Generation of acoustic helical wavefronts using metasurfaces

Hussein Esfahlani\textsuperscript{1,2}, Hervé Lissek\textsuperscript{1}, and Juan R. Mosig\textsuperscript{2}

\textsuperscript{1}Ecole Polytechnique Fédérale de Lausanne, Laboratoire de Traitement des Signaux LTS2, Lausanne, Switzerland.
\textsuperscript{2}Ecole Polytechnique Fédérale de Lausanne, Laboratoire d’Electromagnétisme et d’Antennes LEMA, Lausanne, Switzerland.

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Supplementary Notes

S-parameter retrieval

In the following, we will consider two cylindrical waveguides separated by a device under test (DUT) of thickness $s$, the scattering parameters of which will be assessed. A loudspeaker is used to generate sound in the first (incident) waveguide, the second waveguide being closed with an anechoic termination, and four microphones are placed at locations $x_i$ at both sides of the interface to measure the sound pressures $P_i = P(x_i)$, as shown in Fig. (1). Mathematically, the pressures $P_i$ are defined by:

\begin{align}
P_1 &= (ae^{-jkx_1} + be^{jkx_1})e^{j\omega t} \quad (1a) \\
P_2 &= (ae^{-jkx_2} + be^{jkx_2})e^{j\omega t} \quad (1b) \\
P_3 &= (ce^{-jkx_3} + de^{jkx_3})e^{j\omega t} \quad (1c) \\
P_4 &= (ce^{-jkx_4} + de^{jkx_4})e^{j\omega t} \quad (1d)
\end{align}

where, $k$ represents the wave number in the ambient fluid and an $e^{+j\omega t}$ sign convention has been adopted. The four complex pressures, $P_1$ to $P_4$, represent the superposition of two plane waves, travelling in opposite directions, as depicted in Fig. (1). The four wave amplitude coefficients $(a,b,c,d)$ can be derived from Eq. (1a-1d) as functions of the four pressures:

\begin{align}
a &= \frac{j(P_1e^{jkx_2} - P_2e^{jkx_1})}{2\text{sink}(x_1 - x_2)} \quad (2a)
\end{align}
Let us look at the DUT as a 2-port, with “inward” sound pressures represented by $ae^{-jkx}$ in the incident medium and $de^{+jkx}$ in the transmission medium, and “outward” sound pressure represented by $be^{jkx}$ and $ce^{-jkx}$. Then, the reflected waves amplitudes $(b, c)$ can be related to the incident wave amplitudes $(a, d)$ by the scattering matrix $[S]$. For the measurement setup described in Fig. (1), the corresponding matrix equations is:

$$\begin{pmatrix} b \\ ce^{-jkx} \end{pmatrix}_{\text{Reflected}} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} a \\ de^{jkx} \end{pmatrix}_{\text{Incident}}$$  \hspace{1cm} (3)

where $s$ is the thickness of the sample.

To compute the four scattering parameters $S_{ij}$, two measurement configurations must be considered, corresponding to two different load conditions at the left and right termination of the waveguides, yielding two different sets of amplitudes $(a_1, b_1, c_1, d_1)$ and $(a_2, b_2, c_2, d_2)$. Then, the elements of the $[S]$ matrix are given by:

$$S_{11} = \frac{b_1 d_2 e^{jkx} - b_2 d_1 e^{jkx}}{a_1 d_2 e^{jkx} - a_2 d_1 e^{jkx}}$$  \hspace{1cm} (4a)

$$S_{12} = \frac{a_1 b_2 - a_2 b_1}{a_1 d_2 e^{jkx} - a_2 d_1 e^{jkx}}$$  \hspace{1cm} (4b)

$$S_{21} = \frac{c_1 e^{-jkx} d_2 e^{jkx} - c_2 e^{-jkx} d_1 e^{jkx}}{a_1 d_2 e^{jkx} - a_2 d_1 e^{jkx}}$$  \hspace{1cm} (4c)

$$S_{22} = \frac{a_1 c_2 e^{-jkx} - a_2 c_1 e^{-jkx}}{a_1 d_2 e^{jkx} - a_2 d_1 e^{jkx}}.$$  \hspace{1cm} (4d)

