Deconvolution of Gaussian Random Fields and Application to Radio Astronomy

EDIC SEMESTER PROJECT

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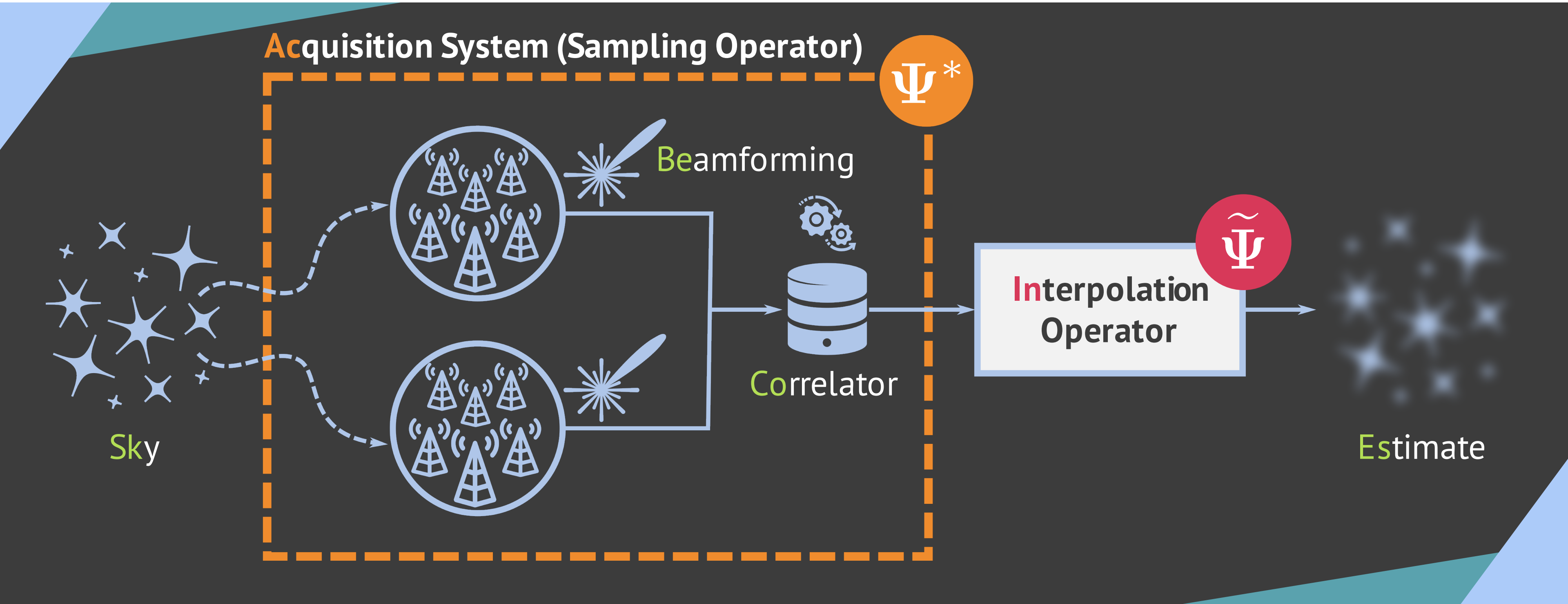
Introduction

- Many scientific applications involve estimating the **intensity field** of some physical phenomenon.
- The field is *sampled* by an **acquisition system**.
- Recovery is performed by *interpolating* the “samples”.
- Sampling followed by interpolation acts as a **projection**.

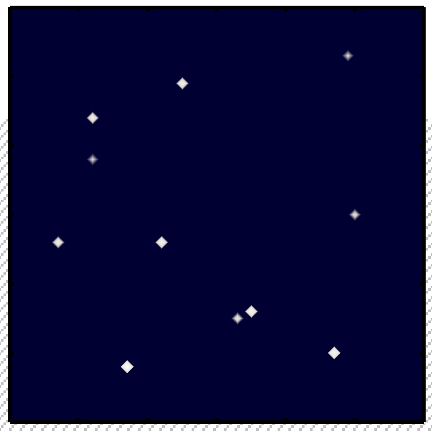


Convolution Artifacts

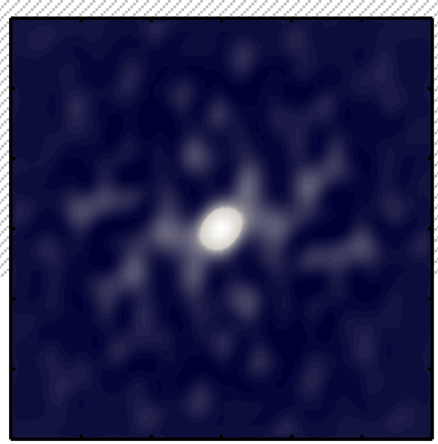
- Interpolating the acquired samples can be seen as convolving the true intensity field with some **point spread function** (specified by the tool).

