Explicit Conditions on Existence and Uniqueness of Load-Flow Solutions in Distribution Networks

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***Abstract*—We present explicit sufficient conditions that guarantee the existence and uniqueness of the load-flow solution for distribution networks with a generic topology (radial or meshed) modeled with positive sequence equivalents. In the problem, we also account for the presence of shunt elements. The conditions have low computational complexity and thus can be efficiently verified in a real system. Once the conditions are satisfied, the unique load-flow solution can be reached by a given fixed point iteration method of approximately linear complexity. Therefore, the proposed approach is of particular interest for modern active distribution network (ADN) setup in the context of real-time control. The theory has been confirmed through numerical experiments.**

***Index Terms*—load flow solution, fixed point method, existence and uniqueness, distribution networks.**

NOMENCLATURE

$v=(v_1,v_2,...,v_N)^T$	v_j is the positive-sequence complex voltage at bus j.
$i=(i_1,i_2,...,i_N)^T$	i_j is the positive-sequence complex nodal current of bus j.
$s=(s_1,s_2,...,s_N)^T$	s_j is the complex nodal power injected into bus j.
bus 0	Slack bus, with v_0=1 p.u.
i_0, s_0	Slack bus complex nodal current and power.
Y	Positive-sequence nodal admittance matrix.
Y_{LL}	Square submatrix of Y, omitting the slack bus.
w_j, $j=1,...,N$	Positive-sequence complex voltage at bus j when s is a zero vector.
$W=\mathrm{diag}(w)$	
$u=W^{-1}v$	Normalized node voltages.
(For any z in $\mathbb{C}$) $\overline{z}$	The complex conjugate of z.

I. INTRODUCTION

THE load-flow problem, which expresses the link between complex node voltages and complex nodal power injections, is one of the main tasks in power system theory and applications. In this paper, we consider a typical case of a distribution network with a generic topology (radial or meshed) that is characterized by a single slack bus, at which the complex voltage is assumed fixed and known, while the rest are PQ buses. Given a vector of nodal power injections into PQ buses, the problem is then to compute the vector of complex node voltages in the network that is *feasible* (i.e., close to 1 p.u. in magnitude). In the rest of the manuscript, we make reference to the load-flow problem formulated for the positive sequence.

Due to the non-linearity of the equations, the existence and uniqueness of the solution to the load-flow problem is not guaranteed in general [1], [2], [3]. There is extensive literature on the subject as detailed in Section II. But for grid control, in order to maintain the system in feasible states, it is essential to provide conditions guaranteeing that the implemented power setpoint leads to a unique and feasible solution of the load-flow problem. Specifically, in active distribution networks (and particularly, microgrids), these conditions are further expected to be both explicitly formulated and verifiable in real time.

There are multiple scenarios that have such expectations. Functions implemented by modern Distribution Management Systems relying on multiple instances of load-flow computations (i.e., centralised optimal controllers or contingency assessment) are certainly the typical application examples. One such example is related to the *islanding maneuver*, namely the disconnection from the main grid due to an intentional or non-intentional decision (e.g., [4]). In particular, with respect to the non-intentional islanding, there is a need to evaluate in real time whether a given resource can serve as a slack for the islanded microgrid [5]. This evaluation is based on verifying whether the currently implemented setpoint leads to a unique and feasible solution of the corresponding load-flow problem. Another practical example is related to performing real-time control of active distribution networks in general. Assuming the knowledge of the current system state (obtained via a corresponding state estimation procedure), a typical task is to decide whether a given collection of power setpoints will result in unique new states that are feasible. This task is especially important in the context of the recently introduced framework for performing real-time control of active distribution networks using explicit power setpoints [6]. Below are the main contributions of this paper:

- We give explicit conditions that guarantee the existence and uniqueness of the load-flow solution for distribution networks with shunt elements. Under the satisfaction of these conditions, the unique solution can be reached by an iterative load-flow method given in this paper, and the feasibility of this unique solution can be conveniently verified.
- Our conditions depend not only on the requested power setpoints but also on the current state of the grid. This allows for refined conditions that are especially useful in a real-time control framework, as mentioned above.
- We also provide conditions in the "classical" setup, where the knowledge of the current system state is absent. In

This work was supported by the SNSF NRP70 "Energy Turnaround".
The authors are with École Polytechnique Fédérale de Lausanne (EPFL), Lausanne, Switzerland.

this case, we show that our results are stronger than those introduced so far in the literature.

- The proposed approach is computationally efficient, with approximately linear complexity. Hence it can be used in the applications that require real-time load-flow computations, such as real-time control and contingency analysis.

We note that it is possible to extend our results to more general three-phase distribution networks, but this is the subject of ongoing work.

The paper is structured as follows. In Section II, we review the related work. In Section III, we present the load-flow problem and its useful equivalent formulation as a fixed point problem. In Section IV, we give our main result, and prove it in Section V. In Section VI, we provide numerical evaluation of our method. Finally, we conclude in Section VII.

II. Related Work

In the last few decades, the existence and uniqueness of the solution to the load-flow problem have been studied from various perspectives.

In [7], conditions for the existence and uniqueness of the solution to reactive power-voltage magnitude problem are given and analyzed. Based on [7], [8] extends the result to active power-voltage angle problem. Under certain assumptions, by decoupling the active and reactive power (i.e., considering a sub-problem of active power with voltage angle, and a sub-problem of reactive power with voltage magnitude), sufficient conditions for load-flow solvability are explored. For balanced radial distribution networks, the uniqueness of a feasible load-flow solution is proved by exploiting the radial structure in [9]. In [10], the result is extended to the unbalanced radial three-phase distribution networks. However, all these results are based on certain assumptions and cannot be generically applied.

Recently, the focus has been moved to fixed point load-flow analysis since the fixed point theorem can guarantee the uniqueness of the load-flow solution. In fact, the first attempt of applying fixed point theorem to power systems dates back to [11], which focused on the study of convergence property of the Newton method. For the latest research, in [12], an efficient fixed point load-flow method is presented for radial distribution networks, but there is no further discussion about the convergence and solvability. Later, in [13], another form of fixed point load-flow method is proposed for balanced distribution network with single slack bus. In the same paper, sufficient conditions are given to guarantee the existence and uniqueness of solution. These sufficient conditions are improved in [14].

In this paper, we use a fixed point formulation of the load-flow problem; then we specify a domain around a specific system state and provide sufficient conditions that guarantee the existence and uniqueness of load-flow solution in this domain. Under the proposed conditions, the unique solution can be reached using the fixed point iteration.

The theory proposed here shares some similarities with the fixed point load-flow methods established in [12], [13] and [14]. But, the method in [12] is a special case of this paper. Furthermore, the sufficient conditions in this paper are more general than the conditions in [13] and [14], and thus improve these results.

III. The Load Flow Problem

We consider a distribution network modeled by its positive sequence equivalents with N PQ buses and one slack bus (in essence, a $V\theta$ bus)[1]. Without loss of generality, we assume that the complex voltage of the slack bus is 1 p.u. Let $v = (v_1, v_2, ..., v_N)^T$ denote the vector of complex node voltages of the PQ buses, $i = (i_1, i_2, ..., i_N)^T$ denote the vector of complex nodal currents into the PQ buses, i_0 denote the complex nodal current into the slack bus, $s = (s_1, s_2, ..., s_N)^T$ denote the vector of complex nodal powers injected into the PQ buses (negative value in real or imaginary part means consumed), and s_0 denote the complex nodal power injected into the slack bus. Also, for any complex number z, we denote its complex conjugate by $\overline{z}$. A similar notation holds for vectors and matrices.

