The impact of the Boussinesq approximation on plasma turbulence in the scrape-off layer

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Why this study?

Plasma turbulence in the scrape-off layer (SOL), where magnetic field lines intersect the vessel, **determines the heat load on the walls, one of the crucial issues on the way towards a fusion reactor**.

What is the Boussinesq approximation and why we use it?

- **Context:** in the SOL is reasonable to use a fluid approximation, in particular the drift-reduced Braginskii equations [1, 2]
- **The Boussinesq approximation** is used in the evaluation of the divergence of the polarisation current:

$$\nabla_\perp \cdot \left[\frac{nc}{B\omega_{ci}}\frac{d}{dt}\left(\boldsymbol{E}_\perp - \frac{\nabla_\perp P_i}{en}\right)\right] \approx \frac{nc}{B\omega_{ci}}\frac{d}{dt}\left(\nabla_\perp \cdot \boldsymbol{E}_\perp - \frac{1}{e}\nabla^2_\perp T_i\right) \quad (1)$$

- **It simplifies the solution of the Poisson equation necessary to evaluate the electric potential**

What do we present in this poster?

1. A **new formulation of the vorticity equation** that allow us to relax the Boussinesq approximation
2. The **energy conservation properties** of the new system of equations
3. **Results of nonlinear 3D turbulent simulations** with and without the Boussinesq approximation with the GBS code [3, 4]

1. New formulation of the vorticity equation Derivation:

1. We **start** from the **ion momentum equation** given in [1] – with $\frac{d}{dt} \equiv \frac{\partial}{\partial t} + (\boldsymbol{v}_i \cdot \nabla)$ –

$$m_i \frac{d}{dt}(n\boldsymbol{v}_i) + m_i(n\boldsymbol{v}_i)(\nabla \cdot \boldsymbol{v}_i) = -\nabla P_i - \nabla \cdot \overline{\overline{\Pi}}_i + Zen\left(\boldsymbol{E} + \frac{1}{c}(\boldsymbol{v}_i \times \boldsymbol{B})\right) - \boldsymbol{R}_i, \quad (2)$$

2. **Hypothesis 1:** $\partial/\partial t \approx (\rho_i^2/L_\perp^2)\,\omega_{ci} \ll \omega_{ci}$. Making use of this ordering and taking the cross product of Eq. (2) with the unit vector $\boldsymbol{b} \Longrightarrow$

$$\boldsymbol{v}_{\perp i0} = \boldsymbol{v}_E + \boldsymbol{v}_{di} = c\frac{\boldsymbol{B} \times \nabla\phi}{B^2} + c\frac{\boldsymbol{B} \times \nabla P_i}{ZenB^2}, \quad \text{with } \phi \text{ the electric potential } (\boldsymbol{E} = -\nabla\phi)$$

3. **The polarisation velocity** ($\boldsymbol{v}_{pol} \equiv \boldsymbol{v}_{\perp i} - \boldsymbol{v}_{\perp i0}$) is approximated with the 1^{st} order $\perp$ velocities:

$$\boldsymbol{v}_{pol0} = \frac{1}{n\omega_{ci}}\boldsymbol{b} \times \frac{d}{dt}(n\boldsymbol{v}_{\perp i0}) + \frac{\left(\nabla \cdot \boldsymbol{v}_{\perp i0} + \nabla_\parallel v_{\parallel i}\right)}{n\omega_{ci}}\boldsymbol{b} \times (n\boldsymbol{v}_{\perp i0}) + \frac{1}{nm_i\omega_{ci}}\boldsymbol{b} \times \nabla \cdot \overline{\overline{\Pi}}_{i0}. \quad (3)$$

This drift approximation allow us to close the system of equations.

