NUMERICAL HOMOGENIZATION AND MODEL ORDER REDUCTION FOR MULTISCALE INVERSE PROBLEMS
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Abstract. A new numerical method based on numerical homogenization and model order reduction is introduced for the solution of multiscale inverse problems. We consider a class of elliptic problems with highly oscillatory tensors that varies on a microscopic scale. We assume that the micro structure is known and seek to recover a macroscopic scalar parametrization of the microscale tensor (e.g. volume fraction). Departing from the full fine scale model that would require mesh resolution for the forward problem down to the finest scale, we solve the inverse problem for a coarse model obtained by numerical homogenization. The input data, i.e., measurement from the Dirichlet to Neumann map, are solely based on the original fine scale model. Furthermore, reduced basis techniques are used to avoid computing effective coefficients for the forward solver at each integration point of the macroscopic mesh. Uniqueness and stability of the effective inverse problem is established based on standard assumptions for the fine scale model and a link with this latter model is established by means of G-convergence. A priori error estimates are established for our method. Numerical experiments illustrate the efficiency of the proposed scheme and confirm our theoretical finding.

Key words. Inverse problems, Homogenization, Multiscale methods, Reduced basis.

AMS subject classifications. 65N21, 65N30, 35R30, 35B40

1. Introduction. Many applications in engineering and the sciences require solving inverse problems for partial differential equations (PDEs). We mention applications in heat conduction, geoscience and wave scattering, medical imaging, etc [13]. In this paper we are interested in PDEs that vary on very fine scale that come, e.g., from heterogeneity in the medium. Assuming that the nature of the micro structure is known, we search for an unknown parametrization of this fine scale structure from the knowledge of measurements coming from the Dirichlet to Neumann map. A typical example is multi-phase medium whose constituents are known but whose volume fraction or its macroscopic orientation are unknown. Classical approaches for such problem would require the resolution of forward problems requiring mesh resolution of the fines scale. Repeated solutions of such high dimensional problem represent a formidable computational challenge and is often not tractable. Using coarse graining techniques and model order reduction can overcome this computational issue for classes of multiscale PDEs. Among such problems we consider the following multiscale elliptic problem. Let \( \Omega \in \mathbb{R}^d, \ d \geq 2 \) be an open, bounded, connected set with sufficiently smooth boundary \( \partial \Omega \) and consider the problem of finding the weak solution \( u^\varepsilon \in H^1(\Omega) \) to

\[
\begin{array}{ll}
- \nabla \cdot (A^\varepsilon \nabla u^\varepsilon) = 0 & \text{in } \Omega, \\
u^\varepsilon = g & \text{on } \partial \Omega,
\end{array}
\]

where \( g \in H^{1/2}(\partial \Omega) \). The tensor \( A^\varepsilon = A^\varepsilon(x), \ x \in \Omega \) belongs to \( \mathcal{M}(\alpha, \beta, \Omega) \), where

\[
\mathcal{M}(\alpha, \beta, \Omega) := \{ A \in L^{\infty}(\Omega, Sym_d) : \alpha|\xi|^2 \leq A(x) \cdot \xi \leq \beta|\xi|^2 \text{ for all } \xi \in \mathbb{R}^d \text{ and almost all } x \in \Omega, \}
\]

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where $\text{Sym}_d$ denotes the class of $d \times d$ real valued symmetric matrices. The superscript in $A^\varepsilon$ indicates that the tensor varies on a fine scale $\varepsilon$ that is much smaller than the size of the computational domain $\Omega$. In turn, the solution of (1.1) itself has variations on such micro scales.

Next, we introduce the Dirichlet to Neumann map associated to the boundary value problem (1.1) as the operator $\Lambda_{A^\varepsilon} : H^{1/2}(\partial \Omega) \to H^{-1/2}(\partial \Omega)$ given by

$$g \mapsto A^\varepsilon \nabla u^\varepsilon \cdot \nu|_{\partial \Omega},$$

where $\nu$ denotes the exterior unit normal to $\partial \Omega$. In this work we are concerned with the inverse conductivity problem of determining $A^\varepsilon$ from the knowledge of the Dirichlet to Neumann map $\Lambda_{A^\varepsilon}$. The inverse conductivity problem, which is also known as electrical impedance tomography (EIT), was proposed first by Calderón [10], and it has gained great popularity in the last decades. Many authors have studied important questions that arise when facing the inverse conductivity problem, such as uniqueness of $A^\varepsilon$ given $\Lambda_{A^\varepsilon}$, the recovery of $A^\varepsilon$ from $\Lambda_{A^\varepsilon}$, and finally the stability or continuity of the inverse map $\Lambda_{A^\varepsilon} \mapsto A^\varepsilon$ [7, 25, 27, 29, 31].

In this paper we are interested in a class of parametrized anisotropic locally periodic multiscale tensor $A^\varepsilon(x) = A(\sigma(x), x/\varepsilon) = A(\sigma(x), y)$ $Y$ periodic in the $y$ variable (here without loss of generality we assume that $Y$ is a cube $Y = (0,1)^d$). The map $(t,x) \mapsto A(t,x/\varepsilon)$ is assumed to be known and $\sigma$ has to be determined. When the tensor does not exhibit a multiscale variation, i.e., for $(t,x) \mapsto A(t,x)$, uniqueness and stability at the boundary were proved by G. Alessandrini and R. Gaburro [6], under some regularity assumptions on the map $(t,x) \mapsto A(t,x)$. While this results is still valid for highly oscillating tensors, the stability estimates will depend on a constant that scales as $O(\varepsilon^{-1})$. In turn, classical numerical techniques such as finite element methods (FEMs) to compute numerically the inverse problem will need scale resolution (i.e. mesh size resolving the smallest scale $\varepsilon$) which represents often a prohibitive cost. Therefore we combine the inverse problem with a coarse grained strategy, which simplifies remarkably the computational effort, but on the other hand introduces additional discrepancies between “reality” (model (1.1), from where the data are obtained) and the model used for inversion.

Homogenization theory [8, 23] ensures that the solution to problem (1.1) converges (in a weak sense) to a homogenized solution $u^0$, solving the elliptic problem

$$
\begin{align*}
- \nabla \cdot (A^0 \nabla u^0) &= 0 & \text{in } \Omega, \\
\quad u^0 &= g & \text{on } \partial \Omega.
\end{align*}
$$

In this work we want to analyse the possibility of retrieving $A^\varepsilon$, in the case where the observed data are obtained from the full multiscale model (1.1), but using as forward model the Dirichlet to Neumann map defined by the homogenized model (1.3). Numerically, the explicit form of $A^0$ is usually not known and can only be recovered at some quadrature points of the macroscopic computational mesh. We therefore rely on the finite element heterogeneous multiscale method (FE-HMM) [1, 2, 4] that recovers such tensors with input data only given by the fine scale model (1.1), by solving appropriate micro problems. Furthermore, as such coarse grain forward solution relies on increasingly accurate micro solutions, we further employ reduced basis techniques to precompute a reduced number of conductivity tensors that are then appropriately interpolated when solving the forward problem following the methodology developed in [3].
We note that the use of the FE-HMM for multiscale inverse problems has first been introduced in [17], where by means of numerical investigation, it is shown that the numerical homogenization can be used for the considered class of multiscale inverse problems. In our paper, we generalize the applicability of the numerical homogenization, we provide a theoretical investigation both on the model problem and on the computational approach for such coarse graining strategy and we further introduce model reduction strategy for the numerical method. We briefly discuss the main contribution of our paper. First, while in [17] it assumed that the scalar parameter $\sigma$ is accurately parametrized by piecewise smooth coefficients, we consider instead general scalar parameter where no specific form of $t \rightarrow A(t, x/\varepsilon)$ is taken into account. Second, assuming that the fine scale inverse problem is well posed, we show that the effective inverse problem is also well posed and we establish stability results independent of the small scale $\varepsilon$. Furthermore, by means of G-convergence we characterize the discrepancy between the fine scale and and coarse scale Dirichlet to Neumann map. As the full Dirichlet to Neumann map is usually not available, we discuss a numerical strategy based on finite measurements of this map. For our more general class of multiscale tensors, regularization is needed and we analyse this strategy in the context of multiscale inverse problems. Finally we provide a new numerical strategy for the inverse problem that combines numerical homogenization and reduced basis techniques. A priori error estimates for the computation of effective boundary fluxes is analysed for this method and convergence of the discrete optimization problem is established.

The outline of the work is as follows. In Section 2 we recall briefly results of uniqueness and stability for the class of inverse problems that we consider. In Section 3 we analyse in detail the coarse grained approach to solve multiscale inverse problem, while the analysis on the regularized problem is performed in Section 4. In Section 5 we describe how the multiscale inverse problem is solved numerically. In particular, we introduce numerical homogenization and we give a priori error estimates for the approximated flux at the boundary. An analysis on the discrete solution of the multiscale inverse problem is also given. Finally in Section 6 we present some numerical results to test our theoretical findings and illustrate our numerical method.

2. Results of uniqueness for the multiscale inverse problem. Let $\Omega$ be an open bounded set in $\mathbb{R}^d$ and let $A'(x)$ be of the form $A(\sigma(x), x/\varepsilon)$, for a certain scalar function $\sigma : \Omega \rightarrow \mathbb{R}$. In [6] uniqueness and stability at the boundary for the inverse conductivity problem for special anisotropic tensors of the form $A(x) = A(\sigma(x), x)$ is proved, in the case where some prior knowledge on the map $(t, x) \mapsto A(t, x)$ is assumed. In particular it is required that the tensor $A(t, x)$, $t \in [\alpha, \beta]$, $0 < \alpha < \beta$, $x \in \Omega$, belongs to some special class of matrix functions that we recall below. In what follows we will use the following norms

$$||M||_{L^p(\Omega)} = \left( \int_{\Omega} \sum_{i=1}^{d} \sum_{j=1}^{d} |m_{ij}(x)|^p \, dx \right)^{1/p}, \quad 1 \leq p < \infty,$$

$$||M||_{L^\infty(\Omega)} = \max_{1 \leq i,j \leq d} \text{ess sup}_{x \in \Omega} |m_{ij}(x)|, \quad p = \infty.$$

for a matrix $M(x) = \{m_{ij}(x)\}_{1 \leq i,j \leq d}$, $x \in \Omega$.

Definition 2.1. [Definition 2.2 in [6]]. Given $p > d$, $E > 0$, and denoting by $\text{Sym}_d$ the class of $d \times d$ real valued symmetric matrices, we say that $A : [\alpha, \beta] \times \Omega \rightarrow \text{Sym}_d$ belongs to $\mathcal{H}$ if the following conditions hold for all $t \in [\alpha, \beta]$, $0 < \alpha < \beta$:
1. \( A \in W^{1,p}([\alpha, \beta] \times \Omega, \text{Sym}_d) \).
2. \( \partial_t A \in W^{1,p}([\alpha, \beta] \times \Omega, \text{Sym}_d) \).
3. \( \text{ess sup}([A(t, \cdot)]_{L^p(\Omega)} + |\nabla_x A(t, \cdot)|_{L^p(\Omega)} + \| \partial_t A(t, \cdot) \|_{L^p(\Omega)} + \| \partial_t \nabla_x A(t, \cdot) \|_{L^p(\Omega)}) \leq E \).
4. Condition of uniform ellipticity:
\[ \alpha|\xi|^2 \leq A(t, x)\xi \cdot \xi \leq \beta|\xi|^2, \quad \text{for a.e. } x \in \Omega \]
\[ \text{and } \forall t \in [\alpha, \beta], \xi \in \mathbb{R}^d. \]
5. Condition of monotonicity with respect to the variable \( t \):
\[ \partial_t A(t, x)\xi \cdot \xi \geq E^{-1}|\xi|^2, \quad \text{for a.e. } x \in \Omega \]
\[ \text{and } \forall t \in [\alpha, \beta], \xi \in \mathbb{R}^d. \]

In [6] the following stability result at the boundary for the unknown function \( \sigma \) has been shown, in the case where \( A(x) = A(\sigma(x), x) \), \( \sigma \in W^{1,p}(\Omega) \), \( A(\cdot, \cdot) \in \mathcal{H} \).

**Theorem 2.2.** [Theorem 2.1 in [6].] Given \( p > d \), let \( \Omega \) be a bounded domain with Lipschitz boundary. Given \( E > 0 \), let \( \sigma_1, \sigma_2 \) satisfy

\[ \alpha \leq \sigma_1(x), \sigma_2(x) \leq \beta \quad \text{for every } x \in \Omega, \]

and

\[ \|\sigma_1\|_{W^{1,p}(\Omega)}, \|\sigma_2\|_{W^{1,p}(\Omega)} \leq E. \]

Let \( A(\cdot, \cdot) \in \mathcal{H} \). Then we have

\[ \|A(\sigma_1(x), x) - A(\sigma_2(x), x)\|_{L^\infty(\partial \Omega)} \leq C\|A_{A(\sigma_1, x)} - A_{A(\sigma_2, x)}\|_{L(H^{1/2}(\partial \Omega), H^{-1/2}(\partial \Omega))}, \]

where \( C \) depends on \( \alpha, \beta, E, p, \) and \( \Omega \) A uniqueness result is also provided in [6].