As known, the nodal powers and nodal currents can be expressed in matrix form as

$$\begin{bmatrix} \overline{s}_0 \\ \overline{s} \end{bmatrix} = \begin{bmatrix} 1 & \\ & \operatorname{diag}(\overline{v}) \end{bmatrix} \begin{bmatrix} i_0 \\ i \end{bmatrix}, \tag{1}$$

$$\begin{bmatrix} i_0 \\ i \end{bmatrix} = Y \begin{bmatrix} 1 \\ v \end{bmatrix}. \tag{2}$$

Here, Y is the $(N+1) \times (N+1)$ nodal admittance matrix of the system.

The *load-flow problem* that we consider is defined as follows: Given the nodal powers s, solve the set of equations (1) and (2) to obtain the nodal voltages v and the power at the slack bus s_0. The node voltages are generally required to be feasible in the sense that all the node voltages have magnitudes close to 1 p.u.

In this paper, we rely on an equivalent formulation of this problem that is known as *implicit Z_{bus} formulation*, see e.g., [17]. First, partition the admittance matrix Y as

$$Y = \begin{bmatrix} Y_{00} & Y_{0L} \\ Y_{L0} & Y_{LL} \end{bmatrix}, \tag{3}$$

where Y_{00} is a number, Y_{0L} is a $1 \times N$ row vector, Y_{L0} is an $N \times 1$ column vector, Y_{LL} is an $N \times N$ matrix. Now, we claim that Y_{LL} is an invertible matrix. This fact was mentioned, e.g., in [13], without a proof; in Appendix A we give a proof that covers general distribution networks. In details, it can include reactive capacitor or inductive banks, since we do not have any restriction on the inclusion of transverse elements. Also, it is possible to include the presence of On Load Tap Changers as we allow the inclusion of transformers in the construction of the system admittance matrix even with complex transformation ratio.

Now define the constant vector w by

$$w \triangleq -Y_{LL}^{-1} Y_{L0} \tag{4}$$

[1] For active distribution networks, distributed generators that provide voltage support are usually deployed via droop control actions (with dead bands) between the local voltage magnitude and the injected/absorbed reactive power. Therefore, their voltage control action can be treated via PQ buses [15], [16].

which, as it can easily be seen, is the *zero-load voltage* of the grid. The implicit Z_{bus} formulation of the load-flow problem is then given by the following proposition; for completeness, we also provide a short proof.

Proposition 1. *The load-flow problem defined above is equivalent to solving for v in*

$$v = w + Y_{LL}^{-1} \operatorname{diag}(\overline{v})^{-1}\overline{s} \triangleq G(v) \tag{5}$$

More precisely (i) if (v, s_0) is a solution to the load-flow problem, then v is a solution of (5) and (ii) if v is a solution of (5), then (v, s_0) with $s_0 = \overline{Y}_{00} + \overline{Y}_{0L}\overline{v}$ is a solution to the load-flow problem.

Proof. (i) Let (v, s_0) be any load-flow solution, if it exists. By (2) and (3), $i = Y_{L0} + Y_{LL}v$. From (1), $i = \operatorname{diag}(\overline{v})^{-1}\overline{s}$ and hence $Y_{LL}^{-1} \operatorname{diag}(\overline{v})^{-1}\overline{s} = -w + v$, which proves that v is a solution of (5).

(ii) Conversely, let v be any solution, if it exists, of (5). Let $i_0 = Y_{00} + Y_{0L}v$, $s_0 = \bar{i}_0$, $i = Y_{L0} + Y_{LL}v$. Then

$$\operatorname{diag}(\overline{v})^{-1}\overline{s} = Y_{LL}v + Y_{L0} = i$$

which shows that (1) and (2) hold. □

Remark 1. This formulation can be viewed as a direct result of the superposition theorem: v is the superposition of the voltages w, resulting from current injections by the slack bus when all other injections are absent ($s_j = 0, j = 1, ..., N$) plus the voltages resulting from current injections due to s when the slack bus injection is absent.

In the subsequent sections, we propose and prove sufficient conditions under which there exists a unique solution to (5), which can be found by the iteration

$$v^{(k+1)} = w + Y_{LL}^{-1} \operatorname{diag}(\overline{v}^{(k)})^{-1}\overline{s}. \tag{6}$$

IV. Main Result

In this section, we give conditions on the complex power injections s which guarantee that iteration (6) converges to the unique solution v of the implicit Z_{bus} formulation of the load-flow problem in the neighborhood of a specific system state[2]. We also provide computational complexity of the method.

Before presenting our method formally, we give a high-level outline. First, we assume the knowledge of a pair $(\hat{v}, \hat{s})$ that satisfies implicit Z_{bus} formulation of the load-flow equations (5). This pair can be interpreted as the current (actual) state of the grid obtained via a measurement and state estimation process. In addition, we are given a desired "next" power setpoint s. Our conditions are thus formulated in terms of $(\hat{v}, \hat{s})$ and s, and will be used to guarantee the existence and uniqueness of the solution v to (5) which is "close" to $\hat{v}$. Finally, we provide conditions on the starting point $v^{(0)}$ from which this solution can be computed using iteration (6).

As mentioned in the introduction, such a procedure is especially useful in the modern ADN setup, where v is continuously estimated and is varying slowly from its current value. In case there is no knowledge of the current state, a trivial choice for $(\hat{v}, \hat{s})$ is $(w, \mathbf{0})$, where w is the zero-load voltage profile (4). For details, see Corollary 1 below.

[2]This is practical for real-world power system operation, since the power setpoints are adjusted in relatively small step sizes in real time. Due to the continuity of physical state transition, the unique solution around the current operating state is of particular interest.

A. Main Theorem

We introduce some further notation. Let $W \triangleq \operatorname{diag}(w)$ and set

$$\xi(s) \triangleq \|W^{-1}Y_{LL}^{-1}\overline{W}^{-1} \operatorname{diag}(\overline{s})\|_\infty, \tag{7}$$

where, for any complex matrix A,

$$\|A\|_\infty \triangleq \max_i \sum_j |A_{ij}|$$

denotes the matrix norm induced by the ℓ_∞ norm. Let

$$u_{min} \triangleq \min_j |\hat{v}_j / w_j| \,. \tag{8}$$

Below is our main result. Its proof is in Section V.

Theorem 1. *Let $\hat{v}$ be a solution to the implicit Z_{bus} formulation of the load-flow problem with complex power injection $\hat{s}$ (i.e. the pair $(\hat{v}, \hat{s})$ satisfies (5)). Consider some other candidate complex power injection s. Assume that*

$$\xi(\hat{s}) < u_{min}^2 \tag{9}$$

$$\text{and} \qquad \Delta \triangleq \left(u_{min} - \frac{\xi(\hat{s})}{u_{min}}\right)^2 - 4\xi(s - \hat{s}) > 0 \tag{10}$$

Let $\mathcal{D} \triangleq \{v : |v_j - \hat{v}_j| \le \rho\,|w_j|\,, j = 1, ..., N\}$ with

$$\rho \triangleq \frac{\left(u_{min} - \frac{\xi(\hat{s})}{u_{min}}\right) - \sqrt{\Delta}}{2}$$

Then there exists a unique solution $v \in \mathcal{D}$ to the implicit Z_{bus} formulation of the load-flow problem with complex power injection s (i.e. such that the pair (v, s) satisfies (5)).

