TECHNICAL REPORT
UNLABELED SENSING: ALGORITHM AND GUARANTEES

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ABSTRACT

It often happens that we are interested in reconstructing a signal from partial measurements. In *Unlabeled sensing* problem, we consider the case in which the order of noisy samples out of a linear measurement system is missing. Thus, the main challenge would be: given a set of unordered sample values, how to recover the data uniquely and at the same time, how to recover it more efficiently. In [1] we considered these two fundamental questions regarding uniqueness and efficiency by designing a robust sampling matrix in linear equation system and providing a novel recovery algorithm. In this document we present the theoretical findings and proofs of the work introduced in [1].

1. NUMBER OF THE SOLUTIONS IN UNIFORMLY SPACED SETTING

$$\boldsymbol{y} = \boldsymbol{P\Phi x} + \boldsymbol{\epsilon}, \tag{1}$$

Theorem 1. *Consider the unlabeled sensing problem in* (1) *with* $\boldsymbol{\epsilon} = \boldsymbol{0}$ *and* N*-equally-spaced sampling vectors. The number of solutions for* (1) *is equal to 2 when* N *is even and equal to* $2N$ *when* N *is odd.*

Proof. We want to show that with the N-equally spaced setting, if N is an odd integer, each point x has exactly $2N$ dual points on the plane and these points are all possible solutions. Set S is constructed such that it contains all possible combinations of N integer numbers from 1 to N. It means that if we give indices to unordered sample values from 1 to N, each set in S shows one possible order for unordered sample set. Suppose that set $M \in S$ shows the correct labeling for sample values. Suppose that y shows correct ordered sample values and also $\Phi(M)$ represents matrix representation of the setting sorted based on set M. In other words,

$$\Phi(M_i) = [\cos(\frac{2\pi}{N}m_i), \sin(\frac{2\pi}{N}m_i)] \tag{2}$$

shows i^{th} row of $\Phi(M)$. Thus, equation $y = \Phi(M)x$ holds. If there is another set L corresponding to the dual point x', following equation must hold:

$$\Phi(M)x = \Phi(L)x'. \tag{3}$$

We can search through all points on the plane in order to find x' or equivalently we can take $x' = cx$ where $c \in (0, \infty)$ and then rotate the whole sampling setting. We can verify that $arg(x) = arg(x')$ and $\|x'\|_2 = |c|\|x\|_2$. By considering different c, we can search through the line that connects point x to origin from zero to infinity. Now we need to rotate this line and complete our search all over the plane. Equivalently, we can rotate the setting with any continuous angle $\eta \in (0, N]$. It can be done by considering $\Phi(L+\eta)$ instead of $\Phi(L)$ in (3):

$$\begin{aligned}
&\Phi(M)x = c\Phi(L+\eta)x\\
&\Phi(M_i)x = c\Phi((L+\eta)_i)x, \quad \forall i \in \{1, \cdots, N\}\\
&\begin{bmatrix}\cos(\frac{2\pi}{N}m_i)\\ \sin(\frac{2\pi}{N}m_i)\end{bmatrix}^T \begin{bmatrix}\cos(\theta_x)\\ \sin(\theta_x)\end{bmatrix} = c\begin{bmatrix}\cos(\frac{2\pi}{N}(l_i+\eta))\\ \sin(\frac{2\pi}{N}(l_i+\eta))\end{bmatrix}^T \begin{bmatrix}\cos(\theta_x)\\ \sin(\theta_x)\end{bmatrix}\\
&\forall i \in \{1, \cdots, N\}.
\end{aligned} \tag{4}$$

In this way, by simplifying (4), for each sample value we can write an equation

$$\cos(\frac{2\pi}{N}m_i - \theta_x) = c\cos(\frac{2\pi}{N}(l_i+\eta) - \theta_x) \quad \forall i \in \{1, 2, \cdots, N\}. \tag{5}$$

By solving (5) and calculating c, η, and L, we can find all possible solutions for N-equally spaced setting. We claim that in (5), $c = 1$ is the only valid norm for dual point of x. In other words, all dual points should have the same norm on the plane. To prove this, first we prove the following lemma.

Lemma 1. $\sum_{n=0}^{N-1}\cos^2(\frac{2\pi}{N}n+\theta) = \frac{N}{2}$.

Proof. The proof is straight forward. We start with left side of equality,

$$\begin{aligned}
&\sum_{n=0}^{N-1}\cos^2(\frac{2\pi}{N}n+\theta) = \sum_{n=0}^{N-1}\frac{1}{2} + \frac{1}{2}\cos(\frac{4\pi}{N}n+2\theta)\\
&= \frac{N}{2} + \frac{1}{2}\sum_{n=0}^{N-1}\cos(\frac{4\pi}{N}n+2\theta)\\
&= \frac{N}{2} + \frac{1}{2}\cos(2\theta)\sum_{n=0}^{N-1}\cos(\frac{4\pi}{N}n) - \frac{1}{2}\sin(2\theta)\sum_{n=0}^{N-1}\sin(\frac{4\pi}{N}n)
\end{aligned} \tag{6}$$