**Theorem 2.3.** [Theorem 2.4 in [6].] Given \( E > 0 \), let be \( \sigma_1 \) and \( \sigma_2 \) two scalar functions satisfying (2.1) and (2.2) with \( p = \infty \), and \( A(\cdot, \cdot) \in \mathcal{H} \). Moreover assume \( A \in W^{1,\infty}([\alpha, \beta] \times \Omega, \text{Sym}_d) \). In addition, suppose that \( \Omega \) can be partitioned into a finite number of Lipschitz domains \( \{\Omega_j\}_{j=1}^N \) such that \( \sigma_1 - \sigma_2 \) is analytic on each \( \Omega_j \). If

\[ A_{A(\sigma_1, x)} = A_{A(\sigma_2, x)} \]

then we have

\[ A(\sigma_1(x), x) = A(\sigma_2(x), x) \quad \text{in } \Omega. \]

The same results hold for matrix functions of the type \( A^\varepsilon(x) = A(\sigma(x), x/\varepsilon) \), for fixed \( \varepsilon \). However in this case the constant \( E \) in Definition 2.1 scales as \( 1/\varepsilon \), and therefore as \( \varepsilon \to 0 \), such results may become useless. Moreover, in numerical experiments, when \( \varepsilon \) is very small, trying to solve the problem numerically by using as model for inversion an approximation of (1.1) is prohibitive in terms of computational cost, and therefore a different strategy has to be preferred. Here we use the framework of homogenization which we briefly review in the next section.
3. Homogenization and analysis of the coarse grained approach. In this section we first briefly describe the asymptotic behaviour of (1.1) as \( \varepsilon \to 0 \), where the matrix \( A^\varepsilon \in M(\alpha, \beta, \Omega) \). We then analyse under which conditions the homogenized tensor \( A^0 \) belongs to \( \mathcal{H} \).

**Definition 3.1.** Let \( \{ A^\varepsilon \} \) be a sequence of matrices in \( M(\alpha, \beta, \Omega) \). We say that it \( G \)-converges to the matrix \( A^0 \in M(\alpha, \beta, \Omega) \) iff for every function \( f \in H^{-1}(\Omega) \), the solution \( u^\varepsilon \) of (1.1) is such that

\[
\begin{align*}
    u^\varepsilon & \rightharpoonup u^0 \quad \text{weakly in } H^1(\Omega), \\
    A^\varepsilon \nabla u^\varepsilon & \rightharpoonup A^0 \nabla u^0 \quad \text{weakly in } (L^2(\Omega))^d,
\end{align*}
\]

where \( u^0 \) is the unique solution of

\[
\begin{align*}
    -\nabla \cdot (A^0 \nabla u^0) &= 0 & \text{in } \Omega, \\
    u^0 &= g & \text{on } \partial \Omega.
\end{align*}
\]

We will sometimes use the notation \( G \lim_{\varepsilon \to 0} A^\varepsilon = A^0 \) to denote the \( G \)-convergence of a family of tensors.

**Theorem 3.2.** One has the following compactness result. Let \( \{ A^\varepsilon \} \) be a sequence of matrices in \( M(\alpha, \beta, \Omega) \). Then there exists a subsequence \( \{ A^{\varepsilon'} \} \) and a matrix \( A^0 \in M(\alpha, \beta, \Omega) \) such that \( \{ A^{\varepsilon'} \} \) \( G \)-converges to \( A^0 \).

In particular we consider the case where \( A^\varepsilon \) is the \( Y \)-periodic matrix defined by

\[
A^\varepsilon(x) = A(x, x/\varepsilon) = A(x, y), \quad A(x, y) \in M(\alpha, \beta, Y),
\]

where

\[
a^{\varepsilon}_{ij}(x) = a_{ij}(x, x/\varepsilon) = a_{ij}(x, y) \text{ is } Y\text{-periodic in } y \forall i, j = 1, \ldots, d,
\]

where \( Y \) denotes the reference unit cell \( (0,1)^d \).

In this particular case we have that the whole sequence \( \{ A^\varepsilon \} \) \( G \)-converges to the tensor \( A^0 \in M(\alpha, \beta, \Omega) \), \( A^0(x) = \{ a^0_{ij}(x) \}_{1 \leq i, j \leq d} \), which is elliptic and it is given by

\[
a^0_{ij}(x) = \frac{1}{|Y|} \int_Y a_{ij}(x, y) \, dy - \frac{1}{|Y|} \sum_{k=1}^d \int_Y a_{ij}(x, y) \frac{\partial \hat{x}_j}{\partial y_k} \, dy \quad \forall i, j = 1, \ldots, d.
\]

The micro functions \( \hat{x}_j \), \( j = 1, \ldots, d. \), are defined to be the unique solutions of the cell problems: find \( \hat{x}_j \in W^{1}_{per}(Y) \) such that

\[
\int_Y A(x, y) \nabla_y \hat{x}_j \cdot \nabla_y v \, dy = \int_Y A(x, y) e_j \cdot \nabla_y v \, dy, \forall v \in W^{1}_{per}(Y),
\]

where \( \{ e_j \}_{j=1}^d \) is the canonical basis of \( \mathbb{R}^d \) and

\[
W^{1}_{per}(Y) = \{ v \in H^{1}_{per}(Y) : \int_Y v \, dy = 0 \},
\]
where \( H^1_{\text{per}}(Y) \) is defined as the closure of \( C^\infty_{\text{per}}(Y) \) (where \( C^\infty_{\text{per}}(Y) \) denotes the subset of \( C^\infty(\mathbb{R}^d) \) of periodic functions in \( Y \)).

In Section 2 we have listed the regularity properties that the map \( (t, x) \mapsto A(t, x/\varepsilon) \) has to satisfy to ensure stability and uniqueness of the inverse problem. However the model we use for inversion is the homogenized one, and therefore as first step we want to analyse under which conditions on \( A^\varepsilon \), the map \( t \mapsto A^0(t) \) satisfies the regularity properties to ensure stability and uniqueness for the inverse problem. First, let us introduce as a corollary of Theorem 2.2 and 2.3 the conditions that \( A^0 \) has to satisfy to ensure stability and uniqueness.

**Corollary 3.3.** Given \( E > 0 \) and \( p > d \), let us consider a \( d \times d \) symmetric matrix valued function \( t \mapsto A(t) \), \( t \in [\alpha, \beta] \), \( 0 < \alpha < \beta \), satisfying the conditions

\[
|\partial_t A(t)| + |\partial_t^2 A(t)| \leq E, \quad \forall t \in [\alpha, \beta].
\]

\[
\alpha|\xi|^2 \leq A(t)\xi \cdot \xi \leq \beta|\xi|^2, \quad \forall t \in [\alpha, \beta], \xi \in \mathbb{R}^d.
\]

\[
\partial_t A(t)\xi \cdot \xi \geq E^{-1}|\xi|^2, \quad \forall t \in [\alpha, \beta], \xi \in \mathbb{R}^d.
\]

Let \( \sigma_1 \) and \( \sigma_2 \) two scalar functions satisfying (2.1)-(2.2). Then we have the following results.

1. The following estimate holds:

\[
\|A(\sigma_1) - A(\sigma_2)\|_{L^\infty(\Omega)} \leq C\|A(\sigma_1) - A(\sigma_2)\|_{L(H^{1/2}(\partial\Omega), H^{-1/2}(\partial\Omega))},
\]

where \( C \) depends on \( \alpha, \beta, E, p, \) and \( \Omega \).

2. Let \( \sigma_1 \) and \( \sigma_2 \) satisfy (2.1) and (2.2) with \( p = \infty \). In addition suppose that \( \Omega \) can be partitioned into a finite number of Lipschitz domains \( \{\Omega_j\}_{j=1}^N \) such that \( \sigma_1 - \sigma_2 \) is analytic on each \( \Omega_j \). If

\[
\Lambda_{A(\sigma_1)} = \Lambda_{A(\sigma_2)}
\]

then we have

\[
A(\sigma_1) = A(\sigma_2) \quad \text{in} \ \Omega.
\]

We also mention the following lemma which establishes a regularity result for the solutions of the cell problems (3.2) with respect to the variable \( t \) [5].

**Lemma 3.4.** Assume that \( A(t, x/\varepsilon) \) is uniformly elliptic and the map \( t \mapsto A(t, x/\varepsilon) \) is of class \( C^1([\alpha, \beta], \mathbb{S}_{\text{sym}}) \). Consider the micro functions \( \hat{\chi}_j(t, y) \), \( j = 1, \ldots, d \), unique solutions of: find \( \hat{\chi}_j \in W^1_{\text{per}}(Y) \) such that

\[
\int_Y A(t, y)\nabla_y \hat{\chi}_j \cdot \nabla_y v \, dy = \int_Y A(t, y)e_j \cdot \nabla_y v \, dy \ \forall v \in W^1_{\text{per}}(Y).
\]

Then the map \( t \mapsto \hat{\chi}_j(t, y) \in W^1_{\text{per}}(Y) \) is of class \( C^1([\alpha, \beta]) \) and satisfies

\[
\partial_t \hat{\chi}_j(t, y) = \phi_j(t, y), \quad \partial_t \nabla_y \hat{\chi}_j(t, y) = \nabla_y \phi_j(t, y),
\]

where \( \phi_j(t, y) \in W^1_{\text{per}}(Y) \) and satisfies

\[
\int_Y A(t, y)\nabla_y \phi_j(t, y) \cdot \nabla_y v \, dy = \int_Y \partial_t A(t, y)(e_j - \nabla_y \hat{\chi}_j(t, y)) \cdot \nabla_y v \, dy \ \forall v \in W^1_{\text{per}}(Y).
\]
THEOREM 3.5. Let \( x/\varepsilon = y \in Y \). Given \( F > 0, p > d \), consider the class of \( d \times d \) symmetric matrix functions \( (t, y) \mapsto A(t, y) \), where \( a_{ij} \) is \( Y \)-periodic, \( \forall i, j = 1, \ldots, d \), \( t \in [\alpha, \beta] \), \( 0 < \alpha < \beta \). Assume

\[
A \in W^{1,\infty}([\alpha, \beta] \times Y, \text{Sym}_d), \quad ||A||_{W^{1,\infty}([\alpha, \beta]; W^{1,\infty}(Y))} \leq F.
\]

\[
\partial_t A \in W^{1,\infty}([\alpha, \beta] \times Y, \text{Sym}_d), \quad ||\partial_t A||_{W^{1,\infty}([\alpha, \beta]; W^{1,\infty}(Y))} \leq F.
\]

\[
|\xi|^2 \leq A(t, y)\xi \cdot \xi \leq |\beta|\xi^2, \quad \text{for a.e. } y \in Y \text{ and } \forall t \in [\alpha, \beta], \xi \in \mathbb{R}^d.
\]

\[
\partial_t A(t, y)\xi \cdot \xi \geq F^{-1}|\xi|^2, \quad \text{for a.e. } y \in Y \text{ and } \forall t \in [\alpha, \beta], \xi \in \mathbb{R}^d.
\]

Then the homogenized tensor \( A^0 \) satisfies (3.3)-(3.5).