Moreover, this solution can be reached using the iterative procedure (6) by starting with any $v^{(0)} \in \mathcal{D}$.

In case there is no knowledge of the current state $(\hat{v}, \hat{s})$, the following corollary can be used.

Corollary 1. *Suppose that the complex power s satisfies $\xi(s) < 0.25$ and let*

$$\mathcal{D}' \triangleq \left\{ v : |v_j - w_j| \le \frac{(1 - \sqrt{1 - 4\xi(s)})\,|w_j|}{2}, j = 1, ..., N \right\}.$$

Then, there exists a unique solution $v \in \mathcal{D}'$ to the implicit Z_{bus} formulation of the load-flow problem (5).

This solution can be reached using the iterative procedure in (6) by starting with any $v^{(0)} \in \mathcal{D}'$.

Proof. We use Theorem 1 with the choice $\hat{v} = w$ and $\hat{s} = \mathbf{0}$. In this case, as $\xi(\mathbf{0}) = 0$, condition (9) is always satisfied. Also, as $u_{min} = 1$, condition (10) becomes $\xi(s) < 0.25$ and ρ is given by $(1 - \sqrt{1 - 4\xi(s)})/2$. □

Under the satisfaction of the proposed conditions, we can further conclude on the feasibility of this unique solution. In details, the unique solution is feasible if domain $\mathcal{D}$ is inside the feasible region, otherwise the unique solution can first be efficiently computed and then verified for feasibility. As we mentioned in the introduction, the success and failure of feasibility check will be useful to help decide the acceptability of a new power setpoint.

We demonstrate the numerical utility of choosing either the method of Theorem 1 or that of Corollary 1 in Section VI.

B. Comparison with Existing Results

In the following theorem, we compare our results to the related recent works of [13] and [14][3].

Theorem 2.

(i) *In the case when no knowledge of the current state is assumed, the conditions of Corollary 1 are strictly weaker than those of [13] and [14]. Thus, the result of Corollary 1 is strictly stronger than that of [13] and [14].*

(ii) *In addition, there exists a given state $(\hat{v}, \hat{s})$ and candidate power injection s that satisfy the condition of Theorem 1 but not that of Corollary 1. Hence, the result of Theorem 1 is strictly stronger than that of [13] and [14].*

Proof. In [13], the following sufficient condition for the unique solution of the load-flow problem was given: $\exists p \in [1, \infty]$ and $q = p/(p-1)$ such that

$$\|W^{-1}Y_{LL}^{-1}\overline{W}^{-1}\|_p^*\|s\|_q < 0.25, \tag{11}$$

where, for any matrix A, $\|A\|_p^* \triangleq \max_h \|A_{h\bullet}\|_p$, and the notation $A_{h\bullet}$ stands for the h-th row of A. This work has been improved in [14] as follows: $\exists p \in [1, \infty]$, $q = p/(p-1)$, and a real-valued diagonal matrix Λ such that

$$\|W^{-1}Y_{LL}^{-1}\overline{W}^{-1}\Lambda\|_p^*\|\Lambda^{-1}s\|_q < 0.25. \tag{12}$$

We next show that the condition of Corollary 1 is weaker (thus the result is stronger). By Holder's inequality with $\frac{1}{p} + \frac{1}{q} = 1$,

$$\begin{aligned}
&\xi(s) = \|W^{-1}Y_{LL}^{-1}\overline{W}^{-1}\operatorname{diag}(\overline{s})\|_\infty \\
&= \max_i \sum_j \left|\left(W^{-1}Y_{LL}^{-1}\overline{W}^{-1}\Lambda\right)_{ij}\right| \left|\left(\Lambda^{-1}s\right)_j\right| \\
&= \sum_j \left|\left(W^{-1}Y_{LL}^{-1}\overline{W}^{-1}\Lambda\right)_{i_{max}j}\right| \left|\left(\Lambda^{-1}s\right)_j\right| \\
&\leq \|\left(W^{-1}Y_{LL}^{-1}\overline{W}^{-1}\Lambda\right)_{i_{max}\bullet}\|_p\|\Lambda^{-1}s\|_q \\
&\leq \|W^{-1}Y_{LL}^{-1}\overline{W}^{-1}\Lambda\|_p^*\|\Lambda^{-1}s\|_q.
\end{aligned} \tag{13}$$

Thus, whenever (11) or (12) is satisfied, we have that $\xi(s) < 0.25$, hence the hypothesis of Corollary 1 is satisfied. We complement this result in Section VI by showing that the converse is not true. This completes the proof of part (i).

Regarding part (ii), we show in Section VI that there exists a given state $(\hat{v}, \hat{s})$ and candidate power injection s, such that the conditions in Theorem 1 are satisfied but $\xi(s) = 0.5770 > 0.25$. Hence, the condition of Corollary 1 is not satisfied, which by part (i) of the current theorem implies that conditions of [13] and [14] are not satisfied either. □

[3]For [14], we extend its original result to the general networks with shunt elements based on the derivations in [13].

C. Computational Complexity

1) The complexity of one iteration: In general, each iteration of (6) can be computed either directly or through solving linear equations. Such procedures usually require $O(N^2)$ computational complexity for a general linear system. But the computational complexity can approximately be $O(N)$ if using LU decomposition with complete Markowitz pivoting [18]. This is because the nodal admittance matrices are structurally sparse and symmetric in general, for which the pivoting reduces the number of fill-ins and preserve the sparsity in LU decomposition [19], [20].

2) The complexity of checking conditions: Generally, complexity of checking conditions is mainly the complexity of computing $\xi(\hat{s})$ and $\xi(s - \hat{s})$, which is $O(N^2)$. But for radial networks, this complexity could be reduced to $O(N)$ by only computing and comparing the rows that correspond to *leaf* nodes[4].

V. Proof of Theorem 1

For the purpose of the proof, we find it useful to parametrize (5) in a different way. Let $u \triangleq W^{-1}v$ denote the *normalized* voltage with respect to an unloaded grid. Then, it is easy to see that (5) is equivalent to

$$u = \mathbf{1} + W^{-1}Y_{LL}^{-1}\overline{W}^{-1}\operatorname{diag}(\overline{u})^{-1}\overline{s} \triangleq \tilde{G}(u), \tag{14}$$

where $\mathbf{1} = (1, 1, ..., 1)^T$ is the unity vector. Clearly, any conditions on u provide corresponding conditions on v using the invertible mapping $v = Wu$. We thus perform the analysis of (14) and the corresponding iteration

$$u^{(k+1)} = \mathbf{1} + W^{-1}Y_{LL}^{-1}\overline{W}^{-1}\operatorname{diag}(\overline{u}^{(k)})^{-1}\overline{s}. \tag{15}$$

From the Banach fixed point theorem [21], if the operator $\tilde{G}$ is a contraction mapping on a metric space $(\tilde{\mathcal{D}}, \tilde{d})$, then there is a unique fixed point u^* in $\tilde{\mathcal{D}}$. Moreover, u^* can be reached by iterative update of $u^{(k+1)} = \tilde{G}(u^{(k)})$ from an arbitrary $u^{(0)}$ in $\tilde{\mathcal{D}}$. In the rest of this section, we show that under the conditions of Theorem 1, operator $\tilde{G}$ is a contraction mapping in the sense that (i) $\tilde{G}$ is a self-mapping of u on a closed set $\tilde{\mathcal{D}}$, and (ii) $\tilde{G}$ has the contraction property: $\|\tilde{G}(u^2) - \tilde{G}(u^1)\|_\infty < \|u^2 - u^1\|_\infty$ for any $u^1, u^2 \in \tilde{\mathcal{D}}$.

