Spectral Methods for Hyperbolic Problems

J. S. Hesthaven

EPFL-SB-MATHICSE-MCSS, Ecole Polytechnique Fédérale de Lausanne
CH-1015 Lausanne, Switzerland

E-mail: Jan.Hesthaven@epfl.ch

We review spectral methods for the solution of hyperbolic problems. To keep the discussion concise, we focus on Fourier spectral methods and address key issues of accuracy, stability, and convergence of the numerical approximations. Polynomial methods are discussed when these lead to qualitatively different schemes as, for instance, when boundary conditions are required. The discussion includes nonlinear stability and the use of filters and post-processing techniques to minimize or overcome the Gibbs phenomenon.

Key Words: Spectral, Pseudospectral, Galerkin, Collocation, Penalty methods, Discontinuous solutions, Gibbs phenomenon, Stability, Filtering, Vanishing viscosity, Skew-Symmetric form

1. INTRODUCTION

The theory and application of spectral methods for the solution of partial differential equations has traditionally focused on problems with a certain amount of inherent regularity of the solutions, e.g., elliptic/parabolic problems. The application that is perhaps most responsible for the widespread use of spectral methods is the incompressible Navier-Stokes equations [8, 11, 52].

At the heart of a spectral method lies the assumption that the solution, $u(x,t)$, to a partial differential equation can be expressed by a series of smooth basis functions as

$$u(x,t) \simeq u_h(x,t) = \sum_{n=0}^{N} \hat{u}_n(t)\phi_n(x) \quad . \tag{1}$$

The choice of the basis $\phi_n(x)$ and the way in which the expansion coefficients $\hat{u}_n(t)$ are computed results in different methods. Let us first assume that $\phi_n(x) : [a,b] \to \mathsf{R}$ is orthogonal in L^2_w such that

$$\hat{u}_n(t) = \frac{1}{\gamma_n} (u, \phi_n)_w \,, \tag{2}$$

where

$$(f,g)_w = \int_a^b f(x)g(x)w(x)\,dx, \quad \gamma_n = (\phi_n, \phi_n)_w \,, \tag{3}$$

and $w(x)$ is an L^1-integrable weight function. This defines the truncated projection of the function $u(x,t)$ as

$$\mathcal{P}_N u(x,t) = \sum_{n=0}^{N} \hat{u}_n(t)\phi_n(x) \quad . \tag{4}$$

To understand the accuracy of this truncated expansion, consider

$$\|u(x,t) - \mathcal{P}_N u(x,t)\|_w^2 = \sum_{n=N+1}^{\infty} \gamma_n \hat{u}_n^2,$$

i.e., the accuracy depends solely on the decay of the expansion coefficients. To understand their behavior, assume that the basis satisfies

$$\left[\frac{d}{dx}q(x)\frac{d}{dx} + \lambda_n w(x)\right]\phi_n(x) = \left[\mathcal{L} + \lambda_n w(x)\right]\phi_n(x) = 0, \quad x \in [a,b]. \tag{5}$$

In the simplest case of $q(x) = w(x) = 1, x \in [0, 2\pi]$, the trigonometric functions, $\phi_n(x) = \exp(inx)$ satisfy this with $\lambda_n = n^2$. For the more general case of $p(x)$ and $w(x)$ with $x \in [-1,1]$, we recover all the classic orthogonal polynomials provided $q(\pm 1) = 0$ [55]. In this case, (5) is the singular Sturm-Liouville problem with $\lambda_n \propto n^2$. Prominent examples of these polynomial basis functions include Legendre and Chebyshev polynomials.

Under the assumption of (5), integration by parts of (2) yields

$$\hat{u}_n = \frac{1}{\gamma_n}(u, \phi_n)_w = \frac{-1}{\gamma_n \lambda_n}\left(\left[uq\phi_n' - u'q\phi_n\right]_a^b + \left(\frac{\mathcal{L}}{w}u, \phi_n\right)_w\right).$$

If we now further assume that the solution u and the basis ϕ_n is periodic in $[a,b]$, as for the trigonometric basis, or $q(a) = q(b) = 0$ as for the polynomial basis, we recover

$$\hat{u}_n = \frac{-1}{\gamma_n \lambda_n}\left(\frac{\mathcal{L}}{w}u, \phi_n\right)_w .$$

Under the assumption that u is sufficiently smooth and periodic, repeating this p times yields

$$\hat{u}_n = \frac{1}{\gamma_n}\left(\frac{-1}{\lambda_n}\right)^p \left(\left(\frac{\mathcal{L}}{w}\right)^p u, \phi_n\right)_w .$$

This we may now bound as

$$|\hat{u}_n| \leq \frac{1}{\gamma_n \lambda_n^p}\|u^{(2p)}\|_w \leq \frac{C}{\gamma_n n^{2p}}\|u^{(2p)}\|_w.$$

This highlights the direct connection between the regularity of the solution u and the decay of the expansion coefficients. This yields the estimate

$$\|u(x,t) - \mathcal{P}_N u(x,t)\|_w^2 \leq C N^{-p}\|u^{(p)}\|_w. \tag{6}$$

In the event that $u(x,t)$ is analytic we recover the remarkable property [57]

$$\|u - \mathcal{P}_N u\|_w \leq C N^{-p}\|u^{(p)}\|_w \sim C\frac{p!}{N^p}\|u\|_w \sim Ce^{-cN}\|u\|_w \ ,$$

known as spectral accuracy or spectral convergence. This is the property that gives name to spectral methods.

The use of spectral methods for the solution of hyperbolic problems has traditionally been viewed as problematic and only more recently have such methods seen a wider use. The reasons for the perceived difficulty are several. Contrary to parabolic and elliptic problems, there is no physical dissipation inherent in the hyperbolic problem. This implies that even minor errors and under resolved phenomena can cause the scheme to become unstable.

Perhaps the most important reason, however, for the slow acceptance of spectral methods for solving hyperbolic conservation laws is the appearance of the Gibbs phenomenon as finite time discontinuities develop in the solution. Left alone, the nonlinear mixing of the Gibbs oscillations with the approximate solution will eventually cause the scheme to become unstable. Moreover, even if stability is maintained for sufficiently long time, the computed solution appears to be only first order accurate in which case the use of a high-order method is questionable. More fundamental issues of conservation and the ability of the scheme to compute the correct entropy solution to conservation laws have also caused considerable concern among practitioners and theoreticians alike.

While many of these issues are genuine and require careful attention, they do not cause the spectral methods to fail if applied correctly. This was indicated already to in early work around 1980 [49, 44, 25] where the first numerical solution of problems with discontinuous solutions and general nonlinear conservation laws were presented.

To understand the potential of spectral methods for solving conservation laws problems, we need to dig deeper into the development and analysis of these methods. To keep the discussion brief we focus on Fourier spectral methods and discuss key developments in this context. However, when appropriate, we revisit qualitative differences induced by the use of a polynomial basis. For further details, in particular for polynomials methods and more complex applications, we refer available texts and reviews, e.g., [3, 2, 18, 63, 5, 24, 53, 42]

The remainder of this chapter is organized as follows. Next, we revisit the spectral expansion and different ways to express this. We also outline the key approximation results of the continuous and discrete expansions for smooth and non-smooth functions. In the subsequent Section 3 we introduce Fourier spectral methods and discuss their stability for linear problems. We shall also discuss polynomial methods and techniques for the imposing general boundary conditions, leading to additional complications. In Section 4 we return to the Fourier spectral methods, now with a focus on nonlinear problems and discuss stability and convergence for such problems. Section 5 discusses ways to overcome the impact of the Gibbs phenomenon on the global accuracy. Throughout the discussion we strive to include sufficient references to allow the reader to pursue more advanced topics.

2. THE SPECTRAL EXPANSION

We focus on spectral methods based on the Fourier expansion

$$\mathcal{P}_N u(x) = \sum_{n=-N}^{N} \hat{u}_n \exp(inx) \quad . \tag{7}$$

Here and in the following we suppress the explicit time-dependency of $u(x,t)$ for simplicity.