$$I_D = I * DB$$

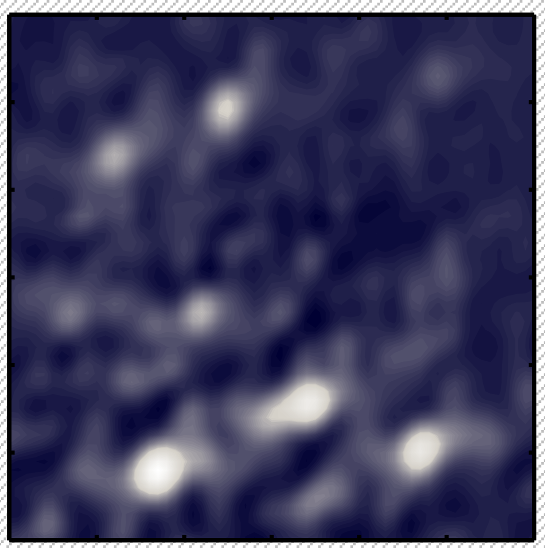
True Sky



Point Spread Function



Dirty Image

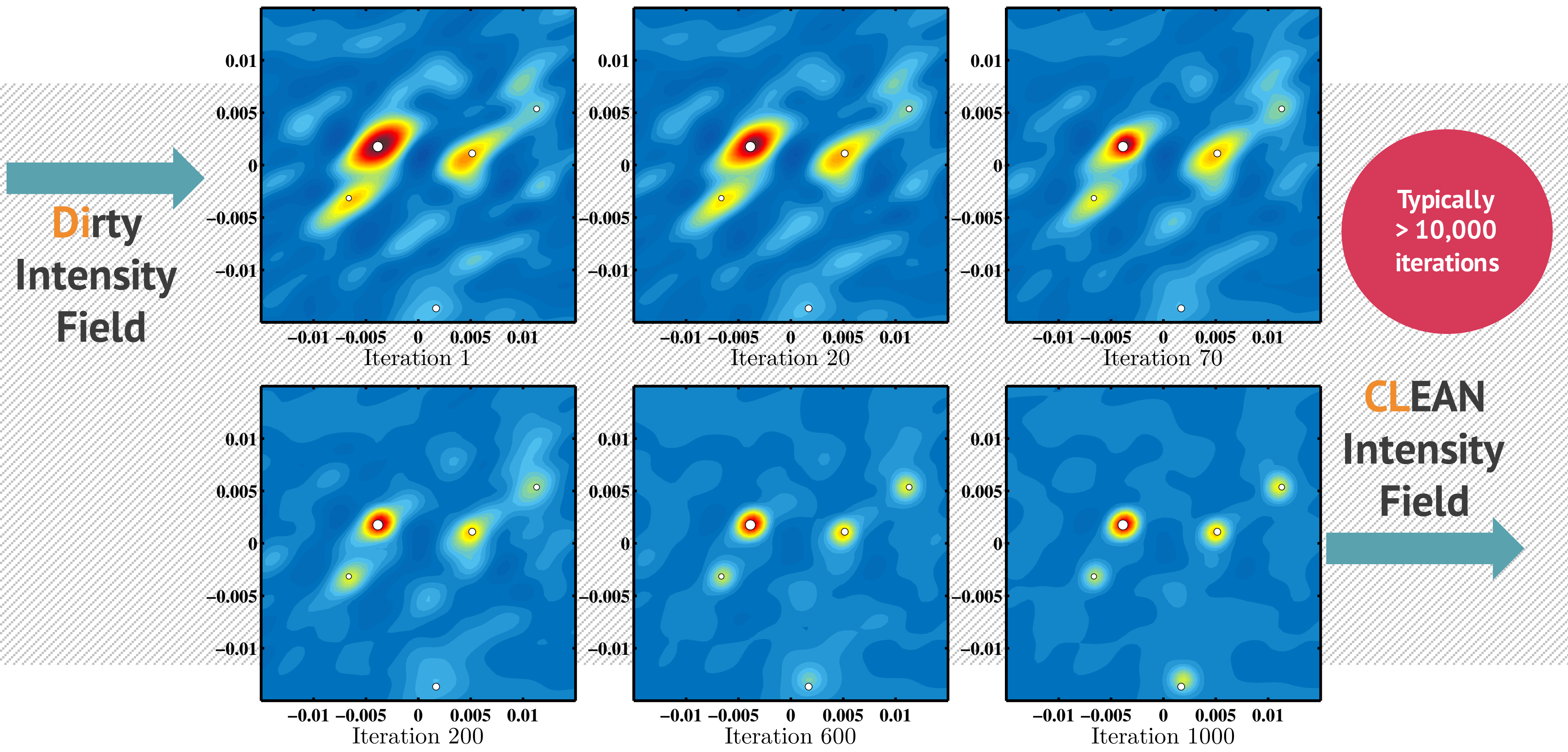


Aliasing artifacts forbid direct identification of sources !

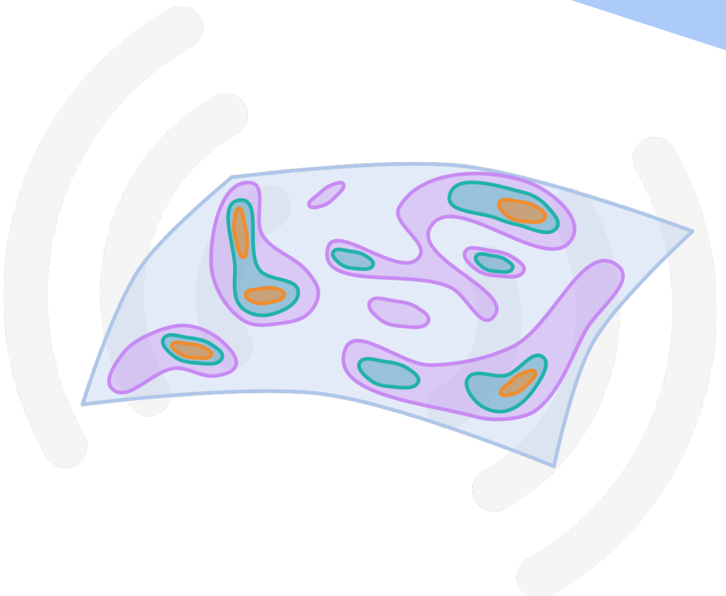
CLEANING the Intensity Field

- Iterative **deconvolution algorithm**: locate strongest source, removes contribution of the dirty beam at that location, iterate with next strongest source.

- **Suboptimal, computationally expensive, nonlinear, no convergence result…**



Random Fields



- Continuous **spatial** random fields

$$\mathcal{S} = \{S(\boldsymbol{r}) : \Omega \to \mathbb{C},\ \boldsymbol{r} \in \mathcal{X}\},$$

with $(\Omega, \mathcal{F}, \mathbb{P})$ some probability space.

- **Gaussian** random fields are such that

$$(S(\boldsymbol{r}_1), \ldots, S(\boldsymbol{r}_n)) \overset{d}{\sim} \mathcal{N}_n, \quad \forall \boldsymbol{r}_1, \ldots, \boldsymbol{r}_n \in \mathcal{X}, \forall n \in \mathbb{N}.$$

- When centered, fully characterized by their **second order moments**
  - Intensity field: $I(\boldsymbol{r}) = \mathbb{E}\left[S(\boldsymbol{r})S^*(\boldsymbol{r})\right], \quad \forall \boldsymbol{r} \in \mathcal{X},$
  - Covariance function: $\kappa(\boldsymbol{r}, \boldsymbol{s}) = \mathbb{E}\left[S(\boldsymbol{r})S^*(\boldsymbol{s})\right], \quad \forall (\boldsymbol{r}, \boldsymbol{s}) \in \mathcal{X}^2.$

Point Source Random Fields

- Consider Gaussian random fields of the form:

$$S(\boldsymbol{r}) = \sum_{q=1}^{Q} \xi_q \, \delta(\boldsymbol{r} - \boldsymbol{r}_q), \quad \forall \boldsymbol{r} \in \mathcal{X},$$

where $\{\boldsymbol{r}_q, \; q = 1, \ldots, Q\} \subset \mathcal{X}$.

- The random amplitudes are such that

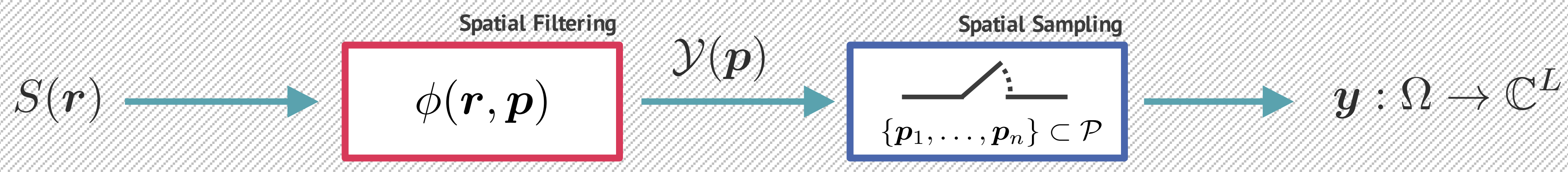
$$\xi_q \overset{i.i.d}{\sim} \mathbb{C}\mathcal{N}(0, \sigma_q^2).$$

- We have

$$I(\boldsymbol{r}) = \sum_{q=1}^{Q} \sigma_q^2 \delta(\boldsymbol{r} - \boldsymbol{r}_q), \qquad \kappa(\boldsymbol{r}, \boldsymbol{s}) = \begin{cases} I(\boldsymbol{r}), & \text{if } \boldsymbol{r} = \boldsymbol{s}, \\ 0, & \text{otherwise.} \end{cases}$$