A. Proof of self-mapping

Lemma 1. *Suppose that the pair $(\hat{v}, \hat{s})$ and the complex power s satisfy* (9) *and* (10). *Then $\tilde{G}$ is a self-mapping of u on*

$$\tilde{\mathcal{D}} \triangleq \{u : |u_i - \hat{u}_i| \leq \rho\} \tag{16}$$

with $$\rho = \frac{\left(u_{min} - \frac{\xi(\hat{s})}{u_{min}}\right) - \sqrt{\Delta}}{2}$$

and $\hat{u}_i = \hat{v}_i / w_i$.

Proof. Since $(\hat{v}, \hat{s})$ satisfies the equation (5), we have that $\hat{u} = \mathbf{1} + W^{-1}Y_{LL}^{-1}\overline{W}^{-1}\operatorname{diag}(\overline{\hat{u}})^{-1}\overline{\hat{s}}$ in addition to (14). Thus,

$$\tilde{G}(u) - \hat{u} = W^{-1}Y_{LL}^{-1}\overline{W}^{-1}\left(\operatorname{diag}(\overline{u})^{-1}\overline{s} - \operatorname{diag}(\overline{\hat{u}})^{-1}\overline{\hat{s}}\right).$$

[4]A leaf node is the terminal one of a lateral, which does not have any subsequent node.

Our goal is to show that there exists a radius r such that if $|u_i - \hat{u}_i| \le r$ then $\left|\tilde{G}(u)_i - \hat{u}_i\right| \le r$ for all i. We have

$$\begin{aligned}\left|\tilde{G}(u)_i - \hat{u}_i\right| &= \left|\sum_j \left(W^{-1}Y_{LL}^{-1}\overline{W}^{-1}\right)_{ij}\left(\frac{\overline{s}_j}{\overline{u}_j} - \frac{\overline{\hat{s}}_j}{\overline{\hat{u}}_j}\right)\right| \\ \le \sum_j &\left|\left(W^{-1}Y_{LL}^{-1}\overline{W}^{-1}\right)_{ij}\frac{\overline{\hat{s}}_j(\overline{\hat{u}}_j - \overline{u}_j) + \overline{\hat{u}}_j(\overline{s}_j - \overline{\hat{s}}_j)}{\overline{u}_j\overline{\hat{u}}_j}\right| \\ \le \sum_j &\left|\left(W^{-1}Y_{LL}^{-1}\overline{W}^{-1}\right)_{ij}\overline{\hat{s}}_j\frac{(\overline{\hat{u}}_j - \overline{u}_j)}{\overline{u}_j\overline{\hat{u}}_j}\right| \\ &+ \sum_j \left|\left(W^{-1}Y_{LL}^{-1}\overline{W}^{-1}\right)_{ij}\frac{(\overline{s}_j - \overline{\hat{s}}_j)}{\overline{u}_j}\right|.\end{aligned}$$

Now, assume that $|u_i - \hat{u}_i| \le r < u_{min}$, where u_{min} is given in (8). Also, by the definition of u_{min}, we have that $|\hat{u}_j| \ge u_{min}$. Therefore, $|\overline{u}_j| \ge u_{min} - r$, and

$$\begin{aligned}&\sum_j \left|\left(W^{-1}Y_{LL}^{-1}\overline{W}^{-1}\right)_{ij}\overline{\hat{s}}_j\frac{(\overline{\hat{u}}_j - \overline{u}_j)}{\overline{u}_j\overline{\hat{u}}_j}\right| \\ \le &\sum_j \left|\left(W^{-1}Y_{LL}^{-1}\overline{W}^{-1}\right)_{ij}\overline{\hat{s}}_j\right|\frac{r}{(u_{min}-r)u_{min}} \le \frac{\xi(\hat{s})r}{(u_{min}-r)u_{min}}.\end{aligned}$$

Similarly,

$$\begin{aligned}&\sum_j \left|\left(W^{-1}Y_{LL}^{-1}\overline{W}^{-1}\right)_{ij}\frac{(\overline{s}_j - \overline{\hat{s}}_j)}{\overline{u}_j}\right| \\ \le &\sum_j \left|\left(W^{-1}Y_{LL}^{-1}\overline{W}^{-1}\right)_{ij}(\overline{s}_j - \overline{\hat{s}}_j)\right|\frac{1}{(u_{min}-r)} \le \frac{\xi(s-\hat{s})}{(u_{min}-r)}.\end{aligned}$$

Combine them and obtain

$$\left|\tilde{G}(u)_i - \hat{u}_i\right| \le \frac{\xi(\hat{s})r}{(u_{min}-r)u_{min}} + \frac{\xi(s-\hat{s})}{(u_{min}-r)}. \tag{17}$$

Therefore, we have a self-mapping if

$$\frac{\xi(\hat{s})r}{(u_{min}-r)u_{min}} + \frac{\xi(s-\hat{s})}{(u_{min}-r)} \le r. \tag{18}$$

It can be re-organized as

$$r^2 - \left(u_{min} - \frac{\xi(\hat{s})}{u_{min}}\right)r + \xi(s-\hat{s}) \triangleq f(r) \le 0. \tag{19}$$

We thus have shown that $\tilde{G}$ is a self-mapping if there exists an $r \in (0, u_{min})$ such that $f(r) \le 0$. Since $f(r)$ is a convex polynomial of degree two and $f(0) = \xi(s-\hat{s}) > 0$, we know there is an interval of such r if (i) the axis of symmetry $\left(u_{min} - \frac{\xi(\hat{s})}{u_{min}}\right)/2 > 0$ and (ii) the discriminant $\Delta = \left(u_{min} - \frac{\xi(\hat{s})}{u_{min}}\right)^2 - 4\xi(s-\hat{s}) > 0$. These two conditions are exactly (9) and (10).

By now, the satisfaction of (9) and (10) gives an interval of r. But we are interested in the smallest possible value of r, which is given by $\rho = \left(\left(u_{min} - \frac{\xi(\hat{s})}{u_{min}}\right) - \sqrt{\Delta}\right)/2$ since it provides a better description of locality for load-flow solution. This completes the proof of the lemma. □

Remark 2. Equivalently, G is a self-mapping of v on $\mathcal{D}$.