Proof. We start by showing (3.3). Let us observe that the homogenized coefficients can be rewritten in the form

\[
a^0_{ij}(t) = \frac{1}{|Y|} \int_Y A(t, y)(e_j - \nabla_y \hat{\chi}_j(t, y)) \cdot (e_i - \nabla_y \hat{\chi}_i(t, y)) \, dy.
\]

Differentiating (3.12) with respect to the variable \( t \) we obtain after a straightforward calculation

\[
\partial_t a^0_{ij}(t) = \frac{1}{|Y|} \int_Y (\partial_t A(t, y))(e_j - \nabla_y \hat{\chi}_j(t, y)) \cdot (e_i - \nabla_y \hat{\chi}_i(t, y)) \, dy.
\]

Then from H"older's inequality we get

\[
\begin{aligned}
\text{ess sup}_{t \in [\alpha, \beta]} |\partial_t a^0_{ij}(t)| \\
\quad \leq \text{ess sup}_{t \in [\alpha, \beta]} ||\partial_t A(t, \cdot)||_{L^\infty(Y)} ||e_j - \nabla_y \hat{\chi}_j(t, \cdot)||_{L^2(Y)} ||e_i - \nabla_y \hat{\chi}_i(t, \cdot)||_{L^2(Y)},
\end{aligned}
\]

and by using Lax-Milgram theorem, triangle inequality and (3.8) we obtain

\[
\begin{aligned}
\text{ess sup}_{t \in [\alpha, \beta]} |\partial_t a^0_{ij}(t)| \\
\quad \leq \text{ess sup}_{t \in [\alpha, \beta]} ||\partial_t A(t, \cdot)||_{L^\infty(Y)} (1 + \alpha^{-1}||A(t, \cdot) \cdot e_j||_{L^\infty(Y)})(1 + \alpha^{-1}||A(t, \cdot) \cdot e_i||_{L^\infty(Y)}) \\
\quad \leq F(1 + \alpha^{-1}F)^2 = C_1.
\end{aligned}
\]

Now, let \( t, s \in [\alpha, \beta] \). From (3.13) and H"older's inequality we have

\[
|\partial_t a^0_{ij}(t) - \partial_t a^0_{ij}(s)| \\
\leq ||\partial_t A(t, \cdot) - \partial_t A(s, \cdot)||_{L^\infty(Y)} ||e_j - \nabla_y \hat{\chi}_j(t, \cdot)||_{L^2(Y)} ||e_i - \nabla_y \hat{\chi}_i(t, \cdot)||_{L^2(Y)} \\
+ ||\partial_t A(s, \cdot)||_{L^\infty(Y)} ||\nabla_y (\hat{\chi}(t, \cdot) - \hat{\chi}(s, \cdot))||_{L^2(Y)} ||e_j - \nabla_y \hat{\chi}_j(t, \cdot)||_{L^2(Y)} \\
+ ||\partial_t A(s, \cdot)||_{L^\infty(Y)} ||e_j - \hat{\chi}_j(s, \cdot)||_{L^2(Y)} ||\nabla_y (\hat{\chi}_i(t, \cdot) - \hat{\chi}_i(s, \cdot))||_{L^2(Y)}.
\]

Now, from the weak definition of the solution of the micro problems, we derive for each \( i = 1, \ldots, d, s, t \in [\alpha, \beta], \forall v \in W^1_{per}(Y) \)

\[
\begin{aligned}
\int_Y A(t, y) \nabla_y (\hat{\chi}_i(t, y) - \hat{\chi}_i(s, y)) \cdot \nabla_y v \, dy = \int_Y (A(t, y) - A(s, y)) e_i \cdot \nabla_y v \, dy \\
+ \int_Y (A(s, y) - A(t, y)) \nabla_y \hat{\chi}_i(s, y) \cdot \nabla_y v \, dy.
\end{aligned}
\]
By choosing \( v = \hat{\chi}_i(t, y) - \hat{\chi}_i(s, y) \), using Hölder’s inequality and (3.8) we obtain

\[
||\nabla_y (\hat{\chi}_i(t, \cdot) - \hat{\chi}_i(s, \cdot))||_{L^2(Y)} \leq \alpha^{-1}||\partial_t A||_{L^\infty([a, b]: L^\infty(Y))} (1 + \alpha^{-1}||A(s, \cdot)e_i||_{L^\infty(Y)}) |t - s| \\
\leq \alpha^{-1} F(1 + \alpha^{-1} F)|t - s| \\
= C_2|t - s|.
\]

Using this latter result and the previous inequality gives

\[
|\partial_t a^0_{ij}(t) - \partial_t a^0_{ij}(s)| \leq (C_1 + 2 F(1 + \alpha^{-1} F) C_2)|t - s| \\
= C_1(1 + 2 \alpha^{-1} F)|t - s| = C_3|t - s|,
\]

and (3.3) follows.

The condition of uniform ellipticity, namely

\[
\alpha |\xi|^2 \leq A^0(t) \xi \cdot \xi \leq \beta |\xi|^2,
\]

for a.e. \( x \in \Omega, \xi \in \mathbb{R}^d \),

follows from a well-known property of the homogenized tensor (for a proof see for example [12]).

Finally we show that the condition of monotonicity with respect to the variable \( t \),

\[
\partial_t A^0(t) \xi \cdot \xi \geq k^{-1} |\xi|^2,
\]

for a.e. \( x \in \Omega, \xi \in \mathbb{R}^d \),

holds. From (3.13), by using the notation \( \varphi_i = (e_i - \nabla_y \hat{\chi}_i(t, y)) \), \( i = 1, \ldots, d \), we have that

\[
\partial_t a^0_{ij}(t) = \frac{1}{|Y|} \int_Y \partial_t A(t, y) \varphi_j \cdot \varphi_i \, dy.
\]

Since \( \partial_t A(t, y) \) is symmetric \( \partial_t A^0(t) \) is also symmetric. Then, given \( \xi \in \mathbb{R}^d \),

\[
\partial_t A^0(t) \xi \cdot \xi
\]

\[
= \frac{1}{|Y|} \int_Y \sum_{j=1}^d \sum_{i=1}^d \xi_i \varphi_i^T \partial_t A(t, y) \xi_j \varphi_j \, dy
\]

\[
= \frac{1}{|Y|} \int_Y \left( \sum_{i=1}^d \xi_i \varphi_i \right)^T \partial_t A(t, y) \left( \sum_{i=1}^d \xi_i \varphi_i \right) \, dy
\]

\[
\geq F^{-1} \int_Y \left| \sum_{i=1}^d \xi_i \varphi_i \right|^2 \, dy \geq 0, \text{ for any } \xi \in \mathbb{R}^d.
\]

In particular this inequality implies that

\[
(3.14) \quad \partial_t A^0(t) \xi \cdot \xi > 0, \text{ for any } \xi \in \mathbb{R}^d,
\]

as can be shown using a simple argument by contradiction. From there we easily derive (3.5) and the proof is complete. \( \square \)

Hence, we established stability and uniqueness for the inverse conductivity problem in the case the measurements at the boundary consist of the Dirichlet to Neumann map \( \Lambda_{A^0(\sigma)} \), under appropriate regularity assumptions. However, as already mentioned in
the introduction, this is not the case we are interested in, since we aim at solving the inverse problem when the data consists of the multiscale Dirichlet to Neumann map \( \Lambda_{A(\sigma,x/\varepsilon)} \). We consider the elliptic equation

\[-\nabla \cdot (A(\sigma(x), x/\varepsilon) \nabla u^\varepsilon) = 0 \quad \text{in } \Omega,\]

and measure the corresponding Dirichlet to Neumann map \( \Lambda_{A(\sigma,x/\varepsilon)} \). However in real experiments we do not have full knowledge of the map \( \Lambda_{A(\sigma,x/\varepsilon)} \). Indeed we would have to know the results of all possible boundary measurements for any Dirichlet function \( g \), which is impossible. In practice, we consider a set of \( N \) experiments, described by a finite set of Dirichlet conditions \( \{g_i\}_{i=1}^N \in H^{1/2}(\partial \Omega) \), and for each of them we measure the corresponding boundary flux. Let us define

\[ A = \{ \mu \in W^{1,\infty}(\Omega) : \alpha \leq \mu(x) \leq \beta, 0 < \alpha < \beta \}, \]

\[ A_{ad} = \{ \mu \in A : ||\mu(x)||_{W^{1,\infty}(\Omega)} \leq E, E > 0 \}. \]

Then we would like to solve the following minimization problem

\[
\min_{\mu \in A_{ad}} \sum_{i=1}^N ||\Lambda_{A(\sigma,x/\varepsilon)}g_i - \Lambda_{A^0(\mu)}g_i||_{H^{-1/2}(\partial \Omega)}^2,
\]

subject to

\[-\nabla \cdot (A^0(\mu(x)) \nabla u^0) = 0 \quad \text{in } \Omega,\]

\[ u^0 = g_i \quad \text{on } \partial \Omega,\]

where \( A^0(\mu(x)) \) is the homogenized tensor corresponding to \( A(\mu(x), x/\varepsilon) \). It is important to remark that \( G \)-convergence of \( A(\sigma(x), x/\varepsilon) \) to \( A^0(\sigma(x)) \) does not imply convergence of the corresponding Dirichlet to Neumann maps in the operator norm. However one verifies that since \( A(\sigma(x), x/\varepsilon) \) \( G \)-converges to \( A^0(\sigma(x)) \), the corresponding Dirichlet to Neumann maps converge weakly.

**Lemma 3.6.** Let us consider a sequence of tensors \( \{A^\varepsilon\} \) in \( M(\alpha, \beta, \Omega) \). Then \( \{A^\varepsilon\} \) \( G \)-converges to \( A^0 \in M(\alpha, \beta, \Omega) \) as \( \varepsilon \to 0 \), if and only if for all \( g \in H^{1/2}(\partial \Omega) \), \( \Lambda_{A^\varepsilon} g \) converges weakly* to \( \Lambda_{A^0} g \) in \( H^{-1/2}(\partial \Omega) \) as \( \varepsilon \to 0 \).

**Proof.** From the definition of \( G \)-convergence we have that, for any \( \psi \in H^1(\Omega) \),

\[
\int_{\Omega} (A^\varepsilon \nabla u^\varepsilon - A^0 \nabla u^0) \cdot \nabla \psi \, dx \to 0 \quad \text{as } \varepsilon \to 0,
\]

where \( u^\varepsilon, u^0 \in H^1(\Omega) \) are the weak solutions of (1.1) and (1.3) respectively. Then, using integration by parts, we obtain that

\[
\langle \Lambda_{A^\varepsilon} g - \Lambda_{A^0} g, \psi \rangle_{H^{-1/2}(\partial \Omega), H^{1/2}(\partial \Omega)} \to 0 \quad \text{as } \varepsilon \to 0,
\]

for each \( g, \psi \in H^{1/2}(\partial \Omega) \). Then

\[
\Lambda_{A^\varepsilon} g \rightharpoonup \Lambda_{A^0} g \quad \text{weakly* in } H^{-1/2}(\partial \Omega).
\]

The reciprocal statement can be proved similarly. \( \square \)
Theorem 3.7. Let us consider a sequence of tensors \( \{ A^\varepsilon(\sigma(x)) \} \) in \( M(\alpha, \beta, \Omega) \), where \( \sigma \in A_{bd} \), such that \( A^\varepsilon(\sigma(x)) \) G-converges to \( A^0(\sigma(x)) \in M(\alpha, \beta, \Omega) \) as \( \varepsilon \to 0 \). For any \( \varepsilon > 0 \) let us consider the minimization problem

\[
\min_{\mu \in A_{bd}} \sum_{i=1}^{N} \| A_{A^\varepsilon(\sigma)} g_i - A_{A^0(\mu)} g_i \|_{H^{-1/2}(\partial\Omega)}^2.
\]

For any \( \varepsilon > 0 \), any minimizing sequence \( \{ \mu_{n_k}^\varepsilon \}_{n \geq 0} \) contains a subsequence \( \{ \mu_{n_k}^\varepsilon \} \) such that

\[
G \lim_{k \to \infty} A^0(\mu_{n_k}^\varepsilon(x)) = A^\varepsilon(\sigma(x)).
\]

Moreover,

\[
G \lim_{\varepsilon \to 0} (G \lim_{k \to \infty} A^0(\mu_{n_k}^\varepsilon(x))) = A^0(\sigma(x)).
\]

Proof. Let us consider the case \( N = 1 \). The proof easily extends to the case where \( N > 1 \). Let us denote

\[
\Phi(\mu_{n_k}^\varepsilon) = \| A_{A^\varepsilon(\sigma)} g - A_{A^0(\mu_{n_k}^\varepsilon)} g \|_{H^{-1/2}(\partial\Omega)}^2,
\]

\[
\inf_{\mu \in A_{bd}} \Phi(\mu) = \overline{\Phi}.
\]

From the minimizing properties of \( \{ \mu_{n_k}^\varepsilon \}_{n \geq 0} \), and the non-negativity of \( \Phi \), it follows that the sequence \( \{ \Phi(\mu_{n_k}^\varepsilon) \}_{n \geq 0} \) is uniformly bounded, i.e., for any \( \delta > 0 \) there exists \( M = M(\delta) \) such that

\[
0 \leq \Phi(\mu_{n_k}^\varepsilon) \leq \overline{\Phi} + \delta \quad \forall n \geq M.
\]

Then there exists a subsequence \( \{ \mu_{n_k}^\varepsilon \} \) such that

\[
A_{A^0(\mu_{n_k}^\varepsilon)} g \rightharpoonup A_{A^\varepsilon(\sigma)} g \quad \text{weakly* in } H^{-1/2}(\partial\Omega),
\]

which implies by Lemma 3.6 that \( A^0(\mu_{n_k}^\varepsilon(x)) \) G-converges to \( A^\varepsilon(\sigma(x)) \), i.e.,

\[
G \lim_{k \to \infty} A^0(\mu_{n_k}^\varepsilon(x)) = A^\varepsilon(\sigma(x)) \quad \forall \varepsilon > 0.
\]

On the other end, we have by assumptions that \( A^\varepsilon(\sigma(x)) \) G-converges to \( A^0(\sigma(x)) \) as \( \varepsilon \to 0 \), and therefore

\[
G \lim_{\varepsilon \to 0} (G \lim_{k \to \infty} A^0(\mu_{n_k}^\varepsilon(x))) = G \lim_{\varepsilon \to 0} A^\varepsilon(\sigma(x)) = A^0(\sigma(x)).
\]

