Cellular Homotopy Excision

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Abstract.

There is a classical "duality" between homotopy and homology groups in that homotopy groups are compatible with *homotopy pullbacks* (every homotopy pullback gives rise to a long exact sequence in homotopy), while homology groups are compatible with *homotopy pushouts* (every homotopy pushout gives rise to a long exact sequence in homology). This last statement is sometimes referred to as the *Mayer-Vietoris* or *excision axiom.* The classical *Blakers-Massey theorem* (or *homotopy excision theorem*) asks to what extent the excision property for homotopy pushouts remains true if we replace homology groups by homotopy groups and gives a range in which the excision property holds.

It does so by estimating the connectivity of a certain comparison map, which is a rather crude measure, as it is just a single number. Since connectivity is a special case of a *cellular inequality*, the hope is that there is a stronger statement hidden behind the connectivity result in terms of such inequalities.

This process of generalising the homotopy excision theorem has been initiated by Chachólski in the 90s, where he proved a more general version for homotopy pushout squares. The caveat was that one had to suspend the comparison map in question first and the goal of our project – which we obtained – was to lose this suspension and then move on to cubical diagrams, rather than squares.

To do so, there are a few basic ingredients that are necessary. We first talk about our abstract approach to derived functors, then construct left Bousfield localisations of combinatorial model categories and finally, generalise the foundational concepts in the theory of closed classes to non-connected spaces.

Keywords. Homotopy excision, Bousfield localisation, Bousfield classes, Cellular classes, Closed classes, Diagrams of spaces, Derived functors

Zusammenfassung.

Die klassische "Dualität" zwischen Homotopie- und Homologiegruppen besagt, dass Homotopiegruppen kompatibel sind mit Homotopie-Pullbacks (jeder solche hat eine assoziierte lange exakte Sequenz von Homotopiegruppen), während Homologiegruppen kompatibel sind mit Homotopie-Pushouts (jeder solche hat eine assoziierte lange exakte Sequenz von Homologiegruppen). Dieses letzte Resultat ist bekannt unter dem Namen *Mayer-Vietoris* oder *Ausschneidungsaxiom*. Im klassischen Theorem von *Blakers-Massey* (auch *Homotopie-Ausschneidungstheorem* genannt) wird die Frage beantwortet, inwieweit das Ausschneidungsaxiom für Homotopiepushouts wahr ist, wenn wir die Homologie- durch Homotopiegruppen ersetzen.

Dies geschieht durch die Abschätzung der Konnektivität einer gewissen Vergleichsabbildung, was ein eher grobes Mass ist, da es aus einer einzelnen Zahl besteht. Da die Konnektivität ein Spezialfall einer *zellulären Ungleichung* ist, liegt die Vermutung nahe, dass sich hinter dem Konnektivitätsresultat eine stärkere Aussage in Form von ebensolchen Ungleichungen versteckt.

Dieser Prozess der Verallgemeinerung des Homotopie-Ausschneidungstheorems wurde in den 90er-Jahren durch Wojciech Chachólski initiiert, der eine allgemeinere Version für viereckige Homotopiepushouts beweisen konnte. Der Vorbehalt dieses Resultats war, dass die Suspension der entsprechende Vergleichsabbildung genommen werden musste. Das Ziel dieser Arbeit – welches auch erreicht wurde – war, zuerst diese Suspension loszuwerden und schliesslich kubische Diagramme zu studieren anstatt Vierecke.

Dafür benötigen wir einige grundlegende Ingredienzen. Als erstes beschreiben wir einen abstrakten Zugang zu derivierten Funktoren. Dann konstruieren wir die linke Bousfield-Lokalisierung einer kombinatorischen Modellkategorie und schliesslich verallgemeinern wir die grundlegenden Konzepte der Theorie der abgeschlossenen Klassen indem wir auch nicht-zusammenhängende Räume zulassen.

Schlüsselwörter. Homotopy excision, Bousfield localisation, Bousfield classes, Cellular classes, Closed classes, Diagrams of spaces, Derived functors

PREFACE

Homotopy Excision

There is a classical "duality" between homotopy and homology groups in that homotopy groups are compatible with *fibre sequences* or *homotopy pullbacks* (every homotopy pullback gives rise to a long exact sequence in homotopy), while homology groups are compatible with *cofibre sequences* or *homotopy pushouts* (every homotopy pushout gives rise to a long exact sequence in homology). This last statement is sometimes referred to as the *Mayer-Vietoris* or *excision axiom.* The classical Blakers-Massey theorem (or *homotopy excision* as it is often called) asks to what extent the excision property remains true if we replace homology groups by homotopy groups and gives a range in which the excision property holds. More specifically, it states that if a space X can be decomposed as $X = U \cup V$ with U, $V \subseteq X$ open and $U \cap V \neq \varnothing$ and such that

$$\pi_i(U, U \cap V) = 0 \qquad \text{and} \qquad \pi_j(V, U \cap V) = 0$$

within ranges $0 < i < p$, $0 < j < q$ (where p, $q \geqslant 1$) then the so-called *excision map* $\pi_n(V, U \cap V) \to \pi_n(X, U)$ is bijective for $1 \leqslant n < p + q - 2$ and surjective for $n = p + q - 2$.

This classical formulation is unnecessarily restrictive (in that it requires U and V be open) and the relative homotopy groups are unhandy when their explicit definition is used. Let us quickly state an easier formulation due to Ellis-Steiner [23] and Goodwillie [30] using homotopically invariant versions of common universal constructions (limits, colimits, fibres). Given a map of spaces $E \to B$ together with a base-point $e \in E$, the *homotopy fibre* (i.e. the homotopy invariant fibre construction) $F := \mathrm{hFib}(E \to B)$ with respect to this base-point fits into the Serre long exact sequence

$$\ldots \to \pi_{n+1}(B) \to \pi_n(F) \to \pi_n(E) \to \pi_n(B) \to \ldots \to \pi_0(E) \to \pi_0(B).$$

More generally, given two maps of spaces $C \to D \leftarrow B$, we can form the *homotopy pullback* $P := \mathrm{holim}(C \to D \leftarrow B)$, which fits into a long exact sequence

$$\ldots \to \pi_{n+1}(D) \to \pi_n(P) \to \pi_n(B \times C) \to \pi_n(D) \to \ldots \to \pi_0(B \times C).$$

With these notions clarified, we can state the classical homotopy excision theorem. For it, we consider a commutative square

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

that is a homotopy pushout, meaning that D is obtained by gluing together B and C along a common subspace A in a non-pathological way. Now, taking the homotopy pullback $P = \mathrm{holim}(C \to D \leftarrow B)$, then by a universal property, there is a comparison map $q\colon A \to P$ and the theorem states that if the homotopy fibres $\mathrm{hFib}(A \to B)$ and $\mathrm{hFib}(A \to C)$ are

p- and q-connected then the homotopy fibre $\mathrm{hFib}(q\colon A \to P)$ of the comparison map is $(p+q)$-connected.

As one direct use of this, by the long exact sequence associated to $\mathrm{hFib}(q)$, we get that $\pi_n(A) \cong \pi_n(P)$ for all $n \leqslant p+q$, which means that in the long exact sequence associated to P, we can replace P by A within this range and we get a new (finite!) long exact sequence

$$\pi_{p+q+1}(B) \times \pi_{p+q+1}(C) \to \pi_{p+q+1}(D) \to \pi_{p+q}(A) \to \pi_{p+q}(B) \times \pi_{p+q}(C) \to \dots$$

$$\dots \to \pi_1(D) \to \pi_0(A) \to \pi_0(B) \times \pi_0(C).$$

This is very useful since, in general, the homotopy pullback P might not be calculable, whereas A is already given.

As an important, less direct application, let us mention the *Freudenthal suspension theorem*, which lies at the foundation of stable homotopy theory. It is implied by the homotopy excision theorem applied to the case where $B = C = CA$ are cones over A and thus contractible. In that case, $D = \Sigma A$ is the suspension of A, while $P = \Omega\Sigma A$ is the loop space of said suspension, and the theorem says that the suspension map

$$\pi_n(A) \to \pi_{n+1}(\Sigma A),\ \left[S^n \xrightarrow{\gamma} A\right] \mapsto \left[S^{n+1} \xrightarrow{\Sigma\gamma} \Sigma A\right]$$

is an isomorphism for all $n \leqslant 2{\cdot}\mathrm{conn}(A)$ and a surjection for $n = 2\,\mathrm{conn}(A)+1$, where $\mathrm{conn}(A)$ is the connectivity of A. Now, this first implies that $\mathrm{conn}(\Sigma^k A) \geqslant \mathrm{conn}(A)+k$, meaning that the connectivity grows with every suspension that we take. But $2\,\mathrm{conn}(\Sigma^k A) \geqslant 2\,\mathrm{conn}(A)+2k$ grows even faster and we conclude that for a fixed $n \in \mathbb{N}$, the sequence

$$\pi_n(A) \to \pi_{n+1}(\Sigma A) \to \pi_{n+2}(\Sigma^2 A) \to \dots$$

stabilises and this stable value $\pi_n^s(A)$ is called the n^{th} stable homotopy group of A.

As a last note for the classical theorem: While we only stated the square case here, the theorem generalises to higher-dimensional *strong homotopy pushouts*, which are higher-dimensional (hyper-)cubical diagrams (see chapter 10 for more details).

Bousfield Localisation

The starting point for cellular homotopy excision – which is the focus of this thesis – is the observation that the connectivity of a space (and hence of a map) can be expressed in terms of so-called *left Bousfield localisations*. This can very much be understood in analogy to the localisation of rings. To wit, given a commutative ring R and a set $M \subseteq R$, we can localise R at M and obtain a new ring $R[M^{-1}]$, where all elements of M are made invertible. Now, given a category $\mathcal{C}$ equipped with a class of weak equivalences, the invertibility of an element is replaced by the contractibility of an object (i.e. its being weakly equivalent to the terminal object). So, given a set M of objects, we want to localise $\mathcal{C}$ at M by adding new weak equivalences to $\mathcal{C}$ in such a way that every object in M becomes contractible.

Of course, this can always be done formally but we would like to have some control over the resulting homotopy category (assuming that was already the case for $\mathcal{C}$). More precisely, if $\mathcal{C}$ comes equipped with a model structure (which means that we have good control over its homotopy category), we would like to localise $\mathcal{C}$ in such a way that we still have a model structure.

To this end, the localisation of the category of spaces first occurred in the works of Quillen [44], Sullivan [47; 48] and Bousfield-Kan [8]. In today's form, they were then developed by Dror Farjoun [25; 26] and Bousfield [6; 7] and subsequently further developed and studied by Chachólski [10; 11; 12]. In its model categorical formulation [33], the *left Bousfield localisation* of a model category (most importantly, that of spaces in the form of simplicial sets) is obtained by adding in the new weak equivalences while keeping the same cofibrations.

Let's come back to the original point that the connectivity of a space is expressible in terms of left Bousfield localisations. Just like for rings, if we make some spaces contractible, they might not be the only ones and, as it turns out, a space X is n-connected iff it becomes contractible when localising the category of spaces at S^{n+1}, the $(n+1)$-sphere. The general motto now is that many theorems that involve connectivities can be reformulated, strengthened and generalised by using localisations. In particular, this should be true for the homotopy excision theorem mentioned at the beginning.

In fact, there is an entire *calculus of acyclic inequalities* hidden behind this. Writing $X > Y$ for the statement that X becomes contractible when localising at Y, we have just mentioned that X is n-connected iff $X > S^{n+1}$. However, connectivity is of course not the only property that is expressible in this way. For example, letting $M(\mathbb{Z}/p\mathbb{Z}, 2)$ be the Moore space, whose only non-trivial reduced integral homology group is $\mathbb{Z}/p\mathbb{Z}$ in dimension 2, one can show that

$$X > M(\mathbb{Z}/p\mathbb{Z}, 2) \quad \text{iff} \quad X \text{ is 1-connected and every } \pi_n(X) \text{ with } n > 2 \text{ is a } p\text{-group.}$$

Now, any acyclic inequality immediately allows us to transfer this property. For example, it is always true that $\Sigma X > X$. This is not too interesting, when just looking at the connectivity (it just tells us that $\operatorname{conn}(\Sigma X) \geqslant \operatorname{conn}(X)$) but if X has the above property, then

$$\Sigma X > X > M(\mathbb{Z}/p\mathbb{Z}, 2)$$

and since “$>$” is transitive, we can conclude that ΣX again has the above property. It is exactly for such properties, other than connectivity, expressible in terms of acyclic inequalities that we think it is worthwhile to generalise classical results (such as homotopy excision), which only involve connectivity.

Main Results

The main objective of this thesis is to show a cellular version of the homotopy excision theorem. The classical version has been proved time and again with different formulations by different people [4; 9; 24; 30] but all of them based solely on connectivity (and the structure of the first non-zero homotopy group). Continuing the work of Chachólski [13], we try to reformulate these results (partially) in terms of closed classes and Bousfield classes. Specifically, as our first main result, we show an acyclic homotopy excision theorem for homotopy pushout squares, which reads as follows.

(9.7.1) **Theorem.** Given a homotopy pushout square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

with comparison map $q\colon A \to \operatorname{holim}(B \to D \leftarrow C)$, then

$$\operatorname{hFib}(q) > \Omega \operatorname{hFib}(f) * \Omega \operatorname{hFib}(g).$$

The big difference to Chachólski's original result [13] is that we directly study the fibres of the comparison map q, whereas he obtains a cellular inequality for the fibres of the suspension Σq. For expository reasons, we intentionally swept some connectivity issues under the rug in the above statement and a lot of work done in this thesis goes into making sense of homotopy fibres (and loop spaces) for non-connected and unpointed spaces in the context of Bousfield classes.

We then move on to prove an analogue of the acyclic Blakers-Massey theorem for cubes instead of squares. Here, just like in Chachólski's original work, we again need to suspend the comparison map (even twice!) in order to be able to establish an acyclic inequality. We also need to impose some connectivity restrictions for our methods to work but are confident that the result should hold even without them.

(11.3.11) **Theorem.** Let $A\colon \square^3 \to \mathbf{sSets}$ be a strong homotopy pushout of connected spaces, with homotopy fibres $F_k := \mathrm{hFib}(A_\varnothing \to A_k)$ and comparison map $q\colon A_\varnothing \to \mathrm{holim}_{\ulcorner^3} A$. As long as the homotopy fibres F_1, F_2 and F_3 are again connected,

$$\mathrm{hFib}(\Sigma^2 q) > \Sigma(\Omega F_1 * \Omega F_2 * \Omega F_3).$$

Organisation of the Thesis

This thesis is split into three different parts. In the first part, we are going to discuss the basics of forming homotopy categories and derived functors from the abstract viewpoint of *relative categories*, which is just a fancy way of saying a category equipped with some class of weak equivalences. No additional structure is assumed and we are only interested in abstract universal properties, while explaining how additional data (such as model structures) can be used for explicit constructions. This approach is more in the spirit of [21] and is necessarily more *derivatoresque* in nature (even though the author only learned about derivators afterwards). For the expert, used to a more explicit approach to derived functors (e.g. via model structures), this is certainly an interesting alternative viewpoint and can help create a link between different such approaches. However, since we are only going to use it in the especially nice context of diagrams of simplicial sets, one shouldn't expect many new results for this particular context.

In the second part, we are going to discuss the construction of left Bousfield localisations of combinatorial model categories. Much of this theory and the results at its core have been considered folklore for a long time but without any one single resource that one could use to learn it. To be able to construct the left Bousfield localisation, we first need to establish the foundations of locally presentable and accessible categories. Good resources are [1] and [39]. Our approach is very much along classical lines, even though we proved a few results not found in the literature, which we were then able to use to facilitate a few proofs later on.

A very good resource for the actual construction of left Bousfield localisations (but for cellular model categories) is of course Hirschhorn's book [33] but we find the combinatorial context more convenient. Here, though, the troubles already start at the basics since an oft-seen reference is Smith's book *Combinatorial model categories* which, to this day, has not appeared. As a replacement, Dugger's work [20] is often cited. However, in anticipation of Smith's book, a lot of details are left out in *op. cit.* In more modern times, there is of course Lurie's tome [37], which, while ingenious, hardly provides a direct path to left Bousfield localisations and operates at a much higher level of abstraction. Then, finally, there is Barkwick's work [2], who, however, is mostly concerned with the enriched case and in which, we were not able to fill in the details of one particular proof.

All in all, we feel that this part might be of great educational interest, even to people not working in (unstable) homotopy theory. For example, left Bousfield localisations have been used successfully in algebraic geometry as well [42]. But again, for the expert, already familiar with the material, there are not too many surprises here.

In the third and final part, we finally come to the proofs of our main results, which we already mentioned above.

As a final comment, even though we included chapters on categorical preliminaries, homotopical preliminaries and model categories, we assume basic familiarity with category theory, homotopy theory (including simplicial sets) and the approach to homotopy theory through model categories. In the respective chapters, we try to cover a few things that are maybe less well-known or done differently by different people or that, while maybe well-known, require some special attention to details.

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Part I

Localisation of Categories

Chapter 1

LOCALISATIONS AND DERIVED FUNCTORS

The content of this chapter started out as a small note in the beginning of the authors PhD, when learning about categorical homotopy theory and realising that there are different approaches to derived functors and that their interplay doesn't seem to be well-documented.

The original goal was to answer the question of why double homotopy pushouts (or more generally, homotopy Kan extensions in a category of diagrams) can be calculated pointwise, even if there is no model structure. This question is usually quickly dismissed as a very simple one but, on close inspection, does have its subtleties, not the least of which being how a homotopy Kan extension should be defined.

In section 1, we first derive a universal property of the (strict) localisation of a category that we are going to need and which is usually only proven for explicit constructions of the localisation (e.g. for the homotopy category of a model category). In section 2, we restate the usual definition of a derived functor and, more importantly, of an absolute derived functor. We then externalise this definition and quickly discuss some class-theoretic issues that arise. In section 3, we slightly generalise the [21] approach for the construction of derived functors and see that the usual construction of derived functors via (co)fibrant replacements is not confined to model categories. In sections 4 and 5, for lack of references, we quickly recall the definition, coherence and Beck-Chevalley interchange condition for the calculus of mates. These are standard tools in the world of derivators but less well-known outside of it. Motivated by [40], we investigate the concept of derived adjunctions in our general context and show the relation between absolute derived functors and adjoints thereof. Similar results were already obtained by [29] but using different techniques. Our proofs, using the external characterisation of absolute Kan extensions, seems much more straightforward than the internal approach taken in *op. cit.* In section 7, we apply this comparison result to the notion of homotopy (co)limits and obtain two equivalent definitions. Finally, sections 8 and 9 contain our main results about the pointwiseness of certain homotopy Kan extensions and we obtain two sufficient criteria for them to be so.

1. Localisations Reviewed

By the localisation of a category $\mathcal{C}$ with respect to a class of morphisms $\mathcal{W}$ is meant the (comparison functor with) the category obtained by formally inverting the morphisms in $\mathcal{W}$.

(1.1) **Definition.** Let $\mathcal{C}$ be a category and $\mathcal{W}$ a class of arrows in $\mathcal{C}$ (which we usually call the *weak equivalences* of $\mathcal{C}$). A *weak localisation* of $\mathcal{C}$ is a functor $H\colon \mathcal{C} \to \mathcal{C}[\mathcal{W}^{-1}]$ sending all arrows in $\mathcal{W}$ to isomorphisms and having the following universal property:

(a) Whenever we have a functor $F\colon \mathcal{C} \to \mathcal{D}$ sending all arrows in $\mathcal{W}$ to isomorphisms, then there is some $\bar{F}\colon \mathcal{C}[\mathcal{W}^{-1}] \to \mathcal{D}$ together with a natural isomorphism $F \cong \bar{F} \circ H$;

(b) For every category $\mathcal{D}$, the precomposition $H^*\colon \mathcal{D}^{\mathcal{C}[\mathcal{W}^{-1}]} \to \mathcal{D}^{\mathcal{C}}$ is fully faithful. That is to say, for any two F, $G\colon \mathcal{C}[\mathcal{W}^{-1}] \to \mathcal{D}$ and any $\tau\colon F \circ H \Rightarrow G \circ H$ there is a unique $\bar{\tau}\colon F \Rightarrow G$ such that $\tau = \bar{\tau}_H$.

(1.2) **Observation.** One easily sees that $\mathcal{C}[\mathcal{W}^{-1}]$ is unique up to equivalence; that the extension $\bar{F}$ in property (a) is unique up to unique isomorphism and the natural isomorphism $F \cong \bar{F} \circ H$ is unique up to unique automorphism of $\bar{F}$.

Most authors will define localisations differently. Namely, $\bar{F}$ in (a) needs to be unique and the natural isomorphism $F \cong \bar{F} \circ H$ is required to be an identity, while property (b) is left out entirely. We call this a *strict localisation*.

(1.3) **Definition.** A *(strict) localisation* of a category $\mathcal{C}$ at a class of arrows $\mathcal{W}$ (again called the *weak equivalences*) is a functor $H\colon \mathcal{C} \to \mathcal{C}[\mathcal{W}^{-1}]$ that sends arrows in $\mathcal{W}$ to isomorphisms and such that every functor $F\colon \mathcal{C} \to \mathcal{D}$ that does so factors uniquely through H as $F = \bar{F} \circ H$. It follows that $\mathcal{C}[\mathcal{W}^{-1}]$ is unique up to isomorphism. We will seldomly have the situation where we are given two different classes of weak equivalences in $\mathcal{C}$ and will thus just write $\operatorname{Ho}\mathcal{C} := \mathcal{C}[\mathcal{W}^{-1}]$, leaving the class $\mathcal{W}$ implicit.

(1.4) **Example.** If every arrow in $\mathcal{W}$ is already an isomorphism then $\mathrm{id}_{\mathcal{C}}\colon \mathcal{C} \to \mathcal{C}$ is a localisation of $\mathcal{C}$ at $\mathcal{W}$. In particular, if $\mathcal{C}$ is a groupoid then $\mathrm{id}_{\mathcal{C}}$ is a localisation at any class of weak equivalences.

(1.5) **Example.** If $\mathcal{C}$ is a model category then the canonical functor $\mathcal{C} \to \operatorname{Ho}\mathcal{C}$ is a localisation of $\mathcal{C}$ with respect to its weak equivalences. In fact, having well-behaved localisations is the reason why the theory of model categories was developed in the first place.

(1.6) **Convention.** If $\mathcal{W}$ is a class of weak equivalences in a category $\mathcal{C}$, one can always add the isomorphisms of $\mathcal{C}$ to $\mathcal{W}$ without changing the localisation and so we shall always assume that isomorphisms are weak equivalences.

It seems curious that part (b) from the definition of a localisation is omitted entirely in the strict version. After all, it has some consequences, e.g. for derived functors, which seem to be important enough to be proven for an explicit construction of a localisation (e.g. [22, 5.9] for the homotopy category of a model category). But using an explicit construction is not necessary as the following proof shows.

(1.7) **Proposition.** If $H\colon \mathcal{C} \to \operatorname{Ho}\mathcal{C}$ is a strict localisation of a category $\mathcal{C}$ at a class $\mathcal{W}$ of morphisms, then, for every category $\mathcal{D}$, the precomposition $H^*\colon \mathcal{D}^{\operatorname{Ho}\mathcal{C}} \to \mathcal{D}^{\mathcal{C}}$ is fully faithful.

Proof. Let $[1]$ be the *interval category* with two objects 0, 1 and exactly one non-identity morphism $i\colon 0 \to 1$. There is the canonical twist isomorphism $(\mathcal{D}^{\mathcal{C}})^{[1]} \cong (\mathcal{D}^{[1]})^{\mathcal{C}}$ so that pairs of functors F, $G\colon \mathcal{C} \to \mathcal{D}$ together with $\tau\colon F \Rightarrow G$ correspond to functors $\mathcal{C} \to \mathcal{D}^{[1]}$. Now given F, $G\colon \operatorname{Ho}\mathcal{C} \to \mathcal{D}$ together with $\tau\colon F \circ H \Rightarrow G \circ H$, these determine

$$T\colon \mathcal{C} \to \mathcal{D}^{[1]} \quad \text{with} \quad (TC)i = \tau_C \text{ and } (Tf)_0 = HFf,\ (Tf)_1 = HGf \text{ for } f\colon C \to C'.$$

Under T, arrows $f \in \mathcal{W}$ are sent to isomorphisms because HFf and GFf are invertible and so we get a unique $\overline{T}\colon \operatorname{Ho}\mathcal{C} \to \mathcal{D}^{[1]}$ such that $T = \overline{T} \circ H$. Under the above twist isomorphism, this corresponds to a unique pair of functors

$$F',\ G'\colon \operatorname{Ho}\mathcal{C} \to \mathcal{D} \quad \text{defined by} \quad F' = \mathrm{ev}_0 \circ \overline{T} \text{ and } G' = \mathrm{ev}_1 \circ \overline{T}$$

together with $\bar{\tau}\colon F' \Rightarrow G'$ defined by $\bar{\tau}_C = (\overline{T}C)i$ that satisfies $\tau = \bar{\tau}_H$. □

(1.8) **Nomenclature.** For brevity reasons, if $\mathcal{C}$ and $\mathcal{D}$ are categories with weak equivalences, we call a functor $F\colon \mathcal{C} \to \mathcal{D}$ *homotopical* iff it preserves weak equivalences. Moreover, by a *natural weak equivalence*, we mean a natural transformation $\tau\colon F \Rightarrow G$, all of whose components are weak equivalences.

Assuming we have an explicit localisation construction (e.g. using zig-zags), we note that Ho is strictly 2-functorial. In particular, any adjunction $F \dashv G$ of homotopical functors yields an adjunction $\operatorname{Ho} F \dashv \operatorname{Ho} G$ between the correspondig homotopy categories.

(1.9) **Observation.** Let $\mathcal{C}$ be a category with weak equivalences and $H\colon \mathcal{C} \to \operatorname{Ho}\mathcal{C}$ a localisation. If an object 0 is initial in $\mathcal{C}$ then $H0$ is initial in $\operatorname{Ho}\mathcal{C}$.

Proof. Both adjoints in $0\colon \{*\} \rightleftarrows \mathcal{C} :!$ are homotopical. □

2. Derived Functors

Consider a category $\mathcal{C}$ with weak equivalences. Time and again, one finds oneself in the situation where one wants to study some $F\colon \mathcal{C} \to \mathcal{D}$ that doesn't map weak equivalences to isomorphisms (so that it cannot be extended along the localisation). Sometimes this is conceived as a defect of F (e.g. for F a [co]limit functor) and at other times as an interesting peculiarity that can be exploited to construct invariants (e.g. using Hom-functors, tensoring or global sections). In either case, the approach is usually to approximate F as well as possible by a functor on $\operatorname{Ho}\mathcal{C}$.

(2.1) **Definition.** Let $H\colon \mathcal{C} \to \operatorname{Ho}\mathcal{C}$ be a localisation of a category with weak equivalences and $F\colon \mathcal{C} \to \mathcal{D}$. Recall that a *left derived functor* $(LF, \lambda\colon LF \circ H \Rightarrow F)$ of F is a right Kan extension of F along H. That is, for every $L\colon \operatorname{Ho}\mathcal{C} \to \mathcal{D}$ together with $\tau\colon L \circ H \Rightarrow F$, there is a unique transformation $\bar{\tau}\colon L \Rightarrow LF$ such that $\lambda \circ \bar{\tau}_H = \tau$. The universal 2-arrow λ is called the *counit* of the left derived functor. Dually (more precisely "*co*-dually", i.e. just reversing 2-cells), one defines a *right derived functor* (RF, ρ) of F, whose universal 2-arrow is called its *unit*.

If $\mathcal{D}$ also comes with a class of weak equivalences and localisation $H'\colon \mathcal{D} \to \operatorname{Ho}\mathcal{D}$, we will usually be more interested in the *total left derived functor*, which is the left derived functor $\mathbb{L}F := L(H' \circ F)$ of $H' \circ F$. Dually for the *total right derived functor* $\mathbb{R}F := R(H' \circ F)$.

(2.2) **Remark.** Being a terminal object in the category $H^* \downarrow F$, a left derived functor (LF, λ) is unique up to unique isomorphism. In particular, for a fixed LF, its counit λ is unique up to precomposition with α_H for some unique automorphism α of LF.

As a special case of this universal property, we obtain that the operations L (resp. $\mathbb{L}$) and R (resp. $\mathbb{R}$) are functorial.

(2.3) **Definition.** Let F, $F'\colon \mathcal{C} \to \mathcal{D}$ have right Kan extensions (LF, λ) and (LF', λ') along some $H\colon \mathcal{C} \to \mathcal{H}$ and $\sigma\colon F \Rightarrow F'$. Then there is a unique $L\sigma := \overline{\sigma \circ \lambda}\colon LF \Rightarrow LF'$ such that $\sigma \circ \lambda = \lambda' \circ (L\sigma)_H$. Dually, if G, $G'\colon \mathcal{C} \to \mathcal{D}$ have left Kan extensions (RG, ρ) and (RG', ρ') along H and $\tau\colon G \Rightarrow G'$ then there is a unique $R\tau := \overline{\rho' \circ \sigma}\colon RG \Rightarrow RG'$ such that $(R\tau)_H \circ \rho = \rho' \circ \tau$. The unicity of these shows that L and R strictly preserve vertical compositions of natural transformations as well as identities. Similarly for the totally derived versions $\mathbb{L}\sigma := L(H'\sigma)$ and $\mathbb{R}\tau := R(H'\tau)$ for $H'\colon \mathcal{D} \to \mathcal{H}'$.

There is the following convenient external characterisation of derived functors. Unfortunately, there are some class-theoretic difficulties involved (cf. the remark below).

(2.4) **Proposition.** If (LF, λ) is a right Kan extension of $F\colon \mathcal{C} \to \mathcal{D}$ along some functor $H\colon \mathcal{C} \to \mathcal{H}$ then

$$\varphi_L\colon \operatorname{Nat}(L, LF) \to \operatorname{Nat}(L \circ H, F),\ \tau \mapsto \lambda \circ \tau_H$$

is a bijection, natural in $L\colon \mathcal{H} \to \mathcal{D}$. Conversely, if there is a functor $LF\colon \mathcal{H} \to \mathcal{D}$ together with a natural family of bijections

$$\varphi_L\colon \operatorname{Nat}(L, LF) \to \operatorname{Nat}(L \circ H, F)$$

indexed by all functors $L\colon \mathcal{H} \to \mathcal{D}$ then $(LF, \varphi_{LF}\mathrm{id}_{LF})$ is a left derived functor of F. Moreover, these two constructions are inverse to each other.

Proof. The map $\tau \mapsto \lambda \circ (H\tau)$ being a bijection is just the universal property of a right Kan extension and naturality is easy. Conversely, if there is a natural family $(\varphi_L)_L$ of bijections as in the proposition, we put $\lambda := \varphi_{LF}\mathrm{id}_{LF}$. Now for any $L\colon \mathcal{H} \to \mathcal{D}$ together with $\tau\colon L \circ H \Rightarrow F$ there is a unique $\bar\tau\colon L \Rightarrow LF$ such that $\varphi_L\bar\tau = \tau$. Chasing id_{LF} around the naturality square

$$\begin{tikzcd}
\operatorname{Nat}(LF, LF) \arrow[r, "\varphi_{LF}"] \arrow[d, "\bar\tau^*"'] & \operatorname{Nat}(LF \circ H, F) \arrow[d, "\bar\tau_H^*"] \\
\operatorname{Nat}(L, LF) \arrow[r, "\varphi_L"'] & \operatorname{Nat}(L \circ H, F)
\end{tikzcd} \tag{2.5}$$

we see that $\bar\tau$ is the unique transformation $L \Rightarrow LF$ such that $\lambda \circ \bar\tau_H = \tau$. □

(2.6) **Remark.** In ordinary NBG class theory, one cannot properly formalise the above proposition because in general, there is no class of all functors $L\colon \mathcal{H} \to \mathcal{D}$. The obvious remedies for this (apart from never using the external characterisation) is to assume the existence of a universe or switch to a higher class theory that has 2-classes, whose relation to ordinary classes is the same as that of classes to sets. Another solution is to observe that the family $(\varphi_L)_L$ is completely determined by φ_{LF} and making the following definition.

(2.7) **Definition.** For $F\colon \mathcal{C} \to \mathcal{D}$, an *(external) right Kan extension* along $H\colon \mathcal{C} \to \mathcal{H}$ is a functor $LF\colon \mathcal{H} \to \mathcal{D}$ together with a bijection $\varphi_{LF}\colon \operatorname{Nat}(LF, LF) \to \operatorname{Nat}(LF \circ H, F)$ such that for every $L\colon \mathcal{H} \to \mathcal{D}$ there is some bijection $\varphi_L\colon \operatorname{Nat}(L, LF) \to \operatorname{Nat}(L \circ H, F)$ making the diagram (2.5) commute. It follows that the φ_L are unique and the proposition tells us that the two definitions of a left derived functor are equivalent. We shall still informally speak of a natural family of bijections although we cannot form an indexing class for it. Dually for *(external) left Kan extensions*. Again if $H\colon \mathcal{C} \to \operatorname{Ho}\mathcal{C}$ is a localisation functor, we speak of *(external) left* and *(external) right derived functors*.

Of particular importance for homotopy theory are the so-called *absolute* Kan extensions, i.e. those preserved by any morphism. Their importance comes from the fact that all derived functors that arise from Quillen adjunctions are absolute and that there is a nice interplay between adjunctions and absolute derived functors.

(2.8) **Definition.** A right Kan extension (LF, λ) of a functor $F\colon \mathcal{C} \to \mathcal{D}$ along $H\colon \mathcal{C} \to \mathcal{H}$ is called *absolute* iff for every $G\colon \mathcal{D} \to \mathcal{E}$, the composite $(G \circ LF, G\lambda)$ is a right Kan extension of $G \circ F$ along H. Dually for left Kan extensions.

(2.9) **Example.** Given two functors $F\colon \mathcal{C} \rightleftarrows \mathcal{D} : G$ then $F \dashv G$ with unit η iff (G, η) is an absolute left Kan extension of $\mathrm{id}_{\mathcal{C}}$ along F (cf. (4.1)). Dually, $F \dashv G$ with counit ε iff (F, ε) is an absolute right Kan extension of $\mathrm{id}_{\mathcal{D}}$ along G.

(2.10) **Example.** If $\mathcal{C}$ and $\mathcal{D}$ are two categories, each equipped with a class of weak equivalences and $F\colon \mathcal{C} \to \mathcal{D}$ is homotopical then $\operatorname{Ho} F\colon \operatorname{Ho}\mathcal{C} \to \operatorname{Ho}\mathcal{D}$ together with the identity transformation is both an absolute total left and an absolute total right derived functor of F.

(2.11) **Remark.** We have already seen that taking Kan extensions preserves vertical compositions of natural transformations. In the absolute case, this is also true for horizontal compositions (which doesn't even make sense in the non-absolute case). To wit, consider two pairs of parallel functors F, $F'\colon \mathcal{C} \to \mathcal{D}$, E, $E'\colon \mathcal{D} \to \mathcal{E}$ together with $\sigma\colon F \Rightarrow F'$, $\tau\colon E \Rightarrow E'$ such that F and F' have absolute right Kan extensions (LF, λ) and (LF', λ') along an $H\colon \mathcal{C} \to \mathcal{H}$. Then $L(\tau \star \sigma) = \tau \star L\sigma$ (where $\star$ is horizontal composition of natural transformations).

Proof. For an arbitrary object $C \in \mathcal{C}$, we easily calculate

$$\begin{aligned} E'\lambda'_C \circ (\tau \star L\sigma)_{HC} &= E'\lambda'_C \circ E'(L\sigma)_{HC} \circ \tau_{(LF)HC} = E'\sigma_C \circ E'\lambda_C \circ \tau_{(LF)HC} \\ &\overset{\tau\text{ nat}}{=} E'\sigma_C \circ \tau_{FC} \circ E\lambda_C = (\tau \star \sigma)_C \circ G\lambda_C. \end{aligned}$$ □

As an obvious next step, we can adapt the external characterisation (2.4) to the case of absolute derived functors (again stated in larger generality).

(2.12) **Proposition.** If (LF, λ) is an absolute right Kan extension of $F\colon \mathcal{C} \to \mathcal{D}$ along some functor $H\colon \mathcal{C} \to \mathcal{H}$ then

$$\varphi_{\mathcal{E},E,L}\colon \operatorname{Nat}(L, E \circ LF) \to \operatorname{Nat}(L \circ H, E \circ F),\ \tau \mapsto E\lambda \circ \tau_H$$

is a bijection natural in $\mathcal{E}$, $E\colon \mathcal{D} \to \mathcal{E}$ and $L\colon \mathcal{H} \to \mathcal{E}$. Conversely, if there is $LF\colon \mathcal{H} \to \mathcal{D}$ together with a family of bijections

$$\varphi_{\mathcal{E},E,L}\colon \operatorname{Nat}(L, E \circ LF) \to \operatorname{Nat}(L \circ H, E \circ F)$$

natural in the category $\mathcal{E}$ and the functor $L\colon \mathcal{H} \to \mathcal{E}$ (naturality in $E\colon \mathcal{D} \to \mathcal{E}$ is automatic) then $(LF, \varphi_{\mathcal{D},\mathrm{id}_{\mathcal{D}},LF}\mathrm{id}_{LF})$ is a total left derived functor of F. Moreover, these two constructions are inverse to each other.

Proof. The bijectivity of $\tau \mapsto E\lambda \circ \tau_H$ is just the universal property of the right Kan extension $(E \circ LF, E\lambda)$ and naturality is easy. For the converse claim, we put $\lambda := \varphi_{\mathcal{D},\mathrm{id}_{\mathcal{D}},LF}\mathrm{id}_{LF}$. For $E\colon \mathcal{D} \to \mathcal{E}$, $L\colon \mathcal{H} \to \mathcal{E}$ and $\tau\colon L \circ H \Rightarrow E \circ F$ there is a unique $\bar{\tau}\colon L \Rightarrow E \circ LF$ such that $\varphi_{\mathcal{E},E,L}\bar{\tau} = \tau$ and chasing id_{LF} around the commutative diagram

$$\begin{tikzcd}
\operatorname{Nat}(LF, LF) \arrow[r, "\varphi_{\mathcal{D},\mathrm{id}_{\mathcal{D}},LF}"] \arrow[d, "E_*"'] & \operatorname{Nat}(LF \circ H, F) \arrow[d, "E_*"] \\
\operatorname{Nat}(E \circ LF, E \circ LF) \arrow[r, "\varphi_{\mathcal{E},E,E\circ LF}"] \arrow[d, "\bar{\tau}^*"'] & \operatorname{Nat}(E \circ LF \circ H, E \circ F) \arrow[d, "\bar{\tau}^*_H"] \\
\operatorname{Nat}(L, E \circ LF) \arrow[r, "\varphi_{\mathcal{E},E,L}"'] & \operatorname{Nat}(L \circ H, E \circ F)
\end{tikzcd}$$

we obtain that indeed, $\bar{\tau}$ is the unique transformation satisfying $\tau = E\lambda \circ \bar{\tau}_H$. □

(2.13) **Remark.** Again the class-theoretic difficulties can be resolved by noting that the family $(\varphi_{\mathcal{E},E,L})_{\mathcal{E},E,L}$ is completely determined by $\varphi_{\mathcal{D},\mathrm{id}_{\mathcal{D}},LF}$.

Because we shall usually work with total derived functors, let us quickly restate this theorem for the total derived case.

(2.14) **Corollary.** Let $\mathcal{C}$, $\mathcal{D}$ be two categories equipped with weak equivalences and $H_{\mathcal{C}}$, $H_{\mathcal{D}}$ the corresponding localisations. Moreover, let $F\colon \mathcal{C} \to \mathcal{D}$ and $\mathbb{L}F\colon \operatorname{Ho}\mathcal{C} \to \operatorname{Ho}\mathcal{D}$. If $\mathbb{L}F$ is an absolute total left derived functor of F with counit λ then $\tau \mapsto E\lambda \circ \tau_{H_{\mathcal{C}}}$ defines a bijection

$$\varphi_{\mathcal{E},E,L}\colon \operatorname{Nat}(L, E \circ \mathbb{L}F) \cong \operatorname{Nat}(L \circ H_{\mathcal{C}}, E \circ H_{\mathcal{D}} \circ F)$$

natural in $\mathcal{E}$, $E\colon \operatorname{Ho}\mathcal{D} \to \mathcal{E}$ and $L\colon \operatorname{Ho}\mathcal{C} \to \mathcal{E}$. Conversely, if there is such a natural family of bijections then $\mathbb{L}F$ is an absolute total left derived functor of F with counit $\varphi_{\operatorname{Ho}\mathcal{D},\mathrm{id}_{\operatorname{Ho}\mathcal{D}},\mathbb{L}F}\mathrm{id}_{\mathbb{L}F}$. These assignments are mutually inverse. Dually, if we have $G\colon \mathcal{D} \to \mathcal{C}$ and $\mathbb{R}G\colon \operatorname{Ho}\mathcal{D} \to \operatorname{Ho}\mathcal{C}$ such that $\mathbb{R}G$ is an absolute total right derived functor of G with unit ρ then $\tau \mapsto \tau_{H_{\mathcal{D}}} \circ E'\rho$ defines a bijection

$$\psi_{\mathcal{E}',E',R}\colon \operatorname{Nat}(E' \circ \mathbb{R}G, R) \cong \operatorname{Nat}(E' \circ H_{\mathcal{C}} \circ G, R \circ H_{\mathcal{D}})$$

natural in $\mathcal{E}'$, $E'\colon \operatorname{Ho}\mathcal{C} \to \mathcal{E}'$, $R\colon \operatorname{Ho}\mathcal{D} \to \mathcal{E}'$ and conversely given such a natural family of bijections, $\mathbb{R}G$ is an absolute total right derived functor of G with unit $\psi_{\operatorname{Ho}\mathcal{C},\mathrm{id}_{\operatorname{Ho}\mathcal{C}},\mathbb{R}G}\mathrm{id}_{\mathbb{R}G}$. Again, these assignments are mutually inverse. □

3. Construction of Derived Functors

Virtually all derived functors that occur "in nature" are constructed by means of resolutions (a.k.a. replacements, a.k.a. approximations) and we shall quickly abstract these constructions to our context. For this, we fix some category $\mathcal{C}$ equipped with a class of weak equivalences and write $H\colon \mathcal{C} \to \operatorname{Ho}\mathcal{C}$ for its localisation.

(3.1) **Convention.** If not stated otherwise, we will always equip a subcategory $\mathcal{C}_0 \subseteq \mathcal{C}$ with the weak equivalences coming from $\mathcal{C}$. I.e. an arrow in $\mathcal{C}_0$ is a weak equivalence iff it is one in $\mathcal{C}$.

(3.2) **Definition.** Following the nomenclature in [21] (and weakening their notion) a *left deformation retract* of $\mathcal{C}$ is a full subcategory $\mathcal{C}_0 \hookrightarrow \mathcal{C}$ with localisation $H_0\colon \mathcal{C}_0 \to \operatorname{Ho}\mathcal{C}_0$ such that there exist

(a) a function $Q\colon \operatorname{Ob}\mathcal{C} \to \operatorname{Ob}\mathcal{C}_0$ and for every $f\colon C \to C'$ in $\mathcal{C}$ a $Qf\colon QC \to QC'$ in $\mathcal{C}_0$ such that H_0Q defines a functor $H_0Q\colon \mathcal{C} \to \operatorname{Ho}\mathcal{C}_0$ that sends weak equivalences to isomorphisms and thus induces a unique $\widetilde{Q}\colon \operatorname{Ho}\mathcal{C} \to \operatorname{Ho}\mathcal{C}_0$ such that $\widetilde{Q} \circ H = H_0Q$;

(b) for each $C \in \mathcal{C}$ a weak equivalence $q_C\colon QC \to C$ such that for every $f\colon C \to C'$

$$\begin{tikzcd} QC \arrow[r, "Qf"] \arrow[d, "q_C"'] & QC' \arrow[d, "q_{C'}"] \\ C \arrow[r, "f"'] & C' \end{tikzcd} \qquad \text{commutes in } \mathcal{C}.$$

The triple $(\mathcal{C}_0, Q, q)$ is then called a *left deformation retraction* (of $\mathcal{C}$ to $\mathcal{C}_0$). Dually, one defines a *right deformation retract*.

(3.3) **Observation.** If the weak equivalences in $\mathcal{C}$ satisfy 2-out-of-3, then the requirement that H_0Q send weak equivalences to isomorphisms is superfluous. Indeed, for $f\colon C \to C'$ in $\mathcal{C}$ we have $q'_C \circ Qf = f \circ q_C$ and so f is a weak equivalence iff Qf is.

(3.4) **Example.** As already mentioned, our notion of a deformation retract is weaker than the one in [21]. There a left deformation retract is defined as as full subcategory $I\colon \mathcal{C}_0 \hookrightarrow \mathcal{C}$ together with a homotopical functor $Q\colon \mathcal{C} \to \mathcal{C}_0$ and a natural weak equivalence $\sigma\colon I \circ Q \Rightarrow \mathrm{id}_{\mathcal{C}}$.

(3.5) **Example.** For $\mathcal{M}$ a model category, the category of cofibrant objects $\mathcal{M}_c$ together with some chosen cofibrant replacements $q_C\colon QC \to C$ (i.e. QC is cofibrant and q_C an acyclic fibration) and chosen lifts $Qf\colon QC \to QC'$ for $f\colon C \to C'$ forms a left deformation retraction (cf. [22, Lemma 5.1]).

(3.6) **Example.** More generally, if $L\colon \mathcal{M} \leftrightarrows \mathcal{C} : R$ is a *left model approximation* in the sense of [15] (left adjoint on the left), we let $\mathcal{C}_0 \subseteq \mathcal{C}$ be the full subcategory comprising all (objects isomorphic to) images of cofibrant objects in $\mathcal{M}$ under L. Choosing a cofibrant replacement Q for $\mathcal{M}$ as in the last example

$$(C \xrightarrow{f} C') \quad \mapsto \quad (LQRC \xrightarrow{LQRf} LQRC') \qquad \text{together with the} \qquad LQRC \xrightarrow{q^\flat_{RC}} C$$

defines a left deformation retraction of $\mathcal{C}$ to $\mathcal{C}_0$. Indeed, the $q^\flat_{RC}$ are weak equivalences because their adjuncts are and by definition of a left model approximation. Moreover, our cofibrant replacement is functorial when passing to the homotopy category because H_0LQR is just

$$\mathcal{C} \xrightarrow{R} \mathcal{M} \xrightarrow{H_{\mathcal{M}}} \operatorname{Ho}\mathcal{M} \xrightarrow{\widetilde{Q}} \operatorname{Ho}\mathcal{M}_c \xrightarrow{\operatorname{Ho}L} \operatorname{Ho}\mathcal{C}_0$$

(where for the last functor, we used that L sends weak equivalences between cofibrant objects to weak equivalences). Finally, the "naturality squares" for the $q^\flat_{RC}\colon LQRC \to C$ commute because their adjunct squares do.

(3.7) **Example.** Again for a model category $\mathcal{C}$, any functorial factorisation $\mathcal{C}^{[1]} \to \mathcal{C}^{[2]}$ into a cofibration followed by a weak equivalence gives rise to a *cofibrant approximation functor* $Q\colon \mathcal{C} \to \mathcal{C}_c$ by factoring the unique arrow $!_C\colon \varnothing \to C$ for $C \in \mathcal{C}$ into $\varnothing \rightarrowtail QC \to C$.

Just like in (3.4) and (3.7), it is usually convenient to require that Q itself be a functor. However, this has the undesirable consequence of needing to have functorial factorisations on a model category for the theory to apply. It is much more important that Q become functorial when passing to the homotopy category.

(3.8) **Proposition.** Let $(\mathcal{C}_0, Q, q)$ be a left deformation retraction of $\mathcal{C}$ and $I\colon \mathcal{C}_0 \hookrightarrow \mathcal{C}$ the inclusion. Then the families $(Hq_C)_{C\in\mathcal{C}}$ and $(H_0q_{C_0})_{C_0\in\mathcal{C}_0}$ define natural isomorphisms

$$\operatorname{Ho}I \circ \widetilde{Q} \cong \mathrm{id}_{\operatorname{Ho}\mathcal{C}} \qquad \text{and} \qquad \widetilde{Q} \circ \operatorname{Ho}I \cong \mathrm{id}_{\operatorname{Ho}\mathcal{C}_0}.$$

Proof. It suffices to check that the two families define natural isomorphisms

$$\operatorname{Ho}I \circ \widetilde{Q} \circ H = \operatorname{Ho}I \circ H_0Q \overset{\cong}{\Rightarrow} H \qquad \text{and} \qquad \widetilde{Q} \circ \operatorname{Ho}I \circ H_0 = H_0Q \circ I \overset{\cong}{\Rightarrow} H_0,$$

which is simple because we already have "naturality squares" in $\mathcal{C}$. For example, the naturality of the first family corresponds to the commutativity of

$$\begin{array}{ccccccccc}
C & & (\operatorname{Ho}I)\widetilde{Q}HC & = & (\operatorname{Ho}I)H_0QC & = & HIQC & = & HQC \xrightarrow{Hq_C} HC \\
{\scriptstyle f}\downarrow & & {\scriptstyle (\operatorname{Ho}I)\widetilde{Q}Hf}\downarrow & & {\scriptstyle (\operatorname{Ho}I)H_0Qf}\downarrow & & {\scriptstyle HIQf}\downarrow & & {\scriptstyle HQf}\downarrow \qquad \downarrow{\scriptstyle Hf} \\
C' & & (\operatorname{Ho}I)\widetilde{Q}HC' & = & (\operatorname{Ho}I)H_0QC' & = & HIQC' & = & HQC' \xrightarrow[Hq_{C'}]{} HC'
\end{array}$$

for all $f\colon C \to C'$ in $\mathcal{C}$. □

(3.9) **Remark.** Because $\operatorname{Ho} I$ has now been shown to be fully faithful (in fact, it is even an equivalence), we will usually view $\operatorname{Ho}\mathcal{C}_0$ as a full subcategory of $\operatorname{Ho}\mathcal{C}$ and under this identification $H_0 = H|_{\mathcal{C}_0}$.

As already mentioned at the beginning of this section, the importance of deformation retracts comes from the fact that we can use them to "deform" functors in order to obtain derived ones.

(3.10) **Definition.** Let $H_{\mathcal{C}}\colon \mathcal{C} \to \operatorname{Ho}\mathcal{C}$ and $H_{\mathcal{D}}\colon \mathcal{D} \to \operatorname{Ho}\mathcal{D}$ be two localisation functors and $F\colon \mathcal{C} \to \mathcal{D}$. A left deformation retract $\mathcal{C}_0 \subseteq \mathcal{C}$ is called a *left F-deformation retract* iff the restriction $F|_{\mathcal{C}_0}\colon \mathcal{C}_0 \to \mathcal{D}$ is homotopical. Consequently, a left deformation retraction $(\mathcal{C}_0, Q, q)$ is called a *left F-deformation retraction* iff $\mathcal{C}_0$ is a left deformation retract of F. Dually for right deformation retracts.

(3.11) **Theorem.** Let $(\mathcal{C}_0, Q, q)$ be a left deformation retraction of $\mathcal{C}$ and $F\colon \mathcal{C} \to \mathcal{D}$ a functor that maps weak equivalences in $\mathcal{C}_0$ to isomorphisms. Then

$$LF\colon \operatorname{Ho}\mathcal{C} \xrightarrow{\widetilde{Q}} \operatorname{Ho}\mathcal{C}_0 \xrightarrow{(F|_{\mathcal{C}_0})^-} \mathcal{D} \quad \text{together with} \quad (Fq_C\colon FQC \to FC)_{C\in\mathcal{C}}$$

is an absolute left derived functor of F. In particular, if $\mathcal{D}$ is also equipped with a class of weak equivalences and $(\mathcal{C}_0, Q, q)$ is a left F-deformation retraction of $\mathcal{C}$ then

$$\mathbb{L}F\colon \operatorname{Ho}\mathcal{C} \xrightarrow{\widetilde{Q}} \operatorname{Ho}\mathcal{C}_0 \xrightarrow{\operatorname{Ho}(F|_{\mathcal{C}_0})} \operatorname{Ho}\mathcal{D} \quad \text{together with} \quad (H_{\mathcal{D}}Fq_C\colon FQC \to FC)_{C\in\mathcal{C}}$$

is an absolute total left derived functor of F.

Proof. Let us write $\bar{F} := (F|_{\mathcal{C}_0})^-\colon \operatorname{Ho}\mathcal{C}_0 \to \mathcal{D}$ for the functor induced by $F|_{\mathcal{C}_0}$. We need to check that the Fq_C are natural in $C \in \mathcal{C}$; i.e. that the square

(3.12)
$$\begin{tikzcd} FQC \arrow[r, "\bar{F}\widetilde{Q}Hf"] \arrow[d, "Fq_C"'] & FQC' \arrow[d, "Fq_{C'}"] \\ FC \arrow[r, "Ff"'] & FC' \end{tikzcd}$$

in $\mathcal{D}$ commutes for all $f\colon C \to C'$ in $\mathcal{C}$, which is clear because $\bar{F}\widetilde{Q}Hf = \bar{F}HQf = FQf$. To see that this transformation is universal, let $G\colon \operatorname{Ho}\mathcal{C} \to \mathcal{D}$ and $\tau\colon G \circ H \Rightarrow F$. If there is $\bar{\tau}\colon G \Rightarrow LF$ such that $Fq \circ \bar{\tau}_H = \tau$ then

$$GHC \xrightarrow{\bar{\tau}_{HC}} LFHC = FQC \xrightarrow{Fq_C} FC \quad = \quad GHC \xrightarrow{\tau_C} FC \qquad \text{for all } C \in \mathcal{C}.$$

For $C \in \mathcal{C}_0$, the arrow Fq_C is invertible and so $\bar{\tau}_{HC} = (Fq_C)^{-1}\tau_C$. For the general case, the naturality of $\bar{\tau}$ gives a commutative diagram in $\mathcal{D}$ as follows:

$$\begin{tikzcd} GHQC \arrow[r, "\bar{\tau}_{HQC}"] \arrow[d, "GHq_C"'] & LFHQC \arrow[r, equal] \arrow[d, "LFHq_C"] & FQQC \arrow[d, "FQq_C"] \\ GHC \arrow[r, "\bar{\tau}_{HC}"'] & LFHC \arrow[r, equal] & FQC \end{tikzcd} .$$

But GHq_C is invertible, so that

$$\bar{\tau}_{HC} = FQq_C \circ \bar{\tau}_{HQC} \circ (GHq_C)^{-1} = FQq_C \circ (Fq_{QC})^{-1} \circ \tau_{QC} \circ (GHq_C)^{-1}.$$

We check that this does indeed define a natural transformation $\bar{\tau}_H : G \circ H \Rightarrow LF \circ H$ (thus determining $\bar{\tau}$) by considering the following commutative diagram for $f : C \to C'$ in $\mathcal{C}$ (where for the last square, we note that while Q is not a functor, $FQ = \bar{F} H_0 Q = \bar{F}\widetilde{Q}H$ is one):

$$\begin{tikzcd} GHC \arrow[r, "(GHq_C)^{-1}"] \arrow[d, "GHf"'] & GHQC \arrow[r, "\tau_{QC}"] \arrow[d, "GHQf"'] & FQC \arrow[r, "(Fq_{QC})^{-1}"] \arrow[d, "FQf"'] & FQQC \arrow[r, "FQq_C"] \arrow[d, "FQQf"] & FQC \arrow[d, "FQf"] \\ GHC' \arrow[r, "(GHq_{C'})^{-1}"'] & GHQC' \arrow[r, "\tau_{QC'}"'] & FQC' \arrow[r, "(Fq_{QC'})^{-1}"'] & FQQC' \arrow[r, "FQq_{C'}"'] & FQC' \end{tikzcd} .$$

Moreover, $\bar{\tau}$ does indeed satisfy $Fq \circ \bar{\tau}_H = \tau$ because if $C \in \mathcal{C}$ then

$$\begin{aligned} Fq_C \circ \bar{\tau}_{HC} &= Fq_C \circ FQq_C \circ (Fq_{QC})^{-1} \circ \tau_{QC} \circ (GHq_C)^{-1} \\ &\overset{Fq\text{ nat}}{=} Fq_C \circ Fq_{QC} \circ (Fq_{QC})^{-1} \circ \tau_{QC} \circ (GHq_C)^{-1} \\ &= Fq_C \circ \tau_{QC} \circ (GHq_C)^{-1} \\ &\overset{\tau\text{ nat}}{=} \tau_C \circ GHq_C \circ (GHq_C)^{-1} \end{aligned}$$

(for the second equality, put $C \rightsquigarrow QC$, $C' \rightsquigarrow C$, $f \rightsquigarrow q_C$ in (3.12) above). Finally, for the absoluteness claim, observe that if $F' : \mathcal{D} \to \mathcal{E}$ is another functor then $F' \circ F$ again maps weak equivalences in $\mathcal{C}_0$ to isomorphisms. □

(3.13) **Example.** As already mentioned in (3.5), the full subcategory of cofibrant objects in a model category $\mathcal{M}$, together with some chosen cofibrant replacements, is a left deformation retract of $\mathcal{M}$. Consequently, a functor $F : \mathcal{M} \to \mathcal{N}$ that sends weak equivalences between cofibrant objects to isomorphisms has an absolute left derived functor in the above manner. By Ken Brown's lemma, a left Quillen functor $F : \mathcal{M} \to \mathcal{N}$ between two model categories (which preserves cofibrations and acyclic cofibrants) has an absolute total left derived functor.

(3.14) **Example.** More generally, in [15], a left model approximation $L : \mathcal{M} \leftrightarrows \mathcal{C} : R$ is called *good* for a functor $F : \mathcal{C} \to \mathcal{D}$ between two categories with weak equivalences iff $F \circ L$ is left Quillen (i.e. sends weak equivalences between cofibrant objects to weak equivalences). This immediately implies that $\mathcal{C}_0 \subseteq \mathcal{C}$ as in (3.6) is a left F-deformation retract and so we obtain $\mathbb{L}F$.

An important instance of this is that if $\mathcal{M}$ is a model category and $F : \mathcal{I} \to \mathcal{J}$ a functor between small categories, there is a Bousfield-Kan left model approximation

$$\mathrm{Fun}^b(\mathbf{N}(\mathcal{I}), \mathcal{M}) \leftrightarrows \mathrm{Fun}(\mathcal{I}, \mathcal{M}),$$

which is good for the functor $F_!$ given by taking the left Kan extension along F. In particular, there is a homotopy left Kan extension functor $\mathbb{L}F_! : \mathcal{M}^{\mathcal{I}} \to \mathcal{M}^{\mathcal{J}}$.

4. The Yoga of Mates

As is well-known, adjunctions are just a special instance of Kan extensions (then again, what isn't?) and the external characterisation makes it really obvious.

(4.1) **Proposition.** Let $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ be two functors. Then $F \dashv G$ with counit ε iff (F, ε) is an absolute right Kan extension of $\mathrm{id}_{\mathcal{D}}$ along G and dually, $F \dashv G$ with unit η iff (G, η) is an absolute left Kan extension of $\mathrm{id}_{\mathcal{C}}$ along F. Indeed, if $F \dashv G$ with unit η and counit ε then $\tau \mapsto E\varepsilon \circ \tau_G$ defines a bijection

$$\varphi_{\varepsilon,L,E} : \mathrm{Nat}(L, E \circ F) \cong \mathrm{Nat}(L \circ G, E) \qquad \text{with inverse} \qquad \sigma_F \circ L\eta \leftarrow\!\shortmid \sigma$$

and this is natural in $\mathcal{E}$, $L\colon \mathcal{C} \to \mathcal{E}$ and $E\colon \mathcal{D} \to \mathcal{E}$. Conversely, given such a natural family of bijections then $F \dashv G$ with unit $\varphi^{-1}_{\mathcal{C},\mathrm{id}_{\mathcal{C}},G}(\mathrm{id}_G)$ and counit $\varphi_{\mathcal{D},F,\mathrm{id}_{\mathcal{D}}}(\mathrm{id}_F)$. These two constructions are inverse to each other. □

Proof. The whole proof is simply about finding the correct naturality conditions to apply but let's do it anyway for the sake of completeness. Starting with an adjunction $(F \dashv G, \eta, \varepsilon)$ and defining $\varphi_{\mathcal{E},L,E}$ as in the claim, the naturality of φ in $\mathcal{E}$ and L is immediate. For the naturality in E, we assume that we have $\alpha\colon E \Rightarrow E'$ and need to show that

$$\begin{tikzcd}
\mathrm{Nat}(L, E\circ F) \arrow[r, "\varphi_{\mathcal{E},L,E}"] \arrow[d, "(\alpha_F)_*"'] & \mathrm{Nat}(L\circ G, E) \arrow[d, "\alpha_*"] \\
\mathrm{Nat}(L, E'\circ F) \arrow[r, "\varphi_{\mathcal{E},L,E'}"'] & \mathrm{Nat}(L\circ G, E')
\end{tikzcd}$$

commutes. Chasing some τ through the square, this means that

$$\alpha \circ E\varepsilon \circ \tau_G = E'\varepsilon \circ \alpha_{FG} \circ \tau_G,$$

which follows from naturality of α. Moreover, the assignments

$$\tau \mapsto E\varepsilon \circ \tau_G \qquad \text{and} \qquad \sigma_F \circ L\eta \mapsfrom \sigma$$

are indeed inverse to each other, because, starting with τ, we have

$$\tau \longmapsto E\varepsilon \circ \tau_G \longmapsto (E\varepsilon \circ \tau_G)_F \circ L\eta \overset{\tau\text{ nat}}{=} E\varepsilon_{FC} \circ EF\eta_C \circ \tau_C \overset{\Delta\text{-id}}{=} \tau_C$$

and similarly the other way around.

Conversely, starting with a natural family of $\varphi_{\mathcal{E},L,E}$, we define $\eta := \varphi^{-1}_{\mathcal{C},\mathrm{id}_{\mathcal{C}},G}(\mathrm{id}_G)$ and $\varepsilon := \varphi_{\mathcal{D},F,\mathrm{id}_{\mathcal{D}}}(\mathrm{id}_F)$ as in the claim and need to verify the triangle identities. For example, to show $\varepsilon_F \circ F\eta = \mathrm{id}_F$ (the other one being similar), we just consider the commutative diagram

$$\begin{tikzcd}
\mathrm{Nat}(\mathrm{id}_{\mathcal{C}}, G\circ F) \arrow[r, "F_*"] \arrow[d, "\varphi_{\mathcal{C},\mathrm{id}_{\mathcal{C}},G}"'] & \mathrm{Nat}(F, F\circ G\circ F) \arrow[r, "(\varepsilon_F)_*"] \arrow[d, "\varphi_{\mathcal{D},F,FG}"'] & \mathrm{Nat}(F, F) \arrow[d, "\varphi_{\mathcal{D},F,\mathrm{id}_{\mathcal{D}}}"'] \\
\mathrm{Nat}(G, G) \arrow[r, "F_*"'] & \mathrm{Nat}(F\circ G, F\circ G) \arrow[r, "\varepsilon_*"'] & \mathrm{Nat}(F\circ G, \mathrm{id}_{\mathcal{D}}),
\end{tikzcd}$$

where the left square commutes by naturality of $\varphi_{\mathcal{E},L,E}$ in the variable $\mathcal{E}$, while the right square commutes by naturality in E. Chasing $\eta = \varphi^{-1}_{\mathcal{C},\mathrm{id}_{\mathcal{C}},G}(\mathrm{id}_G)$ around the diagram, we find

$$\varphi_{\mathcal{D},F,\mathrm{id}_{\mathcal{D}}}(\varepsilon_F \circ F\eta) = \varepsilon = \varphi_{\mathcal{D},F,\mathrm{id}_{\mathcal{D}}}(\mathrm{id}_F)$$

and the claim follows.