The expansion coefficients are obtained directly as

$$\hat{u}_n = \frac{1}{2\pi}\,(u, \exp(inx)) = \frac{1}{2\pi}\int_0^{2\pi} u(x)\exp(-inx)\,dx \quad , \tag{8}$$

through the orthogonality of the basis in the inner product

$$(f,g) = \int_0^{2\pi} f\overline{g}\,dx \quad , \quad \|f\|^2 = \int_0^{2\pi} |f|^2\,dx \quad ,$$

with the associated norm, $\|\cdot\|$.

Once the expansion is known, we can evaluare spatial derivatives of the function as

$$\frac{d^p u(x)}{dx^p} \simeq \frac{d^p \mathcal{P}_N u(x)}{dx^p} = \sum_{n=-N}^{N} (in)^p \hat{u}_n \phi_n(x) = \sum_{n=-N}^{N} \hat{u}_n^{(p)} \phi_n(x) \quad ,$$

i.e., $\hat{u}_n^{(p)} = (in)^p \hat{u}_n$, for the approximation of an arbitrary derivative of a function, given by its Fourier coefficients.

2.1. Smooth problems

We have already discussed the close connection between regularity of the function and accuracy of the truncated Fourier expansion. While the algebra involved is quantitatively different when a different basis and norm is used, the results for a basis comprising classic orthogonal polynomials is qualitative the same as in (6), i.e., there is a direct relationship between the accuracy of the spectral expansion and the regularity of the function being approximated.

Results similar to (6) are also available in higher norms. For the Fourier series we have [38]

$$\|u - \mathcal{P}_N u\|_{W^p} \leq C(p,q) N^{p-q} \|u\|_{W^q} \quad , \tag{9}$$

provided only that $0 \leq p \leq q$. Here, we have the Sobolev norm

$$\|u\|^2_{W^q} = \sum_{s=0}^{q} \|u^{(s)}\|^2 \quad ,$$

to measure the error on the derivatives. The results for the classic polynomials are qualitatively similar [6, 3].

Results for pointwise assuracy are harder to obtain. For the truncated Fourier series one recovers [8]

$$\|u - \mathcal{P}_N u\|_{L^\infty} \leq C(q)(1 + \log N) N^{-q} \|u^{(q)}\|_{L^\infty} \quad ,$$

where the L^∞-norm measures the maximum pointwise error.This indicates that we expect a poor pointwise accuracy for problems with low regularity. This happens both locally, where convergence is lost at discontinuous point, and in the entire domain containing the discontinuity due to the Gibbs phenomenon as discussed in more detail in Section 2.2.

The computation of the Fourier coefficients, $\hat{u}_n$, poses a problem as one cannot in general evaluate the inner product. The natural solution is to introduce a quadrature approximation to (8) on the form

$$\tilde{u}_n = \frac{1}{2N+1} \sum_{j=0}^{2N} u(x_j) \exp(-inx_j) \quad . \tag{10}$$

We recognize this as the trapezoidal rule with the equidistant grid

$$x_j = \frac{2\pi}{2N+1} j, \quad j = 0, \ldots, 2N. \tag{11}$$

This is known as the odd method, due to the odd number of grid points. Historically, an even number of points have been preferred, leading to minor quantitative differences but no qualitative differences. We refer to [38] for through discussion of this.

As N in (10) increases one hopes that $\tilde{u}_n$ is a good approximation to $\hat{u}_n$. To quantify this, we can express $\tilde{u}_n$ using $\hat{u}_n$ as

$$\tilde{u}_n = \hat{u}_n + \sum_{\substack{m=-\infty \\ m \neq 0}}^{m=\infty} \hat{u}_{n+2Nm} \quad ,$$

where the second term is termed the aliasing error. The aliasing error reflects that certain basis components cannot be distinguished on a finite grid, causing highly oscillatory components to be misinterpreted as slowly varying basis components. While the analysis is more complex in the polynomial case, the introduction of the aliasing error by the grid remains qualitatively the same [6, 3].

Understanding the accuracy of the discrete expansion thus reduces to an analysis of the error caused by the aliasing error. For the Fourier basis, the analysis in[44] shows that the aliasing error and the truncation error is of the same order, i.e., the result in (6) carries over to the case of interpolation in the Fourier case. This is likewise the case for the general result (9).

Since the use of the modal expansions requires the introduction of a finite grid one could question the need to consider special basis functions at all. Indeed, given a specific nodal set, x_j, we can construct a global interpolation

$$\mathcal{I}_N u(x) = \sum_{j=0}^{2N} u(x_j) l_j(x) \quad ,$$

where the Lagrange interpolating polynomials, $l_j(x)$, takes the form

$$l_j(x) = \frac{q(x)}{(x - x_j) q'(x_j)} \quad , \quad q(x) = \prod_{j=0}^{2N} (x - x_j) \quad .$$

Clearly, if the x_j's are distinct, $l_j(x)$ is uniquely determined as the polynomial of order $2N$, specified at $2N+1$ points. We can directly explore this to approximate derivatives of $u(x)$. In particular, if we restrict attention to the approximation of the derivative of $u(x)$ at the grid points, x_j, we have

$$\left.\frac{du}{dx}\right|_{x_i} \simeq \left.\frac{d\mathcal{I}_N u}{dx}\right|_{x_i} = \sum_{j=0}^{2N} u(x_j) \left.\frac{dl_j}{dx}\right|_{x_i} = \sum_{j=0}^{2N} u(x_j) \mathrm{D}_{ij} \quad ,$$

where D_{ij} is a differentiation matrix with the entries

$$\mathrm{D}_{ij} = \begin{cases} \frac{(-1)^{i+j}}{2} \left[\sin\left(\frac{\pi}{2N+1} (i-j) \right) \right]^{-1} & i \neq j \\ 0 & i = j \end{cases} \quad . \qquad (12)$$

The global nature of the interpolation implies that the differentiation matrix is full. We also note that D is skew-symmetric, a property that does not carry over to polynomial methods [38].

2.2. Non-smooth problems

If the solution possesses significant regularity we can expect the spectral expansion to be highly efficient as a representation of the solution and its spatial derivatives.

However, for problems with only limited regularity the picture is more complex and the above results do not inform us much about the accuracy of the approximation of such solutions. In particular, if the solution is only piecewise smooth only convergence in mean is ensured while the question of pointwise convergence remains open.

It is by now a classical result that the Fourier series, Eq.(7), in the neighborhood of a point of discontinuity, x_0, behaves as [27]

$$\mathcal{P}_N u\left(x_0+\frac{2z}{2N+1}\right) \sim \frac{1}{2}\left[u(x_0^+)+u(x_0^-)\right]+\frac{1}{\pi}\left[u(x_0^+)-u(x_0^-)\right]\mathrm{Si}(z) \ ,$$

where z is a constant and $\mathrm{Si}(z)$ signifies the Sine-integral. Away from the point of discontinuity, x_0, we recover linear pointwise convergence as $\mathrm{Si}(z) \simeq \frac{\pi}{2}$ for z large. Close to the point of discontinuity, however, we observe that for any fixed value of z, pointwise convergence is lost regardless of the value of N. This non-uniform convergence and loss of pointwise convergence is the celebrated Gibbs phenomenon and the oscillatory behavior of the Sine-integral is the familiar Gibbs oscillations.

As we shall discuss in more detail in Section 5, recent results allow us to dramatically improve on this situation and even completely overcome the Gibbs oscillations to recover an exponentially accurate approximation to a piecewise analytic function, represented by its global expansion.

2.3. The Duality between Modes and Nodes

While there is flexibility in the choice of the quadrature rules, used to compute the discrete expansion coefficients in the modal expansions, and similar freedom in choosing a nodal set on which to base the Lagrange interpolation polynomials, particular choices are awarded by insight.