In (6), the summations are equal to zero, because

$$\begin{aligned}
&\sum_{n=0}^{N-1}\cos(\frac{4\pi}{N}n) + j\sum_{n=0}^{N-1}\sin(\frac{4\pi}{N}n)\\
&= \sum_{n=0}^{N-1}e^{j\frac{4\pi}{N}n} = \frac{1-e^{\frac{j4\pi N}{N}}}{1-e^{j\frac{4\pi}{N}}} = 0.
\end{aligned} \tag{7}$$

The proof is completed by replacing two terms in (6) with zero values which were derived from (7). □

In (5), we raise two sides to the power two and sum up over all values in L and M,

$$\sum_{m_i \in M} \cos^2(\frac{2\pi}{N} m_i - \theta_x) = c^2 \sum_{l_i \in L} \cos^2(\frac{2\pi}{N}(l_i + \eta) - \theta_x). \quad (8)$$

Summation terms in left and right sides of (8) are equal to $N/2$ according to Lemma 1. Therefore, we can conclude that $c = 1$. Now we can simply solve (5) and derive solutions for dual points. There exist two sets of solutions for this equation; the first one is

$$\begin{aligned} &\frac{2\pi}{N} m_i - \theta_x = \frac{2\pi}{N}(l_i + \eta) - \theta_x \\ &\Rightarrow \eta = m_i - l_i \\ &\eta \in \mathbb{Z} \\ &m_i, l_i \in \{1, \cdots, N\}. \end{aligned} \quad (9)$$

In (9), since l_i and m_i are both integer values in set $\{1, \cdots, N\}$, η must be an integer as well. Also, we defined η in interval $(0, N]$, therefore, for this set of solutions, η can only take integer values $\{1, \cdots, N\}$. It means that the equations at most have N valid solutions.

The second solution to (8) is

$$\begin{aligned} &\frac{2\pi}{N} m_i - \theta_x = -\frac{2\pi}{N}(l_i + \eta) + \theta_x \\ &\Rightarrow \eta = \frac{N}{\pi}\theta_x - (m_i + l_i) \\ &m_i, l_i \in \{1, \cdots, N\}. \end{aligned} \quad (10)$$

In (10), we can see that η can only take N different values in interval $(0, N]$. Therefore, (10) has at most N valid solutions. In total, (9) and (10) account for at most $2N$ valid solutions.

In summary, we conclude that for N-equally spaced setting where N is an odd number, there are $2N$ dual points for each point on the plane. We saw that there are two set of solutions for these settings, one was derived by rotating sampling vectors and sample values which corresponds to solutions of (9) and second set of solutions was derived by reflecting the first set over symmetry line of the setting and then rotating them. This set of solutions is corresponding to solutions of (10). We can follow the same steps for the case that N is an even number and derive that in that case there are only two valid solutions for the equations. □

2. UNIQUE SOLUTION OF EXPONENTIALLY-SPACED SAMPLING SETTING

Theorem 2. *Consider the sampling matrix $\boldsymbol{\Phi}$ with $N \geq 4$ exponentially spaced sampling vectors. Suppose that for some $\boldsymbol{x}$ and $\boldsymbol{x}'$, there exists a permutation matrix $\boldsymbol{P} \neq \boldsymbol{I}$ such that $\boldsymbol{\Phi x} = \boldsymbol{P \Phi x}'$. If $\|\boldsymbol{x}\| = \|\boldsymbol{x}'\|$, then $\boldsymbol{x}$ and $\boldsymbol{x}'$ coincide and lie on the bi-sector of two of the sampling vectors.*

Proof. In 2-dimensional vector space, in (1), rows of matrix Φ are simply the sine and cosine of sampling vectors' angles. For instance, $\Phi_i = [\cos(\varphi_i), \sin(\varphi_i)]$. As mentioned before, matrix Φ' is the permuted version of matrix Φ such that we reordered rows of Φ.

$$\begin{aligned} &\|x\| [\cos(\varphi_i), \sin(\varphi_i)] [\cos(\theta_x), \sin(\theta_x)]^T = \\ &\|x'\| [\cos(\varphi_i'), \sin(\varphi_i')] [\cos(\theta_{x'}), \sin(\theta_{x'})]^T \end{aligned} \quad (11)$$

.

Multiplying terms and using geometric relations in (11) results in the following equation,

$$\cos(\varphi_i - \theta_x) = \cos(\varphi_i' - \theta_{x'}). \quad (12)$$

We can easily verify that (12) has two set of solutions,

$$\theta_{x'} = 2l\pi + \varphi_i' - \varphi_i + \theta_x \,, \quad (13)$$

or

$$\theta_{x'} = 2l\pi + \varphi_i' + \varphi_i - \theta_x. \quad (14)$$

We use Lemma (2) to show that only two sampling vectors can satisfy (14) and all the rests satisfy (13). Thus, we conclude for nonuniform-exponential sampling setting, if the solution is not unique, only two vectors satisfy (14) and all other vectors satisfy (13). In that case, only one of the equations in (16). In other words, if (14) holds for i_1 and i_2, then we can write $\theta_{x'} = 2l\pi + \varphi_i' - \varphi_i + \theta_x \,, \forall i \neq i_1, i_2$ then $\varphi_i = \varphi_i' \,, \forall i \neq i_1, i_2$. Substituting on (14) we will have $\theta_{x'} = 2l\pi + \theta_x$. Substituting $\theta_{x'}$ in equation for i_1 and i_2 results in,

$$\theta_x = \frac{\varphi_{i_1} + \varphi_{i_2}}{2} + 2l\pi. \quad (15)$$

It is important to note that in nonuniform-exponential setting, the bisector of every distinct sampling vectors is separated, in other words the bisector of one pair of vectors is not the bisector of any other pairs. We prove this claim in lemma (3).

