Observer Placement for Source Localization: The Effect of Budgets and Transmission Variance

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Abstract—When an epidemic spreads in a network, a key question is where was its source, i.e., the node that started the epidemic. If we know the time at which various nodes were infected, we can attempt to use this information in order to identify the source. However, maintaining observer nodes that can provide their infection time may be costly, and we may have a budget \( k \) on the number of observer nodes we can maintain. Moreover, some nodes are more informative than others due to their location in the network. Hence, a pertinent question arises: Which nodes should we select as observers in order to maximize the probability that we can accurately identify the source?

Inspired by the simple setting in which the node-to-node delays in the transmission of the epidemic are deterministic, we develop a principled approach for addressing the problem even when transmission delays are random. We show that the optimal observer-placement differs depending on the variance of the transmission delays and propose approaches in both low- and high-variance settings. We validate our methods by comparing them against state-of-the-art observer-placements and show that, in both settings, our approach identifies the source with higher accuracy.

I. INTRODUCTION

Regardless of whether a network comprises computers, individuals or cities, in many applications we want to detect whenever any anomalous or malicious activity spreads across the network and, in particular, where the activity originated.\(^1\) We call the spread of such activity an epidemic and the originator the source.

Clearly, monitoring all nodes is not feasible due to cost and overhead constraints: The number of nodes in the network may be prohibitively large and some of them may be unable or unwilling to provide information about their state. Thus, studies have focused on how to estimate the source based on information from a few nodes (called observers). Given a set of observers, many models and estimators for source localization have been developed [26], [21], [32]. However, the selection of observers has not yet received a satisfactory answer: Most of state-of-the-art methods are based on common centrality heuristics (e.g., degree- or betweenness-centrality) or on more advanced heuristic approaches that do not directly optimize source localization (see [32] for a survey) or are limited to simple networks such as trees (e.g., [16]). Moreover, such methods consider only the structure of the network when placing observers. However, depending on the particular epidemic, the expected transmission delay between two nodes, and its variance, can differ widely. We show that different transmission models require different observer placements: This is illustrated in Figure 1: As the variance of the transmission delays changes, the optimal set of observers also changes (see also Figure 2 for a concrete example).

![Fig. 1. Transmission variance \( \sigma \) and optimal observer placement. For \( \sigma \in (0, \sigma_0) \) the transmission delays are effectively deterministic. For \( \sigma \in (\sigma_0, \sigma_1) \) the variance \( \sigma \) affects the accuracy of source localization but the optimal observer placement is still \( O_0 \). For larger \( \sigma \), the optimal observer placement may change, possibly multiple times (\( O_p \) denotes the optimal placement for \( \sigma \in (\sigma_k, \sigma_{k+1}) \) up to \( \sigma = \sigma_F \). For \( \sigma > \sigma_F \) the optimal placement remains the same (\( O_F \)).](image)

The difficulties faced in finding the optimal observers are two-fold. First, computing the likelihood of a node being the source conditional on the available observations can be computationally prohibitive [29], [26]; evaluating the probability of detection given a set of observers is, in general, even harder. Second, the optimal selection of a limited number of observers is NP-hard, even when the transmission times are deterministic. We take a principled approach that begins with considering deterministic transition delays, and build on this intuition in order to develop heuristics for both the low-variance and high-variance regimes.

A. Model and Problem Statement

Our Transmission Model. We assume that the contact network \( G = (V, E, w) \) is known and is weighted. The weight \( w_{uv} \in \mathbb{R}_+ \) of edge \( uv \in E \) is the mean of the transmission delay encoded by the random variable \( X_{uv} \); this is the time it would take for \( u \) to infect \( v \).\(^2\) This transmission model is both natural and versatile as it comprises deterministic transmissions (i.e., if \( X_{uv} = w_{uv} \in \mathbb{R}_+ \) a.s. for all edges \( uv \in E \), which we call zero-variance, and arbitrary random independent transmission models. It naturally captures the SI epidemic model adopted, e.g., in [26], [23] and related SIR/SIS/SEIR models (see [15] and the discussion in [32]). We study, in particular, how the amount of randomness (i.e., the variance of \( X_{uv} \)) in the transmission delays affects the choice of observers for source localization. Towards this, we are the first to separately analyze two different regimes for the amount of randomness in transmission delays: low-

\(^1\)In effect, we wish to answer questions such as what was the origin of a worm in a computer network?; who was the instigator of a false rumor in a social network? and can we identify patient zero of a virulent disease?

\(^2\)For ease of presentation we assume the graph is undirected and \( w_{uv} = w_{vu} \); however our definitions and approach extend straightforwardly to the directed case.
variance and high-variance. A dichotomy exists between the two, and our approach for observer placement differs.

Our Source Estimation. We assume that there is a single source that initiates the epidemic\(^3\) and let \(\mathcal{O} \subseteq V\) (which we will select) be the set of observer nodes. We assume we know the time at which each observer is infected, and refer to this vector of infection times as \(T_{\mathcal{O}}\). This is a standard (see, e.g., [24]) and realistic assumption (for example, clinics keep records of patients and carefully record outbreaks so can provide such information). To identify the source, we use this (and only this) information.