Consider, as an example, the modal expansion, (7), with the expansion coefficients approximated as in (10). Inserting the latter directly into the former yields

$$\begin{aligned}
\mathcal{I}_N u(x) &= \sum_{n=-N}^{N}\left[\frac{1}{2N+1}\sum_{j=0}^{2N} u(x_j)\exp(-inx_j)\right]\exp(inx) \\
&= \sum_{j=0}^{2N} u(x_j)\left[\frac{1}{2N+1}\sum_{n=-N}^{N}\exp\left(in(x-x_j)\right)\right] \\
&= \sum_{j=0}^{2N} u(x_j)\frac{1}{2N+1}\frac{\sin\left(\frac{1}{2}(2N+1)(x-x_j)\right)}{\sin\left(\frac{1}{2}(x-x_j)\right)} = \sum_{j=0}^{2N} u(x_j)h_j(x) \ .
\end{aligned}$$

Hence, provided the expansion coefficients are approximated by the trapezoidal rule, (10), we recover the interpolation. This particular combination of grid points and quadrature rules results in two mathematically equivalent, but computationally very different, ways of expressing the interpolation and hence the computation of spatial derivatives.

A similar result can be recovered for the orthogonal polynomials provided Gauss quadrature nodes are used [8, 38].

3. SPECTRAL METHODS

Let us now turn the attention towards the solution of hyperbolic problems using spectral methods. Prominent examples of problems include Maxwells equations from electromagnetics, the Euler equations of gas-dynamics and the equations of elasticity. For the sake of simplicity we concentrate on methods for the scalar conservation law

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0 \ ,$$

subject to appropriate boundary and initial conditions. For this initial discussion, we focus on problems for which the solution remains smooth and return to the non-smooth case in Section 4.

We assume that the solution is given as

$$u(x,t) \simeq u_h(x,t) = \sum_{n=-N}^{N} \tilde{u}_n(t) \exp(inx),$$

where $\tilde{u}_n(t)$ represent the continuous or the discrete expansion coefficients. We can now define the residual

$$R_h(x,t) = \frac{\partial u_h}{\partial t} + \frac{\partial f(u_h)}{\partial x}.$$

Specifying exactly how this vanishes, hence stating in which sense u_h satisfies the conservation laws, gives rise to different families of methods with subtle differences.

3.1. Galerkin methods

In the Galerkin approach, we require that the residual is orthogonal to the space spanned by the basis functions. For the Fourier case, this results in the scheme

$$\frac{d\hat{u}_n}{dt} = -\frac{1}{2\pi}\int_0^{2\pi} R_h(x,t)\exp(-inx)\,dx.$$

This we can also express as

$$\frac{\partial u_h}{\partial t} + \mathcal{P}_N\left(\frac{\partial f(u_h)}{\partial x}\right) = 0,$$

subject to the initial conditions

$$u_h(x,0) = \mathcal{P}_N u(x,0).$$

One observes that boundary conditions must be reflected in the approximation itself, i.e., in the Galerkin method, each of the basis functions in (1) must satisfy the boundary conditions. For periodic problems, the Fourier series automatically enforces this. However, for non-trivial boundary conditions, this may present a challenge albeit successful schemes have be formulated [27, 53].

The stability of Galerkin schemes is closely related to the wellposedness of the conservation laws in the norm $\|\cdot\|$ [27]. The practical difficulty with the Galerkin scheme is the need to evaluate the projection of the general flux. While this may be possible for certain simple fluxes, e.g. linear or polynomial fluxes, it is clearly not possible for more general cases. In such a case, one can no longer express the scheme without the use of quadratures. However, this introduces a grid, induces aliasing and suggests that we consider collocation methods.

3.2. Collocation methods

To overcome the difficulties associated with exact evaluation of the inner products in the Galerkin method, we can change the statement on the residual. Let us define $2N+1$ distinct collocation points, y_j, and require that the residual vanishes in a pointwise sense

$$R_h(y_j,t) = \left(\frac{\partial u_h}{\partial t} + \frac{\partial f(u_h)}{\partial x} \right)\Bigg|_{y_j} = 0.$$

This results in $2N+1$ equations for the $2N+1$ unknowns. In principle, there are no restrictions on how y_j is chosen although the stability of the scheme is, to some extend, impacted by this [38]. If we make the most natural choice that the interpolation points x_j and the collocation points y_j are the same we recover the classic collocation scheme

$$\mathcal{I}_N \frac{\partial u_h}{\partial t} + \mathcal{I}_N \frac{\partial f(u_h)}{\partial x} = 0,$$

which can also conveniently be expressed as

$$\frac{d}{dt}\boldsymbol{u} + \mathrm{D}\boldsymbol{f} = 0,$$

where $\boldsymbol{u}$ and $\boldsymbol{f}$ represents vectors of the solution and the flux, respectively, evaluated at the grid points.

To understand the stability of collocation schemes for hyperbolic problems, let us consider the linear problem

$$\frac{\partial u}{\partial t} + a(x)\frac{\partial u}{\partial x} = 0 \quad , \tag{13}$$

where $a > 0$ implies a rightward propagating wave and $a < 0$ corresponds to a leftward propagating wave. The Fourier collocation approximation becomes

$$\frac{d}{dt}\boldsymbol{u} + \mathrm{AD}\boldsymbol{u} = 0 \quad , \tag{14}$$

where $\mathrm{A}_{ii} = a(x_i)$ is diagonal.

Define the discrete inner product and L^2-equivalent norm as

$$[f,g]_N = \frac{2\pi}{2N+1}\sum_{i=0}^{2N} f(x_i)g(x_i) \quad , \quad \|f\|_N^2 = [f,f]_N \quad .$$

If we initially assume that $|a(x)| > 0$ [50, 43, 17, 27, 51], it is easy to see that for $\boldsymbol{v} = \mathrm{A}^{-1/2}\boldsymbol{u}$, we recover

$$\frac{d}{dt}\boldsymbol{v} + \mathrm{A}^{1/2}\mathrm{D}\mathrm{A}^{1/2}\boldsymbol{v} = 0 \quad ,$$

such that

$$\frac{1}{2}\frac{d}{dt}\|v_h\|_N^2 = \frac{1}{2}\frac{d}{dt}\boldsymbol{u}^T \mathrm{A}^{-1}\boldsymbol{u} = 0 \ \ ,$$

since $\mathrm{A}^{1/2}\mathrm{D}\mathrm{A}^{1/2}$ is antisymmetric.

For the general case where $a(x)$ changes sign within the computational domain, the situation is more complex. The straightforward way to guarantee stability is to consider the skew-symmetric form [43, 58]

$$\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial a(x)u}{\partial x} + \frac{1}{2}a(x)\frac{\partial u}{\partial x} - \tfrac{1}{2}a_x(x)u(x) = 0 \ \ , \tag{15}$$

with the discrete form

$$\left.\frac{\partial u_h}{\partial t}\right|_{x_j} + \frac{1}{2}\left.\frac{\partial \mathcal{I}_N a(x)u_h}{\partial x}\right|_{x_j} + \frac{1}{2}a(x_j)\left.\frac{\partial u_h}{\partial x}\right|_{x_j} - \tfrac{1}{2}a_x(x_j)u_h(x_j) = 0 \ \ .$$

Stability follows since

$$\frac{1}{2}\frac{d}{dt}\|u_h\|_N^2 \le \tfrac{1}{2}\max_{x\in[0,2\pi]}|a_x(x)|\,\|u_h\|_N^2 \ \ .$$

The disadvantage of the skew-symmetric formulation is a doubling of the computational work.

The question of stability of the simple formulation, (14), for general $a(x)$ remained an open question for a long time, although partial results were known [28]. The difficulty in resolving this issue is associated with the development of very steep spatial gradients which, for a fixed resolution, eventually introduce significant aliasing that affect the stability. By carefully examining the interplay between aliasing, resolution, and stability, it was shown [23] that the Fourier approximation is only algebraically stable [27], i.e.,

$$\|u_h(t)\|_N \le C(t)N\|u_h(0)\|_N \ \ , \tag{16}$$

or weakly unstable. The weak aliasing driven instability spreads from the high modes through the aliasing and results in at most an $\mathcal{O}(N)$ amplification of the Fourier components of the solution. In other words, for well resolved computations where these aliasing components are very small the computation will appear stable for all practical purposes. Furthermore, in [24] it is shown that a weak amount of filtering suffices to control the instability. We return to this in more detail in Section 4.

