Voltage Source Converter MPC with Optimized Pulse Patterns and Minimization of Integrated Squared Tracking Error

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Abstract— Model predictive control schemes for power electronic applications are characterized by a great variety of problem formulations. In this paper, we consider a three phase voltage source converter with an arbitrary number of voltage levels and derive a model predictive control scheme involving a combination of optimized pulse patterns and the integral of squared predicted tracking error as a cost function. We obtain a nonlinear optimization problem with the switching times as optimization variables, and solve it using gradient projection algorithm. To obtain an easier optimization problem to be solved on-line, a linearization around nominal switching instants is performed bringing the problem to a quadratic programming form. Simulation results demonstrating the performance of the derived scheme are provided for the case of a grid-tied converter with LCL filter.

I. INTRODUCTION

Model predictive control (MPC) schemes developed for power electronic applications have demonstrated improved performance compared to the traditional state-of-the-art control in power electronics [1]. The characteristics of power electronic systems, such as the ability of an inverter to supply a finite number of voltage levels and very limited computational power within the sampling intervals, have caused the power electronic MPC schemes to be highly tailored to the application and to be quite different from the standard MPC formulations studied in the control community.

The aforementioned MPC power electronic schemes thus involve a great variety of "problem-tailored" cost functions. For example, the MPC in [2] involves a cost which minimizes the converter switching frequency in order to indirectly reduce the commutation losses of the power converter. Another MPC approach for power electronics [3], involves a cost function which employs superposition of correction pulses to eliminate an error vector over the prediction horizon. In [4], the cost function is formulated so that it directly addresses the commutation losses of a power converter.

The goal of this paper is to develop a controller using the integral square tracking error as a cost function together with optimized pulse patterns (OPPs). The OPPs are pre-computed converter voltage waveforms optimized for high steady-state performance with a fixed number of converter's switchings [5]. Integral squared problem formulations for

power electronic applications were previously considered in [6], [7] in combination with so-called fixed frequency pulse-width modulation (PWM) and in [8] in combination with OPPs. The use of OPPs in this paper instead of PWM allows to further reduce the harmonic distortion caused by the converter. In comparison to [3], [8] which provides MPC schemes as well employing OPPs, our approach involves a cost function that penalizes the squared tracking error along the prediction horizon and is in addition applicable to a greater variety of power electronic systems, though at the expense of more demanding computations.

The development considers the power electronics system in state-space representation and thus covers a wide variety of power electronic configurations. Nevertheless, to facilitate an easier exposition we consider a particular setting common in power electronic applications, which involves a power converter connected to the power grid via an *LCL* filter. By employing OPPs, the discrete decision variables coming from the finite number of converter voltage levels will get eliminated, and the development results in a nonlinear optimization problem with only switching times as continuous decision variables. The optimization problem is solved by applying gradient projection (GP) algorithm operated based on cost function descent. To allow a potentially more convenient optimization problem for an on-line application, a linearization around the OPP switching times is performed which brings the problem to the form of quadratic programming (QP), with decision variables being the deviations from the OPP (nominal) switching times.

II. SYSTEM MODEL

The configuration with power converter interconnected to the power grid via an *LCL* filter is presented on Fig. 1.

A. State-space model of a grid-connected LCL filter

The state of the *LCL* filter is described in stationary abc frame with state vector

$$x_f = [i_{fi,a} \ i_{fi,b} \ i_{fi,c} \ i_{fg,a} \ i_{fg,b} \ i_{fg,c} \ v_{f,a} \ v_{f,b} \ v_{f,c}]' \quad (1)$$

where for each phase $p \in \{a, b, c\}$, $i_{fi,p}$, $i_{fg,p}$ and $v_{f,p}$ denote the inverter current, grid current and capacitor voltage, respectively. The *LCL* filter dynamics are

$$\dot{x}_f(t) = A_f x_f(t) + B_f s(t) + F_f w_g(t) \quad (2)$$

where $s = [s_a, s_b, s_c]' \in \mathbb{R}^3$ is the vector of converter switch signals (further modeled in Section II-B) and $w_g = [w_{ga}, w_{gb}, w_{gc}]' \in \mathbb{R}^3$ is a vector representing the grid voltage (further modeled below). Explicit expressions for the system
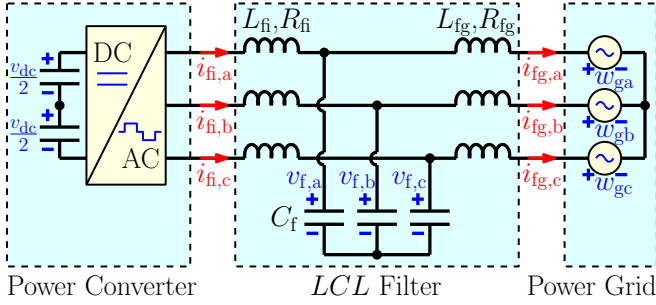
Fig. 1: The configuration with a power converter connected to the power grid via an *LCL* filter.
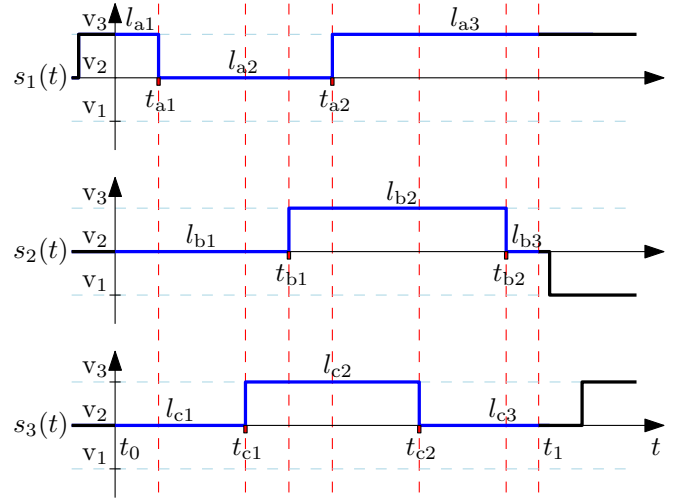