We use maximum likelihood estimation (MLE) to produce an estimate \(\hat{s}\) of the true unknown source \(s^*\) as in [26],\(^4\) i.e.,
\[
\hat{s} \in \text{argmax}_{s \in V} P(T_{\mathcal{O}}|s^* = s)P(s^* = s).
\]

We assume the prior on \(s^*\) is uniform unless otherwise specified (i.e., \(P(s^* = s) = 1/n\) for all nodes \(s \in V\) where \(n = |V|\)).

Our Observer Placement. We assume that we are given a budget \(k\) on the number of observers we can use, and that we must select our observers once and for all. In order to select the best set of observers \(\mathcal{O}\) of size \(k\) we must first define our metric of interest. We consider the two metrics proposed by [16], although variations (including worst-case versions) exist [16]:

1) the success probability \(P_s = P(\hat{s} = s^*)\), and
2) the expected distance between estimated source and real source, i.e., \(E[d(s^*, \hat{s})]\) with \(d\) denoting the distance between two nodes in the network.

The two metrics might require different sets of observers [16], however we show experimentally that maximizing \(P_s\) is a good proxy for minimizing \(E[d(s^*, \hat{s})]\) (see Section III). Hence, due to space constraints, we focus on the minimization of the former.

B. Main Contributions

Low-Variance Regime. When the variance in the transmission delays is low (see Section III), we prove that the set of optimal observers is equal to the optimal set for the zero-variance regime. In the zero- and low-variance regime, both the probability of success \(P_s\) and the expected distance \(E[d(s^*, \hat{s})]\) can be explicitly computed. Despite this seeming simplicity, the problem remains NP-hard. We tackle the problem by using its connection with the well-studied related Double Resolving Set (DRS) problem [7] that minimizes the number of observers for perfect detection.\(^5\) From this connection we find inspiration for our algorithm that, by selecting one observer at a time until the budget is exhausted in order to reach a DRS set, greedily improves \(P_s\).

High-Variance Regime. When the noise in the transmission delays is high (see Section IV), it is no longer negligible and it poses an additional challenge to source localization; in effect, the accumulation of noise from node to node as the epidemic spreads might no longer enable us to distinguish between two potential sources, especially when they are both far from all observers. Hence, we must strengthen the requirements for observer placement in order to ensure that the nodes can be distinguished by observers that are near to them; this nearness is a function of the noise, the budget \(k\), and the network topology. We define a novel objective function that both maximizes the success probability and imposes a uniform spread of observers in the network. Taking inspiration from the low-variance regime, we design an algorithm that greedily maximizes this new objective (see Section IV).

Empirical Results. Our methods perform favourably against state-of-the-art approaches in both the low- and high-variance regimes (see Section V-B). In the appendix of the full version [1], for the low-variance regime, we further compare them against two other natural objective functions; we show that our approach is the best. Moreover, in the empirical results the dichotomy between the low- and high-variance regimes becomes apparent.

II. RELATED WORK

The problem of source localization has been widely studied in recent years, we survey the works that are more relevant to ours and refer the reader to the survey by Jiang et al. [11] for a more complete review of the different approaches.

Transmission delays. Many transmission models for epidemics have been studied [17] and considered for source localization. Although discrete-time transmission delays are common [22], [27], [4], in order to better approximate realistic settings, much work (including ours) adopt continuous-time models with varying distributions for the transmission delays; e.g., exponential [29], [23] or Gaussian [26], [21], [20], [32]. In the same line of the latter class of works, we use truncated Gaussian variables, which gives us the advantage of ensuring that infection delays are strictly positive.

Source localization. Many approaches [34], [27], [30], beginning with the seminal work by Shah and Zaman [29], rely on knowing the state of the entire network at a fixed point in time \(t\); this is often called a complete observation of the epidemic. These models use maximum likelihood estimation (MLE) to estimate the source. The results of [29] have been extended in many ways, for example in the case of multiple sources [23] or to obtain a local source estimator [9]. An alternate line of work considers a complete
observed that the epidemic is relatively short, so a fixed number of observers using several heuristics such as Betweenness-Centrality, Degree-Centrality and Closeness-Centrality and they show that none of these methods are satisfactory. They introduce a new heuristic for the choice of observers, called Coverage-Rate, which is linked to the total number of nodes neighboring observers, and show that an approximated optimization of this metric yields better performance. Connecting the budgeted placement problem to the un-budgeted minimization problem, we prove that our approach outperforms their approach in low-variance settings. More generally, the effect of the variance in the transmission delays is neglected by Zhang et al., leaving open the question of whether their approach works in general. However, we consider Coverage-Rate as one of our baselines.

III. THE LOW-VARIANCE REGIME

In this section, we focus on the low-variance regime. We start by introducing the setting and the definitions we adopt.

