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Approximate cloaking for the full wave equation via change of variables: The Drude–Lorentz model

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ABSTRACT

This paper concerns approximate cloaking by mapping for a full, but scalar wave equation, when one allows for physically relevant frequency dependence of the material properties of the cloak. The paper is a natural continuation of [20], but here we employ the Drude–Lorentz model in the cloaking layer, that is otherwise constructed by an approximate blow up transformation of the type introduced in [10]. The central mathematical problem is to analyze the effect of a small inhomogeneity in the context of this non-local full wave equation.
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RÉSUMÉ

L'article traite du « cloaking » approché par changement de variables pour l'équation des ondes scalaire avec amortissement. Il poursuit l'étude présentée dans [20] par l'étude d'un modèle réaliste et pertinent dans lesquels les coefficients constitutifs du milieu constituant la cape d'invisibilité dépendent de la fréquence. En l'occurence le dispositif est construit par une transformation asymptotiquement singulière analogue à celles introduites dans [10], cependant la région occupée par la cape d'invisibilité est décrite par un modèle de Drude–Lorentz. La question mathématique centrale de l'article est l'analyse de l'effet d'une inhomogénéité de petite taille dans le contexte de l'équation des ondes résultante, non locale avec amortissement.
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1. Introduction

Cloaking by mapping (frequently referred to as transformation optics) was introduced by Pendry, Schurig, and Smith [23] for the Maxwell system, and Leonhardt [12] in the geometric optics setting. These authors used a singular change of variables which blows up a point to a cloaked region. The exact same transformation

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had been used before by Greenleaf, Lassas, and Uhlmann [6] to establish non-uniqueness in the context of the Calderon problem. The singular nature of the cloaks presents various difficulties in practice as well as in theory: (1) they are hard to fabricate and (2) in certain cases the correct definition of the corresponding electromagnetic fields is not obvious. To avoid the use of singular structures, regularized schemes have been proposed in [3,4,10,26,27].

In this paper we analyze approximate cloaking for a full wave equation using transformation optics, where we incorporate the Drude–Lorentz model, see e.g., [8], in the layer constructed by transformation optics. The Drude–Lorentz model takes into account the effect of the oscillations of free electrons on the electric permittivity (by means of a simple harmonic oscillator model). We could have incorporated the same model in other parts of space, to better model conducting metallic elements of these parts as well. For the transformation optics construction we use the approximate scheme introduced in [10], which is based on a transformation blowing up a small ball of radius ε to the cloaked region. When viewed in (complex) frequency domain, the refractive index associated with the Drude–Lorentz model may be extended analytically to the whole upper half plane. As is well known, an immediate consequence of this is causality for the associated non-local wave equation, see [8] and [28], – a property which is most essential for the well-posedness (and the physical relevance) of this equation. Another well known consequence of this analyticity property are the so-called Kramers–Krönig relations between the real and the imaginary part of the refractive index (they are essentially related by Hilbert transforms). However, this fact is not explicitly used in our analysis.

Approximate cloaking schemes for the Helmholtz equation based on the regularized transformations introduced in [10] have been studied extensively in various regimes, see [9,16,17,21]. A related scheme, which (in 3d) blows up a small diameter cylinder to the cloaked region was studied in [18] (see also [5,13,14]). Frequently a (damping) lossy layer is employed inside the transformation cloak. Without this lossy layer, the field inside the cloaked region might depend on the field outside (even for a perfect cloak), and resonance can appear and destroy the cloaking (or approximate cloaking) ability of the pure transformation cloak, see [17].

We next describe the setting in detail. Given $r > 0$, let B_r denote the open ball centered at 0 and of radius r. Let F_ε be the standard transformation $\mathbb{R}^d \to \mathbb{R}^d$, $d = 2, 3$, which blows up the ball B_ε to B_1, equals the identity outside B_2, and is given by

$$F_\varepsilon(x) = \begin{cases} x & \text{if } x \in \mathbb{R}^d \setminus B_2, \\ \Big(\dfrac{2-2\varepsilon}{2-\varepsilon} + \dfrac{|x|}{2-\varepsilon}\Big)\dfrac{x}{|x|} & \text{if } x \in B_2 \setminus B_\varepsilon, \\ \dfrac{x}{\varepsilon} & \text{if } x \in B_\varepsilon. \end{cases} \tag{1.1}$$

Assume that the cloaked region is the ball $B_{1/2}$, the contents of which is characterized by a real, matrix valued function a and a complex function σ. The surrounding cloak contains two parts. In the *time harmonic regime*, these can be described as follows. The *outer* part is the Drude–Lorentz version of the standard layer, generated by the blow up map F_ε. In this layer, occupying $B_2 \setminus B_1$, the material characteristics are given by

$$\big(F_\varepsilon\big)_* I, \ \big(F_\varepsilon\big)_* 1 + \sigma_{1,c} \ , \tag{1.2}$$

where

$$\sigma_{1,c}(k, x) = \frac{\sigma_N}{k_\varepsilon^2 - k^2 - i\sigma_D k} \ . \tag{1.3}$$

While the first part $\big(F_\varepsilon\big)_* 1$ of the refractive index in (1.2) is standard from the transformation optics approach, the second part $\sigma_{1,c}$ is exactly the correction introduced by the Drude–Lorentz model, see e.g., [8, page 331]. Here σ_N and σ_D are material constants which can in principle depend on the space variable x, and $k_\varepsilon > 0$ is the so-called resonant frequency of the Drude–Lorentz model; in a more general model there could be several resonant frequencies $\{k_{i,\varepsilon}\}$, and the corresponding part of the refractive index would be a sum of terms (1.3) ranging over all these frequencies, see e.g., [8, page 310]. In this paper, we use the standard notation

$$F_*A(y) = \frac{DF(x)A(x)DF^T(x)}{|\det DF(x)|}, \quad F_*\Sigma(y) = \frac{\Sigma(x)}{|\det DF(x)|}, \quad x = F^{-1}(y) \ ,$$

for the "pushforward" of a symmetric, matrix valued function, A, and a scalar function, Σ, by the diffeomorphism F. In what follows, we assume for ease of notation that

$$\sigma_N = \sigma_D = 1 \text{ in } B_2 \setminus B_1 \ .$$

The *inner* part of our cloak is a *fixed* damping layer as considered in [16]. This damping (lossy) layer occupies $B_1 \setminus B_{1/2}$, and its material characteristics are given by

$$I, \, 1 + \frac{i}{k} \ .$$

Therefore, in the time harmonic regime, i.e., in frequency domain, the entire medium is characterized by[1]

$$A_c, \Sigma_c := \begin{cases} I, 1 & \text{in } \mathbb{R}^d \setminus B_2 \ , \\ \left(F_\varepsilon\right)_* I, \left(F_\varepsilon\right)_* 1 + \sigma_{1,c} & \text{in } B_2 \setminus B_1 \ , \\ I, 1 + i/k & \text{in } B_1 \setminus B_{1/2} \ , \\ a, \sigma & \text{in } B_{1/2} \ . \end{cases} \tag{1.4}$$

We assume that $a, \sigma \in L^\infty(B_{1/2})$, with

$$\frac{1}{\Lambda}|\xi|^2 \le \langle a\xi, \xi\rangle \le \Lambda|\xi|^2 \ , \quad \frac{1}{\Lambda} \le \Re(\sigma) \le \Lambda \ , \quad \text{and} \quad 0 \le \Im(\sigma) \le \Lambda \ , \tag{1.5}$$

for some positive constant Λ. With this notation, the temporal Fourier transform $\hat{u}_c$ of the field,[2] will be a solution to

$$\operatorname{div}(A_c \nabla \hat{u}_c) + k^2 \Sigma_c \hat{u}_c = -\hat{f} \ .$$

The temporal Fourier transform of a function $v(t, x)$ is given by

$$\hat{v}(k, x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} v(t, x) e^{ikt} \, dt \ .$$

The corresponding field in time domain (for positive time) is the unique weak solution $u_c \in L^\infty\big((0, +\infty); H^1(\mathbb{R}^d)\big)$, with $\partial_t u_c \in L^\infty\big([0, +\infty); L^2(\mathbb{R}^d)\big)$, to the **non-local** wave equation

[1] Notice that the "damping layer", $B_1 \setminus B_{1/2}$, is a bit different from that in [20] where, for any fixed $\gamma > 0$, we used $A_c = I$, $\Sigma_c = \varepsilon^2 + \frac{i}{k\varepsilon^\gamma}$, for $n = 2$, and $A_c = \varepsilon I$, $\Sigma_c = \varepsilon^3 + \frac{i\varepsilon^{1-\gamma}}{k}$, for $n = 3$. This change is, however, not essential – the essential change is in the layer $B_2 \setminus B_1$, with the inclusion of $\sigma_{1,c}$. It would be interesting to investigate whether, in view of the damping present in $\sigma_{1,c}$, the layer $B_1 \setminus B_{1/2}$ is necessary at all.

[2] Where we extend the time domain field by 0 for negative time.

$$\begin{cases} \Sigma_{1,c}\partial^2_{tt}u_c - \operatorname{div}(A_c\nabla u_c) + \Sigma_{2,c}\partial_t u_c + G * \partial_t u_c = f & \text{in } [0,+\infty)\times\mathbb{R}^d, \\ \partial_t u_c(t=0) = u_c(t=0) = 0 & \text{in } \mathbb{R}^d, \end{cases} \tag{1.6}$$

where $f \in L^2\big((0,+\infty)\times\mathbb{R}^d\big)$ with compact support. The definition of weak solutions to (1.6), and the proof of well-posedness of (1.6) is presented in Section 4. The coefficients $\Sigma_{1,c}$ and $\Sigma_{2,c}$ are given by

$$\Sigma_{1,c} = \begin{cases} 1 & \text{in } \mathbb{R}^d\setminus B_2 \ , \\ (F_\varepsilon)_*1 & \text{in } B_2\setminus B_1 \ , \\ 1 & \text{in } B_1\setminus B_{1/2} \ , \\ \sigma & \text{in } B_{1/2} \ , \end{cases} \qquad \Sigma_{2,c} = \begin{cases} 0 & \text{in } \mathbb{R}^d\setminus B_2 \ , \\ 0 & \text{in } B_2\setminus B_1 \ , \\ 1 & \text{in } B_1\setminus B_{1/2} \ , \\ 0 & \text{in } B_{1/2} \ , \end{cases}$$

and $G(t,x)$ is such that

$$\widehat{G}(k,x) = -ik\sigma_{1,c}(k,x) \quad x \in B_2\setminus B_1 \quad .$$

A computation (see, e.g., [8, (7.110)]) shows that

$$G(t,x) = \phi(t)H(t) \ , \tag{1.7}$$

where $H(t)$ denotes the Heaviside function, i.e.,

$$H(t) = \begin{cases} 0 & \text{if } t<0 \ , \\ 1 & \text{otherwise} \ , \end{cases} \tag{1.8}$$

and

$$\phi(t) = \frac{\sqrt{2\pi}}{\gamma_0}\partial_t\Big(e^{-t/2}\sin(\gamma_0 t)\Big) \ , \tag{1.9}$$

with

$$\gamma_0 = \sqrt{k_\varepsilon^2 - 1/4} \ . \tag{1.10}$$

We assume that $k_\varepsilon > 1/2$, so that γ_0 is real and positive.

The presence of the Heaviside function in the formula (1.7) implies **causality** and plays an important role in our analysis; in particular for the proof of well-posedness of u_c, and to establish that the Fourier transform, $\hat{u}_c$, satisfies the outgoing radiation condition.

We only consider zero initial conditions. This is just for ease and simplicity of presentation; indeed, our method would work for the general case, using an approach similar to that in [20].

Given f, the corresponding field in the homogeneous medium without the cloak and the cloaked region is the unique weak solution $u \in L^\infty\big((0,+\infty);H^1(\mathbb{R}^d)\big)$, with $\partial_t u \in L^\infty\big((0,+\infty);L^2(\mathbb{R}^d)\big)$, to the system

$$\begin{cases} \partial^2_{tt}u - \Delta u = f & \text{in } (0,+\infty)\times\mathbb{R}^d \ , \\ \partial_t u(t=0) = u(t=0) = 0 & \text{in } \mathbb{R}^d \ . \end{cases}$$

The extent to which we have succeeded in hiding the contents of $B_{1/2}$ and the cloak itself, should be measured in terms of the difference between u_c and u, outside B_2. The main Theorem of this paper gives an estimate of this difference for the scheme in (1.4).

Theorem 1. *Let $d = 2$ or 3, and let $f \in C^\infty([0, +\infty) \times \mathbb{R}^d)$ be such that $\operatorname{supp} f \subset (0, R) \times (B_R \setminus B_2)$ for some $R > 0$. Suppose $c_* \varepsilon^{-d/2} < k_\varepsilon < C_* \varepsilon^{-K}$ for some positive constant c_*, C_* and $K > d/2$. Given any integer $M \ge 2d + 4K - 2$, there exists a constant C such that*

$$\sup_{0<t<T} \|u_c - u\|_{L^2(B_5 \setminus B_2)} \le C\varepsilon T \|f\|_{C^M([0,R];L^2(B_R))} \qquad \forall T > 0, \quad \text{for } d = 3 \ ,$$

and

$$\sup_{0<t<T} \|u_c - u\|_{L^2(B_5 \setminus B_2)} \le C \frac{1}{|\ln \varepsilon|} T \|f\|_{C^M([0,R];L^2(B_R))} \qquad \forall T > 0, \quad \text{for } d = 2 \ .$$

C depends on R, c_*, C_*, K and M, but is independent of f, ε, k_ε, Λ and T.

We in fact prove the following slightly stronger result:

Theorem 2. *Let $d = 2$ or 3, and let $f \in C^\infty([0, +\infty) \times \mathbb{R}^d)$ be such that $\operatorname{supp} f \subset (0, R) \times (B_R \setminus B_2)$ for some $R > 0$. Suppose $k_\varepsilon > c_* \varepsilon^{-d/2}$ for some positive constant c_*, then*

$$\sup_{0<t<T} \|u_c - u\|_{L^2(B_5 \setminus B_2)} \le C\varepsilon T \|f\| \qquad \forall T > 0, \quad \text{for } d = 3 \ ,$$

and

$$\sup_{0<t<T} \|u_c - u\|_{L^2(B_5 \setminus B_2)} \le C \frac{1}{|\ln \varepsilon|} T \|f\| \qquad \forall T > 0, \quad \text{for } d = 2 \ .$$

Here C is a positive constant depending on R and c_*, but independent of f, ε, k_ε, Λ and T. The norm of f is defined by

$$\|f\| = \|f\|_{k_\varepsilon, \varepsilon} := \int_0^\infty (1 + k^{2d+1}) \|\hat{f}(k, \cdot)\|_{L^2} \, dk + \int_{\lambda_0/\varepsilon}^\infty k^{2d-3} k_\varepsilon^4 \|\hat{f}(k, \cdot)\|_{L^2} \, dk \ ,$$

for some fixed positive constant λ_0, depending only on c_*. Here $\hat{f}$ is the Fourier transform of f with respect to time, f being extended by zero for $t < 0$.

The assumption that $\operatorname{supp} f \subset (0, R) \times (B_R \setminus B_2)$ could be replaced by $\operatorname{supp} f \subset [0, R) \times (B_R \setminus B_2)$ (*i.e.*, f does not have to vanish in a neighborhood of $t = 0$) provided one assumes that $\tilde{f} \in C^\infty((-\infty, +\infty) \times \mathbb{R}^d)$, where $\tilde{f}$ denotes the extension of f by zero for $t < 0$. The condition that f or $\tilde{f}$ be in C^∞ could also be replaced by an assumption about the continuity of only finitely many derivatives. We leave the details to the reader.

Theorem 1 follows directly from Theorem 2 by noting that if $c_* \varepsilon^{-d/2} < k_\varepsilon < C_* \varepsilon^{-K}$, for some $K > d/2$, then

$$\begin{aligned} \|f\| &= \int_0^\infty (1 + k^{2d+1}) \|\hat{f}(k, \cdot)\|_{L^2} + \int_{\lambda_0/\varepsilon}^\infty k^{2d-3} k_\varepsilon^4 \|\hat{f}(k, \cdot)\|_{L^2} \\ &\le C \int_0^\infty (1 + k^{2d+4K-3}) \|\hat{f}(k, \cdot)\|_{L^2} \end{aligned}$$

$$\leq C \left(\int_0^\infty (1+k)^{-2} \right)^{1/2} \left(\int_0^\infty (1+k^{2d+4K-2})^2 \|\hat{f}(k,\cdot)\|_{L^2}^2 \right)^{1/2}$$
$$\leq C\|f\|_{C^M([0,R];L^2(B_R))}$$

for any integer $M \geq 2d + 4K - 2$. Here we used that $\operatorname{supp} f \subset (0, R) \times (B_R \setminus B_2)$ so that the C^M-norm of the extension of f by zero for $t < 0$ is bounded by $\|f\|_{C^M([0,R];L^2(B_R))}$.

The results obtained in this paper are in a slightly different spirit than the ones in [20] (and, of course, for a different problem). The constants in Theorem 2 and Theorem 1 here are independent of Λ, while the ones in [20, Theorems 1 and 2] are not. However, the estimates in [20, Theorems 1 and 2] are uniform in time, while the ones in Theorem 2 and Theorem 1 here are not. The independence of the constants of Λ yields a stronger result about the cloaking effects, since it asserts that the cloak works well for arbitrary objects. Similar results as in [20] (i.e., results that are uniform in time, but not in Λ) would hold in this setting, and results of the type in Theorem 2 and Theorem 1 would hold in the setting of [20].

The approach in this paper borrows several ideas from the approach in [20], and adapts these to the setting considered here. We transform the wave equation into a family of Helmholtz equations by taking the Fourier transform with respect to time. Having established the appropriate near-invisibility estimates for the Helmholtz equations, with explicit frequency dependence, we then essentially invert the Fourier transform. As concerns the Helmholtz equations, we study and compare the model with $\sigma_{1,c}$ in (1.4) and the model without $\sigma_{1,c}$, and establish perturbation estimates in the time harmonic regime. Note that, for the model with $\sigma_{1,c}$, the standard rescaling techniques, as used in [16,17,20,21], do not work. We hence work directly with this model without rescaling (Section 3). The proof is quite delicate, makes use of many ideas from [16,17,20,21], and at a crucial point requires an argument of "removable singularity" (in the proof of Lemma 5). To obtain the estimates in time domain from the estimates in frequency domain, we proceed in a similar, but slightly different way than [20]. We use a simple and helpful idea, also used in [19], by establishing estimates for the difference of the *time derivatives* of u_c and u not for their difference. As a consequence, we avoid the non-standard estimates for very low frequency in [20, Section 2.2]; their proof involved the theory of H-convergence. Moreover, using this idea, we are also able to obtain the independence of Λ for the constants in Theorem 2. As mentioned earlier, another element of our analysis is the (definition of and) verification of well-posedness of u_c (Proposition 1). For this purpose we rely on a non-trivial energy estimate, in the spirit of [19].

