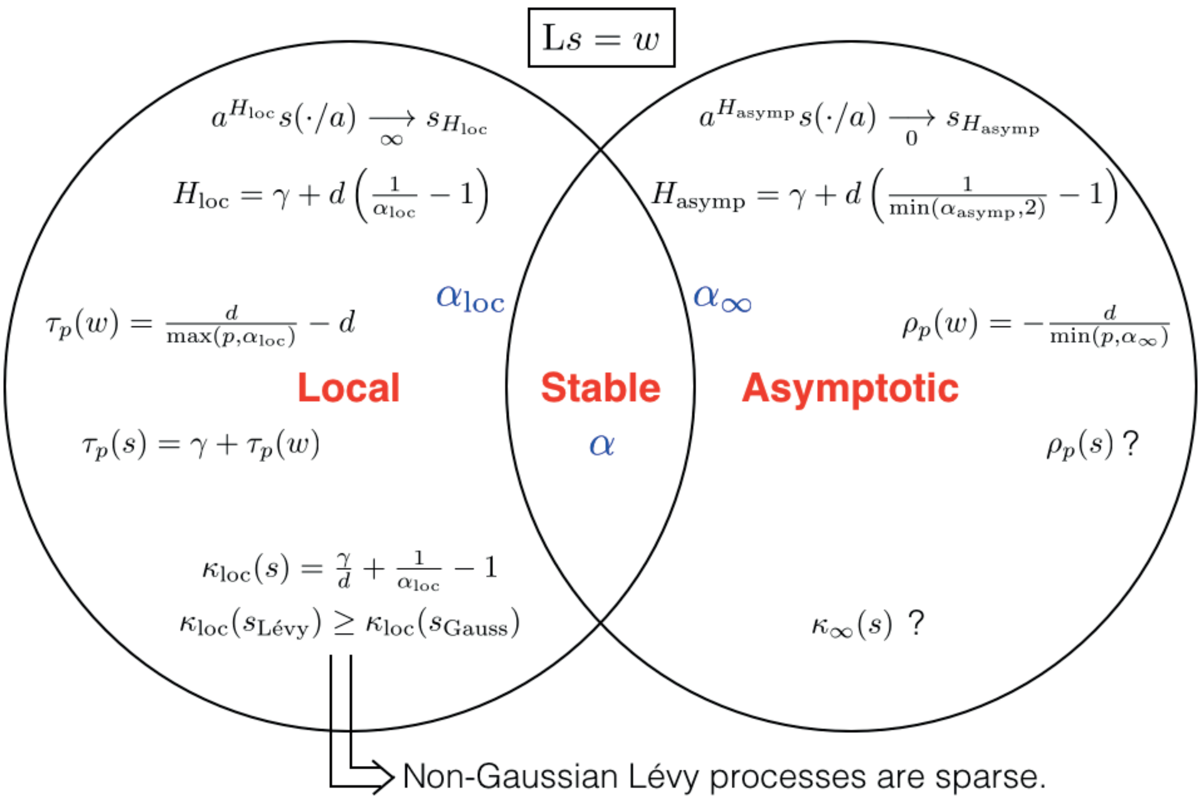
The asymptotic counterpart of our result can be described as follows. We only consider the wavelet coefficients for the scale $j = 0$; that is,

$$\langle s, \Psi_{0,G,\boldsymbol{k}} \rangle, \quad G \in \mathrm{G}^0, \boldsymbol{k} \in \mathbb{Z}^d. \tag{7.4}$$

An adequate notion of asymptotic compressibility could emerge by considering the Besov localization of the sequence (7.4). This calls for further investigations. The asymptotic compressibility has strong connections with the study of the compressibility of i.i.d. random sequences that has been investigated by several authors [Cev09, AUM11, SP12, GCD12]. From these works, it appears that the tail properties of the common law of the sequence determines the compressibility. In particular, heavy-tailed random sequences are more compressible, which corresponds to $\alpha_{\text{asymp}} < \infty$ for infinitely divisible laws. The parameter α_{asymp} again seems to be relevant to order the asymptotic compressibility. This induces the following asymptotic hierarchy, from non-sparse to sparse:

$$\text{Gaussian} \ll \text{Laplace} = \text{Poisson (with finite moments)} \ll \text{non-Gaussian S}\alpha\text{S}.$$