A. Preliminaries

Let $G = (V, E)$ be an undirected weighted network. Assuming $u$ is infected, the weight $w_{uv}$ of edge $uv \in E$ represents the expected time it takes for $u$ to infect $v$. As the network is undirected, we assume $w_{uv} = w_{vu}$ for all $uv \in E$.

We assume that the epidemic is initiated by a single unknown source $s$ at an unknown time $t^*$. If a node $u$ gets infected at time $t_u$, a non-infected neighbor $v$ of $u$ will become infected at time $t_o = t_u + X_{uv}$ where $X_{uv}$ is a random variable with $E[X_{uv}] = w_{uv}$.

The time $t^*$ at which an epidemic starts is unknown. This adds a significant difficulty to the problem because a single observation is not per se informative. Instead, we must use the collection of differences between observed infection times. If the variance is zero or if it is low compared to edge weights, network distances are a good proxy for time delays (see Proposition 1). We refer to this setting as the low-variance regime, as opposed to the high-variance regime in which time delays are highly noisy and network distances no longer work as a proxy for time delays.

Distance vectors and equivalence between nodes. We start with a few definitions. Our setting is similar to [16].

Definition 1 (Equivalence): Let $G = (V, E)$ and $O \subseteq V$ with $|O| = k \geq 2$ be a set of observers on $G$. A node $u$ is said to be equivalent to a node $v$ (which we write $u \sim v$) if and only if, for every $o_i, o_j \in O$

$$d(u, o_i) - d(u, o_j) = d(v, o_i) - d(v, o_j).$$

where $d(x, y)$ is the (weighted) distance between $x$ and $y$. The relation $\sim$ is reflexive, symmetric, and transitive, hence it defines an equivalence relation. Therefore, a set of observers $O$ partitions $V$ in equivalence classes (an example is given in Figure 3). We denote by $q$ the number of equivalence classes and we let $[u]_O$ be the class of $u$, i.e., the set of all nodes that are equivalent to $u$.

For example, on cycles of odd-length $d$ with a budget $k = 2$ in the low-variance setting, any two nodes at distance more than 2 are equivalent with respect to the coverage rate, but only optimal if the observers are at distance $(d - 1)/2$; our approach selects this optimal placement.
When the variance is zero, given an observer set, we can distinguish \( u \) from \( v \) if Equation (1) does not hold for \( u, v \) and a pair of observers \( o_1, o_2 \), i.e., if \([u]_O \neq [v]_O\).

The problem of finding the minimum-size set of nodes \( S \), such that for every \( u, v \) there exist \( s_i, s_j \in S \) for which \( d(u, s_i) − d(u, s_j) \neq d(v, s_i) − d(v, s_j) \) is known as the Double Resolving Set (DRS) Problem \([7]\). Our problem differs from DRS because we focus on the more realistic context in which, due to limited resources, we want to allocate a finite budget in order to maximize the detection probability (as opposed to minimizing the number of observers for perfect detection, which is, in many cases, still prohibitively large). However, the connection between our problem and DRS paves the way for a principled approach.

We now define a distance vector associated with a candidate source, which, as we will prove in Lemma 1, mathematically captures equivalence in a manner that is easy to work with.

**Definition 2 (Distance Vector):** Let \( G = (V, E) \) and \( O \subseteq V \) with \(|O| = k \geq 2\) a set of observers on \( G \). For each candidate source \( s \) the distance vector is \( d_{s,O} \in \mathbb{R}^{k-1} \) with entries \( d(s, o_{i+1}) − d(s, o_1) \) for \( 1 \leq i \leq k − 1 \).

The following lemma, similar in spirit to Lemma 3.1 in [8], shows that, the equality between distance vectors of different candidate sources does not depend on the choice of the reference observer \( o_1 \).

**Lemma 1:** Let \( G = (V, E) \) and \( O \subseteq V \) with \(|O| = k \geq 2\) and let \( u, v \in V \). Then, \([u]_O = [v]_O\) if and only if \( d_{u,O} = d_{v,O} \), independently of the choice of the reference observer \( o_1 \) in Definition 2.

**Estimating the source in the low-variance setting.** We are now ready to describe how we can estimate the source, and define the probability of correct detection in the zero- and low-variance setting, i.e., when \( X_{uv} = w_{uv} \) a.s. for every edge \((u, v)\).

For every observer \( o_i \in O \), denote by \( t_i \) the time at which \( o_i \) gets infected. In the zero-variance setting, the observed infection times of nodes \( o_2, ..., o_K \) with respect to observer \( o_1 \), i.e., the vector \( \tau = t_2 − t_1, ..., t_k − t_1 \), is exactly the distance vector of the unknown source \( s^* \). Then, if for every \( u, v \in V \), \([u]_O \neq [v]_O\), the source can be always correctly identified. We will see in Proposition 1 that this is true also in a more general low-variance framework where we are always able to identify the equivalence class to which the real source belongs.