4. **The stress tensor is:**

$$\nabla \cdot \overline{\overline{\Pi}}_{i0} = \nabla \cdot \overline{\overline{\Pi}}_{FLR_i0} + \nabla \cdot \overline{\overline{\Pi}}_{vis_i0}, \quad (4)$$

with: $\nabla \cdot \overline{\overline{\Pi}}_{vis_i0} = G_0\boldsymbol{\kappa} - \frac{\nabla G_0}{3} + \boldsymbol{B}\nabla_\parallel\left(\frac{G_0}{B}\right)$, the stress function $G_0 = -3\eta_0\left(\nabla_\parallel v_{\parallel i} - \boldsymbol{\kappa} \cdot \boldsymbol{v}_{\perp i0} - \frac{\nabla \cdot \boldsymbol{v}_{\perp i0}}{3}\right)$

5. **Hypothesis 2: magnetic field variation on length scales of order R (tokamak major radius), which is larger compared to the perpendicular turbulent length scale** ($L_\perp/R \ll 1$), this implies: $\overline{\overline{\Pi}}_{FLR_i0} = -m_i n(\boldsymbol{v}_{di} \cdot \nabla)\boldsymbol{v}_i \Longrightarrow$ **'gyro-viscous' cancellation**

6. **Hypothesis 3: plasma quasi-neutrality** ($n = n_e = n_i$). Or, equivalently, we consider the stationary charge conservation equation, $\nabla \cdot \boldsymbol{j} = 0 \Longrightarrow$

$$\nabla \cdot \left(n\boldsymbol{v}_{pol0}\right) + \nabla_\parallel\left(\frac{j_\parallel}{e}\right) + \nabla \cdot (n(\boldsymbol{v}_{di} - \boldsymbol{v}_{de})) = 0 \quad (5)$$

7. Using Hypothesis 2: we neglect the term $\nabla \cdot \boldsymbol{v}_{\perp i0}$ in the polarization expression, Eq. (3) (this term is of order $L_\perp/R \ll 1$). Then the first term on the left hand side of Eq. (5) is:

$$\nabla \cdot \left(n\boldsymbol{v}_{pol0}\right) = -\frac{1}{\omega_{ci}}\frac{\partial\Omega}{\partial t} - \frac{1}{\omega_{ci}}\nabla \cdot [(\boldsymbol{v}_E \cdot \nabla)\boldsymbol{\omega}] - \frac{1}{\omega_{ci}}\nabla \cdot \left[\nabla_\parallel\left(v_{\parallel i}\boldsymbol{\omega}\right)\right] + \frac{1}{m_i\omega_{ci}}\nabla \cdot \left\{\boldsymbol{b} \times \left[G_0\boldsymbol{\kappa} - \frac{\nabla G_0}{3}\right]\right\}, \quad (6)$$

with Ω **the new scalar vorticity:** $\Omega = \nabla \cdot \boldsymbol{\omega} = -\nabla \cdot [\boldsymbol{b} \times (n\boldsymbol{v}_{\perp i0})] = \nabla \cdot (\frac{cn}{B}\nabla_\perp\phi + \frac{c}{ZeB}\nabla_\perp P_i)$ and $\boldsymbol{\omega}$ the perpendicular vector: $\boldsymbol{\omega} = -\boldsymbol{b} \times (n\boldsymbol{v}_{\perp i0}) = \frac{cn}{B}\nabla_\perp\phi + \frac{c}{ZeB}\nabla_\perp P_i$.

8. **From Eqs. (5) and (6), the new formulation of the vorticity equation is:**

$$\frac{\partial\Omega}{\partial t} = -\frac{c}{B}\nabla \cdot \left\{[\phi, \boldsymbol{\omega}]\right\} - \nabla \cdot \left\{\nabla_\parallel\left(v_{\parallel i}\boldsymbol{\omega}\right)\right\} + \omega_{ci}\nabla_\parallel\left(\frac{j_\parallel}{e}\right) + \omega_{ci}\nabla \cdot (n(\boldsymbol{v}_{di} - \boldsymbol{v}_{de})) + \frac{1}{3m_i\omega_{ci}}\frac{B}{2}\left(\nabla \times \left(\frac{\boldsymbol{b}}{B}\right)\right) \cdot \nabla G_0 \quad (7)$$

The $\boldsymbol{E} \times \boldsymbol{B}$ advection term: $(\boldsymbol{v}_E \cdot \nabla)\boldsymbol{\omega} = \frac{c}{B}[\phi, \boldsymbol{\omega}]$ and the viscous term is written as a function of the curvature operator: $\frac{B}{2}\left[\left(\nabla \times \frac{\boldsymbol{b}}{B}\right) \cdot \nabla\right]$.