3.3. Interlude on polynomial methods and boundary conditions

Let us now briefly consider the more general initial boundary value problem

$$\begin{aligned}&\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0 \ \ ,\\ &u(x,0) = g(t) \ \ ,\end{aligned} \tag{17}$$

posed on a finite domain which we take to be $[-1,1]$ without loss of generality. For the problem to be wellposed, we must specify boundary conditions on the form

$$\alpha u(-1,t) = f^-(t) \ , \ \beta u(1,t) = f^+(t) \ .$$

Specification of α and β is related to the fluxfunction, e.g., if

$$x\frac{\partial f}{\partial u} < 0 \ ,$$

at the boundary, information is incoming and a boundary condition must be given. For a system of equations, the equivalent condition is posed through the characteristic variables, i.e., characteristic waves entering the computational domain must be specified and, hence, require a boundary condition to ensure wellposedness of the problem. See [32, 38] for further details.

What separates the polynomial approximation from the trigonometric schemes discussed so far is the need to impose boundary conditions to restrict the numerical solutions, $u_h(x,t)$, to satisfy the boundary conditions.

3.3.1. Strongly Imposed Boundary Conditions

In the classic approach one requires that the boundary conditions are imposed strongly, i.e., exactly. Hence, we sall seek a polynomial, $u_h(x,t)$, that satisfies (17) in a collocation sense at all internal grid points, x_j, as

$$\left.\frac{\partial u_h}{\partial t}\right|_{x_j} + \left.\frac{\partial \mathcal{I}_N f(u_h)}{\partial x}\right|_{x_j} = 0 \ ,$$

while the boundary conditions are imposed exactly

$$\alpha u_h(-1,t) = f^-(t) \ , \ \beta u_h(1,t) = f^+(t) \ .$$

If we again consider the wave-equation, (13), the collocation scheme becomes

$$\left.\frac{\partial u_h}{\partial t}\right|_{x_j} + a(x_j) \left.\frac{\partial u_h}{\partial x}\right|_{x_j} = 0 \ ,$$

at all interior grid points, i.e., for $a>0$, $j \in [1,N]$, while $u_h(x_0,t) = f^-(t)$.

Establishing stability of the collocation scheme is considerably more challenging than for the Fourier collocation method. To expose the source of this difficulty, consider the simple wave equation, (13), with $a(x)=1$ and subject to the conditions

$$u(x,0) = g(x) \ , \ u(-1,t) = 0 \ .$$

A collocation scheme based on the Gauss-Lobatto nodes yields

$$\frac{d}{dt}\boldsymbol{u} = -\tilde{\mathrm{D}}\boldsymbol{u} \ . \tag{18}$$

Here the matrix $\tilde{\mathrm{D}}$ represents the polynomial differentiation matrix [27, 18, 38] modified to enforce the boundary condition strongly, i.e., by introducing zeros in the first row and column.

The strongly enforced boundary condition introduces the first main obstacle as any structure of the differentiation matrix is destroyed. This leaves us with the quadrature formula to establish stability. The straightforward quadrature formula, however, is closely related to the weighted inner product, $(f,g)_{L^2_w}$, in which the polynomials are orthogonal. With the exception of the Legendre polynomials, the norm associated with the inner product is not uniformly equivalent to the usual L^2-norm [27, 38]. This loss of equivalence eliminates the straightforward use of the quadrature rules to establish stability as the corresponding norm is too weak. Thus, the two central techniques utilized for the Fourier methods are not directly applicable to the case of the polynomial collocation methods.

One approach is to construct a new inner product and associated norm, uniformly equivalent to L^2, and subsequently establish stability in this norm. This is the approach taken in [28, 26]. The more general variable coefficient problem, (13), with $a(x)$ being smooth can be addressed using a similar approach. In particular, if $a(x)$ is smooth and uniformly bounded away from zero stability is established in the elliptic norm [28]

$$\frac{1}{2}\frac{d}{dt}\sum_{j=0}^{N-1} v_N^2(x_j)\frac{\tilde{w}_j}{a(x_j)} \leq 0 \ .$$

For the more general case of $a(x)$ changing sign the only known results are based on the skew-symmetric form [7, 38], (15), although numerical experiments suggest that the straightforward Chebyshev collocation approximation of the wave equation with a variable coefficient behaves much as the Fourier approximation discussed above, i.e., if the solution is well resolved, the approximation is stable [27, 28].

3.3.2. Weakly Imposed Boundary Conditions

The conceptual leap that leads one to consider other ways of imposing boundary conditions is the observation that it suffices to impose the boundary conditions to the order of the scheme, i.e., weakly.

This simple idea, put forward in the context of spectral methods in [7] in a weak formulation and in [19, 20] for the strong formulation considered here, has recently been developed further into a flexible technique to impose boundary conditions in pseudospectral approximations to a variety of problems [13, 37, 9, 36, 33, 34, 35].

In this setting, one seeks a polynomial solution, $u_h(x,t)$, to (17) that satisfy

$$\begin{aligned} \frac{\partial u_h}{\partial t} + \mathcal{I}_N \frac{\partial \mathcal{I}_N f(u_h)}{\partial x} = & \quad -\tau^- \alpha Q^-(x)\left[u_h(-1,t) - \tilde{f}^-(t)\right] \\ & \quad -\tau^+ \beta Q^+(x)\left[u_h(1,t) - \tilde{f}^+(t)\right] \ , \end{aligned} \tag{19}$$

where we have introduced the polynomials, $Q^{\pm}(x)$, and the scalars, $\tau^{\pm}$.

To complete the scheme we must specify $Q^{\pm}(x)$ and define an approach by which to specify the scalar parameters, $\tau^{\pm}$. While the latter choice is dictated by requiring semi-discrete stability, the former choice of $Q^{\pm}(x)$ is associated with a great deal of freedom.

As an example, consider the approximation to the constant coefficient wave equation (13)

$$\frac{\partial u_h}{\partial t} + a\frac{\partial u_h}{\partial x} = -\tau^- a Q^-(x)\left[u_h(-1,t) - f(t)\right] \quad ,$$

where $u_h(x,t)$ is based on the Legendre-Gauss-Lobatto points. A viable choice of $Q^-(x)$ is

$$Q^-(x) = \frac{(1-x)P_N'(x)}{2P_N'(-1)} = \begin{cases} 1 & x = -1 \\ 0 & x = x_j \neq -1 \end{cases} \quad ,$$

where x_j refers to the Legendre-Gauss-Lobatto points and $P_N(x)$ is the Legendre polynomial of order N. By requesting that the equation be satisfied in a collocation sense and the scheme be stable, we recover the scheme

$$\left.\frac{\partial u_h}{\partial t}\right|_{x_j} + a\left.\frac{\partial u_h}{\partial x}\right|_{x_j} = -a\frac{N(N+1)}{4}\frac{(1-x_j)P_N'(x_j)}{2P_N'(-1)}\left[u_h(-1,t) - f(t)\right] \quad .$$

Using the accuracy of the quadrature, one easily shows asymptotic stability. Although the boundary condition is imposed only weakly, the approximation is clearly consistent, i.e., if $u_h(x,t) = u(x,t)$ the penalty term vanishes identically. A key difference between the schemes with strongly and weakly imposed boundary conditions is that in the former case, stability is established after construction of the scheme whereas in the latter case, stability is guaranteed as a result of the construction of the scheme.

4. STABILITY AND CONVERGENCE OF NONLINEAR PROBLEMS

Turning to the development of spectral methods for nonlinear problems introduces a number of challenges. First of all, the use of standard energy methods to establish stability is no longer possible except in certain special cases. As a result of this, the question of convergence remains open and must be addressed in a different way.