From all of the discussions here we conclude that with the assumption of $\|x\| = \|x'\|$, $x = x'$ should hold, also only two vectors in Φ and Φ' can be interchanged and x should be on the bisector of these two vectors, the rest of them are the same. □

Lemma 2. *In nonuniform-exponential sampling setting with N vectors,* (12) *holds for $\Phi \neq \Phi'$ if and only if just two sampling vectors satisfy* (14) *and all the rest satisfy* (13). *In this case $x = x'$ and it lies on the bisector of two mentioned vectors.*

Proof. Now we consider different cases:

- More than 2 vectors satisfy (14).
- Only one vectors satisfies (14) and all other remaining vectors satisfy (13).
- Neither of the vectors satisfies (14) and all of them satisfy (13).
- only two vectors satisfy (14) and all others satisfy (13).

First, we will show that in nonuniform-exponential setting it is not possible that more than 2 vectors satisfy (14). We prove this using contradiction. Suppose that (14) holds for three different vectors i_1, i_2, i_3 at the same time. In that case we can write:

$$\begin{aligned} &2l\pi + \varphi_{i_1}' + \varphi_{i_1} - \theta_x = \theta_{x'}, \\ &2l\pi + \varphi_{i_2}' + \varphi_{i_2} - \theta_x = \theta_{x'}, \\ &2l\pi + \varphi_{i_3}' + \varphi_{i_3} - \theta_x = \theta_{x'}. \end{aligned}$$

Subtracting first and second equations and also first and third ones results in the following equations,

$$\begin{aligned} &\varphi_{i_1}' - \varphi_{i_2}' = \varphi_{i_2} - \varphi_{i_1}, \\ &\varphi_{i_1}' - \varphi_{i_3}' = \varphi_{i_3} - \varphi_{i_1}. \end{aligned} \quad (16)$$

Since the spacing between vectors are monotonically increasing, the first equation in (16) is valid only when $\varphi_{i_1}' = \varphi_{i_2}$ and $\varphi_{i_2}' = \varphi_{i_1}$.

At the same time, the second equation holds if and only if $\varphi'_{i_1} = \varphi_{i_3}$ and $\varphi'_{i_3} = \varphi_{i_1}$. This means that $\varphi_{i_2} = \varphi_{i_3}$ which is not possible. This contradicts the assumption of setting with N distinct vectors, therefore no more than 2 vectors can satisfy (14).

No we want to consider the case where only one sampling vector satisfies (14) and all the other vectors satisfy (13). In this case we can say,

$$\begin{aligned} 2l\pi + \varphi'_{i_1} + \varphi_{i_1} - \theta_x &= \theta_{x'}, \\ 2l\pi + \varphi'_i - \varphi_i - \theta_x &= \theta_{x'}\ , \forall i \neq i_1. \end{aligned} \tag{17}$$

From (17) we conclude that $2\theta_x = (\varphi'_{i_1} + \varphi_{i_1}) + (\varphi_i - \varphi'_i)$. Considering this equation for $i = m, n,\ n, m \neq i_1$ we can simply verify that $(\varphi_m - \varphi'_m) = (\varphi_n - \varphi'_n)$. This can be true if and only if all the vectors for Φ and Φ' are the same meaning that $\Phi = \Phi'$ and thus $x = x'$. This contradicts the assumption of non-unique solution. Therefore, it is not possible to have only one sampling vector satisfying (14) and all other satisfying (13).

The remaining task is to show if it is possible that all vectors satisfy (13). In this case similar to previous steps, we can write (13) for two arbitrary distinct vectors like $n, m\ , n \neq m$ and subtract the two equations. We can easily verify that in that case $(\varphi'_m - \varphi'_n) = (\varphi_m - \varphi_n)$. This is true only if $\varphi'_i = \varphi_i,\ \forall i$. Thus we can say $\Phi = \Phi'$ which is not in accordance to our initial assumption. □

Lemma 3. *In the nonuniform-exponential setting with N vectors, lets define $b_{i,j}$ as the bisector angle of two vectors φ_i and φ_j. If $i \neq i'$ or $j \neq j'$ then $b_{i,j} \neq b_{i',j'}$.*

Proof. We know that $b_{i,j} = \frac{\varphi_i + \varphi_j}{2}$. We can easily verify that,

$$b_{i,j} = \pi \frac{2^i + 2^j - 2}{2^N - 1}. \tag{18}$$

We prove the lemma by contradiction. Suppose that $b_{i,j} = b_{i',j'}$. In that case $2^i + 2^j = 2^{i'} + 2^{j'}$. First consider the case that $i = i'$ but $j \neq j'$, it is clear that this equation has no valid solution. Proof for the reversed case when $i \neq i'$ but $j = j'$ is also trivial. It remains to consider the case where $i \neq i'$ and $j \neq j'$. Consider the following definitions,

$$\begin{aligned} T_{\text{max}} &\triangleq \max\{i, j\}, \\ T_{\text{min}} &\triangleq \min\{i, j\}, \\ T'_{\text{max}} &\triangleq \max\{i', j'\}, \\ T'_{\text{min}} &\triangleq \min\{i', j'\}. \end{aligned} \tag{19}$$