Finally, the two assignments $(\eta, \varepsilon) \mapsto \varphi$ and $(\eta, \varepsilon) \mapsfrom \varphi$ as in the claim are mutually inverse, because, starting from (η, ε), constructing $\varphi_{\mathcal{E},L,E}\colon \tau \mapsto E\varepsilon \circ \tau_G$ and taking the associated unit and counit, the new counit is

$$\varphi_{\mathcal{D},F,\mathrm{id}_{\mathcal{D}}}(\mathrm{id}_F) = \mathrm{id}_{\mathcal{D}}\varepsilon \circ (\mathrm{id}_F)_G = \varepsilon,$$

which also shows that the new unit is again η because the counit determines the unit and vice versa. Conversely, starting with φ, and taking the associated unit $\eta := \varphi^{-1}_{\mathcal{C},\mathrm{id}_{\mathcal{C}},G}(\mathrm{id}_G)$ and counit $\varepsilon := \varphi_{\mathcal{D},F,\mathrm{id}_{\mathcal{D}}}(\mathrm{id}_F)$, the associated natural family φ' is defined as

$$\varphi'_{\mathcal{E},L,E}\colon \tau \mapsto E\varepsilon \circ \tau_G = E\varphi_{\mathcal{D},F,\mathrm{id}_{\mathcal{D}}}(\mathrm{id}_F) \circ \tau_G.$$

Now, τ is natural transformation $L \Rightarrow E \circ F$ and by naturality of $\varphi_{\mathcal{E},L,E}$ in the variable L, the square

$$\begin{tikzcd}
\mathrm{Nat}(E \circ F, E \circ F) \arrow[r, "\varphi_{\mathcal{E},E\circ F,E}"] \arrow[d, "\tau^*"'] & \mathrm{Nat}(E \circ F \circ G, E) \arrow[d, "(\tau_G)^*"] \\
\mathrm{Nat}(L, E \circ F) \arrow[r, "\varphi_{\mathcal{E},L,E}"'] & \mathrm{Nat}(L \circ G, E)
\end{tikzcd}$$

commutes. Chasing $\mathrm{id}_{E\circ F}$ through the square, we find that

$$\varphi'_{\mathcal{E},L,E}(\tau) = \varphi_{\mathcal{E},L,E}(\tau \circ \mathrm{id}_{E\circ F}) = \varphi_{\mathcal{E},L,E}(\tau). \qquad \square$$

As always, by dualising (really *op*-dualising, i.e. inverting 1-arrows) we get an alternative external characterisation of adjunctions, which exhibits adjoints as *Kan lifts* and also fixes the counter-intuitive aspect that in the above formula, the left adjoint appears on the right.

(4.2) **Proposition.** Given two functors $F\colon \mathcal{C} \rightleftarrows \mathcal{D} : G$ we have $F \dashv G$ with unit η and counit ε iff $\sigma \mapsto G\sigma \circ \eta_X = \sigma^\sharp$ defines a bijection

$$\mathrm{Nat}(F \circ X, Y) \cong \mathrm{Nat}(X, G \circ Y) \qquad \text{with inverse} \qquad \tau^\flat = \varepsilon_Y \circ F\tau \mapsfrom \tau,$$

which is then natural in $\mathcal{E}$, $X\colon \mathcal{E} \to \mathcal{C}$ and $Y\colon \mathcal{E} \to \mathcal{D}$. Conversely, given such a natural family of bijections then $F \dashv G$ with unit $\psi_{\mathcal{C},\mathrm{id}_{\mathcal{C}},F}(\mathrm{id}_F)$ and counit $\psi^{-1}_{\mathcal{D},G,\mathrm{id}_{\mathcal{D}}}(\mathrm{id}_G)$. These two constructions are inverse to each other.

Combining the two bijections from these propositions leads to the well-known *mating-bijection*

(4.3)
$$\begin{array}{rcl} \sigma & \mapsto & \sigma^{\infty} \\ \mathrm{Nat}(F' \circ X, Y \circ F) & \cong & \mathrm{Nat}(X \circ G, G' \circ Y) \\ {}^{\infty}\tau & \mapsfrom & \tau \end{array}$$

for categories and functors

$$\begin{tikzcd}
\mathcal{C} \arrow[r, shift left, "F"] \arrow[r, phantom, "\perp"] \arrow[d, "X"'] & \mathcal{D} \arrow[l, shift left, "G"] \arrow[d, "Y"] \\
\mathcal{C}' \arrow[r, shift left, "F'"] \arrow[r, phantom, "\perp"] & \mathcal{D}' \arrow[l, shift left, "G'"]
\end{tikzcd},$$

which is natural in X and Y and is explicitly given by

$$\sigma \mapsto G'Y\varepsilon \circ G'\sigma_G \circ \eta'_{XG} = (Y\varepsilon \circ \sigma_G)^\sharp, \qquad (\tau_F \circ X\eta)^\flat = \varepsilon'_{YF} \circ F'\tau_F \circ F'X\eta \mapsfrom \tau.$$

(4.4) **Definition.** We write $\sigma \infty \tau$ and say that $\sigma\colon F'X \Rightarrow YF$ and $\tau\colon XG \Rightarrow G'Y$ are *mates* (we should really say that $\sigma = {}^{\infty}\tau$ is the *left mate* and $\tau = \sigma^{\infty}$ is the *right mate*) iff they correspond to each other under this mating bijection. For the special case where X and Y are identities, we will occasionally use Mac Lane's nomenclature from [38] and speak of *conjugate* transformations.

(4.5) **Remark.** A useful consequence of the mating bijection's naturality in X and Y is that for $\alpha\colon X \Rightarrow X'$, $\beta\colon Y \Rightarrow Y'$ and two squares of natural transformations

$$\begin{tikzcd}
F' \circ X \arrow[r, "\sigma"] \arrow[d, "F\alpha"'] & Y \circ F \arrow[d, "\beta_F"] \\
F' \circ X' \arrow[r, "\sigma'"'] & Y' \circ F
\end{tikzcd}
\qquad
\begin{tikzcd}
X \circ G \arrow[r, "\tau"] \arrow[d, "\alpha_G"'] & G' \circ Y \arrow[d, "G'\beta"] \\
X' \circ G \arrow[r, "\tau'"'] & G' \circ Y'
\end{tikzcd}$$

where the two horizontal pairs are mates, $\beta_F \circ \sigma$ and $G'\beta \circ \tau$ as well as $\sigma' \circ F\alpha$ and $\tau' \circ \alpha_G$ are again mates. Consequently, the left-hand square commutes iff the right-hand square commutes.

Due to a lack of references, we shall briefly establish some standard results about mates, which one expects to be true and whose proofs are purely formal (whence the term *yoga*).

(4.6) **Example.** If X and Y are identities and the two adjunctions $F \dashv G$, $F' \dashv G'$ are the same then $\mathrm{id}_F \circledcirc \mathrm{id}_G$, which is just a complicated way to state the triangle identities. More generally, if the two adjunctions are the same, while X and Y are endofunctors equipped with $\alpha\colon X \Rightarrow \mathrm{id}_{\mathcal{C}}$ and $\beta\colon \mathrm{id}_{\mathcal{D}} \Rightarrow Y$ then by naturality of the mating bijection, $\beta_F \circ F\alpha \circledcirc G\beta \circ \alpha_G$. Dually for the directions of α and β reversed.

(4.7) **Proposition.** In the same situation as in the definition, for two natural transformations $\sigma\colon F'X \Rightarrow YF$ and $\tau\colon XG \Rightarrow G'Y$, the following are equivalent:

(a) $\sigma \circledcirc \tau$, i.e. $\sigma = (\tau_F \circ X\eta)^\flat$ or equivalently $\tau = (Y\varepsilon \circ \sigma_G)^\sharp$;

(b) $\sigma^\sharp = G'\sigma \circ \eta'_X = \tau_F \circ X\eta$ or equivalently $\tau^\flat = \varepsilon'_Y \circ F'\tau = Y\varepsilon \circ \sigma_G$;

(c) for all $C \in \mathcal{C}$ and all $D \in \mathcal{D}$ the rectangle

$$\begin{tikzcd}
\mathcal{D}(FC, D) \arrow[r, "\cong"] \arrow[d, "Y"'] & \mathcal{C}(C, GD) \arrow[d, "X"] \\
\mathcal{D}'(YFC, YD) \arrow[d, "\sigma_C^*"'] & \mathcal{C}'(XC, XGD) \arrow[d, "(\tau_D)_*"] \\
\mathcal{D}'(F'XC, YD) \arrow[r, "\cong"'] & \mathcal{C}'(XC, G'YD)
\end{tikzcd}$$

commutes, where the horizontal arrows are the tuning bijections of the two adjunctions.

Proof. "(a) $\Leftrightarrow$ (b)": Trivial.
"(b) $\Leftrightarrow$ (c)": Condition (b) is just the commutativity of the diagram for $D = FC$ and $C = GD$ respectively. Conversely, this implies the commutativity for all C and D by naturality of the tuning bijection for the adjunction. □

(4.8) **Proposition.** Taking mates is compatible with vertical and horizontal pasting of squares in the following sense:

(a) Given categories and functors

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "X"] \arrow[d, shift right, "F"'] & \mathcal{C}' \arrow[r, "X'"] \arrow[d, shift right, "F'"'] & \mathcal{C}'' \arrow[d, shift right, "F''"'] \\
\mathcal{D} \arrow[r, "Y"'] \arrow[u, shift right, "G"'] \arrow[u, phantom, "\dashv"] & \mathcal{D}' \arrow[r, "Y'"'] \arrow[u, shift right, "G'"'] \arrow[u, phantom, "\dashv"] & \mathcal{D}'' \arrow[u, shift right, "G''"'] \arrow[u, phantom, "\dashv"]
\end{tikzcd}$$

together with natural transformations $\sigma\colon F'X \Rightarrow YF$ and $\sigma'\colon F''X' \Rightarrow Y'F'$ having mates $\tau = \sigma^\circledcirc\colon XG \Rightarrow G'Y$ and $\tau' = \sigma'^\circledcirc\colon X'G' \Rightarrow G''Y'$ respectively then $Y'\sigma \circ \sigma'_X$ and $\tau'_Y \circ X'\tau$ are again mates. In particular, if X, X', Y and Y' are identities then $\sigma \circ \sigma'$ and $\tau' \circ \tau$ are conjugate.

(b) Given categories and functors

$$\begin{tikzcd}
\mathcal{C} \arrow[r, shift left, "F_1"] \arrow[d, "X"'] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, shift left, "G_1"] \arrow[r, shift left, "F_2"] \arrow[d, "Y"'] \arrow[r, phantom, "\perp"] & \mathcal{E} \arrow[l, shift left, "G_2"] \arrow[d, "Z"] \\
\mathcal{C}' \arrow[r, shift left, "F_1'"] \arrow[r, phantom, "\perp"] & \mathcal{D}' \arrow[l, shift left, "G_1'"] \arrow[r, shift left, "F_2'"] \arrow[r, phantom, "\perp"] & \mathcal{E}' \arrow[l, shift left, "G_2'"]
\end{tikzcd}$$

together with natural transformations $\sigma_1\colon F_1'X \Rightarrow YF_1$ and $\sigma_2\colon F_2'Y \Rightarrow ZF_2$ having mates $\tau_1\colon XG_1 \Rightarrow G_1'Y$ and $\tau_2\colon YG_2 \Rightarrow G_2'Z$ then $\sigma_{2F_1} \circ F_2'\sigma_1$ and $G_1'\tau_2 \circ \tau_{1G_2}$ are again mates. In particular, if X, Y and Z are identities then $\sigma' \star \sigma$ and $\tau \star \tau'$ are conjugate. Still more specially, if in addition $F_2 = F_2'$ and σ_2 (whence τ_2) is the identity, we obtain that $F_2\sigma_1$ and τ_{1G_2} are conjugate.

Proof. *Ad* (a): Writing (η,ε), (η',ε') and (η'',ε'') for the unit-counit pairs of the three adjunctions we easily calculate

$$\begin{aligned}
(Y'\sigma \circ \sigma'_X)^\sharp &= G''Y'\sigma \circ G''\sigma'_X \circ \eta''_{X'X} = G''Y'\sigma \circ {\sigma'}^\sharp_X \\
&= G''Y'\sigma \circ \tau'_{F'X} \circ X'\eta'_X \overset{\tau'\text{ nat}}{=} \tau'_{YF} \circ X'G'\sigma \circ X'\eta_X \\
&= \tau'_{YF} \circ X'\sigma^\sharp = \tau'_{YF} \circ X'\tau_F \circ X'X\eta.
\end{aligned}$$

Ad (b): Again, writing (η_1,ε_1), (η_2,ε_2) (η_1',ε_1') and (η_2',ε_2') for the unit-counit pairs this is just a routine calculation:

$$\begin{aligned}
(\sigma_{2F_1} \circ F_2'\sigma_1)^\sharp &= G_1'G_2'\sigma_{2F_1} \circ G_1'G_2'F_2'\sigma_1 \circ (G_1'\eta'_{2F_1'} \circ \eta_1')_X \\
&\overset{\eta_2'\text{ nat}}{=} G_1'G_2'\sigma_{2F_1} \circ G_1'\eta'_{2YF_1} \circ G_1'\sigma_1 \circ \eta'_{1X} \\
&= G_1'\sigma^\sharp_{2F_1} \circ \sigma_1^\sharp = G_1'\tau_{2F_2F_1} \circ G_1'Y\eta_{2F_1} \circ \tau_{1F_1} \circ X\eta_1 \\
&\overset{\tau_1\text{ nat}}{=} G_1'\tau_{2F_2F_1} \circ \tau_{1G_2F_2F_1} \circ XG_1\eta_{2F_1} \circ X\eta_1 \\
&= \big(G_1'\tau_2 \circ \tau_{1G_1}\big)_{F_2F_1} \circ X(G_1\eta_{2F_1} \circ \eta_1).
\end{aligned}$$
□

(4.9) **Corollary.** Given two adjunctions $(F \dashv G, \eta, \varepsilon)$, $(F' \dashv G', \eta', \varepsilon')\colon \mathcal{C} \rightleftarrows \mathcal{D}$ and conjugate transformations $\sigma\colon F \Rightarrow F'$, $\tau\colon G' \Rightarrow G$ then σ is an isomorphism iff τ is one.

Proof. Let $\tau'\colon G' \Rightarrow G$ be the mate of σ^{-1}. Then by point (a) in the proposition (with $F'' = F$, $G'' = G$ and X, X', Y, Y' all identities) $\sigma \circ \sigma^{-1} = \mathrm{id}_F$ and $\tau' \circ \tau$ are mates, so that $\tau' \circ \tau = \mathrm{id}_{G'}$. Similarly, $\sigma^{-1} \circ \sigma = \mathrm{id}_{F'}$ and $\tau \circ \tau'$ are mates, which proves our claim. □

5. Beck-Chevalley Condition

Later on, we will find ourselves in the situation where we have a mating square as in (4.3) with a natural isomorphism $\tau\colon XG \cong G'Y$ but where we would really like its mate $\sigma\colon F'X \Rightarrow YF$ to be an isomorphism.

(5.1) **Definition.** Consider a square of categories and functors together with a natural transformation

$$\begin{tikzcd} \mathcal{C} \arrow[d, "X"'] & \mathcal{D} \arrow[l, "G"'] \arrow[d, "Y"] \arrow[dl, phantom, "\Searrow\tau"] \\ \mathcal{C}' & \mathcal{D}' \arrow[l, "G'"] \end{tikzcd}$$

where G and G' have left adjoints

$(F \dashv G, \eta, \varepsilon)$, $(F' \dashv G', \eta', \varepsilon')$.

We then say that the square satisfies the *Beck-Chevalley condition* (or that it's a *Beck-Chevalley square*) iff the mate ${}^{\infty}\tau$ is an isomorphism $F' \circ X \cong Y \circ F$. Dually, there is the *dual Beck-Chevalley condition*, where we start with F, F' and σ and then require the mate σ^{∞} to be an isomorphism. Usually, τ is some sort of canonical isomorphism and is then often not explicitly mentioned. However, in that case possible confusion can arise if the functors X and Y themselves have left adjoints. In that case, we shall speak of the *horizontal* and *vertical Beck-Chevalley condition* according to whether one considers the horizontal or vertical pairs of adjunctions; the case in the above definition being the horizontal one.

(5.2) **Example.** By (4.9), if X and Y are identities then the square from the definition satisfies the Beck-Chevalley condition iff τ is an isomorphism.

(5.3) **Example.** The mate of τ is explicitly given by $\varepsilon'_{YF} \circ F'\tau_F \circ F'X\eta$; so if τ is an isomorphism and F, G' are fully faithful (i.e. η and ε' are isomorphisms) then the square satisfies the Beck-Chevalley condition.

(5.4) **Example.** More importantly for us, if $\mathcal{I}$, $\mathcal{J}$ are index categories and $\mathcal{C}$ is a category with $\mathcal{I}$-colimits then $\mathcal{C}^{\mathcal{J}}$ has $\mathcal{I}$-colimits, too. A colimit functor is given by

$$\operatorname{colim}\colon \mathcal{C}^{\mathcal{I}\times\mathcal{J}} \cong (\mathcal{C}^{\mathcal{J}})^{\mathcal{I}} \to \mathcal{C}^{\mathcal{J}},\ X \mapsto \Big(J \mapsto \operatorname{colim} X(-,J)\Big)$$

(i.e. colimits are calulcated pointwise). If η' is the unit of the adjunction $\operatorname{colim}\colon \mathcal{C}^{\mathcal{I}} \rightleftarrows \mathcal{C} : \Delta$, whose components are just the universal cocones, then a unit of $\operatorname{colim}\colon \mathcal{C}^{\mathcal{I}\times\mathcal{J}} \rightleftarrows \mathcal{C}^{\mathcal{J}} : \Delta$ is given by $\eta_{X,I,J} := \eta'_{X(-,J),I}$. All in all, the square

$$\begin{tikzcd} \mathcal{C}^{\mathcal{I}\times\mathcal{J}} \arrow[d, "\mathrm{ev}_J"'] & \mathcal{C}^{\mathcal{J}} \arrow[l, "\Delta"'] \arrow[d, "\mathrm{ev}_J"] \arrow[dl, phantom, "\Searrow\mathrm{id}"] \\ \mathcal{C}^{\mathcal{I}} & \mathcal{C} \arrow[l, "\Delta"] \end{tikzcd}$$

satisfies the Beck-Chevalley condition for all $J \in \mathcal{J}$. Indeed, the mate of the identity is again the identity and $\mathrm{ev}_J\eta = \eta'_{\mathrm{ev}_J}$.

(5.5) **Observation.** According to (4.8), horizontal and vertical composites of Beck-Chevalley squares (defined in the obvious manner) are again Beck-Chevalley squares.

The situation gets really interesting when X and Y do have right adjoints. In that case, there is the following interchange law for mates as stated e.g. in [31, p.17, Lemma 1.20].

(5.6) **Theorem. (Beck-Chevalley Interchange)** Consider a square of categories and functors together with a natural transformation $\tau\colon XG \Rightarrow G'Y$ as in the above definition and where all four functors have adjoints

$$(F \dashv G, \eta, \varepsilon), \quad (F' \dashv G', \eta', \varepsilon'), \quad (X \dashv R, \theta, \zeta) \quad \text{and} \quad (Y \dashv S, \theta', \zeta'),$$

so that τ has a horizontal left mate $\sigma: F'X \Rightarrow YF$ as well as a vertical right mate $\rho: GS \Rightarrow RG'$. Then σ and ρ are conjugate natural transformations. In particular, σ is an isomorphism iff ρ is one and thus the square satisfies the horizontal Beck-Chevalley condition iff it satisfies the vertical dual Beck-Chevalley condition.

Proof. Recall that the two mates σ and ρ are defined by

$$G'\sigma \circ \eta'_X = \tau_F \circ X\eta \qquad \text{and} \qquad R\tau \circ \theta_G = \rho_Y \circ G\theta'$$

and we need to check that σ and ρ are conjugate with respect to the composite adjunctions

$$\begin{tikzcd} \mathcal{C} \arrow[r, shift left, "F"] \arrow[r, phantom, "\perp"] \arrow[d, equal] & \mathcal{D} \arrow[l, shift left, "G"] \arrow[r, shift left, "Y"] \arrow[r, phantom, "\perp"] & \mathcal{D}' \arrow[l, shift left, "S"] \arrow[d, equal] \\ \mathcal{C} \arrow[r, shift left, "X"] \arrow[r, phantom, "\perp"] & \mathcal{C}' \arrow[l, shift left, "R"] \arrow[r, shift left, "F'"] \arrow[r, phantom, "\perp"] & \mathcal{D}' \arrow[l, shift left, "G'"] \end{tikzcd} .$$

For this, we just need to take the adjunct of σ, which is

$$\begin{aligned} \sigma^\sharp &= R(G'\sigma \circ \eta'_X) \circ \theta = R(\tau_F \circ X\eta) \circ \theta = R\tau_F \circ RX\eta \circ \theta \\ &= R\tau_F \circ \theta_{GF} \circ \eta = (R\tau \circ \theta_G)_F \circ \eta = (\rho_Y \circ G\theta')_F \circ \eta = \rho_{YF} \circ G\theta'_F \circ \eta \end{aligned}$$

and $G\theta'_F \circ \eta$ is the unit of the upper composite adjunction. □

(5.7) **Corollary.** Consider a square as in (5.1) with $\tau: XG \cong G'Y$ an isomorphism. If Y has a fully faithful right adjoint and X is itself fully faithful and has a right adjoint then the square satisfies the Beck-Chevalley condition.

Proof. The square satisfies the (horizontal) Beck-Chevalley condition iff it satisfies the vertical dual Beck-Chevalley condition. Taking adjoints

$$(X \dashv R, \theta, \zeta)\colon\ \begin{tikzcd} \mathcal{C} \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathcal{C}' \arrow[l, shift left] \end{tikzcd} \qquad \text{and} \qquad (Y \dashv S, \theta', \zeta')\colon\ \begin{tikzcd} \mathcal{D} \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathcal{D}' \arrow[l, shift left] \end{tikzcd}$$

the mate of τ for the vertical adjoints is $RG'\zeta' \circ R\tau_S \circ \theta_{GS}$. But τ is an isomorphism, same as ζ' (since S is fully faithful) and θ (since X is fully faithful). □

(5.8) **Corollary.** Given a mating square with X and Y equivalences

$$\begin{tikzcd} \mathcal{C} \arrow[r, shift left, "F"] \arrow[r, phantom, "\perp"] \arrow[d, "X"'] & \mathcal{D} \arrow[l, shift left, "G"] \arrow[d, "Y"] \\ \mathcal{C}' \arrow[r, shift left, "F'"] \arrow[r, phantom, "\perp"] & \mathcal{D}' \arrow[l, shift left, "G'"] \end{tikzcd} .$$

then $\sigma\colon F'X \Rightarrow YF$ is an isomorphism iff its mate $\tau\colon XG \Rightarrow G'Y$ is so.

Proof. Every equivalence is fully faithful and has a fully faithful left and right adjoint. □

Unfortunately, if the two vertical arrows X and Y in (5.1) have left adjoints rather than right ones, there is no nice interchange law. However, we can still use such adjoints to our advantage.

(5.9) **Theorem.** Again consider a square as in (5.1) with $\tau\colon XG \cong G'Y$ a natural isomorphism and where all four functors have left adjoints

$$(F \dashv G, \eta, \varepsilon), \quad (F' \dashv G', \eta', \varepsilon'), \quad (L \dashv X, \theta, \zeta) \quad \text{and} \quad (M \dashv Y, \theta', \zeta').$$

(a) If X and M are fully faithful then the square satisfies the Beck-Chevalley condition.

(b) If both L and M or both X and Y are fully faithful then the square satisfies the Beck-Chevalley condition iff there is an isomorphism $F'X \cong YF$.

Proof. The mate σ of τ is the unique transformation satisfying $G'\sigma \circ \eta'_X = \tau_F \circ X\eta$ and so it suffices to construct a natural isomorphism σ subject to this equation. For this, we consider the conjugate of τ with respect to the two composite adjunctions, which is the unique isomorphism

$$\rho\colon M \circ F' \cong F \circ L \qquad \text{satisfying} \qquad G'Y\rho \circ G'\theta'_{F'} \circ \eta' = \tau_{FL} \circ X\eta_L \circ \theta.$$

Evaluating this equation at X and postcomposing with $G'YF\zeta$ yields

$$\begin{aligned} G'(YF\zeta \circ Y\rho_X \circ \theta'_{F'X}) \circ \eta'_X &= G'YF\zeta \circ \tau_{FLX} \circ X\eta_{LX} \circ \theta_X \\ &= \tau_F \circ XGF\zeta \circ X\eta_{LX} \circ \theta_X = \tau_F \circ X\eta \circ X\zeta \circ \theta_X = \tau_F \circ X\eta, \end{aligned}$$

so that the mate of τ is $\sigma = YF\zeta \circ Y\rho_X \circ \theta'F'X$. Under the hypotheses of (a), this is an isomorphism since ζ and θ' are invertible, thus proving the first point. For point (b), assume that L and M are fully faithful and there is an isomorphism $F'X \cong YF$. We need to check that $YF\zeta$ is invertible, which is obvious since $YF \cong F'X$ and $X\zeta = \theta_X^{-1}$ is invertible. □

6. Derived Adjunctions

As already mentioned, there is a beauti- and useful interplay between adjunctions and absolute derived functors; the most well-known instance of it perhaps being the famous Quillen adjoint functor theorem. Most proofs of it rely heavily on the explicit construction of a derived functor by means of (co)fibrant replacements whereas our approach really gets down to its bare bones. For this, let us fix two categories $\mathcal{C}$, $\mathcal{D}$ each equipped with a class of weak equivalences and let us write $H_{\mathcal{C}}$ and $H_{\mathcal{D}}$ for the corresponding localisations.

(6.1) **Theorem.** Let $(F \dashv G, \eta, \varepsilon)\colon \mathcal{C} \rightleftarrows \mathcal{D}$ be an adjunction such that G has an absolute total right derived functor $(\mathbb{R}G, \rho)$. If this in turn has a left adjoint $(\dot{F} \dashv \mathbb{R}G, \dot{\eta}, \dot{\varepsilon})$ then $(\dot{F}, \lambda)$ is an absolute total left derived functor where $\lambda := \dot{\varepsilon}_{H_{\mathcal{D}}F} \circ \dot{F}\rho_F \circ \dot{F}H_{\mathcal{C}}\eta$.

Proof. We need to construct a bijection $\mathrm{Nat}(L, E \circ F') \cong \mathrm{Nat}(L \circ H_{\mathcal{C}}, E \circ H_{\mathcal{D}} \circ F)$ natural in $\mathcal{E}$, $E\colon \operatorname{Ho}\mathcal{D} \to \mathcal{E}$ and $L\colon \operatorname{Ho}\mathcal{C} \to \mathcal{E}$. We can do so by

$$\begin{aligned} \mathrm{Nat}(L, E \circ \dot{F}) &\cong \mathrm{Nat}(L \circ \mathbb{R}G, E) && \tau \mapsto E\dot{\varepsilon} \circ \tau_{\mathbb{R}G} \\ &\cong \mathrm{Nat}(L \circ H_{\mathcal{C}} \circ G, E \circ H_{\mathcal{D}}) && \tau \mapsto \tau_{H_{\mathcal{D}}} \circ L\rho \\ &\cong \mathrm{Nat}(L \circ H_{\mathcal{C}}, E \circ H_{\mathcal{D}} \circ F) && \tau \mapsto \tau_F \circ (L \circ H_{\mathcal{C}})\eta \end{aligned}$$

and upon putting $L := \dot{F}$, $E := \mathrm{id}_{\operatorname{Ho}\mathcal{D}}$ and chasing $\mathrm{id}_{\dot{F}}$ through the bijections, the counit λ has indeed the claimed form. □

(6.2) **Theorem.** Let $(F \dashv G, \eta, \varepsilon)\colon \mathcal{C} \rightleftarrows \mathcal{D}$. If F has an absolute total left derived functor $(\mathbb{L}F, \lambda)$ and G has an absolute total right derived functor $(\mathbb{R}G, \rho)$ then we get an adjunction $(\mathbb{L}F \dashv \mathbb{R}G, \dot{\eta}, \dot{\varepsilon})$ where $\dot{\eta}$ is the unique $\mathrm{id}_{\operatorname{Ho}\mathcal{C}} \Rightarrow \mathbb{R}G \circ \mathbb{L}F$ such that $\mathbb{R}G\lambda \circ \dot{\eta}_{H_{\mathcal{C}}} = \rho_F \circ H_{\mathcal{C}}\eta$ and $\dot{\varepsilon}$ is the unique $\mathbb{L}F \circ \mathbb{R}G \Rightarrow \mathrm{id}_{\operatorname{Ho}\mathcal{D}}$ such that $H_{\mathcal{D}}\varepsilon \circ \lambda_G = \dot{\varepsilon}_{H_{\mathcal{D}}} \circ \mathbb{L}F\rho$.

Proof. We need to construct a bijection $\mathrm{Nat}(E' \circ \mathbb{R}G, E) \cong \mathrm{Nat}(E', E \circ \mathbb{L}F)$ natural in $\mathcal{E}$, $E'\colon \operatorname{Ho}\mathcal{C} \to \mathcal{E}$ and $E\colon \operatorname{Ho}\mathcal{D} \to \mathcal{E}$. We can do so by

$$\begin{aligned} \mathrm{Nat}(E' \circ \mathbb{R}G, E) &\cong \mathrm{Nat}(E' \circ H_{\mathcal{C}} \circ G, E \circ H_{\mathcal{D}}) && \sigma \mapsto \sigma_{H_{\mathcal{D}}} \circ E'\rho \\ &\cong \mathrm{Nat}(E' \circ H_{\mathcal{C}}, E \circ H_{\mathcal{D}} \circ F) && \sigma \mapsto \sigma_F \circ E' H_{\mathcal{C}}\eta, \quad EH_{\mathcal{D}}\varepsilon \circ \tau_G \mapsfrom \tau \\ &\cong \mathrm{Nat}(E', E \circ \mathbb{L}F) && E\lambda \circ \tau_{H_{\mathcal{C}}} \mapsfrom \tau \end{aligned}$$

and it plainly follows that $\dot\eta$ and $\dot\varepsilon$ are of the required form. □

The explicit descriptions of the unit and counit in the last theorem is not very enlightening and it might be clearer (although the author is not convinced) to draw the corresponding diagrams:

(6.3)
$$\begin{tikzcd} H_{\mathcal{C}} \arrow[r, "H_{\mathcal{C}}\eta"] \arrow[d, "\dot\eta H_{\mathcal{C}}"'] & H_{\mathcal{C}} \circ G \circ F \arrow[d, "\rho_F"] \\ \mathbb{R}G \circ \mathbb{L}F \circ H_{\mathcal{C}} \arrow[r, "\mathbb{R}G\lambda"'] & \mathbb{R}G \circ H_{\mathcal{D}} \circ F \end{tikzcd} \qquad \begin{tikzcd} \mathbb{L}F \circ H_{\mathcal{C}} \circ G \arrow[r, "\mathbb{L}F\rho"] \arrow[d, "\lambda_G"'] & \mathbb{L}F \circ \mathbb{R}G \circ H_{\mathcal{D}} \arrow[d, "\dot\varepsilon H_{\mathcal{D}}"] \\ H_{\mathcal{D}} \circ F \circ G \arrow[r, "H_{\mathcal{D}}\varepsilon"'] & H_{\mathcal{D}} \end{tikzcd} .$$

Even better, in the situation of the first theorem (6.1), these two diagrams again commute (by naturality of all arrows involved and the triangle identities).

(6.4) **Definition.** By a *derived adjunction* of $(F \dashv G, \eta, \varepsilon)\colon \mathcal{C} \rightleftarrows \mathcal{D}$ we mean an adjunction $(\mathbb{L}F \dashv \mathbb{R}G, \dot\eta, \dot\varepsilon)\colon \operatorname{Ho}\mathcal{C} \rightleftarrows \operatorname{Ho}\mathcal{D}$ together with transformations λ and ρ satisfying

(a) $(\mathbb{L}F, \lambda)$ is an absolute total left derived functor of F;

(b) $(\mathbb{R}G, \rho)$ is an absolute total right derived functor of G;

(c) $\dot\eta$ and $\dot\varepsilon$ are the unique natural transformations making the squares (6.3) commute.

If such a derived adjunction exists, we say that $F \dashv G$ is *derivable.*

With this definition, we can summarise the results of the two theorems above by the following (less precise) corollary.

(6.5) **Corollary.** Let $F \dashv G$ be an adjunction such that an absolute total right derived functor $\mathbb{R}G$ of G exists. Then $F \dashv G$ is derivable if and only if $\mathbb{R}G$ has a left adjoint. □

Also, using the theorems, we can easily study the question when derived functors compose. Unfortunately, they do not in general but at least we can check it on adjoints.

(6.6) **Corollary.** Let $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$ be three categories with weak equivalences and

$$(F \dashv G, \eta, \varepsilon)\colon \mathcal{C} \rightleftarrows \mathcal{D}, \qquad (F' \dashv G', \eta', \varepsilon')\colon \mathcal{D} \rightleftarrows \mathcal{E}$$

with derived adjunctions $(\mathbb{L}F \dashv \mathbb{R}G, \dot\eta, \dot\varepsilon)$ and $(\mathbb{L}F' \dashv \mathbb{R}G', \dot\eta', \dot\varepsilon')$. If $(\mathbb{R}G \circ \mathbb{R}G', \rho'')$ is an absolute total right derived functor of $G \circ G'$ then $\mathbb{L}F' \circ \mathbb{L}F$ is an absolute total left derived functor of $F' \circ F$ with counit

$$(\dot\varepsilon' \circ \mathbb{L}F' \dot\varepsilon_{\mathbb{R}G'})_{H_{\mathcal{E}}F'F} \circ \mathbb{L}F'\mathbb{L}F\rho''_{F'F} \circ \mathbb{L}F'\mathbb{L}F H_{\mathcal{C}}(G\dot\eta_F \circ \eta).$$

Proof. Composing the two adjunctions as well as their derived adjunctions yields

$$(F \circ F \dashv G \circ G', G\eta'_F \circ \eta, \varepsilon' \circ F'\varepsilon_{G'}) \colon \mathcal{C} \rightleftarrows \mathcal{E} \qquad \text{and}$$

$$(\mathbb{L}F' \circ \mathbb{L}F \dashv \mathbb{R}G \circ \mathbb{R}G', \mathbb{R}G\dot{\eta}'_{\mathbb{L}F} \circ \dot{\eta}, \dot{\varepsilon}' \circ \mathbb{L}F'\dot{\varepsilon}_{\mathbb{R}G'}) \colon \operatorname{Ho}\mathcal{C} \rightleftarrows \operatorname{Ho}\mathcal{E}.$$

Now if $(\mathbb{R}G \circ \mathbb{R}G', \rho')$ is an absolute total right derived functor of $G \circ G'$ then by (6.1) $\mathbb{L}F' \circ \mathbb{L}F$ is indeed an absolute total left derived functor of $F' \circ F$ with counit

$$(\dot{\varepsilon}' \circ \mathbb{L}F'\dot{\varepsilon}_{\mathbb{R}G'})_{H_{\mathcal{E}}F'F} \circ \mathbb{L}F'\mathbb{L}F\rho''_{F'F} \circ \mathbb{L}F'\mathbb{L}FH_{\mathcal{C}}(G\eta'_F \circ \eta).$$

□

As a special instance of this corollary, we can consider the case where $\rho'' = \mathbb{R}G\rho' \circ \rho_{G'}$ is the canonical candidate for a unit of $\mathbb{R}G \circ \mathbb{R}G'$. It then follows that a counit of the total left derived functor $\mathbb{L}F' \circ \mathbb{L}F$ is again given by the canonical candidate and vice versa. Note that this result is neither stronger nor weaker than the previous one.

(6.7) **Corollary.** In the same situation as in the last corollary, let us write

$$(\mathbb{L}F, \lambda), \quad (\mathbb{R}G, \rho), \quad (\mathbb{L}F', \lambda') \quad \textit{and} \quad (\mathbb{R}G', \rho')$$

for the absolute total derived functors. Then the composite $(\mathbb{L}F' \circ \mathbb{L}F, \lambda'_F \circ \mathbb{L}F'\lambda)$ is an absolute total left derived functor of $F' \circ F$ if and only if $(\mathbb{R}G \circ \mathbb{R}G', \mathbb{R}G\rho' \circ \rho_{G'})$ is an absolute total right derived functor of $G \circ G'$.

Proof. If $(\mathbb{R}G \circ \mathbb{R}G', \mathbb{R}G\rho' \circ \rho_{G'})$ is an absolute total right derived functor, we apply the last corollary to $\rho'' := \mathbb{R}G\rho' \circ \rho_{G'}$ and the claim follows by a routine (albeit tedious) calculation:

$$\begin{aligned}
&\dot{\varepsilon}'_{H_{\mathcal{E}}F'F} \circ \mathbb{L}F'\dot{\varepsilon}_{\mathbb{R}G'H_{\mathcal{E}}F'F} \circ \mathbb{L}F'\mathbb{L}F\mathbb{R}G\rho'_{F'F} \circ \mathbb{L}F'\mathbb{L}F\rho_{G'F'F} \circ \mathbb{L}F'\mathbb{L}FH_{\mathcal{C}}G\eta'_F \circ \mathbb{L}F'\mathbb{L}FH_{\mathcal{C}}\eta \\
&\overset{\varepsilon\text{ nat}}{=} \dot{\varepsilon}'_{H_{\mathcal{E}}F'F} \circ \mathbb{L}F'\rho_{F'F} \circ \mathbb{L}F'\dot{\varepsilon}_{H_{\mathcal{D}}G'F'F} \circ \mathbb{L}F'\mathbb{L}F\rho_{G'F'F} \circ \mathbb{L}F'\mathbb{L}FH_{\mathcal{C}}G\eta'_F \circ \mathbb{L}F'\mathbb{L}FH_{\mathcal{C}}\eta \\
&\overset{(6.3)}{=} H_{\mathcal{E}}\varepsilon'_{F'F} \circ \lambda'_{G'F'F} \circ \mathbb{L}F'H_{\mathcal{D}}\varepsilon_{G'F'F} \circ \mathbb{L}F'\lambda_{GG'F'F} \circ \mathbb{L}F'\mathbb{L}FH_{\mathcal{C}}G\eta'_F \circ \mathbb{L}F'\mathbb{L}FH_{\mathcal{C}}\eta \\
&\overset{\lambda\text{ nat}}{=} H_{\mathcal{E}}\varepsilon'_{F'F} \circ \lambda'_{G'F'F} \circ \mathbb{L}F'H_{\mathcal{D}}\varepsilon_{G'F'F} \circ \mathbb{L}F'H_{\mathcal{D}}FG\eta'_F \circ \mathbb{L}F'H_{\mathcal{D}}F\eta \circ \mathbb{L}F'\lambda \\
&\overset{\lambda'\text{ nat}}{=} H_{\mathcal{E}}\varepsilon'_{F'F} \circ H_{\mathcal{E}}F'\varepsilon_{G'F'F} \circ H_{\mathcal{E}}F'FG\eta'_F \circ H_{\mathcal{E}}F'F\eta \circ \lambda'_F \circ \mathbb{L}F'\lambda \\
&\overset{\varepsilon\text{ nat}}{=} H_{\mathcal{E}}\varepsilon'_{F'F} \circ H_{\mathcal{E}}F'\bar{\eta}_F \circ H_{\mathcal{E}}F'\varepsilon_F \circ H_{\mathcal{E}}F'F\eta \circ \lambda'_F \circ \mathbb{L}F'\lambda \overset{\Delta\text{-id}}{=} \lambda'_F \circ \mathbb{L}F'\lambda.
\end{aligned}$$

The other direction is dual. □

(6.8) **Definition.** Let $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$ be categories equipped with weak equivalences and let $F \colon \mathcal{C} \to \mathcal{D}$, $F' \colon \mathcal{D} \to \mathcal{E}$ have absolute total left derived functors $(\mathbb{L}F, \lambda)$ and $(\mathbb{L}F', \lambda')$. By abuse of notation, we say that $\mathbb{L}F$ *composes* with $\mathbb{L}F'$ iff $(\mathbb{L}F' \circ \mathbb{L}F, \lambda'_F \circ \mathbb{L}F'\lambda)$ is an absolute total left derived functor of $F' \circ F$. Dually for right derived functors.

(6.9) **Remark.** Clearly, this definition is equivalent to requiring that the composite $F' \circ F$ have a total left derived functor $\big(\mathbb{L}(F' \circ F), \lambda''\big)$ and that the natural transformation

$$\mathbb{L}F' \circ \mathbb{L}F \Rightarrow \mathbb{L}(F' \circ F) \qquad \text{induced by } \lambda'_F \circ \mathbb{L}F'\lambda \text{ be an isomorphism.}$$

(6.10) **Example.** Obviously, if F' is homotopical then $\mathbb{L}F' \cong \operatorname{Ho}F'$, so that $\mathbb{L}F$ and $\mathbb{L}F'$ compose. Also, total left derived functors of left Quillen functors between model categories compose. Dually for right Quillen functors.

(6.11) **Example.** More generally, let $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$ be equipped with weak equivalences and $H_{\mathcal{C}}$, $H_{\mathcal{D}}$, $H_{\mathcal{E}}$ their respective localisations. If we are given

$$\mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{F'} \mathcal{E}$$

together with a left F-deformation retract $\mathcal{C}_0 \subseteq \mathcal{C}$ and a left F'-deformation retract $\mathcal{D}_0 \subseteq \mathcal{D}$ such that $F\mathcal{C}_0 \subseteq \mathcal{D}_0$ then $\mathbb{L}F'$ and $\mathbb{L}F$ compose.

Proof. Let us write $\bar{F} := (F|_{\mathcal{C}_0})^{-}$, $\bar{F}' := (F'|_{\mathcal{D}_0})^{-}$ and fix some left deformation retractions $(\mathcal{C}_0, Q, q)$, $(\mathcal{D}_0, Q', q')$, so that we may choose

$$\mathbb{L}F = \bar{F} \circ \widetilde{Q},\ \mathbb{L}F' = \bar{F}' \circ \widetilde{Q}' \qquad \text{with counits} \qquad \lambda := H_{\mathcal{D}}Fq,\ \lambda' := H_{\mathcal{E}}F'q'.$$

By hypothesis $(\mathcal{C}_0, Q, q)$ is a left $(F'F)$-deformation retraction and $\overline{F'F} := (F'F|_{\mathcal{C}_0})^{-}$ comes with a natural isomorphism $\omega\colon \mathbb{L}F' \circ \mathbb{L}F \cong \overline{F'F} \circ \widetilde{Q}$ determined by the components

$$\omega_{H_{\mathcal{C}}C}\colon \mathbb{L}F'\mathbb{L}FH_{\mathcal{C}}C = H_{\mathcal{E}}F'Q'FQC \xrightarrow{H_{\mathcal{E}}F'q'_{FQC}} H_{\mathcal{E}}F'FQC \qquad \text{for } C \in \mathcal{C}.$$

Moreover, ω is compatible with the claimed counit of $\mathbb{L}F' \circ \mathbb{L}F$ in the sense that

$$H_{\mathcal{E}}F'Fq_C \circ \omega_{H_{\mathcal{C}}C} = H_{\mathcal{E}}F'(Fq_C \circ q'_{FQC}) = H_{\mathcal{E}}F'(q'_{FC} \circ Q'Fq_C) = \lambda'_{FC} \circ \mathbb{L}F'\lambda_C. \qquad \square$$

(6.12) **Corollary.** With the above definition, the last corollary (6.7) can be restated by saying that for two derived adjunctions, the left adjoints compose iff the right adjoints compose.

Finally, we will quickly go one dimension higher and study the connection between mates and derivable adjunctions. For this, we consider two derivable adjunctions $F \dashv G$, $F' \dashv G'$ (as always writing λ, λ', ρ and ρ' for the universal morphisms of $\mathbb{L}F$, $\mathbb{L}F'$, $\mathbb{R}G$ and $\mathbb{R}G'$) together with two homotopical functors X and Y as in the following diagram:

$$\begin{tikzcd} \mathcal{C} \arrow[r, shift left, "F"] \arrow[r, phantom, "\perp"] \arrow[d, "X"'] & \mathcal{D} \arrow[l, shift left, "G"] \arrow[d, "Y"] \\ \mathcal{C}' \arrow[r, shift left, "F'"] \arrow[r, phantom, "\perp"] & \mathcal{D}' \arrow[l, shift left, "G'"] \end{tikzcd} \qquad \text{yielding} \qquad \begin{tikzcd} \operatorname{Ho}\mathcal{C} \arrow[r, shift left, "\mathbb{L}F"] \arrow[r, phantom, "\perp"] \arrow[d, "\operatorname{Ho}X"'] & \operatorname{Ho}\mathcal{D} \arrow[l, shift left, "\mathbb{R}G"] \arrow[d, "\operatorname{Ho}Y"] \\ \operatorname{Ho}\mathcal{C}' \arrow[r, shift left, "\mathbb{L}F'"] \arrow[r, phantom, "\perp"] & \operatorname{Ho}\mathcal{D}' \arrow[l, shift left, "\mathbb{R}G'"] \end{tikzcd}.$$

In addition, we require $\operatorname{Ho}X$ to compose with $\mathbb{L}F'$ and $\operatorname{Ho}Y$ to compose with $\mathbb{R}G'$ (all other reasonable combinations automatically compose as remarked in (6.10)).

(6.13) **Theorem.** In the above situation, if $\sigma\colon F'X \Rightarrow YF$ and $\tau\colon XG \Rightarrow G'Y$ are mates then the induced transformations between the derived functors

$$\mathbb{L}\sigma\colon \mathbb{L}F' \circ \operatorname{Ho}X \Rightarrow \operatorname{Ho}Y \circ \mathbb{L}F \qquad \text{and} \qquad \mathbb{R}\tau\colon \operatorname{Ho}X \circ \mathbb{R}G \Rightarrow \mathbb{R}G' \circ \operatorname{Ho}Y$$

are mates, too (note that for this to even make sense, we need that $\operatorname{Ho}X$ composes with $\mathbb{L}F'$ and that $\operatorname{Ho}Y$ composes with $\mathbb{R}G'$). In particular, for $\mathcal{C} = \mathcal{C}'$, $\mathcal{D} = \mathcal{D}'$ and X, Y identities, if σ and τ are conjugate, so are $\mathbb{L}\sigma$ and $\mathbb{R}\tau$.

Proof. First recall that the composite counits of $\mathbb{L}F' \circ \operatorname{Ho} X$ and $\operatorname{Ho} Y \circ \mathbb{L}F$ are respectively λ'_X and $(\operatorname{Ho} Y)\lambda$ while the composite units of $\operatorname{Ho} X \circ \mathbb{R}G$ and $\mathbb{R}G' \circ \operatorname{Ho} Y$ are $(\operatorname{Ho} X)\rho$ and ρ'_Y. It follows that $\mathbb{L}\sigma$ and $\mathbb{R}\tau$ are defined by the equations

$$(\operatorname{Ho} Y)\lambda \circ (\mathbb{L}\sigma)_{H_{\mathcal{C}}} = H_{\mathcal{D}'}\sigma \circ \lambda'_X \qquad \text{and} \qquad (\mathbb{R}\tau)_{H_{\mathcal{D}}} \circ (\operatorname{Ho} X)\rho = \rho'_Y \circ H_{\mathcal{C}'}\tau.$$

We now need to check that $(\mathbb{L}\sigma)^\sharp = \mathbb{R}G'\mathbb{L}\sigma \circ \dot\eta'_{\operatorname{Ho} X} = (\mathbb{R}\tau)_{\mathbb{L}F} \circ X\dot\eta$, which are natural transformations $\operatorname{Ho} X \Rightarrow \mathbb{R}G' \circ \operatorname{Ho} Y \circ \mathbb{L}F$. By the universal property of a localisation and $\mathbb{L}F$ being an absolute total left derived functor, it suffices to check this after precomposition with $H_{\mathcal{C}}$ and postcomposition with $(\mathbb{R}G' \circ \operatorname{Ho} Y)\lambda$, where then

$$\begin{aligned}
&(\mathbb{R}G' \circ \operatorname{Ho} Y)\lambda \circ (\mathbb{R}G'\mathbb{L}\sigma \circ \dot\eta'_{\operatorname{Ho} X})_{H_{\mathcal{C}}} = \mathbb{R}G'\big((\operatorname{Ho} Y)\lambda \circ \mathbb{L}\sigma_{H_{\mathcal{C}}}\big) \circ \dot\eta'_{H_{\mathcal{C}'}X} \\
&\quad = \mathbb{R}G'(H_{\mathcal{D}'}\sigma \circ \lambda'_X) \circ \dot\eta'_{H_{\mathcal{C}'}X} = \mathbb{R}G' H_{\mathcal{D}'}\sigma \circ (\mathbb{R}G'\lambda' \circ \dot\eta'_{H_{\mathcal{C}'}})_X \\
&\quad = \mathbb{R}G' H_{\mathcal{D}'}\sigma \circ (\rho'_{F'} \circ H_{\mathcal{C}'}\eta')_X = \mathbb{R}G' H_{\mathcal{D}'}\sigma \circ \rho'_{F'X} \circ H_{\mathcal{C}'}\eta'_X \\
&\quad = \rho'_{YF} \circ H_{\mathcal{C}'}G'\sigma \circ H_{\mathcal{C}'}\eta'_X = \rho'_{YF} \circ H_{\mathcal{C}'}(G'\sigma \circ \eta'_X) \\
&\quad = \rho'_{YF} \circ H_{\mathcal{C}'}(\tau_F \circ X\eta) = (\rho'_Y \circ H_{\mathcal{C}'}\tau)_F \circ H_{\mathcal{C}'}X\eta \\
&\quad = (\mathbb{R}\tau)_{H_{\mathcal{D}}F} \circ (\operatorname{Ho} X)\rho_F \circ (\operatorname{Ho} X)H_{\mathcal{C}}\eta = (\mathbb{R}\tau)_{H_{\mathcal{D}}F} \circ (\operatorname{Ho} X)(\rho_F \circ H_{\mathcal{C}}\eta) \\
&\quad = (\mathbb{R}\tau)_{H_{\mathcal{D}}F} \circ (\operatorname{Ho} X)(\mathbb{R}G\lambda \circ \dot\eta_{H_{\mathcal{C}}}) = (\mathbb{R}\tau)_{H_{\mathcal{D}}F} \circ (\operatorname{Ho} X \circ \mathbb{R}G)\lambda \circ (\operatorname{Ho} X)\dot\eta_{H_{\mathcal{C}}} \\
&\quad = (\mathbb{R}G' \circ \operatorname{Ho} Y)\lambda \circ (\mathbb{R}\tau)_{\mathbb{L}FH_{\mathcal{C}}} \circ (\operatorname{Ho} X)\dot\eta_{H_{\mathcal{C}}} \\
&\quad = \mathbb{R}G'(\operatorname{Ho} Y)\lambda \circ \big((\mathbb{R}\tau)_{\mathbb{L}F} \circ (\operatorname{Ho} X)\dot\eta\big)_{H_{\mathcal{C}}}
\end{aligned}$$

□

7. Two Notions of Homotopy Colimits

As an application of our theorems, we are going to show that homotopy colimits in the sense of derivator theory are the same as homotopy colimits in the Quillen model category sense. To make this precise, let us fix the following convention for convenience.

(7.1) **Convention.** Whenever we have a category $\mathcal{C}$ equipped with a class of weak equivalences as well as some index category $\mathcal{I}$ we always equip the diagram category $\mathcal{C}^{\mathcal{I}}$ with the class of pointwise weak equivalences. That is to say, a natural transformation $\tau\colon X \Rightarrow Y$ of diagrams $X, Y : \mathcal{I} \to \mathcal{C}$ is a weak equivalence iff $\tau_I\colon X_I \to Y_I$ is a weak equivalence in $\mathcal{C}$ for all $I \in \mathcal{I}$.

With this natural definition of weak equivalences in a diagram category, one has the problem that in general, colimit and limit functors do not preserve them. The classical example is given by the commutative diagram in **Top**

$$\begin{tikzcd}
D^n \arrow[d] & S^{n-1} \arrow[l, hook'] \arrow[r, hook] \arrow[d, equal] & D^n \arrow[d] \\
* & S^{n-1} \arrow[l, hook'] \arrow[r, hook] & *
\end{tikzcd},$$

where all vertical arrows are (weak) homotopy equivalences but the pushout of the top row is S^n, while the one of the bottom row is $*$ and these spaces are not (weakly) homotopy equivalent. So, in general, we cannot form Ho(colim) and the best we can do is trying to take a derived functor.

(7.2) **Observation.** The constant diagram functor $\Delta\colon \mathcal{C} \to \mathcal{C}^{\mathcal{I}}$ is homotopical and consequently induces a unique $\operatorname{Ho}\Delta\colon \operatorname{Ho}\mathcal{C} \to \operatorname{Ho}(\mathcal{C}^{\mathcal{I}})$ such that $\operatorname{Ho}\Delta \circ H_{\mathcal{C}} = H_{\mathcal{C}^{\mathcal{I}}} \circ \Delta$ (where $H_{\mathcal{C}}$ and $H_{\mathcal{C}^{\mathcal{I}}}$ are the corresponding localisations).

(7.3) **Definition.** For $\mathcal{C}$ a category equipped with weak equivalences and $\mathcal{I}$ an index category a *homotopy colimit functor* is a functor $\operatorname{hocolim}\colon \operatorname{Ho}(\mathcal{C}^{\mathcal{I}}) \to \operatorname{Ho}\mathcal{C}$ left adjoint to $\operatorname{Ho}\Delta$. If $\mathcal{C}$ has $\mathcal{I}$-colimits (so that $\Delta\colon \mathcal{C} \to \mathcal{C}^{\mathcal{I}}$ has a left adjoint colim) a *derived colimit functor* is a total left derived functor $\mathbb{L}\operatorname{colim}$ of colim.

(7.4) **Proposition.** If $\mathcal{C}$ has $\mathcal{I}$-colimits then a homotopy colimit functor exists iff a derived colimit functor exists and they are the same.

Proof. The functor $\Delta\colon \mathcal{C} \to \mathcal{C}^{\mathcal{I}}$ is homotopical and so $\mathbb{R}\Delta = \operatorname{Ho}\Delta$ exists. $\square$

Although, by this result the existence of homotopy colimits is weaker than the existence of derived colimits (one does not need the existence of strict colimits), it seems pathological to have homotopy colimits without strict colimits and we prefer the more restrictive notion.

(7.5) **Definition.** We say that a category $\mathcal{C}$ with weak equivalences *has homotopy colimits of type* $\mathcal{I}$ (or simply that it has $\mathcal{I}$-*homotopy colimits*) iff the adjunction $\operatorname{colim}\colon \mathcal{C}^{\mathcal{I}} \rightleftarrows \mathcal{C} : \Delta$ exists and is derivable. Also, when writing “hocolim” we always mean “$\mathbb{L}\operatorname{colim}$”.

8. Evaluation and Endomorphisms

To say that colimits in a diagram category $\mathcal{C}^{\mathcal{I}}$ are calculated pointwise means that

$$\operatorname{colim} \circ \operatorname{ev}_I = \operatorname{ev}_I \circ \operatorname{colim} \quad \text{(or rather that there is a natural isomorphism)}$$

for all $I \in \mathcal{I}$, where $\operatorname{ev}_I\colon \mathcal{C}^{\mathcal{I}} \to \mathcal{C}$, $X \mapsto X_I$. So in order to prove a similar statement for homotopy colimits, we should try to better understand these evaluation functors.

(8.1) **Proposition.** Let $\mathcal{C}$ be any category and $\mathcal{I}$ an index category. For $I \in \mathcal{I}$ with inclusion $\operatorname{In}_I\colon \{I\} \hookrightarrow \mathcal{I}$, the evaluation functor $\operatorname{ev}_I\colon \mathcal{C}^{\mathcal{I}} \to \mathcal{C}$ has a right adjoint

$$I_*\colon \mathcal{C} \to \mathcal{C}^{\mathcal{I}} \qquad \text{given by} \qquad I_*C = C^{\mathcal{I}(-,I)}$$

(with the obvious arrow map), granted all these powers exist. Moreover, if $\operatorname{End}_{\mathcal{I}}(I) = \{\mathrm{id}_I\}$, then I_* is fully faithful or equivalently, a counit ε of the adjunction is invertible. Dually, ev_I has a left adjoint

$$I_!\colon \mathcal{C} \to \mathcal{C}^{\mathcal{I}} \qquad \text{given by} \qquad I_!C = \mathcal{I}(I,-)\cdot C,$$

granted all these copowers exist and if $\operatorname{End}_{\mathcal{I}}(I) = \{\mathrm{id}_I\}$, then $I_!$ is fully faithful.

Proof. Note that $\operatorname{ev}_I = I^*$ is given by precomposition with $\operatorname{In}_I\colon \{I\} \hookrightarrow \mathcal{I}$ and thus has a right adjoint given by taking the pointwise right Kan extension along I (granted that all these exist)

$$C \mapsto I_*C = \lim(- \downarrow \operatorname{In}_I \to \{C\} \hookrightarrow \mathcal{C}),$$

so that $(I_*C)_{I'} = C^{\pi_0(I' \downarrow \operatorname{In}_I)}$ is the limit of a constant diagram for $I' \in \mathcal{I}$. But $I' \downarrow \operatorname{In}_I$ is a discrete category with objects $\mathcal{I}(I', I)$. The counit's component at $C \in \mathcal{C}$ is $\operatorname{pr}_{\mathrm{id}_I}\colon C^{\operatorname{End}_{\mathcal{I}}(I)} \to C$, which is invertible if $\operatorname{End}_{\mathcal{I}}(I) = \{\mathrm{id}_I\}$. $\square$

(8.2) **Observation.** Given a category $\mathcal{C}$, index categories $\mathcal{I}$, $\mathcal{J}$ and an object $J \in \mathcal{J}$ such that $J_*\colon \mathcal{C} \to \mathcal{C}^{\mathcal{J}}$ as in the proposition exists, then so does $J_*\colon \mathcal{C}^{\mathcal{I}} \to (\mathcal{C}^{\mathcal{I}})^{\mathcal{J}} \cong \mathcal{C}^{\mathcal{I}\times\mathcal{J}}$ (all powers are calculated pointwise) and an easy calculation shows that

$$\begin{tikzcd} \mathcal{C}^{\mathcal{J}} \arrow[r, "\mathrm{ev}_J"] \arrow[d, "\Delta"'] & \mathcal{C} \arrow[d, "\Delta"] \\ \mathcal{C}^{\mathcal{I}\times\mathcal{J}} \arrow[r, "\mathrm{ev}_J"'] & \mathcal{C}^{\mathcal{I}} \end{tikzcd}$$

(filled with the identity) satisfies the dual Beck-Chevalley condition. Indeed, the identity's mate is again the identity, for which we only need to check that $\Delta\varepsilon = \varepsilon'_{\Delta}$, where ε is the counit of the top adjunction and ε' the counit of the bottom one. So let $C \in \mathcal{C}$, $I \in \mathcal{I}$ and then

$$(\varepsilon'_{\Delta C})_I = (\mathrm{pr}_{\mathrm{id}_J})_I\colon \big((\Delta C)^{\mathcal{J}(J,J)}\big)_I = \big((\Delta C)I\big)^{\mathcal{J}(J,J)} = C^{\mathcal{J}(J,J)} \xrightarrow{\mathrm{pr}_{\mathrm{id}_J}} C = (\Delta C)_I$$

while $(\Delta\varepsilon_C)_I = \varepsilon_C = \mathrm{pr}_{\mathrm{id}_J}$. If a right adjoint J_* to ev_J exists but is not calculated pointwise as in the proposition, then the square still satisfies the dual Beck-Chevalley condition, granted that $\mathcal{C}$ has $\mathcal{J}$-colimits. This follows from Beck-Chevalley interchange and (5.4).

In case $\mathcal{C}$ is a model category (in particular bicomplete) and $\mathcal{C}^{\mathcal{J}}$ carries the projective model structure, we even obtain two Quillen adjunctions. The less trivial part of this result can also be found e.g. in [18, Lemme 3.1.12].

(8.3) **Corollary.** If $\mathcal{C}$ is a model category and $\mathcal{J}$ an index category such that the projective model structure on $\mathcal{C}^{\mathcal{J}}$ exists then $J_! \dashv \mathrm{ev}_J \dashv J_*$ are both Quillen adjunctions for all $J \in \mathcal{J}$.

Proof. The adjunction $J_! \dashv \mathrm{ev}_J$ is easy because ev_J preserves fibrations and acyclic fibrations. For the other adjunction, let $f\colon C \to D$ be an (acyclic) fibration in $\mathcal{C}$ and $J' \in \mathcal{J}$. Then

$$J_*(C \xrightarrow{f} D)_{J'} = C^{\mathcal{J}(J',J)} \xrightarrow{f^{\mathcal{J}(J',J)}} D^{\mathcal{J}(J',J)}$$

and a product of (acyclic) fibrations is again an (acyclic) fibration. □

(8.4) **Example.** Let $\mathcal{C}$ be a category with weak equivalences that has a terminal object. If $\mathcal{J}$ is a small preorder, then ev_J has a fully faithful right adjoint J_* and the adjunction

$$\begin{tikzcd} \mathcal{C}^{\mathcal{J}} \arrow[r, shift left, "\mathrm{ev}_J"] \arrow[r, phantom, "\perp"] & \mathcal{C} \arrow[l, shift left, "J_*"] \end{tikzcd} \qquad \text{lifts to} \qquad \begin{tikzcd} \mathrm{Ho}(\mathcal{C}^{\mathcal{J}}) \arrow[r, shift left, "\mathrm{Ho}\,\mathrm{ev}_J"] \arrow[r, phantom, "\perp"] & \mathrm{Ho}\,\mathcal{C} \arrow[l, shift left, "\mathrm{Ho}\,J_*"] \end{tikzcd}$$

for all $J \in \mathcal{J}$. Dually, if $\mathcal{C}$ has an initial object and $\mathcal{J}$ is a preorder then ev_J has a fully faithful left adjoint $J_!$ and the adjunction

$$\begin{tikzcd} \mathcal{C}^{\mathcal{J}} \arrow[r, shift right, "\mathrm{ev}_J"'] \arrow[r, phantom, "\perp"] & \mathcal{C} \arrow[l, shift right, "J_!"'] \end{tikzcd} \qquad \text{lifts to} \qquad \begin{tikzcd} \mathrm{Ho}(\mathcal{C}^{\mathcal{J}}) \arrow[r, shift right, "\mathrm{Ho}\,\mathrm{ev}_J"'] \arrow[r, phantom, "\perp"] & \mathrm{Ho}\,\mathcal{C} \arrow[l, shift right, "\mathrm{Ho}\,J_!"'] \end{tikzcd}.$$

9. Derived Functors on Diagram Categories

A common trick in homotopy theory is to first draw a 3×3-diagram of the form

$$\begin{tikzcd} \bullet & \bullet \arrow[l] \arrow[r] & \bullet \\ \bullet \arrow[u] \arrow[d] & \bullet \arrow[l] \arrow[r] \arrow[u] \arrow[d] & \bullet \arrow[u] \arrow[d] \\ \bullet & \bullet \arrow[l] \arrow[r] & \bullet \end{tikzcd}$$

and then use that the homotopy colimit of this diagram can be calculated as a double homotopy pushout in two ways: rows first or columns first. This is usually taken for granted and under suitably nice hypotheses (e.g. working in a model category with functorial factorisations) very easy to derive from the corresponding result for strict colimits.