4.1. Skew-symmetric form

If we consider the Fourier collocation scheme for Burgers equation

$$\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial u^2}{\partial x} = 0 \quad ,$$

we seeking the approximate solution, $u_h(x,t)$, such that

$$\left.\frac{\partial u_h}{\partial t}\right|_{x_j} + \frac{1}{2}\left.\frac{\partial}{\partial x}\mathcal{I}_N u_h^2\right|_{x_j} = 0 \quad . \tag{20}$$

Note that while the partial differential equation has the equivalent formulation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0 \quad ,$$

for smooth solutions, the corresponding non-conservative Fourier approximation

$$\left.\frac{\partial u_h}{\partial t}\right|_{x_j} + u_h(x_j)\left.\frac{\partial u_h}{\partial x}\right|_{x_j} = 0 \ ,$$

is not equivalent to (20) and may behave differently due to the aliasing.

We cannot establish stability of these scheme using standard means. However, by writing it on skew-symmetric form

$$\frac{\partial u}{\partial t} + \frac{1}{3}\frac{\partial u^2}{\partial x} + \frac{1}{3}u\frac{\partial u}{\partial x} = 0 \ ,$$

stability of the collocation approximation

$$\left.\frac{\partial u_h}{\partial t}\right|_{x_j} + \frac{1}{3}\left.\frac{\partial}{\partial x}\mathcal{I}_N u_h^2\right|_{x_j} + \frac{1}{3}u_h(x_j)\left.\frac{\partial u_h}{\partial x}\right|_{x_j} = 0 \ ,$$

follows directly from the accuracy of the quadrature.

If we consider a general hyperbolic conservation law

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0,$$

one can prove under light conditions on $f(u)$ that it may always be expressed on skew-symmetric form. This extends to many systems. In [56] it is shown that the existence of a skew-symmetric form is guaranteed for any system that has a convex entropy or is symmetrizable. This includes all major systems of conservation laws, e.g., the Euler equations.

This results suggests that one could simply express the conservation law on skew-symmetric form to ensure stability of the scheme. For problems with smooth solutions, this is indeed a powerful technique, although it doubles the computational cost. However, if we recall that for the scalar problem, the quadratic functional u^2 plays the role of both energy and entropy, we realize that the skew-symmetric form conserves entropy. For problems with shocks this is in violation of basic properties of the hyperbolic conservation laws. Hence, the skew-symmetric form is suitable only for problems with smooth solutions or in combination with additional dissipation.

4.2. Filtering for stability

Maintaining stability of the numerical approximation becomes increasingly hard as the discontinuity evolves and generates energy with higher and higher frequency content. This process, amplified by the nonlinear mixing of the Gibbs oscillations and the numerical solution, eventually renders the scheme unstable or, if the scheme is expressed on skew-symmetric form, the solution wildly inaccurate.

Understanding the source of the stability problem, i.e., accumulation of high frequency energy, suggests a possible solution is the introduction of a dissipative mechanism to remove the high frequency components.

A classical way to accomplish this is to modify the original problem by adding artificial dissipation as

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = \varepsilon(-1)^{p+1}\frac{\partial^{2p} u}{\partial x^{2p}} \ .$$

A direct implementation of this, however, may be costly and introduces additional stiffness which limits the stable time step [24, 38].

To seek a different path, let us modify the numerical solution, $u_h(x,t)$, by the use of a spectral filter as

$$\mathcal{F}_N u_h(x,t) = \sum_{n=-N}^{N} \sigma\left(\frac{n}{N}\right) \tilde{u}_n(t) \exp(inx) \quad . \tag{21}$$

To understand the impact of using the filter at regular intervals as a stabilizing mechanism, a procedure first proposed in [49, 44], consider an exponential filter

$$\sigma(\eta) = \exp\left(-\alpha\eta^{2p}\right) \quad .$$

As discussed in Sec. 5.1 this filter allows for a dramatic improvement in the accuracy of the approximation away from points of discontinuity.

To appreciate its impact on stability, consider the generic initial value problem

$$\frac{\partial u}{\partial t} = \mathcal{L}u \quad ,$$

and the Fourier scheme

$$\frac{d}{dt}\boldsymbol{u} = \mathcal{L}_N \boldsymbol{u} \quad .$$

Advancing the solution from $t = 0$ to $t = \Delta t$, followed by filtering, is expressed as

$$\boldsymbol{u}(\Delta t) = \mathcal{F}_N \exp(\mathcal{L}_N \Delta t)\boldsymbol{u}(0) \quad .$$

If we first assume that $\mathcal{L}_N$ represents the constant coefficient hyperbolic problem, i.e., $\mathcal{L} = a\frac{\partial}{\partial x}$, we recover that

$$\tilde{u}_n(\Delta t) = \exp(-\alpha\eta^{2p} + a(ik)\Delta t)\tilde{u}_n(0) \quad , \tag{22}$$

i.e., we are in fact computing the solution to the modified problem

$$\frac{\partial u}{\partial t} = a\frac{\partial u}{\partial x} - \alpha\frac{(-1)^p}{\Delta t N^{2p}}\frac{\partial^{2p} u}{\partial x^{2p}} \quad .$$

The effect of the filter is thus equivalent to that of adding a small dissipative term to the original equation. However, the process of adding the dissipation through the filter is very simple.

For a general $\mathcal{L}$, e.g., with a variable coefficient or a nonlinear flux in which case $\mathcal{F}_N$ and $\mathcal{L}_N$ no longer commute, the modified equation being solved takes the form

$$\frac{\partial u}{\partial t} = \mathcal{L}u - \alpha\frac{(-1)^p}{\Delta t N^{2p}}\frac{\partial^{2p} u}{\partial x^{2p}} + \mathcal{O}(\Delta t^2) \quad ,$$

by viewing the application of the filter as an operator splitting problem [4, 14].

It is clear that the filter has a stabilizing effect, established more rigorously for problem with smooth and nonsmooth initial data in [49, 44, 23] for the Fourier approximation to the general variable coefficient problem, (13). In [24, 38] it is furthermore established that light filtering suffices to stability an aliasing driven instability.

4.3. Vanishing viscosity techniques

The foundation of a convergence theory for the spectral approximations to hyperbolic conservation laws has been laid in [59, 48, 10] for the periodic case and extended in [47] to the Legendre approximation and to the Chebyshev-Legendre scheme in [46, 45].

To outline the basic elements of this convergence theory let us restrict ourselves to the periodic case. For the discrete approximation we must add a dissipative term that is strong enough to stabilize the approximation, yet small enough so as to not ruin the spectral accuracy of the scheme. In [59, 48] the following spectral viscosity method was considered

$$\frac{\partial u_h}{\partial t} + \frac{\partial}{\partial x}\mathcal{P}_N(f(u_h)) = \varepsilon_h(-1)^{p+1}\frac{\partial^p}{\partial x^p}\left[Q_m(x,t) * \frac{\partial^p u_h}{\partial x^p}\right] \quad , \qquad (23)$$

where

$$\frac{\partial^p}{\partial x^p}\left[Q_m(x,t) * \frac{\partial^p u_h}{\partial x^p}\right] = \sum_{m<|n|\leq N} (ik)^{2p}\hat{Q}_n\hat{u}_n \exp(inx) \quad .$$

To ensure stability m should not be taken too big. On the other hand, taking m too small will impact the accuracy in a negative way. An acceptable compromise is

$$m \sim N^\theta \quad , \quad \theta < \frac{2p-1}{2p} \quad .$$

Moreover, the smoothing factors, $\hat{Q}_n$, should only be activated for high modes as

$$\hat{Q}_n = 1 - \left(\frac{m}{|n|}\right)^{\frac{2p-1}{\theta}} \quad ,$$

for $|n| > m$ and $\hat{Q}_n = 1$ otherwise. Finally, we assume that the amplitude of the viscosity is small as

$$\varepsilon_h \sim \frac{C}{N^{2p-1}} \quad .$$

Under these assumptions, one can prove for $p = 1$ that the solution is bounded in $L^\infty[0, 2\pi]$ and obtain the estimate [59]

$$\|u_h\|_{L^2[0,2\pi]} + \sqrt{\varepsilon_h}\left\|\frac{\partial u_h}{\partial x}\right\|_{L^2_{loc}} \leq C \quad .$$

Convergence to the correct entropy solution then follows from compensated compactness arguments [59, 48].