We can rewrite $2^i + 2^j = 2^{i'} + 2^{j'}$ as,

$$2^{T_{\text{max}}} + 2^{T_{\text{min}}} = 2^{T'_{\text{max}}} + 2^{T'_{\text{min}}}.$$

Also this equation can be written as,

$$2^{T_{\text{min}}}(1 + 2^{T_{\text{max}} - T_{\text{min}}}) = 2^{T'_{\text{min}}}(1 + 2^{T'_{\text{max}} - T'_{\text{min}}}).$$

If $T_{\text{min}} > T'_{\text{min}}$ we divide both sides of the equation system to T'_{min}, otherwise we divide it to T_{min}.

$$2^{T_{\text{min}} - T'_{\text{min}}}(1 + 2^{T_{\text{max}} - T_{\text{min}}}) = (1 + 2^{T'_{\text{max}} - T'_{\text{min}}}). \tag{20}$$

In (20) we can see that right side of the equation is odd when left hand side is even. This is not possible so this case is not also possible. We conclude that if $i \neq i'$ or $j \neq j'$, then $b_{i,j} \neq b_{i',j'}$. □

Algorithm 1 Efficient algorithm in 2D

1: **Initialize:**
    $P = \emptyset$, feasible set: $F = \emptyset$.
2: Take every $N(N-1)$ selection of two rows of Φ.
3: Put all $N(N-1)$ matrices with size 2×2 in set P.
4: **for** each $P_i \in P$, $i = 1, 2, \cdots, N(N-1)$ **do**
5:     Solve 2×2 equations $[y_1, y_2]^T = P_i x^i$.
6:     $F \leftarrow F \cup \{x^i\}$.
7: **end for**
8: **for** each sample value y_n, $k = 3, 4, \cdots, N$ **do**
9:     **for** every $x^i \in F$ **do**
10:         Find 2 tangent lines from x^i to the circle $c(0, y_n)$.
11:         Check if any of the two tangent points on the c lies on any of the remaining $N - (n-1)$ rows of Φ.
12:         **if** neither of two tangent points lies on any of the rows **then** Remove x^i from F.
13:         **end if**
14:     **end for**
15: **end for**
16: **return** $x \in F$.

3. POINTS ELIMINATION IN GR ALGORITHM IN 2D CASE

Lemma 4. *In Algorithm 1, at n-th iteration of for loop stated in step 8, when we are considering the n-th sample value y_n, at most $N(N-1)/(n-2)$ points are remaining in F.*

Proof. Suppose that we are evaluating y_3. We have seen in Algorithm 1 that at the beginning we have $N(N-1)$ points in the initial set. At this step, we evaluate every point in F and keep the points in F if and only if one of the tangent points to the circle $c(0, y_3)$ lies on one of the $N-2$ remaining vectors.

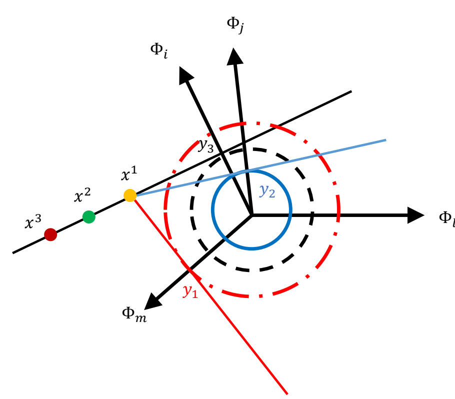


Fig. 1: Collinearity of points in feasible set

For better illustration, in Figure 1, we have 4 vectors, therefore there are 4 perpendicular lines related to sample value y_3. If neither of the points are colinear, then at most 4 points remains in set F at the end of this iteration. However, if we consider colinear points, they can all remain in set F. Now let us check how many colinear points remains at this step. Points x_1, x_2 and x_3 are colinear and they all have projection value y_3 over sampling vector φ_i. Recall that each point is generated by intersecting two tangent lines related to sample values y_1 and y_2. If these points are colinear on sample value y_3, they can not be colinear on neither y_1 or y_2 since two points

Algorithm 2 Combinatorial algorithm 1

1: **for** each $i = 1, 2, \cdots, N!$ **do**
2: Generate $B_i = S_i \Phi$.
3: Check if $y = B_i x$ has a solution:
4: Choose $\widehat{B_i}$, the upper $K \times K$ sub-matrix of B_i.
5: Find $\widehat{x}$ from $\widehat{B_i}\widehat{x} = y$.
6: Put $\widehat{x}$ in the lower $N - K$ equations of $B_i x = y$.
7: **if** $\widehat{x}$ satisfies all $N - K$ equations **then return** $\widehat{x}$ as the solution.
8: **end if**
9: **end for**

can only lie on one line. Therefore, for each remaining colinear point we have two distinct tangent lines which are related to other $N - 1$ vectors (other than φ_i) and we can say ((number of points) $\times 2 = (N-1)$. Number of colinear points corresponding to φ_i is $(N-1)/2$. However, we have N different vectors, so the total number of remaining points can not exceed $N(N-1)/2$. This is the total number of points for the next iteration when we are considering y_4.