When analysing this technique in order to weaken the hypotheses as far as possible, one sees that the whole trick is a combination of two results. First one uses a Fubini type result, which says that a homotopy colimit indexed by $\mathcal{I} \times \mathcal{J}$ can be calculated as a double homotopy colimit and second, one uses that homotopy colimits in a diagram category $\mathcal{C}^{\mathcal{J}}$ can be calculated pointwise (cf. (5.4)). More specifically, the *Fubini theorem* is the following.

(9.1) **Theorem.** If $\mathcal{C}$ is a category with weak equivalences and $\mathcal{I}$, $\mathcal{J}$ are index categories such that $\mathcal{C}^{\mathcal{I}}$ has $\mathcal{J}$-homotopy colimits while $\mathcal{C}$ has $\mathcal{I}$-homotopy colimits. Then $\mathcal{C}$ has $\mathcal{I} \times \mathcal{J}$-homotopy colimits and they are given by

$$\operatorname{Ho}(\mathcal{C}^{\mathcal{I}\times\mathcal{J}}) \cong \operatorname{Ho}\big((\mathcal{C}^{\mathcal{I}})^{\mathcal{J}}\big) \xrightarrow{\operatorname{hocolim}_{\mathcal{J}}} \operatorname{Ho}(\mathcal{C}^{\mathcal{I}}) \xrightarrow{\operatorname{hocolim}_{\mathcal{I}}} \operatorname{Ho}\mathcal{C}.$$

Proof. Look at the right adjoints. □

For the second result recall that if $\mathcal{C}$ is a cocomplete category, then every diagram category $\mathcal{C}^{\mathcal{J}}$ is again cocomplete and the colimits are calculated pointwise. Unfortunately, for homotopy colimits, the matter is not that simple. For instance, the existence of homotopy colimits in a category $\mathcal{C}$ (equipped with a class of weak equivalences) does not imply the existence of homotopy colimits in a diagram category $\mathcal{C}^{\mathcal{J}}$ (with pointwise weak equivalences). However if we assume their existence we can study interactions between homotopy colimits in different categories.

(9.2) **Proposition.** Let $\mathcal{C}$ be a category with weak equivalences, $\mathcal{I}$, $\mathcal{J}$ two index categories and $J \in \mathcal{J}$ such that $J_! \dashv \operatorname{ev}_J \dashv J_*$ exist and are both derivable. If $\mathcal{C}^{\mathcal{J}}$ has $\mathcal{I}$-homotopy colimits then $\mathcal{C}$ has $\mathcal{I}$-homotopy colimits, too.

Proof. We obtain a left adjoint to $\operatorname{Ho}\Delta\colon \operatorname{Ho}\mathcal{C} \to \operatorname{Ho}(\mathcal{C}^{\mathcal{I}})$ by

$$\begin{tikzcd}[column sep=large] \operatorname{Ho}(\mathcal{C}^{\mathcal{I}}) \arrow[r, shift left, "\mathbb{L}J_!"] \arrow[r, phantom, "\perp"] & \operatorname{Ho}\big((\mathcal{C}^{\mathcal{I}})^{\mathcal{J}}\big) \arrow[l, shift left, "\operatorname{Ho}\operatorname{ev}_J"] \cong \operatorname{Ho}(\mathcal{C}^{\mathcal{I}\times\mathcal{J}}) \cong \operatorname{Ho}\big((\mathcal{C}^{\mathcal{J}})^{\mathcal{I}}\big) \arrow[r, shift left, "\operatorname{hocolim}"] \arrow[r, phantom, "\perp"] & \operatorname{Ho}(\mathcal{C}^{\mathcal{J}}) \arrow[l, shift left, "\operatorname{Ho}\Delta"] \arrow[r, shift left, "\operatorname{Ho}\operatorname{ev}_J"] \arrow[r, phantom, "\perp"] & \operatorname{Ho}\mathcal{C} \arrow[l, shift left, "\mathbb{R}J_*"] \end{tikzcd}$$

because $\mathbb{R}J_*$ and $\operatorname{Ho}(\operatorname{ev}_J \circ \Delta)$ compose (cf. (6.10)) and $\operatorname{ev}_J \circ \Delta \circ J_* \cong \Delta$. □

(9.3) **Example.** The conditions of this proposition are satisfied if $\mathcal{C}$ has an initial and a terminal object while J is such that $\mathcal{J}(J', J)$ and $\mathcal{J}(J, J')$ contain at most one element for all $J' \in \mathcal{J}$ (e.g. $\mathcal{J}$ a preorder).

As this proof illustrates, a very basic tactic for establishing results about hocolim is to switch to adjoints and work with $\mathrm{Ho}\,\Delta$ instead. With this in mind, a direct attack on the "pointwiseness" of homotopy colimits in a diagram category might go as follows: If $\mathrm{Ho}\,\Delta$ composes with $\mathbb{R}J_*$ then two composite right adjoints

$$\mathbb{R}J_* \circ \mathrm{Ho}\,\Delta = \mathbb{R}(J_* \circ \Delta) = \mathbb{R}(\Delta \circ J_*) = \mathrm{Ho}\,\Delta \circ \mathbb{R}J_*$$

are the same and so its conjugate is an isomorphism between composite left adjoints

$$\mathrm{Ho}\,\mathrm{ev}_J \circ \mathrm{hocolim} \cong \mathrm{hocolim} \circ \mathrm{Ho}\,\mathrm{ev}_J.$$

Although already a first step, this line of thought only tells us that we can calculate homotopy colimits pointwise on the object level and we cannot conclude anything on the level of arrows. More specifically, if $j\colon J \to J'$ is an arrow we obtain $\mathrm{ev}_j\colon \mathcal{C}^{\mathcal{J}} \to \mathcal{C}^{[1]}$, which induces $\mathrm{Ho}\,\mathrm{ev}_j\colon \mathrm{Ho}(\mathcal{C}^{\mathcal{J}}) \to \mathrm{Ho}(\mathcal{C}^{[1]})$, so that for $X \in \mathrm{Ho}(\mathcal{C}^{\mathcal{I}\times\mathcal{J}})$,

$$(\mathrm{Ho}\,\mathrm{ev}_j)\,\mathrm{hocolim}\,X \in \mathrm{Ho}(\mathcal{C}^{[1]}).$$

On the other hand, if we view $X(-,j)\colon X(-,J) \to X(-,J')$ as a morphism in $\mathrm{Ho}(\mathcal{C}^{\mathcal{I}})$ (i.e. an object in $\mathrm{Ho}(\mathcal{C}^{\mathcal{I}})^{[1]}$) then applying hocolim gives a morphism in $\mathrm{Ho}\,\mathcal{C}$ (i.e. an object in $(\mathrm{Ho}\,\mathcal{C})^{[1]}$) and we need to show that this is the same as

$$(H_{\mathcal{C}})_*^-(\mathrm{Ho}\,\mathrm{ev}_j)\,\mathrm{hocolim}\,X, \quad \text{where } (H_{\mathcal{C}})_*^- \text{ is defined by} \quad \begin{tikzcd} \mathcal{C}^{[1]} \arrow[r, "H_{\mathcal{C}^{[1]}}"] \arrow[dr, "(H_{\mathcal{C}})_*"'] & \mathrm{Ho}(\mathcal{C}^{[1]}) \arrow[d, "(H_{\mathcal{C}})_*^-"] \\ & (\mathrm{Ho}\,\mathcal{C})^{[1]} \end{tikzcd}$$

being commutative. In fact, it is not even clear why $\mathrm{hocolim}\,X(-,j)$ should lie in the image of $H_{\mathcal{C}}$ (up to conjugation with a natural isomorphism). A more informal way to summarise this is that we would like the isomorphism $\mathrm{hocolim} \circ \mathrm{Ho}\,\mathrm{ev}_J \cong \mathrm{Ho}\,\mathrm{ev}_J \circ \mathrm{hocolim}$ to be natural in $J \in \mathcal{J}$; i.e. such that for all $j\colon J \to J'$ in $\mathcal{J}$ the following square in $\mathrm{Ho}\,\mathcal{C}$ is commutative:

$$\begin{tikzcd} \mathrm{hocolim}\,X(-,J) \arrow[r, phantom, "\cong"] \arrow[d, "\mathrm{hocolim}\,X(-,j)"'] & (\mathrm{hocolim}\,X)_J \arrow[d, "(\mathrm{hocolim}\,X)_j"] \\ \mathrm{hocolim}\,X(-,J') \arrow[r, phantom, "\cong"] & (\mathrm{hocolim}\,X)_{J'} \end{tikzcd} .$$

Again inspired by the corresponding strict result in the form of (5.4), the (horizontal) Beck-Chevalley condition comes to our rescue.

(9.4) **Theorem.** Let $\mathcal{C}$ be a category with weak equivalences, $\mathcal{I}$, $\mathcal{J}$ two index categories and $J \in \mathcal{J}$ such that all derived adjunctions in the left-hand square

$$\begin{tikzcd} \mathrm{Ho}((\mathcal{C}^{\mathcal{I}})^{\mathcal{J}}) \cong \mathrm{Ho}((\mathcal{C}^{\mathcal{J}})^{\mathcal{I}}) \arrow[r, shift left, "\mathrm{hocolim}"] \arrow[r, phantom, "\perp"] \arrow[d, shift right, "\mathrm{Ho}\,\mathrm{ev}_J"'] \arrow[d, phantom, "\dashv"] & \mathrm{Ho}(\mathcal{C}^{\mathcal{J}}) \arrow[l, shift left, "\mathrm{Ho}\,\Delta"] \arrow[d, shift right, "\mathrm{Ho}\,\mathrm{ev}_J"'] \arrow[d, phantom, "\dashv"] \\ \mathrm{Ho}(\mathcal{C}^{\mathcal{I}}) \arrow[r, shift left, "\mathrm{hocolim}"] \arrow[r, phantom, "\perp"] \arrow[u, shift right, "\mathbb{R}J_*"'] & \mathrm{Ho}\,\mathcal{C} \arrow[l, shift left, "\mathrm{Ho}\,\Delta"] \arrow[u, shift right, "\mathbb{R}J_*"'] \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Ho}((\mathcal{C}^{\mathcal{I}})^{\mathcal{J}}) \cong \mathrm{Ho}((\mathcal{C}^{\mathcal{J}})^{\mathcal{I}}) \arrow[d, "\mathrm{Ho}\,\mathrm{ev}_J"'] & \mathrm{Ho}(\mathcal{C}^{\mathcal{J}}) \arrow[l, "\mathrm{Ho}\,\Delta"'] \arrow[d, "\mathrm{Ho}\,\mathrm{ev}_J"] \\ \mathrm{Ho}(\mathcal{C}^{\mathcal{I}}) & \mathrm{Ho}\,\mathcal{C} \arrow[l, "\mathrm{Ho}\,\Delta"] \end{tikzcd}$$

exist and $\mathrm{Ho}\,\Delta$ composes with $\mathbb{R}J_*$. Then the right-hand square (filled with the identity) satisfies the (horizontal) Beck-Chevalley condition. In particular, there is an isomorphism

$$\mathrm{hocolim} \circ \mathrm{Ho}\,\mathrm{ev}_J \cong \mathrm{Ho}\,\mathrm{ev}_J \circ \mathrm{hocolim}, \qquad \text{natural in } J \in \mathcal{J}.$$

Proof. By Beck-Chevalley interchange, it suffices to check that the right-hand square satisfies the vertical dual Beck-Chevalley condition. For this, we observe that $\operatorname{Ho}\operatorname{ev}_J$ composes with hocolim since $\operatorname{Ho}\Delta$ composes with $\mathbb{R}J_*$ (cf. (6.12)) and that on the strict level, the right-hand square from the proposition satisfies the Beck-Chevalley condition (cf. (8.2)). Now use (6.13). Finally, the "in particular"-part is just an instance of (4.5). To wit, for $j\colon J \to J'$ in $\mathcal{J}$, the left-hand square below commutes iff the right-hand square does, where the two horizontal pairs are mates:

$$\begin{tikzcd}
\operatorname{hocolim} \circ \operatorname{Ho}\operatorname{ev}_J \arrow[r, "\sigma"] \arrow[d, "\operatorname{hocolim}\operatorname{Ho}\operatorname{ev}_j"'] & \operatorname{Ho}\operatorname{ev}_J \circ \operatorname{hocolim} \arrow[d, "(\operatorname{Ho}\operatorname{ev}_j)_{\operatorname{hocolim}}"'] \\
\operatorname{hocolim} \circ \operatorname{Ho}\operatorname{ev}_{J'} \arrow[r, "\sigma'"'] & \operatorname{Ho}\operatorname{ev}_{J'} \circ \operatorname{hocolim}
\end{tikzcd}
\qquad
\begin{tikzcd}
\operatorname{Ho}\operatorname{ev}_J \circ \operatorname{Ho}\Delta \arrow[r, "\mathrm{id}"] \arrow[d, "(\operatorname{Ho}\operatorname{ev}_j)_{\operatorname{Ho}\Delta}"'] & \operatorname{Ho}\Delta \circ \operatorname{Ho}\operatorname{ev}_j \arrow[d, "(\operatorname{Ho}\Delta)\operatorname{Ho}\operatorname{ev}_j"'] \\
\operatorname{Ho}\operatorname{ev}_{J'} \circ \operatorname{Ho}\Delta \arrow[r, "\mathrm{id}"'] & \operatorname{Ho}\Delta \circ \operatorname{Ho}\operatorname{ev}_{J'}.
\end{tikzcd}$$

□

In some situations (e.g. when using left model approximations for the construction of homotopy colimits as in [15]) it might not be possible to verify the hypotheses of the above theorem. Indeed, it might not even be possible to construct the right adjoint $\mathbb{R}J_*$. However, we may still get away by using the left adjoint $\mathbb{L}J_!$ of $\operatorname{Ho}\operatorname{ev}_J$ rather than $\mathbb{R}J_*$. Admittedly, this approach is (even) less elegant than the one in the theorem.

(9.5) **Proposition.** Let $\mathcal{C}$ be a category with weak equivalences, $\mathcal{I}$, $\mathcal{J}$ two index categories and $J \in \mathcal{J}$ such that all derived adjunctions in the square

$$\begin{tikzcd}
\operatorname{Ho}\big((\mathcal{C}^{\mathcal{I}})^{\mathcal{J}}\big) \cong \operatorname{Ho}\big((\mathcal{C}^{\mathcal{J}})^{\mathcal{I}}\big) \arrow[r, shift left, "\operatorname{hocolim}"] \arrow[r, phantom, "\perp"] \arrow[d, shift left, "\operatorname{Ho}\operatorname{ev}_J"] \arrow[d, phantom, "\dashv"] & \operatorname{Ho}(\mathcal{C}^{\mathcal{J}}) \arrow[l, shift left, "\operatorname{Ho}\Delta"] \arrow[d, shift left, "\operatorname{Ho}\operatorname{ev}_J"] \arrow[d, phantom, "\dashv"] \\
\operatorname{Ho}(\mathcal{C}^{\mathcal{I}}) \arrow[u, shift left, "\mathbb{L}J_!"] \arrow[r, shift left, "\operatorname{hocolim}"] \arrow[r, phantom, "\perp"] & \operatorname{Ho}\mathcal{C} \arrow[u, shift left, "\mathbb{L}J_!"] \arrow[l, shift left, "\operatorname{Ho}\Delta"]
\end{tikzcd}$$

exist and whose units we denote by η, θ, θ' and η' (top to bottom, left to right). Then there is

$$\alpha\colon \operatorname{hocolim} \circ \mathbb{L}J_! \cong \mathbb{L}J_! \circ \operatorname{hocolim}$$

compatible with the composite units (and counits for that matter) in the sense that

$$\operatorname{Ho}(\operatorname{ev}_J \circ \Delta)\alpha \circ (\operatorname{Ho}\operatorname{ev}_J)\eta_{\mathbb{L}J_!} \circ \theta = (\operatorname{Ho}\Delta)\theta'_{\operatorname{hocolim}} \circ \eta'.$$

Proof. Obviously $\operatorname{ev}_J \circ \Delta = \Delta \circ \operatorname{ev}_J$, whence the square of right adjoints commutes. □

While the compatibility formula in this proposition might seem useless at first, note that if the $\mathbb{L}J_!$ are fully faithful (or equivalently, if θ and θ' are isomorphisms), we get a good starting point for comparing η and η'.

(9.6) **Corollary.** Under the hypotheses of (9.5), if the two $\mathbb{L}J_!$ are fully faithful and there is a natural isomorphism $\operatorname{hocolim} \circ \operatorname{Ho}\operatorname{ev}_J \cong \operatorname{Ho}\operatorname{ev}_J \circ \operatorname{hocolim}$ then there is even a natural isomorphism $\beta_J\colon \operatorname{hocolim} \circ \operatorname{Ho}\operatorname{ev}_J \cong \operatorname{Ho}\operatorname{ev}_J \circ \operatorname{hocolim}$ such that $(\operatorname{Ho}\operatorname{ev}_J)\eta = (\operatorname{Ho}\Delta)\beta_J \circ \eta'_{\operatorname{Ho}\operatorname{ev}_J}$.

Proof. The compatibility condition of α at its $(\operatorname{Ho}\operatorname{ev}_J)X$-component for $X \in \operatorname{Ho}\big((\mathcal{C}^{\mathcal{I}})^{\mathcal{J}}\big)$ reads

$$(\operatorname{Ho}\Delta)\Big(\theta'^{-1}_{\operatorname{hocolim}} \circ (\operatorname{Ho}\operatorname{ev}_J)\alpha\Big)_{(\operatorname{Ho}\operatorname{ev}_J)X} \circ (\operatorname{Ho}\operatorname{ev}_J)\eta_{\mathbb{L}J_!(\operatorname{Ho}\operatorname{ev}_J)X} \circ \theta_{(\operatorname{Ho}\operatorname{ev}_J)X} = \eta'_{(\operatorname{Ho}\operatorname{ev}_J)X}$$

and we need to show that $(\mathrm{Ho}\,\mathrm{ev}_J)\eta_{\mathbb{L}J_!\mathrm{ev}_J X} = (\mathrm{Ho}\,\mathrm{ev}_J)\eta_X$ up to composition with suitable natural isomorphisms. For this, let us write ζ for the counit of $\mathbb{L}J_! \dashv \mathrm{Ho}\,\mathrm{ev}_J$ and note that $(\mathrm{Ho}\,\mathrm{ev}_J)\zeta$ is invertible with inverse $\theta_{\mathrm{Ho}\,\mathrm{ev}_J}$. The naturality of η gives a commutative square

$$\begin{tikzcd}
(\mathrm{Ho}\,\mathrm{ev}_J)\mathbb{L}J_!(\mathrm{Ho}\,\mathrm{ev}_J)X \arrow[r, "(\mathrm{Ho}\,\mathrm{ev}_J)\eta_{\mathbb{L}J_!(\mathrm{Ho}\,\mathrm{ev}_J)X}"] \arrow[d, "(\mathrm{Ho}\,\mathrm{ev}_J)\zeta_X"', "\cong"] & (\mathrm{Ho}\,\mathrm{ev}_J)(\mathrm{Ho}\,\Delta)\,\mathrm{hocolim}\,\mathbb{L}J_!(\mathrm{Ho}\,\mathrm{ev}_J)X \arrow[d, "\cong"', "(\mathrm{Ho}\,\mathrm{ev}_J)(\mathrm{Ho}\,\Delta)\,\mathrm{hocolim}\,\zeta_X"] \\
(\mathrm{Ho}\,\mathrm{ev}_J)X \arrow[r, "(\mathrm{Ho}\,\mathrm{ev}_J)\eta_X"'] & (\mathrm{Ho}\,\mathrm{ev}_J)(\mathrm{Ho}\,\Delta)\,\mathrm{hocolim}\,X
\end{tikzcd} ,$$

where the right-hand arrow $(\mathrm{Ho}\,\Delta)(\mathrm{Ho}\,\mathrm{ev}_J)\,\mathrm{hocolim}\,\zeta_X$ is invertible because $\mathrm{Ho}\,\mathrm{ev}_J$ commutes with hocolim, while the left-hand arrow cancels with $\theta_{(\mathrm{Ho}\,\mathrm{ev}_J)X}$ and so $\eta'_{(\mathrm{Ho}\,\mathrm{ev}_J)X}$ equals

$$(\mathrm{Ho}\,\Delta)\Big(\theta'^{-1}_{\mathrm{hocolim}(\mathrm{Ho}\,\mathrm{ev}_J)} \circ (\mathrm{Ho}\,\mathrm{ev}_J)\alpha_{\mathrm{Ho}\,\mathrm{ev}_J} \circ \big((\mathrm{Ho}\,\mathrm{ev}_J)\,\mathrm{hocolim}\,\zeta\big)^{-1}\Big)_X \circ (\mathrm{Ho}\,\mathrm{ev}_J)\eta_X. \qquad \square$$

The last obstacle to overcome before assembling this all is that we need some criterion to decide when the $\mathbb{L}J_!$ are fully faithful. This turns out to be rather simple and it suffices if $J_!$ is so (e.g. as in (8.1)).

(9.7) **Proposition.** Let $\mathcal{C}$ be a category with weak equivalences, $\mathcal{J}$ some index category and $J \in \mathcal{J}$ such that a fully faithful left adjoint $J_!$ to ev_J exists and $J_! \dashv \mathrm{ev}_J$ is derivable. Then $\mathbb{L}J_!$ is fully faithful.

Proof. Choosing compatible units and counits of the derived adjunction, (6.3) gives us

$$\rho_F \circ H_{\mathcal{C}}\eta = (\mathrm{Ho}\,\mathrm{ev}_J)\lambda \circ \dot{\eta}_{H_{\mathcal{C}}},$$

which is an isomorphism because $\mathbb{R}\mathrm{ev}_J = \mathrm{Ho}\,\mathrm{ev}_J$ (i.e. ρ is invertible) and $J_!$ is fully faithful (i.e. η is invertible). Now note that $\mathbb{L}J_!$ and $\mathrm{Ho}\,\mathrm{ev}_J$ compose, whence $\mathrm{Ho}\,\mathrm{ev}_J \circ \mathbb{L}J_! \cong \mathbb{L}(\mathrm{ev}_J \circ J_!)$. But $\mathrm{ev}_J \circ J_! \cong \mathrm{id}_{\mathcal{C}}$, so that $\mathbb{L}(\mathrm{ev}_J \circ J_!) \cong \mathrm{id}_{\mathrm{Ho}\,\mathcal{C}}$ and λ must be invertible. It follows that $\dot{\eta}_{H_{\mathcal{C}}}$ (or equivalently $\dot{\eta}$) is invertible, thus proving our claim. $\square$

(9.8) **Remark.** Obviously, this proposition generalises mutatis mutandis to an arbitrary derivable adjunction with a fully faithful left and a homotopical right adjoint.

(9.9) **Example.** The proposition in particular applies in both of the following cases:

(a) $\mathrm{In}_J \downarrow J' \subseteq J \downarrow \mathcal{J}$ is connected for all $J' \in \mathcal{J}$ and $\mathcal{C}$ has an initial object;

(b) $\mathcal{C}$ is a model category and the projective model structure on $\mathcal{C}^{\mathcal{J}}$ exists.

(9.10) **Theorem.** Let $\mathcal{C}$ be a category with weak equivalences and $\mathcal{I}$, $\mathcal{J}$ two index categories such that for all $J \in \mathcal{J}$, the evaluation functor ev_J has a fully faithful left adjoint $J_!$ and the derived adjunctions in the square

$$\begin{tikzcd}
\mathrm{Ho}\big((\mathcal{C}^{\mathcal{I}})^{\mathcal{J}}\big) \cong \mathrm{Ho}\big((\mathcal{C}^{\mathcal{J}})^{\mathcal{I}}\big) \arrow[r, shift left, "\mathrm{hocolim}"] \arrow[r, phantom, "\perp"] \arrow[d, shift left, "\mathrm{Ho}\,\mathrm{ev}_J"] \arrow[d, phantom, "\dashv"] & \mathrm{Ho}(\mathcal{C}^{\mathcal{J}}) \arrow[l, shift left, "\mathrm{Ho}\,\Delta"] \arrow[d, shift left, "\mathrm{Ho}\,\mathrm{ev}_J"] \arrow[d, phantom, "\dashv"] \\
\mathrm{Ho}(\mathcal{C}^{\mathcal{I}}) \arrow[u, hook, shift left, "\mathbb{L}J_!"] \arrow[r, shift left, "\mathrm{hocolim}"] \arrow[r, phantom, "\perp"] & \mathrm{Ho}\,\mathcal{C} \arrow[l, shift left, "\mathrm{Ho}\,\Delta"] \arrow[u, hook, shift left, "\mathbb{L}J_!"]
\end{tikzcd}$$

exist and there is an isomorphism $\mathrm{hocolim} \circ \mathrm{Ho}\,\mathrm{ev}_J \cong \mathrm{Ho}\,\mathrm{ev}_J \circ \mathrm{hocolim}$. Then there is even a family of isomorphisms

$$\beta_J\colon\ \mathrm{hocolim} \circ \mathrm{Ho}\,\mathrm{ev}_J \cong \mathrm{Ho}\,\mathrm{ev}_J \circ \mathrm{hocolim} \qquad \text{natural in } J.$$

Proof. For $J \in \mathcal{J}$ let us choose β_J as in (9.6) and note that for $j\colon J \to J'$ in $\mathcal{J}$ the arrow $\operatorname{hocolim} X(-,j)$ is the unique $f\colon \operatorname{hocolim} X(-,J) \to \operatorname{hocolim} X(-,J')$ in $\operatorname{Ho}\mathcal{C}$ making

$$\begin{tikzcd}
X(-,J) \arrow[r, "\eta'_{X(-,J)}"] \arrow[d, "X(-,j)"'] & (\operatorname{Ho}\Delta)\operatorname{hocolim} X(-,J) \arrow[d, dotted, "(\operatorname{Ho}\Delta)f"] \\
X(-,J') \arrow[r, "\eta'_{X(-,J')}"'] & (\operatorname{Ho}\Delta)\operatorname{hocolim} X(-,J')
\end{tikzcd}$$

in $\operatorname{Ho}(\mathcal{C}^{\mathcal{I}})$ commute. Extending this square to the right using $\beta_{J,X}$ and $\beta_{J',X}$, the naturality of $\operatorname{Ho}\operatorname{ev}_j$ yields a commutative rectangle of solid arrows in $\operatorname{Ho}(\mathcal{C}^{\mathcal{I}})$

$$\begin{tikzcd}
X(-,J) \arrow[r, "\eta'_{X(-,J)}"] \arrow[d, "X(-,j)"'] & (\operatorname{Ho}\Delta)\operatorname{hocolim} X(-,J) \arrow[r, "(\operatorname{Ho}\Delta)\beta_{J,X}", "\cong"'] \arrow[d, dotted] & (\operatorname{Ho}\Delta)(\operatorname{hocolim} X)J \arrow[d, "(\operatorname{Ho}\Delta)(\operatorname{hocolim} X)j"] \\
X(-,J') \arrow[r, "\eta'_{X(-,J')}"'] & (\operatorname{Ho}\Delta)\operatorname{hocolim} X(-,J) \arrow[r, "\cong", "(\operatorname{Ho}\Delta)\beta_{J',X}"'] & (\operatorname{Ho}\Delta)\operatorname{hocolim} X(-,J')
\end{tikzcd}$$

with the top and bottom composites being $(\operatorname{Ho}\operatorname{ev}_J)\eta_X$ and $(\operatorname{Ho}\operatorname{ev}_{J'})\eta_X$ respectively. Upon defining $f := \beta_{J',X}^{-1} \circ (\operatorname{hocolim} X)j \circ \beta_{J,X}$ the dotted arrow $(\operatorname{Ho}\Delta)f$ renders the entire diagram commutative and the claim follows. □

(9.11) **Example.** If $\mathcal{J}$ is a preorder, $\mathcal{C}$ has an initial and a terminal object and $\mathcal{C}^{\mathcal{J}}$ (whence also $\mathcal{C}$) has $\mathcal{I}$-homotopy colimits then these can be calculated pointwise because

$$\mathbb{L}J_! = \operatorname{Ho} J_! \qquad \text{and} \qquad \mathbb{R}J_* = \operatorname{Ho} J_* \quad (\text{hence } \operatorname{Ho}\Delta \text{ composes with } \mathbb{R}J_*).$$

In particular, double homotopy pushouts can always be calculated pointwise.

(9.12) **Example.** If $\mathcal{C}$ and $\mathcal{C}^{\mathcal{J}}$ are model categories (as always with $\mathcal{C}^{\mathcal{J}}$ carrying the pointwise weak equivalences) such that the projective model structures on $\mathcal{C}^{\mathcal{I}}$ and $(\mathcal{C}^{\mathcal{J}})^{\mathcal{I}}$ exist then $\mathcal{I}$-homotopy colimits in $\mathcal{C}^{\mathcal{J}}$ (whence also in $\mathcal{C}$) exist because $\operatorname{Ho}\Delta$ is a right Quillen functor and so is $\mathbb{R}J_*$, whence they compose. Also, as already seen, the $\mathbb{L}J_!$ are fully faithful and it follows that homotopy colimits in $\operatorname{Ho}(\mathcal{C}^{\mathcal{I}\times\mathcal{J}})$ can be calculated pointwise.

(9.13) **Example.** One particular instances of the last example is the case where $\mathcal{C}$ is cofibrantly generated, where $\mathcal{I}$ and $\mathcal{J}$ can be arbitrary. Another instance is the case where $\mathcal{J}$ is Reedy and $\mathcal{I}$ is a direct category. In particular, double homotopy pushouts in model categories always exist and can be calculated pointwise.

Part II

Left Bousfield Localisation

Chapter 2

CATEGORICAL PRELIMINARIES

In this chapter, we are going to recall and discuss some of the necessary categorical preliminaries. Most of the material is standard and can be found in [38] or [5] with the exception of the results in section 3, which we have taken from [27]. However, some of the results (such as the finality of the Grothendieck construction (4.8)) weren't taken from the literature and some (e.g. (5.1)) are sometimes cited as "well-known" but with unnecessary additional hypotheses.

1. Cocompleteness of Cat

One can easily check that the category **Cat** of small categories is complete. To wit, given a family of (small) categories $(\mathcal{C}_i)_{i\in I}$, their product $\mathcal{C}$ has

$$\operatorname{Ob}\mathcal{C} := \prod_{i\in I}\operatorname{Ob}\mathcal{C}_i \qquad \text{and} \qquad \operatorname{Arr}\mathcal{C} := \prod_{i\in I}\operatorname{Arr}\mathcal{C}_i$$

with the obvious domain, codomain, identity and composition function. Similarly, given two parallel functors F, $G\colon \mathcal{C} \rightrightarrows \mathcal{D}$, their equaliser is the subcategory of $\mathcal{C}$, comprising all objects and arrows that have the same value under both functors.

So, the completeness of **Cat** is pretty straightforward but cocompleteness is another story. Coproducts are easy enough and are obtained by taking disjoint unions of objects and arrows. Coequalisers are where it gets tricky. The problem is that contrary to **Sets**, where quotienting can only make a set smaller, a categorical quotient can make a category bigger, in the sense that it can introduce new arrows. To wit, given a category $\mathcal{C}$ together with an equivalence relation on $\mathcal{C}$ and two arrows $f\colon A \to B$, $g\colon C \to D$ with $B \neq C$ but $[B] = [C]$, the arrows g and f become composable in the quotient. To construct quotients in **Cat**, we need to take a detour via the category of graphs.

(1.1) **Definition.** In this section, the word *graph* shall always mean "directed multigraph with loops". More explicitly, a graph G consists of a class $\operatorname{Ob}G$ of objects (or vertices) and a class $\operatorname{Arr}G$ of arrows that comes with a domain and codomain map dom, cod: $\operatorname{Arr}G \to \operatorname{Ob}G$. Just like for categories, we will abuse notation and simply write $A \in G$ and $f \in G$ instead of $A \in \operatorname{Ob}G$ and $f \in \operatorname{Arr}G$ as long as there is no risk of confusion. Similarly, for two objects A, $B \in G$, we write $f\colon A \to B$ to indicate that an arrow $f \in G$ has domain A and codomain B and we will write $G(A,B) \subseteq \operatorname{Arr}G$ for the class of all such arrows. A *morphism of graphs* $F\colon G \to H$ is just a pair of maps $F\colon \operatorname{Ob}G \to \operatorname{Ob}H$ and $F\colon \operatorname{Arr}G \to \operatorname{Arr}H$ being compatible with the domain and codomain functions. This defines the category **Gra** of small graphs (i.e. those where both the objects and the arrows form a set).

(1.2) **Example.** Every category is a graph by just forgetting about the composition and the role of the identities (but keeping them as arrows). This yields a forgetful functor $U\colon \mathbf{Cat} \to \mathbf{Gra}$.

(1.3) **Observation.** The category **Gra** is complete with products and equalisers being constructed just like in **Cat**.

(1.4) **Proposition.** The functor $U\colon \mathbf{Cat} \to \mathbf{Gra}$ has a left adjoint $\mathcal{P}\colon \mathbf{Gra} \to \mathbf{Cat}$, which is usually called the *path category.*

Proof. A *(directed) path* in a graph G is just a sequence

$$(A_0, f_1, A_1, f_2, \ldots, f_n, A_n) \qquad \text{of some length } n \in \mathbb{N}$$

where every A_i is an object of G and every $f_i\colon A_{i-1} \to A_i$ is an arrow in G. The set of all vertices $\operatorname{Ob} G$ together with the set of all paths in a small graph G carries an obvious category structure, where a path as above has domain A_0 and codomain A_n. Composition is defined by juxtaposition:

$$(A_n, f_{n+1}, \ldots, A_{n+m}) \circ (A_0, f_1, \ldots, A_n) := (A_0, f_1, \ldots, f_n, A_n, f_{n+1}, \ldots, A_{n+m}),$$

so that all paths (A) of length 0 provide the identities. We denote this path category by $\mathcal{P}G$ and note that this construction is functorial in the obvious manner (for $F\colon G \to H$, the functor $\mathcal{P}F$ is just F on the objects and applies F pointwise on the arrows). Now given a small category $\mathcal{C}$, there is a functor $\varepsilon_{\mathcal{C}}\colon \mathcal{P}U\mathcal{C} \to \mathcal{C}$, which is the identity on objects and maps a path $(A_0, f_1, A_1, \ldots, f_n, A_n)$ to $f_n \circ \ldots \circ f_1$. Similarly, for a small graph G, we have a morphism $\eta_G\colon G \to U\mathcal{P}G$, which is again the identity on objects and includes an arrow $f\colon A \to B$ into $U\mathcal{P}G$ as the path (A, f, B) of length 1. These two functors are easily checked to be natural and to satisfy the triangle identities. $\square$

Given a set X and some relation $R \subseteq X \times X$ describing which elements we would like to identify with each other, we can simply replace R by the equivalence relation generated by it (i.e. its reflexive, symmetric, transitive closure) and quotient by that. We can do the analogous thing in the case of a graph.

(1.5) **Definition.** A *graphical equivalence relation* on a graph G is a (not necessarily full) subgraph $E \subseteq G \times G$ such that $\operatorname{Ob} E$ is an equivalence relation on $\operatorname{Ob} G$ and $\operatorname{Arr} E$ is an equivalence relation on $\operatorname{Arr} G$.

(1.6) **Proposition.** Given a graph G and a graphical equivalence relation $E \subseteq G \times G$, there is a graph G/E having

$$\operatorname{Ob}(G/E) := \operatorname{Ob} G / \operatorname{Ob} E, \quad \operatorname{Arr}(G/E) := \operatorname{Arr} G / \operatorname{Arr} E$$

$$\text{with} \quad \operatorname{dom}[f] = [\operatorname{dom} f] \quad \text{and} \quad \operatorname{cod}[f] = [\operatorname{cod} f] \quad \text{for } f \in \operatorname{Arr} G.$$

Moreover, this graph has the universal property that every morphism of graphs $F\colon G \to H$ that is constant on equivalence classes factors uniquely through the projection $G \to G/E$.

Proof. The graph G/E is well-defined, for if we have $f\colon A \to B$ and $g\colon C \to D$ with $(f, g)\colon (A, C) \to (B, D)$ in E, then, by definition, $[\operatorname{dom} f] = [\operatorname{dom} g]$ and $[\operatorname{cod} f] = [\operatorname{cod} g]$. The universal property follows from the usual universal property of quotients in **Sets** by applying it to $\operatorname{Ob} G$ and $\operatorname{Arr} G$. $\square$

Because intersections of equivalence relations are again equivalence relations, it is clear that given a graph G, every subgraph $S \subseteq G \times G$ generates a smallest graphical equivalence relation containing it, just by taking the intersection of all graphical equivalence relations that do. This description is not very explicit though.

(1.7) **Proposition.** Given a graph G and a subgraph $S \subseteq G \times G$, the graphical equivalence relation E generated by S has as objects the equivalence relation generated by $\operatorname{Ob} S$ and as arrows the equivalence relation generated by $\operatorname{Arr} S$.

Proof. Obviously, the equivalence relations generated by $\operatorname{Ob} S$ and the equivalence relation generated by $\operatorname{Arr} S$, will have to be contained in $\operatorname{Ob} E$ and $\operatorname{Arr} E$, respectively. It suffices therefore to show that these define a graph. Given an arrow $f\colon A \to B$ in E, we have $\operatorname{dom}(f, f) = (A, A)$ and $\operatorname{cod}(f, f) = (B, B)$, which both lie in the reflexive closure of $\operatorname{Ob} S$. Similarly, if $(f, g)\colon (A, B) \to (C, D)$ is in S then (A, B), $(C, D) \in \operatorname{Ob} S$ and therefore $\operatorname{dom}(g, f) = (B, A)$ and $\operatorname{cod}(g, f) = (D, C)$ lie in the symmetric closure of $\operatorname{Ob} S$. Finally, given a sequence of arrows $f_0, \ldots, f_n$ with $f_i\colon A_i \to B_i$ and such that all (f_i, f_{i+1}) lie in the reflexive and symmetric closure of $\operatorname{Arr} S$, we get the sequence of objects $A_0, \ldots, A_n$, where all (A_i, A_{i+1}) lie in the reflexive and symmetric closure of $\operatorname{Ob} S$, so that (A_0, A_n) lies in the equivalence relation generated by $\operatorname{Ob} S$. Similarly for the codomain. □

(1.8) **Remark.** In our situation, where we are given two parallel functors F, $G\colon \mathcal{C} \rightrightarrows \mathcal{D}$ and would like to identify every FC with GC and every Ff with Gf, these even form a subcategory of $\mathcal{D} \times \mathcal{D}$ (rather than just a subgraph) but this property is lost upon passing to the graphical equivalence relation generated by it.

(1.9) **Notation.** Given a graph G, together with a subgraph $S \subseteq G \times G$, we will simply write G/S for the quotient of G by the graphical equivalence relation generated by S.

(1.10) **Corollary.** Given a graph G and $S \subseteq G \times G$, the quotient G/S has the universal property that every morphism of graphs $F\colon G \to H$ satisfying $FA = FB$ and $Ff = Fg$ for all $(A, B) \in S$ and $(f, g) \in S$ factors uniquely through the projection $G \to G/S$. □

Back to the definition of coequalisers in **Cat**. Whenever we are given a pair of parallel functors F, $G\colon \mathcal{C} \rightrightarrows \mathcal{D}$, we can consider the subcategory $\mathcal{S} \subseteq \mathcal{D} \times \mathcal{D}$ comprising all pairs of objects and arrows of the form (FC, GC) and (Ff, Gf). We can then form the quotient graph $\mathcal{D}/\mathcal{S}$ but there is no obvious composition law on it. What we need to do instead is to add a free composition law by taking the path category $\mathcal{P}(\mathcal{D}/\mathcal{S})$ and taking a second, special quotient to add the compositions back in that we already had in $\mathcal{D}$.

(1.11) **Definition.** Given a category $\mathcal{C}$, a *congruence* on $\mathcal{C}$ is a subcategory $\mathcal{E} \subseteq \mathcal{C} \times \mathcal{C}$ such that $\operatorname{Ob} \mathcal{E} = \{(C, C) \mid C \in \mathcal{C}\}$ (i.e. $\operatorname{Ob} \mathcal{E}$ is the equality equivalence relation on $\operatorname{Ob} \mathcal{C}$) and such that $\operatorname{Arr} \mathcal{E}$ is an equivalence relation on $\operatorname{Arr} \mathcal{C}$.

(1.12) **Remark.** Our requirement on $\operatorname{Ob} \mathcal{E}$ above means that whenever we have two arrows f, g in $\mathcal{C}$ that become equivalent upon passing to $\mathcal{C}/\mathcal{E}$, they must have the same domain and codomain. The requirement that $\mathcal{E}$ be a subcategory means that all $(\mathrm{id}_C, \mathrm{id}_C)$ lie in $\mathcal{E}$ (which was already implied by the reflexivity of $\operatorname{Arr} \mathcal{E}$) and that equivalence is stable under composition in the sense that if $[f] = [g]$ and $[f'] = [g']$ in $\mathcal{C}/\mathcal{E}$, then also $[f' \circ f] = [g' \circ g]$ whenever this makes sense.

(1.13) **Observation.** Because the object equivalence relation of a congruence is equality, a congruence on a category $\mathcal{C}$ can equivalently be described as a family of equivalence relations $E_{C,D} \subseteq \mathcal{C}(C,D) \times \mathcal{C}(C,D)$ indexed by all pairs of objects C, $D \in \mathcal{C}$ and such that these are compatible with the composition. In fact, it suffices to have the property that whenever we have arrows

$$A \xrightarrow{h} B \underset{g}{\overset{f}{\rightrightarrows}} C \xrightarrow{i} D$$

with $f \; E_{B,C} \; g$ then also $(f \circ h) \; E_{A,C} \; (g \circ h)$ and $(i \circ f) \; E_{B,D} \; (i \circ g)$.

(1.14) **Proposition.** Given a category $\mathcal{C}$ together with a congruence $\mathcal{E}$, then the quotient graph $\mathcal{C}/\mathcal{E}$ comes equipped with the structure of a category turning the universal morphism $\mathcal{C} \to \mathcal{C}/\mathcal{E}$ into a functor, so that

$$\mathrm{id}_{[C]} = [\mathrm{id}_C] \text{ for } C \in \mathcal{C} \quad \text{and} \quad [f] \circ [g] = [f \circ g] \text{ for composable arrows } f,\, g \in \mathcal{C}.$$

With this, every morphism of graphs $\bar{F} \colon \mathcal{C}/\mathcal{E} \to \mathcal{D}$, induced by a functor $F \colon \mathcal{C} \to \mathcal{D}$ that is constant on equivalence classes, is itself a functor.

Proof. The composition is well-defined because $\mathcal{E} \subseteq \mathcal{C} \times \mathcal{C}$ is a subcategory and it is defined for all composable arrows because $\mathrm{Ob}\,\mathcal{E}$ is the equality equivalence relation on $\mathrm{Ob}\,\mathcal{C}$. As for the final claim, $\bar{F}$ preserves identities because

$$\bar{F}[\mathrm{id}_C] = F\mathrm{id}_C = \mathrm{id}_{FC} = \mathrm{id}_{\bar{F}[C]}$$

and compositions because

$$\bar{F}([f] \circ [g]) = \bar{F}[f \circ g] = F(f \circ g) = Ff \circ Fg = \bar{F}[f] \circ \bar{F}[g].$$ □

Just like for graphical equivalence relations, an intersection of congruences is again congruence, so given a category $\mathcal{C}$, every subgraph $S \subseteq \mathcal{C} \times \mathcal{C}$ with $\mathrm{Ob}\,S \subseteq \{(C,C) \mid C \in \mathcal{C}\}$ generates a congruence. It is more explicitly described as follows.

(1.15) **Proposition.** Let $\mathcal{C}$ be a category and $S \subseteq \mathcal{C} \times \mathcal{C}$ a subgraph whose objects satisfy $\mathrm{Ob}\,S \subseteq \{(C,C) \mid C \in \mathcal{C}\}$ then the congruence $\mathcal{E}$ generated by S has objects $\{(C,C) \mid C \in \mathcal{C}\}$ and its arrows are obtained by the following three step process.

(a) First take the reflexive and symmetric closure of $\mathrm{Arr}\,S$, yielding a graph S_1 (with $\mathrm{Ob}\,S_1 = \{(C,C) \mid C \in \mathcal{C}\}$).

(b) Then close S_1 off under compositions, yielding a graph S_2 with $\mathrm{Ob}\,S_2 = \mathrm{Ob}\,S_1$ and where (f,g) is an arrow in S_2 iff there is $n \in \mathbb{N}$ such that f and g decompose as

$$f = f_n \circ \ldots \circ f_0, \quad g = g_n \circ \ldots \circ g_0 \quad \text{with } (f_i, g_i) \in S_1 \text{ for all } i.$$

(c) Finally, take the transitive closure of $\mathrm{Arr}\,S_2$, yielding a subcategory $S_3 \subseteq \mathcal{C} \times \mathcal{C}$ with $\mathrm{Ob}\,S_3 = \mathrm{Ob}\,S_2$ and where an arrow (f,g) lies in S_3 iff there is some $n \in \mathbb{N}$ and sequence of arrows

$$f = h_0, h_1, \ldots, h_n = g \quad \text{with } (h_i, h_{i+1}) \in S_2 \text{ for all } i.$$

Proof. Obviously, S_3 must be contained in the congruence generated by S and so, it suffices to show that S_3 is itself a congruence. First off, $\operatorname{Arr} S_2$ is clearly reflexive and symmetric, which implies that so is $\operatorname{Arr} S_3$ (because taking the transitive closure preserves reflexiveness and symmetry); hence $\operatorname{Arr} S_3$ is an equivalence relation. It remains to check that S_3 is a subcategory. It has all identities because $\operatorname{Arr} S_3$ is reflexive. To check the stability under compositions, let $(f, f')\colon (A, A) \to (B, B)$ and $(g, g')\colon (B, B) \to (C, C)$ be in S_3, which is to say that we find sequences

$$f = h_0, h_1, \ldots, h_m = f' \quad \text{and} \quad g = k_0, k_1, \ldots, k_n = g'$$

with (h_i, h_{i+1}) and (k_j, k_{j+1}) in S_2 for all i, j. By definition of S_2, it follows that every h_i is an arrow from A to B, while every k_j is an arrow from B to C. But now, we obtain a sequence of arrows

$$g \circ f = k_0 \circ h_0, k_0 \circ h_1, \ldots, k_0 \circ h_m, k_1 \circ h_m, \ldots, k_n \circ h_m = g' \circ f',$$

where every pair of successive arrows in the sequence is contained in S_2 because $\operatorname{Arr} S_2$ is reflexive (implying that (k_0, k_0) and (h_m, h_m) lie in it) and closed under compositions. $\square$

(1.16) **Remark.** The graph S_2 in the proposition can alternatively be replaced by the graph S_2' defined as follows.

(b)' $\operatorname{Ob} S_2' = \operatorname{Ob} S_1$ and (f, g) is an arrow in S_2' iff they decompose as $f = i \circ f' \circ h$ and $g = i \circ g' \circ h$ with $(f', g') \in S_1$.

If we then construct S_3' the same way as S_3 but with S_2' instead of S_2, one easily sees that $S_2 \subseteq S_3'$ and hence $S_3' = S_3$. To wit, given arrows f, g that decompose as

$$f = f_n \circ \ldots \circ f_0, \quad g = g_n \circ \ldots \circ g_0 \quad \text{with } (f_i, g_i) \in S_1 \text{ for all } i,$$

we note that f_i and g_i always have the same domain and codomain, giving us a sequence

$$f = (f_n \circ \ldots \circ f_0), (f_n \circ \ldots \circ f_1 \circ g_0), \ldots, (f_n \circ g_{n-1} \circ \ldots \circ g_0), (g_n \circ \ldots \circ g_0) = g$$

in which any pair of successive arrows lies in S_2'.

(1.17) **Definition.** Given a category $\mathcal{C}$ together with a subgraph $S \subseteq \mathcal{C} \times \mathcal{C}$, we define the *categorical quotient* $\mathcal{C} /\!\!/ S$ to be the category obtained as the double quotient $\mathcal{P}(\mathcal{C}/S)/E_{\mathcal{C}}$, where $E_{\mathcal{C}}$ is the congruence generated by all pairs of paths

$$\big(([C], [\mathrm{id}_C], [C]), ([C])\big) \qquad \text{and} \qquad \big(([C], [f], [D], [g], [E]), ([C], [g \circ f], [E])\big)$$

with $C \in \mathcal{C}$ and $f\colon C \to D$, $g\colon D \to E$ in $\mathcal{C}$. This quotient comes with a canonical projection $\mathcal{C} \to \mathcal{C} /\!\!/ S$, which maps an object C to $[C]$ and an arrow $f\colon C \to D$ to $\big[([C], [f], [D])\big]$. This is really just the composition

$$\mathcal{C} \to \mathcal{C}/S \overset{\eta}{\rightarrowtail} \mathcal{P}(\mathcal{C}/S) \to \mathcal{P}(\mathcal{C}/S)/E_{\mathcal{C}},$$

where the two unnamed arrows are the canonical projections, of which the first one is just a morphism of graphs.

(1.18) **Observation.** If S is discrete then the only arrows that are identified with each other are the identities of equivalent objects. In that case, we cannot form new paths by adding in (equivalence classes of) identities, composing or decomposing arrows. So, every path can be reduced to one of minimal length by erasing all identities and forming all possible compositions.

(1.19) **Proposition.** A categorical quotient $\mathcal{C}/\!\!/S$ satisfies the universal property that every functor $F\colon \mathcal{C} \to \mathcal{D}$ that is constant on equivalence classes factors uniquely through the universal projection $\mathcal{C} \to \mathcal{C}/\!\!/S$.

Proof. The factorisation is obtained in three steps:

$$\begin{tikzcd}
\mathcal{C} \arrow[r] \arrow[drrr, bend right, "F"'] & \mathcal{C}/S \arrow[r, tail, "\eta"] \arrow[drr, dotted, "F_1"'] & \mathcal{P}(\mathcal{C}/S) \arrow[r] \arrow[dr, dotted, "F_2"] & \mathcal{P}(\mathcal{C}/S)/E_{\mathcal{C}} \arrow[d, dotted, "F_3"] \\
 & & & \mathcal{D},
\end{tikzcd}$$

where a unique morphism of graphs F_1 exists by the universal property of $\mathcal{C}/S$, a unique functor F_2 exists by the universal property of the free category $\mathcal{P}(\mathcal{C}/S)$ and finally, a unique functor F_3 exists by the universal property of $\mathcal{P}(\mathcal{C}/S)/E_{\mathcal{C}}$. □

Let's give a more explicit description of what the category $\mathcal{C}/\!\!/S$ looks like. Since quotienting by a congruence doesn't affect the objects, we can naturally identify

$$\mathrm{Ob}(\mathcal{C}/\!\!/S) \cong \mathrm{Ob}\,\mathcal{C}/\,\mathrm{Ob}\,S$$

(really the equivalence relation generated by $\mathrm{Ob}\,S$). An arrow from such an object $[C]$ to an object $[D]$ is an equivalence class of paths in $\mathcal{C}/S$ from $[C]$ to $[D]$, where two paths are equivalent iff we can transform one into the other by successively removing/inserting identities (i.e. replacing a subpath of the form $([E], [\mathrm{id}_E], [E])$ by $([E])$ or the other way round) and composing/decomposing arrows (i.e. replacing a subpath of the form $([E_1], [f], [E_2], [g], [E_3])$ by $([E_1], [g \circ f], [E_3])$ or the other way round for composable f, g).

(1.20) **Example.** If we take $P := \{a \leqslant b\}$ to be the trivial lattice and $S := \{(a, b)\} \subseteq P \times P$ then the graphical equivalence relation generated by S identifies a with b on objects and id_a with id_b on arrows. So the quotient graph has a single object with two loops; one corresponding to $[\mathrm{id}_a] = [\mathrm{id}_b]$ and one to $[a \leqslant b]$. Finally, when passing to the categorical quotient, the congruence E_P is just generated by $([a], [\mathrm{id}_a], [a])$ since the only compositions in P involving $a \leqslant b$ is with identities. Hence $P/\!\!/S$ is the monoid $\mathbb{N}$, the category with a single object and freely generated by a single non-trivial endomorphism.

This example shows that the categories of preorders and posets are not closed under categorical colimits (in particular, they are not coreflective in **Cat**). However, colimits in these categories are formed by calculating them in **Cat** and then applying the respective reflectors.

(1.21) **Example.** Modifying the last example slightly, if we again take $P := \{a \leqslant b\}$ but now $S := \{(a, a) \to (a, b)\}$ then the graphical equivalence relation generated by S is all of $P \times P$ and so P/S is the graph with a single object and a single loop, while the categorical quotient $P/\!\!/S$ is the terminal category.

(1.22) **Example.** This time, let $P := \{a \leqslant b, c \leqslant d\}$ be the disjoint union of two trivial lattices and $S := \{(a, c), (b, d)\}$. Then the graphical equivalence relation generated by S identifies a with c and b with d as well as the corresponding identities but no other arrows. Consequently, P/S is the graph with two objects, each one equipped with a loop, and two parallel edges between them. Since, again, the only compositions involving non-trivial morphisms in P are with identities, $P/\!\!/S$ is the categorical circle $\bullet \rightrightarrows \bullet$.

(1.23) **Example.** Similarly, if we also add $(a, c) \leqslant (b, d)$ to S, we obtain the categorical interval $\bullet \to \bullet$.

(1.24) **Example.** As for a slightly more complicated example of a poset, let's consider

$$P\colon \quad \begin{tikzcd} a \arrow[r] \arrow[rrr, bend left] & b & c \arrow[r] & d \end{tikzcd} \qquad \text{with} \qquad S := \{(b, c)\}.$$

The resulting quotient graph is

$$P/S\colon \quad \begin{tikzcd} a \arrow[r] \arrow[rr, bend left] & {[b]} \arrow[r] & d \end{tikzcd} \quad \text{(plus identity loops at each object)}$$

and the categorical quotient has four non-identity arrows (where in the following diagram, the lower half commutes)

$$P/\!\!/S\colon \quad \begin{tikzcd} a \arrow[r] \arrow[rr, bend left, "{(a,d)}"] \arrow[rr, bend right, "{(a,[b],d)=([c],d)\circ(a,[b])}"'] & {[b]} \arrow[r] & d. \end{tikzcd}$$

Let us generalise the previous examples and look at quotients of preorders by some equivalence relation on its objects. We will pay special attention to what happens when we add in arrows to the equivalence relation.

(1.25) **Example.** Let P be a preorder and S an equivalence relation on P (which we view as a discrete subcategory of $P \times P$; i.e. the equivalence relation on $\operatorname{Arr} P$ is just equality). Then the graph P/S has objects P/S and edges $[x] \to [y]$, one for every relation $x' \lesssim y'$ between representatives $x' \mathrel{S} x$ and $y' \mathrel{S} y$. When forming the categorical quotient, observation (1.18) tells us that $P/\!\!/S$ is the category with objects P/S and arrows $[x] \to [y]$ all minimal sequences of pairs $\big((u_0, v_0), (u_1, v_1), \dots, (u_n, v_n)\big)$ with

$$x \mathrel{S} u_0 \lesssim v_0 \mathrel{S} u_1 \lesssim v_1 \mathrel{S} \dots \mathrel{S} u_n \lesssim v_n \mathrel{S} y.$$

Here, minimality means that we never have $u_i = v_i$ or $v_i = u_{i+1}$. In the first case, we can simply leave out the pair and in the second one, replace $(u_i, v_i), (u_{i+1}, v_{i+1})$ by (u_i, v_{i+1}). When passing to the associated preorder, all such sequences $[x] \to [y]$ are identified, so that $[x] \lesssim [y]$ iff there is one.

(1.26) **Example.** As for the other extreme, if we again take P to be a preorder with an equivalence relation S but this time view it as a full subcategory of $P \times P$ then the graph P/S again has objects P/S but there is at most one edge $[x] \to [y]$ between any two objects. More precisely, there is such an edge iff we find representatives $x' \mathrel{S} x$ and $y' \mathrel{S} y$ with $x' \lesssim y'$. In contrast to the discrete case, there seems to be no general description of $P/\!\!/S$ but note that while the graph P/S here has fewer edges than the one in the previous example, there is a directed path $[x] \to [y]$ in the new graph iff there is one in the previous one. In particular, the associated preorders are the same.

(1.27) **Observation.** If P is a preorder (resp. poset) and S an equivalence relation on its underlying set then the quotient preorder (resp. poset) of P by S is independent of how we complete $S \subseteq P \times P$ to a graphical equivalence relation.

With categorical quotients at hand, we can give an explicit (though not generally calculable) description of colimits in **Cat**. But first, let us introduce the following notation.

(1.28) **Notation.** Given a family of categories $(\mathcal{C}_i)_{i\in I}$, the coproduct $\coprod_{i\in I}\mathcal{C}_i$ is just the disjoint union of all objects and arrows. So an object in $\coprod_{i\in I}\mathcal{C}_i$ is really a pair (i,A) with $i\in I$ and $A\in\mathcal{C}_i$. For readability's sake, we simply write A_i instead of (i,A). Similarly for arrows.

The colimit of a diagram $X\colon\mathcal{I}\to\mathbf{Cat}$ is now obtained as the coequaliser of

$$\coprod_{\substack{i\colon I\to J\\ \text{in }\mathcal{I}}} X_I \rightrightarrows \coprod_{I\in\mathcal{I}} X_I,$$

where one morphism maps $(i\colon I\to J, X_I)$ to (I,X_I) by the identity and the other one maps $(i\colon I\to J,X_I)$ to (J,X_J) by X_i. Put differently, the colimit is the categorical quotient of $\coprod_{I\in\mathcal{I}} XI$ by the subgraph S having

$$\operatorname{Ob} S=\{(A_I,((X_i)A)_J)\mid i\colon I\to J\text{ in }\mathcal{I},\ A\in X_I\}$$

and

$$\operatorname{Arr} S=\{(f_I,((X_i)f)_J)\mid i\colon I\to J\text{ in }\mathcal{I},\ f\in\operatorname{Arr}X_I\}$$

So, all in all, the objects of $\operatorname{colim}_{\mathcal{I}} X$ are equivalence classes of objects in $\coprod_{I\in\mathcal{I}}X_I$, where two objects A_I, B_J are equivalent iff there is a zig-zag of arrows in $\mathcal{I}$

$$I=K_0\overset{i_1}{\leftrightarrow}K_1\overset{i_2}{\leftrightarrow}\dots K_{n-1}\overset{i_n}{\leftrightarrow}K_n=J$$

(meaning i_m is either an arrow $K_{m-1}\to K_m$ or the other way round), together with objects $A=C_0,C_1,\dots,C_n=B$ with $C_m\in X_{K_m}$ for all m and fitting into the zig-zag (i.e. for $i_m\colon K_{m-1}\to K_m$ we have $(Xi_m)C_{m-1}=C_m$ or the other way round).

In exactly the same way, we obtain an equivalence relation on the arrows of $\coprod_{I\in\mathcal{I}}X_I$ but these are not the arrows of $\operatorname{colim}_{\mathcal{I}}X$. An arrow in $\operatorname{colim}_{\mathcal{I}}X$ from $[A_I]$ to $[B_J]$ is instead given by an equivalence class of paths

$$\Big([A_I]=[(I_0,C_0)],[(J_1,f_1)],[(I_1,C_1)],[(J_2,f_2)],\dots,[(J_n,f_n)],[(I_n,C_n)]=[B_J]\Big)$$

with $C_m\in X_{I_m}$ and $f_m\in X_{J_m}$ for all m and every f_m is an arrow $f_m\colon D\to E$ for some D, $E\in X_{J_m}$ with $[(J_m,D)]=[(I_{m-1},C_{m-1})]$ and $[(J_m,E)]=[(I_m,C_m)]$. Two such paths are equivalent iff they can be transformed into one another by successively removing/inserting identities and composing/decomposing (representatives of) arrows.

As a final remark, note that while general colimits of categories are hard to calculate, filtered ones are much easier. For clarity and because we are mainly interested in the explicit construction of colimits in **Cat**, we use the explicit construction of colimits in **Sets** as a quotient of a coproduct in the following proposition.

(1.29) **Example.** If $X\colon\mathcal{I}\to\mathbf{Cat}$ is a filtered diagram (see (3.1.1)) then $\operatorname{colim}_I X_I$ can be constructed as follows.

(a) Its objects are $\operatorname{colim}_I\operatorname{Ob}(X_I)$, which in turn is just a quotient of $\coprod_I\operatorname{Ob}(X_I)$, where $A_I\in\operatorname{Ob}X_I$ and $B_J\in\operatorname{Ob}X_J$ are equivalent iff there are some $i\colon I\to K$, $j\colon J\to K$ in $\mathcal{I}$ such that $X_i(A_I)=X_j(B_J)$.

(b) Its morphisms are $\operatorname{colim}_I\operatorname{Arr}(X_I)$, which is constructed analogously.

(c) A morphism $[f_I]$ represented by some $f_I\colon A_I\to B_I$ in X_I has domain $[A_I]$ and codomain $[B_I]$.

(d) Given $f_I\colon A_I\to B_I$ in X_I and $g_J\colon C_J\to D_J$ in X_J with $[B_I]=[C_J]$, there are some $i\colon I\to K$, $j\colon J\to K$ in $\mathcal{I}$ such that $X_i(B_I)=X_j(C_J)$ and the composite $[g_J]\circ[f_I]$ is given by

$$[g_J]\circ[f_I]=[X_j(g_J)]\circ[X_i(f_I)]:=[X_j(g_J)\circ X_i(f_I)].$$

2. Coends as Colimits

In the following section, we shall quickly introduce (co)ends and show how they are related to (co)limits using the twisted arrow category construction. This is only stated as an exercise in [38] and we include a proof for the sake of completeness.