To realize the connection between the spectral viscosity method and the use of filters discussed above, consider the simple case where $f(u) = au$. In this case, the solution to (23) is

$$\hat{u}_n(t) = \exp\left(inat - \varepsilon_h n^2 \hat{Q}_n\right) \hat{u}_n(0) \ , \ \ |n| > m \ ,$$

which is equivalent to the effect of the filtering, albeit with a particular filter function.

For $p \neq 1$ a bound on the $L^\infty[0, 2\pi]$ is no longer known. However, experience suggests that it is better to filter from the first mode but to employ a slower decay of the expansion coefficients, corresponding to taking $p > 1$. This yields the superviscosity [60] method in which one solves

$$\frac{\partial u_h}{\partial t} + \frac{\partial}{\partial x}\mathcal{P}_N f(u_h) = \varepsilon_h (-1)^{p+1} \frac{\partial^{2p} u_h}{\partial x^{2p}} \ ,$$

which is equivalent to the use of a high-order exponential filter.

5. POST PROCESSING TECHNIQUES

A manifestation of the slow and nonuniform convergence of $\mathcal{I}_N u$ for a piecewise smooth functions is the linear decay of the global expansion coefficients, $\tilde{u}_n$. This observation also suggests that one could attempt to modify the global expansion coefficients to enhance the convergence rate of the spectral approximation. The key question to consider is exactly how one should modify the expansion to ensure enhanced convergence to the correct solution.

However, before doing so, it is worth understanding if the emergence of a shock and the Gibbs phenomenon effectively eliminates any hope of maintaining high-order accuracy.

Consider again

$$\frac{\partial u}{\partial t} + a(x)\frac{\partial u}{\partial x} = \frac{\partial u}{\partial t} + \mathcal{L}u = 0.$$

Both $a(x)$ and $u(x,t)$ are considered periodic and $a(x)$ is smooth. We have already established stability of this scheme, possibly by using filtering or through the skew-symmetric form.

We assume that the initial condition, $u(x, 0)$, is non-smooth, resulting in the introduction of the Gibbs phenomenon.

Let us also introduce the adjoint problem

$$\frac{\partial v}{\partial t} - \mathcal{L}^* v = 0,$$

where $(\mathcal{L}u, v) = (u, \mathcal{L}^* v)$. We assume smooth initial conditions for the adjoint problem. A seminal result [1] can be obtained as

$$(u_h(t), v(t)) = (u(t), v(t)) + \varepsilon, \tag{24}$$

where ε is very small and depends only on the smoothness of $v(x,t)$. Since the adjoint problem is smooth, this can be made arbitrarily small. This highlights, at least for the case of a variable coefficient problem with nonsmooth initial conditions, the possibility of recovering a high-order accurate solution, $u_h(t)$. However, this accuracy is not found directly in the solution $u_h(x,t)$, but, rather, in the moments of the solution.

While it is a surprising result, it is also an encouraging result. It clarifies that the Gibbs oscillations may look bad, but they do not destroy the attractive basic properties of the schemes – in particular, the properties related to the highly accurate propagation. This result can be extended to non-smooth solutions and sources [65] and suggests that we consider ways to recover a pointwise spectrally accurate solution from the oscillatory solution which is only pointwise first order accurate. For Burgers equation, extensive computational results in [54] suggest that high-order accuracy is also retained in this case.

5.1. Filtering for accuracy

We consider the filtered approximation, $\mathcal{F}_N u_h(x)$, of the form

$$\mathcal{F}_N u_h(x) = \sum_{n=-N}^{N} \sigma\left(\frac{n}{N}\right) \tilde{u}_n \exp(inx) \quad , \tag{25}$$

where $\sigma(\eta)$ is a real filter function with the following properties [64]

$$\sigma(\eta) = \begin{cases} \sigma(-\eta) & \\ \sigma(0) = 1 & \\ \sigma^{(q)}(0) = 0 & 1 \le q \le 2p-1 \\ \sigma(\eta) = 0 & |\eta| \ge 1 \end{cases} \quad . \tag{26}$$

If $\sigma(\eta)$ has at least $2p-1$ continuous derivatives, $\sigma(\eta)$ is termed a filter of order $2p$.

As the filter is nothing more than a lowpass filter, it is not surprising that the filtered function converges faster than the unfiltered filtered original expansion. To understand exactly how much the filter modifies the convergence rate, assume that $u(x)$ is piecewise C^{2p} with one discontinuity located at $x = \xi$. Let us furthermore assume that the filter is of order $2p$. Then the pointwise error of the filtered approximation is given as [64, 29, 38]

$$|u(x) - \mathcal{F}_N u_N(x)| \le C \frac{1}{N^{2p-1} d(x,\xi)^{2p-1}} K(u) + C \frac{\sqrt{N}}{N^{2p}} \|u^{(2p)}\|_{L^2_B} \quad ,$$

where $d(x,\xi)$ measures the distance from x to the point of discontinuity, ξ, $K(u)$ is uniformly bounded away from the discontinuity and a function of $u(x)$ only. Also $\|\cdot\|_{L^2_B}$ signifies the broken L^2-norm.

While the details of the proof of this result are technical and can be found in [64, 29, 38], the interpretation of the result is simple, and perhaps somewhat surprising. It states that the convergence rate of the filtered approximation is determined solely by the order $2p$ of the filter $\sigma(\eta)$ and the regularity of the function, $u(x)$, away from the point of discontinuity. In particular, if the function $u(x)$ is piecewise analytic and the order of the filter increases with N, one recovers an exponentially accurate approximation to the unfiltered function everywhere except very close to the discontinuity [64, 29]. Partial results for polynomial expansions suggest similar behavior [40].

Spectral filtering of the expansion coefficients remains the most popular way of enhancing the convergence rate. An alternative is to improve the approximation by localizing the approximation close to the point of the discontinuity. This approach, known as physical space filtering, operates directly on the interpolating polynomials rather than the expansion coefficients. This is developed and applied with success in [31, 29, 61, 62].

An alternative approach explores the superior properties of Padé-approximation to address the Gibbs phenomenon by a reprojection. This approach, developed in [15, 16, 41, 39], often yields excellent results, even at the point of discontinuity. However, the nonlinear nature of the Padé approximant makes its application complex.

5.2. Gegenbauer reconstruction

Let us finally outline the key elements of a general theory that establishes the possibility of recovering a piecewise exponentially convergent approximation to a piecewise analytic function, having knowledge of the global expansion coefficients and the position of the discontinuities only.

The basic element of this approach is the identification of a new basis with very special properties and, subsequently, the expansion of the slowly convergent truncated global expansion in this new basis. Provided this new basis satisfies certain conditions, the new expansion has the remarkable property that it is exponentially convergent to the original piecewise analytic function even though its evaluation uses information from the slowly convergent global expansion.

We assume that there exists an interval $[a, b] \subset [0, 2\pi]$ in which $u(x)$ is analytic and, furthermore, that the original truncated expansion is pointwise convergent in all of $[0, 2\pi]$ with the exception of a finite number of points. We introduce the scaled variable

$$\xi(x) = -1 + 2\frac{x-a}{b-a} \quad .$$

Clearly, $\xi : [a, b] \to [-1, 1]$.