The paper is organized as follows. In Section 2, we present results for the model without $\sigma_{1,c}$ and some estimates for $\left(F_\varepsilon^{-1}\right)_* \sigma_{1,c}$. These will be used in the proof of Theorem 2 to obtain estimates in the time harmonic regime, when the frequency is of order at most $1/\varepsilon$. Section 3 provides estimates for u_c in the time harmonic regime for arbitrarily large frequencies. In section 4, we establish the well-posedness of u_c and discuss the outgoing radiation condition for its Fourier transform with respect to time. The required non-trivial energy estimate for u_c is also derived there. Finally, the proof of Theorem 2 is given in Section 5.

2. Preliminaries

In this section we recall some known results, which will be used frequently in this paper, and we derive an estimate related to the model without $\sigma_{1,c}$ in the time harmonic regime, when the frequency is of order at most $1/\varepsilon$. This estimate is an extension of [16, Lemma 2.4]. We also estimate $\left(F_\varepsilon^{-1}\right)_* \sigma_{1,c}$ in various regions. These results will be used in Section 5 in the proof of Theorem 2.

Let U denote a connected smooth open region of $\mathbb{R}^d$ ($d = 2$ or 3) with a bounded complement (this includes $U = \mathbb{R}^d$). Here and in what follows, a solution $v \in H^1_{loc}(U)$ ($d = 2$ or 3) to the Helmholtz equation

$$\Delta v + k^2 v = 0 \text{ in } U \ ,$$

for some $k > 0$, is said to be an outgoing solution (or satisfy the outgoing radiation condition) if

$$\frac{\partial v}{\partial r} - ikv = o\left(r^{-\frac{d-1}{2}}\right) \text{ as } r \to \infty .$$

We shall also need the space $W^1(U)$; it is defined as follows,

$$W^1(U) = \left\{\psi \in L^1_{loc}(U) : \frac{\psi(x)}{\sqrt{1+|x|^2}} \in L^2(U) \text{ and } \nabla\psi \in L^2(U)\right\} \quad \text{for } d = 3 ,$$

and,

$$W^1(U) = \left\{\psi \in L^1_{loc}(U) : \frac{\psi(x)}{\ln(2+|x|)\sqrt{1+|x|^2}} \in L^2(U) \text{ and } \nabla\psi \in L^2(U)\right\} \quad \text{for } d = 2 .$$

Lemma 1. *Let $d = 2$ or 3 and $k > 0$. Suppose $f \in L^2(\mathbb{R}^d)$ with $\operatorname{supp} f \subset B_5$, and let $v_k \in H^1_{loc}(\mathbb{R}^d)$ be the unique outgoing solution to*

$$\Delta v_k + k^2 v_k = f \text{ in } \mathbb{R}^d .$$

Then, for $d = 2$ and $0 \le k \le 1/2$,

$$\|\nabla v_k\|_{L^2(B_6)} + \|v_k\|_{L^2(B_6)} \le C|\ln k|\|f\|_{L^2} ,$$

and for $d = 3$ or for $d = 2$ and $k > 1/2$,

$$\|\nabla v_k\|_{L^2(B_6)} + (k+1)\|v_k\|_{L^2(B_6)} \le C\|f\|_{L^2} .$$

Here C is a positive constant independent of k and f.

Proof. The conclusion in the case $k < k_0$, for arbitrary fixed $k_0 > 0$, follows directly from the properties of the fundamental solution to the Helmholtz equation. The conclusion in the case $k \ge k_0$ can also be obtained from the fundamental solution to the Helmholtz equation. In this case, one can alternately obtain the conclusion using the Morawetz multipliers (see, e.g., [21, Lemma 2 and Proposition 1]). We note that the estimate in [21, Proposition 1] requires a damping layer due to the desire to obtain estimates that are independent of the arbitrary coefficients inside $B_{1/2}$. Since the operator here is $\Delta + k^2$ throughout, there is no need for such a layer. The details are left to the reader. □

We next recall the following result which will be used frequently in this paper. The result is from [16, Lemma 2.2] (see also [21, Lemma 3]).

Lemma 2. *Let $d = 2$ or 3, and let D be a smooth, open bounded subset of $\mathbb{R}^d$ such that $\mathbb{R}^d \setminus D$ is connected. Suppose $0 < k < \tau$, for some fixed $\tau > 0$, and suppose $g_k \in H^{1/2}(\partial D)$. Let $v_k \in H^1_{loc}(\mathbb{R}^d \setminus D)$ be the unique outgoing solution to*

$$\begin{cases} \Delta v_k + k^2 v_k = 0 & \text{in } \mathbb{R}^d \setminus D , \\ v_k = g_k & \text{on } \partial D . \end{cases}$$

Then

$$\|v_k\|_{H^1(B_R \setminus D)} \le C_R \|g_k\|_{H^{1/2}(\partial D)} \quad \text{for any } R > 0 .$$

The constant C_R is independent of k and g_k. Furthermore for any $\varepsilon > 0$ sufficiently small that $D \subset B_{2/\varepsilon}$

$$\begin{cases} \|v_k\|_{L^2(B_{5/\varepsilon}\setminus B_{2/\varepsilon})} \le C\varepsilon^{-1/2}\|g_k\|_{H^{1/2}(\partial D)} & \text{if } \quad d=3 \\ \|v_k\|_{L^2(B_{5/\varepsilon}\setminus B_{2/\varepsilon})} \le C\varepsilon^{-1}\dfrac{|H_0^{(1)}(k/\varepsilon)|}{|H_0^{(1)}(k)|}\|g_k\|_{H^{1/2}(\partial D)} & \text{if } \quad d=2 \, . \end{cases}$$

Here the constant C is independent of k, g_k and ϵ. Finally, if we assume that $g_k \to g$ weakly in $H^{1/2}(\partial D)$ as $k \to 0$, then $v_k \to v$ weakly in $H^1_{loc}(\mathbb{R}^d \setminus D)$ where $v \in W^1(\mathbb{R}^d \setminus D)$ is the unique solution of

$$\begin{cases} \Delta v = 0 & \text{in } \mathbb{R}^d \setminus D \, , \\ v = g & \text{on } \partial D \, . \end{cases}$$

We next establish an estimate for the model without $\sigma_{1,c}$, for frequency at most $1/\varepsilon$.

Lemma 3. *Let $d = 2$ or 3, and let a and σ be in $L^\infty(B_{1/2})$, with*

$$a \text{ real symmetric, uniformly positive definite, and } \Im(\sigma) \ge 0 \, . \tag{2.1}$$

Suppose $0 < \varepsilon < \tau$, and $0 < k < \tau/\varepsilon$ for some fixed, positive constant τ. For $g \in H^{-\frac{1}{2}}(\partial B_1)$ let $v_\varepsilon \in H^1_{loc}(\mathbb{R}^d)$ be the unique outgoing solution to

$$\begin{cases} \Delta v_\varepsilon + \varepsilon^2 k^2 v_\varepsilon = 0 & \text{in } \mathbb{R}^d \setminus \bar{B}_1 \, , \\ \operatorname{div}(A\nabla v_\varepsilon) + k^2 \Sigma v_\varepsilon = 0 & \text{in } B_1 \, , \\ \dfrac{\partial v_\varepsilon}{\partial r}\Big|_{\text{ext}} - \dfrac{1}{\varepsilon^{d-2}}\dfrac{\partial v_\varepsilon}{\partial r}\Big|_{\text{int}} = g & \text{on } \partial B_1 \, . \end{cases} \tag{2.2}$$

Here

$$A = \begin{cases} I & \text{if } x \in \mathbb{R}^d \setminus B_{1/2} \, , \\ a & \text{if } x \in B_{1/2} \, , \end{cases} \qquad \Sigma = \begin{cases} 1 & \text{if } x \in \mathbb{R}^d \setminus B_1 \, , \\ 1 + i/k & \text{if } x \in B_1 \setminus B_{1/2} \, , \\ \sigma & \text{if } x \in B_{1/2} \, . \end{cases}$$

There exists a positive constant C, depending only on d and τ, such that

$$\|v_\varepsilon\|_{H^1(B_5\setminus B_1)} \le C \max\{k^{3-d}, \varepsilon^{d-2}/k\}\|g\|_{H^{-\frac{1}{2}}(\partial B_1)}.$$

Proof of Lemma 3. We follow the strategy in the proof of [16, Lemma 2.4], and consider the case $d = 2$ and $d = 3$ separately.

Case 1: $d = 2$. We first prove

$$\|v_\varepsilon\|_{L^2(B_5\setminus B_1)} \le C \max\{k, 1/k\}\|g\|_{H^{-\frac{1}{2}}(\partial B_1)} \, , \tag{2.3}$$

by contradiction. Suppose this estimate is not true. Then there exist $(g_n) \subset H^{-\frac{1}{2}}(\partial B_1)$, (ε_n), (k_n), (a_n), and (σ_n) such that $0 < \varepsilon_n < \tau$, $0 < k_n < \tau/\varepsilon_n$, a_n and σ_n satisfy (2.1), and

$$\|v_n\|_{L^2(B_5\setminus B_1)} = 1 \quad , \quad \lim_{n\to\infty} \max\{k_n, 1/k_n\}\|g_n\|_{H^{-\frac{1}{2}}(\partial B_1)} = 0 \, . \tag{2.4}$$

Here $v_n \in H^1_{loc}(\mathbb{R}^2)$ is the unique outgoing solution to

$$\begin{cases} \Delta v_n + \varepsilon_n^2 k_n^2 v_n = 0 & \text{in } \mathbb{R}^2 \setminus B_1, \\ \operatorname{div}(A_n \nabla v_n) + k_n^2 \Sigma_n v_n = 0 & \text{in } B_1, \\ \dfrac{\partial v_n}{\partial r}\Big|_{\text{ext}} - \dfrac{\partial v_n}{\partial r}\Big|_{\text{int}} = g_n & \text{on } \partial B_1 \ , \end{cases} \tag{2.5}$$

where A_n and Σ_n are defined the same way as A and Σ, with a and σ replaced by a_n and σ_n. Multiplying the equation for v_n by $\bar{v}_n$ (the conjugate of v_n) and integrating on B_R, we obtain

$$\begin{aligned} \int_{\partial B_R} \partial_r v_n \bar{v}_n - \int_{B_R \setminus B_1} |\nabla v_n|^2 + \varepsilon_n^2 k_n^2 \int_{B_R \setminus B_1} |v_n|^2 \\ - \int_{B_1} \langle A_n \nabla v_n, \nabla \bar{v}_n \rangle + k_n^2 \int_{B_1} \Sigma_n |v_n|^2 = \int_{\partial B_1} g_n \bar{v}_n \ . \end{aligned} \tag{2.6}$$

Letting $R \to \infty$ in (2.6), using the outgoing condition, and considering the imaginary part, we derive that

$$k_n \int_{B_1 \setminus B_{1/2}} |v_n|^2 \le \|g_n\|_{H^{-1/2}(\partial B_1)} \|v_n\|_{H^{1/2}(\partial B_1)} \ . \tag{2.7}$$

By Caccioppoli's inequality, it follows that

$$\begin{aligned} \int_{B_{4/5} \setminus B_{3/5}} |\nabla v_n|^2 &\le C(k_n^2 + 1) \int_{B_1 \setminus B_{1/2}} |v_n|^2 \\ &\le C \max\{k_n, 1/k_n\} \|g_n\|_{H^{-1/2}(\partial B_1)} \|v_n\|_{H^{1/2}(\partial B_1)} \ . \end{aligned}$$

Here and in the remainder of this proof, C denotes a positive constant depending only on d and τ (which might change from one place to another). The above estimate implies that for some $r \in (3/5, 4/5)$ (r depends on n),

$$\int_{\partial B_r} |\nabla v_n|^2 + (1 + k_n^2) \int_{\partial B_r} |v_n|^2 \le C \max\{k_n, 1/k_n\} \|g_n\|_{H^{-1/2}(\partial B_1)} \|v_n\|_{H^{1/2}(\partial B_1)} \ . \tag{2.8}$$

Multiplying the equation for v_n by $\bar{v}_n$ and integrating on $B_5 \setminus B_r$, we have

$$\begin{aligned} \int_{\partial B_5} \partial_r v_n \bar{v}_n - \int_{\partial B_r} \partial_r v_n \bar{v}_n - \int_{B_5 \setminus B_r} |\nabla v_n|^2 + \varepsilon_n^2 k_n^2 \int_{B_5 \setminus B_1} |v_n|^2 \\ + k_n^2 \int_{B_1 \setminus B_r} \Sigma_n |v_n|^2 = \int_{\partial B_1} g_n \bar{v}_n \ . \end{aligned} \tag{2.9}$$

Since $v_n \in H^1_{loc}(\mathbb{R}^2 \setminus B_3)$ is the unique outgoing solution to $\Delta v_n + \varepsilon_n^2 k_n^2 v_n = 0$ in $\mathbb{R}^2 \setminus B_3$ and $\varepsilon_n k_n \le \tau$, it follows that (see, e.g., Lemma 2)

$$\|v_n\|_{H^1(B_6 \setminus B_3)} \le C \|v_n\|_{H^{1/2}(\partial B_3)} \ . \tag{2.10}$$

Since $\Delta v_n + \varepsilon_n^2 k_n^2 v_n = 0$ in $B_5 \setminus B_1$, using the standard theory of elliptic equations, we have that

$$\|v_n\|_{H^{1/2}(\partial B_3)} \le C\|v_n\|_{H^1(B_4\setminus B_2)} \le C\|v_n\|_{L^2(B_5\setminus B_1)} \; . \tag{2.11}$$

A combination of (4.25) and (4.26) yields

$$\|v_n\|_{H^1(B_6\setminus B_3)} \le C\|v_n\|_{L^2(B_5\setminus B_1)} \; . \tag{2.12}$$

Using (2.7), (2.8), and (2.12), we derive from (2.9) that

$$\int\limits_{B_5\setminus B_r} |\nabla v_n|^2 \le C\max\{k_n, 1/k_n\}\|g_n\|_{H^{-1/2}(\partial B_1)}\|v_n\|_{H^{1/2}(\partial B_1)} + C\|v_n\|^2_{L^2(B_5\setminus B_1)} \; . \tag{2.13}$$

We immediately obtain from (2.13) that

$$\begin{aligned}
&\int\limits_{B_5\setminus B_1} |\nabla v_n|^2 + \int\limits_{B_5\setminus B_1} |v_n|^2 \\
&\le C\max\{k_n, 1/k_n\}\|g_n\|_{H^{-1/2}(\partial B_1)}\|v_n\|_{H^{1/2}(\partial B_1)} + C\|v_n\|^2_{L^2(B_5\setminus B_1)} \\
&\le C\left(\max\{k_n, 1/k_n\}\|g_n\|_{H^{-1/2}(\partial B_1)} + \|v_n\|_{L^2(B_5\setminus B_1)}\right)\|v_n\|_{H^1(B_5\setminus B_1)} \; ,
\end{aligned} \tag{2.14}$$

and so, by (2.4)

$$\|v_n\|_{H^1(B_5\setminus B_1)} \le C \; , \quad \text{and} \quad \|v_n\|_{H^{1/2}(\partial B_1)} \le C \; . \tag{2.15}$$

From (2.7) and (2.15), we conclude

$$(1+k_n^2)\int\limits_{B_1\setminus B_{1/2}} |v_n|^2 \le C\max\{k_n, 1/k_n\}\|g_n\|_{H^{-1/2}(\partial B_1)} \; ,$$

so by (2.4)

$$\lim_{n\to\infty}(1+k_n^2)\int\limits_{B_1\setminus B_{1/2}} |v_n|^2 = 0 \; . \tag{2.16}$$

Since, for any $v \in H^1(B_1 \setminus B_r)$,

$$\|v\|^2_{L^2(\partial B_1)} \le C\|v\|_{L^2(B_1\setminus B_r)}\|v\|_{H^1(B_1\setminus B_r)} \; ,$$

(see [7, Lemma 5.5]), it follows from (2.13), (2.15), and (2.16) that

$$\lim_{n\to\infty}\|v_n\|_{L^2(\partial B_1)} = 0 \; .$$

We have (see, e.g., Lemma 2) for any $R > 1$,

$$\|v_n\|_{H^1(B_R\setminus B_1)} \le C_R\|v_n\|_{H^{1/2}(\partial B_1)} \le C_R \; ,$$

where we used the second estimate of (2.15) to obtain the last bound. By extraction of a subsequence (and a diagonalization argument) one might assume that $\varepsilon_n k_n \to \omega \in [0, \tau]$ (since $\varepsilon_n k_n \in [0, \tau]$) and $v_n \to v$ weakly in $H^1_{loc}(\mathbb{R}^2 \setminus B_1)$, $v_n|_{\partial B_1} \to 0$ weakly in $H^{1/2}(\partial B_1)$. By (2.4),

$$\|v\|_{L^2(B_5 \setminus B_1)} = 1 \ , \tag{2.17}$$

and for $\omega > 0$, v is the unique outgoing solution to[3]

$$\begin{cases} \Delta v + \omega^2 v = 0 & \text{in } \mathbb{R}^2 \setminus B_1 \\ v = 0 & \text{on } \partial B_1. \end{cases}$$

Hence $v = 0$, and so we have a contradiction to (2.17). If $\omega = 0$, then by Lemma 2, $v \in W^1(\mathbb{R}^2)$ is the unique such solution to

$$\begin{cases} \Delta v = 0 & \text{in } \mathbb{R}^2 \setminus B_1 \\ v = 0 & \text{on } \partial B_1 \ . \end{cases}$$

Hence $v = 0$, and so again we have a contradiction to (2.17). This verifies the L^2 estimate (2.3). We have, as in (2.13),

$$\int_{B_5 \setminus B_1} |\nabla v_\varepsilon|^2 \le C \max\{k, 1/k\} \|g\|_{H^{-1/2}(\partial B_1)} \|v_\varepsilon\|_{H^{1/2}(\partial B_1)} + C \int_{B_5 \setminus B_1} |v_\varepsilon|^2 \ ,$$

and so by (2.3)

$$\|v_\varepsilon\|_{H^1(B_5 \setminus B_1)} \le C \max\{k, 1/k\} \|g\|_{H^{-1/2}(\partial B_1)}$$

as desired. This completes the proof of the lemma for $d = 2$.