B. Proof of contraction mapping

Lemma 2. *Suppose that the pair $(\hat{v}, \hat{s})$ and the complex power s satisfy* (9) *and* (10). *Then $\tilde{G}$ is a contraction mapping of u on the metric space $(\tilde{\mathcal{D}}, \tilde{d})$, where $\tilde{\mathcal{D}}$ is given in* (16) *and $\tilde{d}$ is defined by the ℓ_∞ norm.*

Proof. As $\tilde{\mathcal{D}}$ is a convex set, there exists a straight path connecting any two points u^1 and u^2 in $\tilde{\mathcal{D}}$. Parameterize the path and denote it by b: $b(t) = u^1 + t(u^2 - u^1)$ for $t \in [0,1]$. Then, we have the relation:

$$\|\tilde{G}(u^2) - \tilde{G}(u^1)\|_\infty = \|\tilde{G}(b(1)) - \tilde{G}(b(0))\|_\infty = \|\int_0^1 \frac{d\tilde{G}\left(b(t)\right)}{dt}dt\|_\infty.$$

By triangular inequality, it holds that

$$\|\tilde{G}(u^2) - \tilde{G}(u^1)\|_\infty \le \int_0^1 \|\frac{d\tilde{G}\left(b(t)\right)}{dt}\|_\infty dt. \tag{20}$$

We view $\mathbb{C}^N$ as an abstract vector space on $\mathbb{R}$ (i.e., of dimension $2N$), equipped with the norm $\|(z_1, ..., z_N)\|_\infty \triangleq \max_{i=1}^N |z_i|$. Note that this is a norm when we view $\mathbb{C}^N$ either as a $\mathbb{C}$-*vector* space or an $\mathbb{R}$-*vector* space. As shown in [22],

$$\tilde{G}(b+h) = \tilde{G}(b) + \tilde{G}'(b)\cdot h + \|h\|_\infty \varepsilon(h) \quad \forall h \in \mathbb{C}^N,$$

where $\tilde{G}'(b) : \mathbb{C}^N \to \mathbb{C}^N$, the differential operator of $\tilde{G}$ at b, is an $\mathbb{R}$-*linear* operator, and "$\cdot$" denotes the action of this operator. Then for the $\tilde{G}$ defined in (14), we have

$$\tilde{G}'(b)\cdot h = -W^{-1}Y_{LL}^{-1}\overline{W}^{-1}\operatorname{diag}\left(\frac{\overline{s}_1}{\overline{b}_1^2}, ..., \frac{\overline{s}_N}{\overline{b}_N^2}\right)\overline{h}.$$

So that, we continue the derivation in (20) and obtain

$$\begin{aligned}&\|\tilde{G}(u^2) - \tilde{G}(u^1)\|_\infty \le \int_0^1 \|\tilde{G}'\left(b(t)\right)\cdot \frac{db(t)}{dt}\|_\infty dt \\ =&\int_0^1 \|W^{-1}Y_{LL}^{-1}\overline{W}^{-1}\operatorname{diag}\left(\frac{\overline{s}_1}{\overline{b}_1^2(t)}, ..., \frac{\overline{s}_N}{\overline{b}_N^2(t)}\right)\frac{d\overline{b}(t)}{dt}\|_\infty dt \\ \le&\int_0^1 \|W^{-1}Y_{LL}^{-1}\overline{W}^{-1}\operatorname{diag}\left(\frac{\overline{s}_1}{\overline{b}_1^2(t)}, ..., \frac{\overline{s}_N}{\overline{b}_N^2(t)}\right)\|_\infty \|u^2 - u^1\|_\infty dt.\end{aligned} \tag{21}$$

Since $b(t)$ is always in $\tilde{\mathcal{D}}$, we have $|b_i(t)| \ge u_{min} - \rho$. Then, by sub-multiplicativity of matrix norm, there is

$$\begin{aligned}&\|W^{-1}Y_{LL}^{-1}\overline{W}^{-1}\operatorname{diag}\left(\frac{\overline{s}_1}{\overline{b}_1^2(t)}, ..., \frac{\overline{s}_N}{\overline{b}_N^2(t)}\right)\|_\infty \\ \le&\|W^{-1}Y_{LL}^{-1}\overline{W}^{-1}\operatorname{diag}\left(\overline{s}_1, ..., \overline{s}_N\right)\|_\infty \|\operatorname{diag}\left(\overline{b}_1^2(t), ..., \overline{b}_N^2(t)\right)^{-1}\|_\infty \\ \le&\frac{\xi(s)}{(u_{min}-\rho)^2}.\end{aligned} \tag{22}$$

Further, observe that from (10),

$$\begin{aligned}\Delta =&(u_{min} - \frac{\xi(\hat{s})}{u_{min}})^2 - 4\xi(s-\hat{s}) \\ =&(u_{min} + \frac{\xi(\hat{s})}{u_{min}})^2 - 4(\xi(\hat{s}) + \xi(s-\hat{s})) > 0.\end{aligned}$$

Hence, we have

$$\begin{aligned}\xi(s) &= \|W^{-1}Y_{LL}^{-1}\overline{W}^{-1}\,\mathrm{diag}\,(\overline{s})\,\|_\infty \\ &=\|W^{-1}Y_{LL}^{-1}\overline{W}^{-1}\,\left(\mathrm{diag}\,(\overline{\hat{s}})+\mathrm{diag}\,(\overline{s}-\overline{\hat{s}})\right)\|_\infty \\ &\leq\|W^{-1}Y_{LL}^{-1}\overline{W}^{-1}\,\mathrm{diag}\,(\overline{\hat{s}})\,\|_\infty+ \\ &\quad\|W^{-1}Y_{LL}^{-1}\overline{W}^{-1}\,\mathrm{diag}\,(\overline{s}-\overline{\hat{s}})\,\|_\infty \\ &=\xi(\hat{s})+\xi(s-\hat{s})<\left(\frac{u_{min}+\frac{\xi(\hat{s})}{u_{min}}}{2}\right)^2 \\ &<\left(\frac{u_{min}+\frac{\xi(\hat{s})}{u_{min}}+\sqrt{\Delta}}{2}\right)^2=(u_{min}-\rho)^2.\end{aligned} \tag{23}$$

Thus, by combining (21), (22) and (23), we obtain

$$\begin{aligned}&\|\tilde{G}(u^2)-\tilde{G}(u^1)\|_\infty \\ &\leq\int_0^1\|W^{-1}Y_{LL}^{-1}\overline{W}^{-1}\,\mathrm{diag}\left(\frac{\overline{s}_1}{\overline{b}_1^2(t)},...,\frac{\overline{s}_N}{\overline{b}_N^2(t)}\right)\|_\infty\|u^2-u^1\|_\infty dt \\ &\leq\frac{\xi(s)}{(u_{min}-\rho)^2}\|u^2-u^1\|_\infty<\|u^2-u^1\|_\infty\end{aligned}$$

which completes the proof of the lemma. □

Remark 3. Equivalently, G is a contraction mapping of v on metric space $(\mathcal{D}, d)$ where d is defined by weighted vector norm $\ell_{W,\infty}$ such that $\|v\|_{W,\infty}\triangleq\|W^{-1}v\|_\infty$.

VI. Numerical Illustration

The proposed conditions have been tested through a large number of experiments on the basis of IEEE models [23]. In this section, we will show numerical results on IEEE 13-feeder (whose structure is illustrated as following in Fig.1), 34-feeder, and 123-feeder models. We adjust them by assuming all power lines are of same type but different length. The model parameters are taken as typical values for medium-voltage cables as in [24]. Note that the shunt elements are included.