The last result does not imply any kind of pointwise convergence for the solution of the inverse problem, but, on the other hand, it ensures that by using such coarse strategy we are close (in the G-limit sense) to the function we want to retrieve. Finally, it is important to remark that, due to the difficulties of working with fractional-order Sobolev spaces when performing numerical experiments, we consider the \( L^2(\partial\Omega) \)-norm to evaluate the distance between data and numerical results produced by the homogenized model. We consider then the problem

\[
\min_{\mu \in A_{bd}} \sum_{i=1}^{N} \| A_{A(\sigma,x/\varepsilon)} g_i - A_{A^0(\mu)} g_i \|_{L^2(\partial\Omega)}^2,
\]

where it is assumed \( g_i \in H^{3/2}(\partial\Omega), \forall i = 1, \ldots, N. \)
4. Tikhonov regularization. Let \( \Phi : A \to \mathbb{R} \) be defined as
\[
\Phi(\mu) = \sum_{i=1}^{N} ||A_{A}(\sigma, x/\varepsilon)g_i - A_{A^0(\mu)}g_i||_{L^2(\partial \Omega)}^2 ,
\]
and let us consider the minimization problem
\[
\overline{\Phi} = \inf_{\mu \in A_{ad}} \Phi(\mu) .
\]
(4.1)

Since \( A_{ad} \) is a closed convex and bounded set in \( W^{1,\infty}(\Omega) \) it is possible to prove that any minimizing sequence \( \{\mu_n\} \) for (4.1) contains a subsequence which weakly converges to \( \overline{\mu} \in A_{ad} \), for which we have \( \overline{\Phi} = \Phi(\overline{\mu}) \). The proof is quite standard. The key point consists in proving that \( \Phi : A \to \mathbb{R} \) is weakly continuous in \( H^1(\Omega) \). Let us recall that \( H^1(\Omega) \) embeds compactly into \( L^r(\Omega) \), \( r \geq 1 \). Then the problem reduces into showing that the map \( \mu \mapsto \Lambda_{A^0(\mu)}g \in L^2(\partial \Omega) \) is continuous with respect to the \( L^r(\Omega) \) topology on the set \( A \), given \( g \in H^{3/2}(\Omega) \). Such result is stated in the lemma below, whose proof is reported in the appendix.

**Lemma 4.1.** Let \( A^0(\cdot) \) satisfy (3.3)-(3.4). Let the sequence \( \{\mu_n\} \subset A \) converge to some \( \mu \in A \) in \( L^r(\Omega) \), \( r \geq 1 \). Then the sequence \( \{\Lambda_{A^0(\mu_n)}g\} \) converges to \( \Lambda_{A^0(\mu)}g \) in \( H^{-1/2}(\partial \Omega) \). If \( u^0 \in H^2(\Omega) \), then the sequence \( \{\Lambda_{A^0(\mu_n)}g\} \) converges to \( \Lambda_{A^0(\mu)}g \) in \( L^2(\partial \Omega) \).

However in numerical experiments we may prefer to adopt indirect methods to ensure stability of the inverse problem instead of directly impose a constraint during the minimization procedure. Among the possible methods to regularize inverse problems we choose Tikhonov regularization (see for example [14,16]). Tikhonov regularization ensures well-posedness by adding to the cost functional a convex variational penalty, so that the new minimization problem reads.

\[
\Psi = \inf_{\mu \in A} \Psi(\mu) \quad \text{(4.2)}
\]

where
\[
\Psi(\mu) = \Phi(\mu) + \gamma R(\mu) , \quad \text{(4.3)}
\]

where \( \gamma \) is the regularization parameter, and \( R \) is the penalty term. Such term induces a priori knowledge on expected conductivity. In what follows we consider \( R(\mu) = ||\mu - \mu_0||_{H^1(\Omega)}^2 \), where \( \mu_0 \) is a prior guess of \( \sigma \). The regularization parameter controls the trade-off between the two terms and has to be properly chosen. The choice of \( \gamma \) represents a problem of considerable interest and will affect how much oscillation is allowed in any minimizing sequence. As the regularization parameter \( \gamma \) varies, we obtain different regularized solutions having properties that vary with \( \gamma \). However how to choose \( \gamma \) is not the main subject of study of this particular work. For sake of completeness we mention that several methods have been proposed in literature, as the Morozov’s discrepancy principle [24,30] or the L-curve method [21,22]. The following theorem is an adaptation of a classical result in non-linear Tikhonov regularization theory (see e.g. [15]).

**Theorem 4.2.** Let \( u^0 \in H^2(\Omega) \), and consider a minimizing sequence \( \{\mu_n\} \) for (4.2). Then it contains a weakly convergent subsequence in \( H^1(\Omega) \) with limit \( \overline{\mu} \in A \) which attains the infimum: \( \overline{\Psi} = \Psi(\overline{\mu}) \).
We have already proved weak continuity of $\Phi : \mathcal{A} \to \mathbb{R}$. From the minimizing property of $\{\mu_n\}$ and the non-negativity of $\Phi$, it follows that the sequence $\{\gamma||\mu_n - \mu_0||_{H^1(\Omega)}^2\}$ is uniformly bounded, i.e., for any $\delta > 0$ there exists $M = M(\delta)$ such that

$$\gamma||\mu_n - \mu_0||_{H^1(\Omega)}^2 \leq \Psi + \delta, \quad \forall n \geq M.$$ 

Hence there exists a subsequence, again denoted $\{\mu_n\}$, such that $\mu_n \rightharpoonup \mu$ weakly in $H^1(\Omega)$. Since $u^0 \in H^2(\Omega)$ the map $\Phi : \mathcal{A} \to \mathbb{R}$ is weakly continuous by Lemma 4.1, together with the weak lower semicontinuity of the penalty term, implies the weak lower semicontinuity of $\Psi : \mathcal{A} \to \mathbb{R}$. Hence

$$\Psi(\mu) \leq \liminf_{n \to \infty} \Psi(\mu_n) \leq \Psi.$$

Since $\Psi(\mu) \geq \Psi$, the result follows.

5. Numerical solution of the inverse problem. In this section we discuss the numerical solution of the inverse conductivity problem. At first we describe the forward solver employed to approximate the homogenized boundary flux. Given the problem

$$- \nabla \cdot (A^\varepsilon \nabla u^\varepsilon) = f \quad \text{in } \Omega,$$

$$u^0 = g \quad \text{on } \partial \Omega,$$

we need an efficient method to evaluate the boundary flux $\Lambda_{A^0} g$ for the homogenized tensor $A^0$. However given $A^\varepsilon$, analytic solutions for the corresponding $A^0$ are usually not available, hence we need numerical homogenization.

5.1. Numerical homogenization and model order reduction techniques.

For the numerical homogenization procedure, we choose the Finite Element Heterogeneous Multiscale Method (FE-HMM) which approximates the homogenized problem originating from (5.1) taking as input only the multiscale data. The FE-HMM it has been studied extensively in literature and for more details we refer to [1, 2, 4]. The method is based on a macro finite element space

$$S^0_l(\Omega, T_H) = \{v^H \in H^1_0(\Omega) : v^H|_K \in \mathcal{P}^l(K), \forall K \in T_H\},$$

where $T_H$ is a partition of $\Omega$ in elements $K$ of diameter $H_K$, and $\mathcal{P}^l(K)$ is the space of polynomials on $K$ of degree at most $l$. Within each macro element $K \in T_H$ we define a quadrature formula $\{x_{K_j}, \omega_{K_j}\}_{j=1}^J$, which is assumed to be exact for polynomials $p(x) \in \mathcal{P}^l(K)$. For each macro element, an approximation of the homogenized tensor on each integration point is needed. Such approximation is obtained by solving a micro problem defined on the sampling domains $K_{\delta_j} = x_{K_j} + (-\delta/2, \delta/2)^d, (\delta \geq \varepsilon)$. For a sampling domain $K_{\delta_j}$ we define a micro finite element space

$$S^q(K_{\delta_j}, T_h) = \{z^h \in W(K_{\delta_j}) : z^h|_T \in \mathcal{P}^q(T) \forall T \in T_h\}.$$ 

The space $W(K_{\delta_j})$ is defined as

$$W(K_{\delta_j}) = W^1_{per}(K_{\delta_j}) = \{z \in H^1_{per}(K_{\delta_j}) ; \int_{K_{\delta_j}} z \, dx = 0\}$$

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in case we ask for periodic coupling, or
\[ W(K_{\delta_j}) = H_0^1(K_{\delta_j}) \]
for a coupling with Dirichlet boundary conditions. Let \( u^H \) be the approximate solution to problem (5.1). Then the numerical method is defined as follows: find \( u^H \in S^l(\Omega, \mathcal{T}_H) \), \( u^H = g \) on \( \partial \Omega \), such that
\[
B_H(u^H, v^H) = F_H(v^H) \quad \forall v^H \in S_0^l(\Omega, \mathcal{T}_H),
\]
where
\[
B_H(v^H, w^H) := \sum_{K \in \mathcal{T}_H} \sum_{j=1}^J \frac{\omega_{K_j}}{|K_{\delta_j}|} \int_{K_{\delta_j}} A^\epsilon \nabla v_{K_j}^h \cdot \nabla w_{K_j}^h \, dx,
\]
and
\[
F_H(v^H) := \sum_{K \in \mathcal{T}_H} \sum_{j=1}^J \frac{\omega_{K_j}}{|K_{\delta_j}|} (fv^H)(x_{K_j}).
\]
In (5.2) \( v_{K_j}^h \) (respectively \( w_{K_j}^h \)) denotes the solution to the micro problem: find \( v_{K_j}^h \) such that \( v_{K_j}^h - v_{lin,j}^H \in S^q(K_{\delta_j}, \mathcal{T}_h) \) and
\[
\int_{K_{\delta_j}} A^\epsilon \nabla v_{K_j}^h \cdot \nabla z^h \, dx = 0 \quad \forall z^h \in S^q(K_{\delta_j}, \mathcal{T}_h),
\]
where \( v_{lin,j}^H := v^H(x_{K_j}) + (x - x_{K_j}) \cdot \nabla v^H(x_{K_j}) \). In particular for piecewise linear functions we have that \( v_{lin,j}^H(x) = v^H(x) \). We conclude this brief section by recalling the convergence estimates for the numerical method which have been extensively studied in the literature \[1, 2\]:
\[
||u^0 - u^H||_{H^1(\Omega)} \leq C \left( H^l + \left( \frac{h}{\epsilon} \right)^{2q} + \epsilon_{MOD} \right),
\]
\[
||u^0 - u^H||_{L^2(\Omega)} \leq C \left( H^{l+1} + \left( \frac{h}{\epsilon} \right)^{2q} + \epsilon_{MOD} \right).
\]
Moreover for the homogenized tensor we have
\[
\sup_{K \in \mathcal{T}_H} ||A^0,h(x) - A^0(x)||_F \leq C \left( H^l + \left( \frac{h}{\epsilon} \right)^{2q} + \epsilon_{MOD} \right),
\]
where \( || \cdot ||_F \) denotes the Frobenius norm and \( A^0,h \) is defined in (5.6). The term \( \epsilon_{MOD} \) is the so called modelling error and does not depend on \( H \) and \( h \). For locally periodic tensors which admit explicit separation between the slow and fast variable, if collocated at the slow variable, this term vanishes (see \[1, 2\]).