Case 2: $d = 3$. We have (see, e.g., Lemma 2)

$$\|v_\varepsilon\|_{H^1(B_5 \setminus B_1)} \le C \|v_\varepsilon\|_{H^{1/2}(\partial B_1)} \ . \tag{2.18}$$

Hence it suffices to prove that

$$\|v_\varepsilon\|_{H^{\frac{1}{2}}(\partial B_1)} \le C \max\{1, \varepsilon/k\} \|g\|_{H^{-\frac{1}{2}}(\partial B_1)} \ . \tag{2.19}$$

We first prove (2.19) by contradiction for $\varepsilon \le \varepsilon_0$, with ε_0 sufficiently small. Suppose this is not true. Then there exist $(g_n) \subset H^{-\frac{1}{2}}(\partial B_1)$, (ε_n), (k_n), (a_n), and (σ_n) such that $0 < \varepsilon_n < \tau$, $0 < k_n < \tau/\varepsilon_n$, a_n and σ_n satisfy (2.1), $\varepsilon_n \to 0$, and

$$\|v_n\|_{H^{\frac{1}{2}}(\partial B_1)} = 1 \ , \quad \lim_{n \to \infty} \max\{1, \varepsilon_n/k_n\} \|g_n\|_{H^{-\frac{1}{2}}(\partial B_1)} = 0 \ . \tag{2.20}$$

Here $v_n \in H^1_{loc}(\mathbb{R}^3)$ is the unique outgoing solution to

$$\begin{cases} \Delta v_n + \varepsilon_n^2 k_n^2 v_n = 0 & \text{in } \mathbb{R}^3 \setminus B_1 \ , \\ \operatorname{div}(A_n \nabla v_n) + k_n^2 \Sigma_n v_n = 0 & \text{in } B_1 \ , \\ \left.\dfrac{\partial v_n}{\partial r}\right|_{\text{ext}} - \dfrac{1}{\varepsilon_n} \left.\dfrac{\partial v_n}{\partial r}\right|_{\text{int}} = g_n & \text{on } \partial B_1 \ , \end{cases} \tag{2.21}$$

[3] The outgoing property of v is just a consequence of the fact that the fundamental solution of the Helmholz equation with frequency $\varepsilon_n k_n$ converges to the fundamental solution of the Helmholtz equation with frequency ω, since $\omega > 0$.

where A_n and Σ_n are defined in the same way as A and Σ, but with a and σ replaced by a_n and σ_n. Since $\|v_n\|_{H^{\frac{1}{2}}(\partial B_1)} = 1$, it follows from (2.18) that

$$\|v_n\|_{H^1(B_5 \setminus B_1)} \leq C \; .$$

In combination with (2.20), (2.21), and the fact that $\varepsilon_n \to 0$ this implies

$$\lim_{n\to\infty} \left\| \frac{\partial v_n}{\partial r}\Big|_{\text{int}} \right\|_{H^{-\frac{1}{2}}(\partial B_1)} = 0 \; . \tag{2.22}$$

Multiplying the equation of v_n by $\bar{v}_n$ and integrating on B_R, we obtain

$$\begin{aligned} \int_{\partial B_R} \partial_r v_n \bar{v}_n - \int_{B_R\setminus B_1} |\nabla v_n|^2 + \varepsilon_n^2 k_n^2 \int_{B_R\setminus B_1} |v_n|^2 \\ - \frac{1}{\varepsilon_n} \int_{B_1} \langle A_n \nabla v_n, \nabla \bar{v}_n \rangle + \frac{k_n^2}{\varepsilon_n} \int_{B_1} \Sigma_n |v_n|^2 = \int_{\partial B_1} g_n \bar{v}_n \; . \end{aligned} \tag{2.23}$$

Letting $R \to \infty$ in (2.23), using the outgoing condition, and considering the imaginary part, we derive from (2.20) and the fact $k_n \varepsilon_n \leq \tau$ that

$$\lim_{n\to+\infty} (1 + k_n^2) \int_{B_1\setminus B_{1/2}} |v_n|^2 = 0 \; . \tag{2.24}$$

Since $\Delta v_n + k_n^2(1 + i/k_n)v_n = 0$ in $B_1 \setminus B_{1/2}$, by Caccioppoli's inequality, we obtain

$$\int_{B_{4/5}\setminus B_{3/5}} |\nabla v_n|^2 \leq C(k_n^2 + 1) \int_{B_1\setminus B_{1/2}} |v_n|^2. \tag{2.25}$$

It follows from (2.24) and (2.25) that there exits $r \in (3/5, 4/5)$ (r depends on n) such that

$$\int_{\partial B_r} |\nabla v_n|^2 + (1 + k_n^2) \int_{\partial B_r} |v_n|^2 \to 0 \quad \text{as } n \to \infty. \tag{2.26}$$

Since $\Delta v_n + k_n^2(1 + i/k_n)v_n = 0$ in $B_1 \setminus B_r$, we have

$$- \int_{B_1\setminus B_r} |\nabla v_n|^2 + (k_n^2 + ik_n) \int_{B_1\setminus B_r} |v_n|^2 = \int_{\partial B_r} \partial_r v_n \bar{v}_n - \int_{\partial B_1} \partial_r v_n \bar{v}_n \; . \tag{2.27}$$

A combination of (2.20), (2.22), (2.24), (2.26), and (2.27) yields

$$\lim_{n\to\infty} \int_{B_1\setminus B_{4/5}} |\nabla v_n|^2 = 0 \; . \tag{2.28}$$

From (2.24) and (2.28), we conclude that

$$\lim_{n\to\infty} \|v_n\|_{H^{\frac{1}{2}}(\partial B_1)} \leq C \lim_{n\to\infty} \|v_n\|_{H^1(B_1\setminus B_{4/5})} = 0 \; .$$

This is a contradiction to (2.20), and thus (2.19) holds under the additional assumption that $\varepsilon \le \varepsilon_0$ for some fixed $0 < \varepsilon_0$, sufficiently small.

It remains to prove (2.19) for $\varepsilon_0 < \varepsilon < \tau$. In this case, we first prove that

$$\|v_\varepsilon\|_{L^2(B_5 \setminus B_1)} \le C \max\{k, 1/k\} \|g\|_{H^{-\frac{1}{2}}(\partial B_1)} , \tag{2.29}$$

by contradiction, and then we show that

$$\|v_\varepsilon\|_{H^1(B_5 \setminus B_1)} \le C \max\{k, 1/k\} \|g\|_{H^{-\frac{1}{2}}(\partial B_1)} . \tag{2.30}$$

We note that since k is bounded ($k < \tau/\varepsilon_0$) and ε is bounded away from zero (2.30) implies (2.19). In the argument by contradiction one may without loss of generality assume the ε_n converge to $\varepsilon_1 > 0$. Thus the system (2.21) is asymptotically similar to the 2d system (2.5), and so the argument of proof proceeds in the same fashion as in the two dimensional case presented above. The details are left to the reader. □

We next provide some useful estimates for $\sigma_{1,\varepsilon}$ which is defined as follows:

$$\sigma_{1,\varepsilon} := \left(F_\varepsilon^{-1}\right)_* \sigma_{1,c} , \quad \text{ in } B_2 \setminus B_\varepsilon . \tag{2.31}$$

Lemma 4. *Assume $k_\varepsilon \ge c_* \varepsilon^{-1}$, for some fixed constant $c_* > 0$. We have*

$$|\sigma_{1,\varepsilon}| \le \frac{C_1}{\varepsilon^{d-1} k_\varepsilon^2} \quad \text{if } k < \frac{c_*}{2}\varepsilon^{-1} ,$$

and,

$$|\sigma_{1,\varepsilon}| \le \frac{C_2}{\varepsilon^{d-1} k} \quad \text{and} \quad \Im(\sigma_{1,\varepsilon}) \ge \frac{c_3 k}{\max\{k_\varepsilon^4, k^4\}} \quad \text{if } k \ge \frac{c_*}{2}\varepsilon^{-1} ,$$

for some positive constants C_1, C_2, c_3 independent of ε, k and k_ε (but dependent on c_*).

Proof. We recall, by (1.3) and the fact that $\sigma_N = \sigma_D = 1$,

$$\sigma_{1,c} = \frac{1}{k_\varepsilon^2 - k^2 - ik} , \tag{2.32}$$

and therefore

$$\Im(\sigma_{1,c}) = \frac{k}{(k_\varepsilon^2 - k^2)^2 + k^2} . \tag{2.33}$$

If $k < \frac{c_*}{2}\varepsilon^{-1}$ then it follows from (2.32) that

$$|\sigma_{1,c}| \le \frac{C}{k_\varepsilon^2} ,$$

since $k_\varepsilon^2 - k^2 > 3k_\varepsilon^2/4$. In this proof, C denotes a positive constant independent of ε, k, and k_ε.

If $k \ge \frac{c_*}{2}\varepsilon^{-1}$, then it follows from (2.32) that

$$|\sigma_{1,c}| \le \frac{C}{k} ,$$

and from (2.33) that

$$\Im(\sigma_{1,c}) \geq \frac{Ck}{\max\{k_\varepsilon^4, k^4\}} \ .$$

Since $\sigma_{1,\varepsilon} = (F_\varepsilon^{-1})_* \sigma_{1,c}$, the estimates in this lemma are now a consequence of the fact that

$$1/C \leq \det D\, F_\varepsilon^{-1} \leq C\varepsilon^{-d+1} \ . \qquad \square$$

3. Stability estimates in the time harmonic regime

Let $\hat{u}_c(k,\cdot)$ be the Fourier transform of $u_c(\cdot, x)$ with respect to t,[4] i.e.,

$$\hat{u}_c(k,x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u_c(t,x) e^{ikt}\, dt \ .$$

Then $\hat{u}_c \in H^1_{loc}(\mathbb{R}^d)$ (for a.e. $k > 0$) is the unique outgoing solution to

$$\operatorname{div}(A_c \nabla \hat{u}_c) + k^2 \Sigma_c \hat{u}_c = -\hat{f} \ ,$$

where (A_c, Σ_c) is given in (1.4) (see Proposition 2 in Section 4).

Define $\tilde{u}_\varepsilon(k,x) = \hat{u}_c(k, F_\varepsilon(x))$. Then $\tilde{u}_\varepsilon \in H^1_{loc}(\mathbb{R}^d)$ is the unique outgoing solution to

$$\operatorname{div}(A_\varepsilon \nabla \tilde{u}_\varepsilon) + k^2 \Sigma_\varepsilon \tilde{u}_\varepsilon = -\hat{f}, \tag{3.1}$$

where

$$A_\varepsilon, \Sigma_\varepsilon = \begin{cases} I, \ 1 & \text{in } \mathbb{R}^d \setminus B_2 \ , \\ I, \ \sigma_\varepsilon(x) := 1 + \sigma_{1,\varepsilon}(x) & \text{in } B_2 \setminus B_\varepsilon \ , \\ \dfrac{1}{\varepsilon^{d-2}} I, \ \dfrac{1}{\varepsilon^d}(1 + i/k) & \text{in } B_\varepsilon \setminus B_{\varepsilon/2} \ , \\ \dfrac{1}{\varepsilon^{d-2}} a(x/\varepsilon), \ \dfrac{1}{\varepsilon^d}\sigma(x/\varepsilon) & \text{in } B_{\varepsilon/2} \ , \end{cases} \tag{3.2}$$

and

$$\sigma_{1,\varepsilon} = \left(F_\varepsilon^{-1}\right)_* \sigma_{1,c} \ . \tag{3.3}$$

In this section, we establish the stability for solutions to (3.1), (3.2) for quite general $\sigma_{1,\varepsilon}$; hence in the remainder of this section, we do **not** assume that $\sigma_{1,\varepsilon}$ is of the form (3.3), but **only** that it satisfies certain bounds. We recall that

$$a \text{ is bounded, uniformly elliptic, and } \sigma \in L^\infty(B_{1/2}) \text{ with } \Im(\sigma) \geq 0 \ . \tag{3.4}$$

The first result of this section concerns the small to moderate frequency regime.

[4] After extending u_c by 0 for $t < 0$.

Lemma 5. *Let $d = 2$ or 3, $\tau > 0$, $0 < \varepsilon, k < \tau$, and $g \in L^2(\mathbb{R}^d)$ with $\operatorname{supp} g \subset B_4 \setminus B_\varepsilon$. Assume that*

$$\|\sigma_{1,\varepsilon}\|_{L^\infty(B_2 \setminus B_\varepsilon)} \le C_0, \quad \text{and} \quad \Im(\sigma_{1,\varepsilon}) \ge 0 \ . \tag{3.5}$$

Let $v_\varepsilon \in H^1_{loc}(\mathbb{R}^d)$ be the unique outgoing solution to

$$\operatorname{div}(A_\varepsilon \nabla v_\varepsilon) + k^2 \Sigma_\varepsilon v_\varepsilon = g \text{ in } \mathbb{R}^d. \tag{3.6}$$

There exists a positive constant C, depending only on d, τ and C_0, such that

$$\|v_\varepsilon\|_{L^2(B_5 \setminus B_\varepsilon)} \le C \max\{1, 1/k\} \|g\|_{L^2} \ . \tag{3.7}$$

Remark 1. In Lemma 5, the support of g is assumed to be inside $B_4 \setminus B_\varepsilon$ not $B_4 \setminus B_2$, since g will be of the form $-k^2 \sigma_{1,\varepsilon} \hat{u}_{1,\varepsilon}$, when we apply this lemma in the proof of Theorem 2. The blow up technique does not work for Lemma 5 due to the presence of $\sigma_{1,\varepsilon} \neq 0$ inside $B_2 \setminus B_\varepsilon$. It is not essential that the support of g be inside $B_4 \setminus B_\varepsilon$, this could be replaced by $B_M \setminus B_\varepsilon$ for any $M > 4$. The constant C in the estimate would depend on M.

Proof. The proof is based on a contradiction argument, in which we use an argument of removable singularity. Suppose (3.7) does not hold. Then there exist $\{k_n\}, \{\varepsilon_n\} \subset (0, \tau)$, $\sigma_{1,n}$, a_n, σ_n, and $\{g_n\}$, $\operatorname{supp} g_n \subset B_4 \setminus B_{\varepsilon_n}$, such that (3.5) holds for $\sigma_{1,n}$, a_n and σ_n satisfy (3.4), and

$$\max\{1, 1/k_n\} \|g_n\|_{L^2} \to 0 \text{ as } n \to \infty, \quad \|v_n\|_{L^2(B_5 \setminus B_{\varepsilon_n})} = 1 \ . \tag{3.8}$$

Here $v_n \in H^1_{loc}(\mathbb{R}^d)$ is the unique outgoing solution to

$$\operatorname{div}(A_n \nabla v_n) + k_n^2 \Sigma_n v_n = g_n \text{ in } \mathbb{R}^d,$$

where A_n, Σ_n are defined in the same way as A_ε, Σ_ε with k, ε, $\sigma_{1,\varepsilon}$, a, and σ replaced by k_n, ε_n, $\sigma_{1,n}$, a_n, and σ_n. Using the outgoing radiation condition, as in (2.24), we obtain,

$$\frac{k_n}{\varepsilon_n^d} \int\limits_{B_{\varepsilon_n} \setminus B_{\varepsilon_n/2}} |v_n|^2 \le \int\limits_{\mathbb{R}^d} |g_n||v_n| \ .$$

Here we also used that $\Im(\sigma_{1,n})$ and $\Im(\sigma_n)$ are non-negative. Since $\operatorname{supp} g_n \subset B_4 \setminus B_{\varepsilon_n}$ and $\|v_n\|_{L^2(B_5 \setminus B_{\varepsilon_n})} = 1$, the above inequality implies that

$$\frac{k_n + 1}{\varepsilon_n^{d-1}} \int\limits_{B_{\varepsilon_n} \setminus B_{\varepsilon_n/2}} |v_n|^2 \le 2\varepsilon_n \max\{1, 1/k_n\} \|g_n\|_{L^2} \ . \tag{3.9}$$

We have

$$\Delta v_n + \frac{k_n^2}{\varepsilon_n^2}(1 + i/k_n) v_n = 0 \text{ in } B_{\varepsilon_n} \setminus B_{\varepsilon_n/2} \ .$$

It follows from Caccioppoli's inequality that

$$\int\limits_{B_{4\varepsilon_n/5} \setminus B_{3\varepsilon_n/5}} |\nabla v_n|^2 \le \frac{C(k_n^2 + 1)}{\varepsilon_n^2} \int\limits_{B_{\varepsilon_n} \setminus B_{\varepsilon_n/2}} |v_n|^2 \ ,$$

and so

$$\frac{\varepsilon_n}{\varepsilon_n^{d-2}(k_n+1)} \int\limits_{B_{4\varepsilon_n/5}\setminus B_{3\varepsilon_n/5}} |\nabla v_n|^2 \le \frac{C(k_n+1)}{\varepsilon_n^{d-1}} \int\limits_{B_{\varepsilon_n}\setminus B_{\varepsilon_n/2}} |v_n|^2 . \tag{3.10}$$

In this proof, C denotes a positive constant depending only on d and τ. From (3.9) and (3.10), we obtain

$$\frac{\varepsilon_n}{\varepsilon_n^{d-2}(k_n+1)} \int\limits_{B_{4\varepsilon_n/5}\setminus B_{3\varepsilon_n/5}} |\nabla v_n|^2 \le C\varepsilon_n \max\{1, 1/k_n\}\|g_n\|_{L^2} . \tag{3.11}$$

A combination of (3.9) and (3.11) now yields

$$\frac{1}{\varepsilon_n^{d-2}} \int\limits_{B_{4\varepsilon_n/5}\setminus B_{3\varepsilon_n/5}} \Big(\frac{k_n+1}{\varepsilon_n}|v_n|^2 + \frac{\varepsilon_n}{k_n+1}|\nabla v_n|^2\Big) \le C\varepsilon_n \max\{1, 1/k_n\}\|g_n\|_{L^2} .$$

It follows that for some $\alpha \in (3\varepsilon_n/5, 4\varepsilon_n/5)$ (α depends on n),

$$\frac{1}{\varepsilon_n^{d-2}} \int\limits_{\partial B_\alpha} \Big(\frac{k_n+1}{\varepsilon_n}|v_n|^2 + \frac{\varepsilon_n}{k_n+1}|\nabla v_n|^2\Big) \le C \max\{1, 1/k_n\}\|g_n\|_{L^2} \to 0 \quad \text{as } n \to \infty . \tag{3.12}$$

Here we used (3.8) for the last convergence assertion. Multiplying the equation for v_n by $\bar{v}_n$, and integrating on $B_5 \setminus B_\alpha$, we have

$$\begin{aligned} &- \int\limits_{B_5\setminus B_{\varepsilon_n}} |\nabla v_n|^2 + k_n^2 \int\limits_{B_5\setminus B_{\varepsilon_n}} (1+\sigma_{1,n})|v_n|^2 - \frac{1}{\varepsilon_n^{d-2}} \int\limits_{B_{\varepsilon_n}\setminus B_\alpha} |\nabla v_n|^2 \\ &+ \frac{k_n^2}{\varepsilon_n^d}\Big(1+\frac{i}{k_n}\Big) \int\limits_{B_{\varepsilon_n}\setminus B_\alpha} |v_n|^2 = \int\limits_{B_5} g_n \bar{v}_n - \int\limits_{\partial B_5} \partial_r v_n \, \bar{v}_n + \frac{1}{\varepsilon_n^{d-2}} \int\limits_{\partial B_\alpha} \partial_r v_n \, \bar{v}_n . \end{aligned} \tag{3.13}$$