Acquisition System

Sampling Operator β^*

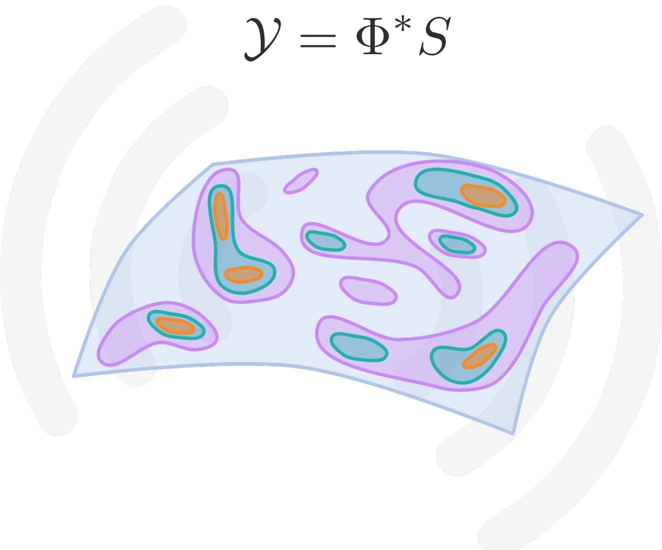


Acquisition System

Sampling Operator β^*

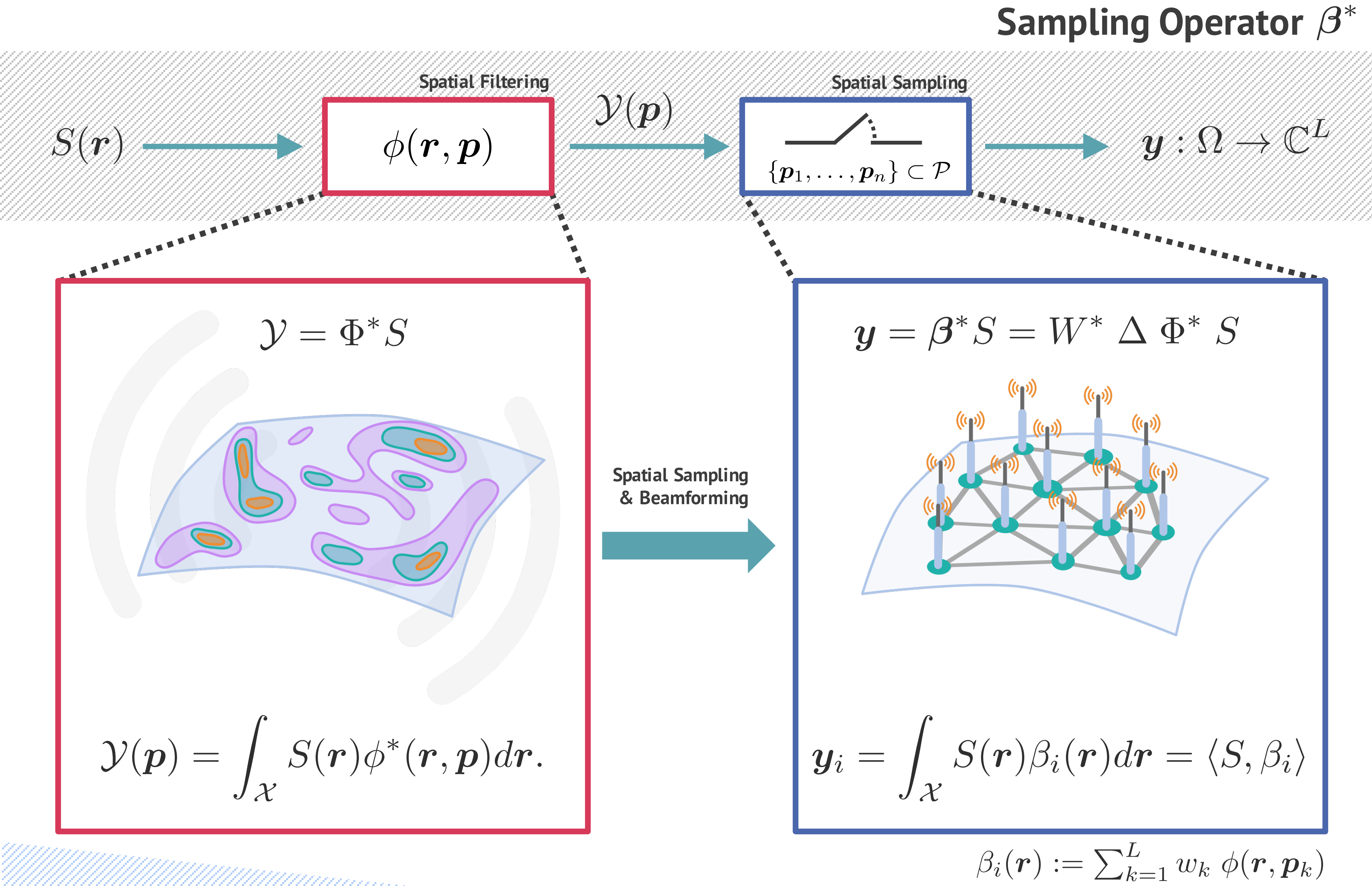
$S(\boldsymbol{r}) \rightarrow$ **Spatial Filtering** $\phi(\boldsymbol{r}, \boldsymbol{p})$ $\xrightarrow{\mathcal{Y}(\boldsymbol{p})}$ **Spatial Sampling** $\{\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n\} \subset \mathcal{P}$ $\rightarrow$ $\boldsymbol{y} : \Omega \rightarrow \mathbb{C}^L$

$$\mathcal{Y} = \Phi^* S$$



$$\mathcal{Y}(\boldsymbol{p}) = \int_{\mathcal{X}} S(\boldsymbol{r}) \phi^*(\boldsymbol{r}, \boldsymbol{p}) d\boldsymbol{r}.$$

Acquisition System



Least-Squares Estimate (Discrete)

- We want to solve the **inverse problem**

$$\boxed{\boldsymbol{y}} = \boldsymbol{\beta}^* \boxed{S},$$

with $\boldsymbol{y} : \Omega \to \mathbb{C}^L$, $\boldsymbol{\beta}^* : \boxed{\mathcal{H}} \to \boxed{\mathbb{C}^L}$, $S : \Omega \times \mathcal{H} \to \mathbb{C}$.

- **Discrete** methods:

$$\underbrace{\begin{bmatrix} \beta_1^*(\boldsymbol{r}_1) & \cdots & \beta_1^*(\boldsymbol{r}_N) \\ \vdots & \ddots & \vdots \\ \beta_L^*(\boldsymbol{r}_1) & \cdots & \beta_L^*(\boldsymbol{r}_N) \end{bmatrix}}_{B^H \in \mathbb{C}^{L \times N}} \underbrace{\begin{bmatrix} S(\boldsymbol{r}_1) \\ \vdots \\ S(\boldsymbol{r}_N) \end{bmatrix}}_{\boldsymbol{s} \in \mathbb{R}^N} = \underbrace{\begin{bmatrix} y_1 \\ \vdots \\ y_L \end{bmatrix}}_{\boldsymbol{y} \in \mathbb{C}^L}$$

- Least-squares estimate given by **pseudo-inverse**

$$\hat{\boldsymbol{s}} = B \boxed{(B^H B)^{-1}} \boldsymbol{y}$$

UNSTABLE
EXPENSIVE
DISCRETE

regularization needed!!