Fig. 2: An example of three-phase switch signal $s(t)$, showing switching times and discrete amplitudes. The example involves a three-level converter ($N_{\text{lvl}} = 3$) and six switching times ($N_{\text{sw}} = 6$) over the prediction horizon $[t_0, t_1]$.

matrices can be found in [9]. The DC side capacitor voltage v_{dc} is assumed constant and is "absorbed" into B_{f}.

The grid voltage $w_{\text{g}} = [w_{\text{ga}}, w_{\text{gb}}, w_{\text{gc}}]'$ is assumed sinusoidal and three-phase symmetric. The voltage can be modeled in $\alpha\beta$ frame [10] as $w_{\text{g}\alpha\beta}(t) = [w_{\text{g}\alpha}(t), w_{\text{g}\beta}(t)]'$ with

$$\dot{w}_{\text{g}\alpha\beta}(t) = R_{2\times2}\, w_{\text{g}\alpha\beta}(t), \qquad R_{2\times2} := \begin{bmatrix} 0 & -\omega_g \\ \omega_g & 0 \end{bmatrix}, \quad (3)$$

where $\omega_g = 2\pi f_g$ is the grid angular frequency.

The state-space model is obtained by combining the *LCL* dynamics (2) and the grid voltage model (3), taking the form

$$\dot{x}_{\text{s}}(t) = A_{\text{s}} x_{\text{s}}(t) + B_{\text{s}} s(t), \quad \text{where} \quad x_{\text{s}} = \begin{bmatrix} x_{\text{f}} & w_{\text{g}\alpha\beta} \end{bmatrix}'. \quad (4)$$

The expressions for the matrices can as well be found in [9].

B. Voltage source converter

The power converter [11] is modeled over the prediction horizon by the sequence of voltage levels which it provides. The naming conventions introduced in the sequel are illustrated with an example in Fig. 2. We assume the components $s_i(t), i \in \{\text{a,b,c}\}$, of the switch vector $s(t) = [s_{\text{a}}(t), s_{\text{b}}(t), s_{\text{c}}(t)]'$ take N_{lvl} distinct values $s_i \in \{v_1, v_2, \ldots, v_{N_{\text{lvl}}}\}$ which correspond to the voltage levels the power converter can suply. In the MPC problem we consider a prediction over the window $[t_0, t_1]$, $t_0 < t_1$, and within it we allow the total number of switches in all three phases be N_{sw}. Thus, $N_{\text{sw}} = N_{\text{a}} + N_{\text{b}} + N_{\text{c}}$ where $N_i, i \in \{\text{a,b,c}\}$ is the number of switchings in phase i.

The transition time of the jth switch in phase i is denoted t_{ij} where $j \in \{1, \ldots, N_i\}$, $i \in \{\text{a,b,c}\}$, and the convention is $t_{ij} \leq t_{ij+1}$. The voltage level applied during the time between t_{ij} and t_{ij+1} is denoted ℓ_{ij+1}. Thus, the switch signals are written

$$s_i(t) = \begin{cases} \ell_{i1}, & t \in [t_0, t_{i1}) \\ \ell_{i2}, & t \in [t_{i1}, t_{i2}) \\ \vdots & \vdots \\ \ell_{iN_i}, & t \in [t_{iN_i-1}, t_{iN_i}) \\ \ell_{iN_i+1}, & t \in [t_{iN_i}, t_1] \end{cases} \quad (5)$$

where $\ell_{ij} \in \{v_1, v_2, \ldots, v_{N_{\text{lvl}}}\}$.

III. COST FUNCTION AND CONSTRAINT SET

The cost function of the controller comprises an integral of squared difference between a reference signal and predicted trajectory. The reference signals corresponding to desired active and reactive power injections to the grid are approximated by sinusoidal shapes. The discrete-valued voltage levels are ruled out from decision variables by using OPPs.