This result can be generalized to other iterations. For instance, when we are considering y_{n-1}, for the points in feasible set, we have already found $n-2$ vectors. Thus, for each perpendicular line, at most there exist $\frac{N-1}{n-2}$ collinear points and for all N perpendicular lines, $\frac{N(N-1)}{n-2}$ points can survive to go to the next iteration. It means that when we are considering y_n, at most $\frac{N(N-1)}{n-2}$ points remain in set F.

In practice, this number is much less than this value since when points are collinear at one step, they can not be collinear at the next step. But here we considered the collinearity for each step independently. This can be viewed as the main reason for reducing the complexity of GR algorithm. In practice, we see that in non-symmetric settings such as random or exponential sampling settings, one or two points remain in the feasible set after considering the third sample value. □

4. COMPLEXITY DERIVATION OF GR ALGORITHM IN HIGHER DIMENSIONAL SPACES COMPARED TO COMBINATORIAL ALGORITHMS

4.1. Combinatorial algorithm 1

Suppose that we have N different sample values in vector y. The order of sample values is not specified. A naïve combinatorial algorithm to reconstruct vector x is as follows. By permuting rows of matrix Φ, we have $N!$ different $B_i = S_i\Phi$, $i = 1, \cdots, N!$. In algorithm 2, at each step we consider matrix B_i and check if there exists a solution for $B_i x = y$. The complexity of each step is as follows.

- In step 4, for a given B_i, inverting a $K \times K$ sub-matrix requires K^3 multiplications.
- In step 5, K^2 multiplications are required for calculating elements of $\widehat{x}$.
- In step 6, checking each equality requires K multiplications. Thus, in total $D(N-K)$ is the complexity of this step.

These steps repeat for each B_i. Therefore, in total the complexity of this algorithm is calculated as follows.

$$C_1 = N! \times [K^3 + K^2 + K(N-K)] = N!(K^3 + NK). \quad (21)$$

Algorithm 3 Combinatorial algorithm 2

1: Choose $\widehat{\Phi}$, the upper $K \times K$ sub-matrix of Φ and invert it.
2: **for** each $i = 1, 2, \cdots, N!$ **do**
3: Generate $y^i = S_i y$.
4: Check if $y^i = \Phi x$ has a solution:
5: Choose $\widehat{y^i}$, the upper $K \times 1$ sub-vector of y^i.
6: Find $\widehat{x}$ from $\widehat{\Phi}\widehat{x} = \widehat{y^i}$.
7: Put $\widehat{x}$ in the lower $N - K$ equations of $\Phi x = y^i$.
8: **if** $\widehat{x}$ satisfies all $N - K$ equations **then return** $\widehat{x}$ as the solution.
9: **end if**
10: **end for**

We conclude that the complexity of Algorithm 2 is $O(K!(D^3 + KD))$.

4.2. Combinatorial algorithm 2

Algorithm 2 considers all possible B_i and for each matrix calculates the inverses of $K \times K$ sub-matrices. In this section we consider another combinatorial algorithm that plays smarter by just considering the inversion procedure once. Now we evaluate the computational complexity for each step of Algorithm 3.

- Step 1 requires one $K \times K$ matrix inversion which takes K^3 operational complexity.
- In step 6, for each i (in total $N!$), we need to compute $\widehat{x} = \widehat{\Phi}^{-1}\widehat{y^i}$, which requires K^2 multiplications.
- In step 7, checking each equality requires K multiplications. Thus, the complexity of this step is $K(N-K)$.

We can calculate the complexity of Algorithm 3 as follows.

$$C_2 = K^3 + N! \times [K^2 + K(N-K)] = K^3 + N!NK. \quad (22)$$

Thus, the total complexity of Algorithm 3 is $O(K^3 + N!NK)$ which is less than Algorithm 2 but still is in order of $N!$.

4.3. GR algorithm for unlabeled sensing in higher dimensional spaces

Algorithm 1 can be extended to higher dimensional spaces. Algorithm 4 illustrates this extension. In this part we are interested to evaluate the complexity of algorithm 4.

- In step 5, in total we have to find the inverse of $N(N-1)\cdots(N-K+1)$ different $K \times K$ matrices. The complexity for solving each equation system is $O(K^3 + K^2)$, where K^3 is the complexity of inversion and K^2 is the complexity order of K multiplications. Therefore, in total the order of the complexity for this step is $O([K^3+K^2][N(N-1)\cdots(N-K+1)])$. Just considering the dominant terms, this step takes $O((K^3+K^2)N^K)$ operations.
- In the step 10, all we have to do is to check if the points that we have already found have the projection equal to one of the sample values over one of the remaining vectors. Checking part will take $(N-K)K$ multiplications and for all the points it will be on order $O([K(N-K)][N(N-1)\cdots(N-K+1)])$. Considering the dominant terms will have $O(KN^{K+1})$.