**Model order reduction.** The FE-HMM, as it is defined, can result in being computationally expensive, since it requires the computation of a cell problem for each
macro element and each macro quadrature point, whose number increases as we refine the macro mesh for an appropriate approximation of the homogenized solution. This is particularly undesirable when solving inverse problems, since typically one needs multiple evaluations of the cost functional for different guesses of the parameter of interest. Here we explain how reduced basis methodology can be combined with FE-HMM to design a new efficient method which drastically reduces the computational effort, by avoiding the repeated solutions of a large number of cell problems. For a detailed description and analysis of the method, called the Reduced Basis Finite Element Heterogeneous Multiscale Method (RB-FE-HMM), we mention [3]. The main idea is the following: instead of computing the micro solutions in each macro element at the given macro quadrature points, during what is called the offline stage we select a small number of carefully precomputed micro solutions to construct a small subspace of micro functions. Then in the online stage each micro solution is obtained as linear combination of the precomputed micro functions. The construction of the subspace is performed using a greedy procedure, and therefore a cheap way to compute residuals, in order to have efficiency of the a posteriori error control, is crucial. We start with the following reformulation of the FE-HMM, which makes a link between the micro problems and the effective tensor:

\[
\frac{1}{|K_{\delta_j}|} \int_{K_{\delta_j}} A^T \nabla v_{K_j}^h \cdot \nabla w_{K_j}^h \, dx = A^{0,h}(x_{K_j}) \nabla v^H(x_{K_j}) \cdot \nabla w^H(x_{K_j}).
\]

We map the domain \(K_{\delta_j}\) into the reference domain \(Y = (0,1)^d\) through \(x = G_{x_{K_j}}(y) = x_{K_j} + \delta(y - 1/2)\). Inserting (5.4) into (5.2) we obtain

\[
B_H(v^H, w^H) := \sum_{K \in T_H} \sum_{j=1}^J \omega_{K_j} A^{0,h}(x_{K_j}) \nabla v^H_{lin,j}(x_{K_j}) \cdot \nabla w^H_{lin,j}(x_{K_j}),
\]

where

\[
(A^{0,h}(x_{K_j}))_{ik} = \int_Y A_{x_{K_j}}(\nabla \chi_{K_j}^{i,h} + e_i) \cdot (\nabla \chi_{K_j}^{k,h} + e_k) \, dy,
\]

where we note that \(A^v(G_{x_{K_j}}(y))\) can be parametrized by \(x_{K_j}\), and therefore we use the notation \(A_{x_{K_j}} = A^v(G_{x_{K_j}}(y))\). Finally, \(\chi_{K_j}^{i,h}\) (respectively \(\chi_{K_j}^{k,h}\)) is the solution of the micro problem

\[
b(\chi_{K_j}^{i,h}, z^h) = \int_Y A_{x_{K_j}} \nabla \chi_{K_j}^{i,h} \cdot \nabla z^h \, dy
\]

\[
= - \int_Y A_{x_{K_j}} e_i \cdot \nabla z^h \, dy =: l_i(z^h) \quad \forall z^h \in S^q(Y, T_h).
\]

In the offline stage we construct a reduced space of \(N\) carefully precomputed micro solutions, which we call \(S_N(Y)\). Details on how the procedure is carried on can be found in [3]. To select the basis functions a greedy algorithm is used. We start by randomly defining the training set \(\Xi_{Train} = \{(x_n, \eta_n) : 1 \leq n \leq N_{Train}, 1 \leq \eta_n \leq d\}, \) where \(T_{\delta_n} = x_n + (-\delta/2, \delta/2)^d\) are sampling domains centred at \(x_n \in \Omega\), while \(\eta_n\) corresponds to the unit vector \(e_{\eta_n}\) of the canonical basis of \(\mathbb{R}^d\). We compute the
first basis function \( \hat{\zeta}_h \) and initialize the reduced space \( S_N(Y) \). Then, successively we continue to add new basis functions to \( S_N(Y) \) until convergence of the a posteriori error is detected. A crucial assumption for efficiently evaluate the a posteriori error is that, for a given \( x_n \in \Omega \), the tensor \( A_{x_n} \) is available in the affine form

\[
A_{x_n}(y) = \sum_{q=1}^{Q} \Theta_q(x_n)A_q(y), \quad \forall y \in Y. \tag{5.8}
\]

However in the case \( A_{x_n} \) is not directly available in the form (5.8), a greedy algorithm, called the empirical interpolation method (EIM), can be applied to obtain an affine approximation of \( A_{x_n} \) [20]. The output of the offline stage is the reduced space

\[
S_N(Y) = \text{span}\{\hat{\zeta}_h, \ldots, \hat{\zeta}_N\}.
\]

Then we define a macro method similar to FE-HMM, with micro functions computed on the reduced space. The method reads: find \( u^{H, RB} \in S_l(\Omega, T_H), u^{H, RB} = g \) on \( \partial \Omega \), such that

\[
B_{H, RB}(u^{H, RB}, v^H) = \int_{\Omega} f v^H \, dx \quad \forall v^H \in S_0^l(\Omega, T_H),
\]

where

\[
B_{H, RB}(v^H, w^H) := \sum_{K \in T_H} \sum_{j=1}^{J} \omega_{Kj} A^{0,N}(x_{K_j}) \nabla v_{lin,j}(x_{K_j}) \cdot \nabla w_{lin,j}(x_{K_j}),
\]

where \( A^{0,N}(x_{K_j})_{ik} \) is the solution of (5.7) in the reduced basis space. Thanks to the affine representation of the tensor \( A^\varepsilon \), solving the micro problems in the reduced space consists with solving an \( N \times N \) linear system, which leads to a great saving of computational effort. The error introduced by this model order reduction is based on the distance between the reduced space \( S_N(Y) \) and \( S_q(Y, T_h) \). Such distance can be quantified by means of the notion Kolmogorov \( N \)-width (see [3]).

5.2. **Approximate boundary flux calculations.** Here we describe a numerical method to approximate the flux at the boundary. This method is based on a Galerkin projection, and is analysed in detail in [28] in its classical finite element formulation. In particular it allows to obtain superconvergence of the approximate flux in the \( L^2(\partial \Omega) \)-norm, and we show how such superconvergence result can be extended in the case where the FE-HMM is used. We start by recalling that the weak solution of the homogenized problem originating from (5.1) is defined as follows: find \( u^0 \in H^1(\Omega) \) with \( u^0 = g \) on \( \partial \Omega \), such that

\[
B(u^0, v) = \int_{\Omega} A^0 \nabla u^0 \cdot \nabla v \, dx = \int_{\Omega} f v \, dx = F(v), \quad \forall v \in H^1_0(\Omega). \tag{5.10}
\]
Similarly, if we consider to approximate \( u^0 \) by means of the FE-HMM with piecewise linear polynomials, we have that the approximate solution \( u^H \) is defined as \( u^H \in S^1(\Omega, T_H) \), \( u^H = g \) on \( \partial\Omega \), such that

\[
B_H(u^H, v^H) = F_H(v^H) \quad \forall v^H \in S^1_0(\Omega, T_H),
\]

where \( B_H(v^H, u^H) \) is defined in (5.2) or (5.5). By definition \( \Lambda_{A^0} g \) is the normal flux at the boundary so that

\[
\Lambda_{A^0} g = A^0 \nabla u^0 \cdot \nu|_{\partial\Omega}.
\]

Multiplying the first equation in (5.1) by a function \( v \in H^1(\Omega) \) and by using Green’s identity we get the following relation for the flux

\[
(5.11) \quad - \int_{\partial\Omega} \Lambda_{A^0} g \cdot v \, ds = \int_{\Omega} A^0 \nabla u^0 \cdot \nabla v \, dx - \int_{\Omega} f v \, dx.
\]

The domain \( \Omega \) is assumed to be a polygonal domain, and we assume that the approximate flux at the corners of \( \Omega \) is specified from direct calculation with the given Dirichlet data. For example, for the case given in Figure 1, this direct calculation is performed as follows:

\[
\nabla u \cdot n_2(x_2) = \nabla g(x_2) \cdot (\alpha t_2 + \beta t_1),
\]

\[
\alpha = (n_2 \cdot n_1) / (t_2 \cdot n_1), \quad \beta = 1 / (t_2 \cdot n_1).
\]

Let us introduce the following subspaces of \( S^1(\Omega, T_H) \):

\[
S^1_c(\Omega, T_H) = \{ v^H \in S^1(\Omega, T_H) : v^H = 0 \text{ at the corners of } \Omega \},
\]

\[
S^1_i(\Omega, T_H) = \{ v^H \in S^1(\Omega, T_H) : v^H = 0 \text{ at the interior nodes of } \Omega \}.
\]

Let us denote as \( S^1_c(\partial\Omega, T_H) \) the finite dimensional space of functions which are restrictions on the boundary \( \partial\Omega \) of functions in \( S^1_c(\Omega, T_H) \). Suppose now we have computed the approximate solutions \( u^H \) and we want to approximate the boundary flux across
\[ \partial \Omega. \text{ Let } \Gamma_j, j = 1, \ldots, n, \text{ be the straight interface portions that defines the boundary } \partial \Omega. \text{ Finally let } X_j(\Omega, \mathcal{T}_H), j = 1, \ldots, n, \text{ denote the strip of all the elements in } \mathcal{T}_H, \text{ which have at least one vertex on } \Gamma_j. \text{ Then following [11, 28] an approximate flux can be constructed progressively for each interface of the boundary, by assembling functions in } S_c^1(\Gamma_j, \mathcal{T}_H), j = 1, \ldots, n, \text{ such that}
\]
\[ - \int_{\partial \Omega} \Lambda^H_{A^0} g \cdot v^H \, ds = B_H(u^H, v^H) - \int_{\Omega} f v^H \, dx \]
\[ \forall v^H \in S_c^1(\Omega, \mathcal{T}_H) \cap S_c^1(\Omega, \mathcal{T}_H), \quad v^H = 0 \text{ on } \partial \Omega \setminus \Gamma_j. \]
Let us remark that \( u^H \) has been already computed, and so constructing \( \Lambda^H_{A^0} g \) leads then to solving a linear system whose unknowns are the values of \( \Lambda^H_{A^0} g \) at the boundary nodes. To obtain an error estimate for the approximate boundary flux we recall the following lemma which relates the functions in \( S_c^1(\Omega, \mathcal{T}_H) \) and their traces on \( \partial \Omega \).

**Lemma 5.1.** Let \( X = X(\Omega, \mathcal{T}_H) \) denote a strip of elements in \( \mathcal{T}_H \), with each element having at least one vertex on \( \partial \Omega \), and let \( v^H \in S_c^1(\Omega, \mathcal{T}_H) \). Then
\[ ||\nabla v^H||_{L^2(X)} \leq CH^{-1/2}||v^H||_{L^2(\partial \Omega)}. \]

Let \( \Pi^H \Lambda^H_{A^0} g \) be the linear interpolant of \( \Lambda^H_{A^0} g \) on \( \partial \Omega \).

**Lemma 5.2.** The following interpolation error estimate holds:
\[ \langle \Lambda^H_{A^0} g - \Pi^H \Lambda^H_{A^0} g, v^H \rangle_{L^2(\partial \Omega)} \leq CH^{3/2}||u^0||_{H^3(\Omega)} ||v^H||_{L^2(\partial \Omega)}. \]

**Proof.** The proof is given in [28], and it is a consequence of the Bramble-Hilbert lemma. \[ \square \]
Following [28] we can then obtain the following theorem which establishes high order convergence to zero of the error for the approximate flux in the \( L^2(\partial \Omega) \)-norm.

**Theorem 5.3.** Consider a quasi-uniform family of macroscopic triangulations \( \{ \mathcal{T}_H \} \). Assume that the coupling between macro and micro meshes follows \( H = \mathcal{O}(h/\varepsilon) \). Let the solution \( u^0 \) of the problem (5.10) be in \( H^3(\Omega) \) and the coefficients \( a_{ij}^0 \in W^{2,\infty}(\Omega) \). Then the approximate boundary flux computed by means of (5.12) satisfies
\[ ||\Lambda^H_{A^0} g - \Lambda^H_{A^0} g||_{L^2(\partial \Omega)} \leq C \left( H^{3/2} + \left( \frac{h}{\varepsilon} \right)^{3/2} \right), \]
where \( C \) is a constant independent on \( H, h, \) and \( \varepsilon \).