Figure 1: Schematic representation of four microphone measurement setup.

\[ \text{Image of diagram} \]
However, for symmetrical ($S_{11} = S_{22}$) and reciprocal ($S_{12} = S_{21}$) networks, the [S] matrix has only two different elements and a single measurement suffices. This is the case for proposed unit-cell where the [S] matrix is given by:

$$S_{11} = S_{22} = \frac{ab - cd}{a^2 - d^2 e^{2jks}}$$  \hspace{1cm} (5a)

$$S_{12} = S_{21} = \frac{ace^{-jks} - bde^{jks}}{a^2 - d^2 e^{2jks}}.$$  \hspace{1cm} (5b)

If an anechoic termination is used as a load, then $d = 0$ and the reflection and transmission coefficients $R$ and $T$ can be derived from equations 5a-5b:

$$R = S_{11} = S_{22} = \frac{b}{a}$$  \hspace{1cm} (6)

$$T = S_{12} = S_{21} = \frac{c}{a} e^{-jks}.$$  \hspace{1cm} (7)

Note that these coefficients are calculated with respect to the two terminal planes of the sample $x = 0$ and $x = s$.

Figure 2 shows a picture of the measurement setup used to retrieve the transmission coefficient of the helicoidal unit-cells.

Figure 2: Four microphone measurement setup.
Transmitted, Reflected and Lost Power

Using the measurement procedure proposed in the preceding section, the reflection, transmission and lost power coefficients ($|R|^2$, $|T|^2$ and $\alpha = 1 - |R|^2 - |T|^2$) have been assessed on the 8 helicoidal unit-cells. As depicted in Fig. (3), the percentages of the reflected and lost power depend on the number of helicoidal turns and increase as unit-cells density increases. While the designed unit-cells have been optimized for best impedance matching, the occurrence of unpredictable losses and reflections in the fabricated prototypes yields the observable discrepancies between the measurements and simulations of sound power transmission coefficient amplitudes (See Fig. 1 in the main manuscript). These discrepancies are relatively small for coarse unit-cells with few number of turns, for which the values of sound power reflection coefficients and losses are much lower than the transmitted one, and increase for denser unit-cells as highlighted by the higher values of reflections and losses. The effect of sound power reflections and losses can be linked to the following physical origins:

- Actual vs. simulated helicoid wall thickness:
  In the numerical simulation, the helicoidal and cylindrical walls are assumed to present a smooth and rigid surfaces and zero thicknesses. Therefore, the only Acoustic pressure field has been accounted for in the COMSOL simulations, where the internal walls of the structure are set to hard boundary conditions, yielding the wall vibration, and the thermoviscous losses have been intentionally discarded. This simplifications are justified by the difficulty to render the actual wall thickness in the geometrical rendering of COMSOL without paying the price of prohibitive computational costs, relative to meshing and numerical processing [1]. However, due to the 3D-printing technology employed for building the prototypes, the actually fabricated helicoidal and cylindrical walls present unpredictable porosity and are not smooth, with random surface states varying around average thickness of $0.5\text{mm}$ and $1\text{mm}$ respectively (considering the fabrication precision, each prototype present varying thicknesses around the average). Such construction tolerances result in increasing the frictions within channels, and also in making the walls vibrate under the incident sound field, thus transferring part of the incident acoustic energy into mechanical energy.

- Thermoacoustic losses not accounted for in the simulations:
  The thickness of the helicoidal walls are of the same order of magnitude as the width of the narrow acoustic paths, especially for the denser unit-cells, and it affects the transmission characteristics. Then, in addition to the disregarded porosity of the prototypes walls, the thermoviscous losses resulting from the ultra-thin labyrinthine pathways is the second dominating factor contributing to decreasing the transmission coefficient.