We assume a prior probability distribution on the location of the source to be given, i.e., \( Q(u) \equiv P(s^* = u) \). As we cannot distinguish between vertices inside \([s^*]_O\) (otherwise they would not be in the same equivalence class), we let our estimated source \( \hat{s} \) be chosen at random from the conditional probability \( Q_{|E^*}(u) \equiv P(s^* = u|u \in E^*) \). Hence the success probability is

\[
\mathcal{P}_s(O) = \sum_{s \in V} P(\hat{s} = s|s^* = s)P(s^* = s)
\]

\[
= \sum_{s \in V} Q(s)P(s) = \sum_{s \in V} \frac{Q(s)}{Q([s]_O)} Q(s),
\]

and is 1 if all equivalence classes are singletons.

In the experimental results in Section V we also look at another relevant metric for the source localization problem, the expected distance (weighted or in hops) between the true and estimated source:

\[
E[d(s^*, \hat{s})] = \sum_{s \in V} P(s^* = s) \sum_{u \in [s]_O} P(\hat{s} = u|s^* = s) d(s, u)
\]

\[
= \sum_{s \in V} \sum_{u \in [s]_O} \frac{Q(s)Q(u)}{Q([s]_O)} d(s, u).
\]

Alternative metrics, including worst-case metrics, also exist [16] (see the appendix of the full version [1] for some examples).

**B. Setting**

For ease of exposition, we focus on the case in which the prior distribution on the position of the source is uniform, hence \( Q(u) = 1/n \) for all \( u \in V \).

**Proposition 1:** Let \( G = (V, E) \) be a network of size \( n \) and \( O \subseteq V \). Call \( \delta = \min_{u,v} |d_{u,O} − d_{v,O}| \) the distance vector. Assume a uniform prior \( Q(u) = 1/n \) for all \( u \in V \) and call \( D \) the maximum distance in hops in any shortest path between any node and any observer.

1) If the zero-variance case, then \( \mathcal{P}_s(O) = q/n \), where \( q \) is the number of equivalence classes for \( O \);

2) If the transmissions are such that for each \( uw \in E \), \( X_{uw} \in [w_{uw} − \varepsilon, w_{uw} + \varepsilon] \), we denote as \( \mathcal{P}_s^\varepsilon(O) \) the probability of success and we define \( \varepsilon_0 = \sup \{ \varepsilon > 0 : \mathcal{P}_s^\varepsilon(O) = \mathcal{P}_s(O) \} \), we have \( \varepsilon_0 > \delta/2D \).

**Proof:**

1) By definition,

\[
\mathcal{P}_s(O) = \sum_{[u]_O} P(\hat{s} = s^*|s^* \in [u])P(s^* \in [u]).
\]

Hence,

\[
\mathcal{P}_s(O) = \sum_{[u]_O} \frac{1}{|[u]_O|} \frac{|[u]_O|}{n} = \frac{1}{n} \sum_{[u]_O} 1 = \frac{q}{n}.
\]

2) Recall that, for \( u, v \in V \), \([u]_O \neq [v]_O\) if and only if \( d_{u,O} \neq d_{v,O} \). Since \( d_{u,O} \neq d_{v,O} \) implies \( |d_{u,O} − d_{v,O}| \geq \delta \), if \( \varepsilon < \delta/2D \), no estimation error is possible between \( u, v \in V \) such that \( d_{u,O} \neq d_{v,O} \).

Hence \( \varepsilon_0 > \delta/2D \).

Note that here \( \varepsilon_0 \) plays the role of \( \sigma_0 \) in Figure 1. Indeed, for \( \varepsilon < \varepsilon_0 \) the variance of the transmission delays does not affect the accuracy of source localization.

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7Our algorithms and observations can be generalized using Equation (2) instead of the simpler formula that we now derive for the uniform case.
If additional conditions on the weights or on the network topology are made, more refined bounds for $\varepsilon_0$ can be derived. For example, in a tree with integer weights, due the uniqueness of the path between two any vertices, the minimum distance (in the infinity norm) between two distance vectors is 2. Hence, in this case, an accumulated variance of less than 1 can be tolerated and we have $\varepsilon_0 > 1/D$.

For the remainder of this section, we will assume $\varepsilon < \delta/2D$, which we call the low-variance case. Independently of the topology of the network $\mathcal{G}$, the success probability $P_s$, as well as other possible metrics of interest, can be computed exactly in polynomial time (see e.g., Equation (2) and (3)). In fact, due to Lemma 1, it is enough to compute the distance vector of Definition 1 for all the nodes. Nonetheless, if we have a budget $k \geq 2$ of nodes that we can choose as observers, finding the configuration that maximizes $P_s$ is an NP-hard problem. This is a direct consequence of the hardness result of Chen et al. [8].

**Theorem 1**: Let $k \geq 2$ be the budget on the number of nodes we can select as observers. Finding $O \subseteq V$ such that $O \in \operatorname{argmax}_{O \subseteq V} P_s(O)$ is NP-hard.