A. Sinusoidal steady-state reference

Assuming a three-phase symmetric and sinusoidal steady-state, the reference $x_r \in \mathbb{R}^9$ is modeled in $\alpha\beta$ coordinates:

$$\dot{x}_{\text{r}\alpha\beta}(t) = R_{6\times6}\, x_{\text{r}\alpha\beta}(t), \quad R_{6\times6} := \text{blkdiag}(R_{2\times2}, R_{2\times2}, R_{2\times2}), \quad (6)$$

where $x_{\text{r}\alpha\beta} = \begin{bmatrix} i_{\text{r,fi}\alpha} & i_{\text{r,fi}\beta} & i_{\text{r,fg}\alpha} & i_{\text{r,fg}\beta} & v_{\text{r,f}\alpha} & v_{\text{r,f}\beta} \end{bmatrix}'$ consists of *LCL* state references and $R_{2\times2}$ is defined in (3). The derivation of the initial value $x_{\text{r}\alpha\beta}(t_0)$ which corresponds to the specified active P_{g} and reactive Q_{g} power to be injected to the grid can be found in [9].

B. Augmented state-space model

To take the model of sinusoidal references (6) into account, we augment the state-space model (4) and obtain

$$\dot{x}(t) = Ax(t) + Bs(t), \quad \text{where} \quad x = \begin{bmatrix} x_{\text{s}} & x_{\text{r}\alpha\beta} \end{bmatrix}' \quad (7)$$

Explicit expressions for the matrices can be found in [9].

Since the goal is to minimize the integral of the squared tracking error, the output is selected to be the deviation of state from the reference trajectory. The tracking error is

$$y(t) = Ex(t), \quad (8)$$

where the expression for E can also be found in [9].

C. Optimized pulse patterns

As described in Section II-B, the switch signal $s(t) \in \mathbb{R}^3$ depends on switching times t_{ij} (continuous decision variables) and voltage levels ℓ_{ij} (discrete decision variables). To avoid optimization over the voltage levels, we take them from an off-line computed steady-state waveform (the OPP) whose fundamental voltage harmonic (the amplitude and phase) corresponds to the desired sinusoidal steady-state voltage.

The OPPs [5], [3] are steady-state converter waveforms obtained by off-line optimization for a specified number of switchings within the period $1/f_g$ and fundamental voltage harmonic. The optimization minimizes a total harmonic distortion (THD) of grid current, the steady-state performance quantifier defined as

$$\mathrm{THD}(I_{\mathrm{fg}}) = \frac{\sqrt{\sum_{k=2}^{\infty} I_{\mathrm{fg},k}^2}}{I_{\mathrm{fg},1}}, \tag{9}$$

where the $I_{\mathrm{fg},k}$ represents the k-th harmonic of the grid current in one of the phases.

D. Predicted state trajectory

To explicitly formulate the solution of the system (7) we need to fix the order of the transitions in all three phases with respect to one another. The single phase voltage levels of $s(t)$ are obtained from an OPP, and it is also necessary to determine what voltage levels are applied to the three phases in between transition times. The sequence of three-phase voltage levels is determined by the sequence of single phase voltage levels ℓ_{ij}, $i \in \{a,b,c\}$ and the sequence of switching times. For example, we may consider the switching order illustrated in Fig. 2 where $N_{\mathrm{sw}} = 6$ and

$$t_0 \le t_{a1} \le t_{c1} \le t_{b1} \le t_{a2} \le t_{c2} \le t_{b2} \le t_1. \tag{10}$$

The corresponding sequence of $N_{\mathrm{sw}}+1$ three-phase voltage levels is

$$\begin{bmatrix} \ell_{a1} \\ \ell_{b1} \\ \ell_{c1} \end{bmatrix}, \begin{bmatrix} \ell_{a2} \\ \ell_{b1} \\ \ell_{c1} \end{bmatrix}, \begin{bmatrix} \ell_{a2} \\ \ell_{b1} \\ \ell_{c2} \end{bmatrix}, \begin{bmatrix} \ell_{a2} \\ \ell_{b2} \\ \ell_{c2} \end{bmatrix}, \begin{bmatrix} \ell_{a3} \\ \ell_{b2} \\ \ell_{c2} \end{bmatrix}, \begin{bmatrix} \ell_{a3} \\ \ell_{b2} \\ \ell_{c3} \end{bmatrix}, \begin{bmatrix} \ell_{a3} \\ \ell_{b3} \\ \ell_{c3} \end{bmatrix}. \tag{11}$$

We now proceed to introduce index vectors which represent the order of the switching times and corresponding three-phase voltage levels. To describe the transition times we introduce the index vector $\bar{I}$ which is a vector of dimension N_{sw} containing the indexes i,j of the switch transitions in a given order:

$$\bar{I} = \begin{bmatrix} ij \end{bmatrix}, \quad i \in \{a,b,c\} \ , \ j \in \{1,\ldots,N_i\}.$$

For example, the index vector $\bar{I}$ corresponding to (10) is

$$\bar{I} = \begin{bmatrix} a1 & c1 & b1 & a2 & c2 & b2 \end{bmatrix}'.$$

To handle the boundary conditions with compact expressions we augment the index vector $\bar{I}$ with zero and one and define

$$I := \begin{bmatrix} 0 & \bar{I}' & 1 \end{bmatrix}'.$$

The augmented index vector I has dimension $N_{\mathrm{sw}}+2$. We let the index denoting the entries of I start with zero so that

$$I_k = \begin{cases} 0, & k = 0 \\ ij, & k \in \{1,\ldots,N_{\mathrm{sw}}\} \ . \\ 1, & k = N_{sw}+1 \end{cases}$$