Here and in what follows, we extend $\sigma_{1,n}$ by 0 in $\mathbb{R}^d \setminus B_2$. Since $v_n \in H^1_{loc}(\mathbb{R}^d \setminus B_{9/2})$ is an outgoing solution to $\Delta v_n + k_n^2 v_n = 0$ in $\mathbb{R}^d \setminus B_{9/2}$, we have (see, e.g., Lemma 2)

$$\|v_n\|_{H^1(B_6\setminus B_{9/2})} \le C\|v_n\|_{H^{1/2}(\partial B_{9/2})} ,$$

and so, by the standard theory of elliptic equations,

$$\|v_n\|_{H^1(B_6\setminus B_{9/2})} \le C\|v_n\|_{L^2(B_5\setminus B_{\varepsilon_n})} . \tag{3.14}$$

Using (3.5), (3.8), (3.9), (3.12), and (3.14), in combination with (3.13), we now obtain

$$\int\limits_{B_5\setminus B_\alpha} |\nabla v_n|^2 \le C . \tag{3.15}$$

Define $u_n \in H^1(B_\alpha)$ as follows

$$\Delta u_n = 0 \text{ in } B_\alpha \quad \text{ and } \quad u_n = v_n \text{ on } \partial B_\alpha .$$

We derive from (3.8) and (3.12) that

$$\int_{B_\alpha} |\nabla u_n|^2 + |u_n|^2 \to 0 \text{ as } n \to \infty \ . \tag{3.16}$$

Indeed, set $w_n(x) = u_n(\alpha x)$ for $x \in B_1$. Then

$$\Delta w_n = 0 \text{ in } B_1 \ ,$$

and

$$\|w_n\|^2_{H^1(\partial B_1)} \le C\Big(\alpha^{1-d} \int_{\partial B_\alpha} |u_n|^2 + \alpha^{3-d} \int_{\partial B_\alpha} |\nabla u_n|^2\Big) \le C \max\{1, 1/k_n\} \|g_n\|_{L^2} \ ,$$

where we used (3.12), and the fact that $3\varepsilon_n/5 < \alpha < 4\varepsilon_n/5$ for the last estimate. It follows that

$$\int_{B_1} |\nabla w_n|^2 + |w_n|^2 \le C \max\{1, 1/k_n\} \|g_n\|_{L^2} \ ,$$

which in terms of u_n yields

$$\alpha^{-d} \int_{B_\alpha} |u_n|^2 + \alpha^{2-d} \int_{B_\alpha} |\nabla u_n|^2 \le C \max\{1, 1/k_n\} \|g_n\|_{L^2} \ .$$

The assertion (3.16) now follows from (3.8). Define

$$V_n = \begin{cases} v_n & \text{in } \mathbb{R}^d \setminus B_\alpha \ , \\ u_n & \text{in } B_\alpha \ . \end{cases}$$

We derive from (3.8), (3.9), (3.15), and (3.16) that

$$\int_{B_5} |\nabla V_n|^2 + |V_n|^2 \le C \ .$$

It follows that (see, e.g., Lemma 2)

$$\|V_n\|_{H^1(B_R \setminus B_4)} \le C_R \|V_n\|_{H^{1/2}(\partial B_4)} \le C_R \|V_n\|_{H^1(B_5)} \le C_R \ ,$$

for any $R \ge 5$, and as a consequence

$$\|V_n\|_{H^1(B_R)} \le C_R$$

for all $R > 0$. After extraction of a subsequence we may thus assume that $k_n \to k_0 \ge 0$, $\varepsilon_n \to \varepsilon_0 \ge 0$, $\alpha \to \alpha_0$ (recall that α depends on n), $\sigma_{1,\varepsilon_n} \to \sigma_1$ weakly in L^2 (σ_1 satisfies (3.5)), and $V_n \to V$ weakly in $H^1_{loc}(\mathbb{R}^d)$.

Suppose $k_0 > 0$. If $\varepsilon_0 = 0$ then V is an outgoing solution to the equation

$$\Delta V + k_0^2(1 + \sigma_1)V = 0 \text{ in } \mathbb{R}^d \setminus \{0\} \ .$$

Since $V \in H^1_{loc}(\mathbb{R}^d)$, it follows that

$$\Delta V + k_0^2(1 + \sigma_1)V = 0 \text{ in } \mathbb{R}^d \ .$$

Therefore, $V = 0$, and we have a contradiction to the fact that $\int_{B_5} |V|^2 = \lim \int_{B_5 \setminus B_{\varepsilon_n}} |V_n|^2 = 1$. Similarly, if $k_0 > 0$ and $\varepsilon_0 > 0$ (and thus $\alpha_0 > 0$), then V is an outgoing solution to

$$\Delta V + k_0^2(1 + \sigma_1)V = 0 \text{ in } \mathbb{R}^d \setminus B_{\alpha_0} . \tag{3.17}$$

It follows from (3.16) that $V = 0$ in B_{α_0}. Hence $V|_{\mathbb{R}^d \setminus B_{\alpha_0}}$ is the unique outgoing solution to (3.17) with $V = 0$ on ∂B_{α_0}, and as a consequence $V = 0$ in all of $\mathbb{R}^d$; we have also arrived at a contradiction.

This leaves the case $k_0 = 0$. We start by considering the case $\varepsilon_0 > 0$ (and thus $\alpha_0 > 0$). By Lemma 2, $V \in W^1(\mathbb{R}^d \setminus B_{\alpha_0})$ is a solution to the equation

$$\Delta V = 0 \text{ in } \mathbb{R}^d \setminus \bar{B}_{\alpha_0} . \tag{3.18}$$

It follows from (3.16) that $V = 0$ in B_{α_0}, and thus V is the unique solution to (3.18), with $V = 0$ on ∂B_{α_0}. Hence $V = 0$ in $\mathbb{R}^d \setminus B_{\alpha_0}$, and as a consequence $V = 0$ in $\mathbb{R}^d$, so we have arrived at a contradiction.

Finally we consider the case $k_0 = \varepsilon_0 = 0$. By Lemma 2, $V \in W^1(\mathbb{R}^d)$ is a solution to the equation

$$\Delta V = 0 \text{ in } \mathbb{R}^d \setminus \{0\} . \tag{3.19}$$

Since $V \in W^1(\mathbb{R}^d)$, it follows that

$$\Delta V = 0 \text{ in } \mathbb{R}^d . \tag{3.20}$$

Thus $V = 0$ in the case $d = 3$, and we have arrived at a contradiction in three dimensions. In two dimensions, we can only at present conclude that V is a constant, due to (3.20). We proceed to prove that $V = 0$ in the case $d = 2$ as well. Set

$$\tilde{v}_n(x) = v_n(\varepsilon_n x) \text{ for } x \in B_1 \setminus B_{4/5} .$$

From (3.8), (3.9), and (3.15), we have

$$\|\tilde{v}_n\|^2_{H^{1/2}(\partial B_1)} \le C \left(\int\limits_{B_1 \setminus B_{4/5}} |\nabla \tilde{v}_n|^2 + |\tilde{v}_n|^2 \right) = C \left(\int\limits_{B_\varepsilon \setminus B_{4\varepsilon/5}} |\nabla v_n|^2 + \varepsilon_n^{-2} |v_n|^2 \right) \le C . \tag{3.21}$$

Since $\lim_{n\to\infty} \|\tilde{v}_n\|_{L^2(B_1 \setminus B_{4/5})} = 0$ by (3.8) and (3.9), it follows from (3.21) that $\tilde{v}_n \to 0$ weakly in $H^1(B_1 \setminus B_{4/5})$, and thus

$$\tilde{v}_n \to 0 \text{ weakly in } H^{1/2}(\partial B_1) . \tag{3.22}$$

Let $v_{1,n} \in H^1_{loc}(\mathbb{R}^2)$ be the unique outgoing solution to

$$\Delta v_{1,n} + k_n^2 v_{1,n} = -k_n^2 \sigma_{1,n} v_n \text{ in } \mathbb{R}^2 .$$

Applying Lemma 1, the regularity theory of elliptic equations, and using (3.5) and (3.8), we have

$$\frac{1}{k_n + 1} \|\nabla^2 v_{1,n}\|_{L^2(B_5)} + \|\nabla v_{1,n}\|_{L^2(B_5)} + (k_n + 1)\|v_{1,n}\|_{L^2(B_5)} \le C k_n^2 (|\ln k_n| + 1) .$$

As a consequence of this and the fact that $k_n \to 0$,

$$\|\nabla v_{1,n}\|_{L^2(B_5)} + \|v_{1,n}\|_{L^\infty(B_5)} \le C k_n^2 (|\ln k_n| + 1) . \tag{3.23}$$

By a rescaling (remember $d = 2$) we get

$$\|\nabla \tilde{v}_{1,n}\|_{L^2(B_5)} + \|\tilde{v}_{1,n}\|_{L^\infty(B_5)} \le C k_n^2 (|\ln k_n| + 1) \ ,$$

with $\tilde{v}_{1,n}(x) = v_{1,n}(\varepsilon_n x)$, and thus

$$\|\tilde{v}_{1,n}\|_{H^{1/2}(\partial B_1)} \le C k_n^2 (|\ln k_n| + 1) \to 0 \ . \tag{3.24}$$

We define

$$w_n = v_n - v_{1,n} \text{ in } \mathbb{R}^2 \setminus B_{\varepsilon_n} \ ,$$

where $w_n \in H^1_{loc}(\mathbb{R}^2 \setminus B_{\varepsilon_n})$ is the unique outgoing solution to

$$\Delta w_n + k_n^2 w_n = 0 \text{ in } \mathbb{R}^2 \setminus B_{\varepsilon_n} \quad \text{ and } \quad w_n = v_n - v_{1,n} \text{ on } \partial B_{\varepsilon_n} \ .$$

Set

$$W_n(x) = w_n(\varepsilon_n x) \text{ for } x \in \mathbb{R}^2 \setminus B_1 \ .$$

Then $W_n \in H^1_{loc}(\mathbb{R}^2 \setminus B_1)$ is the unique outgoing solution to

$$\Delta W_n + k_n^2 \varepsilon_n^2 W_n = 0 \text{ in } \mathbb{R}^2 \setminus B_1 \ , \quad \text{ and } \quad W_n(x) = \tilde{v}_n(x) - \tilde{v}_{1,n}(x) \text{ on } \partial B_1 \ .$$

Applying Lemma 2 for W_n and using (3.22) and (3.24), we have $W_n \to 0$ weakly in $H^1_{loc}(\mathbb{R}^2 \setminus B_1)$, and by interior elliptic regularity estimates

$$\|W_n\|_{H^{1/2}(\partial B_2)} \to 0 \text{ as } n \to 0 \ .$$

Applying Lemma 2 to W_n again and rescaling, we obtain

$$\|w_n\|_{L^2(B_5 \setminus B_4)} \le \frac{C|\ln k_n|}{|\ln(\varepsilon_n k_n)|} \|W_n\|_{H^{1/2}(\partial B_2)} \to 0 \text{ as } n \to \infty \ . \tag{3.25}$$

A combination of (3.23) and (3.25) yields that $v_n \to 0$ in $L^2(B_5 \setminus B_4)$; it follows that $V = 0$ in $B_5 \setminus B_4$, and thus $V = 0$ in all of $\mathbb{R}^2$ (since we already know it must be a constant). We have a contradiction, and the proof is complete. □

The second result in this section deals with the moderate to high frequency regime.

Lemma 6. *Let $d = 2$ or 3, $0 < \varepsilon < 1/2$, and $k > k_0 > 0$ for some constant k_0. Suppose $g \in L^2(\mathbb{R}^d)$ with $\operatorname{supp} g \subset B_4 \setminus B_\varepsilon$ and let $v_\varepsilon \in H^1_{loc}(\mathbb{R}^d)$ be the unique outgoing solution to*

$$\operatorname{div}(A_\varepsilon \nabla v_\varepsilon) + k^2 \Sigma_\varepsilon v_\varepsilon = g \text{ in } \mathbb{R}^d. \tag{3.26}$$

Assume that

$$\|\sigma_{1,\varepsilon}\|_{L^\infty} = \chi_1 \quad \text{ and } \quad \Im(\sigma_{1,\varepsilon}) \ge \chi_2 \ \text{ a.e. in } \ B_2 \setminus B_\varepsilon \ , \tag{3.27}$$

for some $\chi_1 \ge \chi_2 > 0$. There exist two positive constants λ and C, independent of k, ε, χ_1, χ_2, and g such that *i) If $k\chi_1 \le \lambda$, then*

$$\int\limits_{B_5\setminus B_\varepsilon} \left(|\nabla v_\varepsilon|^2 + k^2|v_\varepsilon|^2\right) \le C(k^4+1) \int\limits_{\mathbb{R}^d} |g|^2 \ . \tag{3.28}$$

ii) If $k\chi_1 > \lambda$, then

$$\int\limits_{B_5\setminus B_\varepsilon} \left(|\nabla v_\varepsilon|^2 + k^2|v_\varepsilon|^2\right) \le C\left(k^4 + \frac{k^2\chi_1^4}{\chi_2^2}\right) \int\limits_{\mathbb{R}^d} |g|^2 \ . \tag{3.29}$$

Remark 2. As in the previous lemma, it is not essential that the support of g be inside $B_4 \setminus B_\varepsilon$, this could be replaced by $B_M \setminus B_\varepsilon$ for any $M > 4$. The constants in the estimates would depend on M. We also note that the estimate (3.28) is stronger than the estimate (3.29), since $k > k_0 > 0$. It thus follows immediately that

$$\int\limits_{B_5\setminus B_\varepsilon} \left(|\nabla v_\varepsilon|^2 + k^2|v_\varepsilon|^2\right) \le C\left(k^4 + \frac{k^2\chi_1^4}{\chi_2^2}\right) \int\limits_{\mathbb{R}^d} |g|^2 \ ,$$

for all $k > k_0 > 0$. These facts shall both be used in the proof of Theorem 2.

Proof. The proof is inspired by [21]. To simplify notation we drop the subscript ε from v_ε. Multiplying (3.26) by $\bar{v}$ and integrating on B_R, $R > 1$, we obtain

$$\int\limits_{\partial B_R} \partial_r v \, \bar{v} - \int\limits_{B_R} \langle A_\varepsilon \nabla v, \nabla \bar{v}\rangle + k^2 \int\limits_{B_R} \Sigma_\varepsilon |v|^2 = \int\limits_{B_R} g\bar{v} \ .$$

Letting R go to infinity, using the outgoing condition, and considering the imaginary part, we have

$$k \limsup_{R\to\infty} \int\limits_{\partial B_R} |v|^2 + \frac{k}{\varepsilon^d} \int\limits_{B_\varepsilon \setminus B_{\varepsilon/2}} |v|^2 + k^2\chi_2 \int\limits_{B_2\setminus B_\varepsilon} |v|^2 \le \int\limits_{\mathbb{R}^d} |g||v| \ . \tag{3.30}$$

Since $\Delta v + \frac{k^2}{\varepsilon^2} v + i\frac{k}{\varepsilon^2} v = 0$ in $B_\varepsilon \setminus B_{\varepsilon/2}$ and $k > k_0$, it follows from Caccioppoli's inequality that

$$\int\limits_{B_{4\varepsilon/5}\setminus B_{3\varepsilon/5}} |\nabla v|^2 \le \frac{Ck^2}{\varepsilon^2} \int\limits_{B_\varepsilon\setminus B_{\varepsilon/2}} |v|^2 \ . \tag{3.31}$$

In this proof, C denotes a positive constant independent of ε, k, χ_1, χ_2, and g. It follows from (3.30) and (3.31) that

$$\int\limits_{B_{4\varepsilon/5}\setminus B_{3\varepsilon/5}} |\nabla v|^2 + \frac{k^2}{\varepsilon^2} \int\limits_{B_\varepsilon\setminus B_{\varepsilon/2}} |v|^2 \le C\varepsilon^{d-2} k \int\limits_{\mathbb{R}^d} |g||v| \ .$$

Thus there exists $t \in (3\varepsilon/5, 4\varepsilon/5)$ such that

$$\int\limits_{\partial B_t} |\nabla v|^2 \le C\varepsilon^{d-3} k \int\limits_{\mathbb{R}^d} |g||v| \quad \text{and} \quad \int\limits_{\partial B_t} |v|^2 \le \frac{C\varepsilon^{d-1}}{k} \int\limits_{\mathbb{R}^d} |g||v| \ . \tag{3.32}$$

Applying [21, Lemma 2] with $\alpha = \varepsilon$ and $R > \beta \ge 5$ (β is a fixed constant which will be chosen later), we have[5]

$$\frac{1}{d-1} \int_{B_\beta \setminus B_\varepsilon} |\nabla v|^2 + k^2 |v|^2 \le F_\beta(\varepsilon, v_+) - F_\beta(R, v) + \frac{\beta(3-d)}{2} \int_{B_R \setminus B_\beta} \frac{|v|^2}{r^3}$$
$$+ C \int_{\mathbb{R}^d} |g|(|v'| + |v|) + C \int_{B_2 \setminus B_\varepsilon} k^2 \chi_1 (|v||v'| + |v|^2) \ . \tag{3.33}$$

Here

$$F_\beta(r, v) = -\frac{k^2}{2} \int_{\partial B_r} P_*(r)|v|^2 - \frac{1}{2} \int_{\partial B_r} P_*(r)|v'|^2 + \frac{1}{2} \int_{\partial B_r} Q_*'(r)|v|^2$$
$$- \frac{1}{2} \int_{\partial B_r} Q_*(r)(|v|^2)' + \frac{1}{2} \int_{\partial B_r} P_*(r)|\nabla_{\partial B_r} v|^2 \ ,$$

with

$$P_*(r) = \begin{cases} \dfrac{2\beta}{d-1} & \text{if } \ r > \beta \ , \\ \dfrac{2r}{d-1} & \text{if } \ 0 < r < \beta \ , \end{cases} \qquad \text{and} \quad Q_*(r) = \begin{cases} \dfrac{\beta}{r} & \text{if } \ r > \beta \ , \\ 1 & \text{if } \ 0 < r < \beta \ . \end{cases}$$

Note that

$$F_\beta(r, v) = F(r, v) := -\frac{k^2 r}{d-1} \int_{\partial B_r} |v|^2 - \frac{r}{d-1} \int_{\partial B_r} |v'|^2$$
$$- \frac{1}{2} \int_{\partial B_r} (|v|^2)' + \frac{r}{d-1} \int_{\partial B_r} |\nabla_{\partial B_r} v|^2 \ , \tag{3.34}$$

for $0 < r < \beta$ (where F is independent of β). Since $P_*(r) = \frac{2r}{d-1}$ and $Q_*(r) = 1$ for $0 < r < \beta$,

$$\Re \int_{B_\varepsilon \setminus B_t} (\Delta v + k^2 v) \Big[\frac{2r}{d-1} \overline{v}_r + \overline{v} \Big] = \int_{B_\varepsilon \setminus B_t} \Re \Big[(\Delta v + k^2 v) \Big(\frac{2}{d-1} \, x \cdot \nabla \overline{v} + \overline{v} \Big) \Big].$$

We have[6]

$$\Re \left[(\Delta v + k^2 v) \left(\frac{2}{d-1} x \cdot \nabla \overline{v} + \overline{v} \right) \right] = -\frac{1}{d-1} (|\nabla v|^2 + k^2 |v|^2)$$
$$+ \Re \, \nabla \cdot \ \Big[\frac{2}{d-1} \nabla v \ (x \cdot \nabla \overline{v}) - \frac{1}{d-1} x \ |\nabla v|^2 + \nabla v \ \overline{v} + \frac{k^2}{d-1} x \ |v|^2 \Big]. \tag{3.35}$$

Integrating over the domain $B_\varepsilon \setminus B_t$, we obtain:

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[5] This inequality is a variant of an inequality due to Morawetz and Ludwig [15] (see also [24]).