Least-Squares Estimate (Continuous)

- Why discretizing? Pseudo-inverse of operators also exist! (see **MFSP**)
- **Continuous** least-squares estimate given by

$$\hat{S}(\boldsymbol{r}) = \boldsymbol{\beta}\,(\boldsymbol{\beta}^*\boldsymbol{\beta})^{-1}\,\boldsymbol{y} = \sum_{i=1}^{L} \tilde{y}_i\,\beta_i(\boldsymbol{r}), \quad \forall \boldsymbol{r} \in \mathcal{X}.$$

$\boldsymbol{\kappa} \simeq \mathbf{2}$ **STABLE IN PRACTICE (LOFAR LAYOUT)**

- Numerical stability depends on **sampling strategy and beamforming**.
- Analytical computation possible in radio astronomy

$$(\boldsymbol{\beta}^*\boldsymbol{\beta})_{ij} = \sum_{l,k=1}^{L} w_l^{(i)} w_k^{(j)*} \left(\int_{\mathbb{S}^2} e^{2\pi j \langle \boldsymbol{r}, \boldsymbol{p}_l^{(i)} - \boldsymbol{p}_k^{(j)} \rangle} d\boldsymbol{r}\right), \qquad \int_{\mathbb{S}^2} e^{2\pi j \langle \boldsymbol{r}, \boldsymbol{p} \rangle} d\boldsymbol{r} = 4\pi \mathrm{sinc}(2\pi \|\boldsymbol{p}\|)$$

- Inversion of the Gram matrix **efficiently performed** using *Cholesky factorization, backward and forward substituion.*

Estimation of the Second Order Moments

- The covariance function of the estimated random field is given by

$$\hat{\kappa} = \boldsymbol{\beta}(\boldsymbol{\beta}^*\boldsymbol{\beta})^{-1}\Sigma(\boldsymbol{\beta}^*\boldsymbol{\beta})^{-1}\boldsymbol{\beta}^*,$$

where $\Sigma = \mathbb{E}[\boldsymbol{y}\boldsymbol{y}^H]$.

- Make sense of this equation using **tensor products**:

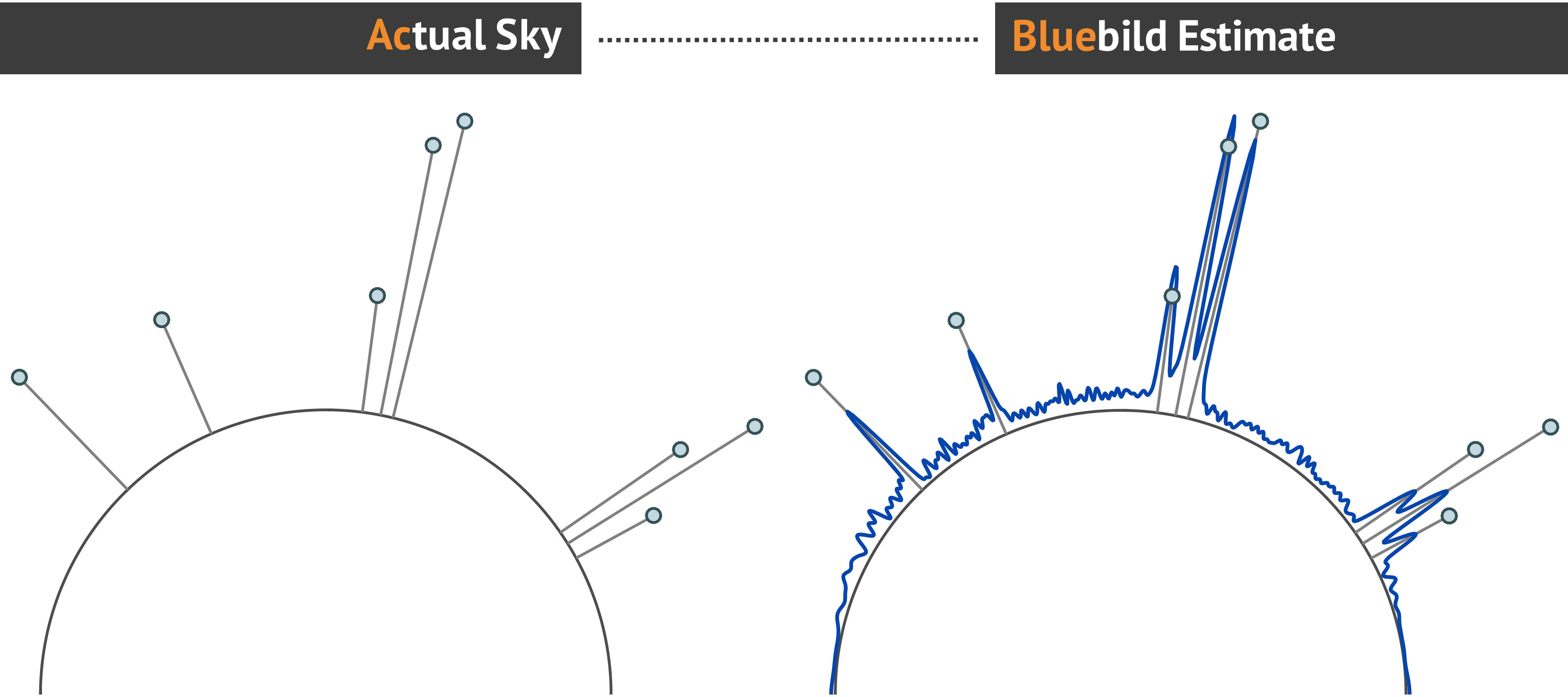
$$\mathrm{vec}(\hat{\kappa}) = \underbrace{(\overline{\boldsymbol{\beta}} \otimes \boldsymbol{\beta})}_{\text{SYNTHESIS}} \underbrace{\left[(\overline{\boldsymbol{\beta}^*\boldsymbol{\beta}})^{-1} \otimes (\boldsymbol{\beta}^*\boldsymbol{\beta})^{-1}\right] \mathrm{vec}(\Sigma)}_{\text{GRAM CORRECTION ON DATA}}.$$

- We finally get

$$\hat{\kappa}(\boldsymbol{r}, \boldsymbol{s}) = \sum_{i,j=1}^{L} \tilde{\sigma}_{ij}\ \beta_i(\boldsymbol{r})\beta_j^*(\boldsymbol{s}), \quad \forall (\boldsymbol{r}, \boldsymbol{s}) \in \mathcal{X}^2,$$

$$\hat{I}(\boldsymbol{r}) = \sum_{i,j=1}^{L} \tilde{\sigma}_{ij}\ \beta_i(\boldsymbol{r})\beta_j^*(\boldsymbol{r}), \quad \forall \boldsymbol{r} \in \mathcal{X}.$$

Example: Point Sources on Circle



> MULTI-RESOLUTION DEMO <

Deconvolution of the Random Field

- Link with the true random field ?

$$\hat{S} = \boldsymbol{\beta}(\boldsymbol{\beta}^*\boldsymbol{\beta})^{-1}\boldsymbol{\beta}^* S.$$

- Orthogonal projection on $\mathcal{R}(\boldsymbol{\beta})$.