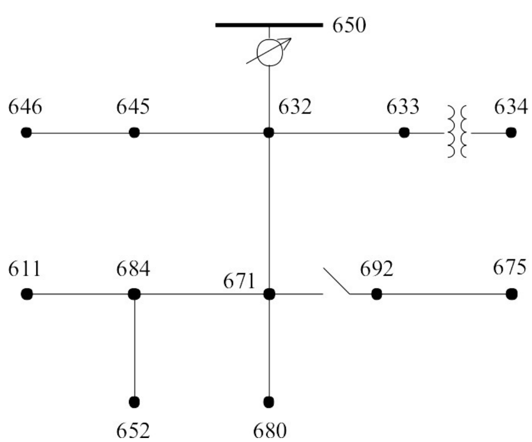

Fig. 1. IEEE 13-feeder grid.

A. Results on IEEE 13-Feeder Model

The power components of the known solution $(\hat{v}, \hat{s})$ are given in Table I; voltage magnitudes are shown on Fig.3. For better expression, first re-number all the nodes. Then take the power injection $\hat{s}=\hat{p}+j\hat{q}$ with normalization base 5MVA for Power and $4.16/\sqrt{3}=2.4$kV for Voltage (which is also the voltage of the slack bus).

TABLE I
Key Parameters

Index	$\hat{p}$(MW)	$\hat{q}$(Mvar)	$\lvert\hat{s}\rvert$(MVA)	e^a
$632\to1$	-0.48	-0.32	0.58	1
$633\to2$	1.28	0.96	1.60	1.05
$634\to3$	-0.72	-0.48	0.87	0.95
$645\to4$	0.96	0.8	1.25	1.03
$646\to5$	-0.96	-0.8	1.25	1.01
$671\to6$	0.64	0.48	0.80	1.05
$692\to7$	-0.8	-0.48	0.93	0.97
$675\to8$	0.64	0.48	0.80	1.04
$684\to9$	-0.64	-0.48	0.80	0.99
$611\to10$	0.32	0.24	0.4	1
$680\to11$	-0.48	-0.32	0.58	1
$652\to12$	0.32	0.24	0.4	1.05

[a] e is an $N\times1$ vector with $e_j=s_j/\hat{s}_j$. Physically, it is the ratio of power changes.

1) Illustration of Main Theorem: Here, for illustration purpose, we apply Theorem 1 to test the candidate power injection s, where $s_j=\hat{s}_je_j$ with $\hat{s}$ and e as in Table I. The computed results are shown in Table II. It is easy to check that the conditions in Theorem 1 are satisfied. In contrast, note that $\xi(s)=0.5770>0.25$, i.e. the method and conditions given in [13] and [14] do not work in this case.

TABLE II
Computed Results

$\xi(\hat{s})$	$\xi(s-\hat{s})$	$\xi(s)$	u_{min}	ρ
0.5692	0.0164	0.5770	1.0050	0.0412

In Fig.2, the circle is of radius $\rho=0.0412$ and represents $\mathcal{D}$ for one coordinate (here for instance, select Node 8).

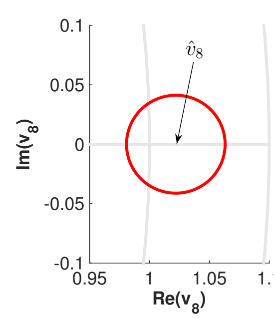

Fig. 2. The domain $\mathcal{D}$ for one coordinate (Node 8).

In Fig.3, the solved voltage magnitudes (i.e., '×') are shown. In the same figure, the Newton-Raphson method (i.e., '○') is used for checking the result. It is well-observed that the method gives out the same solution as Newton-Raphson method. Actually, all the solution coordinates lie in the domain given by our Theorem 1. In addition, the solved solution implies that the new setpoint is acceptable if the feasible region is defined as voltage magnitudes between 0.95 and 1.05 p.u.

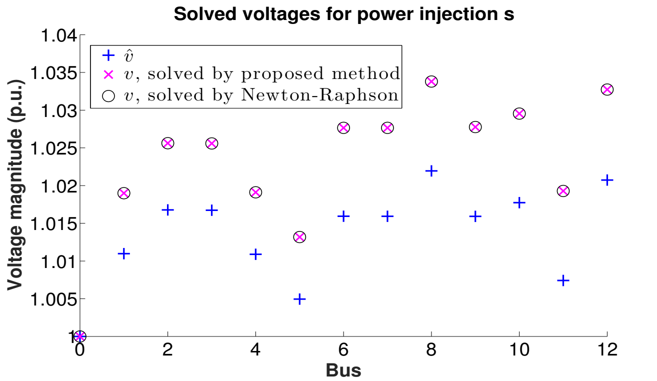

Fig. 3. The voltages of power injection $\hat{s}$ and the computed voltages of power injection s.

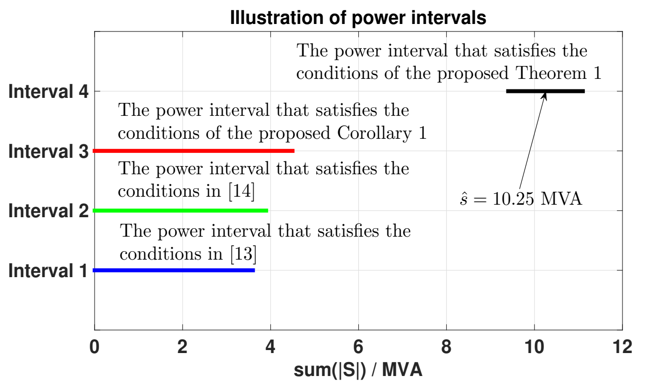

Fig. 4. Intervals of power injection that satisfy the conditions of the proposed Theorem 1, the proposed Corollary 1, the method in [13] and the method in [14] for IEEE 13-feeder model.

2) Continuation Power Flow analysis: In this subsection, we illustrate the range of power injections that are allowed and provided by our Theorem 1 and Corollary 1, using "continuation power flow analysis" [25]. To this end, we do not take the candidate power injections s from Table I but instead we scale them from $\hat{s}$. Specifically, let $s = \kappa\frac{\hat{s}}{\|\hat{s}\|_1}$ with $\kappa \in [0, \infty)$ MVA. In other words, the scaling factor $\kappa = \sum_{i=1}^{N} |s_i|$ is the sum of all apparent power injections. Then,

(i) With ($\hat{v}$,$\hat{s}$): By applying the conditions of the proposed Theorem 1, Interval 4 is obtained in Fig.4. For all the summed power κ in this interval, our conditions (9) and (10) are satisfied.

(ii) Without ($\hat{v}$,$\hat{s}$): Similarly, we can obtain Interval 1 by applying the conditions in [13], Interval 2 by conditions in [14], and Interval 3 by conditions of the proposed Corollary 1. In this example, it is clear that the power interval provided by the proposed method (i.e., Interval 3) covers the power intervals provided by methods in [13] and [14] (i.e., Interval 1 and 2). In other words, the proposed method is (strictly) stronger than the methods in [13] and [14].