The index set I defines the sorted vector of switching times

$$t_I = \begin{bmatrix} t_0 & t_{I_1} & t_{I_2} & \cdots & t_{I_{N_{sw}-1}} & t_{I_{N_{sw}}} & t_1 \end{bmatrix}'. \tag{12}$$

To describe the three-phase voltage levels we introduce the index matrix J defined as

$$J = \begin{bmatrix} J_0,\ldots,J_{N_{\mathrm{sw}}} \end{bmatrix}, \quad \text{where} \quad J_0 = \begin{bmatrix} a1 & b1 & c1 \end{bmatrix}'$$

and the remaining columns are determined by treating the indices a,b,c as values $1,2,3$, respectively, and using the relation

$$J_k = J_{k-1} + e_{(I_k)_1} \cdot (01), \quad k = 1,\ldots,N_{\mathrm{sw}}$$

where $e_j \in \mathbb{R}^3$ is the jth unit vector and where $(I_k)_1$ denotes the first part (the "i") of the kth entry of I. For example, the index matrix J corresponding to (11) is

$$J = \begin{bmatrix} \begin{bmatrix} a1 \\ b1 \\ c1 \end{bmatrix}, \begin{bmatrix} a2 \\ b1 \\ c1 \end{bmatrix}, \begin{bmatrix} a2 \\ b1 \\ c2 \end{bmatrix}, \begin{bmatrix} a2 \\ b2 \\ c2 \end{bmatrix}, \begin{bmatrix} a3 \\ b2 \\ c2 \end{bmatrix}, \begin{bmatrix} a3 \\ b2 \\ c3 \end{bmatrix}, \begin{bmatrix} a3 \\ b3 \\ c3 \end{bmatrix} \end{bmatrix}$$

The index vector J_k is mapped to the voltage levels by

$$\mathscr{L}_k = \mathscr{L}_k(J_k) = \begin{bmatrix} \ell_{(J_k)_1} & \ell_{(J_k)_2} & \ell_{(J_k)_3} \end{bmatrix}'$$

where $(J_k)_i$ denotes the ith entry of J_k.

We are now ready to derive an expression for the system solution. Given a fixed switching order I, we split the prediction window into $N_{\mathrm{sw}}+1$ subintervals according to

$$[t_0,t_1] = [t_0,t_{I_1}) \cup [t_{I_1},t_{I_2}) \cup \cdots \cup [t_{I_{N_{\mathrm{sw}}-1}},t_{I_{N_{\mathrm{sw}}}}) \cup [t_{I_{N_{\mathrm{sw}}}},t_1].$$

We proceed to derive an explicit expression for the (continuous time) state over the subinterval $[t_{I_k},t_{I_{k+1}})$. Define

$$x_k := x(t_{I_k}), \ k = 0,\ldots,N_{\mathrm{sw}}+1.$$

Using the equality

$$e^{Ap}x + \int_0^p e^{A(p-\tau)}B\mathrm{d}\tau = \begin{bmatrix} I_{n\times n} & 0_{n\times 1} \end{bmatrix} e^{\begin{bmatrix} A & B \\ 0_{1\times n} & 0_{1\times 1} \end{bmatrix} p} \begin{bmatrix} x \\ 1 \end{bmatrix}$$

where n is the dimension of the square matrix A, the solution of (7) over the subinterval $[t_{I_k},t_{I_{k+1}})$ can be written as

$$\begin{aligned} x(t) &= e^{A(t-t_{I_k})}x_k + \int_{t_{I_k}}^t e^{A(t-\tau)}B\mathscr{L}_k \mathrm{d}\tau \\ &= Ce^{\bar{A}_k(t-t_{I_k})}\bar{x}_k, \quad t \in [t_{I_k},t_{I_{k+1}}), \ k = 0,\ldots,N_{\mathrm{sw}}, \end{aligned} \tag{13}$$

where we have introduced the definitions

$$C := \begin{bmatrix} I_{n\times n} & 0_{n\times 1} \end{bmatrix}, \quad \bar{A}_k := \begin{bmatrix} A & B\mathscr{L}_k \\ 0_{1\times n} & 0_{1\times 1} \end{bmatrix}, \quad \bar{x}_k := \begin{bmatrix} x_k \\ 1 \end{bmatrix}.$$

The solution (13) can be stated in a form where the state variables x_k are eliminated. It holds

$$x(t) = Ce^{\bar{A}_k(t-t_{I_k})} \prod_{m=0}^{k-1} e^{\bar{A}_m(t_{I_{m+1}}-t_{I_m})}\bar{x}_0, \tag{14}$$

where $t \in [t_{I_k},t_{I_{k+1}}), \ k = 0,\ldots,N_{\mathrm{sw}}$.