**Proof.** Subtracting (5.12) from (5.11) we obtain, for \( j = 1, \ldots, n, \)
\[ (\Lambda^H_{A^0} g - \Lambda^H_{A^0} g, v^H)_{L^2(\partial \Omega)} = B(u^0, v^H) - B_H(u^H, v^H) \]
\[ \forall v^H \in S_c^1(\Omega, \mathcal{T}_H) \cap S_c^1(\Omega, \mathcal{T}_H), \quad v^H = 0 \text{ on } \partial \Omega \setminus \Gamma_j. \] Next we define the bilinear forms
\[ \tilde{B}_H(v^H, w^H) := \sum_{K \in \mathcal{T}_H} |K| \int_K A^0(x_K) \nabla v^H_K \cdot \nabla w^H_K \, dx, \]
and
\[ \overline{B}_H(v^H, w^H) := \sum_{K \in \mathcal{T}_H} \frac{K}{|K_s|} \int_{K_s} A^s \nabla v_K \cdot \nabla w_K \, dx, \]
where \( v_K, w_K \) are the exact solutions to the micro problem (5.3) in the space of functions \( W(K_\delta) \). Hence (5.13) can be estimated by

\[
\langle \Lambda^{A_0} g - \Lambda^{H}_{A_0} g, v^H \rangle_{L^2(\partial \Omega)} \leq \frac{1}{I_1} \left[ B(u^0, v^H) - \bar{B}_H(\Pi^H u^0, v^H) \right]
+ \frac{1}{I_2} \left[ \bar{B}(\Pi^H u^0, v^H) - \bar{B}_H(u^H, v^H) \right]
+ \frac{1}{I_3} \left[ \bar{B}_H(u^H, v^H) - B_H(u^H, v^H) \right],
\]

where \( \Pi^H \) denotes the linear interpolation operator. From [28] it follows that

\[
I_1 \leq C H^2 (\|u^0\|_{H^3(\Omega)} + \|f\|_{H^2(\Omega)}) \|\nabla v^H\|_{L^2(X_j)}.
\]

On the other hand, it is well known [1, 2] that

\[
I_3 \leq C \left( \frac{h}{\varepsilon} \right)^2 \|\nabla u^H\|_{L^2(X_j)} \|\nabla v^H\|_{L^2(X_j)},
\]

where \( C \) is a constant which is independent of \( \delta \) and \( x_K \). The term \( I_2 \) captures the modelling error, which vanishes under the assumption that the locally periodic tensor admits explicit separation between slow and fast variables, and it is collocated at the slow variable. Hence from Lemma 5.1 we get that

\[
\langle \Lambda^{A_0} g - \Lambda^{H}_{A_0} g, v^H \rangle_{L^2(\partial \Omega)} \leq C \left( H^{3/2} + \left( \frac{h}{\varepsilon} \right)^2 H^{-1/2} \right) \|v^H\|_{L^2(\partial \Omega)}.
\]

Then we choose

\[
v^H = \begin{cases} 
\Pi^H \Lambda^{A_0} g - \Lambda^{H}_{A_0} g & \text{on } \Gamma_j, \\
0 & \text{on } \partial \Omega \setminus \Gamma_j,
\end{cases}
\]

where \( \Pi^H \) is the linear interpolant operator that appears in Lemma 5.2. Hence, from the triangle inequality we obtain

\[
\|\Lambda^{A_0} g - \Lambda^{H}_{A_0} g\|_{L^2(\partial \Gamma_j)} \leq C \left( H^{3/2} + \left( \frac{h}{\varepsilon} \right)^2 H^{-1/2} \right),
\]

for \( j = 1 \ldots, n \). Now, for \( H = \mathcal{O}(h/\varepsilon) \) we can finally conclude that

\[
(5.14) \quad \|\Lambda^{A_0} g - \Lambda^{H}_{A_0} g\|_{L^2(\Gamma_j)} \leq C \left( H^{3/2} + \left( \frac{h}{\varepsilon} \right)^{3/2} \right),
\]

for each \( i = j \ldots, n \), and the desired assertion immediately follows. \( \Box \)

We perform some numerical experiments to test the convergence of the method and to observe how the micro error affects the approximate flux. We consider the elliptic problem

\[
- \nabla \cdot (A^\varepsilon \nabla u^\varepsilon) = 0 \quad \text{in } \Omega,
\]

\[
u^\varepsilon = g \quad \text{on } \partial \Omega.
\]
The domain $\Omega$ is defined as
$$
\Omega = \{ x = (x_1, x_2) : 0 < x_1, x_2 < 1 \},
$$
while
$$
g = \sin(\pi(x_1 + x_2)) .
$$

We perform two numerical tests for two different choices of $A^\varepsilon$. In the first experiment we consider
$$
a_{11}(x, x/\varepsilon) = (16(x_1^2 - x_1)(x_2^2 - x_2) + 1)(\cos^2\left(2\pi\frac{x_1}{\varepsilon}\right) + 1),
$$
$$
a_{22}(x, x/\varepsilon) = (16(x_1^2 - x_1)(x_2^2 - x_2) + 1)(\sin^2\left(2\pi\frac{x_2}{\varepsilon}\right) + 2),
$$
$$
a_{12}(x, x/\varepsilon) = a_{21}(x, x/\varepsilon) = 0 .
$$

We compute the approximate flux on the boundary nodes by means of the FE-HMM. We solve the problem for different choices of $N_{mac}$ and $N_{mic}$, where $N_{mac}$ and $N_{mic}$ denote respectively the number of discretization points on each direction of the macro and the micro domain. To compute the error we use as reference solution the one obtained with $N_{mac} = N_{mic} = 64$. Numerical results are shown in Table 1 and Figure 2.

<table>
<thead>
<tr>
<th>$N_{mac}$</th>
<th>$N_{mic}$ = 4</th>
<th>$N_{mic}$ = 8</th>
<th>$N_{mic}$ = 16</th>
<th>$N_{mic}$ = 32</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N_{mac}$ = 4</td>
<td>2.304</td>
<td>0.8864</td>
<td>0.5373</td>
<td>0.4643</td>
</tr>
<tr>
<td>$N_{mac}$ = 8</td>
<td>1.9243</td>
<td>0.5417</td>
<td>0.1601</td>
<td>0.0658</td>
</tr>
<tr>
<td>$N_{mac}$ = 16</td>
<td>1.8896</td>
<td>0.5140</td>
<td>0.1328</td>
<td>0.0331</td>
</tr>
<tr>
<td>$N_{mac}$ = 32</td>
<td>1.8827</td>
<td>0.5087</td>
<td>0.1280</td>
<td>0.0281</td>
</tr>
</tbody>
</table>

Table 1: First experiment, error $||\Lambda_{A^0}g - \Lambda_{A^0\varepsilon}^H g||_{L^2(\partial\Omega)}$ for different choices of $N_{mac}$ and $N_{mic}$.

In the second experiment we consider the tensor
$$
a_{11}(x, x/\varepsilon) = \left(\sqrt{x_1^2 + \sin\left(2\pi\frac{x_1}{\varepsilon}\right) + 1.2}\left(x_1x_2 + \sin\left(4\pi\frac{x_1}{\varepsilon}\right) + 1.5\right)\right)^{-1},
$$
$$
a_{22}(x, x/\varepsilon) = \left(\left(x_1x_2 + \sin\left(5\pi\frac{x_2}{\varepsilon}\right) + 1.2\right)\left(x_2^2 \cos\left(2\pi\frac{x_2}{\varepsilon}\right) + x_1 + 1.5\right)\right)^{-1},
$$
$$
a_{12}(x, x/\varepsilon) = a_{21}(x, x/\varepsilon) = 0 .
$$

As shown in Table 1, 2 and Figure 2, 3, the error converges quadratically with respect to both $N_{mac}$ and $N_{mic}$. In particular, by refining both the macro and the micro meshes, we obtain quadratic convergence rate for the error, in agreement with Theorem 5.3.

5.3. Solving the discrete inverse problem. We consider the discrete inverse problem regularized by means of the Tikhonov method introduced in Section 4. To solve the inverse problem numerically we discretize the domain by using simplicial elements, and we approximate both the macro and micro finite element spaces with linear piecewise polynomials. As forward solver the RB-FE-HMM is used, and we assume that the numerical forward error due to the reduced basis approach is sufficiently
small to be neglected. In other words we assume that the numerical error introduced by the forward solver depends only on the macro discretization. Let us consider a locally periodic tensor $A(\varepsilon(x), x/\varepsilon)$ and assume that $\alpha \leq \sigma(x) \leq \beta$, $\forall x \in \Omega$.

Input parameters to solve the problem are the set of boundary fluxes

$$\Lambda_{A(\sigma,x/\varepsilon)} g_i \quad i = 1, \ldots, N,$$

the range for the parameter $[\alpha, \beta]$, and the matrix function $(t, y) \mapsto A(t, y), t \in [\alpha, \beta]$, $y = x/\varepsilon \in (0,1)^d$. The boundary fluxes in (5.15) are computed by means of FEMs, using a mesh size which we denote as $h_{\text{data}}$. Next we define the discrete admissible set for the solution

$$A^H = \{ \mu^H \in S^1(\Omega; T^H) : \alpha \leq \mu^H \leq \beta \}.$$

In the offline stage we build the reduced space of micro functions which will be then used to perform fast multiple evaluations of the cost functional during the minimization process. Slightly differently from the RB-FE-HMM introduced in Section 5.1, we define the training set as the set of couples

$$\Xi_{\text{Train}} = \{(t_n, \eta_n) : \alpha \leq t_n \leq \beta, 1 \leq \eta_n \leq d, 1 \leq n \leq N_{\text{Train}}\},$$
where each $t_n$ is a value randomly selected from the interval $[\alpha, \beta]$. If, for a given $n$, the parametrization $A(t_n, y)$ is in the affine form

$$A(t_n, y) = \sum_{q=1}^{Q} \Theta_q(t_n) A_q(y),$$

we start building the reduced space $S_N(Y)$ by using the greedy procedure illustrated in Section 5.1. If instead $A(t_n, y)$ is not directly in the form (5.16), we can always obtain an affine approximation of $A(t_n, y)$ by means of the EIM. Finally the discrete minimization problem we want to solve reads: find $\sigma^{\varepsilon,H} \in A^H$ such that

$$\Psi^H(\sigma^{\varepsilon,H}) = \inf_{\mu^H \in A^H} \Psi^H(\mu^H)$$

where

$$\Psi^H(\mu^H) = \sum_{i=1}^{N} \|A_{(x_{\varepsilon},y)}g_i - A^H_{(\mu^H)N}g_i\|_{L^2(\partial\Omega)}^2 + \gamma \|\mu^H - \mu_0\|_{H^1(\Omega)}^2$$

The minimization problem is solved by means of the interior point algorithm. See [9] for details. For each new admissible guess $\mu^H \in A^H$ and for each $i = 1, \ldots, N$ we compute the approximate boundary flux by solving the linear system

$$- \int_{\partial\Omega} A^H_{(\mu^H)N}g_i \cdot v^H \, ds = B_{H,RB}(u^H_{RB}, v^H) \quad \forall u^H \in S_1^1(\Omega, T_H) \cap S_1^1(\Omega, T_H),$$

where $B_{H,RB}(v^H, w^H)$ is defined as in (5.9), with the only difference that now we have

$$(A^H_{(x_{K_j})})_{ik} = \int_Y A^H_{(x_{K_j}),y}(\nabla^{k,N}_{x_{K_j}} + e_i) \cdot (\nabla^{k,N}_{x_{K_j}} + e_k) \, dy.$$
When $\Psi^H$ does not decrease any more, or the gradient of the objective function decreases under a certain tolerance that we choose a priori, the minimization process stops. The set $\mathcal{A}^H$ is finite dimensional and uniformly bounded. Thus, the existence of a minimizer $\sigma_\gamma^H \in \mathcal{A}^H$ to the discrete optimization problem (5.17) is ensured for any $H > 0$ by compactness and norm equivalence of finite dimensional spaces. One question we would like to answer to, is whether the sequence $\{\sigma_\gamma^H\}_{H>0}$ of discrete solutions converges to a minimizer $\sigma_\gamma^c$ of the continuous problem as we refine the mesh. To this end we first state a discrete analogue of Lemma 4.1. In particular, in what follows, we assume that the numerical error in the forward solver only depends on $H$, $h/\varepsilon$, by assuming no error due to the model order reduction.

**Lemma 5.4.** Suppose the assumptions of Theorem 5.3 hold, and let the sequence $\{\mu^H\}_{H>0} \in \mathcal{A}^H \subset \mathcal{A}$ converges in $L^r(\Omega)$, $r \geq 1$, to some $\mu \in \mathcal{A}$ as $H$ tends to zero. The the sequence of finite element approximations $\{\Lambda_{\mathcal{A}_0,N}^H(\mu^H,g)\}_{H>0}$ converges to $\Lambda_{\mathcal{A}_0}^0(\mu)g$ in $L^2(\partial \Omega)$ as $H,h/\varepsilon$ tend to zero.

**Proof.** The desired assertion easily follows from Lemma 4.1 and the estimate (5.14). \(\blacksquare\)

Now, thanks to Lemma 5.4 we can state the convergence of the discrete approximate solutions $\{\sigma_\gamma^H\}_{H>0}$. Let $\sigma_\gamma^c$ be a solution of the regularized inverse problem in the infinite dimension, so that

$$
\Psi(\sigma_\gamma^c) = \min_{\mu \in \mathcal{A}} \Psi(\mu),
$$

where

$$
\Psi(\mu) = \sum_{i=1}^N \|\Lambda_{\mathcal{A}_0}(\sigma,\varepsilon)g_i - \Lambda_{\mathcal{A}_0}(\mu)g_i\|_{L^2(\partial \Omega)}^2 + \gamma \|\mu - \mu_0\|_{H^1(\Omega)}^2
= \Phi(\mu) + \gamma \|\mu - \mu_0\|_{H^1(\Omega)}^2.
$$

**Theorem 5.5.** Let $\{\sigma_\gamma^H\}_{H>0}$ be a sequence of minimizers of the discrete optimization problem (5.17). Then it contains a subsequence that converges to a minimizer of problem (5.18) as $H,h/\varepsilon$ tend to zero. The convergence is weak in $H^1(\Omega)$.