(2.1) **Definition.** Let $\mathcal{I}$ and $\mathcal{C}$ be categories ($\mathcal{I}$ will usually be assumed to be small) and $X\colon \mathcal{I}^{\mathrm{op}} \times \mathcal{I} \to \mathcal{C}$ a diagram. A *wedge* over (or above or to) X consists of an object $W \in \mathcal{C}$, together with a family of arrows $(\omega_I \colon W \to X(I,I))_{I\in\mathcal{I}}$ such that for all arrows $i\colon I \to I'$ in $\mathcal{I}$, the diagram

$$\begin{tikzcd} & W \arrow[dl, "\omega_I"'] \arrow[dr, "\omega_{I'}"] & \\ X(I,I) \arrow[dr, "X(I,i)"'] & & X(I',I') \arrow[dl, "X(i,I')"] \\ & X(I,I') & \end{tikzcd}$$

commutes. A morphism between two wedges $(W,\omega) \to (W',\omega')$ is defined in the obvious way, namely as an arrow $g\colon W \to W'$ in $\mathcal{C}$ such that $\omega'_I \circ g = \omega_I$ for all $I \in \mathcal{I}$; we will usually shorten the notation and write $\omega' \circ g = \omega$ instead. The composition of two morphisms between wedges is defined by the composition in $\mathcal{C}$ and with this the wedges over X form a category. An *end* of X is now simply a universal (i.e. terminal) wedge over X. Dually, a *coend* of X is a universal (i.e. initial) *cowedge* under X. By abuse of language, if (E,ε) is a (co)end of X, we usually call the object E the (co)end of X with the *(co)ending (co)wedge* ε being implicit.

(2.2) **Notation.** The vertex of a (co)end of a diagram X is usually denoted by

$$\int_{I\in\mathcal{I}} X(I,I) \quad \text{for an end} \qquad \text{and} \qquad \int^{I\in\mathcal{I}} X(I,I) \quad \text{for a coend.}$$

Alternative, shorter, notations are $\int_I X(I,I)$ and $\int_{\mathcal{I}} X$. As a (co)end of X is obviously unique up to a unique isomorphism, it does no harm to write "$E \cong \int_I X(I,I)$" for the statement "E is the object of an end of X" (and analogously of coends).

Quite obviously, an end is a special kind of limit, while a coend is a special kind of colimit (as we shall soon see, the converse is also true). Assuming we have a functor $X\colon \mathcal{I}^{\mathrm{op}} \times \mathcal{I} \to \mathcal{C}$, just like we can describe limits by products and equalisers, we can do the same for the end of X

$$\textstyle\int_I X(I,I) \to \prod_{I\in\mathcal{I}} X(I,I) \overset{p}{\underset{q}{\rightrightarrows}} \prod_{i\colon I\to I'} X(I,I'), \quad \text{where} \quad \begin{array}{l} p = \big(X(\mathrm{id}_I, i)\circ \mathrm{pr}_I\big)_{i\colon I\to I'}, \\ q = \big(X(i,\mathrm{id}_{I'})\circ \mathrm{pr}_{I'}\big)_{i\colon I\to I'}. \end{array}$$

Dually, the coend of X fits into a coequaliser diagram

$$\prod_{i\colon I\to I'} X(I',I) \overset{p}{\underset{q}{\rightrightarrows}} \prod_{I\in\mathcal{I}} X(I,I) \to \textstyle\int^I X(I,I), \quad \text{where} \quad \begin{array}{l} p = \big[X(i,\mathrm{id}_I)\circ \mathrm{in}_I\big]_{i\colon I\to I'}, \\ q = \big[X(\mathrm{id}_{I'}, i)\circ \mathrm{in}_{I'}\big]_{i\colon I\to I'}. \end{array}$$

These descriptions, while useful for actual calculations, come with the caveat that the corresponding products (resp. coproducts) in $\mathcal{C}$ need to exist. Usually, we want our categories to be bicomplete anyway, so that this is not really a restriction. However, from a theoretical standpoint, it would be much nicer to describe a (co-)end by an equivalent (co-)limit; i.e. such that one exist if and only if the other one does.

(2.3) **Notation.** Given a diagram $X\colon \mathcal{J}^{\mathrm{op}} \times \mathcal{J} \to \mathcal{C}$ together with a cowedge (W, ω) below X, the cowedge condition tells us that every arrow $i\colon I \to I'$ in $\mathcal{J}$ defines a canonical morphism $X(I', I) \to W$, which we shall denote by

$$\omega_i := \omega_I \circ X(i, I) = \omega_{I'} \circ X(I', i) \qquad \text{(so that } \omega_I = \omega_{\mathrm{id}_I}\text{)}.$$

Dually for wedges.

In fact, this notation is just an alternative description of (co)wedges. Given a diagram $X\colon \mathcal{J}^{\mathrm{op}} \times \mathcal{J} \to \mathcal{C}$, let's say that an object W, together with a family

$$\big(\omega_i\colon X(I', I) \to W\big)_{i\colon I \to I' \text{ in } \mathcal{J}}$$

satisfies the *cowedge-condition* (or is a *cowedge* above X) iff

$$\omega_{\mathrm{id}_I} \circ X(i, I) = \omega_i = \omega_{\mathrm{id}_{I'}} \circ X(I', i) \qquad \text{for all } i\colon I \to I' \text{ in } \mathcal{J}.$$

Dually for *wedges*.

(2.4) **Observation.** This cowedge condition is equivalent to the seemingly stronger requirement that

$$\omega_j \circ X(g, f) = \omega_{g \circ j \circ f} \qquad \text{for all } I \xrightarrow{f} J \xrightarrow{j} J' \xrightarrow{g} I' \text{ in } \mathcal{J}.$$

This is easily verified by noting that $X(g, f) = X(g, I) \circ X(J', f)$ and using the above cowedge condition.

(2.5) **Proposition.** Given a diagram $X\colon \mathcal{J}^{\mathrm{op}} \times \mathcal{J} \to \mathcal{C}$ and an object $W \in \mathcal{C}$, we have the following 1-to-1 correspondence between the two cowedge concepts:

$$\Big\{\big(\omega_I\colon X(I, I) \to W\big)_{I \in \mathcal{J}} \;\Big|\; \omega \text{ cowedge}\Big\} \cong \Big\{\big(\omega_i\colon X(I', I) \to W\big)_{I \xrightarrow{i} I'} \;\Big|\; \omega \text{ cowedge}\Big\}$$

$$(\omega_I)_I \mapsto \big(\omega_I \circ X(i, I)\big)_{I \xrightarrow{i} I'} = \big(\omega_{I'} \circ X(I', i)\big)_{I \xrightarrow{i} I'}$$

$$(\omega_{\mathrm{id}_I})_I \leftarrow\!\shortmid (\omega_i)_i$$

Dually for wedges. □

Hidden behind this alternative wedge condition is the sought for description of a (co)end as a (co)limit and vice versa. For $\mathcal{J}$ a category, let the *twisted arrow category* $\mathcal{J}^{\looparrowright}$ be the category with objects all arrows $i\colon I \to I'$ of $\mathcal{J}$ and a morphism from $i\colon I \to I'$ to $j\colon J \to J'$ being a pair (f, g) of arrows in $\mathcal{J}$ such that the diagram

$$\begin{tikzcd} I \arrow[r, "f"] \arrow[d, "i"'] & J \arrow[d, "j"] \\ I' & J' \arrow[l, "g"] \end{tikzcd}$$

commutes. The composition is then defined in the obvious way. This twisted arrow category comes with a projection functor

$$Q\colon \mathcal{J}^{\looparrowright} \to \mathcal{J}^{\mathrm{op}} \times \mathcal{J}, \qquad \begin{tikzcd} I \arrow[r, "f"] \arrow[d, "i"'] & J \arrow[d, "j"] \\ I' & J' \arrow[l, "g"] \end{tikzcd} \quad \mapsto \quad (I', I) \xrightarrow{(g^{\mathrm{op}}, f)} (J', J),$$

which allows us to relate the coend to the colimit as follows.

(2.6) **Proposition.** For a functor $X\colon \mathcal{I}^{\mathrm{op}}\times\mathcal{I}\to\mathcal{C}$ (with $\mathcal{I}$ small), and $Q\colon \mathcal{I}^{\looparrowright}\to\mathcal{I}^{\mathrm{op}}\times\mathcal{I}$ as above, the category of cowedges below X is isomorphic to the category of cocones below $X\circ Q$ and in particular

$$\int^{\mathcal{I}} X \cong \operatorname*{colim}_{\mathcal{I}^{\looparrowright}} X\circ Q.$$

Proof. This was (2.5) above. The cowedge condition for the $(\omega_i)_i$ just corresponds to the cocone condition with respect to $X\circ Q$. □

In the last proposition, we saw how a coend can be understood as a special kind of colimit. For completeness's sake, let us mention that the converse is true as well.

(2.7) **Proposition.** If $X\colon \mathcal{I}\to\mathcal{C}$ is a diagram and $Q\colon \mathcal{I}^{\mathrm{op}}\times\mathcal{I}\to\mathcal{I}$ the standard projection to the second factor, then the category of cocones below X is equal to the category of cowedges below $X\circ Q$.

Proof. Let $C\in\mathcal{C}$ and $(\gamma_I\colon XI\to C)_{I\in\mathcal{I}}$. If $i\colon I\to J$ is any arrow in $\mathcal{I}$ we have

$$\gamma_I\circ(XQ)(i,I)=\mathrm{id}_{XI}\circ\gamma_I=\gamma_I \qquad\text{and}\qquad \gamma_J\circ(XQ)(J,i)=\gamma_J\circ Xi,$$

so that indeed, (C,γ) is a cocone below X iff it is a cowedge below $X\circ Q$. Moreover, a morphism of cocones below X is the same as a morphism of cowedges below $X\circ Q$. □

3. Preservation, Reflection and Creation of Limits

The calculus of ends and coends is a very useful tool for calculating limits and colimits. Next we will investigate how we can use special functors to reduce the problem of calculating limits in a category $\mathcal{C}$ to calculating them in another (possibly better understood) category. E.g. we get limits and colimit in **Top** by calculating them in **Sets** and then equipping the resulting set with the initial and final topology respectively.

(3.1) **Definition.** Let $\mathcal{I}$ be a category. A functor $F\colon\mathcal{C}\to\mathcal{D}$ is said to

(a) *preserve limits* indexed by $\mathcal{I}$ (or of type $\mathcal{I}$) iff whenever we have a diagram $X\colon\mathcal{I}\to\mathcal{C}$ with a limit (L,λ) then $(FL,F\lambda)$ is a limit of $F\circ X$;

(b) *reflect limits* indexed by $\mathcal{I}$ (or of type $\mathcal{I}$) iff for any diagram $X\colon\mathcal{I}\to\mathcal{C}$ and any cone (C,γ) above X, if $(FC,F\gamma)$ is a limit of $F\circ X$ then (C,γ) is a limit of X;

(c) *create limits* indexed by $\mathcal{I}$ (or of type $\mathcal{I}$) iff for every $X\colon\mathcal{I}\to\mathcal{C}$ and every limit (L,λ) of $F\circ X$ there is exactly one cone (C,γ) above X with $(FC,F\gamma)=(L,\lambda)$ and this is then a limit of X.

We simply say that F preserves/reflects/creates limits iff it does so for all small $\mathcal{I}$ (similarly for "products", "finite limits" etc.). Dually for colimits.

(3.2) **Remark.** Another way to state the first two definitions is that F preserves/reflects limits iff for every $G\colon\mathcal{I}\to\mathcal{C}$, the postcomposition functor $F_*\colon\operatorname{Cone}(G)\to\operatorname{Cone}(F\circ G)$ preserves/reflects terminal objects. This viewpoint shows that the creation of limits is an "evil" notion because it enforces an equality of objects (rather than an isomorphism).

(3.3) **Example.** If $F\colon \mathcal{C} \to \mathcal{D}$ is fully faithful, it reflects limits and colimits. Indeed, let $X\colon \mathcal{I} \to \mathcal{C}$ be a diagram together with a cone (L, λ) such that $(FL, F\lambda)$ is a limit of $F \circ X$ and let (C, γ) be another cone above X. By hypothesis there is a unique $f\colon C \to L$ such that $F\lambda \circ Ff = F\gamma$ which is equivalent to $\lambda \circ f = \gamma$.

(3.4) **Observation.** We already observed that limits are ends and vice versa, so that a functor preserves limits iff it preserves ends.

Obviously, a functor $F\colon \mathcal{C} \to \mathcal{D}$ that creates limits also reflects them. Moreover, it preserves the limits "that exist in $\mathcal{D}$" in the following sense.

(3.5) **Proposition.** If $F\colon \mathcal{C} \to \mathcal{D}$ creates limits of type $\mathcal{I}$ and $\mathcal{D}$ has $\mathcal{I}$-limits then F preserves limits of type $\mathcal{I}$. In particular, if $\mathcal{D}$ is complete and F creates limits, it also preserves limits.

Proof. Let (L, λ) be a limit of some $X\colon \mathcal{I} \to \mathcal{C}$. By hypothesis, a limit (M, μ) of $F \circ X$ exists and because F creates limits of type $\mathcal{I}$, we find a unique cone (C, γ) above X satisfying $(FC, F\gamma) = (M, \mu)$ and this is again a limit of X. But limits are unique to within isomorphism, so that there is an isomorphism $f\colon L \cong C$ such that $\gamma \circ f = \lambda$. After applying F we get $Ff\colon FL \cong FC$ with $\mu \circ Ff = F\gamma \circ Ff = F\lambda$ and so $(FL, F\lambda)$ is a limit of $F \circ X$. $\square$

The classical (and very useful) result about the preservation of limits is that left adjoint functors do preserve colimits and dually, right adjoint functors preserve limits.

(3.6) **Theorem.** If $F\colon \mathcal{C} \rightleftarrows \mathcal{D} :G$ is an adjunction (left adjoint on the left), then F preserves colimits of all types and G preserves limits of all types.

Proof. Let $X\colon \mathcal{I} \to \mathcal{C}$ be a diagram with colimit (L, λ) and let (D, δ) be a cocone below $F \circ X$. That is to say, δ is a family of arrows $\delta_I\colon FX_I \to D$ natural in $I \in \mathcal{I}$ and taking adjuncts gives $\delta_I^\sharp\colon X_I \to GD$ again natural in I. It follows that there is a unique $f\colon L \to GD$ such that $f \circ \lambda = \delta^\sharp$ and taking adjuncts again yields that $f^\flat\colon FL \to D$ is the unique morphism satisfying $f^\flat \circ F\lambda = (f \circ \lambda)^\flat = (\delta^\sharp)^\flat = \delta$. $\square$

(3.7) **Corollary.** In an adjunction $F\colon \mathcal{C} \rightleftarrows \mathcal{D} :G$ (left adjoint on the left), F preserves epimorphisms and G preserves monomorphisms.

Proof. The right adjoint G preserves limits and in particular kernel pairs. $\square$

Even better than in the last theorem, if a left adjoint functor is fully faithful it effectively allows to transfer all questions concerning colimits from one category to the other. More surprisingly (although easier to prove), it also has some implications concerning limits.

(3.8) **Theorem.** Given adjoint functors $F\colon \mathcal{C} \rightleftarrows \mathcal{D} :G$ (left adjoint on the left) with unit η, counit ε and F fully faithful.

(a) Given a diagram $X\colon \mathcal{I} \to \mathcal{C}$ such that $F \circ X$ has a limit, then X has a limit; more specifically, if (L, λ) is a limit of $F \circ X$ then $(GL, \eta_X^{-1} \circ G\lambda)$ is a limit of X.

(b) F preserves and reflects colimits. Even better

(c) A diagram $X\colon \mathcal{I} \to \mathcal{C}$ has a colimit iff $F \circ X$ has one; more specifically, if (L, λ) is a colimit of $F \circ X$ then $(GL, \lambda^\sharp)$ is a colimit of X.

Proof. *Ad* (a): Let (L, λ) be a limit of $F \circ X$, so that $(GL, G\lambda)$ is a limit of $G \circ F \circ X$. But η is an isomorphism because F is fully faithful and so $(GL, \eta_X^{-1} \circ G\lambda)$ is a limit of X.
Ad (b): By the previous theorem and the above example.
Ad (c): "$\Rightarrow$" is implied by (b). Now assume that $F \circ X$ has a colimit (L, λ). Taking adjuncts of the $\lambda_I \colon FX_I \to L$, we get unique $\lambda_I^\sharp \colon X_I \to GL$ such that

$$FX_I \xrightarrow{\lambda_I} L \quad = \quad FX_I \xrightarrow{(\lambda_I^\sharp)^\flat} L \quad = \quad FX_I \xrightarrow{F\lambda_I^\sharp} FGL \xrightarrow{\varepsilon_L} L.$$

By uniqueness of the $\lambda_I^\sharp$, these form a cocone below X, whence the $F\lambda_I^\sharp$ form a cocone below $F \circ X$ and there is a unique $g \colon L \to FGL$ satisfying $g \circ \lambda = F\lambda^\sharp$, from which we get $\varepsilon_L \circ g \circ \lambda = \varepsilon_L \circ F\lambda^\sharp = \lambda$. But λ is a colimiting cocone and it follows that $\varepsilon_L \circ g = \mathrm{id}_L$. Conversely, $g \circ \varepsilon_L = \varepsilon_{FGL} \circ FGg$ since ε is natural and now ε_{FGL} has $F\eta_{GL}$ as its inverse by the triangle identities and the invertibility of η. Hence $F\eta_{GL} \circ g \circ \varepsilon_L = FGg$, which then has $FG\varepsilon_L$ as a retraction; to wit

$$FG\varepsilon_L \circ F\eta_{GL} \circ g \circ \varepsilon_L = FG\varepsilon_L \circ FGg = FG(\varepsilon_L \circ g) = 1_{FGL}.$$

But $FG\varepsilon_L \circ F\eta_{GL} = F(G\varepsilon_L \circ \eta_{GL}) = 1_{FGL}$ again by the triangle identities and so $g \circ \varepsilon_L = 1_{FGL}$. We have thus shown that ε_L is an isomorphism $FGL \cong L$ and even an isomorphism of cocones $(FGL, F\lambda^\sharp) \cong (L, \lambda)$. But F reflects colimits and so $(GL, \lambda^\sharp)$ is a colimit of X. □

(3.9) **Scholium.** Actually, in the proof of point (c), we have shown more. In fact, we showed that the counit's component $\varepsilon_L \colon FGL \to L$ is an isomorphism. Another way to put this is that if $\mathcal{C} \subseteq \mathcal{D}$ is a full coreflective subcategory, it is closed under colimits (up to a canonical isomorphism).

(3.10) **Corollary.** If $\mathcal{D}$ is a full reflective subcategory of a bicomplete category $\mathcal{C}$ with reflection $F \colon \mathcal{C} \to \mathcal{D}$ then $\mathcal{D}$ is again bicomplete. More specifically, the limit of a diagram $X \colon \mathcal{I} \to \mathcal{D}$ can be calculated in $\mathcal{C}$ (and the unit $\lim X \to F \lim X$ is an isomorphism), while the colimit of X in $\mathcal{D}$ is just the reflection of its colimit in $\mathcal{C}$. □

4. Final Functors

In the last two sections, we described some ways how we can calculate the limit of a diagram by manipulating its codomain. Another strategy that can be employed is to simplify the index category and then do the calculation. The archetypical example for this is that if a category $\mathcal{I}$ has a terminal object $*$ and $X \colon \mathcal{I} \to \mathcal{C}$ is a functor then $X_* \cong \operatorname{colim}_{\mathcal{I}} X$ with the colimiting cocone having components $X_{!^C}$, where $!^C \colon C \to *$ is the unique arrow into the terminal object. One can apply the same strategy in other contexts, too, one of the most general being that of a final functor.

Recall from (3.1), that a functor $F \colon \mathcal{C} \to \mathcal{D}$ is said to *reflect colimits* iff we can lift colimits in $\mathcal{D}$ along F. More specifically, given a diagram $D \colon \mathcal{I} \to \mathcal{C}$ together with a cocone $\gamma \colon D \Rightarrow C$ such that $F\gamma \colon F \circ D \Rightarrow FC$ is a colimit, then γ is already a colimit. We now get the notion of a final functor by precomposing with a $G \colon \mathcal{I}' \to \mathcal{I}$ instead of postcomposing with F.

(4.1) **Definition.** A functor $F \colon \mathcal{I}' \to \mathcal{I}$ is called *final* iff for every diagram $X \colon \mathcal{I} \to \mathcal{C}$

(a) $\operatorname{colim}_{\mathcal{I}} X$ exists iff $\operatorname{colim}_{\mathcal{I}'}(X \circ F)$ exists and

(b) a cocone $\gamma \colon X \Rightarrow L$ is colimiting iff $\gamma_F \colon X \circ F \Rightarrow L$ is colimiting.

Under the presence of (a), the point (b) is equivalent to requiring the canonical "change of indexing category" arrow $\operatorname{colim}_{\mathcal{J}'}(X \circ F) \to \operatorname{colim}_{\mathcal{J}} X$ induced by the colimiting cocone below X be an isomorphism if these colimits exist. Now a subcategory $\mathcal{J}' \leqslant \mathcal{J}$ is called *final* iff the inclusion $\mathcal{J}' \subseteq \mathcal{J}$ is final.

(4.2) **Example.** The whole of $\mathcal{J}$ is a final subcategory of $\mathcal{J}$. Moreover, as already seen, if $\mathcal{J}$ has a terminal object $*$ then $\{*\}$ is also a final subcategory of $\mathcal{J}$. A naive guess now would be that it suffices if $*$ is *weakly terminal* in the sense that for every $I \in \mathcal{J}$ there is some (not necessarily unique) $I \to *$ but unfortunately, this is not the case; consider for example b in the index category $a \rightrightarrows b$.

(4.3) **Observation.** The composition of two final functors is final.

Obviously, our definition is not very useful and we should try to find necessary and sufficient criteria for a functor to be final. By our archetypical example $\{*\} \subseteq \mathcal{J}$ of a final subcategory, we should investigate the morphisms out of an arbitrary $K \in \mathcal{J}$ into objects lying in the image of F, leading to the characterisation (4.5) below.

For it, given $F\colon \mathcal{D} \to \mathcal{C}$, $G\colon \mathcal{E} \to \mathcal{C}$, we need to compute $\operatorname{colim} \mathcal{C}(F-, G-)$. As usual in the computation of colimits, we start by taking the coproduct of all $\mathcal{C}(FD, GE)$ with $D \in \mathcal{D}$, $E \in \mathcal{E}$, yielding exactly the objects of $F \downarrow G$. We then look at the family

$$\widetilde{\varepsilon}_{D,E}\colon \mathcal{C}(FD, GE) \to \operatorname{Ob} F \downarrow G,\ a \mapsto (D, a, E) \qquad \text{with } D \in \mathcal{D},\ E \in \mathcal{E}.$$

However, in general these do not form a cocone so that for $f\colon D \to D'$ in $\mathcal{D}$, $g\colon E \to E'$ in $\mathcal{E}$ and $a \in \mathcal{C}(FD', GE)$, we need to identify

$$\widetilde{\varepsilon}_{D',E}(a), \quad \widetilde{\varepsilon}_{D,E}(a \circ Ff), \quad \widetilde{\varepsilon}_{D',E'}(Gg \circ a) \quad \text{and} \quad \widetilde{\varepsilon}_{D,E'}(Gg \circ a \circ Ff).$$

In $F \downarrow G$ the situation is as in the following commutative diagram

$$\begin{tikzcd}
FD' \arrow[d, "a"'] & FD \arrow[l, "Ff"'] \arrow[r, "Ff"] \arrow[d, "a \circ Ff"'] & FD' \arrow[d, "Gg \circ a"] & FD \arrow[l, "Ff"'] \arrow[d, "Gg \circ a \circ Ff"] \\
GE & GE \arrow[l, "G\mathrm{id}_E"] \arrow[r, "Gg"'] & GE' & GE' \arrow[l, "G\mathrm{id}_{E'}"]
\end{tikzcd}$$

from which one can already guess the following result.

(4.4) **Lemma.** Given $F\colon \mathcal{D} \to \mathcal{C}$, $G\colon \mathcal{E} \to \mathcal{C}$ then

$$\operatorname{colim} \mathcal{C}(F-, G-) \cong \pi_0(F \downarrow G),$$

having colimiting cocone $\varepsilon := p \circ \widetilde{\varepsilon}$, with $\widetilde{\varepsilon}$ as above and p the quotient map.

Proof. We have already seen that if two objects of $F \downarrow G$ are equal in the colimit, they are equal in $\pi_0(F \downarrow G)$, too and we only need to check the converse. For this, it suffices to check that for every arrow in $F \downarrow G$, its domain and codomain are equal in the colimit. This is obvious, for given $f\colon D \to D'$ in $\mathcal{D}$ and $g\colon E \to E'$ in $\mathcal{E}$ the diagram

$$\begin{tikzcd}
\mathcal{D}(FD, GE) \arrow[r, "(Gg)_*"] \arrow[dr] & \mathcal{D}(FD, GE') \arrow[d] & \mathcal{D}(FD', GE') \arrow[l, "(Ff)^*"'] \arrow[dl] \\
& \operatorname{colim} \mathcal{C}(F-, G-) &
\end{tikzcd}$$

(with the unnamed arrows being the components of the colimiting cocone) must be commutative and thus, given an arrow $(f, g)\colon (D, a, E) \to (D', a', E')$ in $F \downarrow G$ its domain and codomain are indeed equal in the colimit. $\square$

(4.5) **Proposition.** Given $F\colon \mathcal{I}' \to \mathcal{I}$, the following are equivalent

(a) F is final;

(b) $\pi_0(I \downarrow F) \cong 1$ for all $I \in \mathcal{I}$; i.e. $I \downarrow F$ is non-empty and connected (a.k.a. *cohesive*);

(c) for all $X\colon \mathcal{I} \to \mathcal{C}$, the precomposition functor $F^*\colon \mathrm{Cocone}(X) \to \mathrm{Cocone}(X \circ F)$ is an isomorphism of categories.

Proof. "(a) $\Rightarrow$ (b)": If F is final and $I \in \mathcal{I}$, we consider $\mathcal{I}(I,-) \circ F = \mathcal{I}(I, F-)$. Using (4.4) and the hypothesis, we easily calculate

$$\pi_0(I \downarrow F) \cong \operatorname*{colim}_{\mathcal{J}} \mathcal{I}(I, F-) \cong \operatorname*{colim}_{\mathcal{I}} \mathcal{I}(I,-) \cong \{*\}.$$

"(b) $\Rightarrow$ (c)": Assume we are given a $X\colon \mathcal{I} \to \mathcal{C}$ and a cocone $\gamma\colon X \circ F \Rightarrow C$. We construct another cocone $\widetilde{\gamma}\colon X \Rightarrow C$ such that $\gamma = \widetilde{\gamma}_F$ as follows. For $I \in \mathcal{I}$, we take a $I' \in \mathcal{I}'$ together with an arrow $a\colon I \to FI'$ (which exists because $I \downarrow F \neq \varnothing$) and define

$$XI \xrightarrow{\widetilde{\gamma}_I} C \quad := \quad XI \xrightarrow{Xa} XFI' \xrightarrow{\gamma_{I'}} C.$$

Given two such arrows $a\colon I \to FI'$ and $b\colon I \to FI'$, connectedness of $I \downarrow F$ yields a zig-zag

$$I' = I'_0 \xleftrightarrow{f_1} I'_1 \xleftrightarrow{f_2} I'_2 \ldots I'_{n-1} \xleftrightarrow{f_n} I'_n = I' \qquad \text{in } \mathcal{J}$$

together with arrows $I \to FI'_i$ for $i \in \{1, \ldots, n-1\}$ such that the following diagram commutes

$$\begin{tikzcd}
I \arrow[r, equal] \arrow[d, "a"'] & I \arrow[r, equal] \arrow[d] & I \arrow[d] & \cdots & I \arrow[r, equal] \arrow[d] & I \arrow[d, "b"] \\
FI'_0 \arrow[r, leftrightarrow, "Ff_1"'] & FI'_1 \arrow[r, leftrightarrow, "Ff_2"'] & FI'_2 & & FI'_{n-1} \arrow[r, leftrightarrow, "Ff_n"'] & FI'_n
\end{tikzcd}.$$

and after applying X and composing each $XI \to XFI'_i$ with $\gamma_{I'_i}$, the cocone condition on γ implies that $\widetilde{\gamma}_{I'} \circ Xa = \widetilde{\gamma}_{I'} \circ Xb$ and so our definition of γ_I is independent of the choice of a. Now $\gamma \mapsto \widetilde{\gamma}$ can be extended to a functor by declaring it to be the identity on arrows and this gives us an inverse of F^*.
"(c) $\Rightarrow$ (a)": This is trivial because a colimiting cocone is just an initial cocone. □

(4.6) **Corollary.** Every right adjoint functor $G\colon \mathcal{I}' \to \mathcal{I}$ is final.

Proof. Writing $F\colon \mathcal{I} \to \mathcal{I}'$ for a left adjoint to G, if $I \in \mathcal{I}$, the unit $\eta_I\colon I \to GFI$ is initial in $I \downarrow G$. □

(4.7) **Corollary.** If $\mathcal{C}$ is a category with $\mathcal{I}$-colimits ($\mathcal{I}$ some category) then the diagonal $\Delta\colon \mathcal{C} \to \mathcal{C}^{\mathcal{I}}$ is final. In particular, if $\mathcal{C}$ has binary coproducts then $\Delta\colon \mathcal{C} \to \mathcal{C} \times \mathcal{C}$ is final.

Proof. Given any diagram $X\colon \mathcal{I} \to \mathcal{C}$, the colimiting cocone $X \Rightarrow \operatorname{colim}_{\mathcal{I}} X$ yields an initial object $X \to \Delta(\operatorname{colim}_{\mathcal{I}} X)$ in $X \downarrow \Delta$. In particular, $X \downarrow \Delta$ is cohesive. □

As our last example of a final functor, we are going to consider Grothendieck constructions. Given a diagram $X\colon \mathcal{I} \to \mathbf{Cat}$ and using notation (1.28), there is an obvious functor F from the Grothendieck construction $\int^{\mathcal{I}} X$ to $\operatorname{colim}_{\mathcal{I}} X$ mapping an object (I, A) to $\mathrm{in}_I A = [A_I]$ (where $\mathrm{in}_I\colon X_I \to \operatorname{colim}_{\mathcal{I}} X$ is the universal cocone's component at $I \in \mathcal{I}$) and an arrow $(i, f)\colon (I, A) \to (J, B)$ (i.e. $i\colon I \to J$ and $f\colon (X_i)A \to B$) to

$$\mathrm{in}_J\Big((X_i)A \xrightarrow{f} B\Big) \quad = \quad \Big([A_I] = [((X_i)A)_J], [f_J], [B_J]\Big).$$

(4.8) **Proposition.** For every $X \colon \mathcal{I} \to \mathbf{Cat}$, the functor $F\colon \int^{\mathcal{I}} X \to \operatorname{colim}_{\mathcal{I}} X$ described above is final.

Proof. Given $[A_I] \in \operatorname{colim}_{\mathcal{I}} X$, an object in $[A_I] \downarrow F$ consists of $(J, B) \in \int^{\mathcal{I}} X$, together with a morphism $[A_I] \to [B_J]$ in $\operatorname{colim}_{\mathcal{I}} X$. But every such morphism is represented by a path

$$\Big([A_I] = [(C_0)_{I_0}], [(f_1)_{J_1}], [(C_1)_{I_1}], [(f_2)_{J_2}], \dots, [(f_n)_{J_n}], [(C_n)_{I_n}] = [B_J]\Big),$$

where every $f_m \colon D_m \to E_m$ is an arrow in X_{J_m} such that $\big[(D_m)_{J_m}\big] = \big[(C_{m-1})_{I_{m-1}}\big]$ as well as $\big[(E_m)_{J_m}\big] = \big[(C_m)_{I_m}\big]$. We treat the case $n = 1$, with the general one following by induction. So, our morphism $[A_I] \to [B_J]$ is represented by a path

$$\Big([A_I], [f_K], [B_J]\Big),$$

where $f\colon C \to D$ is a morphism in X_K with $[A_I] = [C_K]$ and $[B_J] = [D_K]$. These equivalences in turn mean that there is a zig-zag of arrows in $\mathcal{I}$

$$I = L_0 \overset{i_1}{\leftrightarrow} L_1 \overset{i_2}{\leftrightarrow} \dots L_{n-1} \overset{i_k}{\leftrightarrow} L_k = K$$

(meaning i_m is either an arrow $L_{m-1} \to L_m$ or the other way round) together with objects $A_I = (E_0)_{L_0}, (E_1)_{L_1}, \dots, (E_k)_{L_k} = C_K$ fitting into the zig-zag (i.e. for $i_m \colon L_{m-1} \to L_m$ we have $(X_{i_m})E_{m-1} = E_m$ or the other way round). Similarly for $B_J \sim D_K$. Again, we assume $k = 1$ with the general case following by induction (similarly for $B_J \sim D_K$). All in all, our situation is as on the left below.

$$\begin{tikzcd}
I \arrow[r, leftrightarrow, "i"] & K \arrow[r, leftrightarrow, "j"] & J \\
A \arrow[r, leftrightarrow] & C \arrow[d, "f"'] & \\
 & D \arrow[r, leftrightarrow] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
{[A_I]} \arrow[d, "\mathrm{id}"'] \arrow[dr] \arrow[drr, bend left=10] \arrow[drrr, bend left=15] & & & \\
{[A_I]} \arrow[r, leftrightarrow, "\cong"'] & {[C_K]} \arrow[r] & {[D_K]} \arrow[r, leftrightarrow, "\cong"'] & {[B_J]} \\
(I, A) \arrow[r, leftrightarrow, "{(i,\mathrm{id})}"'] & (K, C) \arrow[r, "{(\mathrm{id},f)}"'] & (K, D) \arrow[r, leftrightarrow, "{(j,\mathrm{id})}"'] & (J, B)
\end{tikzcd}$$

Explicitly, we have a zig-zag $(i, j)\colon I \leftrightarrow K \leftrightarrow J$ in $\mathcal{I}$, together with objects $A \in X_I$; C, $D \in X_K$; $B \in X_J$ and a morphism $f\colon C \to D$ such that X_i maps A to C (or the other way round) and X_j maps B to D (or the other way round). But this means that we can connect the object $[A_I] \to [B_J]$ in $[A_I] \downarrow F$ to the identity object $[A_I] \to [A_I]$ as depicted on the right above. $\square$

As seen in the previous proof, for a general diagram of categories, there are a lot of zig-zags involved and while every $[A_I] \downarrow F$ can be shown to be connected, it is generally not contractible. However, in an important special case, this is actually true. For this, we use the explicit construction (1.29) of filtered colimits in **Cat**.

(4.9) **Proposition.** If κ is any ordinal (usually a regular cardinal) and

$$\mathcal{C}_0 \hookrightarrow \mathcal{C}_1 \hookrightarrow \dots \hookrightarrow \mathcal{C}_\alpha \hookrightarrow \dots$$

a telescope $\kappa \to \mathbf{Cat}$ (where every $\mathcal{C}_\alpha \hookrightarrow \mathcal{C}_\beta$ is an embedding), then the above functor $F\colon \int^{\alpha<\kappa} \mathcal{C}_\alpha \to \operatorname{colim}_{\alpha<\kappa} \mathcal{C}_\alpha$ is even homotopy final (see (7.4.20)), meaning that every $[A_\alpha]{\downarrow}F$ is contractible.

Proof. For convenience, if $A_\alpha \in \mathcal{C}_\alpha$, we write A_β for its image under $\mathcal{C}_\alpha \to \mathcal{C}_\beta$ (and similarly for morphisms), so that

$$F\colon \int^{\alpha<\kappa} \mathcal{C}_\alpha \to \operatorname*{colim}_{\alpha<\kappa} \mathcal{C}_\alpha, \left(A_\alpha \xrightarrow{f_\beta} B_\beta\right) \mapsto \left([A_\alpha] = [A_\beta] \xrightarrow{[f_\beta]} [B_\beta]\right)$$

(by definition of the Grothendieck construction, $f_\beta\colon A_\beta \to B_\beta$). Now, to see that $[A_\alpha] \downarrow F$ is contractible for every $A_\alpha \in \mathcal{C}_\alpha$, we consider the full subcategory $\big([A_\alpha] \downarrow F\big)_{\geqslant\alpha}$ of all objects $\big(B_\beta, [p]\colon [A_\alpha] \to [B_\beta]\big)$ with $\beta \geqslant \alpha$ and note that the inclusion $\big([A_\alpha] \downarrow F\big)_{\geqslant\alpha} \hookrightarrow [A_\alpha] \downarrow F$ has a left adjoint, which is the identity on $\big([A_\alpha]\downarrow F\big)_{\geqslant\alpha}$ and maps every $\big(B_\beta, [p]\colon [A_\alpha] \to [B_\beta]\big)$ with $\beta < \alpha$ to $\big(B_\alpha, [p]\colon [A_\alpha] \to [B_\alpha]\big)$.

Finally, the category $\big([A_\alpha] \downarrow F\big)_{\geqslant\alpha}$ has an initial object, namely $\big(A_\alpha, [\mathrm{id}]\big)$. For this, we need that the morphisms in the telescope are embeddings, so that every morphism $[p]\colon [A_\alpha] \to [B_\beta]$ with $\beta \geqslant \alpha$ is represented by a unique morphism $p_\beta\colon A_\beta \to B_\beta$ in $\mathcal{C}_\beta$. □

Finally, let us note that the category of sets is in some sense the only base category for which finality of a functor needs to be tested.

(4.10) **Proposition.** A functor $F\colon \mathcal{I} \to \mathcal{J}$ is final iff for every $X\colon \mathcal{J} \to \mathbf{Sets}$ the canonical arrow $\operatorname{colim}_{\mathcal{I}}(X \circ F) \to \operatorname{colim}_{\mathcal{J}} X$ is an isomorphism.

Proof. The direction "⇒" is trivial and for the converse, we consider the adjunction

$$\begin{tikzcd} \mathbf{Sets}^{\mathcal{J}} \arrow[r, shift left, "F^*"] \arrow[r, phantom, "\perp"] & \mathbf{Sets}^{\mathcal{I}} \arrow[l, shift left, "F_*"] \arrow[r, shift left, "\operatorname{colim}_{\mathcal{I}}"] \arrow[r, phantom, "\perp"] & \mathbf{Sets}. \arrow[l, shift left, "\operatorname{Const}_{\mathcal{I}}"] \end{tikzcd}$$

Since, by hypothesis, $\operatorname{colim}_{\mathcal{I}} \circ F^* \cong \operatorname{colim}_{\mathcal{J}}$, we must also have $F_* \circ \operatorname{Const}_{\mathcal{I}} \cong \operatorname{Const}_{\mathcal{J}}$. For the one-point set $* \in \mathbf{Sets}$ and $J \in \mathcal{J}$, this means that

$$* = (\operatorname{Const}_{\mathcal{J}} *)_J \cong \big(F_*(\operatorname{Const}_{\mathcal{I}} *)\big)_J \cong \lim\Big(J \downarrow F \to \mathcal{I} \xrightarrow{*} \mathbf{Sets}\Big) \cong \pi_0(J \downarrow F). \qquad \square$$

5. Colimits in Comma Categories

Nowadays, comma categories are well-known everyday notions. However, after having studied the present literature, let us quickly record the following result about colimits in them, which is sometimes cited with an unnecessary condition that G preserve colimits as well.

(5.1) **Proposition.** Given functors $F\colon \mathcal{C} \to \mathcal{E} \leftarrow \mathcal{D} :G$ such that F preserves colimits indexed by a category $\mathcal{I}$, the projection $F \downarrow G \to \mathcal{C} \times \mathcal{D}$ creates colimits indexed by $\mathcal{I}$.

Proof. Let $X : \mathcal{I} \to F \downarrow G$ be any diagram, which is of the form

$$I \mapsto \Big(X_{I,1}, FX_{I,1} \xrightarrow{x_I} GX_{I,2}, X_{I,2}\Big) \qquad \text{with } X_{I,1} \in \mathcal{C},\ X_{I,2} \in \mathcal{D}.$$

Writing $X_1\colon \mathcal{I} \to F \downarrow G \to \mathcal{C}$ and $X_2\colon \mathcal{I} \to F \downarrow G \to \mathcal{D}$ for the compositions of X with the standard projections, suppose we have colimits

$$\Big(X_{I,1} \xrightarrow{\lambda_I} L\Big)_I \text{ of } X_1\colon \mathcal{I} \to \mathcal{C} \quad \text{and} \quad \Big(X_{I,2} \xrightarrow{\mu_I} M\Big)_I \text{ of } X_2\colon \mathcal{I} \to \mathcal{D}.$$

Now, assume we have any cocone

$$\Big(\gamma_I = (\gamma_{I,1}, \gamma_{I,2})\colon (X_{I,1}, x_I, X_{I,2}) \to (C, q, D)\Big)_I$$

below X that is sent to $\lambda \times \mu$. Then, necessarily $C = L$, $D = M$ and $\gamma_{I,1} = \lambda_I$, $\gamma_{I,2} = \mu_I$ and we need to show that a unique $q\colon FL \to GM$ as above exists. By hypothesis, we know that the $F\lambda_I\colon FX_{I,1} \to FL$ form a colimit of FX_1, while the

$$FX_{I,1} \xrightarrow{x_I} GX_{I,2} \xrightarrow{G\mu_I} GM \qquad \text{form a cocone below } FX_1.$$

Consequently, there is a unique $q\colon FL \to GM$ making all squares

$$\begin{tikzcd} FX_{I,1} \arrow[r, "F\lambda_I"] \arrow[d, "x_I"'] & FL \arrow[d, dotted, "q"] \\ GX_{I,2} \arrow[r, "G\mu_I"'] & GM \end{tikzcd}$$

with $I \in \mathcal{I}$ commute (which is precisely the property required above). Finally, we need to check that the cocone $(\lambda, \mu)\colon (X_1, x, X_2) \Rightarrow (L, q, M)$ is indeed colimiting. If

$$\left(\delta_I = (\delta_{I,1}, \delta_{I,2})\colon (X_{I,1}, x_I, X_{I,2}) \to (C, p, D)\right)_I$$

is any cocone below X, then δ_1 and δ_2 are, respectively, cocones below X_1 and X_2. Consequently, there are induced (unique!) arrows $d_1\colon L \to C$ and $d_2\colon M \to D$. All we need to check is that these define a morphism in $F \downarrow G$; i.e. that the square on the left commutes.

$$\begin{tikzcd} FL \arrow[r, "Fd_1"] \arrow[d, "q"'] & FC \arrow[d, "p"] \\ GM \arrow[r, "Gd_2"'] & GD \end{tikzcd} \qquad \begin{tikzcd} FX_{I,1} \arrow[r, "F\lambda_I"'] \arrow[d, "x_I"'] \arrow[rr, bend left, "F\delta_{I,1}"] & FL \arrow[r, "Fd_1"'] \arrow[d, "q"] & FC \arrow[d, "p"] \\ GX_{I,2} \arrow[r, "G\mu_I"] \arrow[rr, bend right, "G\delta_{I,2}"'] & GM \arrow[r, "Gd_2"] & GD \end{tikzcd}$$

For this, it suffices to show that the square commutes if we precompose it with all the $F\lambda_I$ because these form a colimiting cocone. But this is easy because we can complete everything to a commutative diagram as on the right above. $\square$

(5.2) **Corollary.** Given functors $F\colon \mathcal{C} \to \mathcal{E} \leftarrow \mathcal{D} : G$ such that $\mathcal{C}$ and $\mathcal{D}$ have $\mathcal{I}$-indexed colimits (for some fixed indexing category $\mathcal{I}$) and F preserves these, then $F \downarrow G$ has $\mathcal{I}$-indexed colimits, too (and they are calculated in $\mathcal{C} \times \mathcal{D}$). $\square$

(5.3) **Example.** Taking $F\colon \{*\} \to \mathcal{C}$ to be an object C of a cocomplete category $\mathcal{C}$ and $G := \mathrm{id}_{\mathcal{C}}$, the comma category is just $F \downarrow G \cong C \downarrow \mathcal{C}$. Now, for $\mathcal{I}$ any indexing category, and $X\colon \mathcal{I} \to \{*\}$, the composite $F \circ X$ is constantly C and so, $\mathrm{colim}_{\mathcal{I}}(F \circ X) \cong \pi_0(\mathcal{I}) \cdot C$. It follows that F preserves connected colimits (i.e. ones indexed by connected categories) and hence that all such connected colimits in $C \downarrow \mathcal{C}$ can be calculated in $\mathcal{C}$.

(5.4) **Proposition.** Given $F\colon \mathcal{C} \to \mathcal{D}$ and $D \in \mathcal{D}$, a morphism $s\colon (C, p) \to (C', p')$ is monic in $F \downarrow D$ iff $s\colon C \to C'$ is monic in $\mathcal{C}$.

Proof. The direction "$\Leftarrow$" is easy. For the converse, let f, $g\colon C'' \rightrightarrows C$ be two parallel morphisms in $\mathcal{C}$ such that $s \circ f = s \circ g$. Since $p = p' \circ Fs$, it follows that

$$p \circ Ff = p' \circ Fs \circ Ff = p' \circ Fs \circ Fg = p \circ Fg =: p'',$$

so that f, $g\colon (C'', p'') \rightrightarrows (C, p)$ is a pair of parallel arrows in $F \downarrow D$ and the claim follows. $\square$

6. Generators

Since they are going to play a role in the theory of locally presentable categories, we recall some basic facts about generators in a category.

(6.1) **Definition.** A set $\mathcal{G}$ of objects in a category $\mathcal{C}$ is called *generating* or *separating* iff. for any two parallel arrows f, $f'\colon C \rightrightarrows D$ in $\mathcal{C}$, we have

$$f = f' \;\Leftrightarrow\; (G \xrightarrow{g} C \xrightarrow{f} D) = (G \xrightarrow{g} C \xrightarrow{f'} D) \text{ for all } G \in \mathcal{G} \text{ and } g \in \mathcal{C}(G, C).$$

A generating set $\mathcal{G}$ is said to be *strongly generating* (*extremal* would be a better name) iff, in addition, $\mathcal{G}$ detects proper subobjects in the sense that a monomorphism $S \rightarrowtail C$ in $\mathcal{C}$ is an isomorphism iff every $G \to C$ with $G \in \mathcal{G}$ factors (necessarily uniquely) through it. If a (strongly) generating set $\mathcal{G}$ is actually a singleton $\{G\}$ then the object $G \in \mathcal{C}$ is called a *(strong) generator* or *(strong) separator* of $\mathcal{C}$. Dually for *(strongly) cogenerating sets* and *(strong) cogenerators*.

(6.2) **Example.** If $\mathcal{G}$ is (strongly) generating in $\mathcal{C}$, $F\colon \mathcal{C} \to \mathcal{D}$ and $D \in \mathcal{D}$ then

$$\mathcal{G}' := \{(G, p) \mid G \in \mathcal{G},\, p\colon FG \to D\}$$

is (strongly) generating in $F{\downarrow}D$. The "generating" part is easy and for the strongly generating case, we use that an arrow in $F \downarrow D$ is monic iff its underlying morphism in $\mathcal{C}$ is so (cf. (5.4)).

(6.3) **Example.** In **Sets**, any one-point set (in fact any non-empty set) is a strong generator, while any two-point set (or any set of cardinality $\geqslant 2$) is a strong cogenerator. Similarly, a one-point space (or any non-empty space) is a strong generator in **Top**, while any set of cardinality $\geqslant 2$ equipped with the coarse topology is a strong cogenerator.

(6.4) **Example.** If $\mathcal{I}$ is any small category then the representable functors $\mathcal{I}(-, I)$ with $I \in \mathcal{I}$ are strongly generating in the presheaf category $\mathbf{Sets}^{\mathcal{I}^{\mathrm{op}}}$. Indeed, given a natural transformation $\tau\colon X \Rightarrow Y$ between presheafs X, $Y\colon \mathcal{I}^{\mathrm{op}} \to \mathbf{Sets}$, the naturality of the Yoneda bijection gives us a commutative square

$$\begin{tikzcd}
\mathrm{Nat}(\mathcal{I}(-,I),X) \arrow[r, "\cong"] \arrow[d, "\tau_*"'] & X_I \arrow[d, "\tau_I"] & \alpha \arrow[r, mapsto] \arrow[d, mapsto] & \alpha_I(\mathrm{id}_I) \arrow[d, mapsto] \\
\mathrm{Nat}(\mathcal{I}(-,I),Y) \arrow[r, "\cong"'] & Y_I & \tau \circ \alpha \arrow[r, mapsto] & \tau_I(\alpha_I(\mathrm{id}_I)).
\end{tikzcd}$$

Consequently, if we have two natural transformations $\sigma \neq \tau\colon X \Rightarrow Y$, we find $I \in \mathcal{I}$ such that $\sigma_I \neq \tau_I$ and therefore also

$$\sigma_* \neq \tau_*\colon \mathrm{Nat}(\mathcal{I}(-,I),X) \to \mathrm{Nat}(\mathcal{I}(-,I),Y).$$

But this means exactly that there is some $\alpha\colon \mathcal{I}(-,I) \to X$ such that $\sigma \circ \alpha \neq \tau \circ \alpha$. As for strongness, having a monomorphism $\tau\colon X \rightarrowtail Y$ such that every $\alpha\colon \mathcal{I}(-,I) \Rightarrow Y$ lifts (necessarily uniquely) along it, just means that every $\tau_*\colon \mathrm{Nat}(\mathcal{I}(-,I),X) \to \mathrm{Nat}(\mathcal{I}(-,I),Y)$ (and therefore every τ_I) is a bijection.

(6.5) **Example.** The *interval category* $\mathcal{I} = [1] = \{0 < 1\}$ is a strong generator in **Cat**. Indeed, if F, $G\colon \mathcal{C} \to \mathcal{D}$ are two distinct functors, there is some arrow $f\colon C \to C'$ (maybe an identity) such that $Ff \neq Gf$. Identifying this arrow with the corresponding functor

$f\colon [1] \to \mathcal{C}$, we find that $F \circ f \neq G \circ f$. As for strongness, a functor $S\colon \mathcal{S} \rightarrowtail \mathcal{C}$ is monic iff it is injective on objects and faithful (i.e. injective on arrows). Now, if every $[1] \to \mathcal{C}$ factors through S, then S is also surjective on objects and full. But a fully faithful functor that is bijective on objects is an isomorphism of categories.

Similarly, the *interval groupoid* $\mathcal{J}$ (which has two objects 0, 1 and exactly two non-identity arrows $0 \to 1$, $1 \to 0$) is a strong generator in **Gpd**.

(6.6) **Example.** Being the free group on one generator, the group $\mathbb{Z}$ is a strong generator in **Grp**. The same is true for **Ab** and more generally, if R is a ring, then the free module R is a strong generator of R-**Mod** and **Mod**-R. The category **Grp** doesn't have a cogenerator but **Ab** does, namely $\mathbb{Q}/\mathbb{Z}$. To see this, observe that $\mathbb{Q}/\mathbb{Z}$ is injective (i.e. divisible) and contains elements of every finite order, so that for every abelian group G and every $g \in G$ there is some $f\colon G \to \mathbb{Q}/\mathbb{Z}$ with $f(g) \neq 0$. More generally, for every ring R, the R-module $\mathrm{Hom}_{\mathbb{Z}}(R, \mathbb{Q}/\mathbb{Z})$ (with $(rf)(s) := f(sr)$) is an injective cogenerator in R-**Mod**, which is a consequence of the above special case $R = \mathbb{Z}$ and the fact that $\mathrm{Hom}_{\mathbb{Z}}(R, -)$ is right adjoint to the restriction of scalars R-**Mod** $\to$ **Ab** in combination with the following proposition.

(6.7) **Proposition.** Given an adjunction $F\colon \mathcal{C} \rightleftarrows \mathcal{D} : G$ (left adjoint on the left) with G faithful and $\mathcal{G}$ generating in $\mathcal{C}$, then $F\mathcal{G} := \{FH \mid H \in \mathcal{G}\}$ is generating in $\mathcal{D}$. If G is even fully faithful and $\mathcal{G}$ strongly generating then $F\mathcal{G}$ is strongly generating, too.

Proof. Let f, $f'\colon D \rightrightarrows D'$ be two parallel arrows in $\mathcal{D}$ such that every

$$\begin{tikzcd} FH \arrow[r, "h"] & D \arrow[r, shift left, "f"] \arrow[r, shift right, "f'"'] & D', \end{tikzcd} \qquad \text{with } H \in \mathcal{G} \text{ and } FH \xrightarrow{h} D \text{ arbitrary, is a fork}$$

(i.e. $f \circ h = f' \circ h$). By faithfulness of G, it suffices to check that $Gf = Gf'$, so let $H \in \mathcal{G}$ and $h\colon H \to GD$. We easily calculate

$$(Gf \circ h)^\flat = \varepsilon_{D'} \circ FGf \circ Fh = f \circ \varepsilon_D \circ Fh$$

by naturality of ε. But $\varepsilon_D \circ Fh$ is a morphism of the form $FH \to D$ and so, by hypothesis,

$$f \circ \varepsilon_D \circ Fh = f' \circ \varepsilon_D \circ Fh = (Gf' \circ h)^\flat.$$

So $Gf \circ h = Gf' \circ h$, whence $Gf = Gf'$ because $h\colon H \to GD$ was arbitrary.

As for the strongly generating case, we identify $\mathcal{D}$ with its image in $\mathcal{C}$ and consider $m\colon S \rightarrowtail D$ monic in $\mathcal{D}$ (whence also in $\mathcal{C}$ because the inclusion $\mathcal{D} \hookrightarrow \mathcal{C}$ preserves monomorphisms by (3.7)) such that every $h\colon FH \to D$ with $H \in \mathcal{G}$ factors through it. Now let $h\colon H \to D$ with $H \in \mathcal{G}$. Taking its adjunct, we get a lift

$$\begin{tikzcd} & S \arrow[d, tail, "m"] \\ FH \arrow[ur, dotted, "l"] \arrow[r, "h^\flat"'] & D \end{tikzcd}$$

and taking adjuncts again, we get

$$h = (h^\flat)^\sharp = (m \circ l)^\sharp = m \circ l^\sharp,$$

so that $l^\sharp\colon H \to S$ is a lift of g along m and m is an isomorphism. $\square$

The term *generator* is rather unfortunate but well-established. Some justification for it comes from the second part of the following result.

(6.8) **Proposition.** Given a set $\mathcal{G}$ of objects in a category $\mathcal{C}$ then $\mathcal{G}$ is generating iff

$$\text{for all } C,\, D \in \mathcal{C} \text{ the map } \mathcal{C}(C,D) \to \prod_{\substack{G\in\mathcal{G}\\ a\in\mathcal{C}(G,C)}} \mathcal{C}(G,D),\ f \mapsto (f\circ a)_{G,a} \text{ is injective.}$$

If $\mathcal{C}$ has coproducts this can be internalised, so that then

$$\mathcal{G} \text{ is generating} \quad \text{iff} \quad \text{for all } C \in \mathcal{C} \text{ the morphism } \coprod_{\substack{G\in\mathcal{G}\\ a\in\mathcal{C}(G,C)}} G \xrightarrow{[a]_{G,a}} C \text{ is epi.}$$

Proof. The first part is merely another way to state the definition of a generating set and for the second part we just need to observe that for two parallel arrows $f,\ f'\colon C \rightrightarrows D$ in $\mathcal{C}$ we have $f\circ [a]_{G,a} = f'\circ [a]_{G,a}$ iff $f\circ a = f'\circ a$ for all $G\in\mathcal{G}$ and $a\colon G\to C$. □

Since a strongly generating set $\mathcal{G}$ of a locally small category $\mathcal{C}$ detects proper subobjects and there is only a set of morphisms $G\to C$ with $G\in\mathcal{G}$ for a fixed $C\in\mathcal{C}$, it is not surprising that having a strongly generating set limits the size of $\mathrm{Sub}_{\mathcal{C}}(C)$.

(6.9) **Proposition.** Every locally small category $\mathcal{C}$ with binary pullbacks and a strongly generating set $\mathcal{G}$ is well-powered (meaning that every object in $\mathcal{C}$ has only a set of subobjects, rather than a proper class).

Proof. We need to show that each $\mathrm{Sub}_{\mathcal{C}}(C)$ with $C\in\mathcal{C}$ is a set. For this, consider

$$f\colon \mathcal{M}_C \to \mathfrak{P}\left(\coprod_{G\in\mathcal{G}} \mathcal{C}(G,C)\right),\ m\mapsto \{(G, a\colon G\to C) \mid a \text{ factors through } m\}$$

(where $\mathcal{M}_C$ is the class of all monomorphisms into C). Clearly, if $m \lesssim m'$ then $f(m)\subseteq f(m')$ and so, f factors through $\mathrm{Sub}_{\mathcal{C}}(C)$. Moreover, if $f[m] = f[m']$ for two monomorphisms $m\colon S\rightarrowtail C$, $m'\colon S'\rightarrowtail C$ then also $f[m] = f[m'] = f\big([m]\cap[m']\big)$, where $[m]\cap[m']$ is represented by the pullback $S\cap S'\rightarrowtail C$ of m and m'. But then $S\cap S'\to S$ is an isomorphism because for every $a\colon G\to S$ with $G\in\mathcal{G}$, we have $(G, m\circ a)\in f[m] = f\big([m]\cap[m']\big)$, so that a factors through $S\cap S'$. Similarly, $S\cap S'\to S'$ is an isomorphism and therefore $[m]=[m']$. All in all, $\mathrm{Sub}_{\mathcal{C}}(C)$ is a set because we just constructed an injection into one. □

(6.10) **Example.** Some completeness condition is necessary in the previous proposition. For example, consider the (categorical) cone $X^{\triangleright}$ on some proper class X (viewed as a discrete category) obtained by adding a terminal object $*$. Then $X^{\triangleright}$ is a poset, so every set of objects is generating. Even better, $*$ is a strong generator but $\mathrm{Sub}(*)\cong X+\{*\}$ is a proper class.

Chapter 3

SMALLNESS OF CATEGORIES

For a category $\mathcal{C}$, being small (in the sense that its arrows form a set) usually disqualifies it from being a useful ambient category for anything. The classical example here is Freyd's theorem, which says that any small and complete (or cocomplete) category $\mathcal{C}$ must be a preorder. However, we still wish to gain some control over the size of a category to keep it from being too large. To this end, we are going to introduce and study the notions of compact and small objects, dense subcategories and the property of being locally presentable. All these concepts play fundamental roles in e.g. modern day homotopy theory.

Most of the material in the following chapter is taken from [1] with the exception of the Makkai-Paré theorem (4.16), which – unsurprisingly – is found in [39]. One exception are the results in section 6, which the author didn't find in the standard literature (in [1], only the special case of a presheaf category is mentioned in a remark). Our result (6.4) then allows for easier proofs later on, which are different from the ones given in [1].

Also, by keeping track of our cardinals and in contrast to [1], we are able to be more precise about the level of accessibility of our categories. Finally, on several occasions, we were able to remove some unnecessary hypotheses and filled in quite a few details (proposition (7.14) comes to mind, where half of the work in our proof is described as "obvious" in [1] with the given reason being a hypothesis that is in fact unnecessary and the author of this text does not see how it could be used to facilitate the proof).

1. Filtered Colimits

(1.1) **Definition.** Let κ be an infinite cardinal. A category $\mathcal{I}$ is *κ-filtered* iff

(a) $\mathcal{I}$ is small and $\mathcal{I} \neq \varnothing$;

(b) for every $\mathcal{I} \subseteq \operatorname{Ob}\mathcal{I}$ of cardinality $|\mathcal{I}| < \kappa$ there is some $J \in \operatorname{Ob}\mathcal{I}$ together with a family of arrows $(f_I : I \to J)_{I \in \mathcal{I}}$;

(c) for every set $\mathcal{A}$ of parallel arrows $I \to J$ in $\mathcal{I}$ with $|\mathcal{A}| < \kappa$ there is some $j : J \to K$ such that all composites $j \circ i$ for $i \in \mathcal{A}$ are equal.

By a *κ-filtered diagram* in a category $\mathcal{C}$, we mean a diagram $\mathcal{I} \to \mathcal{C}$ indexed by a small κ-filtered category $\mathcal{I}$ and by a *κ-filtered colimit*, a colimit of a κ-filtered diagram.

(1.2) **Remark.**

- The requirement that κ be infinite is merely for convenience to exclude the degenerate cases $\kappa \in \{0, 1\}$ where the points (b) and (c) are void. In any case, allowing $\kappa \geqslant 2$ finite doesn't add anything new because the resulting requirement is equivalent to being $\aleph_0$-filtered. Also note that some authors (e.g. Hovey, who defines the notion of κ-filtered only for $\mathcal{I}$ a limit ordinal) allow $\mathcal{I}$ and $\mathcal{A}$ in the definition to have cardinality κ.

- Some authors require the cardinal κ in the above definition to be regular. This is not really a restriction because if κ is an infinite cardinal and $\mathcal{I}$ a category, one can prove that $\mathcal{I}$ is κ-directed iff it is $(\mathrm{cf}\,\kappa)$-directed, where $\mathrm{cf}\,\kappa$ is the *cofinality* of κ (i.e. the largest κ' such that κ is κ'-filtered) and $\mathrm{cf}\,\kappa$ is always regular.

(1.3) **Observation.** If $\mathcal{I}$ is κ-filtered and $\kappa' \leqslant \kappa$ then $\mathcal{I}$ is also κ'-filtered. Also, a product of two κ-filtered categories is again κ-filtered.

(1.4) **Example.** Putting $\kappa = \aleph_0$, we obtain the classical notion of a *filtered* category, which is a non-empty category $\mathcal{I}$ satisfying that for every two I, $J \in \mathcal{I}$ there is some $K \in \mathcal{I}$ such that $\mathcal{I}(I, K)$, $\mathcal{I}(J, K) \neq \varnothing$ and where for every two parallel arrows i, $j \colon I \to J$ there is some $k \colon J \to K$ such that $k \circ i = k \circ j$. Similarly, we have the notions of *filtered diagrams, colimits* and *directed posets* where the words "filtered" and "directed" are just short for "$\aleph_0$-filtered" and "$\aleph_0$-directed".

(1.5) **Example.** Every category $\mathcal{I}$ with a terminal object is κ-filtered for every infinite cardinal κ. In particular, the classical example $\boldsymbol{\Delta}$ with objects all non-empty finite ordinals $[n] := \{0, 1, \ldots, n\}$ and morphisms all weakly monotone maps is κ-filtered for every κ.

(1.6) **Example.** If $\mathcal{I} = P$ is a poset (or more generally a preorder), the last condition for P to be κ-filtered is void and the first two conditions simply state that P is non-empty and that every κ' elements of P with $\kappa' < \kappa$ have an upper bound. Such a poset is also called *κ-directed*.

(1.7) **Example.** A special case of the previous example: An ordinal α is κ-filtered iff for all $S \subseteq \alpha$ with $|S| < \kappa$, we must have $\sup S < \alpha$ (some authors add the requirement that α be a limit ordinal).

(1.8) **Example.** Maybe the easiest non-trivial example of a filtered category is the category $\mathcal{I}$ with a single object I and a single non-identity morphism i that is idempotent (i.e. $i \circ i = i$). Obviously, this category is κ-filtered for every infinite cardinal κ. In fact, any finite category that is filtered is automatically κ-filtered for all κ.

Since filtered categories are mainly of interest as indexing categories for colimits, let us give a few examples for such, starting with a trivial one and then using the category from the previous example to get splittings of idempotents.

(1.9) **Example.** Every constant filtered diagram (say constantly C) has a colimit, namely the object C itself (with the identity cocone). To wit, the colimit of a constant diagram (say constantly C) is a copower of C with as many summands as the indexing category has components. Obviously, every filtered category is connected and the claim follows.

(1.10) **Example.** Any retract of an object C is a κ-filtered colimit (for every κ!) of a diagram with C as its only vertex and one idempotent morphism. To wit, if $s \colon C' \to C$ has a retraction r, then $r \colon C \to C'$ is a colimiting cocone for the diagram with a single vertex C and a single non-identity morphism $s \circ r$.