9. **The Poisson equation for the electric potential** ϕ, $\nabla \cdot (\frac{cn}{B}\nabla_\perp\phi) = \Omega - \frac{c}{ZeB}\nabla^2_\perp P_i$, is solved with an efficient parallel multigrid method.

Bibliography

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2. Energy conservation with the new vorticity equation

- Taking into account the continuity, parallel and temperature equations (for ions and electrons) together with the vorticity equation (7) we obtain the expression of the time evolution of the total energy of the system:

$$\frac{d}{dt}\left\{\int dV\left[\frac{nm_i}{2}\left(\boldsymbol{v}^2_{\perp i0} + v^2_{\parallel i}\right) + \frac{nm_e}{2}\left(\frac{j_\parallel}{en}\right)^2 + \frac{3}{2}(p_i + p_e) + \frac{1}{8\pi}(\nabla_\perp\psi)^2\right]\right\} = -\int dV\left[\frac{j^2_\parallel}{\sigma_\parallel} + \frac{G_0^2}{3\eta_0}\right] + \varepsilon \quad (8)$$

with

$$\varepsilon = \int dV\left[m_i\boldsymbol{v}^2_{\perp i0}\left(\frac{c}{2Ze}\nabla p_i + cn\nabla\phi\right)\right] \cdot \left\{\nabla \times \frac{\boldsymbol{b}}{B}\right\} + \int dV\left[\frac{mm_i}{2}\boldsymbol{v}_{pol} \cdot \nabla\left(\boldsymbol{v}^2_{\perp i0} + v^2_{\parallel i}\right) + \frac{3}{2}\boldsymbol{v}_{pol} \cdot \nabla p_i\right]. \quad (9)$$

What do we learn from these equations?

1. The total energy varies because: **Joule, viscous dissipation** and the **approximation made in the drift reduction** of the Braginskii equations (see the ε term Eq. (9))
2. The first term of Eq. (9) is a **curvature term**. **Using Hypothesis 2** we find that this term is smaller than the first term on the left hand side of Eq. (8) by a factor $L_\perp/R \ll 1$
3. The second term of Eq. (9) is of order $(\boldsymbol{v}_{pol} \cdot \nabla)$. Comparing this last term with the corresponding term on the left hand side of Eq. (8), (d/dt), **using Hypothesis 1**: $\boldsymbol{v}_{pol} \cdot \nabla \approx \frac{\rho_i^2}{L_\perp^2}c_s\frac{1}{L_\perp} \approx \frac{\rho_i^2}{L_\perp^2}\frac{\rho_s\omega_{c_i}}{L_\perp} = \frac{\rho_s}{L_\perp}\frac{d}{dt} \ll \frac{d}{dt}$
4. **Therefore, if the dissipation terms can be neglected, the new model conserves the total energy within the ordering used for its deduction**

3. Numerical results

What did we do?

- Turbulent simulations in the SOL, taking into account the Boussinesq (B) and the non-Boussinesq (NB) model
- A safety factor q scan
- We considered cold ions ($\tau = T_{i0}/T_{e0} = 0$) and a hot ion regime ($\tau = 2$)

In what we are interested?

In the SOL pressure typical radial length, defined as $L_P = \left\langle\left|\frac{1}{P}\frac{\partial P}{\partial r}\right|^{-1}\right\rangle$

Why?