- The operator $\boldsymbol{\beta}(\boldsymbol{\beta}^*\boldsymbol{\beta})^{-1}\boldsymbol{\beta}^* : \mathcal{H} \rightarrow \mathcal{H}$ can be seen as a **convolution**

$$\hat{S}(\boldsymbol{r}) = \sum_{i=1}^{L}\left(\int_{\mathcal{S}^2} S(\boldsymbol{s})\beta_i^*(\boldsymbol{s})d\boldsymbol{s}\right)\tilde{\beta}_i(\boldsymbol{r}) = \int_{\mathcal{S}^2} S(\boldsymbol{s})\underbrace{\left(\sum_{i=1}^{L}\beta_i^*(\boldsymbol{s})\tilde{\beta}_i(\boldsymbol{r})\right)}_{\zeta(\boldsymbol{s},\boldsymbol{r})} d\boldsymbol{s}$$

Point Spread Function

- Second order moment is given by

$$\hat{\kappa}(\boldsymbol{r},\boldsymbol{s}) = \int_{\mathcal{S}^2} I(\boldsymbol{u})\zeta(\boldsymbol{r},\boldsymbol{u})\zeta^*(\boldsymbol{s},\boldsymbol{u})d\boldsymbol{u}$$

Deconvolution of the Random Field

- For point source random field, the integral **simplifies**

$$\hat{\kappa}(\boldsymbol{r}, \boldsymbol{s}) = \sum_{q=1}^{Q} \sigma_q^2 \zeta(\boldsymbol{r}, \boldsymbol{r}_q)\zeta^*(\boldsymbol{s}, \boldsymbol{r}_q).$$

- Assume sources' locations **known**, and sample the covariance function

$$\hat{K} = Z\mathrm{diag}(\boldsymbol{\sigma})Z^*,$$
$$\Leftrightarrow \ \mathrm{vech}(\hat{K}) = (\overline{Z} \circ Z)\boldsymbol{\sigma} \quad (+\boldsymbol{\epsilon})$$

- Solve with **weighted least-squares**

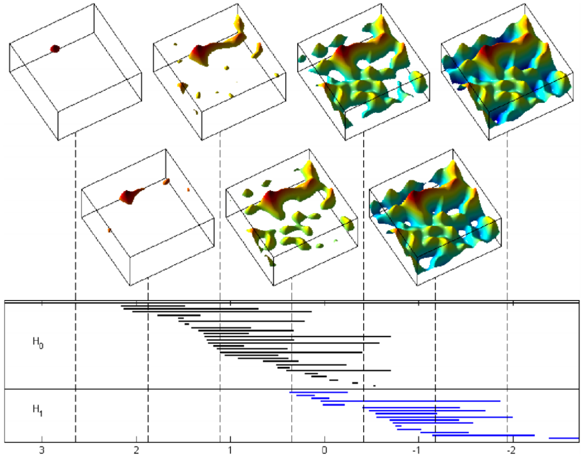
$$\hat{\boldsymbol{\sigma}} = \mathrm{argmin}_{\boldsymbol{\sigma} \in \mathbb{R}^Q} \left\| W^{\frac{1}{2}}(\hat{\boldsymbol{\kappa}} - (\overline{Z} \circ Z)\boldsymbol{\sigma}) \right\|_2^2$$
$$= \left[(\overline{Z} \circ Z)^* W (\overline{Z} \circ Z)\right]^{-1} (\overline{Z} \circ Z)^* W^{\frac{*}{2}} \hat{\boldsymbol{\kappa}}.$$

How to Find Sources Locations ?

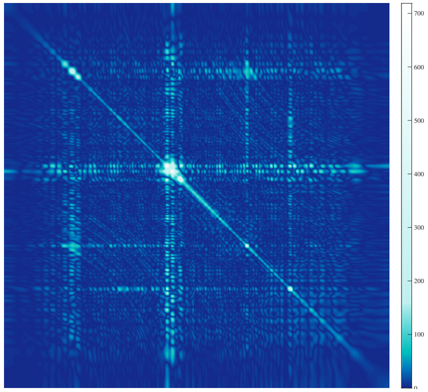
- Brute force: **oversample** the space on a fine grid.
- Computationally intensive, leads to **severe ill-conditioning**…
- Exploit available information !

Intensity Field Estimate

- Local maxima… Too many !
- **Persistent Homology**



Covariance Function Estimate



- **Eigenvectors** inspection

$$f_i = \Phi \alpha_i$$

- **Graph** based methods:
  - Artifacts **clustering**
  - Graph **filtering**

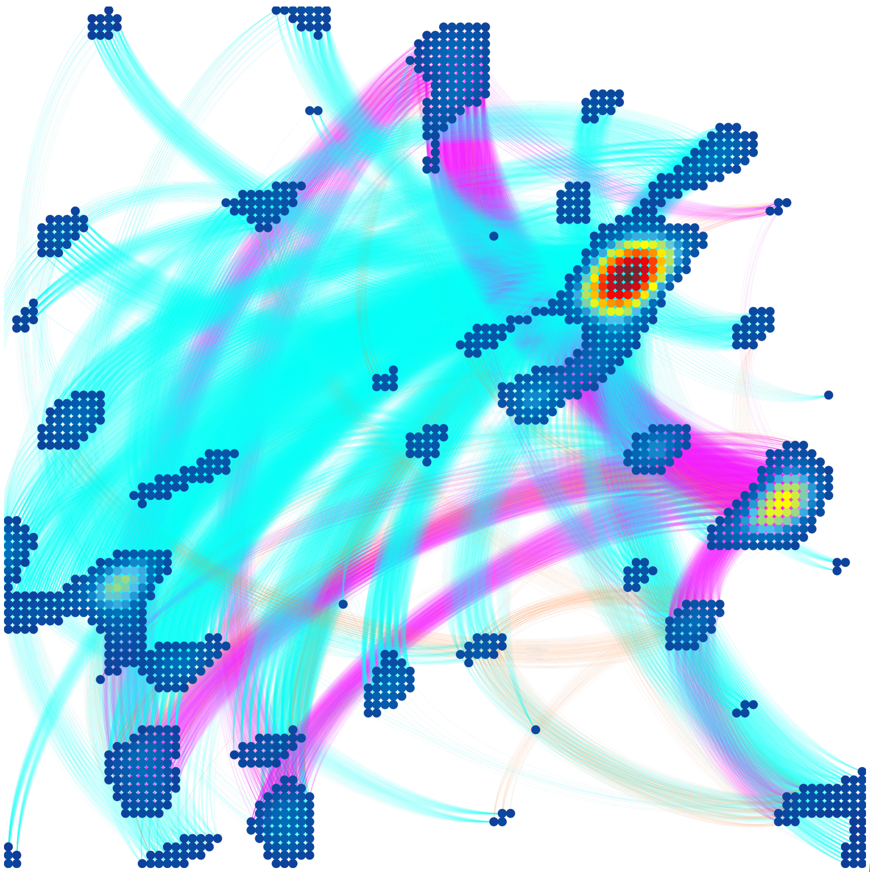
Graph Based Methods

- Define the **affinity matrix** as the absolute value of covariance matrix.
- Interpret points in the field as **nodes** on a graph. Edges are weighted according to the covariance matrix.
- Define the **Laplacian** of the graph as