[6] This is the "Rellich" identity which originates from [15,22,25].

$$\Re \int_{B_\varepsilon \setminus B_t} (\Delta v + k^2 v)\Big[\frac{2r}{d-1}\overline{v}_r + \overline{v}\Big] + \frac{1}{d-1} \int_{B_\varepsilon \setminus B_t} \big(|\nabla v|^2 + k^2 |v|^2\big)$$
$$= \Re \int_{\partial B_\varepsilon} \left(\frac{2\varepsilon}{d-1} \, |v_r|^2 - \frac{\varepsilon}{d-1} \, \big|\nabla v\big|^2 + v_r \, \overline{v} + \frac{k^2 \, \varepsilon}{d-1} \, |v|^2 \right)$$
$$- \Re \int_{\partial B_t} \left(\frac{2t}{d-1} \, |v_r|^2 - \frac{t}{d-1} \, \big|\nabla v\big|^2 + v_r \, \overline{v} + \frac{k^2 \, t}{d-1} \, |v|^2 \right) .$$

It follows that

$$\frac{1}{\varepsilon^{d-2}(d-1)} \int_{B_\varepsilon \setminus B_t} |\nabla v|^2 + k^2 |v|^2 \le -\frac{1}{\varepsilon^{d-2}} F(\varepsilon, v_-) + \frac{1}{\varepsilon^{d-2}} F(t, v)$$
$$+ \frac{C k^2}{\varepsilon^d} \int_{B_\varepsilon \setminus B_t} (\varepsilon |v||v'| + |v|^2) \; . \tag{3.36}$$

Adding (3.33) and (3.36), we obtain

$$\int_{B_\beta \setminus B_\varepsilon} \big(|\nabla v|^2 + k^2 |v|^2\big) + \frac{1}{\varepsilon^{d-2}} \int_{B_\varepsilon \setminus B_t} \big(|\nabla v|^2 + k^2 |v|^2\big)$$
$$\le (d-1) \left(F(\varepsilon, v_+) - F_\beta(R, v) - \frac{1}{\varepsilon^{d-2}} F(\varepsilon, v_-) + \frac{1}{\varepsilon^{d-2}} F(t, v) \right)$$
$$+ \frac{C k^2}{\varepsilon^d} \int_{B_\varepsilon \setminus B_t} (\varepsilon |v||v'| + |v|^2) + \frac{\beta(3-d)(d-1)}{2} \int_{B_R \setminus B_\beta} \frac{|v|^2}{r^3}$$
$$+ C \int_{\mathbb{R}^d} |g|(|v'| + |v|) + C \int_{B_2 \setminus B_\varepsilon} k^2 \chi_1 (|v||v'| + |v|^2). \tag{3.37}$$

We next estimate the first and second lines of the RHS of (3.37). We start with the first line. Using the outgoing condition, we have

$$\limsup_{R \to \infty} -F_\beta(R, v) \le C \beta k^2 \limsup_{R \to \infty} \int_{\partial B_R} |v|^2 \; ,$$

which implies, by (3.30),

$$\limsup_{R \to \infty} -F_\beta(R, v) \le C \beta k \int_{\mathbb{R}^d} |g||v| \; . \tag{3.38}$$

We claim that

$$F(\varepsilon, v_+) - \frac{1}{\varepsilon^{d-2}} F(\varepsilon, v_-) \le C k^2 \int_{\partial B_\varepsilon} |v|^2 \; . \tag{3.39}$$

In fact, if $d = 2$ then there is nothing to prove since $v_+ = v_-$ and $\partial_r v_+ = \partial_r v_-$ on ∂B_ε. Assume $d = 3$. Since $v_+ = v_-$ on ∂B_ε, and $\varepsilon < 1$ we get, by (3.34),

$$F(\varepsilon, v_+) - \frac{1}{\varepsilon} F(\varepsilon, v_-) \le \frac{k^2(1-\varepsilon)}{2} \int_{\partial B_\varepsilon} |v|^2 - \frac{\varepsilon}{2} \int_{\partial B_\varepsilon} |v_+'|^2 + \frac{1}{2} \int_{\partial B_\varepsilon} |v_-'|^2$$
$$- \frac{1}{2} \int_{\partial B_\varepsilon} (|v_+|^2)' + \frac{1}{2\varepsilon} \int_{\partial B_\varepsilon} (|v_-|^2)' \; . \tag{3.40}$$

Using the fact that $v_+' = (1/\varepsilon) v_-'$ (and $\varepsilon < 1$), claim (3.39) follows from (3.40). We next estimate the RHS of (3.39). We have

$$\int_{\partial B_\varepsilon} |v|^2 - (\varepsilon/t)^{d-1} \int_{\partial B_t} |v|^2 \le C \Big(\int_{B_\varepsilon \setminus B_t} |\nabla v|^2 \Big)^{1/2} \Big(\int_{B_\varepsilon \setminus B_t} |v|^2 \Big)^{1/2} .$$

This implies

$$\int_{\partial B_\varepsilon} |v|^2 - (\varepsilon/t)^{d-1} \int_{\partial B_t} |v|^2 \le \frac{Cc}{k^2 \varepsilon^{d-2}} \int_{B_\varepsilon \setminus B_t} |\nabla v|^2 + \frac{C k^2 \varepsilon^{d-2}}{c} \int_{B_\varepsilon \setminus B_t} |v|^2 \; , \tag{3.41}$$

for some small positive constant c, which will be chosen later. A combination of (3.39) and (3.41) yields

$$F(\varepsilon, v_+) - \frac{1}{\varepsilon^{d-2}} F(\varepsilon, v_-)$$
$$\le C k^2 \left[\frac{c}{k^2 \varepsilon^{d-2}} \int_{B_\varepsilon \setminus B_t} |\nabla v|^2 + \frac{k^2 \varepsilon^{d-2}}{c} \int_{B_\varepsilon \setminus B_t} |v|^2 + \int_{\partial B_t} |v|^2 \right] . \tag{3.42}$$

From (3.30) and (3.32), we have

$$\frac{k^4 \varepsilon^{d-2}}{c} \int_{B_\varepsilon \setminus B_t} |v|^2 + k^2 \int_{\partial B_t} |v|^2 \le C \Big(k^3 \varepsilon^{2d-2} + k \varepsilon^{d-1} \Big) \int_{\mathbb{R}^d} |g| |v| \; . \tag{3.43}$$

It follows from (3.42) and (3.43), by choosing c sufficiently small, that

$$F(\varepsilon, v_+) - \frac{1}{\varepsilon^{d-2}} F(\varepsilon, v_-) \le C \Big(k^3 \varepsilon^{2d-2} + k \varepsilon^{d-1} \Big) \int_{\mathbb{R}^d} |g||v| + \frac{1}{3 \varepsilon^{d-2}} \int_{B_\varepsilon \setminus B_t} |\nabla v|^2 \; . \tag{3.44}$$

We also have, by (3.34),

$$F(t, v) \le C \int_{\partial B_t} |v||\nabla v| + \varepsilon |\nabla v|^2 \le C \Big(\varepsilon \int_{\partial B_t} |\nabla v|^2 + \frac{1}{\varepsilon} \int_{\partial B_t} |v|^2 \Big) \; ,$$

which, by (3.32), implies

$$\frac{1}{\varepsilon^{d-2}} F(t, v) \le C k \int_{\mathbb{R}^d} |g||v| \; . \tag{3.45}$$

Here we used that $k \ge k_0 > 0$. Combining (3.38) with (3.44) and (3.45), we reach the following estimate for the first line of the RHS of (3.37)

$$F(\varepsilon, v_+) + \limsup_{R\to\infty} -F_\beta(R, v) - \frac{1}{\varepsilon^{d-2}} F(\varepsilon, v_-) + \frac{1}{\varepsilon^{d-2}} F(t, v)$$
$$\leq C(k^3\varepsilon^{2d-2} + k\varepsilon^{d-1} + \beta k) \int_{\mathbb{R}^d} |g||v| + \frac{1}{3\varepsilon^{d-2}} \int_{B_\varepsilon \setminus B_t} |\nabla v|^2 \ . \tag{3.46}$$

Here we also used that $\beta \geq 5$. We next estimate the second line of the RHS of (3.37). For that purpose

$$\frac{k^2}{\varepsilon^d} \int_{B_\varepsilon\setminus B_t} \left(\varepsilon|v||v'| + |v|^2\right) \leq \int_{B_\varepsilon \setminus B_t} \left(\left[\frac{k^4}{c\varepsilon^d} + \frac{k^2}{\varepsilon^d} \right] |v|^2 + \frac{c}{\varepsilon^{d-2}} |v'|^2 \right)$$

for $c > 0$. Using (3.30) and choosing c sufficiently small, we have

$$C\frac{k^2}{\varepsilon^d} \int_{B_\varepsilon\setminus B_t} \left(\varepsilon|v||v'| + |v|^2\right) \leq C(k^3 + k) \int_{B_\varepsilon\setminus B_t} |g||v| + \frac{1}{3\varepsilon^{d-2}} \int_{B_\varepsilon\setminus B_t} |v'|^2 \ . \tag{3.47}$$

On the other hand, using [21, (2.25)–(2.26)], we have for $d = 2$,

$$\int_{B_R\setminus B_\beta} \frac{|v|^2}{r^3} \leq C \int_\beta^\infty \frac{1}{r^3}\, dr \int_{\partial B_\beta} |v|^2 \leq \frac{C}{\beta^2} \int_{\partial B_\beta} |v|^2 \leq \frac{C}{\beta^2} \int_{B_5\setminus B_4} |v|^2 \ . \tag{3.48}$$

From (3.47) and (3.48), we reach the following estimate for the second line of the RHS of (3.37)

$$\frac{Ck^2}{\varepsilon^d} \int_{B_\varepsilon\setminus B_t} \left(\varepsilon|v||v'| + |v|^2\right) + \frac{\beta(3-d)(d-1)}{2} \int_{B_R\setminus B_5} \frac{|v|^2}{r^3}$$
$$\leq C(k^3 + k) \int_{B_\varepsilon\setminus B_t} |g||v| + \frac{1}{3\varepsilon^{d-2}} \int_{B_\varepsilon\setminus B_t} |v'|^2 + \frac{C}{\beta} \int_{B_5\setminus B_4} |v|^2 \ . \tag{3.49}$$

A combination of (3.37), (3.46) and (3.49) yields (by taking a “limit” of large R)

$$\int_{B_\beta\setminus B_\varepsilon} \left(|\nabla v|^2 + k^2|v|^2\right) + \frac{1}{\varepsilon^{d-2}} \int_{B_\varepsilon\setminus B_t} \left(|\nabla v|^2 + k^2|v|^2\right)$$
$$\leq C(k^3 + \beta k) \int_{\mathbb{R}^d} |g||v| + C \int_{\mathbb{R}^d} |g||v'| + C \int_{B_2\setminus B_\varepsilon} k^2\chi_1(|v||v'| + |v|^2) \ , \tag{3.50}$$

for β sufficiently large. Here we used the fact that $0 < \varepsilon < 1/2$ and $k \geq k_0 > 0$.

<u>Case 1:</u> $k\chi_1 \leq \lambda$. It follows from (3.50) that

$$\int_{B_5\setminus B_\varepsilon} \left(|\nabla v|^2 + k^2|v|^2\right) + \frac{1}{\varepsilon^{d-2}} \int_{B_\varepsilon\setminus B_t} \left(|\nabla v|^2 + k^2|v|^2\right) \leq C(k^4 + 1) \int_{\mathbb{R}^d} |g|^2 \ , \tag{3.51}$$

since

$$(k^3 + k)|g||v| \leq (k^2 + 1)(k + 1)|g||v| \leq \frac{1}{c}(k^2 + 1)^2|g|^2 + c(k + 1)^2|v|^2 \ , \tag{3.52}$$

and

$$|g||v'| \leq \frac{1}{c}|g|^2 + c|v'|^2 \ . \tag{3.53}$$

In (3.51) we have absorbed the remaining terms of the RHS of (3.50) by the LHS (by taking c sufficiently small) since λ can also be chosen sufficiently small.

Case 2: $k\chi_1 \geq \lambda$. It follows from (3.50) that

$$\int_{B_5 \setminus B_\varepsilon} |\nabla v|^2 + k^2|v|^2 + \frac{1}{\varepsilon^{d-2}} \int_{B_\varepsilon \setminus B_t} |\nabla v|^2 + k^2|v|^2 \leq C\left(k^4 + \frac{k^2\chi_1^4}{\chi_2^2}\right) \int_{\mathbb{R}^d} |g|^2, \tag{3.54}$$

since

$$k^2\chi_1(|v||v'| + |v|^2) \leq c|v'|^2 + Ck^4\chi_1^2|v|^2,$$

and, by (3.30),

$$k^4\chi_1^2 \int_{B_2 \setminus B_\varepsilon} |v|^2 \leq \frac{k^2\chi_1^2}{\chi_2} \int_{\mathbb{R}^d} |g||v| \leq \frac{Ck^2\chi_1^4}{\chi_2^2} \int_{\mathbb{R}^d} |g|^2 + c \int_{B_4 \setminus B_\varepsilon} k^2|v|^2.$$

Here we also used (3.52) and (3.53) to treat the remaining terms of the RHS of (3.50) in the same fashion as before. The proof is complete. □

4. Weak solutions and the well-posedness of the non-local wave equations

In this section, we first introduce the notion of weak solutions for the system (1.6) and establish the well-posedness of these. We then outline a proof of the fact that the Fourier transform in time of these solutions solve a corresponding "outgoing" Helmholtz problem for almost every frequency. We start with:

Definition 1. Let $d = 2$ or $d = 3$. We say a function

$$u \in L^\infty\big([0,\infty); H^1(\mathbb{R}^d)\big) \text{ with } \partial_t u \in L^\infty\big([0,\infty); L^2(\mathbb{R}^d)\big)$$

is a weak solution to (1.6) provided $u(0,x) = 0$ in $\mathbb{R}^d$ and for all $t > 0$

$$\begin{aligned}
&-\int_0^t \int_{\mathbb{R}^d} \Sigma_{1,c}(x) \frac{\partial}{\partial s} u(s,x) \frac{\partial}{\partial s} v(s,x)\, dx\, ds + \int_0^t \int_{\mathbb{R}^d} \langle A_c(x) \nabla u(s,x)\,,\, \nabla v(s,x) \rangle\, dx\, ds \\
&\quad + \int_0^t \int_{B_1 \setminus B_{1/2}} \Sigma_{2,c}(x) \frac{\partial}{\partial s} u(s,x)\; v(s,x)\, dx\, ds + \int_0^t \int_{B_2 \setminus B_1} G * \partial_s u(s,x)\, v(s,x)\, dx\, ds \\
&= \int_0^t \int_{\mathbb{R}^d} f(s,x)\, v(s,x)\, dx\, ds \quad ,
\end{aligned} \tag{4.1}$$

for any $v \in L^\infty\big([0,\infty); H^1(\mathbb{R}^d)\big)$ with $\partial_t v \in L^\infty\big([0,\infty); L^2(\mathbb{R}^d)\big)$ and $v(t,x) = 0$.

Note that $u \in C^0\big([0,\infty); L^2(\mathbb{R}^d)\big)$ and so the initial condition $u(0,x) = 0$ makes sense, also note that the initial condition $u_t(0,x) = 0$ is well-defined in a weak sense. It is clear that if $u \in C^2\big([0,+\infty) \times \mathbb{R}^d\big)$ is a weak solution in the sense defined above, then it is a classical solution to (1.6). Our definition is motivated by the standard definition of weak solutions to the wave equation.

The well-posedness of weak solutions to (1.6) is given by the following:

Proposition 1. *Let $d = 2$ or $d = 3$, and let $f \in L^\infty\big([0,\infty) \times L^2(\mathbb{R}^d)\big)$ with compact support in $[0,\infty) \times \mathbb{R}^d$. Then there exists a unique $u \in L^\infty\big([0,\infty); H^1(\mathbb{R}^d)\big)$ with $\partial_t u \in L^\infty\big([0,\infty); L^2(\mathbb{R}^d)\big)$ which is a weak solution to (1.6). Moreover,*

$$E(t,u) \le Ct\|f\|^2_{L^2([0,t]\times\mathbb{R}^d)} \ , \quad \textit{for a.e.} \quad t > 0 \ . \tag{4.2}$$

Here C is a positive constant depending on Λ and ε, but independent of f and t, and

$$E(t,u) := \frac{1}{2}\int\limits_{\mathbb{R}^d} \Big(\Sigma_{1,c}|\partial_t u(t,x)|^2 + \langle A_c \nabla u(t,x), \nabla u(t,x)\rangle \Big) dx. \tag{4.3}$$

The proof is based on a standard Galerkin approach, as part of which we derive a non-trivial energy estimate. Similar ideas were used in [19].