Remark 4. Here, the Λ for the method in [14] is chosen as suggested in [14] $\Lambda_k = 1/\max_h |(W^{-1}Y_{LL}^{-1}\overline{W}^{-1})_{hk}|$. And the norms for methods in [13], [14] are chosen as the best by parameter scanning.

B. Results on Larger Networks

For IEEE 34-feeder model (and respectively 123-feeder model), we directly take the load information in the corresponding IEEE PES standard data sheet as the reference power setpoint s_{ref}. In particular, we (i) take the largest phase power for each bus; (ii) aggregate the distributed load at the farthest bus from the slack. Let $s = \kappa\frac{s_{ref}}{\|s_{ref}\|_1}$ with factor $\kappa = \sum_{i=1}^{N} |s_i|$ being the sum of all apparent power injections.

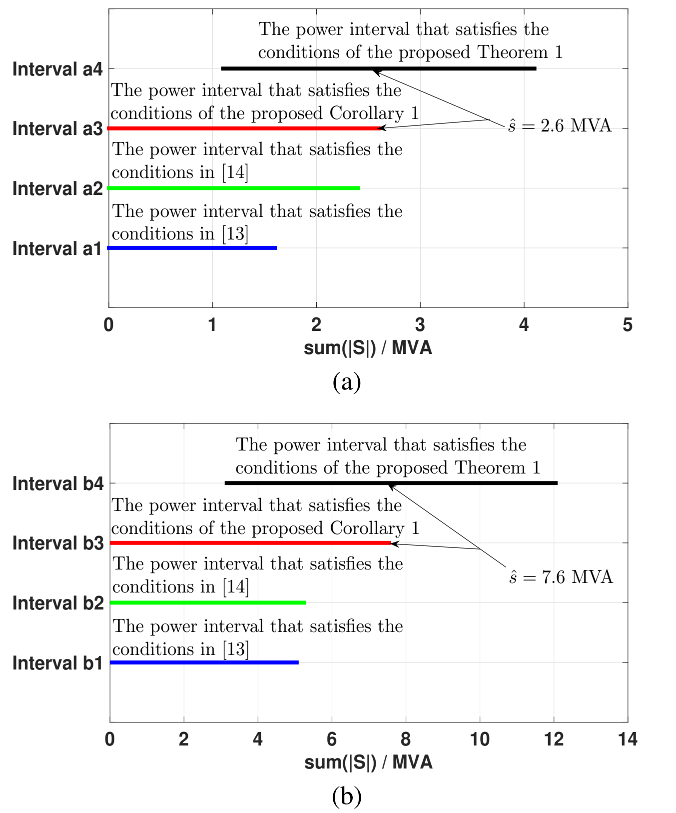

Fig. 5. Intervals of power injection that satisfy the conditions of the proposed Theorem 1, the proposed Corollary 1, the method in [13] and the method in [14] for (a) IEEE 34-feeder model and (b) IEEE 123-feeder model.

1) 34-Feeder Model: The IEEE 34-feeder model is a deep network and contains long transmission lines. Initially, we can assume there is no prior knowledge of the current setpoint and voltage. By applying the conditions in [13], [14] and our Corollary 1, we obtain Interval a1, a2, and a3 in Fig.5(a). It can be observed that Interval a3 provided by our proposed Corollary 1 covers Interval a1 and a2 that are provided by conditions in [13] and [14].

Further, by solving for the voltage with our proposed method at the power setpoint with $\kappa = 2.6$ MVA, we have a new current system state ($\hat{v}$,$\hat{s}$). Then, by applying the proposed Theorem 1, we guarantee the existence and uniqueness of a load-flow solution in a domain around $\hat{v}$ for all power setpoint s whose κ belongs to Interval a4.

2) 123-Feeder Model: The IEEE 123-feeder model is a network with short transmission lines and several long laterals. Similar to the previous examples, with no prior knowledge of current state assumed, we obtain Interval b1, b2, and b3 in Fig.5(b) by applying conditions in [13], [14], and our proposed Corollary 1.

Then, by solving for the voltage with our proposed method at the power setpoint with $\kappa = 7.6$ MVA, we have a pair ($\hat{v}$,$\hat{s}$) that represents the current system state. With this voltage-setpoint pair and the proposed Theorem 1, we guarantee the existence and uniqueness of a load-flow solution in a domain around $\hat{v}$ for all s whose κ belongs to Interval b4.

In the above examples on grids larger than IEEE 13-feeder model, we have solidly demonstrated and verified the fact that (i) the proposed Theorem 1 is general and can be applied to scenarios where conditions in [13], [14], and the proposed Corollary 1 are not applicable; (ii) even for the classical cases without knowledge of current state ($\hat{v}$,$\hat{s}$), the proposed Corollary 1 is always better than previously published works.

VII. Conclusion

We have provided efficient methods and explicit sufficient conditions that guarantee the existence and uniqueness of the load-flow solution for distribution networks with generic topology modeled using their positive sequence equivalents. Our findings improve on all previously known results. The whole theory has been verified in IEEE benchmark grids.

The proposed method is of practical use, as it can easily be deployed in applications for microgrids and distribution networks that require solving load-flows in real time.

In fact, the proposed Theorem 1 and Corollary 1 can be extended to a multi-phase version in a straightforward way, since they are proved using the normalized variables. But for multi-phase situations, the invertibility of matrix Y_{LL} may depend on detailed device modeling. We will systematically extend our method in the next paper based on various device modeling.

Appendix A
Invertibility of Y_{LL}

In circuit theory [26], there are already results on the invertibility of a full admittance matrix which includes the ground as one node. However, these results do not directly apply to Y_{LL}, which is only a sub-matrix of the nodal admittance matrix Y that does not contain ground node. Having considered this fact, we give the proof of the invertibility of Y_{LL} in this appendix.

A. Modeling and the Admittance Matrix

For the non-transformer connection (e.g., transmission lines) between node i and j, the 2×2 longitudinal admittance matrix is

$$\begin{bmatrix} y_{ij} & -y_{ij} \\ -y_{ij} & y_{ij} \end{bmatrix}$$

where y_{ij} (equal to y_{ji}) is the summed admittance of all power lines going directly from node i to node j.

For the transformer connection between node i and j, without loss of generality, let node i be connected to the primary side of this transformer and node j be at the secondary side, the 2×2 admittance matrix is given as

$$\begin{bmatrix} y_{ij}^t & -y_{ij}^t K_{ij}^{-1} \\ -y_{ij}^t \overline{K_{ij}}^{-1} & y_{ij}^t |K_{ij}|^{-2} \end{bmatrix}$$

where y_{ij}^t is the equivalent aggregated admittance on the primary side, complex number K_{ij} is the ratio. Reciprocally, we can denote $y_{ij}^t |K_{ij}|^{-2}$ by y_{ji}^t which is the equivalent aggregated admittance on the secondary side, and K_{ij}^{-1} by K_{ji} which is the inverse ratio. Now, the terms in a general admittance matrix Y including shunt elements can be explicitly written as

$$Y_{ij} = \begin{cases} -y_{ij} & j \in \mathcal{N}(i) \\ -y_{ij}^t K_{ij}^{-1} & j \in \mathcal{N}^t(i) \\ 0 & \text{otherwise} \end{cases}$$

and

$$Y_{ii} = y_{ii}^{shunt} + \sum_{j\in\mathcal{N}(i)} y_{ij} + \sum_{j\in\mathcal{N}^t(i)} y_{ij}^t,$$

where $\mathcal{N}(i)$ is the set of nodes that have direct non-transformer connections with node i, and $\mathcal{N}^t(i)$ is the set of nodes that have direct transformer connections with node i. Here, y_{ii}^{shunt} is the sum of shunt elements around node i.