$$L = D - A.$$

- Perform spectral decomposition

$$L = U \Lambda U^T.$$

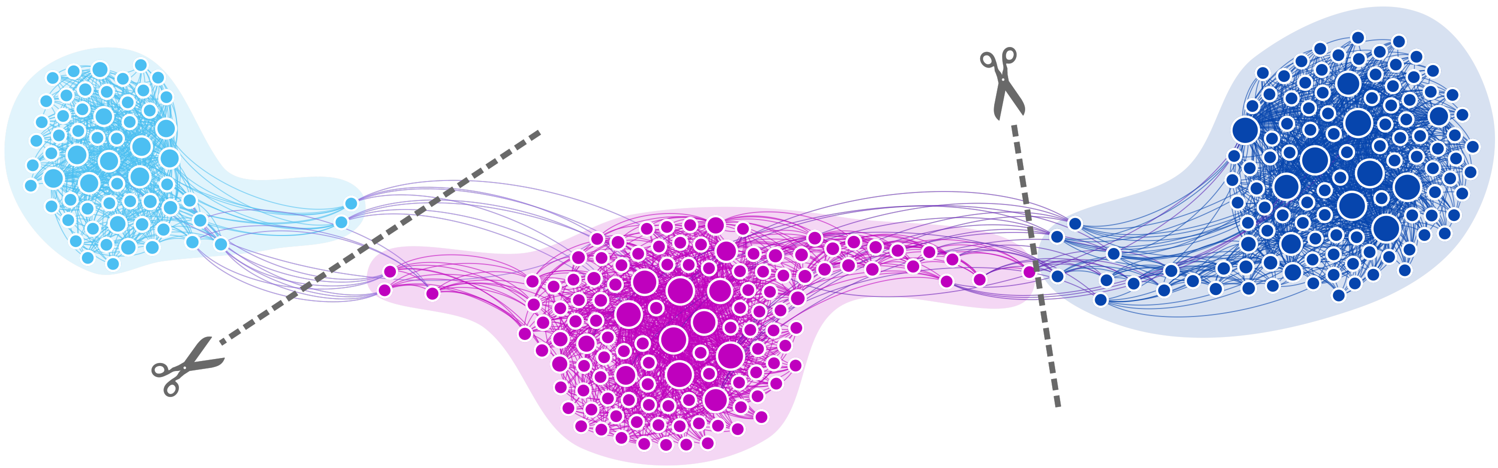


Spectral Clustering

- **Spectral clustering** is solving for

$$\{\tilde{A}_1, \ldots, \tilde{A}_K\} = \arg\min \left\{ \frac{1}{2} \sum_{i=1}^{K} \frac{W(A_i, \bar{A}_i)}{|A_i|} \quad \middle| \quad A_1, \ldots, A_K \in V, \cup_i A_i = V \right\}$$

- Use eigenvectors associated to K smallest eigenvalues as features.
- Perform K-means.