**Proof.** Our proof is inspired from [19]. Here we briefly sketch the main steps to obtain the desired result. We start by noting that the minimizing properties of $\{\sigma_\gamma^H\}_{H>0}$ imply that the sequence $\{\Psi^H(\sigma_\gamma^H)\}_{H>0}$ is uniformly bounded, hence $\{||\sigma_\gamma^H - \mu_0||_{H^1(\Omega)}^2\}_{H>0}$ is uniformly bounded. Then there exists a subsequence, still denoted $\{\sigma_\gamma^H\}_{H>0}$, and some $\sigma_\gamma^c \in \mathcal{A}$, such that $\sigma_\gamma^c \rightharpoonup \sigma_\gamma^c$ weakly in $H^1(\Omega)$. Then from Lemma 5.4 we have that

$$
\Lambda_{\mathcal{A}_0,N}(\sigma_\gamma^c)g \rightharpoonup \Lambda_{\mathcal{A}_0}^0(\sigma_\gamma^c)g \quad \text{in} \ L^2(\partial \Omega),
$$

for any $g$ regular enough. From the weak lower semicontinuity of norms we have that

$$
\Psi(\sigma_\gamma^c) \leq \lim_{H,h/\varepsilon \to 0} \Phi^H(\sigma_\gamma^H) + \liminf_{H \to 0} \gamma ||\sigma_\gamma^H - \mu_0||_{H^1(\Omega)}^2
\leq \liminf_{H,h/\varepsilon \to 0} (\Phi^H(\sigma_\gamma^H) + \gamma ||\sigma_\gamma^H - \mu_0||_{H^1(\Omega)}^2)
\leq \liminf_{H,h/\varepsilon \to 0} \Psi^H(\sigma_\gamma^H).$$

22
It remains now to show that $\sigma_\varepsilon$ is indeed the minimizer of problem (5.18). Since $C^\infty(\Omega)$ is dense in $H^1(\Omega)$, we have that for any $\mu \in \mathcal{A}$, there exists a sequence $\{\mu_\eta\}_{\eta > 0} \subset C^\infty(\Omega) \cap \mathcal{A}$ such that
\[
\lim_{\eta \to 0^+} ||\mu_\eta - \mu||_{H^1(\Omega)} = 0. \tag{5.19}
\]

Let $\Pi^H : \mathcal{A} \to \mathcal{A}^H$ be the linear interpolation operator. The minimizing properties of $\{\sigma_\varepsilon^H\}_{H > 0}$ imply that
\[
\Psi^H(\sigma_\varepsilon^H) \leq \Psi^H(\Pi^H \mu_\eta) \quad \forall \eta > 0.
\]

Letting $H, h/\varepsilon \to 0$, we obtain from the approximation properties of $\Pi^H$, Lemma 5.4, and (5.3), that
\[
\Psi(\sigma_\varepsilon) \leq \Psi(\mu) \quad \forall \mu \in \mathcal{A},
\]
and the desired assertion follows.

**Remark.** The more general case $\mathcal{A}\varepsilon(x) = A(\sigma(x), x, x/\varepsilon)$ could also be treated. In this case we need the matrix function $(t, x, y) \mapsto A(t, x, y)$, $t \in [\alpha, \beta]$, $x \in \Omega$, $y = x/\varepsilon \in (0,1)^d$. Then the algorithm remains unchanged, with the only difference that the training set is defined as the set of triples
\[
\Xi_{\text{Train}} = \{(t_n, T_{\delta_n}, \eta_n) : \alpha \leq t_n \leq \beta, 1 \leq \eta_n \leq d, 1 \leq n \leq N_{\text{Train}}\}.
\]

We end this section by summarizing our algorithm for the computation of multiscale inverse problems using numerical homogenization and model order reduction techniques.

**Algorithm.** Input: $\{g_i\}_{i=1}^N$, $\{\Lambda_{A(\sigma, x/\varepsilon)}g_i\}_{i=1}^N$, $[\alpha, \beta]$, $\gamma$.

1. Offline stage:
   (a) Define training set.
   (b) Perform EIM if necessary.
   (c) Construct reduced space $S^N(Y)$.

2. Online stage: start with an initial guess $\mu_0^H$. For each new guess $\mu^H \in \mathcal{A}^H$:
   (a) For each $i = 1, \ldots, N$ solve:
   \[
   B_{H, RB}(u_i^H, v_i^H) = 0, \quad u_i^H = g_i \text{ on } \partial\Omega,
   \]
   where
   \[
   B_{H, RB}(v_i^H, w_i^H) = \sum_{K \in T_H} \frac{|K|}{|K_\delta|} A_{0,N}(\mu^H(x_K)) \nabla v_i^H(x_K) \cdot w_i^H(x_K).
   \]
   (b) Compute the approximate fluxes $\Lambda_{A(\varepsilon)}g_i$, $i = 1, \ldots, N$ solving 5.12.
   (c) Evaluate $\Psi^H(\mu^H)$ given by
   \[
   \sum_{i=1}^N ||\Lambda_{A(\varepsilon)}g_i - \Lambda_{A(\varepsilon)}^Hg_i||_{L^2(\partial\Omega)}^2 + \gamma ||\mu^H - \mu_0||_{H^1(\Omega)}^2.
   \]
   (d) Stop when convergence to a tolerance is reached.
6. Numerical experiments. In this section we present numerical experiments that illustrate the behaviour of the proposed numerical method for solving inverse problems. At first we consider the case of a 2D affine anisotropic multiscale tensor. The set up of the numerical experiments is as follows. The domain $\Omega$ is defined as

$$\Omega = \{ x = (x_1, x_2) : 0 < x_1, x_2 < 1 \}.$$ 

The data are obtained by solving the full multiscale problem with a very fine mesh by means of FEMs. In what follows we will denote as $h_{\text{data}}$ the mesh size used to obtain the multiscale data. The data consist of approximate boundary fluxes obtained for a prescribed set of Dirichlet conditions $\{g_i\}_{i=1}^N$. We take $N = 2$, and $g_1$ and $g_2$ are chosen as

$$g_1(x) = \sin(\pi(x_1 + x_2)), \quad g_2(x) = \cos(\pi(x_1 + x_2)).$$

The true conductivity is given by

$$a_{11}(\sigma(x), x/\varepsilon) = \sigma(x) \left( \cos^2 \left( \frac{2\pi}{\varepsilon} x_1 \right) + 1 \right),$$

$$a_{22}(\sigma(x), x/\varepsilon) = \sigma(x) \left( \sin^2 \left( \frac{2\pi}{\varepsilon} x_2 \right) + 2 \right),$$

$$a_{12}(\sigma(x), x/\varepsilon) = a_{21}(\sigma(x), x/\varepsilon) = 0,$$

where

$$\sigma(x) = 16(x_1^2 - x_1)(x_2^2 - x_2) + 1.$$ 

In this first example $\sigma$ is a simple smooth parabola, and its profile together with the one of $A^\varepsilon(x)$ is shown in Figure 4. We compute then the approximate solution to the

![Figure 4: The true field $\sigma$ and the two component $a_{11}^\varepsilon$ and $a_{22}^\varepsilon$ of the multiscale tensor ($\varepsilon = 1/8$ for visualization purposes).](image)

multiscale inverse problem $\sigma_{\gamma,H}^\varepsilon$ by solving the minimization problem

$$\min_{\mu \in \mathcal{M}} \sum_{i=1}^N \| A_{A(\sigma,x/\varepsilon)g_i} - A_{A^0,\gamma(\mu^\varepsilon)g_i} \|_{L^2(\Omega)}^2 + \gamma \| \mu^\varepsilon - \mu_0 \|_{H^1(\Omega)}^2.$$
For comparison, let us define as $\sigma_0^\gamma$ the regularized solution to the minimization problem

\[
\min_{\mu \in \mathcal{A}} \sum_{i=1}^{N} ||\Lambda_{A^0} g_i - \Lambda_{A^0} g_i||_{L^2(\partial \Omega)}^2 + \gamma ||\mu - \mu_0||_{H^1(\Omega)}^2,
\]

and let $\sigma_0^{0,H}$ be its discrete approximation. Notice that the forward model used to solve (6.1) is the same as the one used to produce the data, and no coarse grained approach is needed in this case. Hence, in the solution we do not introduce any error due to the discrepancy between the model used to produce the data and one employed to solve the inverse problem. We use then $\sigma_0^{0,H}$ as reference solution, and we compare it with the regularized solutions $\sigma_\varepsilon^{\gamma,H}$ obtained for different values of $\varepsilon$. What we expect is that as $\varepsilon$ gets smaller, $\sigma_\varepsilon^{\gamma,H}$ approximates well $\sigma_0^{0,H}$. Secondly, we test the convergence of $\sigma_\varepsilon^{\gamma,H}$ towards $\sigma_\gamma$ as the mesh size $H$ tends to zero. Indeed we assume the error due to the reduced basis approach to be sufficiently small to be neglected, and thus we assume the numerical error due to discretization as only dependent on the macro discretization. Before showing the results however we show the error between $\sigma$ and $\sigma_0^{0,H}$, obtained for different choices of $\gamma$, while $H$ is fixed to $1/32$, to observe how the choice of $\gamma$ affects the reconstruction procedure. Results are reported in Figure 5. As can be seen from numerical results in Figure 5 the regularized solution we get have properties which vary with the regularization parameter $\gamma$. The larger is the value of $\gamma$, the more regularized is the Tikhonov problem. However for $\gamma$ too large the approximate solution is too regularized, and then far from the true field we want to retrieve. On the other hand, when $\gamma$ becomes too small, the problem becomes more unstable, and much more oscillations will be allowed in the approximate solution. In particular we observe that for our setting a good choice is represented by a regularization parameter $\gamma$ which takes values between 0.01 and 0.005. We collect then data for several different choices of $\varepsilon$, and solve the problem for different mesh sizes $H$. We start by fixing $H = 1/32$ and observe the error between $\sigma_\gamma^{0,H}$ and $\sigma_\varepsilon^{\gamma,H}$

![Figure 5: $L^2(\Omega)$-error and $H^1(\Omega)$-error between $\sigma$ and $\sigma_0^{0,H}$ as function of the regularization parameter $\gamma$ ($H = 1/32$).](image-url)
as $\varepsilon \to 0$. In Figure 6 we can observe the convergence of the approximate solution $\sigma_{\gamma,H}^\varepsilon$ to $\sigma_{\gamma,H}^0$ as $\varepsilon \to 0$. The errors in the $L^2(\Omega)$-norm and $H^1(\Omega)$-norm are both shown. For this experiment $\gamma = 0.01$ and $h_{data} = 1/2048$. For relatively large values of $\varepsilon$, namely $\varepsilon > H$, the error we obtain is relatively large and no convergence is observed. This is due to the fact that, since $\varepsilon > H$, the approximate homogenized flux, which approximates at the best the multiscale flux, is capable of capturing its typical oscillations. Hence, such oscillations will affect the retrieved solution as well. As soon as $\varepsilon < H$ we can instead observe convergence of the error as $\varepsilon \to 0$. However we know that in order to dispose of very accurate data we need $h_{data} << \varepsilon$. Hence if $\varepsilon$ is very small, $h_{data}$ could be not sufficiently small to capture in details the oscillating flux, leading to bad solutions to the inverse problem as shown in Figure 7. To verify convergence with respect to discretization, we fix $\varepsilon = 1/64$, and use the discrete

![Figure 6: Convergence of $\sigma_{\gamma,H}^\varepsilon$ to $\sigma_{\gamma,H}^0$ as $\varepsilon \to 0$, for different choices of $h_{data}$, $\varepsilon = 1/64$, $\gamma = 0.01$, $H = 1/32$.](image)

![Figure 7: Behaviour of the error $||\sigma_{\gamma,H}^0 - \sigma_{\gamma,H}^\varepsilon||_{L^2(\Omega)}$ as $\varepsilon \to 0$, for different choices of $h_{data}$. $\varepsilon = 1/64$, $\gamma = 0.01$, $H = 1/32$.](image)
minimizer $\sigma_{\gamma}^{\varepsilon,H}$ obtained on the finest mesh as reference solution. In Table 3 and in Figure 8a we show the error $||\sigma_{\gamma}^{\varepsilon} - \sigma_{\gamma}^{\varepsilon,H}||_{L^2(\Omega)}$ as $H \to 0$. According to Theorem 5.5 we can observe that the discrete minimizers $\sigma_{\gamma}^{\varepsilon,H}$ converge to the reference solution as the mesh size $H$ decreases. The numerical results are obtained for two different choices of the parameter $\gamma$, while $h_{data} = 1/2048$. In Figure 8b we compare the cross-section of the true field together with the ones of $\sigma_{\gamma}^{0,H}$ and $\sigma_{\gamma}^{\varepsilon,H}$. We can observe that our solution captures well $\sigma_{\gamma}^{0,H}$. Finally in Figure 9 we show the conductivity reconstruction obtained for $h_{data} = 1/2048$, $\varepsilon = 1/64$, $\gamma = 0.01$, $H = 1/32$. It is important to remark that the results showed in Figure 9 are obtained for $\varepsilon = 1/64$ (hence we obtain $\sigma_{\gamma}^{\varepsilon,H}$, $\varepsilon = 1/64$). However, in order to well visualize the results and compare the profile of the multiscale tensor with the one shown in Figure 4, we plot $A(\sigma_{\gamma}^{\varepsilon,H}, x/\varepsilon')$, where $\varepsilon' = 1/8$. We can see a good agreement between our solution and the true tensor shown in Figure 4.