It is obvious that compared to $O(N!(K^3 + NK))$ and $O(K^3 + N!NK)$ which are the complexities of combinatorial algorithms 2 and 3, our efficient algorithm achieves a very reduced complexity $O(DN^{K+1})$.

Algorithm 4 Efficient algorithm in higher dimensional spaces

Initialize:
  $P = \emptyset$, feasible set: $F = \emptyset$
Take every $N(N-1)\cdots(N-K+1)$ selection of K rows of Φ.
Put all $N(N-1)\cdots(N-K+1)$ matrices with size $K \times K$ in set P.
for each $P_i \in P, i = 1, 2, \cdots, N(N-1)\cdots(N-K+1)$ **do**
  Solve $K \times K$ equations $[y_1, \cdots, y_K]^T = P_i x^i$.
  $F \leftarrow F \cup \{x^i\}$.
end for
for each sample value y_n, $n = K+1, \cdots, N$ **do**
  for every $x^i \in F$ **do**
    Check if x^i has the projection value equal to y_n over any of the remaining $N-(n-1)$ rows of Φ.
    if such a row is not found **then** Remove x^i from F.
    end if
  end for
end for
return $x \in F$.

5. DISTANCE THRESHOLD DERIVATION IN GENERALIZED GR ALGORITHM IN NOISY CASES

$$d^{(n,m,p)}_{(k,l,j)} \leq d^{(n,m,p)}_{\text{th}}. \tag{23}$$

for some $p \in \mathbb{N} \setminus \{m, n\}$. Here, $d^{(n,m,p)}_{\text{th}}$ shows the upper bound for the distance from point $x^{n,m}_{k,l}$ to the line L^p_j related to sampling vector φ_p. Point $x^{n,m}_{k,l}$ is derived by intersecting perpendicular lines to sampling vectors φ_n and φ_m with offset value from the origin equal to y_k and y_l. For the target point x, the distance from each line to the point meets this constraint, thus points that have the distance greater than this value from all possible vectors can be easily discarded from feasible set. we have to calculate the $d^{(n,m,p)}_{\text{th}}$ as the upper bound for the distance from one point to the line. Suppose that $x = [x_0, x_1]^T$ is the target point in 2-D plane. x_0 is the projection of point x on the first standard basis e_0 and x_1 is the projection of x on the second standard basis e_1. In addition, suppose that we know that the correct-ordered sample value related to sampling vector φ_m is y_k and correct-ordered sample value related to vector φ_n is y_l. For noiseless y_k and y_l, the following equation holds:

$$x = \frac{1}{\sin(\varphi_n - \varphi_m)} \begin{bmatrix} y_k \sin(\varphi_n) - y_l \sin(\varphi_m) \\ y_l \cos(\varphi_m) - y_k \cos(\varphi_n) \end{bmatrix}. \tag{24}$$

In (24), if we replace y_k and y_l with the noisy sample values $\widehat{y}_k = y_k + \varepsilon_k$ and $\widehat{y}_l = y_l + \varepsilon_l$, we get $\widehat{x}$. This point is the intersection point of noisy sample value lines in GR algorithm.

$$\begin{aligned} \widehat{x} = x^{m,n}_{k,l} &= \frac{1}{\sin(\varphi_n - \varphi_m)} \begin{bmatrix} \widehat{y}_k \sin(\varphi_n) - \widehat{y}_l \sin(\varphi_m) \\ \widehat{y}_l \cos(\varphi_m) - \widehat{y}_k \cos(\varphi_n) \end{bmatrix} \\ &= \frac{1}{\sin(\varphi_n - \varphi_m)} \begin{bmatrix} y_k \sin(\varphi_n) - y_l \sin(\varphi_m) \\ y_l \cos(\varphi_m) - y_k \cos(\varphi_n) \end{bmatrix} \\ &\quad + \frac{1}{\sin(\varphi_n - \varphi_m)} \begin{bmatrix} \varepsilon_k \sin(\varphi_n) - \varepsilon_l \sin(\varphi_m) \\ \varepsilon_l \cos(\varphi_m) - \varepsilon_k \cos(\varphi_n) \end{bmatrix}. \end{aligned}$$

Now we consider that point x has the sample value y_j on sampling vector φ_p. It is easy to derive the equation of the line L^p_j that is orthogonal to φ_p and has distance from origin equal to $\widehat{y}_j$:

$$\sin(\varphi_p)x_1 + \cos(\varphi_p)x_0 - \widehat{y}_j = 0$$

and we know that the distance from point $x = [x_0, x_1]^T$ and line $ax + by - c = 0$ is equal to

$$Distance(ax + by - c = 0, x = [x_0, x_1]^T) = \frac{|ax_0 + bx_1 - c|}{\sqrt{a^2 + b^2}}.$$

Therefore, by replacing x with $x^{m,n}_{k,l}$ and line equation with L^p_j and simplifying the equations, we can write the distance threshold as follows:

$$\begin{aligned} &Distance(L^p_j, x^{m,n}_{k,l}) \\ &= \left| \varepsilon_k \frac{\sin(\varphi_n - \varphi_p)}{\sin(\varphi_n - \varphi_m)} + \varepsilon_l \frac{\sin(\varphi_p - \varphi_m)}{\sin(\varphi_n - \varphi_m)} - \varepsilon_j \right|. \end{aligned}$$