The proof follows straightforwardly with a reduction from the DRS problem (see the appendix of the full version [1]).

**C. Observer Placement**

Our first main contribution in this paper is a solution to the budgeted observer-placement problem. Our approach, presented in Algorithm 1, is specifically designed for the source localization problem and has a simple greedy structure: for every node $v \in V$, initialize $O \leftarrow \{v\}$ and iteratively add to $O$ the node $u$ that maximizes the gain with respect to the success probability until we either run out of budget or $P_s = 1$. Proposition 1 ensures that greedily maximizing the success probability is equivalent to greedily maximizing the number $q$ of equivalence classes. When adding an element to the observer set, the partition in equivalence classes can be updated in linear time, hence the total running time of our algorithm is $O(k n)$. Despite bypassing the NP-hardness of the problem, this might not be sufficient for very large graphs. However, the procedure is extremely parallelizable and well suited, e.g., for Map-Reduce (see, for example, the main for loop and the argmax in the white loop).

**Algorithm 1** (LV-OBS): Observer placement for the low-variance setting.

**Require**: Network $\mathcal{G}$, budget $k$

for $v \in V$ do

\[ O_v \leftarrow \{v\} \]

while $P_s(O_v) \neq 1$ and $O_v < k$ do

\[ u \leftarrow \operatorname{argmax}_{z \in V \setminus O_v} [P_s(O_v \cup \{z\}) - P_s(O_v)] \]

\[ O_v \leftarrow O_v \cup \{u\}. \]

return $\operatorname{argmax}_{v \in V} P_s(O_v)$

The observer placement obtained through Algorithm 1 will be denoted LV-OBS to emphasize the fact that it has been designed for the case in which the variance is absent or very small (LV stands for low-variance regime).

**D. Performance**

As budgeted observer placement (even in the zero-variance setting) is NP-hard, there is no optimal algorithm to compare against. Instead, we evaluate the performance of our algorithm against a set of natural benchmarks that have shown to have good performance in other works [28], [5], [32] (see Section V-B for a discussion of these benchmarks, Figure 7 for the results).

We further compare against two other natural heuristics that also optimize an objective function greedily. The first is an adapted version of the approximation algorithm for the DRS problem proposed by Chen et al. [8] and described in the appendix of the full version [1]. By stopping the greedy process after it selects $k$ nodes, we can adapt in a natural way this approximation algorithm and create a heuristic for the budgeted version. The second is a direct minimization of the expected error distance obtained by Equation (3) with $Q(u) = 1/n$ for all $u \in V$. Comparing all three approaches, our algorithm outperforms the other two (see the appendix of the full version [1] for details).

**IV. THE HIGH-VARIANCE REGIME**

When the variance is not guaranteed to be low, as defined in Section III, computing analytically the success probability - or other metrics of interest - is unfortunately not possible (except for very simple graphs, like the path in the example of Figure 2). Moreover, the estimation of the source is more challenging because the observed infection delay $t_i - t_j$ can be misleading, especially if the corresponding observers $o_i$ and $o_j$ are far from the source. Take, for example, a path of length $L$ where the two leaves are the only two observers and all edges have weight 1. Figure IV shows how the success probability $P_s$ decays faster for increasing values of $L$. Building on this observation, we propose a strategy for observer placement that enforces a controlled distance from a general source node to the observer set.

**A. Diffusion Model and Source Estimation**

For every edge $(u, v)$ the infection delay $X_{uv}$ is distributed as a truncated Gaussian random variable with parameters $(w_{uv}, \sigma_{w_{uv}}, [w_{uv}/2, 3w_{uv}/2])$. More precisely, if $Y_{uv} \sim \mathcal{N}(w_{uv}, \sigma_{w_{uv}})$ is a Gaussian random variable, $X_{uv}$ is obtained by conditioning $Y_{uv}$ with $Y_{uv} \in [w_{uv}/2, 3w_{uv}/2]$. This delay distribution has two advantages with respect to the one of [26], i.e., that $X_{(u,v)} \sim \mathcal{N}(w_{uv}, \sigma_{w_{uv}})$. First, the model admits only strictly positive infection delays. Second, different values of the standard deviation $\sigma$ result

![Fig. 4. (a): Success probability $P_s$ on a path of length $L$ for increasing variance $\sigma$. (b): Counterexample for the converse of Lemma 2; for each pair of observers in $O$, $u$ is not contained in the shortest path between them, yet $O$ is a DRS.](image-url)
in different regimes for the propagation, making our model very versatile. When \( \sigma = 0 \), \( X_w \) boils down to a deterministic value equal to the edge weight \( w_{uv} \); when \( \sigma \) is large, the distribution of \( X_w \) becomes closer to uniform \( U([w_{uv}/2, 3w_{uv}/2]) \). Finally, when \( \sigma \) is strictly positive but small, \( X_w \approx N(w_{uv}, (\sigma w_{uv})^2) \). In the appendix of the full version [1], we explain how an approximated maximum likelihood estimator for the source can be derived in this setting.