<table>
<thead>
<tr>
<th>$H$</th>
<th>$1/4$</th>
<th>$1/8$</th>
<th>$1/16$</th>
<th>$1/32$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$</td>
<td></td>
<td>\sigma_{\gamma}^{\varepsilon} - \sigma_{\gamma}^{\varepsilon,H}</td>
<td></td>
<td>_{L^2(\Omega)}$, $\gamma = 0.01$</td>
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<tr>
<td>$</td>
<td></td>
<td>\sigma_{\gamma}^{\varepsilon} - \sigma_{\gamma}^{\varepsilon,H}</td>
<td></td>
<td>_{L^2(\Omega)}$, $\gamma = 0.001$</td>
</tr>
</tbody>
</table>

Table 3: Errors $||\sigma_{\gamma}^{\varepsilon} - \sigma_{\gamma}^{\varepsilon,H}||_{L^2(\Omega)}$.

(a) Convergence of the error $||\sigma_{\gamma}^{\varepsilon} - \sigma_{\gamma}^{\varepsilon,H}||_{L^2(\Omega)}$ as $H \to 0$, ($\varepsilon = 1/64$, $h_{data} = 1/2048$).

(b) Section of $\sigma$, $\sigma_{\gamma}^{0,H}$, and $\sigma_{\gamma}^{\varepsilon,H}$, at $x_2 = 0.5$, ($\varepsilon = 1/64$, $\varepsilon = 0.01$, $H = 1/32$, $h_{data} = 1/2048$).

Figure 8: Convergence of the $L^2$-error as $H \to 0$ and cross-section of the true field compared to the ones obtained by means of the Tikhonov regularization.

For the second experiment we consider an anisotropic multiscale tensor that is not in
Figure 9: The profile of the reconstructed field $\sigma^{\varepsilon,H}_\gamma$ and the two component $a_{11}(\sigma^{\varepsilon,H}_\gamma, x/\varepsilon')$ and $a_{22}(\sigma^{\varepsilon,H}_\gamma, x/\varepsilon')$ of the multiscale tensor ($\varepsilon = 1/64$, $\gamma = 0.01$, $H = 1/32$, $h_{\text{data}} = 1/2048$) ($\varepsilon' = 1/8$ for visualization purposes).

affine form. We take a tensor of the form

\begin{align*}
a_{11}(\sigma(x), x/\varepsilon) &= \sqrt{\sigma(x) + \sin \left( 4\pi \frac{x_1}{\varepsilon} \right) + 1}, \\
a_{22}(\sigma(x), x/\varepsilon) &= \sqrt{\sigma(x) + \cos^2 \left( 2\pi \frac{x_2}{\varepsilon} \right)}, \\
a_{12}(\sigma(x), x/\varepsilon) &= a_{21}(\sigma(x), x/\varepsilon) = 0,
\end{align*}

where

$$\sigma(x) = \exp \left( -\frac{(x_1 - 2/3)^2 + (x_2 - 2/3)^2}{0.045} \right) + 1.$$ 

This time the parametrized multiscale tensor is not in affine form, and therefore an affine approximation is obtained by means of the EIM. The profile of $\sigma$ and the multiscale tensor for this second experiment are shown in Figure 10. Again the data consist of approximate boundary fluxes obtained for a finite set of Dirichlet conditions. For this experiment we set $N = 3$, and we consider $g_3(x) = \cos(2\pi(x_1 + x_2))$,

while $g_1$ and $g_2$ are the same used in the first numerical experiment. We obtain data by using $h_{\text{data}} = 1/4096$, and we compute the regularized solution $\sigma^{\varepsilon,H}_\gamma$ using $\gamma = 5 \times 10^{-4}$, $\varepsilon = 1/64$, and $H = 1/32$. In Figure 11 we compare the cross-section at $x_2 = 0.67$ of the true field $\sigma$ and the ones of the numerical solutions $\sigma^{H}_\gamma$ and $\sigma^{\varepsilon,H}_\gamma$, while in Figure 12 we show the numerical reconstruction of the two components of the multiscale tensor. As for the first experiment we plot the tensor $A(\sigma^{\varepsilon,H}_\gamma(x), x/\varepsilon')$, $\varepsilon' = 1/8$, in order to have a clear visualization of the numerical results. We can observe good agreement between Figure 10 and Figure 12. In particular we capture well the region where $\sigma$ starts to take higher values. Instead we can observe some difference between the maximum values of $\sigma$ and $\sigma^{\varepsilon,H}_\gamma$. However such error is simply due to the Tikhonov approach, and does not depend on the coarse grained approach we suggest. Indeed it is possible to observe that we obtain a very good approximation of $\sigma^{H}_\gamma$. 

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Figure 10: The true field $\sigma$ and the two component $a_{11}$ and $a_{22}$ of the multiscale tensor considered in the second experiment ($\varepsilon = 1/8$ for visualization purposes).

Figure 11: Section of $\sigma$, $\sigma_0^H$, $H_\gamma$, and $\sigma_\varepsilon^H$, at $x_2 = 0.67$ ($\varepsilon = 1/64$, $\gamma = 5 \times 10^{-4}$, $H = 1/32$, $h_{\text{data}} = 1/4096$).

Acknowledgements. The authors are partially supported by the Swiss national foundation.

Appendix. In this appendix we recall first Meyer’s theorem on regularity of elliptic problems before proving a technical lemma on the continuity of the Dirichlet to Neumann map with respect to parameter, used in Section 3.

Theorem 6.1. (N. G. Meyers, 1963, [18, 26]). Let $\Omega \in \mathbb{R}^d$ be a bounded open set, with a Lipschitz continuous boundary. Let $A \in \mathcal{M}(\alpha, \beta, \Omega)$. There exists a constant $q_1 > 2$, depending on $d$, $\Omega$, $\alpha$ and $\beta$ only, such that if $u$ is the unique weak solution of

$$
- \nabla \cdot (A \nabla u) = f \quad \text{in } \Omega,
$$

$$
u = g \quad \text{on } \partial \Omega,$$

and $f \in W^{-1,q'}(\Omega)$, $g \in W^{1/q,q}(\partial \Omega)$, $1/q' + 1/q = 1$, $q \in [2, q_1)$, then $u \in W^{1,q}(\Omega)$ and there exists a constant $C_1$, depending on $d$, $\Omega$, $\alpha$, $\beta$ and $q$ only, such that

$$
||u||_{W^{1,q}(\Omega)} \leq C_1 (||R_g||_{W^{1,q}(\Omega)} + ||f||_{W^{-1,q'}(\Omega)}),
$$

where $R_g$ denotes the extension of $g$ onto $W^{1,q}(\Omega)$. 29
Proof of Lemma 4.1. It follows from the weak formulation of $u^0(\mu_n)$ and $u^0(\mu)$ that, for all $v \in H^1_0(\Omega)$, we have
\[
\int_\Omega (A^0(\mu)\nabla u^0(\mu) - A^0(\mu_n)\nabla u^0(\mu_n)) \cdot \nabla v \, dx = 0.
\]
Then
\[
\int_\Omega A^0(\mu_n)(\nabla u^0(\mu) - \nabla u^0(\mu_n)) \cdot \nabla v \, dx = \int_\Omega (A^0(\mu_n) - A^0(\mu))\nabla u^0(\mu) \cdot \nabla v \, dx.
\]
By choosing $v = u^0(\mu) - u^0(\mu_n) \in H^1_0(\Omega)$, and using Holder’s inequality we obtain,
\[
\|\nabla u^0(\mu) - \nabla u^0(\mu_n)\|_{L^2(\Omega)} \leq \alpha^{-1} C_1 \|A^0(\mu) - A^0(\mu_n)\|_{L^p(\Omega)} \|R_g\|_{W^{1, q}(\Omega)}
\]
\[
\leq \alpha^{-1} C_1 E \|\mu - \mu_n\|_{L^p(\Omega)} \|R_g\|_{W^{1, q}(\Omega)}
\]
\[
\leq C_2 \|\mu - \mu_n\|_{L^p(\Omega)},
\]
where $q \in [2, \infty)$ and $C_1$ come from Theorem 6.1, and $p$ satisfies $1/p + 1/q = 1/2$. We then set $w = A^0(\mu)\nabla u^0(\mu) - A^0(\mu_n)\nabla u^0(\mu_n)$ and we obtain
\[
\int_\Omega |w|^2 \, dx = \int_\Omega A^0(\mu_n)(\nabla u^0(\mu) - \nabla u^0(\mu_n)) \cdot w \, dx
\]
\[
+ \int_\Omega (A^0(\mu) - A^0(\mu_n))\nabla u^0(\mu) \cdot w \, dx
\]
\[
\leq \|A^0(\mu_n)\|_{L^\infty(\Omega)} \|\nabla u^0(\mu) - \nabla u^0(\mu_n)\|_{L^2(\Omega)} \|w\|_{L^2(\Omega)}
\]
\[
+ \|A^0(\mu) - A^0(\mu_n)\|_{L^p(\Omega)} \|\nabla u^0(\mu)\|_{L^q(\Omega)} \|w\|_{L^2(\Omega)},
\]
and hence, by using (6.2) we get
\[
\|A^0(\mu)\nabla u^0(\mu) - A^0(\mu_n)\nabla u^0(\mu_n)\|_{L^2(\Omega)} \leq (E + \alpha) C_2 \|\mu - \mu_n\|_{L^p(\Omega)}
\]
\[
\leq C_4 \|\mu - \mu_n\|_{L^p(\Omega)},
\]
where the exponent $q \in [2, Q_1)$ comes from Theorem 6.1, and $p$ satisfies $1/p + 1/q = 1/2$. Observing that $w \in H(\Omega, \text{div})$ and using the continuity of the map $w \in H(\Omega, \text{div}) \rightarrow w \cdot n \in H^{-1/2}(\partial\Omega)$, we can finally conclude

$$\|\Lambda_{A^0(\mu)}g - \Lambda_{A^0(\mu_n)}g\|_{H^{-1/2}(\partial\Omega)} \leq C_5 C_4 \|\mu - \mu_n\|_{L^p(\Omega)}.$$  

The desired assertion follows immediately if $r \geq p$. Otherwise, if $r < p$, we can exploit the $L^\infty(\Omega)$ bound of the set $\mathcal{A}$, i.e., for any $\mu \in \mathcal{A}$ we have

$$\int_{\Omega} |\mu|^p \, dx \leq \lambda^{-p} \int_{\Omega} |\mu|^r \, dx.$$  

Assume $g \in H^{3/2}(\partial\Omega)$. Then due to the regularity assumptions on $A^0(t)$, the admissible set $\mathcal{A}$, and $g$, we have that the sequence $A^0(\mu_n) \nabla u^0(\mu_n)$ is uniformly bounded in $(H^1(\Omega))^d$. Then there exists a subsequence $A^0(\mu_{n_r}) \nabla u^0(\mu_{n_r})$ such that

$$A^0(\mu_{n_r}) \nabla u^0(\mu_{n_r}) \rightharpoonup \xi \quad \text{weakly in } (H^1(\Omega))^d,$$

for some $\xi \in (H^1(\Omega))^d$, hence

$$A^0(\mu_{n_r}) \nabla u^0(\mu_{n_r}) \rightarrow \xi \quad \text{strongly in } (L^2(\Omega))^d.$$

But from (6.3) all subsequence $A^0(\mu_{n_r}) \nabla u^0(\mu_{n_r})$ must converge to the same limit, hence

$$A^0(\mu_n) \nabla u^0(\mu_n) \rightharpoonup A^0(\mu) \nabla u^0(\mu) \quad \text{weakly in } (H^1(\Omega))^d,$$

hence

$$A^0(\mu_n) \nabla u^0(\mu_n) \cdot n \rightharpoonup A^0(\mu) \nabla u^0(\mu) \cdot n \quad \text{weakly in } H^{1/2}(\partial\Omega)$$

in the sense of trace, or

$$\Lambda_{A^0(\mu_n)}g \rightharpoonup \Lambda_{A^0(\mu)}g \quad \text{weakly in } H^{1/2}(\partial\Omega)$$

in the sense of trace. Finally the compact injection $H^{1/2}(\partial\Omega) \subset L^2(\partial\Omega)$ yields

$$A^0(\mu_n) \nabla u^0(\mu_n) \cdot n \rightharpoonup A^0(\mu) \nabla u^0(\mu) \cdot n \quad \text{strongly in } L^2(\partial\Omega)$$

in the sense of trace, or $\Lambda_{A^0(\mu_n)}g \rightarrow \Lambda_{A^0(\mu)}g$ strongly in $L^2(\partial\Omega)$ in the sense of trace.\[\Box\]

REFERENCES