(1.11) **Lemma.** The following statements are equivalent for a category $\mathcal{I}$ and an infinite cardinal κ:

(a) $\mathcal{I}$ is κ-filtered;

(b) if $\mathcal{J}$ is a category with a generating set of arrows (in the sense that every arrow in $\mathcal{J}$ is a composite of these) whose cardinality is $< \kappa$ and $X\colon \mathcal{J} \to \mathcal{I}$ then there is a cocone below X;

(c) for every subgraph $\mathcal{J} \subseteq \mathcal{I}$, whose objects and arrows (together or equivalently each) have cardinality $\kappa' < \kappa$ there is a *cocone* below $\mathcal{J}$, i.e. an object $I \in \mathcal{I}$ together with a family of arrows $(\gamma_J\colon J \to I)_{J \in \mathcal{J}}$ such that $\gamma_{J'} \circ f = \gamma_J$ for every $f\colon J \to J'$ in $\mathcal{J}$.

Proof. "(a) $\Rightarrow$ (b)": First note that $\mathcal{J}$ has at most $2 \cdot \kappa'$ many objects, which is still smaller than κ, and we can therefore choose $I \in \mathcal{I}$ together with $(f_J\colon XJ \to I)_{J \in \mathcal{J}}$. Also, choosing a generating set of arrows $\mathscr{J}$ in $\mathcal{J}$ with $|\mathscr{J}| = \kappa' < \kappa$, we can assume that all identities lie in $\mathscr{J}$ (adding them produces a new generating set of cardinality at most $3 \cdot \kappa' < \kappa$). Using the axiom of choice, we pick for every $j\colon J \to J'$ in $\mathscr{J}$ some

$$g_j\colon I \to I'_j \text{ in } \mathcal{I} \qquad \text{such that} \qquad g_j \circ f_{J'} \circ Xj = g_j \circ f_J \text{ for all } j\colon J \to J' \text{ in } \mathscr{J}.$$

Picking another $I'' \in \mathcal{I}$ together with $(h_j\colon I'_j \to I'')_{j \in \mathscr{J}}$ we obtain the family of parallel arrows $(h_j \circ g_j)_{j \in \mathscr{J}}$, for which there is some $i\colon I'' \to I'''$ in $\mathcal{I}$ such that all composites $i \circ h_j \circ g_j$ are equal and the $i \circ h_{\mathrm{id}_J} \circ g_{\mathrm{id}_J} \circ f_J$ then form a cocone below X.
"(b) $\Rightarrow$ (c)": Taking the free category $F\mathcal{J}$ on $\mathcal{J}$ (whose objects are those of $\mathcal{J}$), a cocone below $\mathcal{J}$ as defined in the proposition is simply a cocone below the diagram $F\mathcal{J} \to \mathcal{I}$ mapping an arrow in $\mathcal{J}$ to itself.
"(c) $\Rightarrow$ (a)": Trivial. □

Checking if a full subcategory of filtered one is final is particularly simple. We find that while the following observation is easy, it is good to know and it saves us a few lines in the proof of the following theorem. As usual, we will use the standard characterisation (2.4.5) of final functors.

(1.12) **Lemma.** Let $\mathcal{I}$ be κ-filtered.

(a) A full subcategory $\mathcal{J} \subseteq \mathcal{I}$ is final iff for every $I \in \mathcal{I}$ there is some $J \in \mathcal{J}$ together with a morphism $I \to J$.

(b) Every final full subcategory $\mathcal{J} \subseteq \mathcal{I}$ is itself κ-filtered.

Proof. Ad (a): The direction "$\Rightarrow$" is immediate. For the converse, if we have $I \in \mathcal{I}$ and $p\colon I \to J$, $q\colon I \to J'$ with $J, J' \in \mathcal{J}$ (i.e. objects in $I \downarrow \mathcal{J}$), we use the filteredness of $\mathcal{I}$ to pick a cocone above $J \leftarrow I \to J'$, which consists of $f\colon J \to I'$ and $g\colon J' \to I'$ making the obvious square commute (so that f and g are morphisms in $I \downarrow \mathcal{I}$). Now, by hypothesis, we find $h\colon I' \to J''$ with $J'' \in \mathcal{J}$ and (by fullness of $\mathcal{J}$) $h \circ f$, $h \circ g$ are morphisms in $I \downarrow \mathcal{J}$, showing that p and q lie in the same component.
Ad (b): First, $\mathcal{J} \neq \varnothing$ by definition of finality. Now, if we have a set of objects $\mathscr{J} \subseteq \operatorname{Ob} \mathcal{J} \subseteq \operatorname{Ob} \mathcal{I}$ of cardinality $|\mathscr{J}| < \kappa$, there is some $I \in \mathcal{I}$ together with a family of arrows $(f_J\colon J \to I)_{J \in \mathscr{J}}$. Since $\mathcal{J}$ is final, $I \downarrow \mathcal{J} \neq \varnothing$ and so we find $J' \in \mathcal{J}$ together with $g\colon I \to J'$, yielding a family $(g \circ f_J\colon J \to J')_{J \in \mathscr{J}}$ of morphisms in $\mathcal{J}$ (by fullness). Similarly, if we have a set $\mathcal{A}$ of parallel arrows $J \to J'$ in $\mathcal{J}$ of cardinality $|\mathcal{A}| < \kappa$, we find $i\colon J' \to I$ in $\mathcal{I}$ making all composites $i \circ a$ with $a \in \mathcal{A}$ equal and then again pick $I \to J''$ with $J'' \in \mathcal{J}$. □

(1.13) **Example.** If $\mathcal{I}$ is filtered and $I_0 \in \mathcal{I}$ then the full subcategory $\mathcal{I}_{\geqslant I_0}$ of all objects above I_0 (i.e. all I for which there is some $I_0 \to I$) is final in $\mathcal{I}$.

(1.14) **Theorem.** If κ is a regular cardinal, there is, for every (small) κ-filtered category $\mathcal{I}$, a (small) κ-directed poset P and a final functor $P \to \mathcal{I}$.

Proof. First, let us consider the case where every κ-subgraph (i.e. one whose objects and arrows have cardinality $< \kappa$) of $\mathcal{I}$ is contained in some κ-subgraph with a unique terminal object. The partially ordered set (by definition, $\mathcal{I}$ is small)

$$P := \{\mathcal{J} \subseteq \mathcal{I} \mid \mathcal{J} \text{ is a } \kappa\text{-subgraph and has a unique terminal object}\},$$

ordered by inclusion is directed. Indeed given $\mathcal{J}, \mathcal{J}' \in P$ the union $\mathcal{J} \cup \mathcal{J}'$ is again a κ-subgraph and we can use the κ-filteredness of $\mathcal{I}$ to extend it to one in P, yielding an upper bound. Moreover, we have a functor $F\colon P \to \mathcal{I}$ mapping $\mathcal{J} \in P$ to its unique terminal object and $\mathcal{J} \leqslant \mathcal{J}'$ to the unique morphism $F\mathcal{J} \to F\mathcal{J}'$ in $\mathcal{J}'$. This functor is final by the above lemma, because if $I \in \mathcal{I}$ we always have $\mathrm{id}_I\colon I \to F\{I\}$.

In the general case, if our κ-filtered category $\mathcal{I}$ is arbitrary, we consider $\mathcal{I} \times \kappa$, which, by regularity of κ, is again κ-filtered and has the previously assumed property. In fact, if $\mathcal{J} \subseteq \mathcal{I} \times \kappa$ is a κ-subgraph, we find some cocone $(I, \alpha) \in \mathcal{I} \times \kappa$ below $\mathcal{J}$ and enlarge $\mathcal{J}$ by adding $(I, \alpha + 1)$ to it, as well as all composites of the previously chosen cocone with $(\mathrm{id}_I, \leqslant)\colon (I, \alpha) \to (I, \alpha + 1)$. Finally, the standard projection $\mathcal{I} \times \kappa \to \mathcal{I}$ is final. $\square$

(1.15) **Example.** One might be tempted to think that $\mathcal{I}$ being κ-filtered guarantees that the second case in the proof can never occur. However, consider the category $\mathcal{I}$ from example (1.8) with a single object I and a single non-identity morphism $i\colon I \to I$ and $i \circ i := i$. It is clearly filtered and we are in the second case. In fact, the corresponding final functor constructed in the proof is $\omega \to \mathcal{I}$.

2. Presentable Objects

(2.1) **Definition.** Let κ be an infinite cardinal. An object C of a category $\mathcal{C}$ is called *κ-presentable* iff $\mathcal{C}(C, -)$ preserves κ-filtered colimits. More explicitly, if $X\colon \mathcal{I} \to \mathcal{C}$ is a κ-filtered diagram with colimiting cocone $\lambda\colon X \Rightarrow \operatorname{colim} X$, we require the canonical map

$$\operatorname*{colim}_{\mathcal{I}} \mathcal{C}(C, X) \to \mathcal{C}(C, \operatorname*{colim}_{\mathcal{I}} X)$$

induced by λ to be bijective. More explicitly still, we require that every $f\colon C \to \operatorname{colim} X$

- (surjectivity) factor through some $\lambda_I\colon X_I \to \operatorname{colim}_{\mathcal{I}} X$ and
- (injectivity) for any two such factorisations $f = \lambda_I \circ g = \lambda_J \circ h$ that there be some $K \in \mathcal{I}$ together with $a\colon I \to K$, $b\colon J \to K$ such that $Xa \circ g = Xb \circ h$

(we summarise this by saying that f factors *essentially uniquely* through some X_I). Finally, we call $\aleph_0$-presentable objects *finitely presentable*.

(2.2) **Remark.**

- Because a κ-filtered colimit is also κ'-filtered for all $\kappa' \leqslant \kappa$, it follows that every κ'-presentable object is also κ-presentable.

- Since a colimit is κ-filtered iff it is $(\operatorname{cf}\kappa)$-filtered, we can always assume κ to be regular.
- Assuming κ to be regular, (1.14) tells us that it suffices to consider the case where $\mathcal{I}$ in the definition is a κ-directed poset and for the injectivity condition we can always assume that $I = J$ (and trivially $a = b$) by the poset being filtered.

(2.3) **Convention.** For the rest of this section, the letter κ will always be used to denote an arbitrary regular cardinal.

(2.4) **Example.** Any initial object in a category is finitely presentable.

(2.5) **Example.** In contrast to this, a terminal object need not be finitely presentable. For instance, in the category of $\mathbb{N}$-indexed sequences $\mathbf{Sets}^{\mathbb{N}}$, one can show (analogous to the next example), that such a sequence $X = (X_n)_{n\in\mathbb{N}}$ is κ-presentable iff

$$|X| := \sum_{n\in\mathbb{N}} |X_n| < \kappa.$$

The terminal object is the constant sequence $\Delta_* := (*)_{n\in\mathbb{N}}$, which has $|\Delta_*| = \aleph_0$.

(2.6) **Example.** A set M is κ-presentable iff it has cardinality $< \kappa$. The direction "$\Rightarrow$" is clear because κ is the colimit of $\kappa \to \mathbf{Sets}$, $\alpha \mapsto \alpha$ and κ, being regular, is κ-filtered. Now if $|M| \geqslant \kappa$ there is a surjection $M \twoheadrightarrow \kappa$, which doesn't factor through any $\alpha < \kappa$.

Conversely, if $\mathcal{I}$ is a κ-directed poset, the colimit L of an $X\colon \mathcal{I} \to \mathbf{Sets}$ is a quotient of $\coprod_{I\in\mathcal{I}} X_I$. Now, if $f\colon M \to L$ is any map, we can choose for each $m \in M$ some $I_m \in \mathcal{I}$ as well as $x_m \in X_{I_m}$ with $[x_m] = fm$. Because $\mathcal{I}$ is κ-directed we find $I \geqslant I_m$ for all $m \in M$ and consequently $fm = [x_m] = [X_{(I_m\leqslant I)}x_m]$ and so f factors through X_I. Finally, if f factors through X_I in two different ways, say as

$$M \to X_I,\ m \mapsto x_m \qquad \text{and} \qquad M \to X_I,\ m \mapsto x'_m,$$

then, since $[x_m] = [x'_m] = fm$ in the colimit, we find for each $m \in M$ some $J_m \geqslant I$ such that $X_{(I\leqslant J_m)}x_m = X_{(I\leqslant J_m)}x'_m$ and so, again by κ-filteredness, some $J \geqslant J_m$ for all m and so $X_{(I\leqslant J)}x_m = X_{(I\leqslant J)}x'_m$.

The following example is stated in [1, Example 1.2.(10)] but the proof given there is wrong; as is the one given in the erratum of *op. cit.* Here is our attempt at a correct one.

(2.7) **Example.** A topological space X is κ-presentable iff it is discrete of cardinality $< \kappa$. By the last example, it suffices to show that a presentable space is discrete. Xor this, let κ be any regular cardinal and consider the diagram of spaces $X\colon \kappa \to \mathbf{Top}$ where each X_α has $\kappa + 1$ (the ordinal successor of κ) as its underlying set and all $[\beta, \kappa]$ with $\beta > \alpha$ as its proper open sets, so that the $X_{(\alpha<\beta)} := \mathrm{id}_{\kappa+1}$ are continuous. It follows that $\operatorname{colim}_\kappa X$ again has $\kappa + 1$ as its underlying set but equipped with the coarse topology. Now if Y is any space with a non-open set $M \subseteq Y$, we define $f\colon Y \to \operatorname{colim}_\kappa X$ by mapping M to κ and $Y \setminus M$ to 0. This map does not factor through any X_α because $f^{-1}[\alpha+1, \kappa] = M$ would have to be open.

(2.8) **Example.** For $\mathcal{I}$ a small category, all representable presheaves $\mathcal{I}(-, I)\colon \mathcal{I}^{\mathrm{op}} \to \mathbf{Sets}$ with $I \in \mathcal{I}$ are finitely presentable. Indeed, a diagram $X\colon \mathcal{J} \to \widehat{\mathcal{I}} := \mathbf{Sets}^{\mathcal{I}^{\mathrm{op}}}$ has an exponential adjunct $X^\flat\colon \mathcal{J} \times \mathcal{I}^{\mathrm{op}} \to \mathbf{Sets}$ and by the Yoneda lemma as well as colimits in $\widehat{\mathcal{I}}$ being pointwise

$$\widehat{\mathcal{I}}\Big(\mathcal{I}(-, I), \operatorname*{colim}_J X_J\Big) \cong \Big(\operatorname*{colim}_J X_J\Big)_I = \operatorname*{colim}_J X^\flat_{(J,I)} \cong \operatorname*{colim}_J \widehat{\mathcal{I}}(\mathcal{I}(-, I), X_J).$$

Note however, that there are many more finitely presentable presheaves than just the representable ones. For example, any finite colimit of representable presheaves is finitely presentable by (2.14) below.

(2.9) **Definition.** A category $\mathcal{I}$ is called *κ-small* iff $|\mathrm{Arr}\,\mathcal{I}| < \kappa$ (and thus also $|\mathrm{Ob}\,\mathcal{I}| < \kappa$). As always then, the colimit of a diagram $X\colon \mathcal{I} \to \mathcal{C}$ is *κ-small* iff its indexing category $\mathcal{I}$ is κ-small.

(2.10) **Example.** Still looking at presheaves, we claim that if $\mathcal{I}$ is κ-small and $\kappa' \geqslant \kappa$ regular, then a presheaf $X\colon \mathcal{I}^{\mathrm{op}} \to \mathbf{Sets}$ is κ'-presentable iff $|X_I| < \kappa'$ for all $I \in \mathcal{I}$. Indeed, such a presheaf X is canonically a colimit of representables indexed by its category of elements $\int_{\mathcal{I}} X$, which has objects $\coprod_{I\in\mathcal{I}} X_I$ and whose arrows are of the form $i\colon (I', x\cdot i) \to (I, x)$ with $i\colon I' \to I$ in $\mathcal{I}$ and $x \in X_I$. So, the cardinality of (the set of arrows of) $\int_{\mathcal{I}} X$ is

$$\sum_{I\in\mathcal{I}} |X_I| + \sum_{\substack{i\colon I'\to I\\ \text{non-id}}} |X_I| = \sum_{i\colon I'\to I} |X_I| \leqslant |\mathrm{Arr}\,\mathcal{I}| \cdot \max_I |X_I| < \kappa'$$

as long as every $|X_I|$ is $< \kappa'$. In that case, we have written X as a κ'-small colimit of representables (which are finitely presentable) and can then use (2.14) below. The converse is not immediate (we show a generalisation in (6.4)) but essentially boils down to the fact that the right adjoint $I_*\colon \mathbf{Sets} \to \mathbf{Sets}^{\mathcal{I}}$ of an evaluation functor ev_I with $I \in \mathcal{I}$ preserves κ-filtered colimits.

(2.11) **Example.** The finiteness condition in the last example is necessary. Indeed, let's take $\mathcal{I} := \mathbb{N}$ to be a countably infinite discrete category, so that $\mathcal{I}$-presheaves are just $\mathbb{N}$-indexed families of sets $(X_n)_{n\in\mathbb{N}}$ and morphisms are $\mathbb{N}$-indexed families of maps. We claim that the constant presheaf $X := (*_n)_{n\in\mathbb{N}}$, with every $*_n = \{*\}$ a one-point set, is pointwise finite but not finitely presentable. Indeed, for $k \in \mathbb{N}$, letting $Y_k := \big(\{0,\dots,k\}\big)_{n\in\mathbb{N}}$ be constantly $\{0,\dots,k\}$, we have a filtered diagram

$$Y_0 \hookrightarrow Y_1 \hookrightarrow Y_2 \hookrightarrow \dots \qquad \text{with colimit} \qquad Y_\infty = (\mathbb{N})_{n\in\mathbb{N}}.$$

But the morphism $f = (f_n)_{n\in\mathbb{N}}\colon X \to Y_\infty$ with $f_n\colon *_n \mapsto n$ doesn't factor through any Y_n.

(2.12) **Example.** A group G is finitely presentable in our sense iff it is finitely presentable in the classical sense. More generally, a group G is κ-presentable iff it has a presentation $G \cong F/K$ where both F and K are free on a set of generators of cardinality $< \kappa$ (i.e. they are κ-generated).

A well-known fact (and a defining property of filtered colimits) about **Sets** is that finite limits commute with filtered colimits, which can be generalised to κ-small limits and κ-filtered colimits. But first, let's remind ourselves, what it formally means for limits and colimits to commute. The easiest formulation in our situation would probably be that for $\mathcal{I}$ κ-filtered,

$$\operatorname*{colim}_{\mathcal{I}}\colon \mathbf{Sets}^{\mathcal{I}} \to \mathbf{Sets} \qquad \text{preserves } \kappa\text{-small limits.}$$

But let's be more explicit. Consider

$$X\colon \mathcal{I} \times \mathcal{J} \to \mathbf{Sets}.$$

Taking its exponential adjunct $\mathcal{I} \to \mathbf{Sets}^{\mathcal{J}}$ and composing with $\lim_J\colon \mathbf{Sets}^{\mathcal{J}} \to \mathbf{Sets}$, we get a new functor $\mathcal{I} \to \mathbf{Sets}$, $I \mapsto \lim_J X_{(I,J)}$. Similarly, we have $\mathcal{J} \to \mathbf{Sets}$, $J \mapsto \mathrm{colim}_I\, X_{(I,J)}$.

Because (co)limits in presheaf categories are calculated pointwise, we can alternatively obtain these functors as

$$\lim_J \left(\mathcal{J} \xrightarrow{X} \mathbf{Sets}^{\mathcal{I}}\right) \qquad \text{and} \qquad \operatorname*{colim}_I \left(\mathcal{I} \xrightarrow{X} \mathbf{Sets}^{\mathcal{J}}\right) \qquad \text{respectively.}$$

We now again respectively take the limit and colimit of these two functors and obtain a canonical morphism

$$\operatorname*{colim}_I \lim_J X_{(I,J)} \to \lim_J \operatorname*{colim}_I X_{(I,J)}$$

as follows. Let's first introduce names for all (co)limiting (co)cones involved (where, by abuse of notation, we use I and J both as names for objects in $\mathcal{I}$ and $\mathcal{J}$ respectively as well as variables over which we take the colimit and limit):

$$\lim_J X_{(I,J)} \xrightarrow{\lambda_{I,J}} X_{(I,J)}, \quad X_{(I,J)} \xrightarrow{\gamma_{I,J}} \operatorname*{colim}_I X_{(I,J)},$$

$$\lim_J X_{(I,J)} \xrightarrow{\gamma'_I} \operatorname*{colim}_I \lim_J X_{(I,J)}, \quad \lim_J \operatorname*{colim}_I X_{(I,J)} \xrightarrow{\lambda'_J} \operatorname*{colim}_I X_{(I,J)}$$

(where $\lambda_{I,J}$ and $\gamma_{I,J}$ are natural in $I \in \mathcal{I}$ and $J \in \mathcal{J}$ respectively). Putting these together in a diagram, we obtain

$$\begin{tikzcd}
X_{(I,J)} \arrow[d, "\gamma_{I,J}"'] & \lim_J X_{(I,J)} \arrow[l, "\lambda_{I,J}"'] \arrow[r, "\gamma'_I"] \arrow[d, dotted] & \operatorname{colim}_I \lim_J X_{(I,J)} \arrow[dl, dotted] \\
\operatorname{colim}_I X_{(I,J)} & \lim_J \operatorname{colim}_I X_{(I,J)} \arrow[l, "\lambda'_J"] &
\end{tikzcd}$$

where the dotted arrows are the unique ones making the diagram commute for all I, J. To wit, the $\gamma_{I,J} \circ \lambda_{I,J}$ form a cone above $\operatorname{colim}_I X_{(I,J)}$ because the $\lambda_{I,J}$ are already a cone and $\gamma_{I,J}$ is natural in J. The induced vertical dotted arrows then form a cocone below $\lim_J X_{(I,J)}$ by the naturality of $\lambda_{I,J}$ in I, the cocone condition on $\gamma_{I,J}$ and uniqueness of the vertical dotted arrow. Now, we say that $\mathcal{I}$-colimits commute with $\mathcal{J}$-limits iff for each X, the oblique dotted arrow thus obtained is an isomorphism.

(2.13) **Lemma.** A category $\mathcal{I}$ is κ-filtered iff κ-small limits in **Sets** commute with $\mathcal{I}$-colimits.

Proof. "$\Rightarrow$": Assume $\mathcal{I}$ is κ-filtered, $X\colon \mathcal{I} \times \mathcal{J} \to \mathbf{Sets}$ is a diagram with $\mathcal{J}$ κ-small and let's describe the canonical morphism, which must be an isomorphism. Elements of $\lim_J X_{(I,J)}$ (for a fixed I) are families $x = (x_J)_{J\in\mathcal{J}}$ with $x_J \in X_{(I,J)}$ and such that $j_* x_{J'} = x_J$ for all $j\colon J' \to J$ in $\mathcal{J}$ (or more concisely $\lim_J X_{(I,J)} \cong \operatorname{Nat}(*, X_{(I,-)})$). Now the colimit $\operatorname{colim}_I \lim_J X_{(I,J)}$ is obtained by taking the disjoint union of all these limits and taking the quotient with respect to the equivalence relation $\sim$ generated by

$$(I', (x_J)_{J\in\mathcal{J}}) \sim (I, (i_* x_J)_{J\in\mathcal{J}}) \qquad \text{for all } i\colon I' \to I \text{ in } \mathcal{I}.$$

On the other hand, each $\operatorname{colim}_I X_{(I,J)}$ (with $J \in \mathcal{J}$ fixed) is obtained from $\coprod_I X_{(I,J)}$ by taking the quotient with respect to the equivalence relation $\sim$ generated by $(I', x) \sim (I, i_* x)$ for all $i\colon I' \to I$ in $\mathcal{I}$. With this, $\lim_J \operatorname{colim}_I X_{(I,J)}$ comprises all families

$$[I_J, x_J]_{J\in\mathcal{J}} \qquad \text{with } x_J \in X_{(I_J,J)} \text{ and such that} \qquad [I_{J'}, j_* x_{J'}] = [I_J, x_J]$$

for all $j\colon J' \to J$ in $\mathcal{J}$ (in particular $j_* x_{J'} \in X_{(I_{J'},J')}$ and the above expression is well-defined).

Before showing the bijectiveness of the canonical map, note that by the filteredness of $\mathcal{I}$, two (I,x), (I',x') are equivalent iff there is a diagram $i\colon I \to I'' \leftarrow I' : i'$ in $\mathcal{I}$ such that $i_* x = i'_* x'$. Indeed, (I,x) and (I',x') are equivalent iff there is a sequence of generating equivalences

$$(I,x) = (I_0,x_0) \sim (I_1,x_1) \sim \ldots \sim (I_n,x_n) = (I',x')$$

with each one induced by some $I_k \to I_{k+1}$ or $I_{k+1} \to I_k$. This gives us a zig-zag of arrows in $\mathcal{I}$ and by (1.11), there is some cocone (I'',γ) below this zig-zag. By an easy induction on n, it then follows that $(\gamma_0)_* x = (\gamma_n)_* x'$. Now, back to the canonical map

$$\operatorname*{colim}_I \lim_J X_{(I,J)} \to \lim_J \operatorname*{colim}_I X_{(I,J)} \quad \text{given by} \quad \big[I,(x_J)_{J\in\mathcal{J}}\big] \mapsto [I,x_J]_{J\in\mathcal{J}},$$

which is well-defined because $x_J \in X_{(I,J)}$ for all $J \in \mathcal{J}$ and if $i\colon I' \to I$ is a morphism in $\mathcal{I}$ (and so $\big[I',(x_J)_J\big] = \big[I,(i_* x_J)_J\big]$) then $[I', i_* x_J] = [I, x_J]$ for each $J \in \mathcal{J}$.

To see that this map is surjective, take any family $[I_J, x_J]_{J\in\mathcal{J}}$ and note that because $|\operatorname{Ob}\mathcal{J}| < \kappa$ we find $I \in \mathcal{I}$ together with a family of morphisms $(i_J\colon I_J \to I)_{J\in\mathcal{J}}$, so that $[I_J,x_J] = [I,(i_J)_* x_J]$ for all $J \in \mathcal{J}$. Moreover, $[I,(i_J)_* x_J]_{J\in\mathcal{J}}$ is an element of $\lim_J \operatorname{colim}_J X_{(I,J)}$ because given $j\colon J' \to J$ in $\mathcal{J}$, we have

$$\big[I, j_*(i_{J'})_* x_{J'}\big] = \big[I,(i_{J'})_* j_* x_{J'}\big] = [I_{J'}, j_* x_{J'}] = [I_J, x_J] = \big[I,(i_J)_* x_J\big].$$

With this, we see that $[I_J,x_J]_{J\in\mathcal{J}}$ is the image of $\big[I, \big((i_J)_* x_J\big)_{J\in\mathcal{J}}\big]$ under the canonical map. As for injectivity, assume that $\big[I,(x_J)_J\big]$, $\big[I',(x'_J)_J\big]$ are mapped to the same value; i.e. $[I,x_J] = [I',x'_J]$ for all $J \in \mathcal{J}$, which, as noted above, means that for each $J \in \mathcal{J}$, there are $i_J\colon I \to I''$, $i'_J\colon I' \to I''$ such that $(i_J)_* x_J = (i'_J)_* x'_J$. These are fewer than $2\kappa = \kappa$ such morphisms and so, there is even I''' and $i\colon I \to I'''$, $i'\colon I' \to I'''$ with $i_* x_J = i'_* x'_J$ for all $J \in \mathcal{J}$. Then

$$\big[I,(x_J)_J\big] = \big[I''',(i_* x_J)_J\big] = \big[I''',(i'_* x'_J)\big] = \big[I',(x'_J)_J\big]$$

and so, the canonical map is injective.
"$\Leftarrow$": Assuming, $\mathcal{I}$-colimits in **Sets** commute with κ-small limits, $\mathcal{I}$ cannot be empty because otherwise, taking $\mathcal{J} := \varnothing$, there is the unique functor $X\colon \mathcal{I} \times \mathcal{J} = \varnothing \to \mathbf{Sets}$, which has

$$\operatorname*{colim}_{\mathcal{I}} \lim_{\mathcal{J}} X = \varnothing \qquad \text{but} \qquad \lim_{\mathcal{J}} \operatorname*{colim}_{\mathcal{I}} X \cong *.$$

Now, given a family of objects $(I_j)_{j\in K}$ indexed by a set K of cardinality $|K| < \kappa$, we consider

$$X := \big[\mathcal{I}(I_k,-)\big]_{k\in K}\colon \coprod_{k\in K} \mathcal{I} = K \times \mathcal{I} \to \mathbf{Sets}.$$

Because the colimit of a representable functor is a point, we easily calculate

$$\lim_k \operatorname*{colim}_I X_{(k,I)} = \lim_k \operatorname*{colim}_I \mathcal{I}(I_k,I) \cong \lim_k * \cong \prod_{k\in K} * \cong *.$$

On the other hand

$$\operatorname*{colim}_I \lim_k X_{(k,I)} = \operatorname*{colim}_I \prod_{k\in K} \mathcal{I}(I_k,I) \qquad \text{is a quotient of} \qquad \coprod_{I\in\mathcal{I}} \prod_{k\in K} \mathcal{I}(I_k,I).$$

Since this has to be a one-point set, we find at least one family $\big(I,(a_k\colon I_k \to I)_{k\in K}\big)$, which proves the second axiom of a κ-filtered category. Finally, given a family $(a_k\colon I' \to I)_{k\in K}$ of parallel arrows, indexed by a set K of cardinality $|K| < \kappa$, let $\mathcal{J}$ be the category with two objects a, b and for each $k \in K$ a morphism $k\colon a \to b$. Now consider the diagram

$$X^\sharp\colon \mathcal{J} \to \mathbf{Sets}^{\mathcal{I}},\ a \mapsto \mathcal{I}(I, -),\ b \mapsto \mathcal{I}(I', -),\ k \mapsto a_k^*,$$

which again has $\lim_{\mathcal{J}} \operatorname{colim}_{\mathcal{I}} X \cong \lim_{\mathcal{J}} * \cong *$, whereas $\operatorname{colim}_{\mathcal{I}} \lim_{\mathcal{J}} X$ is a quotient of

$$\coprod_{I'' \in \mathcal{I}} \lim_{\mathcal{J}} X_{(I'',-)} \cong \coprod_{I'' \in \mathcal{I}} \{a\colon I' \to I'' \mid \text{the } a_j \circ a \text{ with } j \in J \text{ are all equal}\}\,.$$

In particular, there must be some $a\colon I' \to I''$ such that the $a_j \circ a$ are all equal. □

(2.14) **Proposition.** A κ-small colimit of κ-presentable objects is κ-presentable.

Proof. Let $X : \mathcal{I} \to \mathcal{C}$, $Y : \mathcal{J} \to \mathcal{C}$ be two diagrams with $\mathcal{I}$ κ-small, each X_I κ-presentable and $\mathcal{J}$ κ-filtered. Then by the above lemma

$$\begin{aligned}\mathcal{C}\left(\operatorname*{colim}_I X_I, \operatorname*{colim}_J Y_J\right) &\cong \lim_I \mathcal{C}\left(X_I, \operatorname*{colim}_J Y_J\right) \cong \lim_I \operatorname*{colim}_J \mathcal{C}(X_I, Y_J) \\ &\cong \operatorname*{colim}_J \lim_I \mathcal{C}(X_I, Y_J) \cong \operatorname*{colim}_J \mathcal{C}\left(\operatorname*{colim}_I X_I, Y_J\right).\end{aligned}$$

□

(2.15) **Corollary.** A retract (a.k.a. split quotient, a.k.a. split subobject) of a κ-presentable object C is again κ-presentable.

Proof. Given $s\colon C' \to C$ with a retraction $r\colon C \to C'$, the object C' is the colimit of the diagram with the single vertex C and the single arrow $s \circ r$. It is also the coequaliser of $s \circ r\colon C \to C$ and id_C. Alternatively, there is the following direct proof.

Given a κ-filtered diagram $X : \mathcal{I} \to \mathcal{C}$ as well as $s\colon C \to D$ and $r\colon D \to C$ with D κ-presentable and $r \circ s = \mathrm{id}_C$ we get a commutative diagram

$$\begin{tikzcd}
\operatorname{colim}_{\mathcal{I}} \mathcal{C}(C, X) \arrow[r] \arrow[d, "\operatorname{colim} r^*"'] & \mathcal{C}(C, \operatorname{colim}_{\mathcal{I}} X) \arrow[d, "r^*"] \\
\operatorname{colim}_{\mathcal{I}} \mathcal{C}(D, X) \arrow[r, "\cong"] \arrow[d, "\operatorname{colim} s^*"'] & \mathcal{C}(D, \operatorname{colim}_{\mathcal{I}} X) \arrow[d, "s^*"] \\
\operatorname{colim}_{\mathcal{I}} \mathcal{C}(C, X) \arrow[r] & \mathcal{C}(C, \operatorname{colim}_{\mathcal{I}} X)
\end{tikzcd} ,$$

whose vertical composites are identities. From this, we can extract an inverse for the top arrow. □

(2.16) **Example.** We have already seen that standard n-simplices (i.e. representables) $\Delta[n] \in \mathbf{sSets}$ are finitely presentable and with the above proposition, it follows that boundaries $\partial\Delta[n]$ and horns $\Lambda^k[n]$ are again finitely presentable (though not representable) because they are finite colimits of representables.

Finally, our key fact (2.14) above also helps us understand how presentability interacts with forming product categories (though this can just as easily be achieved without). Clearly, an object in a coproduct of categories $\coprod_{i \in I} \mathcal{C}_i$ is κ-presentable iff it is so as an object of the corresponding summand. The same is not quite true for products and we need to account for the size of the indexing set.

(2.17) **Proposition.** Given a family of categories $(\mathcal{C}_i)_{i\in I}$ with $|I| < \kappa$ then an object $(C_i)_{i\in I} \in \prod_{i\in I} \mathcal{C}_i$ is κ-presentable iff it is pointwise κ-presentable (i.e. iff every $C_i \in \mathcal{C}_i$ is κ-presentable).

Proof. This is essentially due to the fact that colimits in $\prod_i \mathcal{C}_i$ are calculated pointwise. But first, note that the statement is vacuously true if any of the $\mathcal{C}_i$ are empty; so let's assume $\mathcal{C}_i \neq \varnothing$ for all i.

Now, if $(C_i)_{i\in I}$ is κ-filtered then it is pointwise so because any κ-filtered diagram $X\colon \mathcal{J} \to \mathcal{C}_{i_0}$ with a colimit lifts to a diagram in $\prod_i \mathcal{C}_i$ by making it constant away from i_0 (which we can do since the $\mathcal{C}_i$ are non-empty) and then the colimit in $\mathcal{C}_{i_0}$ lifts to a colimit in $\prod_i \mathcal{C}_i$ whose coefficient away from i_0 is the previously chosen constant object there (using that filtered indexing categories are connected).

Conversely, if $C = (C_i)_{i\in \mathcal{I}}$ is pointwise κ-presentable and $X\colon \mathcal{J} \to \prod_i \mathcal{C}_i$ a κ-filtered diagram we again use the fact that colimits in $\mathcal{C} := \prod_i \mathcal{C}_i$ are calculated pointwise and get

$$\begin{aligned}\mathcal{C}(C, \operatorname*{colim}_J X_J) = \prod_i \mathcal{C}_i(C_i, \operatorname*{colim}_J X_{J,i}) &\cong \prod_i \operatorname*{colim}_J \mathcal{C}_i(C_i, X_{J,i}) \\ &\cong \operatorname*{colim}_J \prod_i \mathcal{C}_i(C_i, X_i) = \operatorname*{colim}_J \mathcal{C}(C, X_J),\end{aligned}$$

where in the second to last isomorphism, we used that κ-small limits in **Sets** commute with κ-filtered colimits. □

(2.18) **Remark.** In (6.4) we are going to prove a similar claim for arbitrary index categories (not just discrete ones) but with a fixed (locally κ-presentable) base category.

3. Locally Presentable Categories

(3.1) **Definition.** Given an infinite cardinal κ, a category $\mathcal{C}$ is called *κ-accessible* iff it is locally small, has κ-filtered colimits and a set $\mathcal{R}$ of κ-presentable objects (which we shall call a *set of κ-generators* and identify with the full subcategory it generates) such that each object in $\mathcal{C}$ is a κ-filtered colimit of objects in $\mathcal{R}$. If $\mathcal{C}$ is even cocomplete, we call it *locally κ-presentable.*

As before, in case $\kappa = \aleph_0$, we speak of *finitely accessible* and *locally finitely presentable* categories. Finally, we simply call $\mathcal{C}$ *accessible* (resp. *locally presentable* iff it is κ-accessible (resp. locally κ-presentable) for some κ.

(3.2) **Remark.**

- Again, as in the definition of κ-filtered categories and κ-presentable objects, we can always replace κ by $\operatorname{cf}\kappa$ and will thus assume that κ is regular for what follows.
- Under this assumption, we can, by (1.14), equivalently require that each object in $\mathcal{C}$ be a κ-directed colimit of objects in $\mathcal{R}$.
- While not obvious from the definition, every locally κ-presentable category is also locally κ'-presentable for every regular cardinal $\kappa' \geqslant \kappa$, which we are going to show in (3.28). This is not true for accessible categories, though.
- Not every object in a locally κ-presentable category is κ-presentable (cf. the lemma (3.21) below). However, because every κ-presentable object is also κ'-presentable for all $\kappa' \geqslant \kappa$ and every object is a (small) colimit of κ-presentable ones, (2.14) implies that every object is presentable.

(3.3) **Example.** Given a family $(\mathcal{C}_i)_{i\in I}$ of κ-accessible categories with $|I| < \kappa$ then their product $\mathcal{C} := \prod_{i\in I} \mathcal{C}_i$ is again κ-accessible. This is due to the fact that the κ-presentable objects in $\mathcal{C}$ are the pointwise κ-presentable ones (as shown in (2.17)) and colimits in $\mathcal{C}$ are calculated pointwise.

(3.4) **Example.** As we will show in (3.31), every presheaf category $\widehat{\mathcal{I}} := \mathbf{Sets}^{\mathcal{I}^{\mathrm{op}}}$ is locally finitely presentable. This is not trivial because, while all representable functors are finitely presentable (cf.(2.8)) and every presheaf P is canonically a colimit of representables, the category of elements of a presheaf P (which indexes this canonical colimit) is usually not filtered. The problem here is that the coproduct of two representable presheaves as well as the coequaliser of two transformations between representables, while still finitely presentable, are generally not representable.

(3.5) **Example.** The category **Grp** of groups is locally finitely presentable (every group is the colimit of its finitely generated subgroups). A generating set of finitely presentables is given by $\{\mathbb{Z}^{*n} \mid n \in \mathbb{N}\}$, which contains for every $n \in \mathbb{N}$ a free group on n generators.

(3.6) **Example.** The category of modules over a ring R is locally finitely presentable (every module is the colimit of its finitely generated submodules). Just like for the category of groups, a generating set of finitely presentables is given by $\{R^n \mid n \in \mathbb{N}\}$.

Of course, there is a more general theme hidden behind the two previous examples, namely that the category of Eilenberg-Moore algebras over an accessible monad is accessible.

(3.7) **Example.** One can show that the category of fields is finitely accessible but not locally presentable.

(3.8) **Example.** As seen in (2.7), presentable topological spaces are discrete and so, the category of topological spaces cannot be accessible (since every object in an accessible category is presentable). This is one of the reasons why sometimes, simplicial sets are preferred in algebraic topology; although there are other ways to remedy the situation (e.g. the category of so-called Δ-spaces).

Before even starting to investigate properties of κ-accessible and locally κ-presentable categories, let us quickly establish a canonical way of writing an object as a colimit of κ-presentable ones (the definition only tells us that there is a way).

(3.9) **Definition.** Given a (regular) cardinal κ and a category $\mathcal{C}$, we write $\mathcal{C}_\kappa$ for the full subcategory of κ-presentable objects.

(3.10) **Proposition.** If $\mathcal{C}$ is κ-accessible with a set $\mathcal{R}$ of κ-generators (which we identify with the corresponding full subcategory) and $C \in \mathcal{C}$, then

(a) $\mathcal{R} \downarrow C$ and $\mathcal{C}_\kappa \downarrow C$ are κ-filtered (with the caveat that the latter is not small);

(b) if $\mathcal{C}$ is even locally κ-presentable then $\mathcal{C}_\kappa \downarrow C$ even has κ-small colimits;

(c) for $\mathcal{R} \downarrow C \to \mathcal{R}$ and $\mathcal{C}_\kappa \downarrow C \to \mathcal{C}_\kappa$ the canonical projection functors,

$$C \cong \operatorname{colim}(\mathcal{R} \downarrow C \to \mathcal{R} \hookrightarrow \mathcal{C}) \cong \operatorname{colim}(\mathcal{C}_\kappa \downarrow C \to \mathcal{C}_\kappa \hookrightarrow \mathcal{C})$$

with colimiting cocones given by $\left(P \xrightarrow{p} C\right)_{(P,p)}$.

Proof. Ad (b): Colimits in $\mathcal{C}_\kappa \downarrow C$ are calculated in $\mathcal{C}_\kappa$ (cf. (2.5.1)) and we have already shown in (2.14) that κ-small colimits of κ-presentables are again κ-presentable.
Ad (a): We write C as a colimit of some κ-filtered diagram $X\colon \mathcal{I} \to \mathcal{C}$ with values in $\mathcal{R}$ and colimiting cocone $(\lambda_I\colon X_I \to C)_{I\in\mathcal{I}}$. By definition of a κ-presentable object, every $p\colon P \to C$ with $P \in \mathcal{R}$ factors essentially uniquely through some λ_I. Noting that the λ_I can be viewed as elements of $\mathcal{R}\downarrow C$, this, together with the κ-filteredness of $\mathcal{I}$ implies the κ-filteredness of $\mathcal{R}\downarrow C$. The same proof goes through for $\mathcal{C}_\kappa \downarrow C$ replacing $\mathcal{R}$ everywhere by $\mathcal{C}_\kappa$.
Ad (c): Note that (a) together with (1.12) implies that the full subcategory of all λ_I is final in $\mathcal{R}\downarrow C$ (as well as in $\mathcal{C}_\kappa \downarrow C$). □

(3.11) **Remark.** Technically, $\mathcal{C}_\kappa$ above is not κ-filtered because it is not small. However, we will show in (3.23) below that it is essentially small and can then replace it by a (small) skeleton $\mathrm{Sk}(\mathcal{C}_\kappa)$.

Another (commonly found) way of stating the above proposition is to use the notion of a dense functor (or in our case, a dense subcategory).

(3.12) **Definition.** A functor $F\colon \mathcal{I} \to \mathcal{C}$ is *dense* iff each $C \in \mathcal{C}$ is a colimit

$$C \cong \operatorname{colim}\left(F\downarrow C \xrightarrow{Q} \mathcal{I} \xrightarrow{F} \mathcal{C}\right) \qquad \text{(where } Q \text{ is the projection functor)}$$

with colimiting cocone $\left(FI \xrightarrow{p} C\right)_{(I,p)\in F\downarrow C}$. In other words, F is dense iff the pointwise left Kan extension $F_!F$ of F along itself exists and is isomorphic to the identity. Similarly, a subcategory $\mathcal{S} \subseteq \mathcal{C}$ is *dense* iff $\mathcal{S} \hookrightarrow \mathcal{C}$ is dense.

(3.13) **Notation.** If need arises to give the *canonical diagram* $F\downarrow C \to \mathcal{I} \to \mathcal{C}$ from the definition a name, we shall denote it by Can^F_C or $\mathrm{Can}^{\mathcal{I}}_C$ if $F\colon \mathcal{I} \to \mathcal{C}$ happens to be the inclusion of a subcategory.

(3.14) **Example.** If the empty subcategory lies dense in $\mathcal{C}$ then either $\mathcal{C} = \varnothing$ or $\mathcal{C} = \{*\}$.

(3.15) **Example.** As was just shown, if $\mathcal{C}$ is κ-accessible then $\mathcal{C}_\kappa \subseteq \mathcal{C}$ is dense.

We are going to show later in (5.4) that every locally presentable category is actually complete. For now, let us record how the above proposition implies the existence of a terminal object and – more importantly – its level of presentability.

(3.16) **Proposition.** Every locally κ-presentable $\mathcal{C}$ has a terminal object. If $\mathcal{C}$ even has a set of κ-generators $\mathcal{R}$ (which we identify with the corresponding full subcategory) such that

$$|\mathrm{Arr}\,\mathcal{R}| < \kappa \qquad \text{(i.e. } |\mathrm{Ob}\,\mathcal{R}| < \kappa \text{ and } |\mathcal{C}(R,S)| < \kappa \text{ for all } R,\, S \in \mathcal{R}\text{)}$$

then the terminal object is κ-presentable.

Proof. Every object $C \in \mathbf{C}$ can be written as the colimit

$$C \cong \operatorname{colim}\left(\mathcal{R}\downarrow C \xrightarrow{Q} \mathcal{R} \hookrightarrow \mathcal{C}\right),$$

where Q is the standard projection. But then Q induces a morphism between colimits

$$Q_*\colon C \longrightarrow \operatorname{colim}(\mathcal{R} \hookrightarrow \mathcal{C}) =: W,$$

meaning that $W \in \mathcal{C}$ is weakly terminal. Since $\mathcal{C}$ is cocomplete, this implies the existence of a terminal object $*$, given by the coequaliser of all endomorphisms $W \to W$. We now note that $\mathcal{R} \downarrow * \cong \mathcal{R}$ (via the standard projection), meaning that

$$* \cong \operatorname{colim}\left(\mathcal{R} \downarrow * \xrightarrow{\cong} \mathcal{R} \hookrightarrow \mathcal{C}\right) \cong \operatorname{colim}(\mathcal{R} \hookrightarrow \mathcal{C}) \cong W$$

and $W \cong *$ was already terminal. Finally, if $|\operatorname{Arr} \mathcal{R}| < \kappa$, we have just presented $*$ as a κ-small colimit of κ-presentables. □

Next, let us find some properties and characterisations of accessible or even locally presentable categories. The fact that every object in a κ-accessible category $\mathcal{C}$ is a κ-filtered colimit of κ-presentable ones allows us to easily characterise arbitrary κ-filtered colimits in $\mathcal{C}$, which is helpful for future purposes.

(3.17) **Proposition.** Given a κ-filtered diagram $X : \mathcal{I} \to \mathcal{C}$ in a κ-accessible category $\mathcal{C}$, a cocone $\lambda \colon X \Rightarrow C$ below X is a colimit iff for all κ-presentables $D \in \mathcal{C}$ (or equivalently for all D in a set of κ-generators)

(a) every morphism $D \to C$ factors through some λ_I and

(b) if two morphisms $f, g \colon D \to XI$ have $\lambda_I \circ f = \lambda_I \circ g$ then there is some $i \colon I \to J$ in $\mathcal{I}$ such that $Xi \circ f = Xi \circ g$.

Proof. The direction "⇒" is by definition of a κ-presentable object. Conversely, the two hypotheses from the proposition say that at least on κ-presentable objects, the two functors $\mathcal{C}(-, C)$ and $\operatorname{colim}_I \mathcal{C}(-, XI)$ agree. Recalling that limits and colimits in the metacategory $\mathbf{Sets}^{\mathcal{C}}$ are calculated pointwise, we have for every κ-presentable object $D \in \mathcal{C}$

$$\mathcal{C}(D, C) \cong \operatorname*{colim}_I \mathcal{C}(D, XI) \cong \mathcal{C}(D, \operatorname*{colim}_I XI).$$

For an arbitrary $D \in \mathcal{D}$, we pick a set $\mathcal{R}$ of κ-presentable objects as in the definition and write D as a colimit of a diagram $Y : \mathcal{J} \to \mathcal{C}$ with $\mathcal{J}$ κ-filtered and $YJ \in \mathcal{R}$ for every $J \in \mathcal{J}$. Using that contravariant Hom-functors map colimits to limits, we get

$$\mathcal{C}(D, C) \cong \lim_J \mathcal{C}(YJ, C) \cong \lim_J \mathcal{C}(YJ, \operatorname*{colim}_I XI) \cong \mathcal{C}(D, \operatorname*{colim}_I XI).$$

But this means that $\mathcal{C}(-, C) \cong \mathcal{C}(-, \operatorname{colim}_I XI)$ and hence $C \cong \operatorname{colim}_I XI$. □

(3.18) **Corollary.** Given a κ-accessible category $\mathcal{C}$ with a set of κ-generators $\mathcal{R}$, the Hom-functors $\mathcal{C}(R, -)$ with $R \in \mathcal{R}$ jointly reflect κ-filtered colimits.

Proof. This is just a restatement of the above proposition. □

A first obvious observation (which is true in any category with filtered colimits) about accessible categories is the following.

(3.19) **Observation.** If $\mathcal{C}$ is accessible, it has split idempotents. That is to say, for every $i \colon C \to C$ with $i \circ i = i$, there are $s \colon D \rightarrowtail C$, $r \colon C \twoheadrightarrow D$ such that $r \circ s = \mathrm{id}_D$ and $s \circ r = i$.

Proof. By (1.10), $r \colon C \to D$ is the colimiting cocone of the (infinitely filtered) diagram in $\mathcal{C}$ with C as a single vertex and the idempotent i as the unique non-identity morphism. □

Next, we are going to give alternative characterisations of locally κ-presentable categories and conclude that every locally κ-presentable category is also locally κ'-presentable for every $\kappa' \geqslant \kappa$ regular, which is not immediately obvious from the definition (and in fact wrong for accessible categories).

(3.20) **Proposition.** A set $\mathcal{R}$ of κ-generators in an accessible category is strongly generating (more generally, a dense full subcategory is always strongly generating).

Proof. Given two parallel arrows $C \rightrightarrows C'$ in $\mathcal{C}$ such that $R \to C \rightrightarrows C'$ is a fork for every $R \in \mathcal{R}$ and every $R \to C$, we write C as a colimit of a diagram $X \colon \mathcal{I} \to \mathcal{C}$, all of whose objects lie in $\mathcal{R}$. But now $XI \to C \rightrightarrows C'$ is a fork for each $I \in \mathcal{I}$, where the $XI \to C$ are the components of the universal cocone. By the universal property of the colimit then the two parallel arrows $C \rightrightarrows C'$ agree.

This shows that $\mathcal{R}$ is generating. To see that it is strongly so (see definition (2.6.1)), let $m \colon S \rightarrowtail C$ be a monomorphism such that each $R \to C$ with $R \in \mathcal{R}$ factors (necessarily uniquely) through it and again, write C as a colimit of a diagram $X \colon \mathcal{I} \to \mathcal{C}$, all of whose objects lie in $\mathcal{R}$. Then each component $XI \to C$ of the universal cocone factors (uniquely) through S and these factorisations form a cocone below X. So there is an induced morphism $C \to S$, which, by universality, must be a retraction of m. Therefore, m is an isomorphism. □

(3.21) **Lemma.** If $\mathcal{C}$ is κ-accessible and $\mathcal{R} \subseteq \operatorname{Ob}\mathcal{C}$ a set of κ-generators, then a $C \in \mathcal{C}$ is κ-presentable iff it is a retract of some $R \in \mathcal{R}$.

Proof. The direction "$\Leftarrow$" was (2.15). For the converse, we write $C \cong \operatorname{colim} X$ as a colimit of a κ-filtered diagram $X \colon \mathcal{I} \to \mathcal{C}$ of objects in $\mathcal{R}$. Since C is κ-presentable, id_C factors through some XI, so that C is a retract of XI. □

(3.22) **Notation.** In the following proof, given some object C in a category, we write $\operatorname{Mono}(C)$ for the class of monomorphisms with codomain C, which is preordered by $s \lesssim s'$ iff there is some (necessarily unique) f such that $s = s' \circ f$. As always with a preorder, this defines an equivalence relation $\sim$ on $\operatorname{Mono}(C)$, where $s \sim s'$ iff $s \lesssim s'$ as well as $s' \lesssim s$ and the quotient $\operatorname{Sub}(C) := \operatorname{Mono}(C)/{\sim}$ is the poset of subobjects of C. Finally, we write $\operatorname{Split}(C) \subseteq \operatorname{Sub}(C)$ for the subposet of all split subobjects of C.

(3.23) **Proposition.** For a regular cardinal κ and a locally small category $\mathcal{C}$ with κ-filtered colimits, being locally κ-presentable is equivalent to requiring that

(a) every object in $\mathcal{C}$ be a κ-filtered colimit (or equivalently, by (1.14), κ-directed) of κ-presentable ones and

(b) there be only a set of isomorphism classes of κ-presentable objects.

Proof. "$\Leftarrow$": Immediate; take $\mathcal{R}$ to be a complete set of representatives for the isomorphism classes of κ-presentable objects.
"$\Rightarrow$": Point (a) by definition and for (b) the lemma tells us that the κ-presentable objects are just the retracts of elements in $\mathcal{R}$. So we need to show that in a locally small category, the isomorphism classes of retracts of a given object C form a set. To see this, we consider

$$i \colon \operatorname{Split}(C) \to \mathfrak{P}(\operatorname{End}(C)),\ [s] \mapsto \{s \circ r \mid r \text{ retraction of } s\},$$

which is clearly well-defined. Now, i is injective because given split subobjects $s\colon D \rightarrowtail C$ and $s'\colon D' \rightarrowtail C$ with $i(s) = i(s')$, we pick a retraction r of s, so that there is a retraction r' of s' with $s \circ r = s' \circ r'$. Now, we form the commutative diagram

$$\begin{tikzcd}
 & & D \arrow[d, tail, "s"] \arrow[dr, equal] \arrow[drr, bend left, tail, "s"] & & \\
C \arrow[r, two heads, "r'"] \arrow[urr, bend left, two heads, "r"] \arrow[drr, bend right, two heads, "r'"'] & D' \arrow[r, tail, "s'"] \arrow[dr, equal] & C \arrow[r, two heads, "r"] \arrow[d, two heads, "r'"] & D \arrow[r, tail, "s"] & C \\
 & & D' \arrow[urr, bend right, tail, "s'"'] & &
\end{tikzcd},$$

(we really only need the right half) proving that $s \lesssim s'$ via $r' \circ s$. Analogously, $s' \lesssim s$. $\square$

(3.24) **Corollary.** If $\mathcal{C}$ is κ-accessible then $\mathcal{C}_\kappa$ is essentially small and consequently, the set $\mathcal{R}$ of κ-generators in definition (3.1) can always be taken to be a complete set of representatives for the isomorphism classes of κ-presentable objects. $\square$

As a converse to (3.20) above, we are going show that a locally small cocomplete category is locally κ-presentable iff it has a strongly generating set of κ-presentable objects. Here, it is essential to have all colimits and the analogous claim for accessible categories is not true.

(3.25) **Theorem.** A locally small cocomplete category $\mathcal{C}$ is locally κ-presentable iff it has a strongly generating set of κ-presentable objects.

Proof. The direction "$\Rightarrow$" was (3.20). For the converse, we take a strongly generating set $\mathcal{G}$ of κ-presentable objects (which we identify with the full subcategory it defines) and denote by $\mathcal{R}$ its closure under κ-small colimits. Formally, $\mathcal{R} = \bigcup_{\alpha<\kappa} \mathcal{R}_\alpha$ where $\mathcal{R}_0 := \mathcal{G}$, $\mathcal{R}_\beta := \bigcup_{\alpha<\beta} \mathcal{R}_\alpha$ for $\beta < \kappa$ a limit ordinal and $\mathcal{R}_{\alpha+1}$ is a set of representatives for the isomorphism classes in

$$\{C \in \mathcal{C} \mid \text{there is } \mathcal{I}\ \kappa\text{-small and } X\colon \mathcal{I} \to \mathcal{R}_\alpha \subseteq \mathcal{C} \text{ with } C \cong \operatorname{colim} X\}$$

(to be even more formal, $\mathcal{I}$ above is still a class variable but we can always replace it by an isomorphic category whose objects and Hom-sets are subsets of κ). It follows from the regularity of κ that the set $\mathcal{R}$ is indeed closed under κ-small colimits (in particular, it is κ-filtered) and by (2.14), each of its objects is κ-presentable. Now, for $C \in \mathcal{C}$, we form the canonical diagram

$$X\colon \mathcal{R}\downarrow C \to \mathcal{R} \hookrightarrow \mathcal{C},$$

and observe that $\mathcal{R}\downarrow C$ is again closed under κ-small colimits (and in particular κ-filtered) because these are just calculated in $\mathcal{C}$. Taking X's colimit (L, λ), we shall show that the morphism $l\colon L \to C$ induced by the canonical cocone $\gamma\colon X \Rightarrow C$ with components $\gamma_{(R,p)} := p$ is an isomorphism. For this, it suffices to show that it is monic because $\mathcal{G}$ is strongly generating and every $p\colon G \to C$ with $G \in \mathcal{G}$ factors through $l\colon L \rightarrowtail C$ as $p = \gamma_{(G,p)} = l \circ \lambda_{(G,p)}$.

So let f, $g\colon B \to L$ such that $l \circ f = l \circ g$. Because $\mathcal{G}$ is generating, it suffices to consider the case where $B \in \mathcal{G}$. Since L is a κ-filtered colimit and B is κ-presentable, both f and g factor through some component of the universal cocone λ. That is to say, there is some $(A, q\colon A \to C) \in \mathcal{R}\downarrow C$ together with morphisms f', $g'\colon B \to A$ such that

$$f = \lambda_{(A,q)} \circ f' \qquad \text{and} \qquad g = \lambda_{(A,q)} \circ g'$$

(we can assume that f and g factor through the same component by filteredness of $\mathcal{R}\downarrow C$). Now, let $e\colon A \to E$ be the coequaliser of f' and g'. Because $\mathcal{R}$ is closed under κ-small colimits, we have $E \in \mathcal{R}$ and because

$$q \circ f' = l \circ \lambda_{(A,q)} \circ f' = l \circ f = l \circ g = l \circ \lambda_{(A,q)} \circ g' = q \circ g',$$

there is a unique $q^\flat\colon E \to C$ such that $q = q^\flat \circ e$. But now $(E, q^\flat) \in \mathcal{R}\downarrow C$ and

$$\lambda_{(A,q)} = \lambda_{(A,q^\flat \circ e)} = \lambda_{(A,q^\flat)} \circ e$$

by the cocone condition and so

$$f = \lambda_{(A,q)} \circ f' = \lambda_{(A,q^\flat)} \circ e \circ f' = \lambda_{(A,q^\flat)} \circ e \circ g' = \lambda_{(A,q)} \circ g' = g. \qquad \square$$

(3.26) **Scholium.** If $\mathcal{G}$ is a strongly generating set of κ-presentables in a locally small cocomplete category $\mathcal{C}$ then a set of κ-generators consists of all κ-small colimits (in $\mathcal{C}$) of diagrams with values in $\mathcal{G}$.

(3.27) **Example.** This scholium is noteworthy because in general, a strongly generating set of κ-presentables is not a set of κ-generators. For example, if $R \neq 0$ is any ring then R is a strong generator in R-**Mod** (which is locally finitely presentable). However, $\{R\}$ is not a set of $\aleph_0$-generators. If this were so, then there would be a surjection $r\colon R \to R^2$ by (3.21) and composing this with a standard projection $R^2 \twoheadrightarrow R$ gives a surjection $R \twoheadrightarrow R$. Since an R-linear surjection $R \twoheadrightarrow R$ is an isomorphism, this means that r has a retraction. But then r is itself an isomorphism, as is either one of the standard projections $R^2 \twoheadrightarrow R$. This is a contradiction because we assumed that $R \neq 0$.

(3.28) **Corollary.** A locally κ-presentable category $\mathcal{C}$ is also locally κ'-presentable for every regular cardinal $\kappa' \geqslant \kappa$.

Proof. By the theorem, $\mathcal{C}$ has a strongly generating set of κ-presentable objects, which are also κ'-presentable and we can apply the theorem again. $\square$

Since we know that in a locally κ-presentable category the full subcategory $\mathcal{C}_\kappa$ of κ-presentables is essentially small, the above corollary implies that in fact, every $\mathcal{C}_{\kappa'}$ is essentially small. Since every object in $\mathcal{C}$ is presentable, this has the noteworthy consequence that

$$\mathcal{C} = \operatorname*{colim}_{\kappa} \mathcal{C}_\kappa = \bigcup_{\kappa} \mathcal{C}_\kappa \qquad \text{(where } \kappa \text{ ranges over all regular cardinals)}$$

is a directed colimit (albeit no a small one) of essentially small categories.

(3.29) **Corollary.** If $\mathcal{C}$ is locally κ-presentable and $C \in \mathcal{C}$, then the comma category $\mathcal{C}\downarrow C$ is locally κ-presentable, too.

Proof. By (2.5.1), colimits in $\mathcal{C}\downarrow C$ are calculated in $\mathcal{C}$ and in particular, $\mathcal{C}\downarrow C$ is cocomplete. By this very reason, it is clear that an object $(D,p) \in \mathcal{C}\downarrow C$ with $D \in \mathcal{C}$ κ-presentable, is itself so. Furthermore, one easily verifies that if $\mathcal{G}$ is a strongly generating set for $\mathcal{C}$, then all $(G,p) \in \mathcal{C}\downarrow C$ with $G \in \mathcal{G}$ form a strongly generating set for $\mathcal{C}\downarrow C$. $\square$

(3.30) **Remark.** One can also show that the comma categories $C \downarrow \mathcal{C}$ are locally κ-presentable, too. In fact, in (7.14), we are going to show more generally that any comma category $F \downarrow G$ with F, G sufficiently nice is itself κ-accessible, which will imply the two special cases mentioned here.

(3.31) **Corollary.** Every presheaf category $\mathbf{Sets}^{\mathcal{J}^{\mathrm{op}}}$ is locally finitely presentable with a set of $\aleph_0$-generators given by all finite colimits of represented presheaves.

Proof. By (2.6.4), the representable presheaves are strongly generating and they are finitely presentable by (2.8). □

(3.32) **Corollary.** If $\mathcal{C}$ is locally κ-presentable and $\mathcal{A} \subseteq \mathcal{C}$ is a full reflective subcategory closed under κ-filtered colimits then $\mathcal{A}$ is again locally κ-presentable.

Proof. First off, $\mathcal{A}$ is cocomplete by (2.3.10). Next, writing $F\colon \mathcal{C} \to \mathcal{A}$ for the reflection functor, observe that if $C \in \mathcal{C}$ is κ-presentable, its reflection FC is κ-presentable in $\mathcal{A}$. Indeed, given a κ-filtered diagram $X\colon \mathcal{I} \to \mathcal{A}$, we observe that its colimit in $\mathcal{C}$ is contained in $\mathcal{A}$ by hypothesis and get

$$\begin{aligned}\mathcal{A}(FC, \operatorname*{colim}_I XI) = \mathcal{C}(FC, \operatorname*{colim}_I XI) &\cong \mathcal{C}(C, \operatorname*{colim}_I XI)\\ &\cong \operatorname*{colim}_I \mathcal{C}(C, XI) = \operatorname*{colim}_I \mathcal{A}(FC, XI).\end{aligned}$$

Now, given a strongly generating set $\mathcal{R}$ of κ-presentable objects in the category $\mathcal{C}$, their reflections $F\mathcal{R} := \{FR \mid R \in \mathcal{R}\}$ form a strongly generating set (by (2.6.7)) of κ-presentable objects in $\mathcal{A}$. □

Note that this last corollary does not specify a set of κ-generators for the full reflective subcategory $\mathcal{A}$ and going through the proofs, it seems like we need to take the set of all κ-small colimits of reflections FR with R in some set of κ-generators in $\mathcal{C}$. But we can do better.