### B. Observer Placement

First, we formalize why distances between observers are important: If \( o_i, o_j \) are two observers and the source is \( s^* \in \mathcal{P}(o_i, o_j) \), then

\[
\text{var}(t_i - t_j) \approx \sigma^2 \sum_{(uv) \in \mathcal{P}(o_i, o_j)} w_{uv}^2
\]

where \( \mathcal{P}(x, y) \) denotes the shortest path from \( x \) to \( y \), written as a sequence of edges. Although we cannot control \( \sigma \), we can control the path length between observers.\(^8\) We make use of the following sufficient condition for a set to be a DRS, i.e., for an observer set to guarantee optimal source detection.

**Lemma 2:** Let \( G = (V, E) \) be a network, \( \mathcal{O} \subseteq V \). If for every \( u \in V \) there exist \( o_1, o_2 \in \mathcal{O} \) such that there is a unique shortest path \( \mathcal{P}(o_1, o_2) \) between \( o_1 \) and \( o_2 \) and \( u \in \mathcal{P}(o_1, o_2) \), then \( \mathcal{O} \) is a DRS for \( G \).

**Proof:** Let \( u, v \in V \setminus \mathcal{O} \). We will prove that there exist \( o_1, o_2 \in \mathcal{O} \) such that the pair \( (u, v) \) is resolved by \( (o_1, o_2) \), i.e., \( d(v, o_1) - d(u, o_1) \neq d(v, o_2) - d(u, o_2) \). Let \( o_1, o_2 \in \mathcal{O} \) such that \( u \) appears in the unique shortest path \( \mathcal{P}(o_1, o_2) \) and \( o_3, o_4 \in S \) such that \( v \) appears in the unique shortest path \( \mathcal{P}(o_3, o_4) \). If \( v \in \mathcal{P}(o_1, o_2) \) or \( u \in \mathcal{P}(o_3, o_4) \) then \( u \) and \( v \) are resolved by, respectively, \( (o_1, o_2) \) or \( (o_3, o_4) \). Take \( v \notin \mathcal{P}(o_1, o_2) \) and \( u \notin \mathcal{P}(o_3, o_4) \). In this case, \( \{o_1, o_2\} \neq \{o_3, o_4\} \). Let us suppose without loss of generality that \( o_1 \notin \{o_3, o_4\} \). We look only at the case where \( (o_1, o_2) \) does not resolve \( (u, v) \) and prove that the pair is indeed resolved by two vertices in \( \mathcal{O} \). Since \( (o_1, o_2) \) does not resolve \( (u, v) \), there exists \( e \in \mathbb{R} \) such that \( d(v, o_1) - d(u, o_1) = d(v, o_2) - d(u, o_2) \). Since the unique shortest path between \( o_1 \) and \( o_2 \) goes through \( u \) we have that \( c > 0 \). We prove that either \( (o_1, o_2) \) or \( (o_3, o_4) \) resolves \( (u, v) \). If this was not the case, we would have the following equalities:

\[
c = d(v, o_1) - d(u, o_1) = d(v, o_3) - d(u, o_3)
\]

\[
c = d(v, o_1) - d(u, o_1) = d(v, o_4) - d(u, o_4).
\]

Since \( c > 0 \), \( d(v, o_3) > d(u, o_3) \) and \( d(v, o_4) > d(u, o_4) \) giving a contradiction with \( v \) (and not \( u \)) being on the shortest path \( \mathcal{P}(o_3, o_4) \). We conclude that \( (u, v) \) are resolved by either \( (o_1, o_2) \) or \( (o_3, o_4) \).

The converse of this lemma is not true: If \( \mathcal{O} \) double resolves \( \mathcal{G} \), it is not even true that for every node \( u \) there must exist \( o_1, o_2 \in \mathcal{O} \) such that \( u \) is contained in some shortest path between \( o_1 \) and \( o_2 \) of (see the Example in Figure IV).

**Path covering strategy.** We take Lemma 2 as a basis for deriving a path covering strategy for observer placement. In practice, the condition about the uniqueness of the shortest path is too strong and excludes many potentially useful observer nodes\(^9\). This is why we relax the condition of Lemma 2 and we prefer, when the shortest path is not unique, to select one arbitrarily. Let \( S \subseteq V \) be a set of observers and \( L \) a positive integer: We call \( P_L(S) \) the set of nodes that lie on a shortest path of length at most \( L \) between any two observers in the set \( S \). Given a budget \( k \), and a positive integer \( L \), we denote by \( S^{*}_{k,L} \) the set of \( k \) vertices that maximize the cardinality of \( P_L(S) \). We call \( L \) the length constraint for the observer placement because we consider an observer to be useful for source localization only if it is within distance \( L \) from another observer. \( S^{*}_{k,L} \) can be approximated greedily as in Algorithm 2.\(^{10}\)