We are not aware of the exact value of noise, but here we assume that the noise is bounded and we know the upper bound of the noise values. In other words, we know that $|\varepsilon_i| \leq \varepsilon_{max}, \forall i \in \mathbb{N}$. It is easy to derive $d^{(n,m,p)}_{\text{th}}$, the distance upper bound as written in following equation.

$$\begin{aligned} &d^{(n,m,p)}_{\text{th}} \leq \frac{\varepsilon_{max}}{|\sin(\varphi_n - \varphi_m)|} \times \\ &[|\sin(\varphi_n - \varphi_p)| + |\sin(\varphi_p - \varphi_m)| + |\sin(\varphi_n - \varphi_m)|]. \end{aligned} \tag{25}$$

This completes the derivation of the distance constraint for generalized GR. We can use this threshold to eliminate invalid labellings from feasible set in generalized GR.

6. NOISE ROBUSTNESS OF THE GENERALIZED GR ALGORITHM AND BOUND DERIVATION

Theorem 3. *Consider the exponentially-spaced vector setting with $\|\boldsymbol{\epsilon}\|_\infty \leq \epsilon_{max}$. If there exist two solutions $\boldsymbol{x}$ and $\boldsymbol{x}'$ with different labelings, then*

$$\|\boldsymbol{x} - \boldsymbol{x}'\| \leq \frac{4\epsilon_{max}}{\cos(\frac{2^{N-1}-1}{2^N-1}\pi)}.$$

Proof. Based on Lemma (5) which will be presented later, all the sampling vectors cover one side of the plane. In this way we can make sure that positive and negative vectors are separated and we can find the smallest and largest angles by searching among one group. The next task is to find the smallest and the largest angles between sampling vectors. We call them $\delta_{\min}$ and $\delta_{\max}$ respectively.

$$\delta_{\min} = \min_{i,j}\{\delta_{i,j}\} = \varphi_1 - \varphi_0 = \delta_{1,0} = (\frac{2\pi}{2^N - 1}). \tag{26}$$

$$\delta_{\max} = \max_{i,j}\{\delta_{i,j}\} = \varphi_{N-1} - \varphi_0 = \delta_{N-1,0} = 2\pi(\frac{2^{N-1} - 1}{2^N - 1}). \tag{27}$$

Based on Lemma (6), if two sample values are interchangeable then their distance from each other is less than $4\epsilon_{\max}$. Suppose that two sample values have the distance less than $4\epsilon_{\max}$. In this case, according to other sample values there might be another solution to the equation system $y = \Phi x + \epsilon$ with different labeling. Note that different labellings result from the interchangeable sample values. Thus we are interested to find the farthest distance between the consistent regions of two interchangeable labellings related to two solutions. We can easily verify that the intersection of two $4\epsilon_{\max}$ region over two distinct sampling vectors is a symmetric diamond on the bisector of two mentioned vectors. Two cases might happen:

This diamond might be vertical or horizontal. In other words, if $\delta_{i,j}$ which is the angle between φ_i and φ_j is greater than $\pi/2$ then the diamond is vertical otherwise if $\delta_{i,j} > \pi/2$ the diamond is horizontal. In the former, the farthest points in the diamond are the top and bottom vertices and in the latter when the diamond is horizontal, then the farthest points are right and left vertices. When the diamond is horizontal, maximum distance among all possible points in the plane can be achieved when the point is around the bisector of the smallest angle between two vectors, δ_{min} and similarly when the diamond is vertical the maximum distance is achieved for the points around the bisector of the largest angle between two sampling vectors, δ_{max}. Lemma (7) and Lemma (8) provide us with upper bounds for maximum distances. From this results we can conclude that

$$\|x - x'\| \leq \max\{d_H, d_V\}. \tag{28}$$

We need to find the maximum value between d_H and d_V. We can rewrite d_H as follows,

$$d_H = \frac{4\epsilon_{\text{max}}}{\sin(\frac{\pi}{2^N-1})} = \frac{4\epsilon_{\text{max}}}{\cos(\frac{\pi}{2} - \frac{\pi}{2^N-1})}. \tag{29}$$

We define $\theta_H \triangleq \frac{\pi}{2} - \frac{\pi}{2^N-1} = \pi\frac{2^{N-1}-1.5}{2^N-1}$ and $\theta_V \triangleq \pi\frac{2^{N-1}-1}{2^N-1}$. Obviously, $0 < \theta_H < \theta_V < \pi/2$, thus $0 < \cos(\theta_V) < \cos\theta_H < 1$. From Lemma (7) and (8) we conclude that $d_H < d_V$ and thus $d_V = \max\{d_H, d_V\}$. Substituting the result in (28) we get,

$$\|x - x'\| \leq \frac{4\epsilon_{\text{max}}}{\cos(\pi\frac{2^{N-1}-1}{2^N-1})}.$$

□

Lemma 5. *In nonuniform-exponential sampling setting the angle between every two vectors is less that* π*.*

Proof. We can verify that $\varphi_0 = 0$ and

$$\varphi_{N-1} = 2\pi\frac{2^{N-1}-1}{2^N-1} = \pi\frac{2^{N-1}-1}{2^{N-1}-0.5} = \pi(1 - \frac{1}{2^N-1}) < \pi.$$