(3.33) **Proposition.** If $\mathcal{C}$ is locally κ-presentable with a set of κ-generators $\mathcal{R}$ and $\mathcal{A} \subseteq \mathcal{C}$ a full reflective subcategory closed under κ-filtered colimits then $\mathcal{A}$ is again locally κ-presentable with a set of κ-generators given by $\{FR \mid R \in \mathcal{R}\}$ for $F\colon \mathcal{C} \to \mathcal{A}$ a reflector.

Proof. We have already proved the first part in the corollary. Now if $A \in \mathcal{A}$, there is some κ-filtered diagram $X\colon \mathcal{I} \to \mathcal{C}$ in $\mathcal{R}$, whose colimit is A. Since F is a left adjoint, we get that

$$A \cong FA \cong F \operatorname*{colim}_{\mathcal{I}} X \cong \operatorname*{colim}_{\mathcal{I}} F \circ X,$$

which is a κ-filtered colimit in $\mathcal{A}$. □

4. Level of Accessibility

We know that within a κ-accessible category $\mathcal{C}$, every object is presentable and every object can be written as a κ-filtered colimit of κ-presentables (which we call a *presentation* of the object in question). The question we ask now is how the level of presentability of some object is reflected in the indexing category for its presentation. We have already made the following "soft" observation in the previous section and are afterwards going to strengthen it gradually, leading to the Makkai-Paré theorem.

(4.1) **Observation.** For all regular cardinals $\kappa \leqslant \kappa'$, an object of a locally κ-presentable category $\mathcal{C}$ is κ'-presentable iff it is a retract of a κ'-small colimit of elements in $\mathcal{R}$ (in particular of κ-presentables).

Proof. The direction "$\Leftarrow$" was (2.14). As for the converse: From (3.28), we know that $\mathcal{C}$ is also locally κ'-presentable and inspecting the proof, a set of κ'-generators is given by all κ'-small colimits (taken in $\mathcal{C}$) of diagrams in $\mathcal{R}$. Now, we can apply (3.21), which tells us that every κ'-presentable object on $\mathcal{C}'$ is in fact a retract of elements in $\mathcal{R}'$. $\square$

In a first step, we are going to strengthen this observation (and in doing so, make it usable in the accessible context) by showing that, as long as κ' is "big enough" relative to κ (which, in most applications, is no real restriction), we can even assume the diagram in the last proposition to be κ-filtered. This is an important conclusion because presentations of objects are really κ-filtered diagrams rather than arbitrary ones (cf. the definition of a locally presentable category).

(4.2) **Definition.** For a cardinal κ and a set X, we write

$$\mathfrak{P}_\kappa(X) := \{S \subseteq X \mid |S| < \kappa\}$$

for the poset (ordered by inclusion) of subsets of X with cardinality $< \kappa$. Now, for κ infinite and κ' another cardinal, we write $\kappa \trianglelefteq \kappa'$ iff $\kappa \leqslant \kappa'$ and for every set X of cardinality $|X| < \kappa'$ (it suffices to consider $\kappa \leqslant |X| < \kappa'$ by remark (4.5) below), the poset $\mathfrak{P}_\kappa(X)$ has a final subset of cardinality $< \kappa'$. With this, we say that κ is *sharply smaller* than κ' and write $\kappa \triangleleft \kappa'$ iff $\kappa \trianglelefteq \kappa'$ and $\kappa < \kappa'$.

(4.3) **Remark.** Because κ is infinite, the poset $\mathfrak{P}_\kappa(X)$ is directed and so finality of a subset $\mathfrak{I} \subseteq \mathfrak{P}_\kappa(X)$ is easy to check using (1.12): $\mathfrak{I}$ is final iff every $S \subseteq X$ of cardinality $|S| < \kappa$ is contained in some $I \in \mathfrak{I}$.

Before seeing some examples of sharp inequalities, let use record the following elementary properties for the sake of completeness.

(4.4) **Proposition.** The relation $\trianglelefteq$ is reflexive and transitive (i.e. a partial order).

Proof. If κ is any infinite cardinal and X a set of cardinality $|X| < \kappa$ then $\mathfrak{P}_\kappa(X) = \mathfrak{P}(X)$, which has a top (hence final) element; namely X. As for transitivity, if $\kappa \triangleleft \lambda \triangleleft \mu$ and X is any set of cardinality $|X| < \mu$ then $\mathfrak{P}_\lambda(X)$ as a final subset $\mathfrak{I}$ of cardinality $|\mathfrak{I}| < \mu$. Now, since every $I \in \mathfrak{I}$ has cardinality $|I| < \lambda$, every $\mathfrak{P}_\kappa(I)$ with $I \in \mathfrak{I}$ has some final subset $\mathfrak{K}_I$ of cardinality $|\mathfrak{K}_I| < \lambda$. We now claim that

$$\mathfrak{L} := \bigcup_{I \in \mathfrak{I}} \mathfrak{K}_I \qquad \text{(which has cardinality } |\mathfrak{L}| \leqslant |\mathfrak{I}| \cdot \sup_{I \in \mathfrak{I}} |\mathfrak{K}_I| \leqslant |\mathfrak{I}| \cdot \lambda < \mu\text{)}$$

is final in $\mathfrak{P}_\kappa(X)$. Indeed, if $S \subseteq X$ has cardinality $|S| < \kappa < \lambda$ then there is some $I \in \mathfrak{I}$ with $S \subseteq I$ and therefore some $K \in \mathfrak{K}_I$ with $S \subseteq K$. $\square$

(4.5) **Remark.** As seen in the proof of reflexivity, in the definition of the $\trianglelefteq$-relation, it suffices to consider sets X whose cardinality satisfies $\kappa \leqslant |X| < \kappa'$ because if $|X| < \kappa$, then $\mathfrak{P}_\kappa(X) = \mathfrak{P}(X)$ has X as a top element.

In the above definition, we required κ to be infinite but the definition obviously also makes sense for finite κ. The reason why we excluded this is that, just like for the definition of filtered categories, allowing for finite cardinals just adds two degenerate cases (namely 0 and 1) as well as one trivial case for every other finite cardinal.

(4.6) **Example.** If we allow for finite cardinals, then $0 \trianglelefteq \kappa$ for every cardinal κ because $\mathfrak{P}_0(X) = \varnothing$ and $1 \trianglelefteq \kappa$ for every cardinal $\kappa \geqslant 2$ (no reflexivity!) because $\mathfrak{P}_1(X) = \{\varnothing\}$. On the other hand, if $n > 1$ is finite then $n \trianglelefteq \kappa$ iff $\kappa = n$ or κ is infinite.

For the infinite case, it suffices to consider $\kappa = \aleph_0$ by transitivity and example (4.8) below. But this is easy because if X is finite, so is $\mathfrak{P}_n(X)$ and we can take all of it as a finite final subset.

As for the case where κ is finite, too, we first note that $\mathfrak{P}_\kappa(X)$ is not directed and so remark (4.3) above does not hold anymore. Now, if $n < m$ and X is any set of cardinality $m-1$ then $\mathfrak{P}_n(X)$ has

$$\binom{m-1}{n} \text{ maximal elements and } \binom{m-1}{n-1} \text{ pairwise intersections of such.}$$

Any final subset $\mathfrak{I}$ of $\mathfrak{P}_n(X)$ necessarily contains the maximal elements but also needs to contain the pairwise intersections because any $(M \cap M') \downarrow \mathfrak{I}$ with M, $M' \in \mathfrak{P}_n(X)$ maximal must be connected. But this means that $\mathfrak{I}$ has at least $\binom{m}{n} \geqslant m$ elements.

(4.7) **Example.** We have $\kappa \triangleleft \kappa^+$ for every regular cardinal κ. To wit, if X is any set of cardinality $\kappa \leqslant |X| < \kappa^+$ (i.e. $|X| = \kappa$), we pick a bijection $\kappa \cong X$, $\alpha \mapsto x_\alpha$ and find that the collection $\mathfrak{I} \subseteq \mathfrak{P}_\kappa(X)$ of all sets of the form

$$I_\lambda := \{x_\alpha \mid \alpha < \lambda\} \subseteq X \qquad \text{with } \lambda < \kappa$$

is final in $\mathfrak{P}_\kappa(X)$. Indeed, if $S \subset X$ has $|S| < \kappa$ then the set of indices α with $x_\alpha \in S$ has a supremum $\beta < \kappa$ by regularity, implying that $S \subseteq I_{\beta+1}$. Finally, by definition, $|\mathfrak{I}| = \kappa < \kappa^+$.

(4.8) **Example.** We have $\aleph_0 \trianglelefteq \kappa$ for every infinite cardinal κ. To wit, if $\aleph_0 < \kappa$ and X is a set of cardinality $\aleph_0 \leqslant |X| < \kappa$, we write $\mathfrak{P}_{\aleph_0}(X)$ as a union

$$\mathfrak{P}_{\aleph_0}(X) = \bigcup_{n \in \mathbb{N}} \mathfrak{P}_n(X).$$

Obviously, we have a surjection $X^n \twoheadrightarrow \mathfrak{P}_n(X)$, given by $(x_1, \ldots, x_n) \mapsto \{x_1, \ldots, x_n\}$, so that $|\mathfrak{P}_n(X)| \leqslant |X^n| = |X|$ and thus

$$|\mathfrak{P}_{\aleph_0}(X)| \leqslant \aleph_0 \cdot |X| = |X| < \kappa.$$

Therefore, we can take all of $\mathfrak{P}_{\aleph_0}(X)$ as a final subset of cardinality $< \kappa$.

We can generalise the approach of this last example to arbitrary cardinals, the main problem being finding the correct hypotheses.

(4.9) **Lemma.** Given infinite cardinals $\kappa < \kappa'$ with κ' regular and such that $\lambda'^\lambda < \kappa'$ for all $\lambda < \kappa$ and $\lambda' < \kappa'$ then $\kappa \triangleleft \kappa'$.

Proof. We follow the same proof as in the last example. To wit, for X a set of cardinality $|X| < \kappa'$, we filter $\mathfrak{P}_\kappa(X)$ by

$$\mathfrak{P}_\kappa(X) = \bigcup_{\lambda < \kappa} \mathfrak{P}_\lambda(X)$$

and have an obvious surjection $X^\lambda \twoheadrightarrow \mathfrak{P}_{=\lambda}(X)$, given by $(x_\alpha)_{\alpha<\lambda} \mapsto \{x_\alpha\}_{\alpha<\lambda}$, which implies that $|\mathfrak{P}_{=\lambda}(X)| \leqslant |X|^\lambda < \kappa'$. With this

$$|\mathfrak{P}_\kappa(X)| \leqslant \kappa \cdot \sup_{\lambda<\kappa} |X|^\lambda < \kappa \cdot \kappa' = \kappa',$$

where in the last inequality, we used the regularity of κ'. □

(4.10) **Proposition.** If $\kappa \leqslant \kappa'$ are infinite cardinals then $\kappa \triangleleft (2^{\kappa'})^+$.

Proof. Taking $\lambda < \kappa$ and $\lambda' < (2^{\kappa'})^+$, we have

$$\lambda'^{\lambda} \leqslant 2^{\kappa' \cdot \lambda} = 2^{\kappa'} < (2^{\kappa'})^+$$

and we can apply the lemma since any successor cardinal is regular. $\square$

The reason we proved this proposition is the following corollary to it, which is routinely used in applications of the theory of locally presentable or accessible categories.

(4.11) **Corollary.** For any infinite cardinal κ, there are arbitrarily large regular cardinals κ' such that $\kappa \trianglelefteq \kappa'$. $\square$

After having seen examples of sharp inequalities between cardinals and having proved that we can always pick arbitrarily large sharply larger cardinals, let's move on to their main purposes, which are the Makkai-Paré theorem for locally presentable categories and the result (4.14) that if $\kappa \trianglelefteq \kappa'$, then every κ-accessible category is also κ'-accessible.

(4.12) **Lemma.** If $\kappa \triangleleft \kappa'$ are regular cardinals and $\mathcal{I}$ a κ-filtered category, then every κ'-small subcategory of $\mathcal{I}$ is contained in a κ'-small subcategory that is also κ-filtered.

Proof. Starting with a κ'-small subcategory $\mathcal{J}_0 \subseteq \mathcal{I}$, we construct an increasing transfinite sequence of κ'-small subcategories $(\mathcal{J}_\alpha)_{\alpha<\kappa}$, whose union $\mathcal{J}_\kappa := \bigcup_{\alpha<\kappa} \mathcal{J}_\alpha$ is going to be κ-filtered. By transfiniteness of the sequence, we mean that $\mathcal{J}_\beta = \bigcup_{\alpha<\beta} \mathcal{J}_\alpha$ for every limit ordinal $\beta < \alpha$ (which is κ'-small by regularity of κ') and we only need to treat the successor case.

So suppose we have constructed the κ'-small category $\mathcal{J}_\alpha$. By hypothesis, $\mathfrak{P}_\kappa(\operatorname{Arr} \mathcal{J}_\alpha)$ has a final subset $\mathfrak{J}_\alpha$ of cardinality $|\mathfrak{J}_\alpha| < \kappa'$ and replacing every $J \in \mathfrak{J}_\alpha$ (which is just a collection of arrows in $\mathfrak{J}_\alpha$) by the subcategory it generates, we can assume that the elements of $\mathfrak{J}_\alpha$ are subcategories.

Now, because $\mathcal{I}$ is κ-filtered, we can choose a cocone $\gamma_{\mathcal{J}} \colon \mathrm{In}_{\mathcal{J}} \Rightarrow I_{\mathcal{J}}$ in $\mathcal{I}$ below every $\mathcal{J} \in \mathfrak{J}_\alpha$ (where $\mathrm{In}_{\mathcal{J}} \colon \mathcal{J} \hookrightarrow \mathcal{I}$ is the inclusion) and we define $\mathcal{J}_{\alpha+1} \subseteq \mathcal{I}$ to be the subcategory generated by

$$\operatorname{Arr} \mathcal{J}_\alpha \cup \bigcup_{J \in \mathfrak{J}_\alpha} \{\gamma_{\mathcal{J},J} \colon J \to I_{\mathcal{J}} \mid J \in \mathcal{J}\},$$

which is a set of cardinality $\leqslant \max\{|\operatorname{Arr} \mathcal{J}_\alpha|, |\mathfrak{J}_\alpha|, \kappa\} < \kappa'$, so that $\mathcal{J}_{\alpha+1}$ is indeed κ'-small.

Finally, $\mathcal{J}_\kappa := \bigcup_{\alpha<\kappa} \mathcal{J}_\alpha$ is κ'-small by regularity of κ' and to see that it is κ-filtered, let $X \colon \mathcal{K} \to \mathcal{J}_\kappa$ be a κ-small diagram. Since $\mathcal{K}$ is κ-small and κ is regular, the set of all indices α such that $X_k \in \operatorname{Arr} \mathcal{J}_\alpha$ for some $k \in \operatorname{Arr} \mathcal{K}$ has a supremum β, meaning that the diagram X factors through $\mathcal{J}_\beta$. By finality, there is some $\mathcal{J} \in \mathfrak{J}_\beta$ such that $X(\operatorname{Arr} \mathcal{K}) \subseteq \mathcal{J}$, meaning that X even factors through $\mathcal{J}$. But now, the $\left(\gamma_{\mathcal{J},X_K} \colon X_K \to I_{\mathcal{J}}\right)_{K \in \mathcal{K}}$ form a cocone in $\mathcal{J}_{\beta+1} \subseteq \mathcal{J}_\kappa$ below X $\square$

(4.13) **Proposition.** If $\kappa \triangleleft \kappa'$ are regular cardinals and $\mathcal{I}$ a κ-filtered category, then the poset

$$P := \{\mathcal{J} \subseteq \mathcal{I} \mid \mathcal{J} \text{ is a } \kappa'\text{-small } \kappa\text{-filtered subcategory}\}$$

(ordered by inclusion) is κ'-directed. Moreover, $\mathcal{I} = \bigcup_{\mathcal{J} \in P} \mathcal{J} = \operatorname{colim}_{\mathcal{J} \in P} \mathcal{J}$ (where the colimit is taken in **Cat**). In particular, if $\mathcal{C}$ is any category and $X \colon \mathcal{I} \to \mathcal{C}$, we have an isomorphism

$$\operatorname*{colim}_{\mathcal{J} \in P} \operatorname*{colim}_{\mathcal{J}} X|_{\mathcal{J}} \xrightarrow{\cong} \operatorname*{colim}_{\mathcal{I}} X$$

(assuming these colimits exist) induced by the inclusions $\mathcal{J} \hookrightarrow \mathcal{I}$ (or rather by the cocone consisting of all $\operatorname{colim}_{\mathcal{J}} X|_{\mathcal{J}} \to \operatorname{colim}_{\mathcal{I}} X$ induced by inclusions).

Proof. Given a family $(\mathcal{J}_\alpha)_{\alpha<\kappa}$ in P, the union $\bigcup_{\alpha<\kappa} \operatorname{Arr} \mathcal{J}_\alpha$ is of cardinality $< \kappa'$, so that the category it generates is κ'-small and therefore, by the lemma, contained in one belonging to P. As for the second part of the proposition, any morphism in $\mathcal{I}$ can be viewed as a category, which then (again by the lemma) belongs to some $\mathcal{J} \in P$. The union of all categories in P is the colimit in **Cat** because P is directed and the final claim is just Thomason's theorem. $\square$

With this proposition, we have finally arrived at our first improvement over the initial "soft" observation. Given $\kappa \trianglelefteq \kappa'$, not only does this corollary show that every κ-accessible category is also κ'-accessible; it even gives a set of κ'-generators.

(4.14) **Corollary.** If $\kappa \triangleleft \kappa'$ are regular cardinals and $\mathcal{C}$ a κ-accessible category with a set of κ-generators $\mathcal{R}$ then $\mathcal{C}$ is also κ'-accessible with a set of κ'-generators $\mathcal{R}'$ given by all κ'-small κ-filtered colimits (taken in $\mathcal{C}$) of diagrams in $\mathcal{R}$.

Proof. First off, $\mathcal{R}'$ is indeed a set because every κ'-small category can be chosen to have objects and Hom-sets in κ', so that there are at most $2^{\kappa'}$-many. Now, if $C \in \mathcal{C}$ is any object, we find a κ-filtered diagram $X \colon \mathcal{I} \to \mathcal{R}$ whose colimit (in $\mathcal{C}$) is C. We now construct the κ'-directed poset P as in the proposition and get that

$$C \cong \operatorname*{colim}_{\mathcal{I}} X \cong \operatorname*{colim}_{\mathcal{J}\in P} \operatorname*{colim}_{\mathcal{J}} X|_{\mathcal{J}}.$$

But every $\mathcal{J} \in P$ is κ'-small and κ-filtered, so that every $\operatorname{colim}_{\mathcal{J}} X|_{\mathcal{J}}$ is in $\mathcal{R}'$. $\square$

As we are going to show below, the set $\mathcal{R}'$ from the corollary is actually the set of all κ'-presentables. In fact, this is exactly the Makkai-Paré theorem. First, let us record the following improved "soft" observation.

(4.15) **Observation.** If $\kappa \trianglelefteq \kappa'$ are regular cardinals and $\mathcal{C}$ a locally κ-presentable category with a set of κ-generators $\mathcal{R}$ then an object in $\mathcal{C}$ is κ'-presentable iff it is a retract of a κ'-small κ-filtered colimit (taken in $\mathcal{C}$) of a diagram in $\mathcal{R}$.

Proof. The proof is the same as in (4.1) but using the set of κ'-generators $\mathcal{R}'$ from the corollary. $\square$

Now, finally, let's come to our main goal for this section, which says that in this observation, we can actually forget about the retracts as they add nothing new. This makes sense intuitively, since, as already observed in (1.10), retracts of an object C are κ-filtered finite limits (with κ arbitrary) of diagrams with C as a single vertex and a single idempotent non-identity morphism. The main technicality now is to mix such a diagram with a presentation for C.

(4.16) **Theorem. (Makkai-Paré)** If $\kappa \trianglelefteq \kappa'$ are regular cardinals and $\mathcal{C}$ a locally κ-presentable category with a set of κ-generators $\mathcal{R}$ then an object in $\mathcal{C}$ is κ'-presentable iff it is a κ'-small κ-filtered colimit (taken in $\mathcal{C}$) of a diagram in $\mathcal{R}$

Proof. If $\kappa = \kappa'$, the claim is trivial since κ-presentables are retracts of objects in $\mathcal{R}$, which can be expressed as κ-filtered finite colimits of diagrams in $\mathcal{R}$. So let's assume $\kappa \triangleleft \kappa'$.

By the last observation, it suffices to check that any retract of a κ'-small κ-filtered colimit is again one; so let $X \colon \mathcal{I} \to \mathcal{C}$ be a κ'-small κ-filtered diagram of κ-presentables with colimiting cocone $\lambda \colon X \Rightarrow \operatorname{colim}_{\mathcal{I}} X =: A$ and $s \colon B \to A$ a morphism having a retraction $r \colon A \to B$.

Out of $\mathcal{I}$, we now construct two new categories $\mathcal{J}$ and $\mathcal{K}$, both of which are going to have the same objects as $\mathcal{I}$; i.e. $\operatorname{Ob}\mathcal{I} = \operatorname{Ob}\mathcal{J} = \operatorname{Ob}\mathcal{K}$. A morphism $I \to I'$ in $\mathcal{J}$ is a morphism $f\colon X_I \to X_{I'}$ in $\mathcal{C}$ such that

$$\begin{tikzcd} X_I \arrow[rr, "f"] \arrow[d, "\lambda_I"'] & & X_{I'} \arrow[d, "\lambda_{I'}"] \\ A \arrow[dr, "r"'] & & A \arrow[dl, "r"] \\ & B & \end{tikzcd}$$

commutes (with composition of morphisms done in $\mathcal{C}$). In particular, every Xi with $i\colon I \to I'$ in $\mathcal{I}$ belongs to $\mathcal{J}(I, I')$. We have an obvious diagram $Y\colon \mathcal{J} \to \mathcal{C}$ given on objects by $Y_I := X_I$ and which sends a morphism $f\colon I \to I'$ to $f\colon X_I \to X_{I'}$. Because every X_I is κ-presentable and $A = \operatorname{colim}_{\mathcal{I}} X$ is a κ-filtered colimit, we can pick, for every $I \in \mathcal{I}$, an object $FI \in \mathcal{I}$ together with a morphism $f_I\colon X_I \to X_{FI}$ in $\mathcal{C}$ such that

$$\begin{tikzcd} X_I \arrow[rr, "f_I"] \arrow[d, "\lambda_I"'] & & X_{FI} \arrow[d, "\lambda_{FI}"] \\ A \arrow[r, "r"'] & B \arrow[r, "s"'] & A \end{tikzcd}$$

commutes. In particular, f_I is a morphism $I \to FI$ in $\mathcal{J}$ because $r \circ s = \mathrm{id}_B$. With this, we define $\mathcal{K}$ to be the subcategory of $\mathcal{J}$ generated by all Xi and all f_I and claim that $Y|_{\mathcal{K}}\colon \mathcal{K} \to \mathcal{C}$ is a κ'-small κ-filtered diagram with colimit B. Clearly, $\mathcal{K}$ is κ'-small because its morphisms are represented by words in the Xi and f_I (though two words might describe the same morphism), meaning that

$$|\operatorname{Arr}\mathcal{K}| \leqslant \aleph_0 \cdot |\operatorname{Arr}\mathcal{I}| \cdot |\operatorname{Ob}\mathcal{I}| < \kappa'.$$

For later usage, let's say that a morphism is an *f-morphism* iff it is representable by a word containing at least one f_I. As for the colimit, let's show that the

$$Y_I = X_I \xrightarrow{\lambda_I} A \xrightarrow{r} B \qquad \text{with } I \in \mathcal{I} \text{ form a colimiting cocone below } Y|_{\mathcal{K}}.$$

This is a cocone by definition of $\mathcal{J}$. For the universal property, we first note that its components are jointly epi because r is epi and the λ_I form a colimiting cocone. Now, if

$$\big(\gamma_I\colon Y_I = X_I \to C\big)_{I\in\mathcal{I}} \qquad \text{is another cocone below } Y|_{\mathcal{K}},$$

it suffices to show that it factors through the $r \circ \lambda_I$ with uniqueness being implied by their being jointly epi. Since every $Xi\colon X_I \to X_{I'}$ with $i\colon I \to I'$ in $\mathcal{I}$ belongs to $\mathcal{K}$, the γ_I also form a cocone below X; whence there is a unique morphism $g\colon A \to C$ such that $\gamma_I = g \circ \lambda_I$. We now claim that we also have $\gamma_I = g \circ s \circ r \circ \lambda_I$, for which we use the f_I:

$$g \circ s \circ r \circ \lambda_I = g \circ \lambda_{FI} \circ f_I = \gamma_{FI} \circ f_I = \gamma_I,$$

where for the last equality, we used the cocone condition on γ. Finally, we need to check κ-filteredness of $\mathcal{K}$, which is going to take up the bulk part of this proof.

Given a family $(I_\alpha)_\alpha$ of fewer than κ objects in $\mathcal{K}$ (i.e. in $\mathcal{I}$), then, by κ-filteredness of $\mathcal{I}$, we find an object I together with a family of arrows $(i_\alpha\colon I_\alpha \to I)_\alpha$ in $\mathcal{I}$, which gives us the family $(Xi_\alpha\colon I_\alpha \to I)_\alpha$ in $\mathcal{K}$.

The case of a family of parallel morphisms is much more complicated. The idea is to show that every f-morphism (those representable by a word containing at least one f_I)

$I \to I'$ in $\mathcal{K}$ can be composed with a suitable $Xi'\colon I' \to I''$ to end up being of the form $Xi \circ f_I\colon I \to FI \to I''$ for some $i\colon FI \to I''$ in $\mathcal{I}$. Since every morphism in $\mathcal{K}$ is represented by a word in the Xi and f_I, we first show that by post-composing with a suitable Xi'

(a) every f_I can be moved to the back of the word (becoming the first arrow) and

(b) every $f_{FI} \circ f_I$ can be replaced by f_I alone.

This then allows for

(c) an inductive argument to show that, after post-composing with a suitable Xi', every f-morphism is of the form $Xi \circ f_I$, and, finally,

(d) the use of the κ-filteredness of $\mathcal{I}$.

Ad (a): Suppose we are given $i\colon I \to I'$ in $\mathcal{I}$, and let's consider the arrow $f_{I'} \circ Xi$. We wish to obtain two arrows $i'\colon FI \to I''$ and $i''\colon FI' \to I''$ such that

$$Xi'' \circ f_{I'} \circ Xi = Xi' \circ f_I \qquad \text{i.e.} \qquad \begin{tikzcd} X_I \arrow[r, "f_I"] \arrow[d, "Xi"'] & X_{FI} \arrow[dr, "Xi'"] & \\ & & X_{I''} \\ X_{I'} \arrow[r, "f_{I'}"'] & X_{FI'} \arrow[ur, "Xi''"'] & \end{tikzcd} \qquad \text{commutes.}$$

For this, we observe that the arrow

$$X_I \xrightarrow{Xi} X_{I'} \xrightarrow{\lambda_{I'}} A \xrightarrow{r} B \xrightarrow{s} A$$

factors through two different components of the colimiting cocone; namely $\lambda_{FI'}$ and λ_{FI}:

$$\lambda_{FI'} \circ f_{I'} \circ Xi \overset{(1)}{=} s \circ r \circ \lambda_{I'} \circ Xi \overset{(2)}{=} s \circ r \circ \lambda_I \overset{(3)}{=} \lambda_{FI} \circ f_I,$$

where (1) and (3) are, respectively, by definition of $f_{I'}$ and f_I, while (2) is just the cocone property of λ. Because A is a κ-filtered colimit and X_I is κ-presentable, it follows that there is a pair of arrows i' and i'' as required above. Note that this is the first time where we really need the injectivity condition from the definition of a κ-presentable object.

Ad (b): Similarly, every $X_I \xrightarrow{\lambda_I} A \xrightarrow{r} B \xrightarrow{s} A$ factors through two different components of the colimiting cocone; namely λ_{FI} and λ_{F^2I}:

$$\begin{tikzcd} X_I \arrow[rr, "f_I"] \arrow[d, "\lambda_I"'] & & X_{FI} \arrow[rr, "f_{FI}"] \arrow[d, "\lambda_{FI}"'] & & X_{F^2I} \arrow[d, "\lambda_{F^2I}"'] \\ A \arrow[r, two heads, "r"'] & B \arrow[r, tail, "s"'] \arrow[rr, bend right, "\mathrm{id}_B"'] & A \arrow[r, two heads, "r"'] & B \arrow[r, tail, "s"'] & A\,. \end{tikzcd}$$

So (again by the injectivity condition in the definition of the κ-presentability of X_I), there are two arrows $i\colon FI \to I'$ and $i'\colon F^2I \to I'$ in $\mathcal{I}$ such that

$$Xi' \circ f_{FI} \circ f_I = Xi \circ f_I \qquad \text{i.e.} \qquad \begin{tikzcd} X_I \arrow[r, "f_I"] \arrow[d, "f_I"'] & X_{FI} \arrow[dr, "Xi"] & \\ & & X_{I'} \\ X_{FI} \arrow[r, "f_{FI}"'] & X_{F^2I} \arrow[ur, "Xi'"'] & \end{tikzcd} \qquad \text{commutes.}$$

Ad (c): Given an f-morphism $k\colon I \to I'$ (i.e. one representable by a word containing at least one f_I), we wish to find some $I'' \in \mathcal{I}$ together with $i\colon I \to I''$ and $i'\colon I' \to I''$ in $\mathcal{I}$ such that

$$\begin{tikzcd} I \arrow[r, "k"] \arrow[d, "f_I"'] & I' \arrow[d, "Xi'"] \\ FI \arrow[r, "Xi"'] & I'' \end{tikzcd}$$

commutes. For this, we represent k by a word w in the Xi and the f_I. Replacing two adjacent Xi, Xi' by $X(i \circ i')$ and inserting $X\mathrm{id}_I$ where necessary, we can assume that w is a word of alternating Xi and f_I and that starts and ends with an Xi (not necessarily the same one). We now continue by induction on the number of f_I in w. If w contains only a single f_I, it is of the form

$$I \xrightarrow{k} I' \quad = \quad I \xrightarrow{Xi_1} I_1 \xrightarrow{f_{I_1}} FI_1 \xrightarrow{Xi_2} I'$$

and claim is just point (a) above. So assume w contains two or more f_I; meaning that k is of the form

$$I \xrightarrow{k} I' \quad = \quad I \xrightarrow{k'} I_1 \xrightarrow{f_{I_1}} FI_1 \xrightarrow{Xi_1} I_2 \xrightarrow{f_{I_2}} FI_2 \xrightarrow{Xi_2} I'$$

where k' is represented by the final part w' of w (which has two f_I fewer than w). We now complete this string of arrows to a commutative diagram (where all unnamed arrows are of the form Xi for some suitable i in $\mathcal{I}$):

$$\begin{tikzcd} I \arrow[r, "k'"] & I_1 \arrow[r, "f_{I_1}"] \arrow[dd, "f_{I_1}"'] \arrow[ddr, phantom, "(3)"] & FI_1 \arrow[r, "Xi_1"] \arrow[d, "f_{FI_1}"'] \arrow[dr, phantom, "(1)"] & I_2 \arrow[r, "f_{I_2}"] & FI_2 \arrow[r, "Xi_2"] \arrow[d] \arrow[dr, phantom, "(2)"] & I' \arrow[d] \\ & & F^2 I_1 \arrow[rr] \arrow[d] \arrow[drr, phantom, "(4)"] & & I_3 \arrow[r] & I_4 \arrow[d] \\ & FI_1 \arrow[r] & I_5 \arrow[rrr] & & & I_6 \end{tikzcd}.$$

Squares (1) and (3) are by point (b), while squares (2) and (4) are by $\mathcal{I}$ being filtered. So, all in all, we get that

$$I \xrightarrow{k} I' \to I_4 \to I_6 \quad = \quad I \xrightarrow{k'} I_1 \xrightarrow{f_{I_1}} FI_1 \to I_5 \to I_6,$$

where the composite on the right is represented by a word with one fewer f_I.

Ad (d): Let $(k_\alpha\colon I \to I')_\alpha$ be a family of $< \kappa$ parallel arrows in $\mathcal{K}$. Without loss of generality, we can assume that they are f-morphisms. Otherwise, we just compose them with $f_{I'}$. For every α, we use (c) to pick some $I''_\alpha \in \mathcal{I}$ together with $i_\alpha\colon FI \to I''_\alpha$ and $i'_\alpha\colon I' \to I''_\alpha$ such that

$$\begin{tikzcd} I \arrow[r, "k_\alpha"] \arrow[d, "f_I"'] & I' \arrow[d, "Xi'_\alpha"] \\ FI \arrow[r, "Xi_\alpha"'] & I''_\alpha \end{tikzcd}$$

commutes in $\mathcal{K}$. By κ-filteredness of $\mathcal{I}$, we find $I'' \in \mathcal{I}$ together with arrows

$$(j_\alpha\colon I''_\alpha \to I'')_\alpha, \qquad j\colon FI \to I'', \qquad j'\colon I' \to I''$$

in $\mathcal{I}$ such that all diagrams

$$\begin{tikzcd} FI \arrow[r, "i_\alpha"] \arrow[dr, "j"'] & I''_\alpha \arrow[d, "j_\alpha"] & I' \arrow[l, "i'_\alpha"'] \arrow[dl, "j'"] \\ & I'' & \end{tikzcd}$$

in $\mathcal{I}$ commute. With this,

$$Xj' \circ k_\alpha = Xj_\alpha \circ Xi'_\alpha \circ Xk_\alpha = Xj_\alpha \circ Xi_\alpha \circ f_I = Xj \circ f_I,$$

which is independent of α. □

After having gone to considerable lengths to prove this theorem, an obvious next question would be if every morphism between κ'-presentable objects lifts to a natural transformation between κ-filtered κ'-small diagrams of κ-presentables. This is indeed the case we will show later on.

5. Localisations of Presheaf Categories

For every small category $\mathcal{I}$, we have a Yoneda embedding $\mathcal{I} \to \mathbf{Sets}^{\mathcal{I}^{\mathrm{op}}}$, which can be seen as the free cocompletion of $\mathcal{I}$. For a large category $\mathcal{C}$, there is no such thing because formally, a functor $\mathcal{C} \to \mathbf{Sets}$ is a proper class and so we cannot assemble them into a new class. Of course, if one is willing to accept sufficiently strong large cardinal axioms, one just switches up one universe and even if one is not willing to, sometimes, we can get away with using metacategories.

In this section, we are going to see how this can be overcome, at least for locally presentable categories and in doing so, show that every locally presentable category can be obtained as a full reflective subcategory (a.k.a. a reflective localisation) of a presheaf category closed under sufficiently filtered colimits. This then implies its completeness and well-bipoweredness.

(5.1) **Proposition.** Let $\mathcal{C}$ be a category and $\mathcal{A} \subseteq \mathcal{C}$ a small full subcategory. Then the Yoneda functor $Y\colon \mathcal{C} \to \mathbf{Sets}^{\mathcal{A}^{\mathrm{op}}} =: \widehat{\mathcal{A}}$, $C \mapsto \mathcal{C}(-, C)$

(a) is fully faithful iff $\mathcal{A} \subseteq \mathcal{C}$ is dense (see (3.12));

(b) preserves κ-filtered colimits iff every $A \in \mathcal{A}$ is κ-presentable in $\mathcal{C}$.

Proof. Ad (a): Given $C, C' \in \mathcal{C}$, we observe that morphisms between YC and YC' correspond to cocones under the canonical diagram $\mathcal{A} \downarrow C \to \mathcal{C}$ with vertex C'. Indeed, a natural transformation $\tau\colon YC \Rightarrow YC'$ has, for each $A \in \mathcal{A}$, a component

$$\tau_A\colon \mathcal{C}(A, C) \to \mathcal{C}(A, C') \qquad \text{such that} \qquad \tau_A(f \circ a) = \tau_B(f) \circ a$$

for every $a\colon A \to B$ in $\mathcal{A}$ and $f\colon B \to C$ in $\mathcal{C}$. On the other hand, a cocone under $\mathcal{A} \downarrow C \to \mathcal{C}$ with vertex C' is a family of arrows $\big(\gamma_{(A,f)}\colon A \to C'\big)_{(A,f)\in\mathcal{A}\downarrow C}$ such that every commutative triangle on the left (with A, $B \in \mathcal{A}$) gives us a commutative triangle on the right

$$\begin{tikzcd} A \arrow[rr, "a"] \arrow[dr, "f\circ a"'] & & B \arrow[dl, "f"] \\ & C & \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} A \arrow[rr, "a"] \arrow[dr, "\gamma_{(A,f\circ a)}"'] & & B \arrow[dl, "\gamma_{(B,f)}"] \\ & C' & \end{tikzcd} .$$

So all in all, we get a bijection (where we identify an object C' of $\mathcal{C}$ with the corresponding constant factor $\mathcal{A} \downarrow C \to \mathcal{C}$)

$$\mathrm{Nat}(YC, YC') \cong \mathrm{Cocone}(\mathcal{A} \downarrow C \to \mathcal{C}, C') = \mathrm{Nat}(\mathcal{A} \downarrow C \to \mathcal{C}, C'),$$

given by

$$\tau \mapsto (\tau_A f)_{(A,f)\in \mathcal{A}\downarrow C} \qquad \text{and} \qquad \Big(\gamma_{(A,-)}\Big)_{A\in\mathcal{A}} \hookleftarrow \gamma.$$

Next, consider the commutative diagram

$$\begin{tikzcd} \mathcal{C}(C,C') \arrow[rr] \arrow[d, hook] \arrow[drr] & & \mathrm{Nat}(YC,YC') \arrow[d, "\cong"] \\ \mathrm{Nat}(C,C') \arrow[r] & \mathrm{Cocone}(\mathcal{A}\downarrow C \to \mathcal{C}, C'), & \end{tikzcd}$$

where the bottom arrow is induced by the canonical natural transformation $\mathrm{Can}_C^{\mathcal{A}} \Rightarrow C$. Now, for Y to be fully faithful means that for all C' the top arrow is a bijection and for $\mathcal{A}$ to be dense means that $C \cong \mathrm{colim}(\mathcal{A} \downarrow C \to \mathcal{C})$, which, by definition of a colimit, is to say that for all C', the diagonal arrow is a bijection.

Ad (b): Since colimits in $\mathbf{Sets}^{\mathcal{A}^{\mathrm{op}}}$ are objectwise, Y preserves κ-filtered colimits iff every

$$\mathrm{Ev}_A \circ Y = \mathcal{C}(A,-)\colon \mathcal{C} \to \mathbf{Sets},\ C \mapsto \mathcal{C}(A,C)$$

with $A \in \mathcal{A}$ does so. □

For the next corollary, recall that if $\mathcal{C}$ is a κ-accessible category with a set of κ-generators $\mathcal{R}$ (which we identify with the full subcategory it generates), then $\mathcal{R}$ lies dense in $\mathcal{C}$, as was shown in (3.10). Usually, we can take $\mathcal{R} = \mathrm{Sk}(\mathcal{C}_\kappa)$ to just be a skeleton of $\mathcal{C}_\kappa$, the full subcategory of κ-presentable objects.

(5.2) **Corollary.** If $\mathcal{C}$ is κ-accessible and $\mathcal{R}$ a set of κ-generators, then $\mathcal{C}$ is a full subcategory of $\mathbf{Sets}^{\mathcal{R}^{\mathrm{op}}}$ via the Yoneda embedding

$$\mathcal{C} \hookrightarrow \mathbf{Sets}^{\mathcal{R}^{\mathrm{op}}},\ C \mapsto \mathcal{C}(-,C)$$

and is closed under κ-filtered colimits (up to repletion). □

(5.3) **Proposition.** If $\mathcal{C}$ is a cocomplete category and $\mathcal{A} \subseteq \mathcal{C}$ a small full subcategory then the Yoneda functor $Y\colon \mathcal{C} \to \mathbf{Sets}^{\mathcal{A}^{\mathrm{op}}}$ is right adjoint to

$$F\colon \mathbf{Sets}^{\mathcal{A}^{\mathrm{op}}} \to \mathcal{C},\ X \mapsto \mathrm{colim}\left(\int_{\mathcal{A}} X \xrightarrow{Q} \mathcal{A} \hookrightarrow \mathcal{C}\right),$$

where Q is the projection functor.

Proof. The functor F is just the Yoneda extension of $\mathcal{A} \hookrightarrow \mathcal{C}$. □

(5.4) **Corollary.** Every locally κ-presentable category $\mathcal{C}$ with a set of κ-generators $\mathcal{R}$ is a full reflective subcategory of $\mathbf{Sets}^{\mathcal{R}^{\mathrm{op}}}$ (via the Yoneda embedding) closed under κ-filtered colimits. In particular, it is bicomplete and well-powered.

Proof. This is a combination of the last two propositions and the fact that $\mathcal{R}$ lies dense in $\mathcal{C}$. The bicompleteness then transfers from $\mathbf{Sets}^{\mathcal{R}^{\mathrm{op}}}$ and the well-poweredness is by (2.6.9). □

(5.5) **Corollary.** A category is locally κ-presentable iff it is (equivalent to) a full reflective subcategory closed under κ-filtered colimits of some presheaf category.

Proof. We just showed "$\Rightarrow$" and the converse is by (3.31), (3.28) and (3.32). $\square$

Note that with this corollary, we have found a characterisation of locally κ-presentable categories, whereas in the κ-accessible case (5.2), we only have a necessary condition. Strengthening this condition in such a way that it also becomes sufficient (and thus gives us a similar characterisation of κ-accessible categories) will be the main goal of section 8 below.

(5.6) **Corollary.** A category $\mathcal{I}$ is κ-filtered iff for every locally κ-presentable category, κ-small limits in it commute with $\mathcal{I}$-colimits (i.e. $\operatorname{colim}_{\mathcal{I}} \colon \mathcal{C}^{\mathcal{I}} \to \mathcal{C}$ preserves κ-small limits).

Proof. For the direction "$\Leftarrow$", it suffices that κ-small limits in **Sets** (which is locally finitely presentable) commute with $\mathcal{I}$-colimits, which was shown in (2.13). Conversely, if $\mathcal{C}$ is any locally κ-presentable category then limits (in particular κ-small ones) and κ-filtered colimits are calculated in $\mathbf{Sets}^{\mathrm{Sk}(\mathcal{C}_\kappa)^{\mathrm{op}}}$. There in turn, they are calculated pointwise and the claim follows since we already know it for **Sets**. $\square$

6. Diagram Categories

We already showed in (3.31) that presheaf categories are locally finitely presentable. We are now going to see more generally that if $\mathcal{C}$ is locally κ-presentable and $\mathcal{I}$ any small category then $\mathcal{C}^{\mathcal{I}}$ is again locally κ-presentable. In fact, by reflectively embedding $\mathcal{C}$ into a presheaf category, this becomes trivial.

(6.1) **Proposition.** If $\mathcal{C}$ is a locally κ-presentable category and $\mathcal{I}$ small then $\mathcal{C}^{\mathcal{I}}$ is again locally κ-presentable.

Proof. The meta-2-functor $-^{\mathcal{I}} \colon \mathbf{CAT} \to \mathbf{CAT}$ preserves full reflective subcategories and maps presheaf categories to presheaf categories. $\square$

In order to get an analogue of the Makkai-Paré-theorem for morphisms, we are also going to study how presentability in such a diagram category relates to pointwise presentability. In general, these are not equivalent as is apparent from example (2.11). However, for small enough indexing categories (most notably for $\mathcal{I} = [1]$), they are.

(6.2) **Nomenclature.** For convenience, given an infinite cardinal κ, let us call a diagram $X \colon \mathcal{I} \to \mathcal{C}$ *pointwise κ-presentable* iff every X_I with $I \in \mathcal{I}$ is κ-presentable.

Obviously, if we want to analyse how presentability in a diagram category relates to pointwise presentability, evaluation functors are going to play some role. Recall that if $\mathcal{C}$ is a category, $\mathcal{I}$ a (small) index category and $I \in \mathcal{I}$ then the evaluation functor

$$\mathrm{ev}_I = I^* \colon \mathcal{C}^{\mathcal{I}} \to \mathcal{C},\ X \mapsto X_I \qquad \text{is just precomposition with } \mathrm{In}_I \colon \{I\} \hookrightarrow \mathcal{I}$$

and so has a right adjoint given by right Kan extension (assuming these exist), which turns out to be

$$I_* \colon \mathcal{C} \to \mathcal{C}^{\mathcal{I}} \qquad \text{with} \qquad I_* C = C^{\mathcal{I}(-,I)}.$$

To wit, by the limit formula, the right Kan extension is $I_*C = \lim(- \downarrow \mathrm{In}_I \to \{C\} \hookrightarrow \mathcal{C})$, so that $(I_*C)I' = C^{\pi_0(I' \downarrow \mathrm{In}_I)}$ is the limit of a constant diagram for $I' \in \mathcal{I}$. But $I' \downarrow \mathrm{In}_I$ is a discrete category with objects $\mathcal{I}(I', I)$ and the above formula follows.

(6.3) **Lemma.** Let κ be a regular cardinal. If $\mathcal{C}$ is a locally κ-presentable category, $\mathcal{I}$ a κ-small indexing category and $I \in \mathcal{I}$, then I_* preserves κ-filtered colimits.

Proof. Recall that colimits in $\mathcal{C}^{\mathcal{I}}$ are calculated pointwise. So let $X\colon \mathcal{J} \to \mathcal{C}$ be a κ-filtered diagram and $I' \in \mathcal{I}$. Then

$$\left(I_* \operatorname{colim}_{J\in\mathcal{J}} X_J\right)_{I'} \cong \left(\operatorname{colim}_{J\in\mathcal{J}} X_J\right)^{\mathcal{I}(I',I)} \overset{(*)}{\cong} \operatorname{colim}_{J\in\mathcal{J}} X_J^{\mathcal{I}(I',I)} \cong \operatorname*{colim}_{J\in\mathcal{J}} I_* X_J,$$

where, for the isomorphism $(*)$, we used that κ-small limits in $\mathcal{C}$ commute with κ-filtered colimits. $\square$

(6.4) **Proposition.** Let κ be a regular cardinal, $\mathcal{C}$ a category with κ-filtered colimits and $\mathcal{I}$ a κ-small indexing category.

(a) If a diagram $X \in \mathcal{C}^{\mathcal{I}}$ is pointwise κ-presentable, then it is κ-presentable.

(b) If $\mathcal{C}$ is even locally κ-presentable, then the converse holds, too.

Proof. Ad (a): By the Nat-formula, we have

$$\operatorname{Nat}(X,Y) \cong \int_{I\in\mathcal{I}} \mathcal{C}(X_I, Y_I) \qquad \text{for } X,\, Y\colon \mathcal{I} \to \mathcal{C}.$$

Using the twisted arrow category (2.2.6), this end can be calculated as a κ-small limit. Now, let $X\colon \mathcal{I} \to \mathcal{C}$ be pointwise κ-presentable and $Y\colon \mathcal{J} \to \mathcal{C}^{\mathcal{I}}$ a κ-filtered diagram in $\mathcal{C}^{\mathcal{I}}$. Since κ-filtered colimits in $\mathcal{C}^{\mathcal{I}}$ are calculated pointwise and κ-small limits in **Sets** commute with κ-filtered colimits, we easily calculate that

$$\begin{aligned}\operatorname{Nat}\left(X, \operatorname*{colim}_{J\in\mathcal{J}} Y_J\right) &\cong \int_{I\in\mathcal{I}} \mathcal{C}\left(X_I, \left(\operatorname*{colim}_{J\in\mathcal{J}} Y_J\right)_I\right) \cong \int_{I\in\mathcal{I}} \mathcal{C}\left(X_I, \operatorname*{colim}_{J\in\mathcal{J}} Y_{J,I}\right)\\ &\overset{(*)}{\cong} \int_{I\in\mathcal{I}} \operatorname*{colim}_{J\in\mathcal{J}} \mathcal{C}(X_I, Y_{J,I}) \cong \operatorname*{colim}_{J\in\mathcal{J}} \int_{I\in\mathcal{I}} \mathcal{C}(X_I, Y_{J,I})\\ &\cong \operatorname*{colim}_{J\in\mathcal{J}} \operatorname{Nat}(X, Y_J),\end{aligned}$$

where we used the pointwise κ-presentability of X for $(*)$.
Ad (b): Assume that there is some $I \in \mathcal{I}$ with X_I not κ-presentable, meaning that there is some κ-filtered diagram $Y\colon \mathcal{J} \to \mathcal{C}$ together with a morphism $f\colon X_I \to \operatorname{colim}_{\mathcal{J}} Y$ that does not factor (essentially uniquely) through any Y_J. By the lemma above $I_* \operatorname{colim}_{\mathcal{J}} Y \cong \operatorname{colim}_{\mathcal{J}} I_* Y$ and using the natural tuning bijection

$$\mathcal{C}(X_I, \operatorname{colim}_{\mathcal{J}} Y) \cong (\mathcal{C}^{\mathcal{I}})(X, I_* \operatorname{colim}_{\mathcal{J}} Y) \cong (\mathcal{C}^{\mathcal{I}})(X, \operatorname{colim}_{\mathcal{J}} I_* Y),$$

we find that $f^\sharp\colon X \to \operatorname{colim}_{\mathcal{J}} I_* Y$ does not factor (essentially uniquely) through any $(I_* Y)_J$, which contradicts our hypothesis that X is κ-presentable. $\square$

(6.5) **Remark.** Since every locally κ-presentable category is also locally κ'-presentable for all regular $\kappa' \geqslant \kappa$ and every κ-small indexing category is also κ'-small, this proposition actually tells us that for $\kappa' \geqslant \kappa$, κ'-presentability is the same as pointwise κ'-presentability.

With this result, we can finally prove the “arrow version” of the Makkai-Paré theorem, telling us that we can lift morphisms between objects in a locally presentable category to morphisms between presentations.

(6.6) **Corollary. (Makkai-Paré** [bis]**)** Let $\kappa \trianglelefteq \kappa'$ be regular cardinals, $\mathcal{C}$ a locally κ-presentable category and $f\colon C \to D$ a morphism between κ'-presentable objects in $\mathcal{C}$. Then there is a κ-filtered κ'-small category $\mathcal{I}$ together with a morphism $\varphi\colon X \Rightarrow Y$ of $\mathcal{I}$-indexed diagrams $X, Y\colon \mathcal{I} \to \mathcal{C}_\kappa \subseteq \mathcal{C}$ and isomorphisms $\operatorname{colim}_{\mathcal{I}} X \cong C$, $\operatorname{colim}_{\mathcal{I}} Y \cong D$ (with the colimits taken in $\mathcal{C}$) making the following square commute:

$$\begin{tikzcd} \operatorname{colim}_{\mathcal{I}} X \arrow[r, phantom, "\cong"] \arrow[d, "\operatorname{colim}_{\mathcal{I}} \varphi"'] & C \arrow[d, "f"] \\ \operatorname{colim}_{\mathcal{I}} Y \arrow[r, phantom, "\cong"] & D \end{tikzcd} .$$

Proof. The category $\mathcal{C}^{[1]}$ is locally κ-presentable and $f\colon C \to D$ (viewed as an object therein) is κ'-presentable because it is pointwise so. By the original Makkai-Paré theorem, we find a κ-filtered κ'-small category $\mathcal{I}$ together with a diagram $Z\colon \mathcal{I} \to \left(\mathcal{C}^{[1]}\right)_\kappa = (\mathcal{C}_\kappa)^{[1]}$ such that $\operatorname{colim}_{\mathcal{I}} Z \cong f$. Using the exponential adjunction, this is really just a transformation $\varphi\colon X \to Y$ between diagrams $X, Y\colon \mathcal{I} \to \mathcal{C}_\kappa$ and we are done. □

7. Accessible Functors

(7.1) **Definition.** Given any infinite cardinal κ (as always, we can even assume κ to be regular), a functor $F\colon \mathcal{C} \to \mathcal{D}$ between κ-accessible categories is called *κ-accessible* iff it preserves κ-filtered colimits. As always, a functor is simply called *accessible* iff it is accessible for some κ; it is *finitely accessible* iff it is $\aleph_0$-accessible.

(7.2) **Example.** By definition, an object $C \in \mathcal{C}$ is κ-presentable iff $\mathcal{C}(C, -)\colon \mathcal{C} \to \mathbf{Sets}$ is κ-accessible (we need $\mathcal{C}$ to be κ-accessible by definition).

(7.3) **Example.** Obviously, every left adjoint functor between κ-accessible categories is κ-accessible because left adjoints preserve all colimits. As it turns out in (7.13), right adjoints are also accessible.

(7.4) **Example.** As a special case of this, we showed in (6.3), that the right adjoint to an evaluation functor $\mathcal{C}^{\mathcal{I}} \to \mathcal{C}$ (with $\mathcal{C}$ locally κ-presentable, which we only need for the construction of the right adjoint) is accessible.

(7.5) **Example.** Every constant functor between κ-accessible categories is κ-accessible since the colimit of a constant filtered diagram is just the object itself.

(7.6) **Example.** As observed in (3.3), given a family $(\mathcal{C}_i)_{i\in I}$ of κ-accessible categories with $|I| < \kappa$, their product $\mathcal{C} := \prod_{i\in I} \mathcal{C}_i$ is again κ-accessible. Since colimits in $\mathcal{C}$ are calculated pointwise, the standard projections $\mathrm{Pr}_i\colon \mathcal{C} \to \mathcal{C}_i$ are κ-accessible. Moreover, since the κ-presentable objects of $\mathcal{C}$ are the pointwise κ-presentable ones (as shown in (2.17)), these standard projections also preserve κ-presentable objects.

(7.7) **Example.** As for a more concrete example, most forgetful functors for algebraic structures (e.g. $\mathbf{Grp} \to \mathbf{Sets}$) are finitely accessible. This follows directly from (7.13) below and the fact that the free structure (e.g. group) on a finite set is finitely presentable. Of course, there is a more general theory hidden behind this, namely that of accessible monads.

(7.8) **Remark.** Obviously, κ-accessible functors compose (i.e. their composite is again κ-accessible) and for $\kappa \trianglelefteq \lambda$ regular, every κ-accessible functor is also λ-accessible (because κ-accessible categories are also λ-accessible and λ-filtered colimits are also κ-filtered).

When it comes to accessible functors, the preservation of filtered colimits is only one piece of the puzzle. What is going to be important, too, is the preservation of presentability. To this end, we have the following so-called uniformisation theorem.

(7.9) **Theorem. (Uniformisation Theorem)** Let $F\colon \mathcal{C} \to \mathcal{D}$ be κ-accessible (κ regular) and $\mathcal{R}$ a set of κ-generators in $\mathcal{C}$. If $\kappa' \trianglerighteq \kappa$ is such that every FR with $R \in \mathcal{R}$ is κ'-presentable then F preserves κ''-presentable objects for all regular $\kappa'' \trianglerighteq \kappa'$.

Proof. By (4.14) (and (3.21)), every κ''-presentable $C \in \mathcal{C}$ is a retract of a κ''-small κ-filtered colimit (in $\mathcal{C}$) of a diagram in $\mathcal{R}$ and because F is κ-accessible, FC is a retract of a κ''-small colimit (in $\mathcal{D}$) of a diagram in $F\mathcal{R}$. Because all objects in $F\mathcal{R}$ are κ'-presentable, and $\kappa'' \geqslant \kappa'$, it follows that FC is κ''-presentable. □

(7.10) **Example.** As a special case of uniformisation, let's fix a κ-accessible $\mathcal{C}$ with a set of κ-generators $\mathcal{R}$ and consider the Yoneda embedding (which is κ-accessible)

$$\mathcal{C} \hookrightarrow \mathbf{Sets}^{\mathcal{R}^{\mathrm{op}}},\ C \mapsto \mathcal{C}(-, C).$$

Because, by (2.8), representable presheaves in $\mathbf{Sets}^{\mathcal{R}^{\mathrm{op}}}$ are finitely presentable (i.e. they must be representable by an object in $\mathcal{R}$!), the Yoneda embedding maps objects of $\mathcal{R}$ to κ-presentables and thus (by uniformisation) preserves κ'-presentables for all $\kappa' \trianglerighteq \kappa$ regular.

(7.11) **Remark.** This example has an interesting consequence (which we will use later on). Let's use the same notation as in the example and identify $\mathcal{R}$ with the full subcategory it generates. We know from (6.4), that for every regular $\lambda > |\operatorname{Arr}\mathcal{R}|$, a presheaf in $X \in \mathbf{Sets}^{\mathcal{R}^{\mathrm{op}}}$ is λ-presentable iff it is pointwise so, which just means that $|X_R| < \lambda$ for all $R \in \mathcal{R}$. By the previous example then, it follows that for every regular $\lambda \trianglerighteq \kappa$ with $\lambda > |\operatorname{Arr}\mathcal{R}|$, we have

$$|\mathcal{C}(R, C)| < \lambda \qquad \text{for every } R \in \mathcal{R} \text{ and } C\ \lambda\text{-presentable.}$$

As we have seen very early on in the theory of accessible categories (3.18), the Hom-functors ranging over a set of generators work as "test functions" for filtered colimits. This easily transfers to accessible functors.

(7.12) **Proposition.** Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor between κ-accessible categories, $\mathcal{R}$ a set of κ-generators for $\mathcal{D}$ and $\lambda \trianglerighteq \kappa$ regular. Then F is λ-accessible iff every composite

$$\mathcal{D}(R, F-)\colon \mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{\mathcal{D}(R,-)} \mathbf{Sets}$$

with $R \in \mathcal{R}$ is λ-accessible.

Proof. The direction "⇒" is obvious because accessible functors compose. As for the other direction: Since the $\mathcal{D}(R, -)$ with $R \in \mathcal{R}$ jointly reflect κ-filtered (and a fortiori λ-filtered) colimits, it follows that F is λ-accessible. □

(7.13) **Proposition.** Given an adjunction $F\colon \mathcal{C} \rightleftarrows \mathcal{D} : G$ between κ-accessible categories (κ regular and left adjoint on the left),

(a) F is κ-accessible and

(b) G is λ-accessible for $\lambda \trianglerighteq \kappa$ regular such that F preserves λ-presentable objects.

In particular (by uniformisation), G is accessible, albeit not necessarily κ-accessible.

Proof. Point (a) is clear because F preserves all colimits. For point (b), picking a set of κ-generators $\mathcal{R}$ for $\mathcal{C}$, it suffices, by the last proposition, to show that every $\mathcal{C}(R, G-)$ with $R \in \mathcal{R}$ is λ-accessible. So, let $X\colon \mathcal{I} \to \mathcal{D}$ be a λ-filtered diagram. For $R \in \mathcal{R}$, we easily calculate that

$$\begin{aligned}\mathcal{C}(R, G(\operatorname*{colim}_I X_I)) \cong \mathcal{D}(FR, \operatorname*{colim}_I X_I) &\overset{(*)}{\cong} \operatorname*{colim}_I \mathcal{D}(FR, X_I)\\ &\cong \operatorname*{colim}_I(R, GX_I) \cong \mathcal{C}(R, \operatorname*{colim}_I GX_I),\end{aligned}$$

where for $(*)$, we used that FR is λ-accessible. $\square$

As a next step, we are going to show an adjoint functor theorem in the context of accessible functors (and thus accessible categories). For this, we are first going to study comma categories of accessible functors, which, as a corollary, gives us Freyd's Solution Set Condition.

(7.14) **Proposition.** Given two κ-accessible functors $F\colon \mathcal{C} \to \mathcal{E} \leftarrow \mathcal{D} : G$ with F preserving κ-presentable objects, the comma category $F \downarrow G$ together with the standard projection $F \downarrow G \to \mathcal{C} \times \mathcal{D}$ are κ-accessible. In particular (using uniformisation), comma categories of accessible functors are accessible.

Proof. First off, by (2.5.1), $F \downarrow G$ has κ-filtered colimits (created by the standard projection $F \downarrow G \to \mathcal{C} \times \mathcal{D}$, which is thus going to be κ-accessible) because $\mathcal{C}$ and $\mathcal{D}$ have κ-filtered colimits and F preserves them. Secondly, we note that, since colimits in $F \downarrow G$ are calculated in $\mathcal{C}$ and $\mathcal{D}$ and F as well as G preserve them, it follows that an object (C, p, D) in $F \downarrow G$ with C and D both κ-presentable, is itself κ-presentable. So, choosing sets of κ-generators $\mathcal{R}$, $\mathcal{S}$ for $\mathcal{C}$ and $\mathcal{D}$, respectively, it suffices to show that every object (C, p, D) in $F \downarrow G$ is a κ-filtered colimit of objects (R, r, S) with $R \in \mathcal{R}$ and $S \in \mathcal{S}$.

So, let $(C, p, D) \in F \downarrow G$ be arbitrary and fix κ-directed diagrams $X\colon I \to \mathcal{C}$, $Y\colon J \to \mathcal{D}$ (where I and J are posets!) that take values in $\mathcal{R}$ and $\mathcal{S}$, respectively, and whose colimits are C and D. Denoting the colimiting cocones by $\lambda\colon X \Rightarrow C$ and $\mu\colon Y \Rightarrow D$, respectively, we form a new poset K, whose elements are all triples (i, q, j) with $i \in I$, $j \in J$, $q\colon FX_i \to GY_j$ compatible with the colimiting cocones and $p\colon FC \to GD$; i.e. such that

$$\begin{tikzcd} FX_i \arrow[r, "F\lambda_i"] \arrow[d, "q"'] & FC \arrow[d, "p"] \\ GY_j \arrow[r, "G\mu_j"'] & GD \end{tikzcd}$$

commutes (and thus $(\lambda_i, \mu_j)\colon (X_i, q, Y_j) \to (C, p, D)$ is a morphism in $F \downarrow G$). The ordering is the obvious one, namely $(i, q, j) \leqslant (i', q', j')$ iff $i \leqslant i'$, $j \leqslant j'$ and

$$\begin{tikzcd} FX_i \arrow[r, "FX_{i\leqslant i'}"] \arrow[d, "q"'] & FX_{i'} \arrow[d, "p"] \\ GY_j \arrow[r, "GY_{j\leqslant j'}"'] & GY_{j'} \end{tikzcd}$$

commutes (i.e. $(X_{i\leqslant i'}, Y_{j\leqslant j'})\colon (X_i, q, Y_j) \to (X_{i'}, q', Y_{j'})$ is a morphism in $F \downarrow G$). To prove our claim, it suffices to do two things:

(a) We need to check that K is κ-directed and

(b) we need to show how $(C, p, D) \in F \downarrow G$ can be obtained as K-indexed colimit.