E. Integral of squared predicted tracking error

We now formulate the cost representing the integral of the squared predicted deviation between *LCL* states $x_f(t)$ and the references $x_r(t)$ over the prediction horizon $[t_0, t_1]$:

$$
J_c = \int_{t_0}^{t_1} y(t)' Q y(t) dt = \sum_{k=0}^{N_{sw}} \int_{t_{I_k}}^{t_{I_{k+1}}} y(t)' Q y(t) dt
$$
$$
= \sum_{k=0}^{N_{sw}} \bar{x}_k' \left(\int_{t_{I_k}}^{t_{I_{k+1}}} e^{\bar{A}_k'(t-t_{I_k})} \left(E'C'QCE \right) e^{\bar{A}_k(t-t_{I_k})} dt \right) \bar{x}_k.
\tag{15}
$$

Using the results of [12], the cost function becomes

$$
J_c = \sum_{k=0}^{N_{sw}} \bar{x}_k' \, N_k \, \bar{x}_k
\tag{16}
$$

where

$$
N_k = F_{k3}' G_{k2}, \qquad k = 0, \ldots, Nsw,
\tag{17}
$$

with F_{k3} and G_{k2} obtained by first computing

$$
\hat{A}_k = \begin{bmatrix} -\bar{A}_k' & C'E'QEC \\ 0_{(n+1)\times(n+1)} & \bar{A}_k \end{bmatrix}
\tag{18}
$$

and then taking F_{k3} and G_{k2} as submatrices of

$$
e^{\hat{A}_k(t_{I_{k+1}} - t_{I_k})} = \begin{bmatrix} F_{k2} & G_{k2} \\ 0_{(n+1)\times(n+1)} & F_{k3} \end{bmatrix}.
\tag{19}
$$

F. Cost function

The cost function depends on the continuous decision variables t_{ij} within the prediction horizon $[t_0, t_1]$ denoted as

$$
\bar{t} = \begin{bmatrix} t_{a1} & \ldots & t_{aN_a} & t_{b1} & \ldots & t_{bN_b} & t_{c1} & \ldots & t_{cN_c} \end{bmatrix}'.
\tag{20}
$$

The times t_{ij} are selected so that their sequence and number correspond to the nominal switching times obtained from the OPP. The cost has the form

$$
J_{tot}(\bar{t}) = J_c(\bar{t}) + J_d(\bar{t})
\tag{21}
$$

where J_c is the integrated squared tracking error from (16), and J_d is a quadratic penalty on deviations of switching times t_{ij} from their corresponding nominal switching times t_{ij}^* of the off-line computed OPP:

$$
J_d = \sum_{i=1}^{3} \sum_{j=1}^{N} q \left(t_{ij} - t_{ij}^* \right)^2,
\tag{22}
$$

with $q \geq 0$. It should be noted that evaluation of the cost term J_c requires sorting of the vector $\bar{t}$ with switching times to the vector t_I with monotonically non-decreasing times, as previously described in Section III-D.

G. Constraint set

The constraint involved in the optimization problem imposes the fixed order of the switching times in each of the three phases idependently:

$$
t_0 \leq t_{p1}, \quad t_{p1} + \delta \leq t_{p2}, \quad \ldots, \quad t_{pN_p} + \delta \leq t_1, \quad p \in \{a, b, c\}
\tag{23}
$$

where δ is the minimal allowed separation between any two consecutive switching times in each phase, except between t_0 and t_{i1}, $i \in \{a, b, c\}$.

IV. SOLUTION APPROACHES

The control optimization problem consisting of the cost (21) and the constraint set (23) is solved in two ways. The first applies GP algorithm, and the second performs linearization of the state vectors $\bar{x}_k$ to obtain a QP form.

A. Gradient projection

By denoting with X the poytopic constraint set defined by (23), the GP algorithm [13] takes the form

$$
\bar{t}_{k+1} = P_X \left(\bar{t}_k - s_{gp,k} \nabla J_{tot}(\bar{t}_k) \right),
\tag{24}
$$

where $\bar{t}_k \in \mathbb{R}^{N_{sw}}$ is the vector of switching times as in (20) at k-th iteration of gradient projection algorithm, $P_X : \mathbb{R}^{N_{sw}} \to \mathbb{R}^{N_{sw}}$ denotes the projection on the set $X \subset \mathbb{R}^{N_{sw}}$, and $s_{gp,k} > 0$ is the stepsize at the iteration k chosen so that there holds

$$
J(\bar{t}_k) - J(\bar{t}_{k+1}) \geq \sigma \nabla J(\bar{t}_k)' (\bar{t}_k - \bar{t}_{k+1}),
\tag{25}
$$

with $\sigma \in (0, 1)$. The stepsize $s_{gp,k}$ satisfying (25) at iteration k can be found by using backtracking, *i.e.*, by examining for some fixed $\beta \in (0, 1)$ and $s_{init} > 0$ the sequence of values $\{s_{init}, \beta s_{init}, \beta^2 s_{init}, \beta^3 s_{init}, \ldots\}$ and taking as s_k the largest one for which (25) holds.