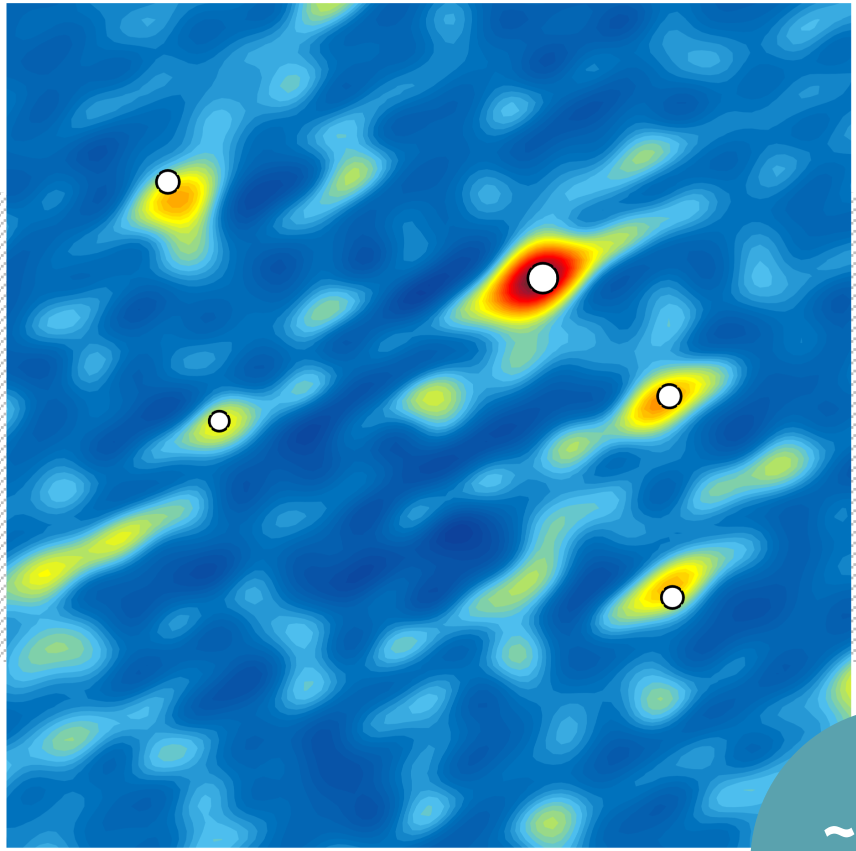


Application to Radio Astronomy

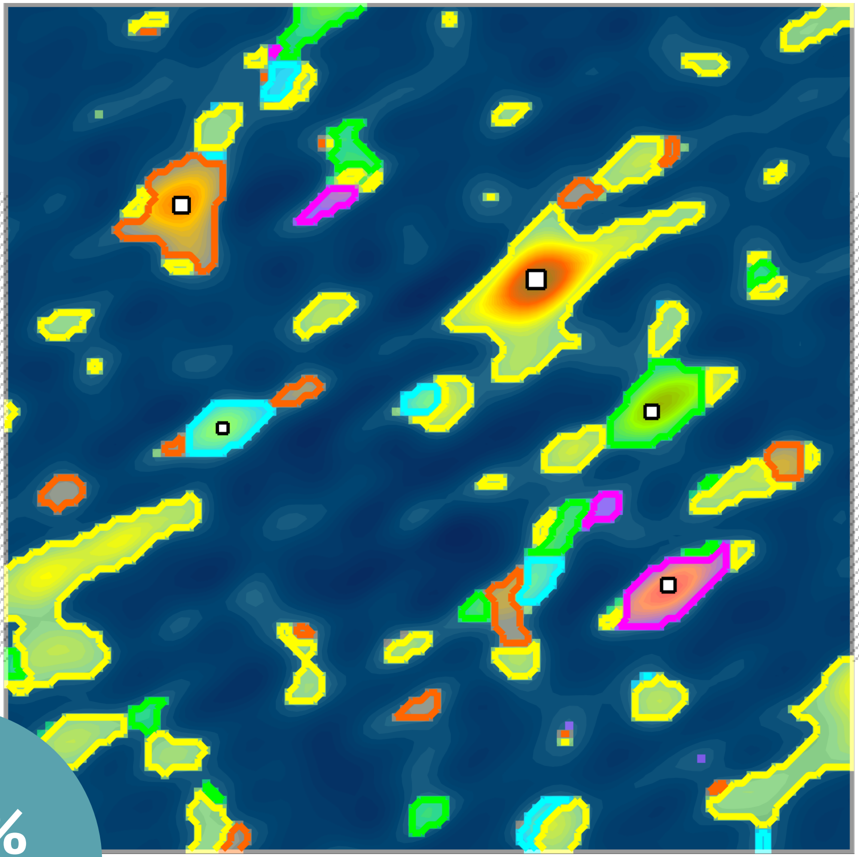
6 sources

PSNR=-23 dB

Dirty Image



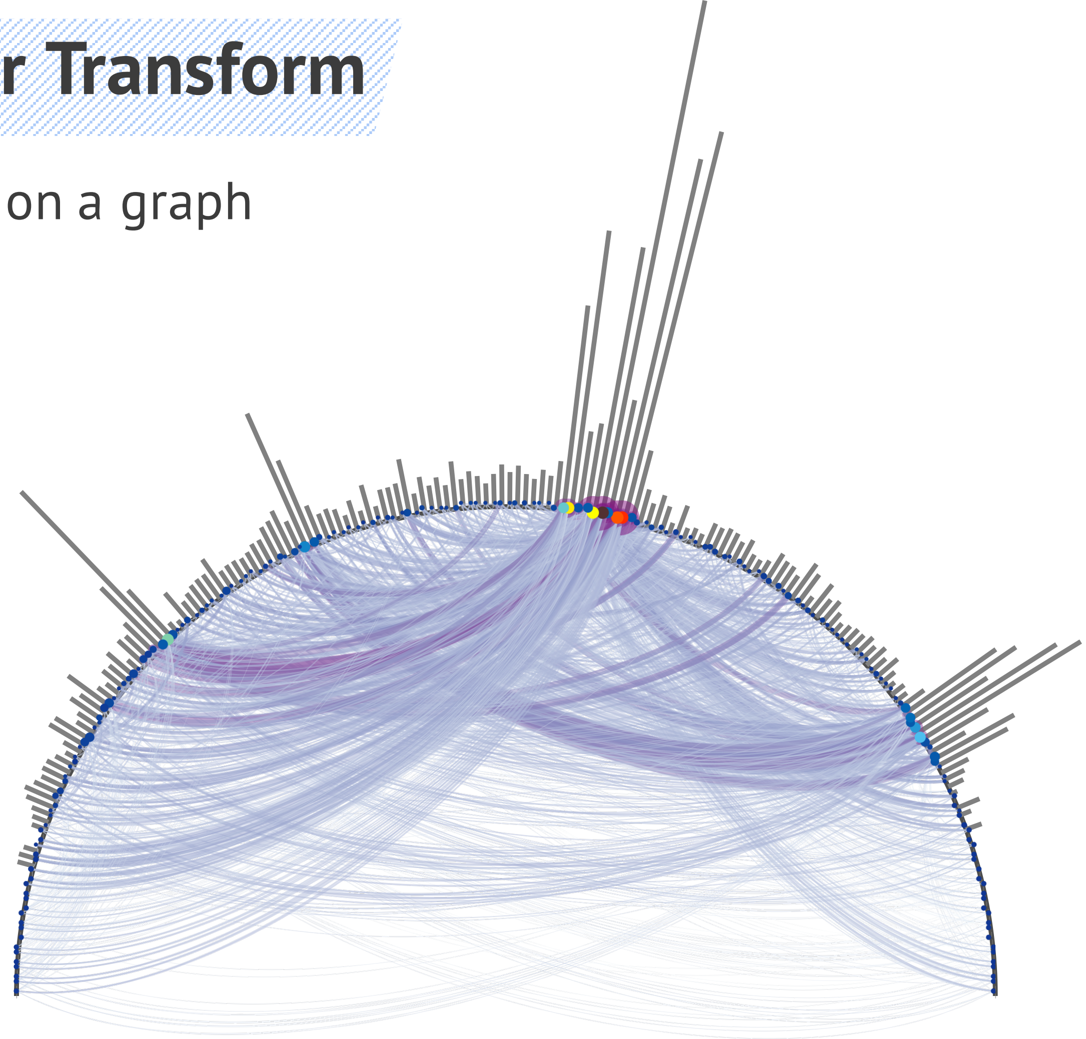
Result of Clustering



~87% accurate

Graph Fourier Transform

- Define signal on a graph

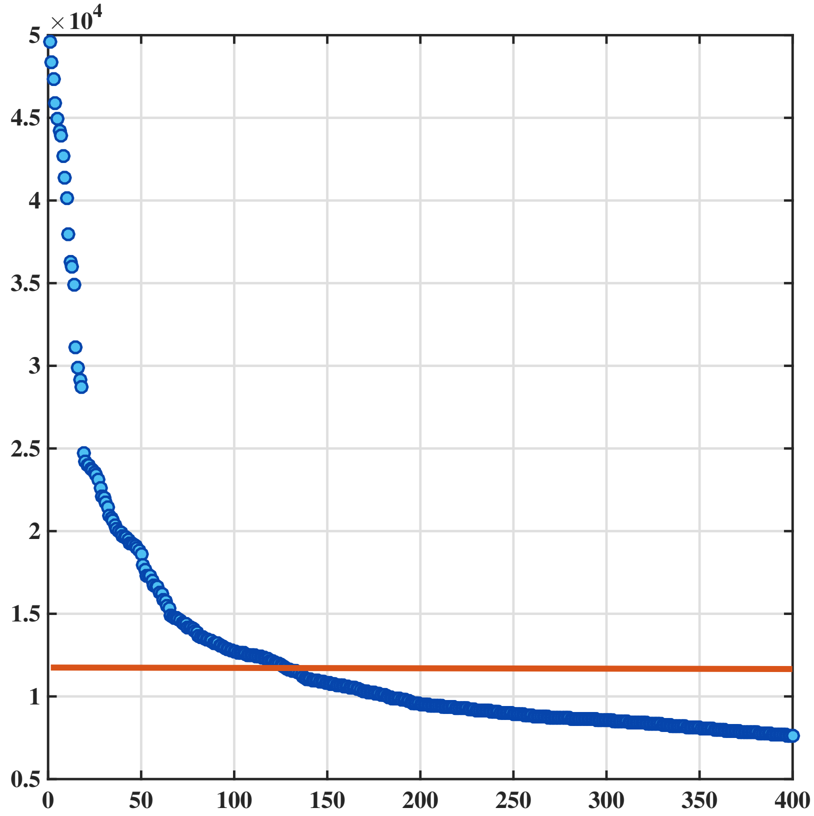


- Graph Fourier transform can be defined as

$$\hat{I} = U^T I, \qquad \text{where} \qquad L = U \Lambda U^T$$

Graph Fourier Transform

- High eigenvalues correspond to high frequency correspond to high frequencies, low eigenvalues to low frequencies