Now define a new basis, $\psi_n^\lambda(\xi)$, which is orthogonal in the weighted inner product, $(\cdot,\cdot)_w^\lambda$ where λ signifies that the weight, $w(x)$, may depend on λ, i.e.,

$$\left(\psi_k^\lambda, \psi_n^\lambda\right)_w^\lambda = \|\psi_n^\lambda\|_{L_w^2}^2 \delta_{kn} = \gamma_n^\lambda \delta_{kn} \quad .$$

Furthermore, we require that if $v(\xi)$ is analytic then

$$\mathcal{P}_\lambda v(\xi) = \sum_{n=0}^{\lambda} \frac{1}{\gamma_n^\lambda} \left(v, \psi_n^\lambda\right)_w^\lambda \psi_n^\lambda(\xi) \quad ,$$

is pointwise exponentially convergent as λ increases, i.e.,

$$\|v - \mathcal{P}_\lambda v\|_{L^\infty} \le Ce^{-c\lambda} \quad ,$$

with $c > 0$. This is simply the statement of exponential convergence for a polynomial expansion of a analytic function.

A final condition sets this basis apart and is central in order to overcome the Gibbs phenomenon. We require that there exists a number $\beta < 1$, such that for $\lambda = \beta N$ we have

$$\left| \frac{1}{\gamma_n^\lambda} \left(\phi_k(x(\xi)), \psi_n^\lambda(\xi) \right)_w^\lambda \right| \|\psi_n^\lambda\|_{L^\infty} \leq \left(\frac{\alpha N}{k} \right)^\lambda \quad , \tag{27}$$

for $k > N$, $n \leq \lambda$ and $\alpha < 1$. The interpretation of this condition is that the projection of the high modes of ϕ_k onto the basis, ψ_n^λ, is exponentially small in the interval, $\xi \in [-1, 1]$. In other words, by reexpanding the slowly decaying ϕ_n-based global expansion in the local ψ_n^λ-basis, an exponentially accurate local approximation is recovered. Moreover, this can be achieved everywhere in the domain where $u(x)$ is analytic. This latter condition on ψ_n^λ is termed the Gibbs condition to emphasize its close connection to the resolution of the Gibbs phenomenon [29, 30].

Provided only that the ψ_n^λ-basis, termed the Gibbs complementary basis, is complete we recover the key result

$$\left\| u(x) - \sum_{n=0}^{\lambda} \frac{1}{\gamma_n^\lambda} \left(\mathcal{P}_N u, \psi_n^\lambda \right)_w^\lambda \psi_n^\lambda(\xi(x)) \right\|_{L^\infty} \leq C \exp(-cN) \quad ,$$

where $\lambda = \beta N$ and $u(x)$ is analytic in the interval $[a, b]$.

In other words, if a Gibbs complementary basis exists it is possible to reconstruct a piecewise exponentially convergent approximation to a piecewise analytic function from the information contained in the original slowly converging global approximation. The only additional piece of information needed is the location of the points of discontinuity. The Gibbs phenomenon can be overcome.

A constructive approach to the identification of the complementary basis is currently unknown. The existence of such a basis, however, has been established by carefully examining the properties of the basis

$$\psi_n^\lambda(\xi) = C_n^{(\lambda)}(\xi) \quad ,$$

where $C_n^\lambda(\xi)$ represent the Gegenbauer polynomials, also known as the symmetric Jacobi polynomials or the ultraspherical polynomials [55].

Using the Fourier basis, it must be established that

$$\left| \frac{1}{\gamma_n^\lambda} \left(\phi_k, \psi_n^\lambda \right)_w^\lambda \right| \leq \left(\frac{\alpha N}{k} \right)^\lambda \quad ,$$

for $k > N$, $0 < \alpha < 1$, and $n \leq \beta N = \lambda$. However, for this basis the inner product allows an exact evaluation

$$\frac{1}{\gamma_n^\lambda} \left(\phi_k, \psi_n^\lambda \right) = i^n \Gamma(\lambda) \left(\frac{2}{\pi k \varepsilon} \right)^\lambda (n + \lambda) J_{n+\lambda}(\pi \varepsilon k) \quad ,$$

with $J_\nu(x)$ being the Bessel function and $\varepsilon = b - a$ measures the width of the interval. Using the properties of the Bessel function and the Stirling formula for the asymptotic of the Γ-function, the Gibbs condition is satisfied if [29]

$$\beta = \frac{2\pi\varepsilon}{27} \quad .$$

This establishes the existence of a Gibbs complementary basis to the Fourier basis [29, 30].

The extension to the polynomial case follows a similar approach and the Gegenbauer polynomials again play the role as the complementary basis[29, 30].

The reconstruction of piecewise smooth solutions to conservation laws as a post processing technique has been exploited in [12, 14, 21, 22].

REFERENCES

1. Saul Abarbanel, David Gottlieb, and Eitan Tadmor. *Spectral methods for discontinuous problems*, volume 177974. Langley Research Center, NASA, 1985.
2. Guo Ben-Yu. *Spectral methods and their applications*. World Scientific, 1998.
3. Christine Bernardi and Yvon Maday. Spectral methods. *Handbook of numerical analysis*, 5:209–485, 1997.
4. John P Boyd. Two comments on filtering (artificial viscosity) for chebyshev and legendre spectral and spectral element methods: preserving boundary conditions and interpretation of the filter as a diffusion. *Journal of Computational Physics*, 143(1):283–288, 1998.
5. John P Boyd. *Chebyshev and Fourier spectral methods*. Courier Corporation, 2001.
6. C Canuto and A Quarteroni. Approximation results for orthogonal polynomials in sobolev spaces. *Mathematics of Computation*, 38(157):67–86, 1982.
7. C Canuto and A Quarteroni. Error estimates for spectral and pseudospectral approximations of hyperbolic equations. *SIAM Journal on Numerical Analysis*, 19(3):629–642, 1982.
8. Claudio Canuto, M Yousuff Hussaini, Alfio Quarteroni, and Thomas A Zang. *Spectral methods in fluid dynamics*. Springer-Verlag, New York, 1988.
9. Mark H Carpenter and David Gottlieb. Spectral methods on arbitrary grids. *Journal of Computational Physics*, 129:74–86, 1996.
10. Gui Qiang Chen, Qiang Du, and Eitan Tadmor. Spectral viscosity approximations to multidimensional scalar conservation laws. *Mathematics of Computation*, 61(204):629–643, 1993.
11. Michel O Deville, Paul F Fischer, and Ernest H Mund. *High-order methods for incompressible fluid flow*, volume 9. Cambridge University Press, 2002.
12. Wai Sun Don. Numerical study of pseudospectral methods in shock wave applications. *Journal of Computational Physics*, 110(1):103–111, 1994.
13. Wai Sun Don and David Gottlieb. The chebyshev-legendre method: implementing legendre methods on chebyshev points. *SIAM Journal on Numerical Analysis*, 31(6):1519–1534, 1994.
14. Wai Sun Don and David Gottlieb. Spectral simulation of supersonic reactive flows. *SIAM journal on numerical analysis*, 35(6):2370–2384, 1998.
15. Tobin A Driscoll and Bengt Fornberg. A Padé-based algorithm for overcoming the Gibbs phenomenon. *Numerical Algorithms*, 26(1):77–92, 2001.
16. L Emmel, Sidi Mahmoud Kaber, and Yvon Maday. Padé–Jacobi filtering for spectral approximations of discontinuous solutions. *Numerical Algorithms*, 33(1-4):251–264, 2003.
17. Bengt Fornberg. On a fourier method for the integration of hyperbolic equations. *SIAM Journal on Numerical Analysis*, 12(4):509–528, 1975.
18. Daniele Funaro. *Polynomial approximation of differential equations*, volume 8. Springer Science & Business Media, 2008.
19. Daniele Funaro and David Gottlieb. A new method of imposing boundary conditions in pseudospectral approximations of hyperbolic equations. *Mathematics of Computation*, 51(184):599–613, 1988.