Ad (a): Let $K' \subseteq K$ be of cardinality $|K'| < \kappa$. Because I is κ-directed, all the i of elements in K' have an upper bound $i_0 \in I$ and all the j have an upper bound $j_0 \in J$. Now, since X_{i_0} is κ-presentable, which F preserves, and G preserves κ-filtered colimits, the composite $FX_{i_0} \to FC \to GD \cong \operatorname{colim}_j GY_j$ factors through some GY_j and without loss of generality, we assume that it factors through GY_{j_0}, say by $q_0 \colon FX_{i_0} \to GY_{j_0}$. More precisely, the square on the right in

$$\begin{tikzcd}
FX_i \arrow[r, "FX_{i\leqslant i_0}"] \arrow[d, "q"'] & FX_{i_0} \arrow[r, "F\lambda_{i_0}"] \arrow[d, "q_0"'] & FC \arrow[d, "p"'] \\
GY_j \arrow[r, "GY_{j\leqslant j_0}"'] & GY_{j_0} \arrow[r, "G\mu_{j_0}"'] & GD
\end{tikzcd}$$

commutes. However, the square on the left need not commute for $(i, q, j) \in K'$; i.e. (i_0, q_0, j_0) is not an upper bound for K'. What we do know is that the outer rectangle commutes and so, the left hand square composed with $G\mu_{j_0}$ commutes; or more symbolically

$$G\mu_{j_0} \circ q_0 \circ FX_{i\leqslant i_0} = G\mu_{j_0} \circ GY_{j\leqslant j_0} \circ q.$$

That is to say, the square provides two different factorisations of $FX_i \to GD$ through GY_{j_0}. Since FX_i is κ-presentable and GD is κ-filtered, we therefore find some $j_0' \geqslant j_0$ such that the square on the left above commutes when composed with $GY_{j_0} \to GY_{j_0'}$ (this is the injectivity condition in the definition of a κ-presentable object). We therefore obtain a commutative diagram of solid arrows

$$\begin{tikzcd}
FX_i \arrow[r, "FX_{i\leqslant i_0}"] \arrow[dd, "q"'] & FX_{i_0} \arrow[r, "F\lambda_{i_0}"] \arrow[d, "q_0"'] & FC \arrow[dd, "p"'] \\
 & GY_{j_0} \arrow[d, "GY_{j_0\leqslant j_0'}"'] \arrow[dr, dotted, bend left, "G\mu_{j_0}"] & \\
GY_j \arrow[r, "GY_{j\leqslant j_0'}"'] \arrow[ur, dotted, bend left, "GY_{j\leqslant j_0}"] & GY_{j_0'} \arrow[r, "G\mu_{j_0'}"'] & GD
\end{tikzcd}$$

and so, writing $q_0' := GY_{j_0\leqslant j_0'} \circ q_0$ for the vertical composite, we showed $(i, q, j) \leqslant (i_0, q_0', j_0')$. Still, we fixed $(i, q, j) \in K'$ and so (i_0, q_0', j_0') is not an upper bound for K' yet. But since $|K'| < \kappa$ and J is κ-directed, we can repeat this process, choose a j_0' for every $(i, q, j) \in K'$ and take an upper bound of those in J. This finishes the proof of the κ-directedness of K.

Ad (b): We would like to write (C, p, D) as a K-indexed colimit, for which we note that by definition of K, there is a canonical diagram $Z \colon K \to F \downarrow G$, given by

$$(i, q, j) \leqslant (i', q', j') \quad \mapsto \quad (X_i, q, Y_j) \xrightarrow{(X_{i\leqslant i'}, Y_{j\leqslant j'})} (X_{i'}, q', Y_{j'}).$$

Again by definition of K, this diagram comes with a canonical cocone over (C, p, D), whose component at (i, q, j) is (λ_i, μ_j) and this cocone is in fact colimiting. To see this, we use (2.5.1), which says that since F preserves κ-filtered colimits, they are calculated pointwise in $F \downarrow G$ (and p is going to be the unique map out of $FC \cong \operatorname{colim}_{(i,q,j)\in K} FX_i$). So, it suffices to check that

(b)' $\operatorname{colim}_{(i,q,j)\in K} X_i \cong C$ with the colimiting cocone having λ_i as its (i, q, j)-component;

(b)" similarly for D.

This is not quite as trivial as it sounds.

Ad (b)': Let $\left(\gamma_{i,q,j}\colon X_i \to C'\right)_{(i,q,j)\in K}$ be any cocone below $K \to F\downarrow G \to \mathcal{C}$. Out of this, we construct a cocone $\gamma'\colon X \Rightarrow C'$ (and thus the sought for induced morphism $C \to C'$) by noting that for every $i \in I$ there is some $(i,q,j) \in K$ because FX_i is κ-presentable and $GD \cong \operatorname{colim}_{j\in J} GY_j$ a κ-filtered colimit, so that $FX_i \to GD$ factors through some GY_j. Picking any such (i,q,j), we define $\gamma'_i := \gamma_{i,q,j}$. This is independent of q and j because if we have (i,q,j) and $(i,q',j') \in K$, then (by the injectivity condition from the definition of the κ-presentability of FX_i), there is some (i,q'',j'') above (i,q,j) and (i,q',j'), so that, since $(i,q,j) \leqslant (i,q'',j'')$ and $(i,q',j') \leqslant (i,q'',j'')$ are mapped to id_{X_i}, we get $\gamma_{i,q,j} = \gamma_{i,q'',j''} = \gamma_{i,q',j'}$.

Ad (b)'': To show that $\operatorname{colim}_{(i,q,j)\in K} Y_j \cong D$ (with the colimiting cocone having μ_j as its component at any $(i,q,j) \in K$), we let $\left(\delta_{i,q,j}\colon Y_j \to D'\right)_{(i,q,j)\in K}$ be any cocone below $K \to F\downarrow G \to \mathcal{D}$. Picking any $i_0 \in I$ and, as before, completing it to an $(i_0,q_0,j_0) \in K$, we define a cocone

$$\delta'\colon Y|_{J_{\geqslant j_0}} \Rightarrow D' \qquad \text{by} \qquad \delta'_j := \delta_{i_0,GY_{j_0\leqslant j}\circ q_0,j}.$$

To see that this gives us the required induced morphism $\operatorname{colim}_{J_{\geqslant j_0}} Y \overset{?}{\cong} D \to D'$, we first note that $J_{\geqslant j_0} \subseteq J$ and $K_{\geqslant (i_0,q_0,j_0)}$ are, respectively, final in J and K (by (1.12)), so that indeed $\operatorname{colim}_{J_{\geqslant j_0}} Y \cong D$ and we write $g\colon D \to D'$ for the morphism induced by δ'. Now, we just need to check that

$$\text{(7.15)} \qquad \begin{tikzcd} Y_j \arrow[r, "\mu_j"] \arrow[dr, bend right, "\delta_{i,q,j}"'] & D \arrow[d, "g"] \\ & D' \end{tikzcd} \qquad \text{commutes for all } (i,q,j) \geqslant (i_0,q_0,j_0)$$

(which we only know for $i = i_0$ and $q = GY_{j_0\leqslant j} \circ q_0$, so that $\delta_{i,q,j} = \delta'_j$). But this is easy because the commutative diagram

$$\begin{tikzcd} FX_{i_0} \arrow[r] \arrow[d, "q_0"'] & FX_i \arrow[d, "q"] \\ GY_{j_0} \arrow[r] & GY_j \end{tikzcd} \qquad \begin{array}{c}\text{corresponding to} \\ (i_0,q_0,j_0) \leqslant (i,q,j) \\ \text{can be rewritten as}\end{array} \qquad \begin{tikzcd} FX_{i_0} \arrow[r] \arrow[d, "q_0"'] & FX_i \arrow[dd, "q"] \\ GY_{j_0} \arrow[d] & \\ GY_j \arrow[r, equal] & GY_j \end{tikzcd} .$$

This, in turn, corresponds to $(i_0, GY_{j_0\leqslant j} \circ q_0, j) \leqslant (i,q,j)$, implying that

$$\delta'_j = \delta_{i_0,GY_{j_0\leqslant j}\circ q_0,j} = \delta_{i,q,j}$$

and the triangle (7.15) does indeed commute. □

(7.16) **Remark.** As observed, every object $(C,p,D) \in F\downarrow G$ with C and D κ-presentable is itself κ-presentable. However, the converse is not true. For example, (C, id_C) is initial (and thus finitely presentable) in $C\downarrow\mathcal{C}$ even though C need not be finitely presentable.

(7.17) **Scholium.** Under the same hypotheses as in the proposition, if $\mathcal{R}$ and $\mathcal{S}$ are sets of κ-generators for $\mathcal{C}$ and $\mathcal{D}$, respectively, then the set of all $(C,p,D) \in F\downarrow G$ with $C \in \mathcal{R}$ and $D \in \mathcal{S}$ is a set of κ-generators.

(7.18) **Corollary.** If $\mathcal{C}$ is κ-accessible and $C \in \mathcal{C}$ then $\mathcal{C}\downarrow C$ is again κ-accessible. If C is even κ-presentable, then $C\downarrow\mathcal{C}$ is also κ-accessible.

Proof. By definition $\mathcal{C}\downarrow C \cong \mathrm{id}_{\mathcal{C}}\downarrow C$ (where $C\colon\{*\}\to\mathcal{C}$) and $C\downarrow\mathcal{C}\cong C\downarrow \mathrm{id}_{\mathcal{C}}$. This also explains the additional requirement for under-categories because in C is not κ-presentable, the functor $C\colon\{*\}\to\mathcal{C}$ does not preserve κ-presentable objects. □

Later on, we will show that categories of diagrams with an accessible base are themselves accessible. For now, let us record the following special case.

(7.19) **Corollary.** If $\mathcal{C}$ is κ-accessible, so is its arrow category $\mathcal{C}^{[1]}$ (and the arrows with κ-presentable domain and codomain form a set of κ-generators in $\mathcal{C}^{[1]}$).

Proof. We apply the proposition to $\mathrm{id}_{\mathcal{C}}\downarrow\mathrm{id}_{\mathcal{C}}=\mathcal{C}\downarrow\mathcal{C}\cong\mathcal{C}^{[1]}$. □

(7.20) **Corollary.** If $\mathcal{C}$ is κ-accessible with a set of κ-generators $\mathcal{R}$, then an $F\colon\mathcal{C}\to\mathcal{D}$ that preserves κ-filtered colimits is uniquely determined (up to isomorphism) by its restriction to the full subcategory $\mathcal{R}\subseteq\mathcal{C}$.

Proof. By definition, every morphism in $\mathcal{C}$ (viewed as an object in $\mathcal{C}^{[1]}$) is a κ-filtered colimit of a diagram with values in $\mathcal{R}^{[1]}$. Since F preserves κ-filtered colimits, the values of all morphisms under F are thus determined (up to isomorphism). □

(7.21) **Theorem. (Accessible Adjoint Functor Theorem)** Every accessible functor $G\colon\mathcal{D}\to\mathcal{C}$ satisfies the solution set condition (i.e. every $C\downarrow G$ has a weakly initial subset of objects). Consequently, an accessible functor has a left adjoint iff it preserves limits.

Proof. For $C\in\mathcal{C}$, we use uniformisation to pick κ such that C is κ-presentable and G κ-accessible. By the above proposition then, $C\downarrow G$ is κ-accessible. Therefore, all its κ-presentable objects form a weakly initial set because every other object can be written as a colimit of such. The final claim then is just by Freyd's adjoint functor theorem. □

(7.22) **Corollary.** A κ-accessible category $\mathcal{C}$ is

(a) complete iff (b) cocomplete iff (c) locally κ-presentable.

Proof. The implication (b) $\Rightarrow$ (c) is by definition and (c) $\Rightarrow$ (a) was part of (5.4). As for (a) $\Rightarrow$ (b), we pick a skeleton $\mathrm{Sk}(\mathcal{C}_\kappa)$ for the κ-presentable objects in $\mathcal{C}$. By (5.1) (and because presheaf categories are locally finitely presentable), the Yoneda functor

$$Y\colon\mathcal{C}\to\mathbf{Sets}^{\mathrm{Sk}(\mathcal{C}_\kappa)^{\mathrm{op}}},\ C\to\mathcal{C}(-,C)$$

is fully faithful and κ-accessible. Now, if $\mathcal{C}$ is complete, Y clearly preserves limits because the Hom-functors $\mathcal{C}(A,-)$ do. From the accessible adjoint functor theorem then, we get a reflection along Y and the claim follows by (3.32). □

8. Pure Subobjects

In (5.5), we characterised locally κ-presentable categories as being exactly the full reflective subcategories of presheaf categories, closed under κ-filtered colimits. For a κ-accessible category $\mathcal{C}$, we only know from (5.2) that it is necessarily a full subcategory of a presheaf category, closed under κ-filtered colimits via the Yoneda embedding

$$\mathcal{C}\hookrightarrow\mathbf{Sets}^{\mathrm{Sk}(\mathcal{C}_\kappa)^{\mathrm{op}}},\ C\mapsto\mathcal{C}(-,C).$$

However, this condition is far from sufficient for κ-accessibility and the goal of this section is to amend this condition to get a characterisation similar to that of locally κ-filtered categories. Again, just like in previous section, let us fix some regular cardinal κ.

(8.1) **Definition.** A morphism $i\colon A \to B$ in a category $\mathcal{C}$ is called *κ-pure* iff every commutative square of solid arrows in $\mathcal{C}$

$$\begin{tikzcd} R \arrow[r, "f"] \arrow[d, "p"'] & A \arrow[d, "i"] \\ S \arrow[r, "g"'] \arrow[ur, dotted, "d"] & B \end{tikzcd}$$

with R, S κ-presentable has an *upper diagonal filler*, meaning a $d\colon D \to A$ such that $f = d \circ p$ (but not necessarily $g = i \circ d$).

(8.2) **Observation.** While we allowed $\mathcal{C}$ in the definition to be any category, we are only interested in the accessible case. There, it is noteworthy that if $\mathcal{C}$ is κ-accessible with a set of κ-generators $\mathcal{R}$, it suffices to check the condition from the definition for R, $S \in \mathcal{R}$.

Proof. By (7.17) and (7.19), $\mathcal{C}^{[1]}$ is κ-accessible with a set of κ-generators given by $\mathcal{R}^{[1]}$. With this, using the standard lemma (3.21), every morphism between κ-presentables in $\mathcal{C}$ is a retract of a morphism between objects in $\mathcal{R}$ and if we can construct an upper diagonal filler for some morphism p, we can do so for every retract of p (as is easily verified). □

(8.3) **Example.** Every split monomorphism (i.e. every morphism that has a retraction) is κ-pure for all κ. To wit, if i in the above square has a retraction r, then $d := r \circ h$ is an upper diagonal filler.

(8.4) **Example.** As we shall show in (8.8) below, every pure morphism is necessarily monic. Combining this with the previous example and noting that every monomorphism in **Sets** is split, we conclude that in **Sets**, the pure morphisms are precisely the injections.

(8.5) **Remark.** Clearly, every κ-pure morphism is also κ'-pure for $\kappa' \geqslant \kappa$ regular. Moreover, one easily checks that κ-pure morphisms are closed under composition, retracts (in $\mathcal{C}^{[1]}$) and satisfy the following descent condition: If $j \circ i$ is κ-pure, then i is κ-pure.

Not only are pure morphisms closed under retracts (which is a special case of a filtered colimit) but also under filtered colimits, as it turns out.

(8.6) **Proposition.** The class of κ-pure morphisms, viewed as a full subcategory of the arrow category $\mathcal{C}^{[1]}$, is closed under arbitrary κ-filtered colimits.

Proof. Let $\mathcal{J} \to \mathcal{C}^{[1]}$, $J \mapsto i_J$ be a κ-filtered colimit of κ-pure morphisms with colimiting cocone $\big((\lambda_J, \mu_J)\colon i_J \to i\big)_J$ and $(f, g)\colon p \to i$ any commutative square as in the definition with $p\colon R \to S$ having a κ-presentable domain and codomain. From (6.4), we know that p (as an object of $\mathcal{C}^{[1]}$) is κ-presentable and so, (f, g) factors as

$$(f,g)\colon p \xrightarrow{(f',g')} i_J \xrightarrow{(\lambda_J,\mu_J)} i \qquad \text{i.e.} \qquad \begin{tikzcd} R \arrow[r, "f'"] \arrow[d, "p"'] & A_J \arrow[r, "\lambda_J"] \arrow[d, "i_J"'] & A \arrow[d, "i"'] \\ S \arrow[r, "g'"'] & B_J \arrow[r, "\mu_J"'] & B \end{tikzcd}$$

for some $J \in \mathcal{J}$ and we can construct the upper diagonal filler in the left square. □

In the above example, we saw that split monomorphisms are pure. Conversely, we can show that every pure morphism is monic. In fact, within a locally κ-presentable category, one can even show that they are regularly monic but since we are only concerned with the accessible case here, we shall not pursue this. The proof that every pure morphism is monic is not very hard, though its easiness is somewhat clouded by the fact that we work in an arrow category. Let us therefore formalise the key step (which is not dependent on our working in an arrow category) as a small lemma.

(8.7) **Lemma.** Let $\mathcal{C}$ be a κ-accessible category with a set of κ-generators $\mathcal{R}$. Then any cofork $P \rightrightarrows A \to B$ in $\mathcal{C}$ with P κ-presentable lifts to a cofork $P \rightrightarrows R \to S$ with R, $S \in \mathcal{R}$, meaning that it can be completed to a diagram

$$\begin{tikzcd} P \arrow[r, shift left, "f'"] \arrow[r, shift right, "g'"'] \arrow[d, equal] & R \arrow[r, "j"] \arrow[d, "p"'] & S \arrow[d, "q"'] \\ P \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & A \arrow[r, "i"'] & B \end{tikzcd},$$

where the rows are coforks and the top and bottom square on the left as well as the square on the right commute (and, as already mentioned, R, $S \in \mathcal{R}$).

Proof. The object P is κ-presentable and f', g' are just factorisations of f and g through the canonical κ-filtered diagram (indexed by $\mathcal{R}\downarrow A$) over A. But now, f' and g' are two different factorisations of $i \circ f = i \circ g$ through the canonical κ-filtered diagram over B and (again by κ-presentability of P), we get the required j. □

(8.8) **Proposition.** Every κ-pure morphism $i\colon A \to B$ in a κ-accessible category $\mathcal{C}$ is monic and we can therefore speak of *κ-pure subobjects.*

Proof. Because the κ-presentable objects generate $\mathcal{C}$, it suffices to show that if we have a pair of parallel arrows f, $g\colon G \rightrightarrows A$ with $i \circ f = i \circ g$ and G κ-presentable, then $f = g$. For this, we write $h := i \circ f = i \circ g$ and consider the cofork

$$\Big(G \xrightarrow{\mathrm{id}_G} G\Big) \overset{(f,h)}{\underset{(g,h)}{\rightrightarrows}} \Big(A \xrightarrow{i} B\Big) \xrightarrow{(i,\mathrm{id}_B)} \Big(B \xrightarrow{\mathrm{id}_B} B\Big)$$

in $\mathcal{C}^{[1]}$. By (7.19), this arrow category is κ-accessible with a set of κ-generators given by all arrows with a κ-presentable domain and codomain (or rather in some skeleton $\mathrm{Sk}(\mathcal{C}_\kappa)$ to get a set). By the lemma then, we can lift this cofork to some

$$\begin{tikzcd} \Big(G \xrightarrow{\mathrm{id}} G\Big) \arrow[r, shift left, "(f',h_1)"] \arrow[r, shift right, "(g',h_2)"'] \arrow[d, equal] & \Big(R \xrightarrow{r} R'\Big) \arrow[r, "(j,j')"] \arrow[d, "(p,p')"'] & \Big(S \xrightarrow{s} S'\Big) \arrow[d, "(q,q')"'] \\ \Big(G \xrightarrow{\mathrm{id}} G\Big) \arrow[r, shift left, "(f,h)"] \arrow[r, shift right, "(g,h)"'] & \Big(A \xrightarrow{i} B\Big) \arrow[r, "(i,\mathrm{id})"'] & \Big(B \xrightarrow{\mathrm{id}} B\Big) \end{tikzcd},$$

with R, R', S, S' κ-presentable. Now we have a commutative square of solid arrows

$$\begin{tikzcd} R \arrow[r, "p"] \arrow[d, "s\circ j"'] & A \arrow[d, "i"] \\ S' \arrow[r, "q'"'] \arrow[ur, dotted, "d"] & B \end{tikzcd}$$

and can pick an upper diagonal filler d because A is κ-pure and R, S' are κ-presentable. Finally, we easily calculate

$$f = p \circ f' = d \circ s \circ j \circ f' = d \circ s \circ j \circ g' = p \circ g' = g.$$

□

(8.9) **Proposition.** A κ-accessible functor $F\colon \mathcal{C} \to \mathcal{D}$ preserves κ-pure subobjects.

Proof. Let $i\colon A \to B$ be κ-pure in $\mathcal{C}$ and

$$\begin{tikzcd} R \arrow[r, "f"] \arrow[d, "p"'] & FA \arrow[d, "Fi"] \\ S \arrow[r, "g"'] & FB \end{tikzcd}$$

a commutative square in $\mathcal{D}$ with R, S κ-presentable. Using the κ-accessibility (7.19) of the arrow category $\mathcal{C}^{[1]}$, we can write i as a colimit of a κ-filtered diagram $\mathcal{J} \to \mathcal{C}^{[1]}$, $J \mapsto i_J$ where each i_J has a κ-presentable domain and codomain. Because $p \in \mathcal{D}^{[1]}$ is κ-presentable (cf. (6.4)) and F preserves κ-filtered colimits (in particular $Fi \cong \operatorname{colim}_J Fi_J$), the morphism $(f, g)\colon p \to Fi$ factors through some Fi_J, yielding a commutative diagram

$$\begin{tikzcd} R \arrow[r] \arrow[d, "p"'] & FA_J \arrow[r] \arrow[d, "Fi_J"'] & FA \arrow[d, "Fi"'] \\ S \arrow[r] & FB_J \arrow[r] & FB \end{tikzcd} .$$

Because i is κ-pure and A_J, B_J κ-presentable, we can construct an upper diagonal filler for the right square in $\mathcal{C}$ and then apply F to it, so that the composite with $S \to FB_J$ is the required upper diagonal filler for the original square. □

With the notion of a pure subobject, we can amend the condition for an accessible category to be a full subcategory closed under filtered colimits in some presheaf category. As we will show later, this is already the sought for sufficient condition for accessibility. For this, let us introduce some terminology

(8.10) **Definition.** A (not necessarily full) subcategory $\mathcal{C} \subseteq \mathcal{D}$ is called *κ-accessibly embedded* iff it is closed under κ-filtered colimits. If $\mathcal{C} \subset \mathcal{D}$ is not full, this means two things: Given a κ-filtered diagram $X\colon \mathcal{I} \to \mathcal{C}$ together with a colimiting cocone $\lambda\colon X \Rightarrow L$ in $\mathcal{D}$

(a) $\lambda\colon X \Rightarrow L$ is even a cocone in $\mathcal{C}$ and (b) it is colimiting there, too.

This latter point means that if $\gamma\colon X \Rightarrow C$ is another cocone in $\mathcal{C}$, then the induced morphism $L \to C$ is required to belong to $\mathcal{C}$, too. As always, a subcategory is *accessibly embedded* iff it is κ-accessibly embedded for some (without loss of generality regular) κ.

(8.11) **Remark.** Every accessibly embedded subcategory is closed under retracts using (1.10). Also, the closure under κ-filtered colimits is to be understood as implying repleteness (i.e. closure under isomorphisms). Finally, κ-accessible embeddings reflect κ-presentability (though they don't need to preserve it) and if $\mathcal{C} \subseteq \mathcal{D}$ is κ-accessibly embedded, it is also κ'-accessibly embedded for all $\kappa' \geqslant \kappa$ regular.

Note that if $\mathcal{C} \subseteq \mathcal{D}$ is an accessibly embedded subcategory of an accessible category $\mathcal{D}$, this does not imply that $\mathcal{C}$ is itself accessible. However, if it is, then by uniformisation, we can find some κ such that the embedding $\mathcal{C} \hookrightarrow \mathcal{D}$ is κ-accessible and preserves κ'-presentability for all $\kappa' \trianglerighteq \kappa$.

(8.12) **Proposition.** If $\mathcal{C} \subseteq \mathcal{D}$ is a κ-accessibly embedded full subcategory such that the inclusion $\mathcal{C} \hookrightarrow \mathcal{D}$ is κ-accessible (in particular, $\mathcal{C}$ must be κ-accessible) and preserves κ-presentable objects, then $\mathcal{C}$ is closed under κ-pure subobjects.

Proof. Let's write $\mathcal{R}_{\mathcal{D}}$ for some complete set of representatives of κ-presentables in $\mathcal{D}$ and $\mathcal{R}_{\mathcal{C}} := \mathcal{R}_{\mathcal{D}} \cap \mathcal{C}$. Then $\mathcal{R}_{\mathcal{C}}$ is a complete set of representatives of κ-presentables in $\mathcal{C}$ because $\mathcal{C} \hookrightarrow \mathcal{D}$ preserves κ-presentability (and because $\mathcal{C}$ is replete). Now, given a κ-pure subobject $i\colon D \to C$ in $\mathcal{D}$ with $C \in \mathcal{C}$, we write D as a canonical κ-filtered colimit

$$D \cong \operatorname{colim}(\mathcal{R}_{\mathcal{D}} \downarrow D \to \mathcal{R}_{\mathcal{D}} \hookrightarrow \mathcal{D})$$

and show that $\mathcal{R}_{\mathcal{C}} \downarrow D \subseteq \mathcal{R}_{\mathcal{D}} \downarrow D$ is final. For this, we note that because κ-filtered colimits in $\mathcal{C}$ are calculated in $\mathcal{D}$ (and in particular the canonical κ-filtered colimit for C), we have

$$C \cong \operatorname{colim}(\mathcal{R}_{\mathcal{C}} \downarrow C \to \mathcal{R}_{\mathcal{C}} \hookrightarrow \mathcal{D}).$$

Now, for every $r\colon R \to D$ with $R \in \mathcal{R}_{\mathcal{D}}$, the composite $i \circ r\colon R \to C$ factors through this canonical κ-filtered diagram (in $\mathcal{C}$!) for C, meaning that we find some commutative square

$$\begin{tikzcd} R \arrow[r, "r"] \arrow[d, dotted, "f"'] & D \arrow[d, "i"] \\ S \arrow[r, dotted, "s"'] & C \end{tikzcd}$$

with $S \in \mathcal{R}_{\mathcal{C}}$. Because i is κ-pure and S is κ-presentable in $\mathcal{D}$, we can pick an upper diagonal filler $l\colon S \to D$. But this then means that $f\colon (R, r) \to (S, s)$ is a morphism in $\mathcal{R}_{\mathcal{D}} \downarrow D$ and so, by (1.12) $\mathcal{R}_{\mathcal{C}} \downarrow D \subseteq \mathcal{R}_{\mathcal{D}} \downarrow D$ is indeed final. By the same lemma, $\mathcal{R}_{\mathcal{C}} \downarrow D$ is also κ-filtered and since $\mathcal{C}$ is closed under κ-filtered diagrams, we have $D \in \mathcal{C}$. □

(8.13) **Corollary.** For $\mathcal{C}$ κ-accessible, with some set of κ-generators $\mathcal{R}$, the repletion of $\mathcal{C} \subseteq \mathbf{Sets}^{\mathcal{R}^{\mathrm{op}}}$ is closed under κ-filtered colimits and κ'-pure subobjects for all $\kappa' \trianglerighteq \kappa$ regular.

Proof. We note that the Yoneda embedding

$$\mathcal{C} \hookrightarrow \mathbf{Sets}^{\mathcal{R}^{\mathrm{op}}},\ C \mapsto \mathcal{C}(-, C)$$

is not only κ-accessible (and thus κ'-accessible) but also preserves κ'-presentable objects, as observed in (7.10). Now the claim follows from the proposition. □

Our next step to a characterisation of accessible subcategories is to show that every accessible category has enough pure subobjects, meaning the following.

(8.14) **Proposition.** Let $\mathcal{C}$ be κ-accessible with a set $\mathcal{R}$ of κ-generators (which we identify with the full subcategory it generates). Then, if $\lambda \triangleright \kappa$ is regular such that

$$\lambda > |\operatorname{Arr} \mathcal{R}| \qquad (\text{i.e. } |\operatorname{Ob} \mathcal{R}| < \lambda \text{ and } |\mathcal{C}(R, S)| < \lambda \text{ for all } R,\ S \in \mathcal{R}),$$

every morphism $f\colon C \to B$ with C λ-presentable factors through a κ-pure $i\colon A \rightarrowtail B$, where A is again λ-presentable.

Proof. Using the Yoneda embedding, we identify $\mathcal{C}$ with its repletion in $\mathcal{D} := \mathbf{Sets}^{\mathcal{R}^{\mathrm{op}}}$. As before ((5.2), (7.10), (8.11) and the above corollary (8.13)), for every $\mu \trianglerighteq \kappa$ regular, $\mathcal{C} \subseteq \mathcal{D}$ is closed under μ-filtered colimits, μ-pure subobjects and an object in $\mathcal{C}$ is μ-presentable in $\mathcal{C}$ iff it is so in $\mathcal{D}$.

Now, given $f\colon C \to B$, we construct a κ-pure subobject $i\colon A \rightarrowtail B$ through which f factors by "cellular approximation". To wit, we construct a transfinite sequence of morphisms in $\mathcal{D}$ through which f factors

$$(i_\alpha\colon A_\alpha \to B)_{\alpha\leqslant\kappa}$$

and i_κ is going to be κ-pure. As our initial step, we put $A_0 := C$ and $i_0 := f$. Also, for $\beta \leqslant \kappa$ a limit ordinal, we simply put $A_\beta := \operatorname{colim}_{\alpha<\beta} A_\alpha$ (in $\mathcal{D}$!), which leaves the successor case. Assuming we have already constructed $i_\alpha\colon A_\alpha \to B$, we consider the set $\mathcal{S}_\alpha$ of all spans

(8.15) $\quad S \xleftarrow{r} R \xrightarrow{f} A_\alpha$ $\quad$ with R, $S \in \mathcal{R}$ and that can be completed to a commutative square

$$\begin{tikzcd} R \arrow[r, "f"] \arrow[d, "r"'] & A_\alpha \arrow[d, "i_\alpha"] \\ S \arrow[r, "g_{r,f}"'] & B \end{tikzcd}.$$

Picking one such completion $g_{r,f}$ for every span $(r,f) \in \mathcal{S}_\alpha$, we define $A_{\alpha+1}$ to be the pushout (in $\mathcal{D}$!) of

$$\coprod_{\substack{(r\colon R\to S,f)\\ \text{in } \mathcal{S}_\alpha}} S \xleftarrow{\coprod_{(r,f)} r} \coprod_{\substack{(r\colon R\to S,f)\\ \text{in } \mathcal{S}_\alpha}} R \xrightarrow{[f]_{(r,f)}} A_\alpha$$

with $i_{\alpha+1}\colon A_{\alpha+1} \to B$ the morphism induced by i_α and all the $g_{r,f}$. To check the κ-purity of i_κ, it suffices to construct an upper diagonal filler for all commutative squares

$$\begin{tikzcd} R \arrow[r, "f"] \arrow[d, "r"'] & A_\kappa \arrow[d, "i_\kappa"] \\ S \arrow[r, "g"'] & B \end{tikzcd} \qquad \text{with } R,\ S \in \mathcal{R} \text{ (see (8.2)).}$$

Given such a square, we note that because R is κ-presentable (in $\mathcal{C}$, whence in $\mathcal{D}$) and $A_\kappa = \operatorname{colim}_{\alpha<\kappa} A_\alpha$ is a κ-filtered colimit in $\mathcal{D}$, $f\colon R \to A_\kappa$ factors through some A_α, say as $f'\colon R \to A_\alpha$, so that $(r, f') \in \mathcal{S}_\alpha$. By definition, we have a chosen completion $g_{r,f'}$ of (r, f') to a square (8.15) as above (though not necessarily $g_{r,f'} = g$) and so, by definition, $g_{r,f'}\colon S \to B$ factors through $i_{\alpha+1}\colon A_{\alpha+1} \to B$, say as $g'_{r,f'}\colon S \to A_{\alpha+1}$. Now, this factorisation can be completed to an upper diagonal filler $S \to A_{\alpha+1} \to A_\kappa$. This need not be a diagonal filler for the entire square because $g_{r,f'}$ might be different from g. With the κ-purity of A_κ checked, we conclude that $A := A_\kappa$ is the sought for subobject and lies in $\mathcal{C}$ because $\mathcal{C}$ is closed under κ-pure subobjects.

For the final claim, assume that C is λ-presentable for some λ as in the proposition and let's check that every $A_\alpha \in \mathcal{D}$ is λ-presentable. This is clear for $A_0 = C$ because $\lambda \rhd \kappa$ and it is also clear for $\alpha \leqslant \kappa$ a limit ordinal because $A_\alpha = \operatorname{colim}_{\beta<\alpha} A_\beta$ is a λ-small colimit of λ-presentables.

Finally, for the successor case, assume that A_α is λ-presentable. We recall from (7.11) that a presheaf $X \in \mathcal{D}$ is λ-presentable iff it is pointwise so (i.e. $|X_R| < \lambda$ for all $R \in \mathcal{R}$). This is in particular the case for A_α (i.e. $|\mathcal{C}(R, A_\alpha)| < \lambda$ for all $R \in \mathcal{R}$). With this, the number of spans $(r, f)\colon S \leftarrow R \to A_\alpha$ with R, $S \in \mathcal{R}$ is

$$|\mathcal{C}(S,R)| + |\mathcal{C}(R, A_\alpha)| < \lambda + \lambda = \lambda.$$

Consequently, by definition, $A_{\alpha+1}$ is a λ-small colimit of λ-presentables. $\square$

(8.16) **Proposition.** Let $\mathcal{C}$ be κ-accessible with a set $\mathcal{R}$ of κ-generators and, just like before, let $\lambda \rhd \kappa$ be regular such that $|\mathrm{Arr}\,\mathcal{R}| < \lambda$. Then the wide subcategory

$$\mathrm{Pure}_\kappa(\mathcal{C}) \subseteq \mathcal{C}$$

of κ-pure morphisms is a κ-accessibly embedded λ-accessible (non-full!) subcategory and the inclusion $\mathrm{Pure}_\kappa(\mathcal{C}) \subseteq \mathcal{C}$ is λ-accessible.

Proof. Let us first check that $\mathrm{Pure}_\kappa(\mathcal{C}) \subseteq \mathcal{C}$ is closed under κ-directed colimits. So, let $X\colon I \to \mathrm{Pure}_\kappa(\mathcal{C})$ be κ-directed (i.e. I is a poset) with colimiting cocone $(\lambda_i\colon X_i \to L)_{i\in I}$ in $\mathcal{C}$. Since κ-pure subobjects are closed under κ-filtered colimits (in $\mathcal{C}^{[1]}$, as shown in (8.6)), it follows that every λ_i is κ-pure. Indeed, fixing $i_0 \in I$, all $X_{i_0} \to X_i$ with $i \in I_{\geqslant i_0}$ form a κ-filtered diagram in $\mathcal{C}^{[1]}$, whose colimit is $\lambda_{i_0}\colon X_{i_0} \to L$ because $I_{\geqslant i_0} \subseteq I$ is final (see (1.13)). This means that λ is indeed a cocone in $\mathrm{Pure}_\kappa(\mathcal{C})$.

To see that $\lambda\colon X \Rightarrow L$ is even a colimiting cocone in $\mathrm{Pure}_\kappa(\mathcal{C})$, we consider another cocone $\gamma\colon X \Rightarrow C$ below X in $\mathrm{Pure}_\kappa(\mathcal{C})$ and need to check that the induced morphism $g\colon L \to C$ is κ-pure. So consider a commutative square as on the left

$$\begin{tikzcd} R \arrow[r] \arrow[d] & L \arrow[d, "g"] \\ S \arrow[r] & C \end{tikzcd} \qquad \begin{tikzcd} R \arrow[rr] \arrow[dd] \arrow[dr, dotted] & & L \arrow[dd, "g"] \\ & X_i \arrow[ur, "\lambda_i"] \arrow[dr, "\gamma_i"'] & \\ S \arrow[rr] & & C \end{tikzcd}$$

with R, S κ-presentable. Since L is a κ-directed colimit, the morphism $R \to L$ factors through some X_i (as depicted on the right above) and because γ_i is κ-pure, we find an upper diagonal filler $S \to X_i$, so that the composite with λ_i is an upper diagonal filler for the entire square.

As for λ-accessibility, we have just checked that $\mathrm{Pure}_\kappa(\mathcal{C}) \subseteq \mathcal{C}$ is κ-accessibly embedded. In particular, it has κ-filtered colimits (which are calculated in $\mathcal{C}$) and for all $\mu \geqslant \kappa$ regular, μ-presentability in $\mathcal{C}$ implies μ-presentability in $\mathrm{Pure}_\kappa(\mathcal{C})$ (as noted in (8.11)). Now, given any object B, because $\lambda \rhd \kappa$, $\mathcal{C}$ is λ-accessible and the indexing category $\mathrm{Sk}(\mathcal{C}_\lambda) \downarrow B$ for the canonical diagram therefore λ-filtered. By (1.12), together with the above proposition, the full subcategory

$$\mathcal{P} \subseteq \mathrm{Sk}(\mathcal{C}_\lambda) \downarrow B \text{ of all } \kappa\text{-pure subobjects } A \rightarrowtail B \text{ (with } A \ \lambda\text{-presentable in } \mathcal{C})$$

is final and itself λ-filtered. This proves the λ-accessibility of $\mathrm{Pure}_\kappa(\mathcal{C})$ because of the descent condition for κ-purity: If $j \circ i$ is κ-pure then so is i. In particular does the canonical diagram $\mathcal{P} \hookrightarrow \mathrm{Sk}(\mathcal{C}_\lambda) \downarrow B \to \mathcal{C}$ for B lie in $\mathrm{Pure}_\kappa(\mathcal{C})$. □

Recall that the original goal for this section was to have a characterisation of accessible subcategories, analogous to the locally presentable case:

(8.17) **Proposition.** Let $\mathcal{C}$ be κ-accessible with a set $\mathcal{R}$ of κ-generators and, just like before, let $\lambda \rhd \kappa$ be regular such that $|\mathrm{Arr}\,\mathcal{R}| < \lambda$. Then the following are equivalent for a λ-accessibly embedded subcategory $\mathcal{D} \subseteq \mathcal{C}$:

(a) $\mathcal{D}$ is λ-accessible;

(b) $\mathcal{D}$ is closed under λ-pure subobjects;

(c) $\mathcal{D}$ is closed under κ-pure subobjects.

Proof. The implication "(a) $\Rightarrow$ (b)" was (8.12) and "(b) $\Rightarrow$ (c)" is trivial because κ-purity implies λ-purity. All that remains to check is "(c) $\Rightarrow$ (a)". Because $\mathrm{Pure}_\kappa(\mathcal{C}) \subseteq \mathcal{C}$ is a κ-accessibly embedded λ-accessible wide subcategory, every $D \in \mathcal{D}$ can be written as a λ-filtered colimit (calculated in $\mathcal{C}$) of κ-pure subobjects that are λ-presentable (in $\mathcal{C}$, whence in $\mathcal{D}$). Since $\mathcal{D}$ is closed under κ-pure subobjects, this diagram lies in $\mathcal{D}$ and because $\mathcal{D} \subseteq \mathcal{C}$ is λ-accessibly embedded, its colimiting cocone in $\mathcal{C}$ is also colimiting in $\mathcal{D}$. $\square$

(8.18) **Corollary.** If $\mathcal{C}$ is accessible and $\mathcal{D} \subseteq \mathcal{C}$ an accessibly embedded full subcategory then $\mathcal{D}$ is accessible iff it is closed under μ-pure subobjects for some regular μ.

Proof. Assume $\mathcal{C}$ is κ-accessible (κ regular) with a set of κ-generators $\mathcal{R}$ and that $\mathcal{D} \subseteq \mathcal{C}$ is λ-accessibly embedded (λ regular). We now pick a regular κ' such that

$$\kappa' \rhd \kappa, \qquad \kappa' \geqslant \lambda, \qquad \kappa' \geqslant \mu \qquad \text{and} \qquad \kappa' > |\mathrm{Arr}\,\mathcal{R}|$$

(this can be done e.g. using (4.10)).Since $\mathcal{D} \subseteq \mathcal{C}$ is λ-accessibly embedded and $\kappa' \geqslant \lambda$, it is also κ'-accessibly embedded. Similarly, since $\mathcal{D}$ is closed under μ-pure subobjects and $\kappa' \geqslant \mu$, it is also closed under κ'-pure subobjects and the proposition applies. $\square$

The following two corollaries are really the main justification for our characterisation, as we are going to apply them to combinatorial model categories.

(8.19) **Corollary.** If $\mathcal{C}$ is accessible and $(\mathcal{D}_i)_{i\in I}$ is a small family of accessibly embedded accessible full subcategories in $\mathcal{C}$, then so is their intersection $\bigcap_{i\in I} \mathcal{D}_i$. $\square$

(8.20) **Corollary.** If $F\colon \mathcal{C} \to \mathcal{D}$ is accessible and $\mathcal{D}' \subseteq \mathcal{D}$ an accessibly embedded accessible full subcategory, then so is its full preimage $F^{-1}\mathcal{D}' \subseteq \mathcal{C}$.

Proof. Assume $\mathcal{C}$ and $\mathcal{D}$ are κ-accessible with sets of κ-generators $\mathcal{R}$, $\mathcal{S}$ and take $\lambda \rhd \kappa$ regular with $\mathcal{D}'$ λ-accessible and $|\mathrm{Arr}\,\mathcal{R}|$, $|\mathrm{Arr}\,\mathcal{S}| < \lambda$ (so that $\mathcal{D}'$ is closed under λ-pure subobjects). To show the λ-accessibility of $\mathcal{C}' := F^{-1}\mathcal{D}'$, we just need to check that $\mathcal{C}'$ is closed under λ-pure subobjects. But given $i\colon A \rightarrowtail B$ λ-pure in $\mathcal{C}$ with $B \in \mathcal{C}'$, we know from (8.9), that the image Fi is again λ-pure and because $FB \in \mathcal{D}'$, we must have $FA \in \mathcal{D}'$; i.e. $A \in \mathcal{C}'$. $\square$

(8.21) **Remark.** Lurie shows that if $F\colon \mathcal{C} \to \mathcal{D}$ is a κ-accessible functor between locally presentable categories then the full preimage $F^{-1}\mathcal{D}'$ of any κ-accessibly embedded κ-accessible full subcategory $\mathcal{D}'$ is again κ-accessibly embedded κ-accessible. However, he uses a different definition of local presentability and accessibility, where he puts no presentability restrictions on the objects of a set of κ-generators.

Chapter 4

FACTORISATION SYSTEMS

Recall that a model structure on a category consists of three classes $\mathcal{W}$, $\mathcal{F}$ and $\mathcal{C}$ such that $\mathcal{W}$ satisfies the 2-out-of-3-property and such that $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ and $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ are *weak factorisation systems* (see (4.1)). The separation of results about weak factorisation systems and about model structures is not always made very clear in the present literature and we feel that it is worthwhile to first study weak factorisation systems on their own.

To do so, we first need to talk about classes defined by lifting properties and their closure properties (i.e. *saturation*). After then having defined (weak) factorisation systems and having shown the usual characterisation of such, we will follow the standard approach using small objects for their construction. These are closely related to presentable objects and indeed, this approach is at its most powerful when carried out within a locally presentable category.

1. Transfinite Composition

When localising **Top** at the weak homotopy equivalences, the idea is that every space is weakly homotopic to a CW-complex and that between CW-complexes, weak equivalences are easily understood (they are just the ordinary homotopy equivalences). To generalise this approach, we first need to study transfinite compositions in an arbitrary (cocomplete) category because after all, a CW-complex is just a transfinite composition of cell attachments. So for the rest of this section, let us fix a cocomplete category $\mathcal{C}$. All of the material could be developed equally well in a non-cocomplete context by adding suitable existence requirements everywhere.

(1.1) **Definition.** For $\lambda > 0$ an ordinal (or a well-ordered set) a λ-*sequence* in $\mathcal{C}$ is a functor $X\colon \lambda \to \mathcal{C}$ that preserves filtered colimits and is usually written as

$$X_0 \to X_1 \to \ldots \to X_\alpha \to \ldots.$$

We collectively refer to such sequences indexed by some λ as *transfinite sequences*. Because X is assumed to preserve filtered colimits, for every limit ordinal $\beta < \lambda$ the induced arrow

$$\operatorname*{colim}_{\alpha<\beta} X_\alpha = \operatorname{colim} X|_\beta \quad \longrightarrow \quad X_\beta$$

is an isomorphism and this makes the study of λ-sequences accessible to transfinite induction. If X_λ is any colimit of X, we call the 0-component of the colimiting cocone

$$X_0 \to X_\lambda$$

the *(transfinite) composition* or the *(transfinite) composite* of the sequence X. If $\mathcal{S}$ is a class of arrows in $\mathcal{C}$ and each $X_\alpha \to X_{\alpha+1}$ (for $\alpha + 1 < \lambda$) lies in $\mathcal{S}$, we say that $X_0 \to X_\lambda$ is a (transfinite) composition of arrows in $\mathcal{S}$. This need not imply that every $X_0 \to X_\lambda$ lie in $\mathcal{S}$ but if this is the case for every transfinite sequence X in $\mathcal{S}$, we say that $\mathcal{S}$ is *closed under transfinite composition*.

(1.2) **Remark.** The requirement that X preserve filtered colimits is equivalent to X preserving all colimits except the initial object and this in turn is equivalent to the requirement that if $\beta < \lambda$ is a limit ordinal then the induced arrow $\operatorname{colim}_{\alpha<\beta} X_\alpha \to X_\beta$ be an isomorphism.

(1.3) **Remark.** The notation $X_0 \to X_1 \to \ldots \to X_\alpha \to \ldots$ is a bit deceptive because one is tempted to think that we only have arrows of the form $X_\alpha \to X_{\alpha+1}$ when in fact, we have an arrow $X_\alpha \to X_\beta$ for all $\alpha \leqslant \beta$. This is the reason why in the above definition, we took a class of arrows S instead of a subcategory of $\mathcal{C}$, which could lead to confusions because then a transfinite sequence $X\colon \lambda \to \mathcal{C}$ of arrows in a subcategory $\mathcal{D}$ would not be the same as a transfinite sequence $X\colon \lambda \to \mathcal{D}$.

(1.4) **Observation.** Given a transfinite sequence $X\colon \lambda \to \mathcal{C}$ and $\beta < \lambda$, β is terminal in $[0,\beta]$ and so for $X' := X|_{[\beta,\lambda[}$ the restriction $\mathrm{Cocone}(X) \to \mathrm{Cocone}(X')$ is an isomorphism of categories, whose inverse is precomposition with the $X_\alpha \to X_\beta$ for $\alpha < \beta$. In particular, any transfinite composition $X_\beta \to \operatorname{colim} X'$ can be precomposed with $X_0 \to X_\beta$ to give a transfinite composition $X_0 \to X_\beta \to \operatorname{colim} X' \cong \operatorname{colim} X$ and conversely, any composition $X_0 \to \operatorname{colim} X$ factors as $X_0 \to X_\beta \to \operatorname{colim} X \cong \operatorname{colim} X'$.

(1.5) **Example.** Given an ordinal λ and a λ-indexed family of objects $(C_\alpha)_{\alpha<\lambda}$ in $\mathcal{C}$, coproducts over initial segements from a transfinite sequence

$$X\colon \lambda \to \mathcal{C},\ \beta \mapsto \coprod_{\alpha<\beta} C_\alpha$$

with all $X_\beta \to X_{\beta'}$ (for $\beta \leqslant \beta' < \lambda$) being standard inclusions; i.e. induced by the

$$\left(C_\alpha \xrightarrow{\mathrm{in}_\alpha} \coprod_{\alpha<\beta'} C_\alpha \right)_{\alpha<\beta}.$$

Indeed, let $\lambda' < \lambda$ be a limit ordinal, D any object and $(\gamma_\alpha\colon X_\alpha \to D)_{\alpha<\lambda'}$ a cocone. Every component of it, consists of components itself, say $\gamma_\alpha = [\gamma_{\alpha,\beta}]_{\beta<\alpha}$ and the cocone condition just says that $\gamma_{\alpha,\beta} = \gamma_{\alpha',\beta}$ for all $\beta < \alpha \leqslant \alpha'$. So, all in all, γ is just a collection of morphisms $C_\alpha \to X_{\lambda'}$ (say $\gamma_{\alpha+1,\alpha}$) and $\operatorname{colim}_{\alpha<\lambda'} X_\alpha$ is indeed the coproduct $\coprod_{\alpha<\lambda'} C_\alpha$.

As one would expect, transfinite compositions interact well with isomorphisms, identities and compositions, even again transfinite ones. At the same time, unlike ordinary compositions, transfinite ones are insensitive to inserting or deleting isomorphisms because they are defined by a universal property. The reader who is willing to accept this on faith (or deems it trivial) can safely skip to the next section.

(1.6) **Proposition.** A transfinite composition of isomorphisms is an isomorphism. More specifically, if $X\colon \lambda \to \mathcal{C}$ is a transfinite sequence where each $X_\alpha \to X_{\alpha+1}$ (with $\alpha+1 < \lambda$) is an isomorphism and $X_\lambda := \operatorname{colim} X$ then the components of a colimiting cocone are again isomorphisms. In particular, a composition $X_0 \to X_\lambda$ is again an isomorphism and conversely, any isomorphism with domain X_0 (e.g. id_{X_0}) is a composition of X because transfinite compositions are only defined up to isomorphism.

Proof. By transfinite induction over λ. If the proposition is true for all $\beta < \lambda$ then every $X_0 \to X_\beta$ is an isomorphism because X preserves colimits. Moreover, they form a cone above X and hence the inverses $X_\beta \to X_0$ form a cocone under X. It is obviously universal because any other cocone $\varphi\colon X \Rightarrow Y$ induces $\varphi_0\colon X_0 \to Y$. All in all, $\operatorname{colim} X = X_\lambda \cong X_0$. □

(1.7) **Corollary.** Isomorphisms can be eliminated from transfinite sequences. Specifically, if $X\colon \lambda \to \mathcal{C}$ is a transfinite sequence, we define an equivalence relation $\sim$ on λ by decreeing that for $\alpha < \beta$ we have

$$\alpha \sim \beta \quad \text{iff} \quad X_\gamma \to X_{\gamma+1} \text{ is an isomorphism for all } \alpha \leqslant \gamma < \beta.$$

Then $\lambda/{\sim}$ is again well-ordered and X induces a unique transfinite sequence $X'\colon \lambda/{\sim} \to \mathcal{C}$ to within isomorphism such that $X \cong X' \circ p$, where $p\colon \lambda \twoheadrightarrow \lambda/{\sim}$ is the standard projection. Moreover, p induces an equivalence of categories $\operatorname{Cocone}(X) \simeq \operatorname{Cocone}(X')$.

Proof. One readily sees that an equivalence class $S \subseteq \lambda$ of $\sim$ is either a closed interval $[\min S, \sup S]$ or of the form $[\min S, \lambda[$. It follows that the order on λ induces a well-order on $\lambda/{\sim}$ by defining $S < S'$ iff $\min S < \min S'$. Similarly, X induces a functor $X'\colon \lambda/{\sim} \to \mathcal{C}$ by sending S to $X_{\min S}$ and $p^*\colon \operatorname{Cocone}(X') \to \operatorname{Cocone}(X)$ has an inverse that sends a cocone $(\varphi_\alpha)_{\alpha<\lambda}$ to $(\varphi_{\min S})_{S\in\lambda/\sim}$. Now observe that whenever $\alpha \sim \beta$, the last proposition tells us that $X_\alpha \to X_\beta$ is an isomorphism. □

(1.8) **Remark.** Obviously, we can also selectively eliminate isomorphisms (instead of all of them) by considering some other equivalence relation $\sim$. As is visible from the proof, we only need its equivalence classes to be of the form $[\alpha, \beta]$ or $[\alpha, \lambda[$ and that if $\alpha \sim \beta$ then $X_\gamma \to X_{\gamma+1}$ is an isomorphism for all $\alpha \leqslant \gamma < \beta$.

(1.9) **Corollary.** If $X\colon \lambda \to \mathcal{C}$ is a transfinite sequence and $\lambda < \lambda'$ then X can be extended to a transfinite $X'\colon \lambda' \to \mathcal{C}$ satisfying that the restriction $\operatorname{Cocone}(X') \to \operatorname{Cocone}(X)$ is an isomorphism. In particular, X' has the same compositions as X.

Proof. We obviously put $X'|_\lambda := X$, $X'_\lambda := \operatorname{colim} X$ and $X'|_{\{\alpha \mid \lambda<\alpha<\lambda'\}} := \operatorname{Const}_{X_\lambda}$ constant. It follows from the previous proposition and (1.4) that this is a transfinite sequence and that the restriction $\operatorname{Cocone}(X') \to \operatorname{Cocone}(X)$ is an isomorphism. □

By this corollary, if we have a (set-indexed) family $(X^i\colon \lambda_i \to \mathcal{C})_{i\in I}$ of transfinite sequences, we can always assume that $\lambda_i = \sup_{j\in I} \lambda_j = \bigcup_{j\in I} \lambda_j$ for all $i \in I$ by extending each X^i as in the corollary.

(1.10) **Proposition.** Let $X\colon \lambda \to \mathcal{C}$ be a transfinite sequence such that each $X_\beta \to X_{\beta+1}$ with $\beta + 1 < \lambda$ is a composition of a transfinite sequence $Y^\beta\colon \mu_\beta \to \mathcal{C}$. If we well-order

$$\nu := \{(\beta, \gamma) \mid \beta < \lambda,\ \gamma < \mu_\beta\}$$

by the lexicographic order, we gain a new transfinite sequence $Y\colon \nu \to \mathcal{C}$ by

$$(\beta, \gamma) < (\beta', \gamma') \mapsto \begin{cases} Y^\beta_\gamma \to Y^\beta_{\gamma'} & \beta = \beta' \\ Y^\beta_\gamma \to Y^{\beta+1}_0 = X_{\beta+1} \to X_{\beta'} = Y^{\beta'}_0 \to Y^{\beta'}_{\gamma'} & \beta < \beta'. \end{cases}$$

Furthermore, the restriction $\operatorname{Cocone}(Y) \to \operatorname{Cocone}(X)$, induced by $\lambda \cong \lambda \times \{0\} \hookrightarrow \nu$, is an isomorphism of categories.

Proof. Notice that $\{(\beta,0)\}_{\beta<\lambda}$ is final in ν and so $\mathrm{Cocone}(Y) \to \mathrm{Cocone}(X)$ is indeed an isomorphism of categories as claimed. As for transfiniteness, if $(\beta,\gamma) \in \nu$ is a limit element with $\gamma > 0$ then $(\beta,0)$ is terminal in $\nu_{\leqslant(\beta,0)}$, so that $\operatorname{colim}_{(\alpha_1,\alpha_2)<(\beta,\gamma)} Y^{\alpha_1}_{\alpha_2} = \operatorname{colim}_{\alpha<\gamma} Y^{\beta}_{\alpha}$ and Y^{β} preserves filtered colimits. If $\gamma = 0$ and $\beta = (\beta-1)+1$ is a successor ordinal then we can argue analogously with $Y^{\beta-1}$ instead of Y^{β}. Finally, if $\gamma = 0$ and β is a limit ordinal then $\operatorname{colim}_{(\alpha_1,\alpha_2)<(\beta,0)} Y^{\alpha_1}_{\alpha_2} = \operatorname{colim}_{\alpha<\beta} Y^{\alpha}_0 = \operatorname{colim}_{\alpha<\beta} X_\alpha$ by the initial remark. □

(1.11) **Remark.** In the statement of the proposition, it is sometimes convenient to use (1.7) (or rather (1.8)) and take the quotient of ν by the equivalence relation that identifies $(\beta,\gamma-1)$ with $(\beta+1,0)$ whenever $\gamma = (\gamma-1)+1$ is a successor ordinal because for example if we split up $X_0 \to X_1 \to X_2$ into $X_0 \to X_1$ and $X_1 \to X_2$ and reassemble them according to the proposition, we get $X_0 \to X_1 \to X_1 \to X_2$ with the middle arrow the identity.

2. Lifting Properties

Most students that have had an introduction to algebraic topology (or more specifically homotopy theory) probably feel that lifting properties are really important, while at the same time being confused by the sheer number of them. In this section, we are going to fix a concise notation to work with them and prove some elementary results. We will always be working in some fixed category $\mathcal{C}$.

(2.1) **Definition.** Given two morphisms $f\colon C \to C'$, $g\colon D \to D'$ in a category $\mathcal{C}$, we write $f \pitchfork g$ and say that f is *left transverse* to g (or has the *left lifting property* with respect to g) or that g is *right transverse* to f (or has the *right lifting property* with respect to f) iff every commutative square of solid arrows

$$\begin{tikzcd} C \arrow[r, "a"] \arrow[d, "f"'] & D \arrow[d, "g"] \\ C' \arrow[r, "b"'] \arrow[ur, dotted, "d"] & D' \end{tikzcd}$$

has a *diagonal filler* d making the diagram commute. If in addition, this diagonal filler is always unique, we write $f \perp g$ and say that f is *left orthogonal* to g or that g is *right orthogonal* to f. More generally, for two classes of arrows $\mathcal{A}$, $\mathcal{B}$ in $\mathcal{C}$, we write $\mathcal{A} \pitchfork \mathcal{B}$ and $\mathcal{A} \perp \mathcal{B}$ to indicate respectively that $a \pitchfork b$ and $a \perp b$ for all $a \in \mathcal{A}$, $b \in \mathcal{B}$. Finally, given a class $\mathcal{A}$ of arrows in $\mathcal{C}$, we write

(a) ${}^{\pitchfork}\mathcal{A} := \mathcal{A}\text{-proj} := \{f \in \mathcal{C} \mid f \pitchfork g\ \forall g \in \mathcal{A}\}$ for the class of $\mathcal{A}$-*projective arrows*;

(b) $\mathcal{A}^{\pitchfork} := \mathcal{A}\text{-inj} := \{g \in \mathcal{C} \mid f \pitchfork g\ \forall f \in \mathcal{A}\}$ for the class of $\mathcal{A}$-*injective arrows*;

(c) $\mathcal{A}\text{-fib} := ({}^{\pitchfork}\mathcal{A})^{\pitchfork}$ for the class of $\mathcal{A}$-*fibrations*;

(d) $\mathcal{A}\text{-cof} := {}^{\pitchfork}(\mathcal{A}^{\pitchfork})$ for the class of $\mathcal{A}$-*cofibrations*;

(e) ${}^{\perp}\mathcal{A} := \{f \in \mathcal{C} \mid f \perp g\ \forall g \in \mathcal{A}\}$;

(f) $\mathcal{A}^{\perp} := \{g \in \mathcal{C} \mid f \perp g\ \forall f \in \mathcal{A}\}$.

(2.2) **Definition.** Similarly, for an arrow $f\colon C \to C'$ and an object D in a category $\mathcal{C}$, we write $f \pitchfork D$ and say that f is *left transverse* to D iff every arrow $a\colon C \to D$ factors through f:

$$\begin{tikzcd} C \arrow[r, "a"] \arrow[d, "f"'] & D \\ C' \arrow[ur, dotted] & \end{tikzcd} .$$

Note that if $\mathcal{C}$ has a terminal object $*$, this is equivalent to requiring that $f \pitchfork (D \to *)$. Just like before, if in addition, the above factorisation is unique, we write $f \perp D$ and say that f is *left orthogonal* to D. Dually for $D \pitchfork f$, $D \perp f$.

(2.3) **Observation.** The two pairs of operations $({}^{\pitchfork}-, -^{\pitchfork})$ and $({}^{\perp}-, -^{\perp})$ both form an antitone Galois connection

$$\mathfrak{P}(\operatorname{Arr}\mathcal{C}) \rightleftarrows \mathfrak{P}(\operatorname{Arr}\mathcal{C}) \qquad \text{(where } \mathfrak{P} \text{ should be understood at the meta-level)},$$

meaning that they are order-reversing and $\mathcal{A} \subseteq {}^{\pitchfork}(\mathcal{A}^{\pitchfork})$, $\mathcal{A} \subseteq ({}^{\pitchfork}\mathcal{A})^{\pitchfork}$ for all classes of arrows $\mathcal{A}$ (similarly for $\perp$) or equivalently $\mathcal{A} \subseteq {}^{\pitchfork}\mathcal{B}$ iff $\mathcal{B} \subseteq \mathcal{A}^{\pitchfork}$ for any two classes of arrows $\mathcal{A}$, $\mathcal{B}$. In particular, ${}^{\pitchfork}\mathcal{A} = {}^{\pitchfork}(({}^{\pitchfork}\mathcal{A})^{\pitchfork}) = {}^{\pitchfork}(\mathcal{A}\text{-fib})$ and $\mathcal{A}^{\pitchfork} = \left({}^{\pitchfork}(\mathcal{A}^{\pitchfork})\right)^{\pitchfork} = (\mathcal{A}\text{-cof})^{\pitchfork}$ (similarly for $\perp$).

(2.4) **Example.** If $\mathcal{A}$ is any class of arrows, then any isomorphism in $\mathcal{C}$ is contained in each of the six classes from (2.1).

(2.5) **Example.** For an arrow a in a category $\mathcal{C}$

$$a \pitchfork a \;\Leftrightarrow\; a \perp a \;\Leftrightarrow\; a \in {}^{\pitchfork}\mathcal{C} \;\Leftrightarrow\; a \in {}^{\perp}\mathcal{C} \;\Leftrightarrow\; a \text{ is an isomorphism}$$

(where we identified $\mathcal{C}$ with its class of arrows) and dually for right transverse/orthogonal.

(2.6) **Example.** For a category $\mathcal{C}$, let's write $\mathcal{M}$ for the class of all its monomorphisms and $\mathcal{E}$ for the class of all its epimorphisms. A monomorphism in $\mathcal{E}^{\pitchfork}$ (or equivalently $\mathcal{E}^{\perp}$) is called *strong*. Dually, an epimorphism in ${}^{\pitchfork}\mathcal{M}$ (or equivalently ${}^{\perp}\mathcal{M}$) is again called *strong*.

(2.7) **Example.** Let $\mathcal{J}$ be the interval groupoid with two objects $0, 1$ and exactly two non-identity morphisms $0 \to 1$, $1 \to 0$. A functor $F\colon \mathcal{C} \to \mathcal{D}$ that is right transverse to $\{0\} \hookrightarrow \mathcal{J}$ is usually called an *isofibration*. Explicitly, this means that for every isomorphism $g\colon D \to D'$ in $\mathcal{D}$ and every $C \in \mathcal{C}$ with $FC = D$ there is some isomorphism $f\colon C \to C'$ in $\mathcal{C}$ such that $Ff = g$.

A fundamental observation now is that classes of arrows determined by lifting properties have nice closure properties; namely, given any class $\mathcal{A}$ of arrows, ${}^{\pitchfork}\mathcal{A}$ and ${}^{\perp}\mathcal{A}$ are closed under coproducts, pushouts along arbitrary arrows, retracts and transfinite compositions.