Is related to the power deposition on the limiter or divertor targets, the heat flux: $\Gamma \propto P\,c_s$

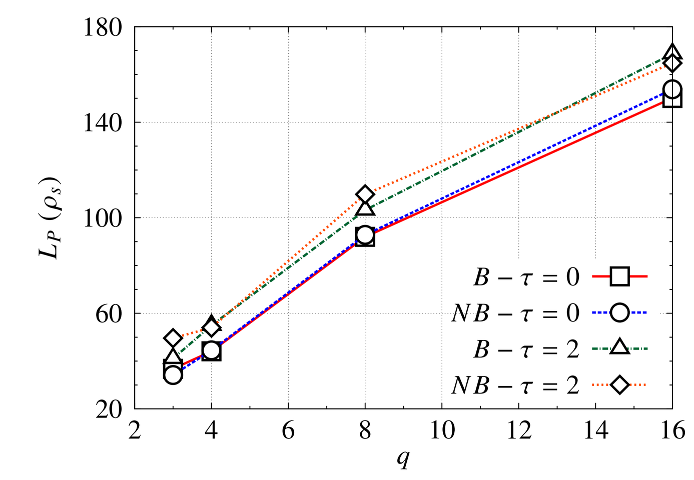

Figure 1: Radial pressure length (ρ_s units) as a function of the safety factor q.

What do we find? (see Fig. 1)

1. For $\tau = 0$ the difference in the L_P value between the B and the NB model is of a few percent
2. **L_P is 10% larger for the NB model compared to B if $q = 3$ and $\tau = 2$ are considered**

So now, what do we find for the case $q = 3$ and $\tau = 2$? (see Figs. 2, 3 and 4)

1. **In Fig. 2 a flattening of the pressure profile is visible for the NB case**
2. **The enhancement of the turbulent transport explains the flattening of P, or increase of L_P**
3. In Fig. 3 the snapshot of the pressure field shows turbulent structures that are larger for NB
4. In Fig. 4:
   a. For NB: the standard deviation and skewness have larger values
   b. For NB: the pressure spectrum shows stronger fluctuations with lower poloidal mode numbers

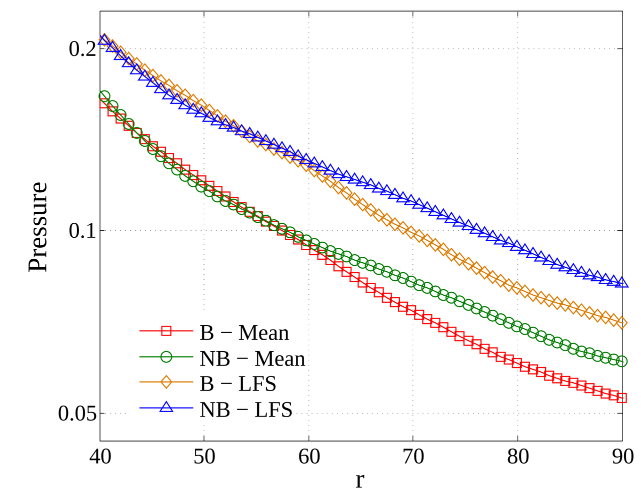

Figure 2: Radial pressure profile (semi-log), mean profile and profile at the low field side (LFS), $q = 3$ and $\tau = 2$.