20. Daniele Funaro and David Gottlieb. Convergence results for pseudospectral approximations of hyperbolic systems by a penalty-type boundary treatment. *Mathematics of Computation*, 57(196):585–596, 1991.
21. Anne Gelb and David Gottlieb. The resolution of the Gibbs phenomenon for spliced functions in one and two dimensions. *Computers & Mathematics with Applications*, 33(11):35–58, 1997.
22. Anne Gelb and Eitan Tadmor. Enhanced spectral viscosity approximations for conservation laws. *Applied Numerical Mathematics*, 33(1):3–21, 2000.
23. Jonathan Goodman, Thomas Hou, and Eitan Tadmor. On the stability of the unsmoothed fourier method for hyperbolic equations. *Numerische Mathematik*, 67(1):93–129, 1994.
24. David Gottlieb and Jan S Hesthaven. Spectral methods for hyperbolic problems. *Journal of Computational and Applied Mathematics*, 128(1):83–131, 2001.
25. David Gottlieb, Liviu Lustman, and Steven A Orszag. Spectral calculations of one-dimensional inviscid compressible flows. *SIAM Journal on Scientific and Statistical Computing*, 2(3):296–310, 1981.
26. David Gottlieb, Liviu Lustman, and Eitan Tadmor. Stability analysis of spectral methods for hyperbolic initial-boundary value systems. *SIAM journal on numerical analysis*, 24(2):241–256, 1987.
27. David Gottlieb and Steven A Orszag. *Numerical analysis of spectral methods: theory and applications*, volume 26. Siam, 1977.
28. David Gottlieb, Steven A Orszag, and Eli Turkel. Stability of pseudospectral and finite-difference methods for variable coefficient problems. *mathematics of computation*, 37(156):293–305, 1981.
29. David Gottlieb and Chi-Wang Shu. On the gibbs phenomenon and its resolution. *SIAM review*, 39(4):644–668, 1997.
30. David Gottlieb and Chi-Wang Shu. A general theory for the resolution of the gibbs phenomenon. *ATTI DEI CONVEGNI LINCEI-ACCADEMIA NAZIONALE DEI LINCEI*, 147:39–48, 1998.
31. David Gottlieb and Eitan Tadmor. Recovering pointwise values of discontinuous data within spectral accuracy. In *Progress and Supercomputing in Computational Fluid Dynamics*, pages 357–375. Springer, 1985.
32. Bertil Gustafsson, Heinz-Otto Kreiss, and Joseph Oliger. *Time dependent problems and difference methods*, volume 24. John Wiley & Sons, 1995.
33. Jan S Hesthaven. A stable penalty method for the compressible Navier–Stokes equations: II. One-dimensional domain decomposition schemes. *SIAM Journal on Scientific Computing*, 18(3):658–685, 1997.
34. Jan S Hesthaven. A stable penalty method for the compressible Navier–Stokes equations: III. Multidimensional domain decomposition schemes. *SIAM Journal on Scientific Computing*, 20(1):62–93, 1998.
35. Jan S Hesthaven. Spectral penalty methods. *Applied Numerical Mathematics*, 33(1):23–41, 2000.
36. Jan S Hesthaven and D Gottlieb. Stable spectral methods for conservation laws on triangles with unstructured grids. *Computer Methods in Applied Mechanics and Engineering*, 175(3):361–381, 1999.
37. Jan S Hesthaven and David Gottlieb. A stable penalty method for the compressible Navier-Stokes equations: I. Open boundary conditions. *SIAM Journal on Scientific Computing*, 17(3):579–612, 1996.
38. Jan S Hesthaven, Sigal Gottlieb, and David Gottlieb. *Spectral methods for time-dependent problems*, volume 21. Cambridge University Press, 2007.
39. Jan S Hesthaven, Sidi Mahmoud Kaber, and Laura Lurati. Padé-Legendre interpolants for Gibbs reconstruction. *Journal of Scientific Computing*, 28(2-3):337–359, 2006.
40. Jan S Hesthaven and Robert Kirby. Filtering in Legendre spectral methods. *Mathematics of Computation*, 77(263):1425–1452, 2008.
41. Sidi Mahmoud Kaber and Yvon Maday. Analysis of some Padé–Chebyshev approximants. *SIAM journal on numerical analysis*, 43(1):437–454, 2005.
42. George Karniadakis and Spencer Sherwin. *Spectral/hp element methods for computational fluid dynamics*. Oxford University Press, 2013.

43. Heinz-Otto Kreiss and Joseph Oliger. Comparison of accurate methods for the integration of hyperbolic equations. *Tellus*, 24(3):199–215, 1972.

44. Heinz-Otto Kreiss and Joseph Oliger. Stability of the fourier method. *SIAM Journal on Numerical Analysis*, 16(3):421–433, 1979.

45. Heping Ma. Chebyshev–legendre spectral viscosity method for nonlinear conservation laws. *SIAM Journal on Numerical Analysis*, 35(3):869–892, 1998.

46. Heping Ma. Chebyshev–legendre super spectral viscosity method for nonlinear conservation laws. *SIAM Journal on Numerical Analysis*, 35(3):893–908, 1998.

47. Yvon Maday, Sidi M Ould Kaber, and Eitan Tadmor. Legendre pseudospectral viscosity method for nonlinear conservation laws. *SIAM Journal on Numerical Analysis*, 30(2):321–342, 1993.

48. Yvon Maday and Eitan Tadmor. Analysis of the spectral vanishing viscosity method for periodic conservation laws. *SIAM Journal on Numerical Analysis*, 26(4):854–870, 1989.

49. Andrew Majda, James McDonough, and Stanley Osher. The fourier method for nonsmooth initial data. *Mathematics of Computation*, 32(144):1041–1081, 1978.

50. Steven A Orszag. Comparison of pseudospectral and spectral approximation. *Studies in Applied Mathematics*, 51(3):253–259, 1972.

51. Joseph E Pasciak. Spectral and pseudospectral methods for advection equations. *Mathematics of Computation*, 35(152):1081–1092, 1980.

52. Roger Peyret. *Spectral methods for incompressible viscous flow*, volume 148. Springer Science & Business Media, 2013.

53. Jie Shen, Tao Tang, and Li-Lian Wang. *Spectral methods: algorithms, analysis and applications*, volume 41. Springer Science & Business Media, 2011.

54. Chi-Wang Shu and Peter S Wong. A note on the accuracy of spectral method applied to nonlinear conservation laws. *Journal of Scientific Computing*, 10(3):357–369, 1995.

55. Gabor Szego. *Orthogonal polynomials*, volume 23. American Mathematical Soc., 1939.

56. Eitan Tadmor. Skew-selfadjoint form for systems of conservation laws. *Journal of Mathematical Analysis and Applications*, 103(2):428–442, 1984.

57. Eitan Tadmor. The exponential accuracy of Fourier and Chebyshev differencing methods. *SIAM Journal on Numerical Analysis*, 23(1):1–10, 1986.

58. Eitan Tadmor. Stability analysis of finite difference, pseudospectral and Fourier-Galerkin approximations for time-dependent problems. *SIAM review*, 29(4):525–555, 1987.

59. Eitan Tadmor. Convergence of spectral methods for nonlinear conservation laws. *SIAM Journal on Numerical Analysis*, 26(1):30–44, 1989.

60. Eitan Tadmor. Shock capturing by the spectral viscosity method. *Computer Methods in Applied Mechanics and Engineering*, 80(1-3):197–208, 1990.

61. Eitan Tadmor and Jared Tanner. Adaptive mollifiers for high resolution recovery of piecewise smooth data from its spectral information. *Foundations of Computational Mathematics*, 2(2):155–189, 2002.

62. Eitan Tadmor and Jared Tanner. Adaptive filters for piecewise smooth spectral data. *IMA journal of numerical analysis*, 25(4):635–647, 2005.

63. Lloyd N Trefethen. *Spectral methods in MATLAB*, volume 10. Siam, 2000.

64. Hervé Vandeven. Family of spectral filters for discontinuous problems. *Journal of Scientific Computing*, 6(2):159–192, 1991.

65. Jens Zudrop and Jan S Hesthaven. Accuracy of high order and spectral methods for hyperbolic conservation laws with discontinuous solutions. *SIAM Journal of Numerical Analysis*, c:a–b, 2015.