**Algorithm 2 (HV-OBS):** Observer placement for the high-variance setting.

**Require:** Network \( G(V, E) \), budget \( k \), length constraint \( L \)

\[n \leftarrow |G|\]

**for** \( v \in V \) **do**

\[O_v \leftarrow v\]

**while** \(|P_L(O_v)| \neq n \) **and** \( O_v < k \) **do**

\[u \leftarrow \arg \max_{v \in V \setminus O_v} \{|P_L(O_v \cup \{z\})| - |P_L(O_v)|\}\]

\[O_v \leftarrow O_v \cup \{u\}\]

**return** \( \arg \max_{v \in V} |P_L(O_v)| \)

We will refer to the observer placement produced by Algorithm 2 as \( HV-OBS(L) \) to emphasize that it is designed for the high-variance case.

**Comparison with Algorithm 1.** Note that taking \( L \) equal to the maximum weighted distance \( \Delta \) does not make Algorithm 2 equivalent to Algorithm 1, i.e., we do not obtain LV-OBS. To see how the two algorithms could give different results, take a cycle of odd length \( d \) with a leaf node \( \ell \) added as a neighbor to an arbitrary node \( v \) and assume to start the algorithm with initial set \( \{v\} \). At the first step, the two algorithms will make the same choice, choosing one of the two nodes that is at distance \((d - 1)/2 \) from \( v \). At the second step however, LV-OBS will add \( \ell \) (a DRS contains all leaves [8]), whereas Algorithm 2 will add a node on the cycle. This observation is key to our results because it explains why Algorithm 2 results in a more uniform (and hence variance-resistant) observer placement with respect to LV-OBS. HV-OBS operates a trade-off between the average distance to the observers and the maximization of \( P_L \).

**Choice of the \( L \) parameter.** How could one optimally set \( L \)? Needless to say, the optimal \( L \) depends on the network topology and on the available budget: Clearly, for a larger budget a smaller \( L \) is preferred.

\(^8\)Experimentally we see that in many practical situations two shortest paths differ only by a few nodes and the majority of nodes on the path are resolved by the two extreme nodes.

\(^9\)The running time of Algorithm 2 is \( O(n^2k^2) \), however, as with the low-variance case, this is highly parallelizable and hence tractable even for large networks.
The cardinality of $P_L(O)$ is a good proxy for the performance of $O$. The value $|P_L|$ is increasing in $L$ and reaches its maximum for $L$ equal to the maximum weighted distance $(L = \Delta)$. For small $L$, $|P_L^{\{HV-OBS\}}| < |P_L^{\{LV-OBS\}}|$ but for $L$ large enough this is no more the case. See Figure 5 for an example. Our empirical results suggest that $L$ should be chosen as the maximum for which $|P_L^{\{HV-OBS\}}| \leq |P_L^{\{LV-OBS\}}|$. The key property of HV-OBS with respect to LV-OBS is that observers are spread more uniformly without losing too much in terms of success probability $P_s$: Figure 6 shows $|P_L^{\{HV-OBS\}}|$ and $P_s$ as a function of $L$.

LV-OBS and HV-OBS can give drastically different observers (see the appendix of the full version [1] for an example).

V. Empirical results

We purposely run our experiments on three very different real-world networks that, in addition to being relevant examples of networks for epidemic spread, display different characteristics in terms of size, diameter, clustering coefficient and average degree (see Table I), enabling us to test the performance of our methods on various topologies.

A. Datasets

The three networks we consider are:

- Friend & Families (F & F). This is a dataset containing phone calls, SMS exchanges and bluetooth proximity, among a community living in the proximity of a university campus [3]. We select the largest connected component of individuals who took part in the experiment during its whole duration. The edges are weighted, according to the number of phone calls, SMSs, and bluetooth contacts.

- Facebook-like Message Exchange (FB) [25]. As the individuals included in this dataset were living on the same university-campus, the number of messages exchanged is likely to be a good measure of in-person interaction. We selected links on which at least one message was sent in both directions and individuals that had more than 1 contact.

- California Road Network (CR) [2]. In order to obtain a single connected component and remove points that effectively represent the same location, we collapsed the points falling within a distance of 2 km. Moreover we iteratively deleted all leaves. The diameter of this network is very large compared with that of the other two networks. The edges are weighted according to a rescaled version of the real distance (measured in km).

In all three networks, edges are given (non-unit) integer weights, which is realistic in many applications as the expected transmission delays are known only up to some level of precision. Integer weights do not simplify the estimation of the source; in fact, this makes it more difficult to distinguish between vertices. For example, if the edges of the CR network were weighted according to the Euclidean distance between the two endpoints, LV-OBS would use only a very small portion of the budget and the comparison would not be meaningful.

B. Comparison against Benchmarks

We compare HV-OBS against the following benchmarks:

1) LV-OBS: this is our solution for the low-variance case (see Section III).