We can see that the smallest and the largest angles lie into the interval $[0, \pi)$. □

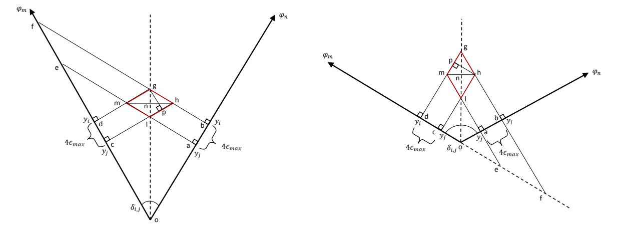
In this part we want to provide a very simple lemma regarding the maximum distance between interchangeable sample values.

Lemma 6. *If* $\|\epsilon\| \leq \epsilon_{max}$ *and* $\exists y_i, y_j,\ i \neq j$ *such that* $|y_i - y_j| \leq 4\epsilon_{max}$ *then solutions for* $y = \Phi x + \epsilon$ *might be interchangeable.*

Proof. If the distance between two sample values is less than $4\epsilon_{\text{max}}$ then these two samples are interchangeable. The reason for this claim is that $\pm\epsilon_{\text{max}}$ around each noisy sample value is the possible region that the noiseless sample value lies. If two noisy sample values are closer than $4\epsilon_{\text{max}}$ then they both might be translated to the same noiseless sample values or equivalently, their related noiseless sample values can be interchangeable. □

In this section we want to find the farthest points in the horizontal and vertical diamonds which is translated to the maximum distance between valid solutions of unlabeled sensing problem with nonuniform-exponential sampling setting.

Lemma 7. *In nonuniform-exponential sampling setting with* N *vectors, if* $\delta_{i,j} < \pi/2$ *then the distance between the farthest points in the horizontal diamond bisector region is upper bounded by* $d_H = \frac{4\epsilon_{max}}{\sin(\frac{\pi}{2^N-1})}$*.*



(a) Horizontal diamond interchangeable region (b) Vertical diamond interchangeable region

Fig. 2: Maximum distance between two valid regions with interchangeable labellings

Proof. In figure 2, we use geometry to derive the upper bound for the distance. We can simply verify that $\overset{\triangle}{fhc} \sim \overset{\triangle}{fbo}$, therefore from equivalence of angles of two triangles we can say that $\widehat{ghp} = \delta_{i,j}$. $gp = 4\epsilon_{\text{max}}$. Thus in $\overset{\triangle}{ghp}$ we can write $gh = \frac{gp}{\sin(\delta_{i,j})} = \frac{4\epsilon_{\text{max}}}{\sin(\delta_{i,j})}$. On the other hand, in $\overset{\triangle}{ghn}$ we have $hn = gh\cos(\delta_{i,j}/2) \triangleq d$. We can easily derive that $d = \frac{2\epsilon_{\text{max}}}{\sin(\delta_{i,j}/2)}$. Following this derivation we can write $d_H = mh = 2d = \frac{4\epsilon_{\text{max}}}{\sin(\delta_{i,j}/2)}$. By substituting $\delta_{i,j} = \delta_{\text{max}}$,the largest angle between sampling vectors, we will have,

$$d_H = 2d = \frac{4\epsilon_{\text{max}}}{\sin(\frac{\pi}{2^N-1})}.$$

□

Lemma 8. *In nonuniform-exponential sampling setting with* N *sampling vectors, if* $\delta_{i,j} > \pi/2$ *then the distance between the farthest points in the vertical diamond bisector region is upper bounded by* $d_V = \frac{4\epsilon_{max}}{\cos(\pi\frac{2^{N-1}-1}{2^N-1})}$*.*

Proof. Figure 2 illustrates the geometry of this region. In $\overset{\diamond}{bodg}$ we can write $\widehat{aoc} + \widehat{mgh} = 180$. On the other hand, in $\overset{\diamond}{mghl}$ we have $\widehat{ghl} + \widehat{mgh} = 180$. From these two equalities we conclude that $\widehat{ghl} = \delta_{i,j}$. In $\overset{\triangle}{pgh}$ we can derive the equality $\widehat{mgh} + \widehat{ghp} = 90$ and we know that $\widehat{mgh} = 180 - \delta_{i,j}$ so we can conclude that $\widehat{ghp} = \delta_{i,j} - 90$. From triangular equalities we can say $gh = \frac{hp}{\cos(\delta_{i,j}-90)} = \frac{4\epsilon_{\text{max}}}{\sin(\delta_{i,j})}$. On the other hand, in $\overset{\triangle}{ghn}$, we can write $gn = gh\sin(\delta_{i,j}/2) \triangleq d$. We can easily verify that $d = \frac{2\epsilon_{\text{max}}}{\cos(\delta_{i,j}/2)}$. In this way maximum distance between two interchangeable consistent regions can be achieved by considering $\delta_{i,j} = \delta_{\text{max}}$.

$$d_V = 2d = \frac{4\epsilon_{\text{max}}}{\cos(\pi\frac{2^{N-1}-1}{2^N-1})}.$$

□

References

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