- Presence of residual powder in the prototypes:
  The SLS fabrication process results in leaving residual material powder
inside the channels of the labyrinthine paths. While almost all the resid-
ual powder can be blown away for coarse unit-cells using compressed air,
it is difficult if not impossible for denser twisted shapes. Consequently
the paths are not cleaned properly in denser unit-cells and the remain-
ing powder can block the channels increasing sound power reflection and
absorption.

The last identified sources of discrepancies are dominant for the denser unit-
cells, and the whole mentioned problems are the consequences of the 3D printing
technology.

The most straightforward way to minimize the influence of the aforemen-
tioned problems consists in increasing the global length of each unit-cell $h$ while
increasing the thickness, which results in reducing fabrication problems and er-
rors due to the lower density of helicoidal turns, and consequently lower value
of absorption and reflection. However, this solution presents the critical short-
coming of increasing the global thickness of the structure, which contradicts
the metasurface denomination. Another option consists in choosing an alterna-
tive piecewise function $f(\tau)$ for the modulation of helicoid to decrease both the
sound power reflection and lost power coefficients. This solution has not been
investigated here however.

Instead, rather than focusing on achieving unitary sound power transmission
coefficient on all unit-cells, which is finally not required to obtain a Bessel beam
(doughnut shaped) with helical phase front, a uniform transmission coefficient
among unit-cells can be targeted. This can be done by partially blocking the
output of each unit-cell, to the price of a degradation of the overall transmission
performance. Then, the proposed design still preserves the thickness criterion
of acoustic metasurfaces, transforming an incident plane wave into a helicoidal
wave, to the price of a relative deterioration of the efficiency in terms of power
transmission.
Figure 3: Measured transmitted, reflected and lost power in the helicoidal unit-cells.
Mathematical definitions of the proposed helicoidal unit-cells

The helicoid can be mathematically described as the following parametric equation:

\[ \vec{r}(\rho, t) = \langle x, y, z \rangle = \langle \rho \cos(2\pi \int_0^t f(\tau)d\tau), \rho \sin(2\pi \int_0^t f(\tau)d\tau), bt \rangle, \]

where \( \rho \) is the radius of helicoid, \( b \) is the constant rate of gradual displacement along z-axis, and for \( b = 1 \), \( t \) defines the height of the unit-cell. The spatial modulation function of the helicoid is \( f(\tau) = f_c + f_\Delta x_m(t) \), where \( f_c \) is the average spatial frequency of twists, \( f_\Delta \) is the deviation from \( f_c \) and \( x_m(t) \) is a piecewise function allowing changing the spatial variations. Therefore, \( f_c \) controls the phase of the transmission coefficient, whereas \( f_\Delta \) and \( x_m \) have an influence on the amplitude of the transmission coefficient, acting on the impedance matching. The mathematical definitions of the helicoidal unit-cells used for acoustic OAM have been summarized in Table. 1.
Table 1: Mathematical definitions of helcoidal Unit-cells.