2) BC (Betweenness Centrality): This is a popular method for placing observers for source-localization (see, e.g., [21] and [28], where it emerges as the best heuristic for observer placement among those tested). It consists of the $k$ nodes having the largest BC, which is defined, for all $u \in V$ as

$$BC(u) = \frac{\sum_{y \in V, y \neq x} \sigma_{x,y}(u)}{\sigma_{x,y}}$$

where $\sigma_{x,y}$ is the number of shortest paths between $x$ and $y$ and $\sigma_{x,y}(u)$ is the number of those paths that passes through $u$.

3) Coverage-rate (Coverage) [32]: This approach maximizes the number of nodes that have an observer as a neighbor, i.e.,

$$C(O) = \frac{|\bigcup_{o \in O} N_o|}{n},$$

\footnote{The roads that cross the state border are not completely tracked in this dataset and terminate with a leaf. Some other leaves might represent remote locations, not necessarily close to the borders, but their influence on the epidemic should anyway be very low.}
where $N_o$ denotes the set of neighbors of $o$ and $n = |V|$. It has been shown to outperform several heuristics with a diffusion model and an estimation setting that are very similar to ours.

4) **K-MEDIAN**: this is the optimal placement for the closely-related problem of maximizing the detectability of a flow [5]. The K-MEDIAN placement is the set of $k$ nodes $O$ such that

$$O = \arg\min_{|O|=k} \sum_{s \in V} \min_{o \in O} d(s, o).$$

Determining the K-MEDIAN of a network is NP-hard [13]; we use a greedy heuristic for K-MEDIANS.

**C. Experimental Results**

We estimate $P_s$ and $E[d(s^*, \tilde{s})]$ for different values of the variance $\sigma$. We generate epidemics by using each node in turn as the source. For the FB and CR datasets, we run 5 simulations per node and variance level; and for the F & F dataset, as the network is smaller, we run 20 simulations per node and variance level. For the FB and CR datasets, we estimate the source based on the first 20 observations only: Given the large size of the network, it would be unrealistic to wait for all the network to get infected before running the algorithm. The results for $P_s$ are displayed in Figure 7. An approximation of the value $\sigma_1$, above which HV-OBS outperforms LV-OBS, is marked with a vertical line. For the expected distance (weighted and in hops), see the appendix of the full version [1].

We first take as budget for the observers the minimum budget for which $P_s(\text{LV-OBS}) = 1$. This corresponds to $k \sim 9\%$ for the F & F dataset, $k \sim 9\%$ for the CR network and $k \sim 5\%$ for the FB dataset. This is the setting in which we expect the improvement of HV-OBS over LV-OBS to be especially strong: For smaller values of $k$ we expect LV-OBS to be nearly optimal even in the high-variance regime because we do not have enough budget to contrast both the topological undistinguishability among nodes (what LV-OBS is designed for) and the accumulation of variance (what HV-OBS is designed for). For the F & F and the CR networks, we also experiment with smaller percentages of observers and consistently find an improvement of HV-OBS over LV-OBS in the high-variance regime: Below a certain amount of variance $\sigma_1$, LV-OBS performs better than HV-OBS for any choice of the parameter $L$, whereas above $\sigma_1$ a calibrated choice of $L$ leads to a significant improvement. Such $L$ stays constant for all $\sigma > \sigma_1$, i.e., with the notation of Figure 1 we have $\sigma_1 = \sigma_{FF}$. For the FB dataset instead, probably due to the low diameter with respect to the number of nodes, we observe that HV-OBS does not improve on LV-OBS for any value of $L$. Both LV-OBS and HV-OBS systematically outperform the baseline heuristics for observer placement that we described in Section V-B. For the CR dataset the performance of Betweenness Centrality is particularly poor and the results are not shown. The Coverage Rate heuristic outperforms Betweenness Centrality on all three networks (confirming what found by by Zhang et al. [32]) but is
consistently less effective than K-Medians and our methods.

D. Robustness

To measure the robustness of our approach, we consider an alternate transmission model, and we measure whether, without making any changes, our observer placement still performs well. For every edge \( uv \in E \) with weight \( w_{uv} \), we take \( X_{uv} \sim \text{Unif}([1 - \varepsilon]w_{uv}, [1 + \varepsilon]w_{uv}) \). We find comparable results (see the appendix of the full version [1]); they suggest that our observer placement is not dependent on the exact transmission model and that the variance of the transmission delays is really a key factor for a good observer placement.

VI. CONCLUSION & FUTURE WORK

In this work, we have taken a principled approach towards budgeted observer placement for source localization. We are the first to have observed a dichotomy between the low and high-variance regimes, and we developed complementary approaches for both. We have evaluated our approaches against state-of-the-art and alternative heuristics and find that the performance of our algorithms is favourable.

One natural extension would account for two stages of observation; in the first stage, as in this work, we select a set of observers to monitor the network. In the next stage, once an epidemic begins, we deploy additional observers in the relevant region of the network. This would pave the way for other types of adaptive models, including ones where we not only observe a node but can act to immunize it or in which we can move the observers as required.

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