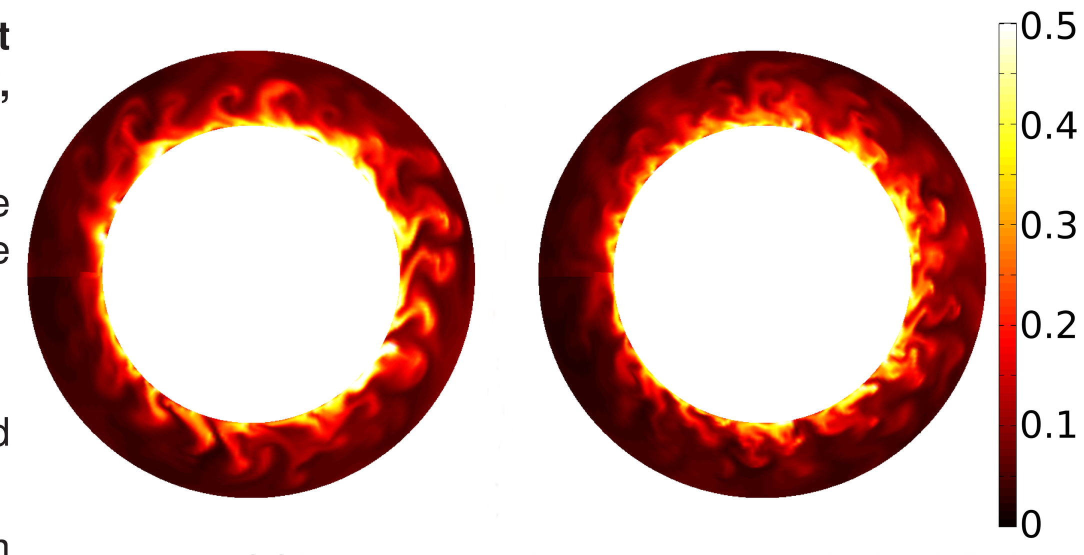

Figure 3: SOL snapshot of the pressure field for NB (left) and B (right) models, $q = 3$ and $\tau = 2$.

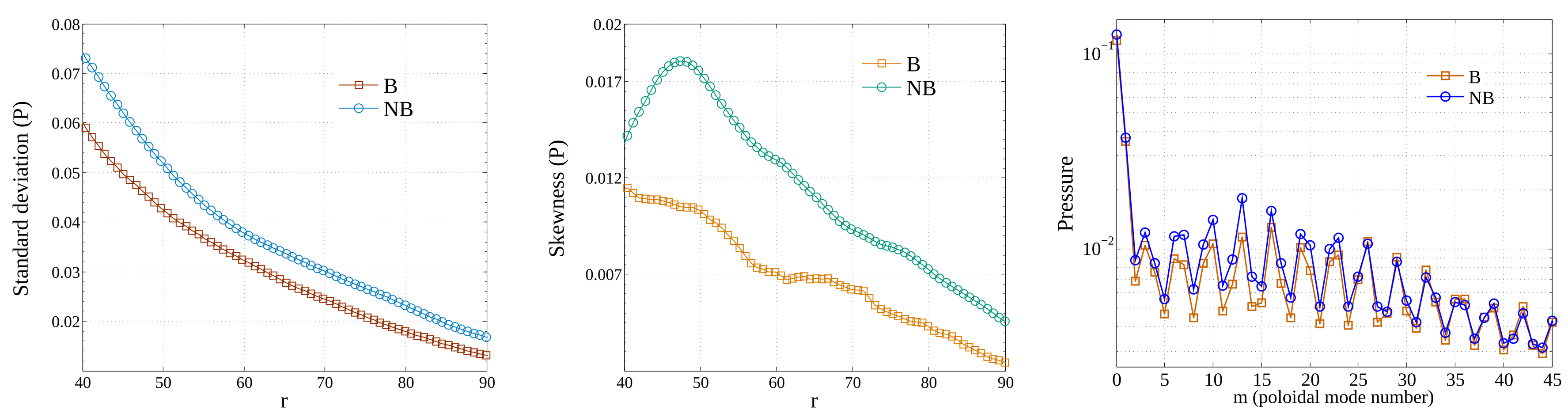

Figure 4: Standard deviation profile (left), skewness profile (center) and poloidal mode number spectrum (right) of the pressure field, for $q = 3$ and $\tau = 2$.

Pending questions: Is the difference between B and NB models the result of a change in the linear growth rate of the main instability? And/Or the result of a more complex nonlinear mechanism?

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