Proof. We first establish the existence of a weak solution by an approximate (Galerkin) approach. Let $(\varphi_j)_{j=1}^\infty \subset C_c^\infty(\mathbb{R}^d)$ be an orthonormal basis in $H^1(\mathbb{R}^d)$. For $m \in \mathbb{N}$, consider u_m of the form

$$u_m = \sum_{j=1}^m d_{m,j}(t)\ \varphi_j(x) \quad , \quad d_{m,j} \in C^2\big([0,\infty)\big) \ , \tag{4.4}$$

satisfying

$$\begin{aligned} &\frac{d^2}{ds^2}\int\limits_{\mathbb{R}^d} \Sigma_{1,c}(x) u_m(s,x)\ \varphi_j(x)\, dx + \int\limits_{\mathbb{R}^d} \langle A_c \nabla u_m(s,x), \nabla \varphi_j(x)\rangle\, dx \\ &\quad + \frac{d}{ds}\int\limits_{B_1\setminus B_{1/2}} \Sigma_{2,c}(x) u_m(s,x)\ \varphi_j(x)\, dx + \int\limits_{B_2\setminus B_1} G * \partial_s u_m(s,x)\ \varphi_j(x)\, dx \\ &= \int\limits_{\mathbb{R}^d} f(s,x)\ \varphi_j(x)\, dx \ , \end{aligned} \tag{4.5}$$

for $j = 1, \ldots, m$, and

$$d_{m,j}(0) = d'_{m,j}(0) = 0 \qquad \text{for } j = 1, \ldots, m \ . \tag{4.6}$$

Since $(\varphi_j)_j$ are linearly independent, the $(n \times n)$ matrix M given by $M_{i,j} = \langle \varphi_i, \varphi_j\rangle_{L^2(\mathbb{R}^d)}$ is invertible. Therefore, the existence and uniqueness of u_m follow by a standard argument, for example, one can use the theory of Volterra equations (see, e.g., [2, Theorem 2.1.1]).

We now derive an estimate for u_m. Multiplying (4.5) by $d'_{m,j}(s)$, summing up with respect to j, integrating on $[0,t]$ with respect to s, and using (4.6) we obtain

$$\begin{aligned} &E(t,u_m) + \int_0^t \int\limits_{B_2\setminus B_1} G * \partial_s u_m(s,x)\ \partial_s u_m(s,x)\, dx\, ds + \int_0^t \int\limits_{B_1\setminus B_{1/2}} \Sigma_{2,c}|\partial_s u_m|^2\, ds\, dx \\ &= \int_0^t \int\limits_{\mathbb{R}^d} f(s,x)\ \partial_s u_m(s,x)\, dx\, ds \ . \end{aligned} \tag{4.7}$$

We claim that

$$\int_0^t \int_{B_2 \setminus B_1} G * \partial_s u_m(s,x)\, \partial_s u_m(s,x)\, dx\, ds \geq 0 \quad \text{for a.e. } t > 0\ . \tag{4.8}$$

Indeed, define

$$U(s,x) = \begin{cases} \partial_s u_m(s,x) & \text{if } 0 < s < t\ , \\ 0 & \text{if } s \geq t\ , \end{cases}$$

and extend U by zero for $s \leq 0$. Then

$$\begin{aligned} \int_0^t \int_{B_2 \setminus B_1} G * \partial_s u_m(s,x)\, \partial_s u_m(s,x)\, dx\, ds &= \int_{-\infty}^{\infty} \int_{B_2 \setminus B_1} G * U\, U\, dx\, ds \\ = \int_{-\infty}^{\infty} \int_{B_2 \setminus B_1} \widehat{G * U} \overline{\widehat{U}}\, dx\, dk &= 2\Re \int_0^{\infty} \int_{B_2 \setminus B_1} \widehat{G * U}\, \overline{\widehat{U}}\, dx\, dk \\ = 2 \int_0^{\infty} \int_{B_2 \setminus B_1} \Re(\widehat{G}) |\widehat{U}|^2\, dx\, dk &= 2 \int_0^{\infty} \int_{B_2 \setminus B_1} \frac{k^2}{(k_\varepsilon^2 - k^2)^2 + k^2} |\widehat{U}|^2\, dx\, dk \ \geq 0\ , \end{aligned}$$

by the definition of G. This establishes (4.8). From (4.7) and (4.8), we arrive at

$$E(t, u_m) \leq \int_0^t \int_{\mathbb{R}^d} f(s,x)\, \partial_s u_m(s,x)\, dx\, ds\ . \tag{4.9}$$

It follows from (4.9) that

$$E(t, u_m) \leq \Big(\int_0^t \int_{\mathbb{R}^d} |\partial_s u_m(s,x)|^2\, dx\, ds \Big)^{1/2} \Big(\int_0^t \int_{\mathbb{R}^d} |f(s,x)|^2\, dx\, ds \Big)^{1/2}\ ,$$

which implies

$$E(t, u_m) \leq C \Big(\int_0^t E(s, u_m)\, ds \Big)^{1/2} \Big(\int_0^t \int_{\mathbb{R}^d} |f(s,x)|^2\, dx\, ds \Big)^{1/2}. \tag{4.10}$$

Here and in the remainder of this proof, C denotes a positive constant which depends on ε and Λ, but is independent of f, t, and m. We derive from (4.10) that

$$E(t, u_m) \leq Ct \int_0^t \int_{\mathbb{R}^d} |f(s,x)|^2\, dx\, ds\ . \tag{4.11}$$

Hence, for any fixed $T > 0$, there exists a subsequence of (u_m) (which is also denoted by u_m for notational ease) such that $u_m \to u$ weakly star in $L^\infty\big([0,T], H^1(\mathbb{R}^d)\big)$ and $\partial_t u_m \to \partial_t u$ weakly star in $L^\infty\big([0,T], L^2(\mathbb{R}^d)\big)$. It is clear that $u(0,x) = \partial_t u(0,x) = 0$, and that u satisfies (4.1) for any v of the form $v(s,x) = \varphi_j(x)\psi(s)$, $\psi \in C^1([0,\infty))$, $\psi(t) = 0$. By a standard linearity and approximation argument it follows that u satisfies (4.1) for any $v \in L^\infty\big([0,\infty), H^1(\mathbb{R}^d)\big)$ with $\partial_t v \in L^\infty\big([0,\infty), L^2(\mathbb{R}^d)\big)$ and $v(t,x) = 0$. In other words, u is a weak solution to (1.6). To see that u is unique, it suffices to prove that if $w \in L^\infty\big([0,T], H^1(\mathbb{R}^d)\big)$, with $\partial_t w \in L^\infty\big([0,T], L^2(\mathbb{R}^d)\big)$, $w(0,x) = \partial_t w(0,x) = 0$, and w satisfies (4.1) with $f = 0$ then w is identically zero. We have

$$- \int_0^t \int_{\mathbb{R}^d} \Sigma_{1,c}(x)\partial_s w(s,x)\, \partial_s v(s,x)\, dx\, ds + \int_0^t \int_{\mathbb{R}^d} A_c(x)\nabla w(s,x)\nabla v(s,x)\, dx\, ds$$

$$+ \int_0^t \int_{B_1\setminus B_{1/2}} \Sigma_{2,c}(x)\partial_s w(s,x)\; v(s,x)\, dx\, ds + \int_0^t \int_{B_2\setminus B_1} G * \partial_s w(s,x)\, v(s,x)\, dx\, ds = 0\ ,$$

for all $v \in L^\infty\big([0,\infty), H^1(\mathbb{R}^d)\big)$ with $\partial_t v \in L^\infty\big([0,\infty), L^2(\mathbb{R}^d)\big)$ and $v(t,x) = 0$. After integration by parts, this implies

$$- \int_0^t \int_{\mathbb{R}^d} \Sigma_{1,c}(x)\partial_s w(s,x)\partial_s v(s,x)\, dx\, ds + \int_0^t \int_{\mathbb{R}^d} A_c(x)\nabla w(s,x)\nabla v(s,x)\, dx\, ds$$

$$- \int_0^t \int_{B_1\setminus B_{1/2}} \Sigma_{2,c}(x)w(s,x)\; \partial_s v(s,x)\, dx\, ds - \int_0^t \int_{B_2\setminus B_1} G * w(s,x)\partial_s v(s,x)\, dx\, ds = 0\ , \tag{4.12}$$

for all $v \in L^\infty\big([0,\infty), H^1(\mathbb{R}^d)\big)$ with $\partial_t v \in L^\infty\big([0,\infty), L^2(\mathbb{R}^d)\big)$ *and* $v(t,x) = 0$. Setting

$$v(s,x) = \int_s^t w(\tau,x)\, d\tau\ ,$$

substituting v in (4.12), and using the fact that $\partial_s v(s,x) = -w(s,x)$, we obtain

$$\int_0^t \int_{\mathbb{R}^d} \Sigma_{1,c}(x)\partial_s w(s,x)\; w(s,x)\, dx\, ds - \int_0^t \int_{\mathbb{R}^d} A_c(x)\partial_s \nabla v(s,x)\nabla v(s,x)\, dx\, ds$$

$$+ \int_0^t \int_{B_1\setminus B_{1/2}} \Sigma_{2,c}(x)|w(s,x)|^2\, dx\, ds + \int_0^t \int_{B_2\setminus B_1} G * w(s,x)w(s,x)\, dx\, ds = 0\ .$$

It follows that

$$\frac{1}{2}\int_{\mathbb{R}^d} \left(\Sigma_{1,c}|w(t,x)|^2 + A_c(x)|\nabla v(0,x)|^2\right)\, dx$$

$$+ \int_0^t \int_{B_1\setminus B_{1/2}} \Sigma_{2,c}(x)|w(s,x)|^2\, dx\, ds + \int_0^t \int_{B_2\setminus B_1} G * w(s,x)w(s,x)\, dx\, ds = 0\ ,$$

which in particular yields

$$\int_{\mathbb{R}^d} \Sigma_{1,c} |w(t,x)|^2 \, dx = 0 \ ,$$

or

$$w(t,x) = 0 \quad \text{for a.e. } x \in \mathbb{R}^d \text{ and all } t \geq 0 \ .$$

Here we used the fact that

$$\int_0^t \int_{B_2 \setminus B_1} G * w(s,x) w(s,x) \, dx \, ds \geq 0 \ ,$$

cf. (4.8). This establishes the uniqueness of the weak solution u. The proof is complete. □

Let $\hat{u}_c(k,x)$ be the Fourier transform of u_c with respect to time, i.e.,[7]

$$\hat{u}_c(k,x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u_c(t,x) e^{ikt} \, dt.$$

We have

Proposition 2. *Let $d = 2$ or $d = 3$, and let $f \in L^2\big([0,+\infty) \times \mathbb{R}^d\big)$ with compact support. Suppose $u_c \in L^\infty\big([0,+\infty); H^1(\mathbb{R}^d)\big)$ with $\partial_t u_c \in L^\infty\big([0,+\infty); L^2(\mathbb{R}^d)\big)$ is the unique weak solution to (1.6). Then $\hat{u}_c(k,\cdot) \in H^1_{loc}(\mathbb{R}^d)$ is the unique outgoing solution to the equation*

$$\operatorname{div}(A_c \nabla \hat{u}_c) + k^2 \Sigma_c \hat{u}_c = -\hat{f} \ , \tag{4.13}$$

for a.e. $k > 0$. Moreover,

$$k\hat{u}_c(k,x) \in L^2_{loc}\big([0,+\infty) \times \mathbb{R}^d\big) \ .$$

We recall that $\hat{u}_c$ denotes the Fourier transform of u_c (u_c is extended by 0 for $t < 0$).

Outline of Proof. The proof of the first fact is similar to the one of [20, Theorem A1], and is based on the so called limiting absorption principle. A key ingredient, as in [20, (A9)], is the technique from the proof of Proposition 1 where the energy estimate was established. The fact that $k\hat{u}_c \in L^2_{loc}\big([0,+\infty) \times \mathbb{R}^d\big)$ is obtained as follows. Let $u_\delta \in L^\infty\big([0,+\infty); H^1(\mathbb{R}^d)\big)$ with $\partial_t u_\delta \in L^\infty\big([0,+\infty); L^2(\mathbb{R}^d)\big)$ be the unique weak solution[8] to

$$\begin{cases} \Sigma_{1,c} \partial^2_{tt} u_\delta - \operatorname{div}(A_c \nabla u_\delta) + \Sigma_{2,c} \partial_t u_\delta + G * \partial_t u_\delta + \delta \partial_t u_\delta = f & \text{in } [0,+\infty) \times \mathbb{R}^d \ , \\ \partial_t u_\delta(t=0) = u_\delta(t=0) = 0 & \text{in } \mathbb{R}^d \ . \end{cases} \tag{4.14}$$

Then, as in the proof of Proposition 1,

$$\delta \int_0^t \int_{\mathbb{R}^d} |\partial_t u_\delta|^2 \leq C \ . \tag{4.15}$$

[7] We extend u_c by 0 for $t < 0$.
[8] The definition of weak solutions for the equation of u_δ is similar to the one for the equation of u_c.

This implies $\partial_t u_\delta \in L^2([0,+\infty); L^2(\mathbb{R}^d))$, and thus $k\hat{u}_\delta \in L^2([0,+\infty); L^2(\mathbb{R}^d)) = L^2([0,\infty) \times \mathbb{R}^d)$. Here $\hat{u}_\delta$ denotes the Fourier transform of u_δ (u_δ is extended by 0 for $t < 0$). As in the proof of [20, Theorem A1],[9] for almost every $k > 0$, $\hat{u}_\delta(k,) \in H^1(\mathbb{R}^d)$ is the unique solution to

$$\operatorname{div}(A_c \nabla \hat{u}_\delta(k,\cdot)) + k^2 \Sigma_c \hat{u}_\delta(k,\cdot) + i\delta k \hat{u}_\delta(k,\cdot) = -\hat{f}(k,\cdot).$$

Fix $k_0 > 0$ arbitrary. We have, for $0 \le k \le k_0$ and for $0 < \delta < 1$,

$$\|\hat{u}_\delta(k,\cdot)\|_{L^2(B_R)} \le C k^{-1} \|\hat{f}(k,\cdot)\|_{L^2(B_R)} \ , \tag{4.16}$$

for some positive constant C, independent of δ and k, see Lemma 7 below. Since f has compact support,

$$\|\hat{f}(k,\cdot)\|_{L^2(B_R)} \le C\|f\|_{L^2} \ . \tag{4.17}$$

A combination of (4.15), (4.16), and (4.17) yields

$$\Big(k\hat{u}_\delta(k)\Big)_{0<\delta<1} \text{ is bounded in } L^2_{loc}([0,+\infty) \times \mathbb{R}^d). \tag{4.18}$$

By the limiting absorption principle (see e.g. [11, Section 4.6]) we have, for almost every $k > 0$,

$$\hat{u}_\delta(k,\cdot) \to V(k,\cdot) \text{ weakly in } H^1_{loc}(\mathbb{R}^d) \ , \tag{4.19}$$

where $V(k,x) \in H^1_{loc}(\mathbb{R}^d)$ is the unique outgoing solution to (4.13). On the other hand (see e.g., the proof of [20, Theorem A1] in particular [20, (A13)]),

$$k\hat{u}_\delta(k,x) \text{ converges to } kV(k,x) \text{ in the distributional sense on } \mathbb{R} \times \mathbb{R}^d \ . \tag{4.20}$$

Since $V(k,\cdot) = \hat{u}_c(k,\cdot)$, $k > 0$, we derive from (4.18), (4.19), and (4.20) that

$$k\hat{u}_c \in L^2_{loc}([0,+\infty) \times \mathbb{R}^d) \ .$$

The proof is complete. □

In the proof of Proposition 2, we used a simple consequence of the following lemma:

Lemma 7. *Given $g \in L^2(\mathbb{R}^d)$ with $\operatorname{supp} g \subset B_{R_0}$, $0 < \delta < 1$, and $0 < k < k_0$, let $v_{k,\delta} \in H^1(\mathbb{R}^d)$ be the unique solution to*

$$\operatorname{div}(A_c \nabla v_{k,\delta}) + (k^2 \Sigma_c + ik\delta) v_{k,\delta} = g \text{ in } \mathbb{R}^d \ .$$

For any $R_1 > 0$, there exists a positive constant C_{R_1} independent of k, δ, and g such that

$$\|v_{k,\delta}\|_{L^2(B_{R_1})} \le C_{R_1}(|\ln k|^2 + 1)\|g\|_{L^2} \quad \text{for } d = 2 \ , \tag{4.21}$$

and

$$\|v_{k,\delta}\|_{L^2(B_{R_1})} \le C_{R_1}\|g\|_{L^2} \quad \text{for } d = 3 \ . \tag{4.22}$$

[9] More precisely, [20, (A10) and the following paragraph].