B. The Invertibility

If the grid is viewed as a graph where buses are vertices and power lines are edges, then a new graph can be generated by eliminating node 0. Suppose that the new graph has c connected components, then by carefully re-numbering each node, Y_{LL} can be written as a c-block diagonal matrix. In this way, Y_{LL} is invertible iff all blocks are invertible. Thus, if we can show an arbitrary one of these components invertible, then the invertibility of Y_{LL} is proved. Thus, without loss of generality, assume that the new graph itself be one connected component.

First, denote this undirected graph as $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. In addition, let $\mathcal{V}^{slack} \subseteq \mathcal{V}$ be the set of nodes that are originally connected to the slack bus; $\mathcal{G}^t = (\mathcal{V}^t, \mathcal{E}^t)$ be the subgraph that contains all the transformer edges and corresponding endpoints; $\mathcal{G}_m = (\mathcal{V}_m, \mathcal{E}_m)$, $m \in \{1, ..., M\}$ be all the M connected components in $(\mathcal{V}, \mathcal{E} \setminus \mathcal{E}^t)$.

Let x be an N-by-1 vector such that $Y_{LL}x = 0$, and for all $i \in \mathcal{V}^{slack}$ define

$$\tilde{y}_{i0} = \begin{cases} y_{i0} & \text{non-transformer connection} \\ y_{i0}^t & \text{transformer connection} \end{cases}$$

Then, we have

$$\begin{aligned} x^H Y_{LL} x &= \sum_{i,j\in\mathcal{V}} \overline{x}_i (Y_{LL})_{ij} x_j \\ &= \sum_{i=1}^{N} \sum_{j:(i,j)\in\mathcal{E}\setminus\mathcal{E}^t} y_{ij}\overline{x}_i(x_i - x_j) + \sum_{i=1}^{N} \sum_{j:(i,j)\in\mathcal{E}^t} y_{ij}^t \overline{x}_i (x_i - K_{ij}^{-1} x_j) \\ &\quad + \sum_{i\in\mathcal{V}^{slack}} \tilde{y}_{i0}|x_i|^2 + \sum_{i\in\mathcal{V}} y_{ii}^{shunt}|x_i|^2 \end{aligned}$$

For the first term, we have

$$\begin{aligned}
&\sum_{i=1}^{N}\sum_{j:(i,j)\in\mathcal{E}\setminus\mathcal{E}^t} y_{ij}\overline{x}_i(x_i-x_j)\\
=&\sum_{i=1}^{N}\sum_{j>i:(i,j)\in\mathcal{E}\setminus\mathcal{E}^t} y_{ij}\overline{x}_i(x_i-x_j)+\sum_{i=1}^{N}\sum_{j<i:(i,j)\in\mathcal{E}\setminus\mathcal{E}^t} y_{ij}\overline{x}_i(x_i-x_j)\\
=&\sum_{i=1}^{N}\sum_{j>i:(i,j)\in\mathcal{E}\setminus\mathcal{E}^t} y_{ij}\overline{x}_i(x_i-x_j)+\sum_{i=1}^{N}\sum_{j>i:(i,j)\in\mathcal{E}\setminus\mathcal{E}^t} y_{ji}\overline{x}_j(x_j-x_i)\\
=&\sum_{i=1}^{N}\sum_{j>i:(i,j)\in\mathcal{E}\setminus\mathcal{E}^t} (y_{ij}\overline{x}_i(x_i-x_j)+y_{ij}\overline{x}_j(x_j-x_i))\\
=&\sum_{i<j:(i,j)\in\mathcal{E}\setminus\mathcal{E}^t} y_{ij}|x_i-x_j|^2
\end{aligned}$$

Similarly, for the second term, we have

$$\begin{aligned}
&\sum_{i=1}^{N}\sum_{j:(i,j)\in\mathcal{E}^t} y^t_{ij}\overline{x}_i(x_i-K_{ij}^{-1}x_j)\\
=&\sum_{i=1}^{N}\sum_{j>i:(i,j)\in\mathcal{E}^t} y^t_{ij}\overline{x}_i(x_i-K_{ij}^{-1}x_j)+\sum_{i=1}^{N}\sum_{j<i:(i,j)\in\mathcal{E}^t} y^t_{ij}\overline{x}_i(x_i-K_{ij}^{-1}x_j)\\
=&\sum_{i=1}^{N}\sum_{j>i:(i,j)\in\mathcal{E}^t} y^t_{ij}\overline{x}_i(x_i-K_{ij}^{-1}x_j)+\sum_{i=1}^{N}\sum_{j>i:(i,j)\in\mathcal{E}^t} y^t_{ji}\overline{x}_j(x_j-K_{ji}^{-1}x_i)\\
=&\sum_{i=1}^{N}\sum_{j>i:(i,j)\in\mathcal{E}^t} (y^t_{ij}\overline{x}_i(x_i-K_{ij}^{-1}x_j)+y^t_{ij}\overline{K}_{ij}^{-1}\overline{x}_j(K_{ij}^{-1}x_j-x_i))\\
=&\sum_{i<j:(i,j)\in\mathcal{E}^t} y^t_{ij}|x_i-K_{ij}^{-1}x_j|^2
\end{aligned}$$

So that,

$$\begin{aligned}
&x^HY_{LL}x\\
=&\sum_{i<j:(i,j)\in\mathcal{E}\setminus\mathcal{E}^t} y_{ij}|x_i-x_j|^2+\sum_{i<j:(i,j)\in\mathcal{E}^t} y^t_{ij}|x_i-K_{ij}^{-1}x_j|^2\\
&+\sum_{i\in\mathcal{V}^{slack}}\tilde{y}_{i0}|x_i|^2+\sum_{i\in\mathcal{V}} y_{ii}^{shunt}|x_i|^2=0
\end{aligned}$$

Since $\Re\tilde{y}_{i0}>0$ for all $i\in\mathcal{V}^{slack}$, $\Re y_{ii}^{shunt}$ non-negative for all $i\in\mathcal{V}$, and $\Re y_{ij},\Re y^t_{ij}>0$ for all i,j s.t. $(i,j)\in\mathcal{E}$, we have

1) $x_i=0$ for all $i\in\mathcal{V}^{slack}$;
2) $x_i=x_j$ for all $i,j\in\mathcal{V}_m$ given any $m\in\{1,...,M\}$;
3) $x_i=K_{ij}^{-1}x_j$ for all i,j s.t. $(i,j)\in\mathcal{E}^t$.

Because $\mathcal{G}$ is connected, it can be obtained that

- By above 1 and 2, there exists at least one m s.t. $x_i=0$ for all $i\in\mathcal{V}_m$.
- By 2 and 3, the zero value will propagate throughout $\mathcal{G}$.

Thus, the vector x must be a zero vector, which implies Y_{LL} has a trivial null space and hence is invertible.

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