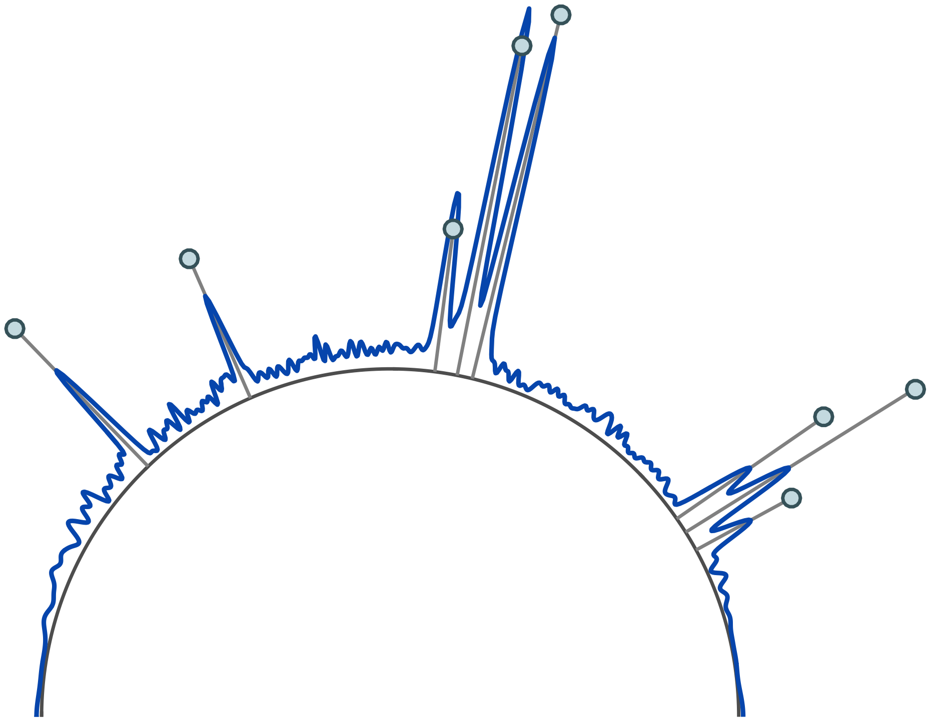


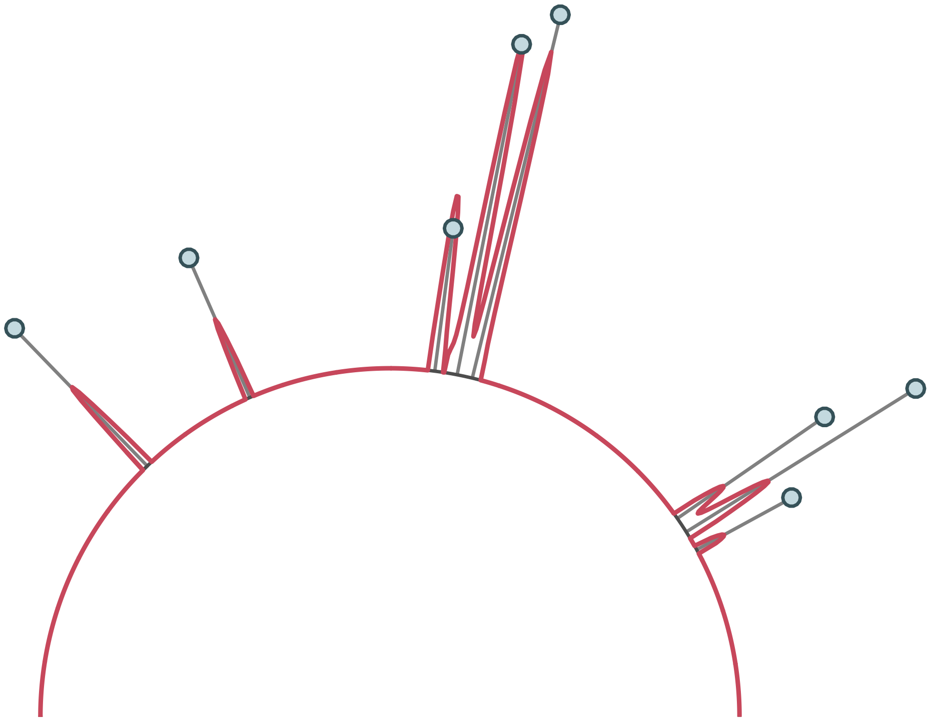
PERFORM HIGH-PASS FILTERING

$$\tilde{I} = \hat{I} \times f$$

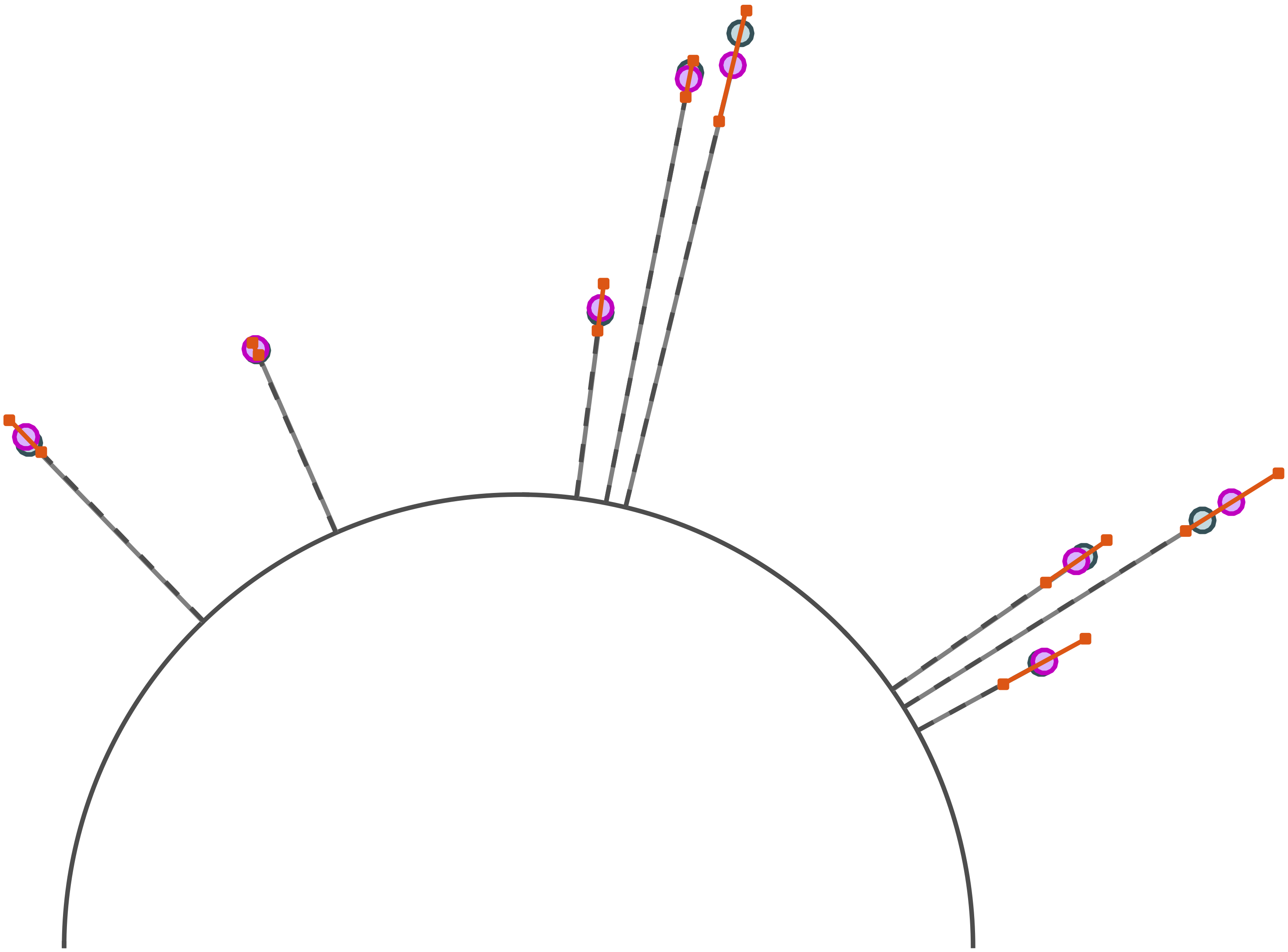
Example: Graph Fourier Filtering

Before Filtering .... **After Filtering**





Example: After Deconvolution



Deconvolution of Arbitrary Random Field

- What about aribitrary random fields? Integral does not simplify.
- Consider an operator formulation

$$\hat{\kappa}(\boldsymbol{r}, \boldsymbol{s}) = \int_{\mathcal{S}^2} I(\boldsymbol{u}) \zeta(\boldsymbol{r}, \boldsymbol{u}) \zeta^*(\boldsymbol{s}, \boldsymbol{u}) d\boldsymbol{u}$$
$$\Leftrightarrow \hat{\kappa} = \mathcal{M}^* \kappa.$$

- Solution given by

$$\hat{\hat{\kappa}} = \mathcal{M}(\mathcal{M}^* \mathcal{M})^{-1})\hat{k}.$$

- In practice, sample one side to compute the inverse

$$\hat{K} = \Delta \mathcal{M}^* \kappa, \qquad \hat{\hat{\kappa}} = \mathcal{M} \Delta (\Delta \mathcal{M}^* \mathcal{M} \Delta)^{-1})\hat{K}.$$

Conclusion and future work

- We have proposed a new interpolation algorithm to produce continuous least-squares estimate of the random field
- Orders of magnitude faster !
- Aliasing artifacts remain.
- Deconvolution methods for point source random fields have been proposed
- Graph-based methods are very successful
- Deconvolution for arbitrary random field needs to be address (at the continuous level)