Proof. Under the conditions $k < k_0$ and $\delta < 1$, the estimates (4.21) and (4.22) follow from a standard contradiction argument if k is bounded below by a positive constant. Lemma 8 below implies these same estimates for sufficiently small k, and the proof is complete. □

Lemma 8. *Let $A \in [L^\infty(\mathbb{R}^d)]^{d\times d}$ and $\Sigma \in L^\infty(\mathbb{R}^d)$, $d = 2, 3$ be such that A is uniformly elliptic, $A = I$ in $\mathbb{R}^d \setminus B_2$, $\Sigma = 1$ in $\mathbb{R}^d \setminus B_2$, $\Re(\Sigma)$ is strictly positive, and $\Im(\Sigma) \ge 0$. Given $g \in L^2(\mathbb{R}^d)$ with $\operatorname{supp} g \subset B_{R_0}$, and $0 < \varepsilon\ ,\ \delta < 1$, let $v_{\varepsilon,\delta} \in H^1(\mathbb{R}^d)$ be the unique solution to*

$$\operatorname{div}(A\nabla v_{\varepsilon,\delta}) + (\varepsilon^2\Sigma + i\delta)v_{\varepsilon,\delta} = g \text{ in } \mathbb{R}^d\ .$$

For any $R_1 > 0$, there exist two positive constants c and C independent of ε and δ such that if $0 < \varepsilon,\ \delta < c$ then

$$\|v_{\varepsilon,\delta}\|_{L^2(B_{R_1})} \le C|\ln\varepsilon|^2\|g\|_{L^2} \quad \text{for } d = 2\ ,$$

and

$$\|v_{\varepsilon,\delta}\|_{L^2(B_{R_1})} \le C\|g\|_{L^2} \quad \text{for} \quad \text{for } d = 3\ .$$

Proof. The proof of this lemma for $d = 3$ is simpler than for $d = 2$. Essentially the proof for $d = 3$ follows along the first third of the argument for $d = 2$. For this reason we only present the proof for $d = 2$. We may without loss of generality suppose $R_1 > R_0$ (if not, simply increase R_1), and for simplicity of notation we use $R_0 = 4$, $R_1 = 5$. The proof proceeds by contradiction. Suppose there exist a sequence $\varepsilon_n \to 0$, a sequence $\delta_n \to 0$, and a sequence $(g_n) \subset L^2(\mathbb{R}^2)$ such that $\operatorname{supp} g_n \subset B_4$, and

$$\lim_{n\to\infty} |\ln\varepsilon_n|^2\|g_n\|_{L^2} = 0, \quad \text{and} \quad \|v_n\|_{L^2(B_5)} = 1\ . \tag{4.23}$$

Here $v_n \in H^1(\mathbb{R}^2)$ is the unique solution to

$$\operatorname{div}(A\nabla v_n) + (\varepsilon_n^2\Sigma + i\delta_n)v_n = g_n \text{ in } \mathbb{R}^2\ .$$

Multiplying this equation by $\bar{v}_n$ and integrating the obtained expression on B_5, we have

$$\int_{B_5} \langle A\nabla v_n, \nabla\bar{v}_n\rangle - \int_{B_5} (\varepsilon_n^2\Sigma + i\delta_n)|v_n|^2 = -\int_{B_5} g_n\bar{v}_n + \int_{\partial B_5} \frac{\partial v_n}{\partial r}\bar{v}_n\ . \tag{4.24}$$

Since $\Delta v_n + (\varepsilon_n^2 + i\delta_n)v_n = 0$ in $\mathbb{R}^2 \setminus B_4$ and $v_n \in H^1(\mathbb{R}^2)$, it follows that[10]

$$\|v_n\|_{H^1(B_R\setminus B_{9/2})} \le C_R\|v_n\|_{H^{1/2}(\partial B_{9/2})} \le C_R\|v_n\|_{L^2(B_5\setminus B_4)} \le C_R \quad \text{for } R > 9/2\ . \tag{4.25}$$

We derive from (4.23) and (4.24) that

$$\int_{B_5} |\nabla v_n|^2 \le C\ . \tag{4.26}$$

A combination of (4.25) and (4.26) yields

[10] One can use (4.27). below to derive this property.

$$\|v_n\|_{H^1(B_R)} \le C_R \quad \forall\, R > 0\ .$$

Thus (after extraction of a subsequence) $v_n \to v$ in $L^2_{loc}(\mathbb{R}^2)$, where $v \in W^1(\mathbb{R}^2)$ is a solution to[11]

$$\operatorname{div}(A\nabla v) = 0 \text{ in } \mathbb{R}^2\ .$$

It is clear that $v = \alpha$ for some (complex) constant α. For $d = 3$ the proof would proceed similarly until this point, where we could automatically conclude that α is zero, and we would have reached a contradiction. In the two dimensional case it requires the following additional argument to show that α is zero. Since $\Delta v_n + \hat{\varepsilon}_n^2 v_n = 0$ in $\mathbb{R}^2 \setminus B_4$ with $\hat{\varepsilon}_n^2 = \varepsilon_n^2 + i\delta_n$ and $\Im(\hat{\varepsilon}_n) > 0$, $v_n \in H^1(\mathbb{R}^2)$ can be represented as

$$v_n(x) = \sum_{l=-\infty}^{\infty} a_{l,n} H_l^{(1)}(\hat{\varepsilon}_n|x|)e^{il\theta} \quad |x| > 4\ , \tag{4.27}$$

where $H_l^{(1)}$ is the Hankel function of the first kind of order l. This implies

$$v_n = v_{0,n} + v_{1,n} \quad |x| > 4\ , \tag{4.28}$$

where

$$v_{0,n} = a_{0,n} H_0^{(1)}(\hat{\varepsilon}_n|x|), \quad \text{ and } \quad v_{1,n} = \sum_{l \neq 0} a_{l,n} H_l^{(1)}(\hat{\varepsilon}_n|x|)e^{il\theta}, \quad |x| > 4\ . \tag{4.29}$$

By orthogonality, it is clear that for any $R > 4$,

$$\|v_{0,n}\|_{H^1(B_R\setminus B_4)} + \|v_{1,n}\|_{H^1(B_R\setminus B_4)} \le C\|v_n\|_{H^1(B_R\setminus B_4)}\ .$$

After extraction of a subsequence, we may assume that $v_{0,n} \to \alpha_0$ in $L^2_{loc}(\mathbb{R}^2 \setminus B_4)$ and $v_{1,n} \to v_1$ in $L^2_{loc}(\mathbb{R}^2 \setminus B_4)$ for some (complex) constant α_0 and some $v_1 \in L^2_{loc}(\mathbb{R}^2 \setminus B_4)$. Therefore,

$$\alpha = v = \alpha_0 + v_1 \quad |x| > 4\ .$$

This implies that v_1 is constant on $\{|x| > 4\}$. It follows that $v_1 = 0$ for $|x| > 4$ since

$$\int_{B_6\setminus B_5} v_1 = \lim_{n\to\infty} \int_{B_6\setminus B_5} v_{1,n} = 0\ .$$

As a consequence

$$\lim_{n\to\infty} v_n = \lim_{n\to\infty} v_{0,n} = \alpha_0 \qquad \text{in } L^2_{loc}(\mathbb{R}^2 \setminus B_4)\ .$$

We have

$$\int_{\partial B_R} \partial_r v_n \bar{v}_n = \int_{B_R} \langle A\nabla v_n, \nabla \bar{v}_n\rangle - \int_{B_R} (\varepsilon_n^2\Sigma + i\delta_n)|v_n|^2 + \int_{B_R} g_n \bar{v}_n\ .$$

If we let $R \to \infty$ and consider only the imaginary part, then we obtain

[11] The proof is similar to the one of Lemma 2.

$$\Im \int_{\mathbb{R}^2} (\varepsilon_n^2 \Sigma + i\delta_n)|v_n|^2 = \Im \int_{\mathbb{R}^2} g_n \bar{v}_n \ . \tag{4.30}$$

Due to the fact that $\hat{\varepsilon}^n$ has a positive imaginary part we have that $v_{0,n} \in H^1(\mathbb{R}^2)$ (actually it decreases exponentially at ∞), and so

$$\int_{\partial B_5} \partial_r v_{0,n} \bar{v}_{0,n} = - \int_{\mathbb{R}^2 \setminus B_5} |\nabla v_{0,n}|^2 + \int_{\mathbb{R}^2 \setminus B_5} (\varepsilon_n^2 + i\delta_n)|v_{0,n}|^2 \ ,$$

which leads to

$$\begin{aligned} \Im \int_{\partial B_5} \partial_r v_{0,n} \bar{v}_{0,n} &= \Im \int_{\mathbb{R}^2 \setminus B_5} (\varepsilon_n^2 + i\delta_n)|v_{0,n}|^2 = \int_{\mathbb{R}^2 \setminus B_5} \delta_n |v_{0,n}|^2 \\ &\leq \int_{\mathbb{R}^2 \setminus B_5} \delta_n |v_n|^2 \leq \Im \int_{\mathbb{R}^2} (\varepsilon_n^2 \Sigma + i\delta_n)|v_n|^2 \ . \end{aligned}$$

For the last two inequalities we used the orthogonality of the decomposition (4.28), and the facts that $\Sigma = 1$ in $\mathbb{R}^2 \setminus B_5$ and $\Im \Sigma \geq 0$. In combination with (4.23) and (4.30) this gives

$$\left| \Im \int_{\partial B_5} \partial_r v_{0,n} \bar{v}_{0,n} \right| \leq \|g_n\|_{L^2} \|v_n\|_{L^2(B_5)} \leq \|g_n\|_{L^2} \ . \tag{4.31}$$

A simple calculation, based on (4.29) and the well-known asymptotics of the Hankel function $H_0^{(1)}$ for small argument (see e.g., [1, page 360]), gives

$$c|a_{0,n}|^2 \leq \left| \Im \int_{\partial B_5} \partial_r v_{0,n} \bar{v}_{0,n} \right| \ ,$$

and so, in combination with (4.31), and (4.23) we get

$$|\ln \varepsilon_n|^2 |a_{0,n}|^2 \leq C |\ln \varepsilon_n|^2 \|g_n\|_{L^2} \to 0 \text{ as } n \to \infty \ .$$

This estimate and the formula (4.29) for $v_{0,n}$ now yields

$$\lim_{n \to \infty} v_{0,n} = 0 \quad \text{on any bounded subset of } \mathbb{R}^2 \setminus B_4 \ .$$

Accordingly we have $\alpha = \alpha_0 = 0$, and so it follows that the v_n converge to 0 in $L^2_{loc}(\mathbb{R}^2)$. We have thus reached a contradiction to the fact that $\|v_n\|_{L^2(B_5)} = 1$, and the proof is complete. □

5. Proof of Theorem 2

The proof is related to that in [20], however, we shall estimate $\partial_t u_c - \partial_t u$ as a way of getting to $u_c - u$. This idea was also used in [19]. Let $\hat{u}_c(k, \cdot)$ be the Fourier transform of u_c with respect to time. By Proposition 2, for a.e. $k > 0$, $\hat{u}_c(k, \cdot) \in H^1_{loc}(\mathbb{R}^d)$ is the unique outgoing solution to

$$\operatorname{div}(A_c \nabla \hat{u}_c) + k^2 \Sigma_c \hat{u}_c = -\hat{f} \ , \tag{5.1}$$

where (A_c, Σ_c) is given in (1.4). Moreover,

$$k\hat{u}_c \in L^2_{loc}\big([0, +\infty) \times \mathbb{R}^d\big) \ . \tag{5.2}$$

As before we introduce $\tilde{u}_\varepsilon(k, x) = \hat{u}_c(k, F_\varepsilon(x))$. Then $\tilde{u}_\varepsilon \in H^1_{loc}(\mathbb{R}^d)$ is the unique outgoing solution to

$$\operatorname{div}(A_\varepsilon \nabla \tilde{u}_\varepsilon) + k^2 \Sigma_\varepsilon \tilde{u}_\varepsilon = -\hat{f} \text{ in } \mathbb{R}^d \ . \tag{5.3}$$

Here

$$A_\varepsilon, \Sigma_\varepsilon = \begin{cases} I, \ 1 & \text{in } \mathbb{R}^d \setminus B_2 \ , \\ I, \ \sigma_\varepsilon(x) = 1 + \sigma_{1,\varepsilon}(x) & \text{in } B_2 \setminus B_\varepsilon, \\ \dfrac{1}{\varepsilon^{d-2}} I, \ \dfrac{1}{\varepsilon^d}(1 + i/k) & \text{in } B_\varepsilon \setminus B_{\varepsilon/2} \ , \\ \dfrac{1}{\varepsilon^{d-2}} a(x/\varepsilon), \ \dfrac{1}{\varepsilon^d}\sigma(x/\varepsilon) & \text{in } B_{\varepsilon/2} \ , \end{cases} \tag{5.4}$$

and

$$\sigma_{1,\varepsilon} = \left(F_\varepsilon^{-1}\right)_* \sigma_{1,c} \ .$$

Recall that $\sigma_{1,c}$ is given by (1.3) with $\sigma_D = \sigma_N = 1$. Let $\tilde{u}_{1,\varepsilon} \in H^1_{loc}(\mathbb{R}^d)$ be the unique outgoing solution to

$$\operatorname{div}(A_{1,\varepsilon} \nabla \tilde{u}_{1,\varepsilon}) + k^2 \Sigma_{1,\varepsilon} \tilde{u}_{1,\varepsilon} = -\hat{f} \ ,$$

with

$$A_{1,\varepsilon}, \Sigma_{1,\varepsilon} = \begin{cases} I, 1 & \text{in } \mathbb{R}^d \setminus B_\varepsilon \ , \\ \dfrac{1}{\varepsilon^{d-2}} I, \dfrac{1}{\varepsilon^d}(1 + i/k) & \text{in } B_\varepsilon \setminus B_{\varepsilon/2} \ , \\ \dfrac{1}{\varepsilon^{d-2}} a(x/\varepsilon), \dfrac{1}{\varepsilon^d}\sigma(x/\varepsilon) & \text{in } B_{\varepsilon/2} \ . \end{cases}$$

Finally, let $\hat{u}(k, x)$ be the Fourier transform of u with respect to time; $\hat{u}(k, \cdot) \in H^1_{loc}(\mathbb{R}^d)$ is the unique outgoing solution to

$$\Delta \hat{u} + k^2 \hat{u} = -\hat{f} \text{ in } \mathbb{R}^d \ .$$

We first estimate

$$\int_0^{1/\varepsilon} k \|\tilde{u}_{1,\varepsilon} - \hat{u}\|_{L^2(B_5 \setminus B_2)} \, dk \ .$$

For this purpose, let $\tilde{U}_{1,\varepsilon}(k, \cdot) \in H^1_{loc}(\mathbb{R}^d)$ be the unique outgoing solution to

$$\begin{cases} \Delta \tilde{U}_{1,\varepsilon} + k^2 \tilde{U}_{1,\varepsilon} = -\hat{f} & \text{in } \mathbb{R}^d \setminus B_\varepsilon \ , \\ \tilde{U}_{1,\varepsilon} = 0 & \text{in } B_\varepsilon \ , \end{cases}$$

and define, in all of $\mathbb{R}^d$,

$$w_{1,\varepsilon}(k,\cdot) = \tilde{U}_{1,\varepsilon}(k,\cdot) - \hat{u}(k,\cdot) \quad \text{and} \quad w_{2,\varepsilon} = \tilde{u}_{1,\varepsilon}(k,\cdot) - \tilde{U}_{1,\varepsilon}(k,\cdot)\ .$$

Then $w_{1,\varepsilon}(k,\cdot) \in H^1_{loc}(\mathbb{R}^d)$ is the unique outgoing solution to

$$\begin{cases} \Delta w_{1,\varepsilon} + k^2 w_{1,\varepsilon} = 0 & \text{in } \mathbb{R}^d \setminus B_\varepsilon\ , \\ w_{1,\varepsilon} = -\hat{u} & \text{in } B_\varepsilon\ , \end{cases} \tag{5.5}$$

and $w_{2,\varepsilon} \in H^1_{loc}(\mathbb{R}^d)$ is the unique outgoing solution to

$$\begin{cases} \Delta w_{2,\varepsilon} + k^2 w_{2,\varepsilon} = 0 & \text{in } \mathbb{R}^d \setminus B_\varepsilon\ , \\ \nabla \cdot (A_{1,\varepsilon} \nabla w_{2,\varepsilon}) + k^2 \Sigma_{1,\varepsilon} w_{2,\varepsilon} = 0 & \text{in } B_\varepsilon\ , \\ \dfrac{\partial w_{2,\varepsilon}}{\partial r}\Big|_{\text{ext}} - \dfrac{1}{\varepsilon^{d-2}} \dfrac{\partial w_{2,\varepsilon}}{\partial r}\Big|_{\text{int}} = -\dfrac{\partial \tilde{U}_{1,\varepsilon}}{\partial r} & \text{on } \partial B_\varepsilon\ . \end{cases} \tag{5.6}$$

We first estimate $w_{1,\varepsilon}$. By Lemma 1 and the theory of regularity of elliptic equations, we have, for $d = 3$ or ($d = 2$ and $k > 1/2$)

$$\frac{1}{k+1}\|\nabla^2 \hat{u}(k,\cdot)\|_{L^2(B_2)} + \|\nabla \hat{u}(k,\cdot)\|_{L^2(B_2)} + (k+1)\|\hat{u}(k,\cdot)\|_{L^2(B_2)} \le C\|\hat{f}(k,\cdot)\|_{L^2} \tag{5.7}$$

and for ($0 < k < 1/2$ and $d = 2$)

$$\|\nabla^2 \hat{u}(k,\cdot)\|_{L^2(B_2)} + \|\nabla \hat{u}(k,\cdot)\|_{L^2(B_2)} + \|\hat{u}(k,\cdot)\|_{L^2(B_2)} \le C|\ln k|\|\hat{f}(k,\cdot)\|_{L^2}. \tag{5.8}$$

Here and in the remainder of this proof, C denotes a positive constant independent of ε, k, and f. Since $\Delta \hat{u}(k,\varepsilon\cdot) + k^2\varepsilon^2 \hat{u}(k,\varepsilon\cdot) = 0$ in B_2, it follows that

$$\int_{B_1} |\nabla \hat{u}(k,\varepsilon\cdot)|^2 \le C \max\{1, \varepsilon^2 k^2\} \int_{B_2} |\hat{u}(k,\varepsilon\cdot)|^2 \le C\|\hat{u}(k,\cdot)\|^2_{L^\infty(B_2)}\ ,$$

for $0 < k < 1/\varepsilon$. Using (5.7) and (5.8), we derive that for $0 < k < 1/\varepsilon$,

$$\|\hat{u}(k,\varepsilon\cdot)\|_{H^{1/2}(\partial B_1)} \le C(k+1)\|\hat{f}(k,\cdot)\|_{L^2} \text{ for } d = 3\ , \tag{5.9}$$

and

$$\|\hat{u}(k,\varepsilon\cdot)\|_{H^{1/2}(\partial B_1)} \le C(k+1)\varphi(k)\|\hat{f}(k,\cdot)\|_{L^2} \text{ for } d = 2\ . \tag{5.10}$$

Here

$$\varphi(k) = 1 \text{ if } k > 1/2\ , \quad \text{and} \quad \varphi(k) = |\ln k| \text{ if } 0 < k < 1/2\ .$$

Applying Lemma 2 and rescaling, we have for $0 < k < 1/\varepsilon$,

$$\|w_{1,\varepsilon}(k,\cdot)\|_{L^2(B_5\setminus B_2)} \le C\varepsilon(k+1)\|\hat{f}(k,\cdot)\|_{L^2} \text{ for } d = 3\ , \tag{5.11}$$

and

$$\|w_{1,\varepsilon}(k,\cdot)\|_{L^2(B_5\setminus B_2)} \le C\frac{|H_0^{(1)}(k)|}{|H_0^{(1)}(k\varepsilon)|}(k+1)\varphi(k)\|\hat{f}(k,\cdot)\|_{L^2} \text{ for } d = 2\ . \tag{5.12}$$

Since

$$\frac{|H_0^{(1)}(k)|}{|H_0^{(1)}(k\varepsilon)|} \le C\frac{\min\{k^{-1/2}\,,\,|\ln k|+2\}}{|\ln(k\varepsilon)|+2} \quad \text{for } 0<k\varepsilon<1 \quad ,$$

and thus

$$\frac{|H_0^{(1)}(k)|}{|H_0^{(1)}(k\varepsilon)|} \le C\begin{cases} \dfrac{|\ln k|+2}{|\ln\varepsilon|+2}\,, & k\le 1\,, \\ \dfrac{1}{|\ln\varepsilon|+2}\,, & 1\le k\le 1/\varepsilon\,, \end{cases}$$

we have

$$k\frac{|H_0^{(1)}(k)|}{|H_0^{(1)}(k\varepsilon)|}\varphi(k) \le C\frac{k(|\ln k|^2+1)}{|\ln\varepsilon|+2} \quad , \tag{5.13}$$

for $0<k<1/\varepsilon$. It now follows from (5.11) and (5.12) that, in the range $0<k<1/\varepsilon$,

$$k\|w_{1,\varepsilon}(k,\cdot)\|_{L^2(B_5\setminus B_2)} \le C\varepsilon k(k+1)\|\hat f(k,\cdot)\|_{L^2} \text{ for } d=3\,, \tag{5.14}$$

and

$$k\|w_{1,\varepsilon}(k,\cdot)\|_{L^2(B_5\setminus B_2)} \le \frac{C}{|\ln\varepsilon|}k(k+1)\left(|\ln k|^2+1\right)\|\hat f(k,\cdot)\|_{L^2} \text{ for } d=2\,. \tag{5.15}$$