<table>
<thead>
<tr>
<th>Unit-cell</th>
<th>(d[\text{mm}])</th>
<th>(h[\text{mm}])</th>
<th>(N)</th>
<th>(f_c = \frac{N}{h})</th>
<th>(f_m = \frac{1}{h})</th>
<th>(k)</th>
<th>(f_\Delta = k(f_c - f_m))</th>
<th>(x_m(t))</th>
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<tbody>
<tr>
<td>I</td>
<td>30</td>
<td>100</td>
<td>10</td>
<td>0.1</td>
<td>0.01</td>
<td>0.9</td>
<td>0.081</td>
<td>(N)</td>
</tr>
<tr>
<td>II</td>
<td>30</td>
<td>100</td>
<td>9</td>
<td>0.09</td>
<td>0.01</td>
<td>0.7</td>
<td>0.056</td>
<td>(N)</td>
</tr>
<tr>
<td>III</td>
<td>30</td>
<td>100</td>
<td>7.8</td>
<td>0.078</td>
<td>0.01</td>
<td>1</td>
<td>0.068</td>
<td>(N)</td>
</tr>
<tr>
<td>IV</td>
<td>30</td>
<td>100</td>
<td>6.6</td>
<td>0.066</td>
<td>0.01</td>
<td>1</td>
<td>0.056</td>
<td>(N)</td>
</tr>
<tr>
<td>V</td>
<td>30</td>
<td>100</td>
<td>5</td>
<td>0.05</td>
<td>0.01</td>
<td>0.9</td>
<td>0.036</td>
<td>(N)</td>
</tr>
<tr>
<td>VI</td>
<td>30</td>
<td>100</td>
<td>3.6</td>
<td>0.036</td>
<td>0.01</td>
<td>0.1</td>
<td>0.0026</td>
<td>(N)</td>
</tr>
<tr>
<td>VII</td>
<td>30</td>
<td>100</td>
<td>2.5</td>
<td>0.025</td>
<td>0.01</td>
<td>0</td>
<td>0</td>
<td>(N)</td>
</tr>
<tr>
<td>VIII</td>
<td>30</td>
<td>100</td>
<td>1</td>
<td>0.01</td>
<td>0.01</td>
<td>0</td>
<td>0</td>
<td>(N)</td>
</tr>
</tbody>
</table>

\[\vec{r}(\rho,t) = <x,y,z>,\]
\[
x = \rho \cos(2\pi \int_0^t f(\tau) d\tau) = \rho \cos(2\pi \int_0^t [f_c + f_\Delta x_m(t)] d\tau) = \rho \cos(2\pi f_c t + 2\pi f_\Delta \int_0^t x_m(t) d\tau) \quad 0 < t < h
\]
\[
y = \rho \sin(2\pi \int_0^t f(\tau) d\tau) = \rho \sin(2\pi \int_0^t [f_c + f_\Delta x_m(t)] d\tau) = \rho \sin(2\pi f_c t + 2\pi f_\Delta \int_0^t x_m(t) d\tau) \quad -\frac{d}{2} < \rho < \frac{d}{2}
\]
\[
z = bt
\]
Measurement setup for the OAM Metasurface

The measurement setup for our proposed acoustic OAM is shown in Fig. 4 and the corresponding dimensions have been summarized in Table 2.

![Figure 4: Schematic representation of the acoustic OAM measurement setup.](image)

**Table 2: OAM measurement setup dimensions.**

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Value (mm)</th>
<th>Symbol</th>
<th>Value (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D_{out}$</td>
<td>200</td>
<td>$h$</td>
<td>100</td>
</tr>
<tr>
<td>$D_{in}$</td>
<td>194</td>
<td>$\theta_1$</td>
<td>45°</td>
</tr>
<tr>
<td>$t_w$</td>
<td>3</td>
<td>$\theta_2$</td>
<td>22.5°</td>
</tr>
<tr>
<td>$t_b$</td>
<td>3.5</td>
<td>$R_1$</td>
<td>45</td>
</tr>
<tr>
<td>$t_a$</td>
<td>4</td>
<td>$R_2$</td>
<td>81.50</td>
</tr>
<tr>
<td>$t_p$</td>
<td>8</td>
<td>$r_1$</td>
<td>88</td>
</tr>
<tr>
<td>$d$</td>
<td>30</td>
<td>$r_2$</td>
<td>66</td>
</tr>
<tr>
<td>$d_m$</td>
<td>7</td>
<td>$r_3$</td>
<td>44</td>
</tr>
<tr>
<td>$L$</td>
<td>800</td>
<td>$r_4$</td>
<td>22</td>
</tr>
</tbody>
</table>

The diameter of the holes holding the unit-cells is chosen to be 0.8mm bigger than $d = 30$mm (unit-cell diameter) in order to fit the helicoidal unit-cells in the support holes.
Figure 5: Acoustic OAM measurement setup.
References