In both cases, the Gaussian law is the least sparse. Non-Gaussian innovation models are therefore *sparse*, in the sense that they are sparser than Gaussian, both locally and asymptotically. However, what makes an innovation model sparse differs whether it is observed from a local (characterized by α_{loc}) or an asymptotic (captured by α_{asymp}) point of view.



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Probability theory	Random processes Stochastic differential equations
Functional analysis	Approximation Theory Splines and wavelets

Publications

Journal

1. **J. Fageot**, M. Unser, and J.P. Ward. The n-term approximation of periodic generalized Lévy processes. *arXiv preprint arXiv:1702.03335*, 2017.

2. **J. Fageot**, V. Uhlmann, and M. Unser. Gaussian and sparse processes are limits of generalized Poisson processes. *Submitted*.

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Conference

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3. V. Uhlmann, **J. Fageot**, H. Gupta, and M. Unser. Statistical optimality of Hermite splines. In *Proceedings of the Eleventh International Workshop on Sampling Theory and Applications (SampTA15), Washington DC, USA*, pages 226-230, 2015.

4. **J. Fageot**, E. Bostan, and M. Unser. Statistics of wavelet coefficients for sparse self-similar images. In *Proceedings of the 2014 International Conference on Image Processing (ICIP2014),* pages 6096-6100, 2016.

5. E. Bostan, **J. Fageot**, U.S. Kamilov, and M. Unser. MAP estimators for self-similar sparse stochastic models. In *Proceedings of the Tenth International Workshop on Sampling Theory and Applications (SampTA13), Bremen, Germany*, pages 197-199, 2013.

Teaching

Teaching Assistant for the courses Signals and Systems I & II at EPFL (autumns 2012 to 2016)

Part of the team that won 2013 Education Award from EPFL's Life Sciences Section

Students Supervised

2017 Benjamin Beck (co-advised with Prof. A. Depeursinge and Prof. M. Unser)
Semester Project: *Complete detection strategy with 3D steerable filters*

2016 Lilian Besson (co-advised with Prof. M. Unser)
Master Project: *A theoretical study of steerable homogeneous operators, and applications*

2016 Shayan Aziznejad (co-advised with Prof. M. Unser)
Summer internship: *Sharp smoothness and decay rate of Lévy noise*

2016 Sahar Hanna (co-advised with Dr. Zs. Püspöki and Prof. M. Unser)
Semester Project: *Java optimization package development*

2015 Alireza Fallah (co-advised with Prof. M. Unser)
Summer internship: *Besov regularity of Lévy noise*

2015 Bertrand Vermot (co-advised with V.Uhlmann and Prof. M. Unser)
Semester Project: *Corner Detection based on PDE Methods*

2015 Miryam Chaabouni (co-advised with V. Uhlmann, D. Schmitter and Prof. M. Unser)
Semester Project: *Design of active contour models using NURBS*

2014 Kilian Thomas (co-advised with Zs. Püspöki and Prof. M. Unser)
Semester Project: *Junction detection in biomedical micrographs*

2013 Morteza Ashraphijuo (co-advised with Prof. M. Unser)
Summer internship: *Wavelet approximation of compound Poisson processes*

2013 Thomas Pumir (co-advised with Prof. M. Unser)
Master Project: *Investigation of the multiscale statistics of natural and biomedical images*

Professional Experience

2012-ongoing *Biomedical Imaging Group, École polytechnique fédérale de Lausanne (EPFL), Lausanne, Switzerland*
Researcher and teaching assistant

2007-2010 *Lycée St Louis & Lycée Louis le Grand, Paris, France*
Teaching assistant for Mathematics in preparatory schools

Personal

Chess player at competition level since 1997.
Amateur runner and tennis/badminton player.