Iteration of GP in (24) requires the gradient $\nabla J_{tot}(\bar{t}_k)$ of the cost (21). To compute the gradient of the term $J_c(\bar{t})$ at $\bar{t}$, it is necessary to perform the conversion $\bar{t} \to t_I$, *i.e.*, to arrange the switching times in a non-decreasing order as explained in III-D. Then for the computation of the partial derivative with respect to a switching time t_{ij} in vector $\bar{t}$, we use its corresponding time t_{I_m} in vector t_I. Since $\bar{t}$ and t_I are essentially two vectors containing information about the same switching times, with slight abuse of notation in the sequel we will ocasionally be writing $J(t_I)$ instead of $J(\bar{t})$.

By using the chain rule, the partial derivative of $J_c(t_I)$ in (16) with respect to t_{I_j}, $j \in \{1, \ldots, N_{sw}\}$, is given by

$$
\frac{\partial J_c(t_I)}{\partial t_{I_j}} = \sum_{k=0}^{N_{sw}} \left(\frac{\partial \bar{x}_k'}{\partial t_{I_j}} N_k \bar{x}_k + \bar{x}_k' \frac{\partial N_k}{\partial t_{I_j}} \bar{x}_k + \bar{x}_k' N_k \frac{\partial \bar{x}_k}{\partial t_{I_j}} \right),
\tag{26}
$$

and it as well represents the partial derivative of $J_c(\bar{t})$ with respect to the coordinate of $\bar{t}$ corresponding to t_{I_j}.

The gradient expression (26) involves the partial derivatives $\partial \bar{x}_k / \partial t_{I_j}$ and $\partial N_k / \partial t_{I_j}$, whose expressions will be derived now. The expressions for the partial derivatives $\partial \bar{x}_k / \partial t_{I_j}$ are obtained from (14), and have the form:

- For $k > j + 1$

$$
\frac{\partial \bar{x}_k}{\partial t_{I_j}} = \prod_{m=j+1}^{k-1} e^{\bar{A}_m(t_{I_{m+1}} - t_{I_m})} (-\bar{A}_j) \prod_{m=0}^{j} e^{\bar{A}_m(t_{I_{m+1}} - t_{I_m})} \bar{x}_0 +
$$
$$
\prod_{m=j}^{k-1} e^{\bar{A}_m(t_{I_{m+1}} - t_{I_m})} \bar{A}_{j-1} \prod_{m=0}^{j-1} e^{\bar{A}_m(t_{I_{m+1}} - t_{I_m})} \bar{x}_0.
\tag{27}
$$

- For $k = j + 1$

$$
\frac{\partial \bar{x}_k}{\partial t_{I_j}} = (-\bar{A}_j) \prod_{m=0}^{j} e^{\bar{A}_m(t_{I_{m+1}} - t_{I_m})} \bar{x}_0 +
$$
$$
e^{\bar{A}_j(t_{I_{j+1}} - t_{I_j})} \bar{A}_{j-1} \prod_{m=0}^{j-1} e^{\bar{A}_m(t_{I_{m+1}} - t_{I_m})} \bar{x}_0.
\tag{28}
$$

- For $k = j$

$$\frac{\partial \bar{x}_k}{\partial t_{I_j}} = \bar{A}_{k-1} \prod_{m=0}^{k-1} e^{\bar{A}_m(t_{I_{m+1}} - t_{I_m})} \bar{x}_0. \qquad (29)$$

- For $k < j$

$$\frac{\partial \bar{x}_k}{\partial t_{I_j}} = 0. \qquad (30)$$

The expressions for the partial derivatives $\partial N_k / \partial t_{I_j}$ are obtained using (17)-(19). By using (19) and denoting $m := n+1$ where $n = 17$ is the order of the system (7), the N_k in (17) can be expressed as

$$N_k = \begin{bmatrix} 0 & I \end{bmatrix} e^{\hat{A}'_k(t_{I_{k+1}} - t_{I_k})} \begin{bmatrix} 0 & 0 \\ I & 0 \end{bmatrix} e^{\hat{A}_k(t_{I_{k+1}} - t_{I_k})} \begin{bmatrix} 0 \\ I \end{bmatrix}, \qquad (31)$$

where the I and 0 denote $I_{m \times m}$ and $0_{m \times m}$. By introducing

$$T_1 := \begin{bmatrix} 0 & I \end{bmatrix}, \quad T_2 := \begin{bmatrix} 0 & 0 \\ I & 0 \end{bmatrix}, \quad T_3 := \begin{bmatrix} 0 \\ I \end{bmatrix}, \qquad (32)$$

the partial derivatives of N_k matrices are:

- For $k = j$

$$\frac{\partial N_k}{\partial t_{I_j}} = T_1 \cdot \left(-\hat{A}'_k\right) \cdot e^{\hat{A}'_k(t_{I_{k+1}} - t_{I_k})} \cdot T_2 \cdot e^{\hat{A}_k(t_{I_{k+1}} - t_{I_k})} \cdot T_3 +$$
$$T_1 \cdot e^{\hat{A}'_k(t_{I_{k+1}} - t_{I_k})} \cdot T_2 \cdot \left(-\hat{A}_k\right) \cdot e^{\hat{A}_k(t_{I_{k+1}} - t_{I_k})} \cdot T_3. \qquad (33)$$