We next estimate $w_{2,\varepsilon}$. Applying Lemma 3, we have, for $0<k<1/\varepsilon$,

$$\|w_{2,\varepsilon}(k,\varepsilon\cdot)\|_{L^2(B_5\setminus B_1)} \le C\max\{1,\varepsilon/k\}\left\|\frac{\partial}{\partial r}\left(\tilde U_{1,\varepsilon}(k,\varepsilon\cdot)\right)\right\|_{H^{-1/2}(\partial B_1)} \text{ for } d=3\,, \tag{5.16}$$

and

$$\|w_{2,\varepsilon}(k,\varepsilon\cdot)\|_{L^2(B_5\setminus B_1)} \le C\max\{k,1/k\}\left\|\frac{\partial}{\partial r}\left(\tilde U_{1,\varepsilon}(k,\varepsilon\cdot)\right)\right\|_{H^{-1/2}(\partial B_1)} \text{ for } d=2\,. \tag{5.17}$$

For $0<\varepsilon k<1$, the standard trace estimate, and a classical interior elliptic estimate, yield

$$\|w_{2,\varepsilon}(k,\varepsilon\cdot)\|_{H^{1/2}(\partial B_2)} \le C\|w_{2,\varepsilon}(k,\varepsilon\cdot)\|_{H^1(B_4\setminus B_{3/2})} \le C\|w_{2,\varepsilon}(k,\varepsilon\cdot)\|_{L^2(B_5\setminus B_1)}$$

and so by use of Lemma 2, (5.16), (5.17) and a scaling argument, it follows that

$$\begin{aligned}\|w_{2,\varepsilon}(k,\cdot)\|_{L^2(B_5\setminus B_2)} &= \varepsilon^{3/2}\|w_{2,\varepsilon}(k,\varepsilon\cdot)\|_{L^2(B_{5/\varepsilon}\setminus B_{2/\varepsilon})} \\ &\le C\varepsilon\|w_{2,\varepsilon}(k,\varepsilon\cdot)\|_{L^2(B_5\setminus B_1)} \\ &\le C\varepsilon\max\{1,\varepsilon/k\}\left\|\frac{\partial}{\partial r}\left(\tilde U_{1,\varepsilon}(k,\varepsilon\cdot)\right)\right\|_{H^{-1/2}(\partial B_1)} \text{ for } d=3,\end{aligned} \tag{5.18}$$

and

$$\begin{aligned}\|w_{2,\varepsilon}(k,\cdot)\|_{L^2(B_5\setminus B_2)} &= \varepsilon\|w_{2,\varepsilon}(k,\varepsilon\cdot)\|_{L^2(B_{5/\varepsilon}\setminus B_{2/\varepsilon})}\\ &\le C\frac{|H_0^{(1)}(k)|}{|H_0^{(1)}(\varepsilon k)|}\|w_{2,\varepsilon}(k,\varepsilon\cdot)\|_{L^2(B_5\setminus B_1)}\\ &\le C\frac{|H_0^{(1)}(k)|}{|H_0^{(1)}(\varepsilon k)|}\max\{k,1/k\}\Big\|\frac{\partial}{\partial r}\left(\tilde U_{1,\varepsilon}(k,\varepsilon\cdot)\right)\Big\|_{H^{-1/2}(\partial B_1)} \text{ for } d=2\ . \end{aligned}\tag{5.19}$$

We have

$$\Big\|\frac{\partial}{\partial r}\left(\tilde U_{1,\varepsilon}(k,\varepsilon\cdot)\right)\Big\|_{H^{-1/2}(\partial B_1)} \le \Big\|\frac{\partial}{\partial r}\big(w_{1,\varepsilon}(k,\varepsilon\cdot)\big)\Big\|_{H^{-1/2}(\partial B_1)} + \Big\|\frac{\partial}{\partial r}\big(\hat u(k,\varepsilon\cdot)\big)\Big\|_{H^{-1/2}(\partial B_1)}\ .$$

Applying Lemma 2 to $w_{1,\varepsilon}(k,\varepsilon\cdot)$ and using (5.9) and (5.10), we obtain, for $0<k<1/\varepsilon$,

$$\Big\|\frac{\partial}{\partial r}\left(\tilde U_{1,\varepsilon}(\varepsilon\cdot)\right)\Big\|_{H^{-1/2}(\partial B_1)} \le C(k+1)\|\hat f(k,\cdot)\|_{L^2} \text{ for } d=3,$$

and

$$\Big\|\frac{\partial}{\partial r}\left(\tilde U_{1,\varepsilon}(\varepsilon\cdot)\right)\Big\|_{H^{-1/2}(\partial B_1)} \le C(k+1)\varphi(k)\|\hat f(k,\cdot)\|_{L^2} \text{ for } d=2.$$

It now follows from (5.18) and (5.19) that, for $0<k<1/\varepsilon$,

$$k\|w_{2,\varepsilon}\|_{L^2(B_5\setminus B_2)} \le C\varepsilon(k+1)\max\{k,\varepsilon\}\|\hat f(k,\cdot)\|_{L^2} \text{ for } d=3\ ,\tag{5.20}$$

and

$$k\|w_{2,\varepsilon}\|_{L^2(B_5\setminus B_2)} \le \frac{C(k+1)}{|\ln\varepsilon|}\left(|\ln k|^2+1\right)\max\{1,k^2\}\|\hat f(k,\cdot)\|_{L^2} \text{ for } d=2\ .\tag{5.21}$$

For the last estimate we also used (5.13). A combination of (5.14), (5.15), (5.20), and (5.21) yields

$$\int_0^{1/\varepsilon} k\|\tilde u_{1,\varepsilon}-\hat u\|_{L^2(B_5\setminus B_2)}\,dk \le C\varepsilon\int_0^{1/\varepsilon}(k+1)^2\|\hat f(k,\cdot)\|_{L^2} \le C\varepsilon\|f\| \quad \text{if } d=3\ ,\tag{5.22}$$

and

$$\begin{aligned}\int_0^{1/\varepsilon} k\|\tilde u_{1,\varepsilon}-\hat u\|_{L^2(B_5\setminus B_2)}\,dk &\le \frac{C}{|\ln\varepsilon|}\int_0^{1/\varepsilon}(|\ln k|^2+1)(1+k)^3\ \|\hat f(k,\cdot)\|_{L^2}\\ &\le \frac{C}{|\ln\varepsilon|}\|f\| \quad \text{if } d=2\ ,\end{aligned}\tag{5.23}$$

where $\|f\|$ is the norm introduced in the statement of Theorem 2.

We next estimate $\|\tilde u_\varepsilon(k,\cdot)-\tilde u_{1,\varepsilon}(k,\cdot)\|_{L^2(B_5\setminus B_2)}$ for k of order up to $1/\varepsilon$. We already know that $\Im(\sigma_{1,\varepsilon})>0$ for $k>0$, and from Lemma 4 and the fact that $k_\varepsilon > c_*/\varepsilon^{d/2}$ we have

$$|\sigma_{1,\varepsilon}| \le \frac{C}{\varepsilon^{d-1}k_\varepsilon^2} \le C_0\varepsilon\ ,\tag{5.24}$$

for $0 < k < \frac{c_*}{2}\varepsilon^{-1}$. Applying Lemma 5 and the first part of Lemma 6 to $\tilde{u}_\varepsilon - \tilde{u}_{1,\varepsilon}$ (with $g = -k^2\sigma_{1,\varepsilon}\tilde{u}_{1,\varepsilon}$) we obtain

$$k\|\tilde{u}_\varepsilon(k,\cdot) - \tilde{u}_{1,\varepsilon}(k,\cdot)\|_{L^2(B_5\setminus B_2)} \le C(k^2+1)k^2 \sup|\sigma_{1,\varepsilon}| \|\tilde{u}_{1,\varepsilon}(k,\cdot)\|_{L^2(B_2\setminus B_\varepsilon)} \tag{5.25}$$

for $0 < k < \frac{\lambda}{C_0\varepsilon}$ (λ is the constant from Lemma 6). A combination of (5.24) and (5.25) yields

$$k\|\tilde{u}_\varepsilon(k,\cdot) - \tilde{u}_{1,\varepsilon}(k,\cdot)\|_{L^2(B_5\setminus B_2)} \le Ck^2\varepsilon(k^2+1)\|\tilde{u}_{1,\varepsilon}(k,\cdot)\|_{L^2(B_2\setminus B_\varepsilon)} \ , \tag{5.26}$$

for $0 < k < \lambda_0/\varepsilon$, with $\lambda_0 = \min\{1, c_*/2, \lambda/C_0\}$. Similarly, applying Lemma 5 and the first part of Lemma 6 to the function $\hat{u}_{1,\varepsilon}$ (with $g = -f$ and coefficients $A_{1,\varepsilon}$, $\Sigma_{1,\varepsilon}$, *i.e.*, A_ε, Σ_ε with $\sigma_{1,\varepsilon} = 0$) we obtain

$$k\|\tilde{u}_{1,\varepsilon}(k,\cdot)\|_{L^2(B_5\setminus B_2)} \le C(k^2+1)\|\hat{f}(k,\cdot)\|_{L^2} \ , \tag{5.27}$$

for $0 < k < \lambda_0/\varepsilon$. A combination of (5.26) and (5.27) yields

$$\int_0^{\lambda_0/\varepsilon} k\|\tilde{u}_\varepsilon - \tilde{u}_{1,\varepsilon}\|_{L^2(B_5\setminus B_2)}\,dk \le C\varepsilon \int_0^{\lambda_0/\varepsilon} (k+1)^5\|\hat{f}(k,\cdot)\|_{L^2} \le C\varepsilon\|f\| \ . \tag{5.28}$$

We now consider the regime $k > \lambda_0/\varepsilon$. From the second part of Lemma 6, and the remark following, we have

$$k\|\tilde{u}_\varepsilon(k,\cdot)\|_{L^2(B_5\setminus B_2)} \le C\Big(k^2 + \frac{k\chi_1^2}{\chi_2}\Big)\|\hat{f}(k,\cdot)\|_{L^2} \le \frac{C}{\lambda_0}\varepsilon\Big(k^3 + \frac{k^2\chi_1^2}{\chi_2}\Big)\|\hat{f}(k,\cdot)\|_{L^2} \ . \tag{5.29}$$

On the other hand, using Lemma 1 we have

$$k\|\hat{u}(k,\cdot)\|_{L^2(B_5\setminus B_2)} \le C\|\hat{f}(k,\cdot)\|_{L^2} \le \frac{C}{\lambda_0}\varepsilon k\|\hat{f}(k,\cdot)\|_{L^2} \ , \tag{5.30}$$

for $k > \lambda_0/\varepsilon$. Lemma 4 yields

$$\frac{k^2\chi_1^2}{\chi_2} \le Ck^2\frac{1}{\varepsilon^{2(d-1)}k^2}\frac{\max\{k_\varepsilon^4, k^4\}}{k} \le C\frac{\max\{k_\varepsilon^4, k^4\}}{k\varepsilon^{2(d-1)}} \le C(k^{2d+1} + k^{2d-3}k_\varepsilon^4) \ , \tag{5.31}$$

for $k > \lambda_0/\varepsilon$. We derive from (5.29), (5.30), and (5.31) that

$$\int_{\lambda_0/\varepsilon}^{\infty} k\|\tilde{u}_\varepsilon - \hat{u}\|_{L^2(B_5\setminus B_2)}\,dk \le C\varepsilon \int_{\lambda_0/\varepsilon}^{\infty} (k^{2d+1} + k^{2d-3}k_\varepsilon^4)\|\hat{f}(k,\cdot)\|_{L^2} \ ,$$

or

$$\int_{\lambda_0/\varepsilon}^{\infty} k\|\tilde{u}_\varepsilon - \hat{u}\|_{L^2}\,dk \le C\varepsilon\|f\| \ . \tag{5.32}$$

A combination of (5.22), (5.23), (5.28), and (5.32) now gives

$$\int_0^{\infty} k\|\tilde{u}_\varepsilon - \hat{u}\|_{L^2(B_5\setminus B_2)}\,dk \le C\varepsilon\|f\| \quad \text{if } d = 3 \ ,$$

and

$$\int_0^\infty k\|\tilde{u}_\varepsilon - \hat{u}\|_{L^2(B_5\setminus B_2)}\,dk \le \frac{C}{|\ln\varepsilon|}\|f\| \quad \text{if } d = 2 \ .$$

Therefore, since $\tilde{u}_\varepsilon(k,\cdot) = \hat{u}_c(k,\cdot)$ outside B_2 (and since u_c and u are real, so that $\hat{u}_c(-k,\cdot) - \hat{u}(-k,\cdot) = \overline{\hat{u}_c(k,\cdot) - \hat{u}(k,\cdot)}$) it follows that

$$\sup_{t>0}\|\partial_t u_c(t,\cdot) - \partial_t u(t,\cdot)\|_{L^2(B_5\setminus B_2)} \le C\varepsilon\|f\| \quad \text{if } d = 3 \ ,$$

and

$$\sup_{t>0}\|\partial_t u_c(t,\cdot) - \partial_t u(t,\cdot)\|_{L^2(B_5\setminus B_2)} \le \frac{C}{|\ln\varepsilon|}\|f\| \quad \text{if } d = 2 \ .$$

From this we conclude

$$\sup_{0<t<T}\|u_c - u\|_{L^2(B_5\setminus B_2)} \le CT\varepsilon\|f\| \quad \text{if } d = 3 \ ,$$

and

$$\sup_{0<t<T}\|u_c - u\|_{L^2(B_5\setminus B_2)} \le \frac{CT}{|\ln\varepsilon|}\|f\| \quad \text{if } d = 2 \ .$$

The proof of Theorem 2 is complete. □

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References

[1] M. Abramowitz, I. Stegun, Handbook of Mathematical Functions, National Bureau of Standards Applied Mathematics Series, vol. 55, 1964.
[2] T.A. Burton, Volterra Integral and Differential Equations, Mathematics in Science and Engineering, vol. 167, Academic Press, Inc., Orlando, FL, 1983.
[3] W. Cai, U.K. Chettiar, A.V. Kildishev, V.M. Shalaev, Optical cloaking with metamaterials, Nat. Photonics 1 (2007) 224–227.
[4] A. Greenleaf, Y. Kurylev, M. Lassas, G. Uhlmann, Improvement of cylindrical cloaking with the SHS lining, Opt. Express 15 (2007) 12717.
[5] A. Greenleaf, Y. Kurylev, M. Lassas, G. Uhlmann, Electromagnetic wormholes via handlebody constructions, Commun. Math. Phys. 281 (2008) 369–385.
[6] A. Greenleaf, M. Lassas, G. Uhlmann, On nonuniqueness for Calderon's inverse problem, Math. Res. Lett. 10 (2003) 685–693.
[7] H. Haddar, P. Joly, H-M. Nguyen, Generalized impedance boundary conditions for scattering by strongly absorbing obstacles: the scalar case, Math. Models Methods Appl. Sci. 15 (2005) 1273–1300.
[8] J.D. Jackson, Classical Electrodynamics, third ed., John Willey, 2001.
[9] R.V. Kohn, D. Onofrei, M.S. Vogelius, M.I. Weinstein, Cloaking via change of variables for the Helmholtz equation, Commun. Pure Appl. Math. 63 (2010) 973–1016.
[10] R.V. Kohn, H. Shen, M.S. Vogelius, M.I. Weinstein, Cloaking via change of variables in electric impedance tomography, Inverse Probl. 24 (2008) 015016.

[11] R. Leis, Initial-Boundary Value Problems in Mathematical Physics, B. G. Teubner/John Wiley-Sons, Ltd., Stuttgart/Chichester, 1986.
[12] U. Leonhardt, Optical conformal mapping, Science 312 (2006) 1777–1780.
[13] U. Leonhardt, T. Tyc, Broadband invisibility by non-Euclidean cloaking, Science 323 (2009) 110–112.
[14] J. Li, H. Liu, L. Rondi, G. Uhlmann, Regularized transformation-optics cloaking for Helmholtz equation: from partial cloak to full cloak, Commun. Math. Phys. 335 (2015) 671–712.
[15] C.S. Morawetz, D. Ludwig, An inequality for the reduced wave operator and the justification of geometrical optics, Commun. Pure Appl. Math. 21 (1968) 187–203.
[16] H-M. Nguyen, Cloaking via change of variables for the Helmholtz equation in the whole space, Commun. Pure Appl. Math. 63 (2010) 1505–1524.
[17] H-M. Nguyen, Approximate cloaking for the Helmholtz equation via transformation optics and consequences for perfect cloaking, Commun. Pure Appl. Math. 65 (2012) 155–186.
[18] H-M. Nguyen, On a regularized scheme for approximate acoustic cloaking using transformation optics, SIAM J. Math. Anal. 45 (2013) 3034–3049.
[19] H-M. Nguyen, L. Nguyen, Generalized impedance boundary conditions for scattering by strongly absorbing obstacles for the full wave equation: the scalar case, Math. Models Methods Appl. Sci. 25 (2015) 1927–1960.
[20] H-M. Nguyen, M.S. Vogelius, Approximate cloaking for the full wave equation via change of variables, SIAM J. Math. Anal. 44 (2012) 1894–1924.
[21] H-M. Nguyen, M.S. Vogelius, Full range scattering estimates and their application to cloaking, Arch. Ration. Mech. Anal. 203 (2012) 769–807.
[22] L.E. Payne, H.F. Weinberger, New bounds for solutions of second order elliptic partial differential equations, Pac. J. Math. 8 (1958) 551–573.
[23] J.B. Pendry, D. Schurig, D.R. Smith, Controlling electromagnetic fields, Science 312 (2006) 1780–1782.
[24] B. Perthame, L. Vega, Morrey–Campanato estimates for Helmholtz equations, J. Funct. Anal. 164 (1999) 340–355.
[25] F. Rellich, Darstellung der Eigenwerte von $\Delta u + \lambda u = 0$ durch ein Randintegral, Math. Z. 46 (1940) 635–636.
[26] Z. Ruan, M. Yan, C.M. Neff, M. Qiu, Ideal cylindrical cloak: perfect but sensitive to tiny perturbations, Phys. Rev. Lett. 99 (2007) 113903.
[27] D. Schurig, J.J. Mock, J. Justice, S.A. Cummer, J.B. Pendry, A.F. Starr, D.R. Smith, Metamaterial electromagnetic cloak at microwave frequencies, Science 314 (2006) 1133628.
[28] J.S. Toll, Causality and the dispersion relation: logical foundations, Phys. Rev. 104 (1956) 1760–1770.