- For $k+1 = j$

$$\frac{\partial N_k}{\partial t_{I_j}} = T_1 \cdot \hat{A}'_k \cdot e^{\hat{A}'_k(t_{I_{k+1}} - t_{I_k})} \cdot T_2 \cdot e^{\hat{A}_k(t_{I_{k+1}} - t_{I_k})} \cdot T_3 +$$
$$T_1 \cdot e^{\hat{A}'_k(t_{I_{k+1}} - t_{I_k})} \cdot T_2 \cdot \hat{A}_k \cdot e^{\hat{A}_k(t_{I_{k+1}} - t_{I_k})} \cdot T_3. \qquad (34)$$

- For $k < j-1$ or $k > j$

$$\frac{\partial N_k}{\partial t_{I_j}} = 0_{m \times m}. \qquad (35)$$

On the other hand, the partial derivative of the function $J_d(t_I)$ in (21) with respect to t_{I_j}, $j \in \{1, \ldots, N_{sw}\}$, is given by

$$\frac{\partial J_d(t_I)}{\partial t_{I_j}} = 2q t_{I_j}. \qquad (36)$$

An appropriate choice of the initial iterate $\bar{t}_0 \in \mathbb{R}^{N_{sw}}$ for GP can be obtained by using the vector composed of the nominal switching times

$$\bar{t}^* = \begin{bmatrix} t^*_{a1} & \cdots & t^*_{aN_a} & t^*_{b1} & \cdots & t^*_{bN_b} & t^*_{c1} & \cdots & t^*_{cN_c} \end{bmatrix}'. \qquad (37)$$

which correspond to the employed OPP.

B. Quadratic programming

In order to obtain a quadratic approximation of the cost term $J_c(t_I)$ in (16) with a possibility of reducing evaluations of the matrix exponentials online, the vectors $\bar{x}_k$ are linearized around the nominal switching times t^*_I from OPP:

$$\bar{x}_k(t_I) \approx \bar{x}_k(t^*_I) + \nabla \bar{x}_k(t^*_I)\Delta t_I, \quad \forall k \in \{0, \ldots, N_{sw}+1\}, \qquad (38)$$

where $\Delta t_I = \begin{bmatrix} \Delta t_{I_1} & \cdots & \Delta t_{I_{N_{sw}}} \end{bmatrix}'$ with elements $\Delta t_{I_j} = t_{I_j} - t^*_{I_j}$, $\forall j \in \{1, \ldots, N_{sw}\}$, and the Jacobian $\nabla \bar{x}_k(t^*_I)$ has the form

$$\nabla \bar{x}_k(t^*_I) = \begin{bmatrix} \frac{\partial \bar{x}_k}{\partial t_{I_1}}(t^*_I) & \cdots & \frac{\partial \bar{x}_k}{\partial t_{I_{N_{sw}}}}(t^*_I) \end{bmatrix}, \quad \forall k \in \{0, \ldots, N_{sw}+1\},$$

where the partial derivatives $\partial \bar{x}_k / \partial t_{I_j}$ are given in (27)-(30).

By introducing the linearizations (38) into the expression for J_c (16), its quadratic approximation is obtained:

$$J_{cQP} = \sum_{k=0}^{N_{sw}} \left(\bar{x}_k(t^*_I) + \nabla \bar{x}_k(t^*_I)\Delta t_I \right)' \cdot N_k \cdot \left(\bar{x}_k(t^*_I) + \nabla \bar{x}_k(t^*_I)\Delta t_I \right)$$
$$= \Delta t'_I \, H_{QP} \, \Delta t_I + h'_{QP} \, \Delta t_I + c_{QP}, \qquad (39)$$

where H_{QP}, h_{QP} and c_{QP} are obtained by sorting the terms.

By taking into account as well the term J_d, the quadratic approximation of the cost J_{tot} in (21) is

$$J_{totQP} = J_{cQP} + \sum_{k=1}^{N_{sw}} q \left(\Delta t_{I_k} \right)^2. \qquad (40)$$

Since the vectors $\bar{x}_k$ are linearized around the nominal switching times t^*_I, the approximation J_{cQP} is valid only when the sequence of times $\bar{t}_I$ is kept unchanged from the sequence of t^*_I. This is imposed by introducing the sequence constraint $t_0 \leq t_{I_1} \leq t_{I_2} \leq \cdots \leq t_{I_{N_{sw}-1}} \leq t_{I_{N_{sw}}} \leq t_1$, and by modifying it to also incorporate the minimal separation between the switching times δ we obtain the constraint

$$t_0 \leq t_{I_1}, \quad t_{I_1} + \delta \leq t_{I_2}, \quad \ldots, \quad t_{I_{N_{sw}}} + \delta \leq t_1. \qquad (41)$$

where $t_{I_j} = t^*_{I_j} + \Delta t_{I_j}$, $\forall j \in \{1, \ldots, N_{sw}\}$, as previously defined. Notice that this sequence constraint is a subset of the constraint (23), which thus can be removed.

Finally, since the linearizations (38) are valid only for small values of deviations Δt_I, a box constraint is imposed;

$$l_b \leq \Delta t_{I_k} \leq u_b, \qquad \forall k \in \{1, 2, \ldots, N_{sw}\}. \qquad (42)$$

In summary, the optimization problem with quadratic approximation of the cost involves the quadratic cost function (40) and the constraint set composed of sequence and box constraint (41) and (42), respectively. It can be noticed that the constraint set is a polytope.

V. SIMULATION RESULTS

The performance of the developed controller in steady-state and transients is explored by Matlab simulation, examining both the GP and QP solution approach. The system and controller parameters can be found in [9].

A. Steady-state performance of GP and QP approach

The steady-state performance is measured by using total harmonic distortion, as defined in (9). The average of the THD in each of the three phases is considered. The steady-state simulations have been run for a range of referent apparent powers to be injected to the grid of the form

$$S_g = \kappa(P_{g,nom} + j Q_{g,nom}), \qquad (43)$$

where the paramater κ is varied from -1 to $+1$ in steps of 0.2, and we select $P_{g,nom} = Q_{g,nom} = 8\,\text{MVA}$. The values of the THD obtained with GP and QP approach are given

in Fig 3. It can be noticed that the QP approach can cause a reduction of steady-state performance which is at some operating points more significant than on the others.
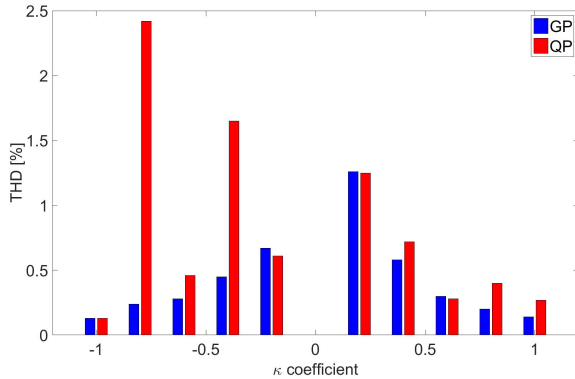

Fig. 3: Steady-state THD values obtained with GP and QP.

## B. Transient performance of GP and QP approach

The transient performance is tested by applying step changes of reference injection power. The reference power is again specified as in (43), and a step change is introduced by the value of κ. The simulation results obtained by changing $\kappa = 1$ to values 0.6 and 0 are given on Fig. 4. It can be noticed from simulation results that for step changes of smaller sizes, the GP and QP solution approach give similar transient behaviours. For larger step changes, the QP approach has worse performance since the corrections Δt_I that it can apply are limited by the box constraint.

VI. Conclusion

In this paper we have developed an MPC scheme for power electronics application which employs integral squared tracking error as its cost function and combines it with OPPs. The combination of these two ingredients results in a controller characterized by a great transient behaviour and steady-state performance. The obtained optimization problem is approached by using GP algorithm and by linearizing the cost function which brought the problem the form of QP. The performance of the controller is examined by Matlab simulations.

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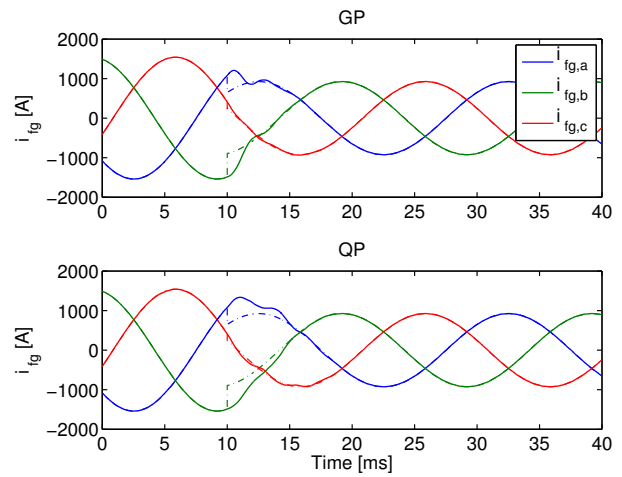
(a) Transient from $\kappa = 1$ to $\kappa = 0.6$.
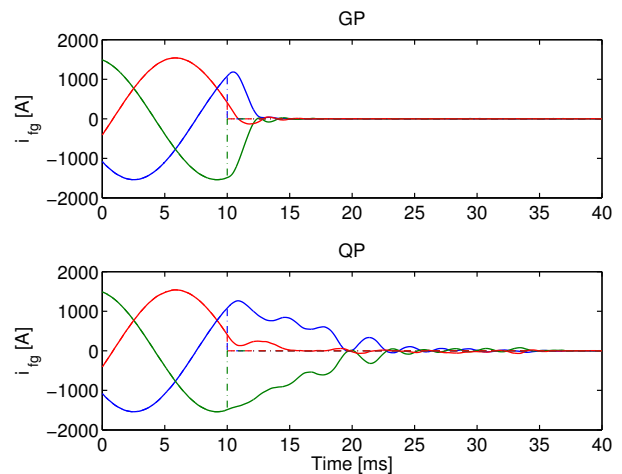
(b) Transient from $\kappa = 1$ to $\kappa = 0$.

Fig. 4: Transients of grid current i_{fg} with GP and QP, obtained by changing at $t = 10$ ms the parameter $\kappa = 1$ in equation (43) to the value (a) $\kappa = 0.6$, (b) $\kappa = 0$.

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