(2.8) **Definition.** We call two arrows in $\mathcal{C}$ *isomorphic* iff they are isomorphic as objects in $\mathcal{C}^{\to}$. Similarly, an arrow $f\colon X \to Y$ in a category $\mathcal{C}$ is called a *retract* of an arrow $g\colon X' \to Y'$ iff it is a retract in the arrow category $\mathcal{C}^{\to}$ of $\mathcal{C}$. More generally (at least for $\mathcal{C}$ cocomplete where we can put $A = \varnothing$) if we also have arrows $i\colon A \to X$ and $i'\colon A \to X'$ we say that f is a retract of g *relative to* (i, i') (or just *relative to* A if i and i' are clear from the context) iff there is a commutative diagram

$$\begin{tikzcd}
& A \arrow[dl, "i"'] \arrow[d, "i'"] \arrow[dr, "i"] & \\
X \arrow[r] \arrow[d, "f"'] & X' \arrow[r] \arrow[d, "g"] & X \arrow[d, "f"] \\
Y \arrow[r] & Y' \arrow[r] & Y
\end{tikzcd},$$

in which the composites of the two rows are identities. Dually, one defines a retract *corelative* to (p, p') where $p\colon Y \to Q$, $p'\colon Y' \to Q$. For $X = X'$ we say that f is a *strong retract* of g iff it is a retract relative to $(\mathrm{id}_X, \mathrm{id}_X)$ (so that the top row in the above diagram consists of identities only). And dually for a *costrong retract*.

(2.9) **Example.** If $f = g \circ h$ with g an isomorphism then f and h are isomorphic:

$$\begin{tikzcd}
C \arrow[r, equal] \arrow[d, "h"'] & C \arrow[d, "f"] \\
D' \arrow[r, "g"'] & D
\end{tikzcd}.$$

Dually, if $f = g \circ h$ with h an isomorphism then f and g are isomorphic.

(2.10) **Example.** Given an arrow $f\colon C \to D$ together with a retract $s\colon C' \to C$ (i.e. s has a retraction r) then the restriction $g := f \circ s$ is a costrong retract of f if $f = g \circ r$ (i.e. f is an extension of g by means of r):

$$\begin{tikzcd}
C' \arrow[r, "s"] \arrow[d, "g"'] & C \arrow[r, "r"] \arrow[d, "f"] & C' \arrow[d, "g"] \\
D \arrow[r, equal] & D \arrow[r, equal] & D
\end{tikzcd}.$$

Dually, given a retract $s\colon D' \to D$ with retraction r then f is a strong retract of $g := r \circ f$ if f factorises through D' as $f = s \circ g$.

(2.11) **Example.** As a special case of the last example, given a retract $s\colon C \to D$ with retraction $r\colon D \to C$ then r is a strong retract of $s \circ r$ and s is a costrong retract of $s \circ r$:

$$\begin{tikzcd}
D \arrow[r, equal] \arrow[dd, "r"'] & D \arrow[r, equal] \arrow[d, "r"'] & D \arrow[dd, "r"] \\
& C \arrow[d, "s"'] & \\
C \arrow[r, "s"'] & D \arrow[r, "r"'] & C
\end{tikzcd}, \qquad
\begin{tikzcd}
C \arrow[r, "s"] \arrow[dd, "s"'] & D \arrow[r, "r"] \arrow[d, "r"'] & C \arrow[dd, "s"] \\
& C \arrow[d, "s"'] & \\
D \arrow[r, equal] & D \arrow[r, equal] & D
\end{tikzcd}$$

For convenience, let's introduce names for classes of arrows (such as ${}^{\pitchfork}\mathcal{A}$ or ${}^{\perp}\mathcal{A}$) that are closed under certain constructions. The following is a version of definition [27, IV.2.1] due to Gabriel and Zisman, which we adapted to our more general context.

(2.12) **Definition.** A class of arrows $\mathcal{A}$ is called *cellularly saturated* iff it contains all isomorphisms and is closed under cobase change (i.e. pushouts along arbitrary arrows) and transfinite compositions. It is called *saturated* iff it is in addition closed under retracts.

(2.13) **Remark.** In the definition, "all isomorphisms" can be replaced by "all identities" if at least one of the other two requirements for a cellularly saturated class are read properly in the sense that *any* cobase change or *any* composition must lie in $\mathcal{A}$ (not just those obtained by a fixed colimit functor).

(2.14) **Example.** Within any category $\mathcal{C}$, the class of all isomorphisms is saturated. The closure under cobase change is easy and the closure under transfinite compositions was (1.6). Finally, given a commutative diagram

$$\begin{tikzcd} A \arrow[r, "s"] \arrow[d, "f"'] & B \arrow[r, "r"] \arrow[d, "g"', "\cong"] & A \arrow[d, "f"'] \\ A' \arrow[r, "s'"'] & B' \arrow[r, "r'"'] & A' \end{tikzcd}$$

with the two horizontal composites identities and g an isomorphism, then $r \circ g^{-1} \circ s'$ is an inverse for f.

Of course the adjective *cellular* comes from the motivating example of relative cell complexes in **Top**, which are morphisms obtained by gluing a (possibly infinite) number of cells (not necessarily in order of their dimension). Having this in mind, we should also require closedness under coproducts but alas, this is a consequence of the other two requirements (when assuming the axiom of choice), which is shown in the next section. For the time being, we will just show it anyway because a direct verification is easy in our case.

(2.15) **Proposition.** If $\mathcal{C}$ is a category and $\mathcal{A}$ a class of arrows in $\mathcal{C}$ then ${}^{\pitchfork}\mathcal{A}$ and ${}^{\perp}\mathcal{A}$ are saturated (the axiom of choice is needed for ${}^{\pitchfork}\mathcal{A}$).

Proof. The closure under coproducts is immediate. Indeed, let $(i_k \colon A_k \to B_k)_{k \in K}$ be a family of arrows in ${}^{\pitchfork}\mathcal{A}$ (or ${}^{\perp}\mathcal{A}$), whose coproduct $i := \coprod_k i_k \colon \coprod_k A_k \to \coprod_k B_k$ exists and consider a commutative square of solid arrows on the left

$$\begin{tikzcd} \coprod_{k\in K} A_k \arrow[r, "{[a_k]_{k\in K}}"] \arrow[d, "\coprod_k i_k"'] & X \arrow[d, "f"] \\ \coprod_{k\in K} B_k \arrow[r, "{[b_k]_{k\in K}}"'] \arrow[ur, dotted, "d"'] & Y \end{tikzcd} \qquad \begin{tikzcd} A_k \arrow[r, "a_k"] \arrow[d, "i_k"'] & X \arrow[d, "f"] \\ B_k \arrow[r, "b_k"'] \arrow[ur, dotted, "d_k"'] & Y \end{tikzcd}$$

where $f \in \mathcal{A}$. In the case of ${}^{\perp}\mathcal{A}$ there is a unique diagonal filler d_k, one for each $k \in K$, as on the right and in the case of ${}^{\pitchfork}\mathcal{A}$, we use the axiom of choice. These then assemble to a diagonal filler $d := [d_k]_{k\in K}$ as on the left. Next, suppose we are given a commutative diagram of solid arrows

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & B \arrow[r] \arrow[d, "j"] & C \arrow[d, "f"] \\ D \arrow[r] \arrow[urr, dotted] & E \arrow[r] \arrow[ur, dotted] \arrow[ul, phantom, "\ulcorner", very near start] & F \end{tikzcd}$$

with $i \in {}^{\pitchfork}\mathcal{A}$ (or ${}^{\perp}\mathcal{A}$) and $f \in \mathcal{A}$. From this we get a diagonal filler $D \dashrightarrow C$ for the outer rectangle (unique in the case of ${}^{\perp}\mathcal{A}$) and the universal property of the pushout induces a (unique) diagonal filler $E \dashrightarrow C$ for the right-hand square. Finally, $E \dashrightarrow C \to F = E \to F$ follows from the universal property of the pushout. More specifically, we have

$$D \longrightarrow E \dashrightarrow C \xrightarrow{f} F \;=\; D \dashrightarrow C \xrightarrow{f} F \quad \text{and} \quad B \xrightarrow{j} E \dashrightarrow C \xrightarrow{f} F \;=\; B \longrightarrow C \xrightarrow{f} F$$

and the same for $E \dashrightarrow C \xrightarrow{f} E$ replaced by $E \to F$. For retracts, consider a commutative diagram of solid arrows in which $r_1 \circ s_1 = \mathrm{id}_C$, $r_2 \circ s_2 = \mathrm{id}_D$, $i \in {}^{\pitchfork}\mathcal{A}$ (or ${}^{\perp}\mathcal{A}$) and $f \in \mathcal{A}$

$$\begin{tikzcd} C \arrow[r, "s_1"] \arrow[d, "j"'] & A \arrow[r, "r_1"] \arrow[d, "i"'] & C \arrow[r, "a"] \arrow[d, "j"'] & X \arrow[d, "f"] \\ D \arrow[r, "s_2"'] & B \arrow[r, "r_2"'] \arrow[urr, dotted, "d"] & D \arrow[r, "b"'] \arrow[ur, dotted, "e"'] & Y \end{tikzcd} .$$

By hypothesis, we find a (unique) diagonal filler $d\colon B \dashrightarrow X$ for the right-hand rectangle. We then put $e := d \circ s_2$, so that

$$e \circ j = d \circ s_2 \circ j = d \circ i \circ s_1 = a \circ r_1 \circ s_1 = a$$

and similarly

$$f \circ e = f \circ d \circ s_2 = b \circ r_2 \circ s_2 = b.$$

In the case of ${}^{\perp}\mathcal{A}$, this diagonal filler e is unique because if e' is any one such, we must have $r_2 \circ e' = d$ (by uniqueness of d) and therefore $e' = s_2 \circ r_2 \circ e' = s_2 \circ d = e$.

Finally, let $X\colon \lambda \to \mathcal{C}$ be a transfinite sequence with composition X_λ and every $X_\alpha \to X_{\alpha+1}$ in ${}^{\pitchfork}\mathcal{A}$ (resp. ${}^{\perp}\mathcal{A}$), $a\colon A \to B$ an arrow in $\mathcal{A}$ and $f\colon X_0 \to A$, $g\colon X_\lambda \to B$ such that $a \circ f = g \circ (X_0 \to X_\lambda)$. The construction of a diagonal filler is equivalent to constructing a cocone $(\varphi_\beta\colon X_\beta \to A)_{\beta \leqslant \lambda}$ all of whose components φ_β make the following diagram commute

$$\begin{tikzcd} X_0 \arrow[rr, "f"] \arrow[d] & & A \arrow[d] \\ X_\beta \arrow[r] \arrow[urr, dotted, "\varphi_\beta"] & X_\lambda \arrow[r, "g"'] & B \end{tikzcd} .$$

We can define such a cocone by transfinite recursion over $\beta \leqslant \lambda$. Obviously, we put $\varphi_0 := f$ and if $\beta = (\beta - 1) + 1$ is a successor, we can choose a lift φ_β as in the following diagram:

$$\begin{tikzcd} X_{\beta-1} \arrow[rr, "\varphi_{\beta-1}"] \arrow[d] & & A \arrow[d] \\ X_\beta \arrow[r] \arrow[urr, dotted, "\varphi_\beta"] & X_\lambda \arrow[r, "g"'] & B \end{tikzcd} .$$

Finally, if β is a limit ordinal, we let φ_β be the unique arrow induced by the $(\varphi_\alpha)_{\alpha<\beta}$. Clearly, if each $X_\alpha \to X_{\alpha+1}$ lies in ${}^{\perp}\mathcal{A}$ then the φ_α are unique and furthermore, as already indicated, any diagonal filler $d\colon X_\lambda \to A$ determines a cocone, so that φ_λ above is unique. □

(2.16) **Remark.** The last part in the above proof wasn't a mere transfinite recursion. In fact, it was a form of transfinite dependent choice because each successor step involved the choice of a lift. Formally, for every successor ordinal $\beta = (\beta - 1) + 1 \leqslant \lambda$ and every $\varphi\colon X_{\beta-1} \to A$ the class (or set for $\mathcal{C}$ locally small)

$$C_{\beta,\varphi} := \left\{ \psi\colon X_\beta \to A \;\middle|\; \begin{tikzcd} X_{\beta-1} \arrow[rr, "\varphi"] \arrow[d] & & A \arrow[d] \\ X_\beta \arrow[r] \arrow[urr, dotted, "\psi"] & X_\lambda \arrow[r, "g"'] & B \end{tikzcd} \text{ commutes} \right\}$$

is non-empty and we can choose $(\psi_{\beta,\varphi})_{\beta,\varphi} \in \prod_{\beta,\varphi} C_{\beta,\varphi}$. Now in the successor case of the above proof, we can simply define $\varphi_\beta := \psi_{\beta,\varphi_{\beta-1}}$.

(2.17) **Remark.** Note that in the dual claim, $\mathcal{A}^{\pitchfork}$ and $\mathcal{A}^{\perp}$ are closed under products, base change, retracts but not transfinite compositions. Rather, they are closed under reverse transfinite compositions; i.e. taking limits of cofiltered limit preserving functors $X\colon \lambda^{\mathrm{op}} \to \mathcal{C}$ with each $X_\alpha \to X_{\alpha+1}$ respectively in $\mathcal{A}^{\pitchfork}$ and $\mathcal{A}^{\perp}$.

(2.18) **Corollary.** For any class of arrows $\mathcal{A}$, the classes ${}^{\pitchfork}\mathcal{A}$, ${}^{\perp}\mathcal{A}$, $\mathcal{A}^{\pitchfork}$ and $\mathcal{A}^{\perp}$ are replete (as classes of objects in $\mathcal{C}^{\to}$).

Proof. Any commutative square of solid arrows

$$\begin{tikzcd} C \arrow[r, "a"] \arrow[d, "f"'] & C' \arrow[r, dotted, "a^{-1}"] \arrow[d, "f'"] & C \arrow[d, dotted, "f"] \\ D \arrow[r, "b"'] & D' \arrow[r, dotted, "b^{-1}"'] & D \end{tikzcd}$$

with a, b isomorphisms and f' in one of the four classes from the proposition can be completed to a rectangle exhibiting f as a retract of f'. □

3. Saturated Classes

Recall from (2.12), that a class $\mathcal{A}$ of arrows is cellularly saturated iff it contains all isomorphisms and is closed under cobase changes and transfinite compositions. It is even saturated iff it is also closed under retracts.

(3.1) **Definition.** Given a class $\mathcal{A}$ of arrows in a category, its *cellular saturation* (if it exists) is the smallest cellularly saturated class containing $\mathcal{A}$. Similarly its *saturation* (if it exists) is the smallest saturated class containing $\mathcal{A}$.

In this section, we are first going to show that cellularly saturated classes are also closed under coproducts (which we already alluded to in the previous section). We then go on to explicitly construct (cellular) saturations of any class of arrows. Again, let us fix some cocomplete category $\mathcal{C}$, in which we are working. On several occasions, we will need the following trivial observation.

(3.2) **Observation.** Since colimits in $\mathcal{C}$ commute, if we are given an ordinal λ and a commutative pushout square of sequences $\lambda \to \mathcal{C}$

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & Y' \arrow[d] \\ X \arrow[r] \arrow[ur, phantom, "\ulcorner"] & Y \end{tikzcd}$$

(i.e. it is a componentwise pushout)

with X', Y' and X transfinite, then Y is transfinite, too, and the square of colimits is again a pushout.

To prove the closure of cellularly saturated classes under coproducts, we are going to show more generally how cobase changes of coproducts can be obtained as transfinite compositions. The idea is to simply well-order the indexing set and glue on the summands successively instead of all at once. So, starting from a span

$$\begin{tikzcd} \coprod_{k\in K} C_k \arrow[r, "f={[f_k]_k}"] \arrow[d, "\coprod_k i_k"'] & Y_0 \\ \coprod_{k\in K} D_k & \end{tikzcd} ,$$

we well-order K using the axiom of choice, identify it with its order type λ and make X_0 the initial vertex of a diagram

$$Y\colon \lambda \to \mathcal{C},\ \beta \mapsto \operatorname{colim}\left(\coprod_{\alpha<\beta} D_\alpha \xleftarrow{\coprod_\alpha i_\alpha} \coprod_{\alpha<\beta} C_\alpha \xrightarrow{[f_\alpha]_\alpha} Y_0\right) =: Y_\beta$$

with all $Y_\beta \to Y_{\beta'}$ induced by the obvious morphisms of spans. For $\alpha < \beta$, let us write $g_\alpha^\beta\colon D_\alpha \to Y_\alpha$ for the induced morphism into the pushout so that $g_\alpha^{\beta'} = Y(\beta \leqslant \beta') \circ g_\alpha^\beta$ for $\alpha < \beta \leqslant \beta'$.

(3.3) **Proposition.** Cobase changes of coproducts can be calculated as a transfinite composition, gluing one summand at a time. More precisely, in the above situation, Y is transfinite and every $Y_{\alpha+1}$ fits into a pushout square

$$\begin{tikzcd} C_\alpha \arrow[r, "f_\alpha"] \arrow[d, "i_\alpha"'] & Y_0 \arrow[r] & Y_\alpha \arrow[d] \\ D_\alpha \arrow[rr, "g_\alpha^{\alpha+1}"'] & & Y_{\alpha+1} \end{tikzcd} .$$

Proof. The transfiniteness of Y follows immediately from the above observation (3.2) where we take $Y' := Y_0$ to be constant and X', X the transfinite sequences of initial segments (cf. (1.5)). The second claim is an easy diagrammatic exercise but we shall do it nevertheless. We need to show that if we have a pushout square

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "j"'] & Z \arrow[d, "\mathrm{in}_2"] \\ B \arrow[r, "\mathrm{in}_1"'] & P \arrow[ul, phantom, "\ulcorner" very near start] \end{tikzcd}$$

then a rectangle of the form

$$\begin{tikzcd} A' \arrow[r, "h'"] \arrow[d, "j'"'] & Z \arrow[r, "\mathrm{in}_2"] & P \arrow[d, "\mathrm{in}'_2"] \\ B' \arrow[rr, "\mathrm{in}'_1"'] & & Q \end{tikzcd} \quad \text{is a pushout iff} \quad \begin{tikzcd} A + A' \arrow[r, "{[h,h']}"] \arrow[dd, "j+j'"'] & Z \arrow[d, "\mathrm{in}_2"] \\ & P \arrow[d, "\mathrm{in}'_2"] \\ B + B' \arrow[r, "{[\mathrm{in}'_2\circ \mathrm{in}_1, \mathrm{in}'_1]}"'] & Q \end{tikzcd} \quad \text{is one.}$$

For this, we fix an object E and observe that in both cases, a cocone consists of three arrows

$$B \xrightarrow{\varphi} E, \quad B' \xrightarrow{\psi} E \quad \text{and} \quad Z \xrightarrow{\chi} E,$$

which, for the left-hand rectangle, are required to satisfy

$$\varphi \circ j = \chi \circ h \text{ (to get } [\varphi,\chi]\colon P \to E) \qquad \text{and} \qquad \psi \circ j' = [\varphi,\chi] \circ \mathrm{in}_2 \circ h' = \chi \circ h',$$

while, for the right-hand rectangle, they need to satisfy

$$[\varphi,\psi] \circ (j + j') = \chi \circ [h, h']; \qquad \text{i.e. again} \qquad \varphi \circ j = \chi \circ h \text{ and } \psi \circ j' = \chi \circ h'.$$

So the two spans have isomorphic pushouts and assuming that the two arrows $\mathrm{in}'_1\colon B' \to Q$ and $\mathrm{in}'_2 = [\mathrm{in}'_{2,1}, \mathrm{in}'_{2,2}]\colon P \to Q$ are colimiting, they correspond to the (again colimiting) cocone $[\mathrm{in}'_{2,1}, \mathrm{in}'_1]\colon B + B' \to Q$ and $\mathrm{in}'_{2,2}\colon Z \to Q$ as claimed. □

(3.4) **Corollary.** Granted the axiom of choice, every cellularly saturated class $\mathcal{A}$ is closed under coproducts. More explicitly, given a set-indexed family $(i_k\colon C_k \to D_k)_{k\in K}$ of arrows in $\mathcal{A}$ then $\coprod_{k\in K} i_k\colon \coprod_{k\in K} C_k \to \coprod_{k\in K} D_k$ lies again in $\mathcal{A}$.

Proof. We use the axiom of choice to well-order K and apply the above proposition to the case $X_0 := \coprod_{k\in K} D_k$ and $f = \coprod_{k\in K} i_k$. □

Next, let us construct the cellular saturation of a class $\mathcal{A}$. If one is willing to work within a Grothendieck universe, one can simply define it as the intersection of all cellularly saturated "classes" containing $\mathcal{A}$ (which is what is usually done in the literature [27; 35]). Indeed, such a class always exists (e.g. ${}^{\pitchfork}(\mathcal{A}^{\pitchfork})$) and so we do stay inside our universe. However, this approach has two downsides. Firstly, it is not an explicit description and secondly, it cannot be formulated within NBG class theory.

(3.5) **Definition.** Let $\mathcal{A}$ be a class of arrows in $\mathcal{C}$. A *relative $\mathcal{A}$-cell complex* is a transfinite composition of $\mathcal{A}$-pushouts. That is, it's a composition $X_0 \to X_\lambda$ of a transfinite sequence $X\colon \lambda \to \mathcal{C}$ such that for each α with $\alpha + 1 < \lambda$ there is a pushout square

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f_\alpha"'] & X_\alpha \arrow[d, "X_{\alpha<\alpha+1}"] \\ B \arrow[r] & X_{\alpha+1} \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

with $f_\alpha \in \mathcal{A}$. We denote the class of all relative $\mathcal{A}$-cell complexes by $\mathrm{Cell}(\mathcal{A})$. An *(absolute) $\mathcal{A}$-cell complex* is an $X \in \mathcal{C}$ such that the unique arrow $0 \to X$ from the initial object is a relative $\mathcal{A}$-cell complex.

(3.6) **Example.** By putting $\lambda := 1$, we immediately see that any isomorphism is a relative $\mathcal{A}$-cell complex for any class of arrows $\mathcal{A}$. To wit, a diagram $X\colon 1 \to \mathcal{C}$ is just a single object X_0 (so that there is no cell complex condition to check) and any isomorphism $X_0 \cong C$ is a transfinite composition (i.e. a colimit) of X.

(3.7) **Example.** Also, if $X\colon \lambda \to \mathcal{C}$ is a transfinite composition of arrows in $\mathcal{A}$ then a composite $X_0 \to X_\lambda$ is obviously a relative $\mathcal{A}$-cell complex. In particular $\mathcal{A} \subseteq \mathrm{Cell}(\mathcal{A})$.

(3.8) **Example.** More concretely, if $\mathcal{C} = \mathbf{Top}$ and $\mathcal{A} = \{S^{n-1} \hookrightarrow D^n \mid n \in \mathbb{N}\}$ (with $S^{-1} := \varnothing$) then $\mathrm{Cell}(\mathcal{A})$ is the class of relative cell complexes in the usual sense.

We shall show next that $\mathrm{Cell}(\mathcal{A})$ is the cellular saturation of $\mathcal{A}$. As noted in the first two examples above, $\mathcal{A}$ as well as all isomorphisms lie in $\mathrm{Cell}(\mathcal{A})$. Moreover, $\mathrm{Cell}(\mathcal{A})$ is clearly closed under transfinite compositions by (1.10) and we only need to check that it is closed under cobase change.

(3.9) **Proposition. (Transfinite Pushout Lemma)** Let $X\colon \lambda \to \mathcal{C}$ be a transfinite sequence, $Y\colon \lambda \to \mathcal{C}$ and $\varphi\colon X \Rightarrow Y$. If $Y \cong Y_0 +_{X_0} X$ and the $\varphi_\alpha\colon X_\alpha \to Y_\alpha \cong Y_0 +_{X_0} X_\alpha$ are the canonical morphisms then Y is a transfinite sequence and every square

$$\begin{tikzcd} X_\alpha \arrow[r, "\varphi_\alpha"] \arrow[d] & Y_\alpha \arrow[d] \\ X_\beta \arrow[r, "\varphi_\beta"'] & Y_\beta \end{tikzcd}$$

with $\alpha < \beta < \lambda$ is a pushout. Conversely, if Y is a transfinite sequence and the above square is a pushout for all $\beta = \alpha + 1 < \lambda$ then $Y \cong Y_0 +_{X_0} X$ and the φ_α are the canonical morphisms.

Proof. All squares from the proposition are pushouts by the ordinary pushout lemma and the transfiniteness of Y is a special case of the observation (3.2) above, where $X' = X_0$ and $Y' = Y_0$ are constant. The converse claim is by transfinite induction on $\beta < \lambda$, again using the pushout lemma for β a successor and else

$$Y_\beta \cong \operatorname*{colim}_{\alpha<\beta} Y_\alpha \cong \operatorname*{colim}_{\alpha<\beta}(Y_0 +_{X_0} X_\alpha) \cong Y_0 +_{X_0} X_\beta,$$

where the first isomorphism is by hypothesis, the second by the inductive hypothesis and the last by what we have already shown, namely that $\alpha \mapsto Y_0 +_{X_0} X_\alpha$ is transfinite. □

(3.10) **Proposition.** The class $\mathrm{Cell}(\mathcal{A})$ is closed under cobase change.

Proof. Let $X_0 \to X_\lambda$ a composition of a transfinite sequence $X\colon \lambda \to \mathcal{C}$ of $\mathcal{A}$-pushouts and $f\colon X_0 \to Y_0$. By the transfinite pushout lemma, the functor $Y\colon \lambda \to \mathcal{C}$ defined by $Y_\alpha := Y_0 +_{X_0} X_\alpha$ is a transfinite sequence and

$$Y_0 +_{X_0} \operatorname{colim} X = Y_0 +_{X_0} \operatorname*{colim}_{\alpha<\lambda} X_\alpha \cong \operatorname*{colim}_{\alpha<\lambda}(Y_0 +_{X_0} X_\alpha) = \operatorname{colim} Y,$$

so that the claim follows by transfinite induction on λ and the above observation that $\mathrm{Cell}(\mathcal{A})$ is closed under transfinite compositions. □

(3.11) **Corollary.** The class $\mathrm{Cell}(\mathcal{A})$ is the cellular saturation of $\mathcal{A}$ (i.e. the smallest cellularly saturated class containing $\mathcal{A}$). □

Finally, we go even a step further and construct the saturation $\mathrm{Sat}(\mathcal{A})$ of a class $\mathcal{A}$ in the obvious manner.

(3.12) **Definition.** For a class of arrows $\mathcal{A}$, we define $\mathrm{Sat}(\mathcal{A})$ to be the class of all retracts of arrows in $\mathrm{Cell}(\mathcal{A})$.

Again, $\mathrm{Sat}(\mathcal{A})$ clearly contains $\mathcal{A}$ as well as all isomorphisms and it is closed under retracts (the retract of a retract is a retract). So we just need to check that it is closed under cobase change and transfinite compositions.

(3.13) **Proposition.** $\mathrm{Sat}(\mathcal{A})$ is closed under cobase change and transfinite compositions (where for transfinite compositions, the axiom of choice is needed).

Proof. For every arrow $X_0 \to X_\lambda$ with a retract $A \to B$ and every $A \to C$ we consider

$$\begin{tikzcd}
A \arrow[rr] \arrow[dr] \arrow[dd] & & X_0 \arrow[rr] \arrow[dr] \arrow[dd] & & A \arrow[dr] \arrow[dd] & \\
& C \arrow[rr, crossing over] & & Y_0 \arrow[rr, dotted] & & C \arrow[dd] \\
B \arrow[rr] \arrow[dr] & & X_\lambda \arrow[rr] \arrow[dr] & & B \arrow[dr] & \\
& D \arrow[rr] \arrow[from=uu, crossing over] & & Y_\lambda \arrow[rr, dotted] \arrow[from=uu, crossing over] & & D
\end{tikzcd}$$

with $D := B +_A C$, $Y_0 := X_0 +_A C$, $Y_\lambda := X_\lambda +_{X_0} Y_0$ (so by the pushout lemma, all sides of the left-hand cube except possibly the front and back are pushouts) and the dotted arrows are determined by the requirement that all squares must commute and all horizontal composites be identities. It follows that the cobase change $C \to D$ of $A \to B$ is a retract of $Y_0 \to Y_\lambda$, which in turn is a relative $\mathcal{A}$-cell complex if $X_0 \to X_\lambda$ is one.

As for transfinite compositions, let $A_0 \to A_1 \to \ldots \to A_\lambda$ be a transfinite sequence of arrows in $\mathrm{Sat}(\mathcal{A})$, so that for each $A_\alpha \to A_{\alpha+1}$ we can choose a diagram

$$\begin{tikzcd}
A_\alpha \arrow[r, "i_\alpha"] \arrow[d] & X_\alpha \arrow[r, "p_\alpha"] \arrow[d, "c_\alpha"] & A_\alpha \arrow[d] \\
A_{\alpha+1} \arrow[r, "j_\alpha"'] & Y_\alpha \arrow[r, "q_\alpha"'] & A_{\alpha+1}
\end{tikzcd}$$

with $c_\alpha \colon X_\alpha \to Y_\alpha$ a relative $\mathcal{A}$-cell complex and the horizontal composites being identities. We obtain a new relative $\mathcal{A}$-cell complex $Z_0 \to Z_1 \to \ldots \to Z_\lambda$ by starting with $Z_0 := X_0$ and consecutively gluing the $X_\alpha \to Y_\alpha$ onto Z_α. More formally, we use a transfinite recursion to define the transfinite sequence Z together with a natural transformation $r\colon Z \Rightarrow A$ and two families of morphisms

$$(f_\alpha \colon X_\alpha \to Z_\alpha)_{\alpha<\lambda}, \qquad (g_\alpha \colon Y_\alpha \to Z_{\alpha+1})_{\alpha<\lambda}$$

such that $r_\alpha \circ f_\alpha = p_\alpha$, $r_{\alpha+1} \circ g_\alpha = q_\alpha$, $g_\alpha \circ j_\alpha = f_{\alpha+1} \circ i_{\alpha+1}$ and

$$\begin{tikzcd}
X_\alpha \arrow[r, "f_\alpha"] \arrow[d, "c_\alpha"'] & Z_\alpha \arrow[d] \\
Y_\alpha \arrow[r, "g_\alpha"'] & Z_{\alpha+1} \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

is a pushout for all $\alpha < \lambda$. Note that the last two conditions imply the naturality of $k := f \circ i \colon A \Rightarrow Z$ because the diagram

$$\begin{tikzcd}
A_\alpha \arrow[r, "i_\alpha"] \arrow[d] & X_\alpha \arrow[r, "f_\alpha"] \arrow[d, "c_\alpha"] & Z_\alpha \arrow[d] \\
A_{\alpha+1} \arrow[r, "j_\alpha"'] \arrow[dr, bend right, "i_{\alpha+1}"'] & Y_\alpha \arrow[r, "g_\alpha"'] & Z_{\alpha+1} \\
& X_{\alpha+1} \arrow[ur, bend right, "f_{\alpha+1}"'] &
\end{tikzcd}$$

commutes. To get the recursion started, we put $Z_0 := X_0$, $f_0 := \mathrm{id}_{X_0}$ and $r_0 := p_0$. For the successor case, $Z_{\alpha+1}$ and $g_{\alpha+1}$ are already defined by the pushout square and we define $r_{\alpha+1}$, $f_{\alpha+1}$ by

$$\begin{tikzcd}
X_\alpha \arrow[r, "f_\alpha"] \arrow[d, "c_\alpha"'] & Z_\alpha \arrow[r, "r_\alpha"] \arrow[d] & A_\alpha \arrow[dd] \\
Y_\alpha \arrow[r, "g_\alpha"'] \arrow[drr, bend right, "q_\alpha"'] & Z_{\alpha+1} \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, dotted, "r_{\alpha+1}"] & \\
& & A_{\alpha+1}
\end{tikzcd}
\quad \text{and} \quad f_{\alpha+1} \colon X_{\alpha+1} \xrightarrow{p_{\alpha+1}} A_{\alpha+1} \xrightarrow{j_\alpha} Y_\alpha \xrightarrow{g_\alpha} Z_{\alpha+1}.$$

By definition, $r_{\alpha+1} \circ g_\alpha = q_\alpha$ and plainly $r_{\alpha+1} \circ f_{\alpha+1} = p_{\alpha+1}$, $f_{\alpha+1} \circ i_{\alpha+1} = g_\alpha \circ j_\alpha$. For a limit ordinal $\gamma \leqslant \lambda$, we obviously put $Z_\gamma := \mathrm{colim}_{\alpha<\gamma} Z_\alpha$ and $r_\gamma \colon Z_\gamma \to A_\gamma$ the morphism between the colimits induced by $(r_\alpha)_{\alpha<\gamma}$. Finally, we define $f_\gamma \colon X_\gamma \to Z_\gamma$ to be the composite

$$f_\gamma \colon X_\gamma \xrightarrow{p_\gamma} A_\gamma \xrightarrow{k_\gamma} Z_\gamma,$$

where k_γ is induced by the $k_\alpha := f_\alpha \circ i_\alpha$. The only condition we need to check in the limit case is $r_\gamma \circ f_\gamma = p_\gamma$, which is easy because $r_\gamma \circ f_\gamma = r_\gamma \circ k_\gamma \circ p_\gamma$ and $r_\gamma \circ k_\gamma = \mathrm{id}_{A_\gamma}$ as $r_\alpha \circ k_\alpha = r_\alpha \circ f_\alpha \circ i_\alpha = p_\alpha \circ i_\alpha = \mathrm{id}_{A_\alpha}$ for all $\alpha < \gamma$.

Since every $Z_\alpha \to Z_{\alpha+1}$ is a relative $\mathcal{A}$-cell complex, so is the transfinite composition $Z_0 \to Z_\lambda$ and $A_0 \to A_\lambda$ is a retract of it by means of the natural transformations $k = f \circ i \colon A \Rightarrow Z$, $r \colon Z \Rightarrow A$, which satisfy $r \circ k = \mathrm{id}_A$. □

(3.14) **Corollary.** Granted the axiom of choice, $\mathrm{Sat}(\mathcal{A})$ is the saturation of $\mathcal{A}$. □

4. Factorisation Systems

Factorisation systems play an all-important role for the localisation of categories and hence in modern homotopy theory.

(4.1) **Definition.** A *weak factorisation system* on a category $\mathcal{C}$ is a pair $(\mathcal{L}, \mathcal{R})$ of two classes of arrows in $\mathcal{C}$ such that

(a) every arrow f in $\mathcal{C}$ factors as $f = p \circ i$ for some $i \in \mathcal{L}$, $p \in \mathcal{R}$;

(b) $\mathcal{R} = \mathcal{L}^{\pitchfork}$ and $\mathcal{L} = {}^{\pitchfork}\mathcal{R}$ (so in fact each of these classes determines the weak factorisation system).

It is a *(orthogonal) factorisation system* iff instead of (b) we have

(b)' $\mathcal{R} = \mathcal{L}^{\perp}$ and $\mathcal{L} = {}^{\perp}\mathcal{R}$.

(4.2) **Observation.** Given a (weak) factorisation system $(\mathcal{L}, \mathcal{R})$ on a category $\mathcal{C}$ then $\mathcal{L} \cap \mathcal{R} = \mathcal{I}$ is the class of all isomorphisms in $\mathcal{C}$ by (2.4) and (2.5). Furthermore, if $(\mathcal{L}, \mathcal{R})$ is a (weak) factorisation system on $\mathcal{C}$ then $(\mathcal{R}^{\mathrm{op}}, \mathcal{L}^{\mathrm{op}})$ is a (weak) factorisation system on $\mathcal{C}^{\mathrm{op}}$.

(4.3) **Observation.** One thing that distinguishes non-weak factorisation systems from weak ones is that for full-fledged factorisation systems, factorisations are unique up to a unique isomorphism. More precisely, given a factorisation system $(\mathcal{L}, \mathcal{R})$ and an arrow $f \colon A \to B$ factoring as $f = r \circ l = r' \circ l'$ with l, $l' \in \mathcal{L}$ and r, $r' \in \mathcal{R}$, then there is a unique dotted morphism g as in the following diagram making everything commute and this is an isomorphism

$$\begin{tikzcd} & C \arrow[dd, dotted, "g"] \arrow[dr, "r"] & \\ A \arrow[ur, "l"] \arrow[dr, "l'"'] & & B. \\ & C' \arrow[ur, "r'"'] & \end{tikzcd}$$

Indeed, the existence of a unique g follows from $l \perp r'$ and then $l' \perp r$ provides an inverse.

It is not immediately clear from the definition that every factorisation system is also a weak factorisation system. We address this question after giving some examples and constructions.

(4.4) **Example.** For every category $\mathcal{C}$ we always have the two trivial factorisation systems $(\mathcal{C}, \mathcal{I})$ and $(\mathcal{I}, \mathcal{C})$ where $\mathcal{I}$ is the class of all isomorphisms in $\mathcal{C}$.

(4.5) **Example.** Consider a functor $F \colon \mathcal{C} \to \mathcal{D}$ together with $D \in \mathcal{D}$ and let's write $P \colon F \downarrow D \to \mathcal{C}$ for the canonical projection. Then any (weak) factorisation system $(\mathcal{L}, \mathcal{R})$ on $\mathcal{C}$ gives us a (weak) factorisation system $(\mathcal{L} \downarrow D, \mathcal{R} \downarrow D)$ on $F \downarrow D$ defined as

$$\mathcal{L} \downarrow D := P^{-1}\mathcal{L} \qquad \text{and} \qquad \mathcal{R} \downarrow D := P^{-1}\mathcal{R}.$$

Dually for $D \downarrow F$.

Proof. Any arrow $f\colon (C,p) \to (C',p')$ can be factored in $\mathcal{C}$ as $f = C \xrightarrow{l} C'' \xrightarrow{r} C'$ with $l \in \mathcal{L}$ and $r \in \mathcal{R}$. These determine the object $(C'', p' \circ Fr)$ in $F \downarrow D$ and we obtain the sought for factorisation $f = r \circ l$ in $F \downarrow D$:

$$\begin{tikzcd} FC \arrow[r, "Fl"] \arrow[dr, "p"'] & FC'' \arrow[r, "Fr"] \arrow[d] & FC' \arrow[dl, "p'"] \\ & D & \end{tikzcd}.$$

Now given a square of solid arrows in $F \downarrow D$

$$\begin{tikzcd} (A,a) \arrow[r, "f"] \arrow[d, "l"'] & (X,x) \arrow[d, "r"] \\ (B,b) \arrow[r, "g"'] \arrow[ur, dotted, "d"] & (Y,y) \end{tikzcd}$$

there is a diagonal filler d iff there is one in $\mathcal{C}$. Indeed, given $d\colon B \to X$ such that $d \circ l = f$ and $r \circ d = g$, we get that

$$b = y \circ Fg = y \circ Fr \circ Fd = x \circ Fd.$$

□

(4.6) **Example.** Maybe the simplest and most familiar non-trivial example is the factorisation system $(\mathcal{E}, \mathcal{M})$ on **Sets** with $\mathcal{E}$ the class of all surjections and $\mathcal{M}$ the class of all injections. Note that there is no choice axiom needed for this.

(4.7) **Example.** Curiously enough, when assuming the axiom of choice, $(\mathcal{M}, \mathcal{E})$ is also a weak factorisation system (albeit not a factorisation system). If not assuming choice then at least $(\mathcal{M}, \mathcal{E}_{\text{split}})$ is a weak factorisation system, where $\mathcal{E}_{\text{split}}$ is the class of split epimorphisms.

The following example has been alluded to in a comment of Goodwillie's (see `http://mathoverflow.net/questions/29635`) and can be considered folklore by now.

(4.8) **Example.** With $\mathcal{E}$, $\mathcal{M} \subseteq$ **Sets** as before, let $\mathcal{I}$ be the class of isomorphisms and

$$\mathcal{V} := \{\varnothing \hookrightarrow A \mid A \neq \varnothing\}$$

the proper inclusions of $\varnothing$. Then apart from $(\mathbf{Sets}, \mathcal{I})$, $(\mathcal{I}, \mathbf{Sets})$, $(\mathcal{E}, \mathcal{M})$ and $(\mathcal{M}, \mathcal{E})$, there are only two other weak factorisation systems on **Sets**, namely

$$(\mathcal{M} \setminus \mathcal{V}, \mathcal{E} \cup \mathcal{V}) \qquad \text{and} \qquad (\mathbf{Sets} \setminus \mathcal{V}, \mathcal{V} \cup \mathcal{I}),$$

which are easily checked to be weak factorisation systems. On the other hand, to see that these are all one makes the observations that

(a) $\mathcal{E} \subseteq \mathcal{L}$ iff $\mathcal{M} \supseteq \mathcal{R}$ iff $\mathcal{L}$ contains a non-injective map;

(b) $\mathcal{M} \setminus \mathcal{V} \subseteq \mathcal{L}$ iff $\mathcal{E} \cup \mathcal{V} \supseteq \mathcal{R}$ iff $\mathcal{L}$ contains a non-surjective map;

(c) $p \in \mathcal{E}$ iff $i \pitchfork p$ for some $i \in \mathcal{V}$;

Now given any weak factorisation system $(\mathcal{L}, \mathcal{R})$ with $\mathcal{L} \neq \mathcal{I}$, there is some non-injective or non-surjective map in $\mathcal{L}$. In the first case $\mathcal{E} \subseteq \mathcal{L}$ by (a) and if even $\mathcal{E} = \mathcal{L}$ then $\mathcal{R} = \mathcal{M}$. However, if $\mathcal{L}$ also contains a non-surjective map then even $\mathbf{Sets} \setminus \mathcal{V} \subseteq \mathcal{L}$ by (b). With this either $\mathcal{L} = \mathbf{Sets} \setminus \mathcal{V}$ (and so $\mathcal{R} = \mathcal{V} \cup \mathcal{I}$) or $\mathcal{L} \cap \mathcal{V} \neq \varnothing$ in which case $\mathcal{R} \subseteq \mathcal{E}$ by (c), whence $\mathcal{R} \subseteq \mathcal{E} \cap (\mathcal{V} \cup \mathcal{I}) = \mathcal{I}$ and $\mathcal{L} = \mathbf{Sets}$.

On the other hand, if $\mathcal{L}$ contains some non-surjective map but only injections then again $\mathcal{M} \setminus \mathcal{V} \subseteq \mathcal{L}$ by (b) and so $\mathcal{R} \subseteq \mathcal{E} \cup \mathcal{V}$. Now either $\mathcal{L} = \mathcal{M} \setminus \mathcal{V}$ and we are done or $\mathcal{V} \cap \mathcal{L} \neq \varnothing$, in which case $\mathcal{R} \subseteq \mathcal{E}$ by (c) and so $\mathcal{M} \subseteq \mathcal{L}$ and therefore $\mathcal{M} = \mathcal{L}$.

In the theory of model categories, people usually use a different (but equivalent) description of a weak factorisation system, which only requires $\mathcal{R} \subseteq \mathcal{L}^{\pitchfork}$ and $\mathcal{L} \subseteq {}^{\pitchfork}\mathcal{R}$ but adds the requirement that $\mathcal{L}$ and $\mathcal{R}$ be closed under retracts.

(4.9) **Proposition. (Retract Argument)** If $f = p \circ i$ then $f \pitchfork p$ iff f is a strong retract of i while $f \perp p$ iff p is an isomorphism. Dually, $i \pitchfork f$ iff f is a costrong retract of p while $i \perp f$ iff i is an isomorphism.

Proof. Assume $f \pitchfork p$ first and choose a lift s in the left-hand diagram, yielding the diagram on the right.

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "f"'] & Y \arrow[d, "p"] \\ Z \arrow[r, equal] \arrow[ur, dotted, "s"] & Z \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, equal] \arrow[d, "f"'] & X \arrow[r, equal] \arrow[d, "i"] & X \arrow[d, "f"] \\ Z \arrow[r, "s"'] & Y \arrow[r, "p"'] & Z \end{tikzcd} .$$

Conversely, if we have a diagram as depicted on the right above (exhibiting f as strong retract of i) then every square of solid arrows below on the left has a dotted diagonal filler.

$$\begin{tikzcd} X \arrow[r, "a"] \arrow[d, "f"'] & Y \arrow[d, "p"] \\ Z \arrow[r, "b"'] \arrow[ur, dotted, "s\circ b"] & Z \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "i"] \arrow[d, "i"'] & Y \arrow[d, "p"] \\ Y \arrow[r, "p"'] & Z \end{tikzcd}$$

Now if even $f \perp p$, we construct s as above and then $s \circ p$ is a diagonal filler of the right-hand square above. By uniqueness then $s \circ p = \mathrm{id}_Y$ and so $s = p^{-1}$. Conversely, if p is an isomorphism then clearly $f \perp p$. □

(4.10) **Corollary.** Let $(\mathcal{L}, \mathcal{R})$ be two classes of arrows in a category $\mathcal{C}$ such that every arrow f in $\mathcal{C}$ factors as $f = p \circ i$ with $p \in \mathcal{R}$, $i \in \mathcal{L}$. Then

(a) $(\mathcal{L}, \mathcal{R})$ is a weak factorisation system iff $\mathcal{L} \pitchfork \mathcal{R}$ and $\mathcal{L}$, $\mathcal{R}$ are closed under retracts iff $\mathcal{L} \pitchfork \mathcal{R}$, $\mathcal{L}$ is closed under strong retracts and $\mathcal{R}$ is closed under costrong retracts.

(b) $(\mathcal{L}, \mathcal{R})$ is a factorisation system iff $\mathcal{L} \perp \mathcal{R}$ and $\mathcal{L}$, $\mathcal{R}$ are closed under retracts iff $\mathcal{L} \perp \mathcal{R}$ and $\mathcal{L}$, $\mathcal{R}$ are replete (as classes of objects in $\mathcal{C}^{\to}$) iff $\mathcal{L} \perp \mathcal{R}$, $\mathcal{L}$ is closed under composition with isomorphisms from the left and $\mathcal{R}$ is closed under composition with isomorphisms from the right.

Proof. The non-trivial parts of "$\Rightarrow$" follow from (2.15). For the converses, let $f = p \circ i \in \mathcal{L}^{\pitchfork}$ (resp. $\mathcal{L}^{\perp}$) with $p \in \mathcal{R}$ and $i \in \mathcal{L}$. By the retract argument, f is a costrong retract of p in case (a) and hence $f \in \mathcal{R}$; while in case (b), i is even an isomorphism and so $f \in \mathcal{R}$. Dually for $f \in {}^{\pitchfork}\mathcal{R}$ (resp. ${}^{\perp}\mathcal{R}$). □

(4.11) **Corollary.** Every factorisation system $(\mathcal{L}, \mathcal{R})$ is also a weak one. $\square$

(4.12) **Corollary.** Let $(\mathcal{L}, \mathcal{R})$ be two classes of arrows such that $\mathcal{L} \pitchfork \mathcal{R}$ and every arrow f can be factored into $f = p \circ i$ with $i \in \mathcal{L}$, $p \in \mathcal{R}$. Then $(\mathcal{L}\text{-cof}, \mathcal{R}\text{-fib}) = \left({}^{\pitchfork}(\mathcal{L}^{\pitchfork}), ({}^{\pitchfork}\mathcal{R})^{\pitchfork}\right)$ is a weak factorisation system. If even $\mathcal{L} \perp \mathcal{R}$ then $\left({}^{\perp}(\mathcal{L}^{\perp}), ({}^{\perp}\mathcal{R})^{\perp}\right)$ is a factorisation system.

Proof. Because ${}^{\pitchfork}-$ and $-^{\pitchfork}$ are order reversing and $\mathcal{L} \subseteq {}^{\pitchfork}\mathcal{R}$, we get ${}^{\pitchfork}(\mathcal{L}^{\pitchfork}) \subseteq {}^{\pitchfork}(({}^{\pitchfork}\mathcal{R})^{\pitchfork})$. Moreover, ${}^{\pitchfork}(\mathcal{L}^{\pitchfork})$ and $({}^{\pitchfork}\mathcal{R})^{\pitchfork}$ are closed under retracts by (2.15). Similarly for $\perp$. $\square$

If $(\mathcal{L}, \mathcal{R})$ is a weak factorisation system on $\mathcal{C}$ and given a commutative square of arrows in $\mathcal{C}$ on the left below then upon factoring f and g we can always fill in a dotted arrow in the right making everything commute (because $l_f \pitchfork r_g$).

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "a"'] & C' \arrow[d, "a'"] \\ D \arrow[r, "g"'] & D' \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} C \arrow[r, "l_f"] \arrow[d, "a"'] & C'' \arrow[r, "r_f"] \arrow[d, dotted] & C' \arrow[d, "a'"] \\ D \arrow[r, "l_g"'] & D'' \arrow[r, "r_g"'] & D' \end{tikzcd}$$

However, since $(\mathcal{L}, \mathcal{R})$ is just a weak factorisation system, this dotted arrow is not unique (it is if the factorisation system is non-weak). While uniqueness is usually too strict and not necessary anyway, in some situations, we at least need the factorisations to be functorial.

Since the definition of a functorial factorisation given in the widely used book [34] was “wrong” (or at least inadequate), let us recall the correct one here.

(4.13) **Definition.** For $\mathcal{C}$ a category, the increasing map $\delta^1 \colon [1] \to [2]$ that misses 1 induces a functor $(\delta^1)^* \colon \mathcal{C}^{[2]} \to \mathcal{C}^{[1]}$, which is just composition in $\mathcal{C}$. Now a *functorial factorisation* on $\mathcal{C}$ is a functor

$$F \colon \mathcal{C}^{[1]} \to \mathcal{C}^{[2]} \qquad \text{such that} \qquad \mathcal{C}^{[1]} \xrightarrow{F} \mathcal{C}^{[2]} \xrightarrow{(\delta^1)^*} \mathcal{C}^{[1]} \quad = \quad \mathcal{C}^{[1]} \xrightarrow{\mathrm{id}} \mathcal{C}^{[1]}.$$

It is often convenient to identify F with the pair $\left((\delta^2)^* \circ F, (\delta^0)^* \circ F\right)$, where δ^0 and δ^2 are the increasing maps $[1] \to [2]$ that miss 0 and 2, respectively, and we will usually call a pair (ι, π) that arises in this manner a functorial factorisation. We call *functorial (weak) factorisation system* a (weak) factorisation system $(\mathcal{L}, \mathcal{R})$, together with a functorial factorisation $F \colon \mathcal{C}^{[1]} \to \mathcal{C}^{[2]}$ factoring every arrow into a composite of one in $\mathcal{L}$, followed by one in $\mathcal{R}$.

(4.14) **Example.** If $(\mathcal{L}, \mathcal{R})$ is an orthogonal factorisation system then every choice of factorisations, one for each morphism, automatically defines a functorial factorisation.

5. Small Objects

Small objects are a generalisation of presentable objects, which is sufficient for the construction of weak factorisation systems. This construction is called the small object argument, which we shall see in the next section. For the entire section, we fix a locally small cocomplete category $\mathcal{C}$. We recall the following definition from chapter 3.

(5.1) **Definition.** Let κ be an infinite cardinal. Recall that an ordinal λ is said to be *κ-directed* iff for all $S \subseteq \lambda$ with $|S| < \kappa$, we must have $\sup S < \lambda$; i.e. S has an upper bound in λ. Intuitively speaking, an ordinal λ is κ-directed if it can't be reached by subsets of cardinality $< \kappa$. An infinite cardinal κ that is κ-directed is also called *regular*. As usual, a diagram $X \colon \lambda \to \mathcal{C}$ is called *κ-directed* iff its domain λ is κ-directed. Finally, given a limit ordinal λ, the smallest cardinal κ such that λ is κ-directed is called the *cofinality* of λ and denoted by $\mathrm{cf}\,\lambda$.

(5.2) **Example.** One can show that for every limit ordinal λ, we have $\operatorname{cf}\operatorname{cf}\lambda = \operatorname{cf}\lambda$. In other words, every cardinal of the form $\operatorname{cf}\lambda$ is regular.

(5.3) **Remark.** One can show that a limit ordinal λ is κ-directed iff it is $(\operatorname{cf}\kappa)$-directed and as just observed, $\operatorname{cf}\kappa$ is regular. So, when speaking of κ-directed ordinals, we can always assume κ to be regular.

(5.4) **Example.** If κ is an infinite cardinal, its cardinal successor κ^+ is regular. In fact, κ^+ is the least κ^+-directed limit ordinal because $|\lambda| \leqslant \kappa$ for every ordinal $\lambda < \kappa^+$. To check that κ^+ is regular, we consider $S \subseteq \kappa^+$ of cardinality $|S| \leqslant \kappa$ and we only need to check that $\sup S < \kappa^+$, i.e. that $|\sup S| = |\bigcup S| \leqslant \kappa$.

To see this, we will need an immediate generalisation of the familiar fact that a countable union of countable sets is countable. Unfortunately, just like this familiar fact, the generalisation relies heavily on the axiom of choice.

(5.5) **Lemma.** $|\bigcup M| \leqslant |M| \cdot \sup\{|X| \mid X \in M\}$ for every $M \neq \varnothing$.

Proof. Let $\kappa := |M|$, $\kappa' := \{\sup|X| \mid X \in M\}$ and choose an enumeration $\kappa \cong M$, $\alpha \mapsto X_\alpha$. Furthermore, using the axiom of choice, choose for every $X_\alpha \in M$ a surjection $\kappa' \twoheadrightarrow X_\alpha$, $\beta \mapsto x_{\alpha,\beta}$. This gives us a surjection

$$\kappa \times \kappa' \to \bigcup M,\ (\alpha, \beta) \mapsto x_{\alpha,\beta}$$

and thus $|\kappa \times \kappa'| = \kappa \cdot \kappa' \leqslant |\bigcup M|$. $\square$

(5.6) **Definition.** Let $\mathcal{C}$ be a locally small cocomplete category, $C \in \mathcal{C}$, $\mathcal{I}$ a class of arrows in $\mathcal{C}$ and κ a cardinal. Given a κ-directed transfinite sequence $X\colon \lambda \to \mathcal{C}$ of arrows in $\mathcal{I}$

$$X_0 \xrightarrow{a_0} X_1 \xrightarrow{a_1} \ldots \to X_\alpha \xrightarrow{a_\alpha} \ldots,$$

we can apply $\mathcal{C}(C, -)$ to get a functor $\lambda \to \textbf{Sets}$. The colimiting cocone $X \Rightarrow \operatorname{colim} X$ in $\mathcal{C}$ induces a cocone below $\mathcal{C}(C, X-)$ to the vertex $\mathcal{C}(C, \operatorname{colim} X)$ and thus a unique map

(5.7) $$\operatorname{colim}_\alpha \mathcal{C}(C, X_\alpha) \to \mathcal{C}(C, \operatorname{colim}_\alpha X_\alpha) \quad \text{such that} \quad \begin{tikzcd} \mathcal{C}(C, X_\beta) \arrow[r] \arrow[dr] & \operatorname{colim}_\alpha \mathcal{C}(C, X_\alpha) \arrow[d, dotted] \\ & \mathcal{C}(C, \operatorname{colim}_\alpha X_\alpha) \end{tikzcd}$$

commutes for all $\beta < \lambda$. The object C is called *κ-small relative to $\mathcal{I}$* iff this map is a bijection for all κ-directed sequences X of arrows in $\mathcal{I}$. It is *small relative to $\mathcal{I}$* iff it is κ-small relative to $\mathcal{I}$ for some κ. If $\mathcal{I} = \mathcal{C}$, we usually omit the "relative to $\mathcal{C}$" part and simply speak of *κ-small* and *small* objects.

(5.8) **Remark.** Just like for presentable objects, we take note of the following points.

- The ordinal λ is κ-directed iff it is $(\operatorname{cf}\kappa)$-directed and $\operatorname{cf}\kappa$ is regular. So we can (and shall) always assume κ to be regular.
- If $\kappa \leqslant \kappa'$ and C is a κ-small object (relative to some class $\mathcal{I}$ of arrows), it is also κ'-small since every κ'-directed transfinite sequence is also κ-directed.

(5.9) **Convention.** For the rest of this chapter, the letter κ always denotes an arbitrary regular cardinal.

Let us restate this definition in a manner, internal to $\mathcal{C}$ (which also effectively removes the local smallness condition). For this, recall that

$$\operatorname*{colim}_{\alpha<\lambda} \mathcal{C}(C, X_\alpha) = \left(\coprod_{\alpha<\lambda} \mathcal{C}(C, X_\alpha) \right) / \sim,$$

where $\sim$ is the equivalence relation generated by $f \sim (X_\beta \to X_{\beta'}) \circ f$ for $f \in \mathcal{C}(C, X_\beta)$ and $\beta \leqslant \beta'$ arbitrary. Now the map from the definition is bijective iff for all $f \in \mathcal{C}(C, \operatorname{colim}_\alpha X_\alpha)$ there is a unique $[f'] \in \operatorname{colim}_\alpha \mathcal{C}(C, X_\alpha)$ such that $a_{\beta,\lambda} \circ f' = f$, where $f' \in \mathcal{C}(C, X_\beta)$ and $a_{\beta,\lambda} \colon X_\beta \to \operatorname{colim}_\alpha X_\alpha$ is the β-component of the colimiting cocone. Internally in $\mathcal{C}$ this means

- Surjectivity: Every $f \colon C \to \operatorname{colim}_\alpha X_\alpha$ factors as $\begin{tikzcd} C \arrow[r, "f'"] \arrow[dr, "f"'] & X_\beta \arrow[d, "a_{\beta,\lambda}"] \\ & \operatorname{colim}_\alpha X_\alpha \end{tikzcd}$ for some $\beta < \lambda$.

- Injectivity: Given $\begin{tikzcd} & & C \arrow[dl, "f'"'] \arrow[dr, "f''"] & & \\ \ldots \arrow[r] & X_\beta \arrow[r] \arrow[dr, "a_{\beta,\lambda}"'] & \ldots \arrow[r] & X_{\beta'} \arrow[r] \arrow[dl, "a_{\beta',\lambda}"] & \ldots \\ & & \operatorname{colim}_\alpha X_\alpha & & \end{tikzcd}$ with the outer square commutative. Then there is $\gamma \geqslant \beta'$ such that $(X_\beta \to X_\gamma) \circ f' = (X_{\beta'} \to X_\gamma) \circ f''$

(5.10) **Example.** Every presentable object is small.

(5.11) **Proposition.** A κ-small colimit of κ-small objects (relative to some class of arrows $\mathcal{I}$) is again κ-small.

Proof. Same as (3.2.14). □

(5.12) **Corollary.** Retracts of κ-small objects (relative to some class of arrows $\mathcal{I}$) are again κ-small. □

(5.13) **Proposition.** If an object $D \in \mathcal{C}$ has a retract that is κ-small relative to a class $\mathcal{I}$ of arrows, then D is itself κ-small relative to $\mathcal{I}$.

Proof. Given a κ-directed transfinite sequence $X \colon \lambda \to \mathcal{C}$ of arrows in $\mathcal{I}$ as well as $s \colon C \to D$ and $r \colon D \to C$ with C κ-small relative to $\mathcal{I}$ and $r \circ s = \mathrm{id}_C$ we get a commutative diagram

$$\begin{tikzcd} \operatorname{colim} \mathcal{C}(C, X) \arrow[r] \arrow[d, "\operatorname{colim} r^*"'] & \mathcal{C}(C, \operatorname{colim} X) \arrow[d, "r^*"] \\ \operatorname{colim} \mathcal{C}(D, X) \arrow[r, "\cong"] \arrow[d, "\operatorname{colim} s^*"'] & \mathcal{C}(D, \operatorname{colim} X) \arrow[d, "s^*"] \\ \operatorname{colim} \mathcal{C}(C, X) \arrow[r] & \mathcal{C}(C, \operatorname{colim} X) \end{tikzcd}$$

whose vertical composites are identities. So, $\operatorname{colim} \mathcal{C}(C, X) \to \mathcal{C}(C, \operatorname{colim} X)$ is a retract of an isomorphism and hence itself one by (2.14). □

(5.14) **Example.** As shown in (3.2.6), a set is κ-small iff it has cardinality $< \kappa$.

6. Cellular Approximation and the Small Object Argument

Just like we generalised the concept of a (relative) CW-complex in definition (3.5), we can generalise the cellular approximation theorem to this context. As the name suggests, the idea is to approximate an arbitrary arrow $f\colon X \to Y$ by a relative $\mathcal{A}$-cell complex for $\mathcal{A}$ some fixed set (rather than a possibly proper class) of arrows in a cocomplete category $\mathcal{C}$.

Recalling that relative $\mathcal{A}$-cell complexes are just transfinite compositions of $\mathcal{A}$-cobase changes, the idea for approximating $f\colon X \to Y$ is to fix some limit ordinal λ (the choice of this will be discussed later) and define the $\mathcal{A}$-cellular approximation for f to be the composition of the transfinite sequence, where at each step, we glue on "all $\mathcal{A}$-cells possible".

(6.1) **Definition.** For a set of arrows $\mathcal{A}$ in a locally small cocomplete category $\mathcal{C}$ and an arbitrary arrow $f\colon X \to Y$, we define the $\mathcal{A}$-*cellular successor* of f to be the factorisation

$$X \xrightarrow{\iota_1(f)} C_1(f) \xrightarrow{\pi_1(f)} Y$$

of f obtained as follows. Let $S(f) := \coprod_{i\in\mathcal{A}} \mathrm{Hom}_{\mathcal{C}^{[1]}}(i,f)$ be the set that consists of all triples

$$s = (i_s, g_s, h_s) \quad \text{fitting into a commuting square} \quad \begin{array}{ccc} A_s & \xrightarrow{g_s} & X \\ {\scriptstyle i_s}\downarrow & & \downarrow{\scriptstyle f} \\ B_s & \xrightarrow[h_s]{} & Y \end{array} \quad \text{and with } i_s \in \mathcal{A}.$$

We define $\iota_1(f)\colon X \to C_1(f)$ to be the pushout

$$\begin{tikzcd} \coprod_{s\in S(f)} A_s \arrow[r, "{[g_s]_s}"] \arrow[d, "\amalg_s i_s"'] & X \arrow[d, "\iota_1(f)"] \\ \coprod_{s\in S(f)} B_s \arrow[r] & C_1(f) \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd} ,$$

while $\pi_1(f)$ is the unique arrow induced by f and $[h_s]_s$.

(6.2) **Definition.** Let $\mathcal{A}$ be a set of arrows in a locally small cocomplete category $\mathcal{C}$ and $f\colon X \to Y$ an arrow. We define a transfinite sequence $C(f)\colon \mathrm{Ord} \to \mathcal{C}$ (where Ord is the poset of all ordinals), together with a cocone $\pi(f)\colon C(f) \Rightarrow Y$ recursively by $C_0(f) := X$, $\pi_0(f) = f$ and for λ a limit ordinal necessarily $C_\lambda(f) := \mathrm{colim}_{\alpha<\lambda} C_\alpha(f)$ and $\pi_\lambda(f)$ the unique arrow induced by $\big(\pi_\alpha(f)\big)_{\alpha<\lambda}$. Finally, in the successor case, we define

$$C_{\alpha+1}(f) := C_1\big(\pi_\alpha(f)\big), \quad \pi_{\alpha+1}(f) := \pi_1\big(\pi_\alpha(f)\big) \quad \text{and } C(\alpha < \alpha+1) := \iota_1\big(\pi_\alpha(f)\big).$$

Now for λ any (usually limit) ordinal and $\iota_\lambda(f)\colon X = C_0(f) \to C_\lambda(f)$, the pair $\big(\iota_\lambda(f), \pi_\lambda(f)\big)$ is called an $\mathcal{A}$-*cellular approximation (of length λ)* of f. By construction $\iota_\lambda(f) \in \mathrm{Cell}(\mathcal{A})$ and $\pi_\lambda(f) \circ \iota_\lambda(f) = f$.

(6.3) **Proposition.** For $\mathcal{A}$ a set of arrows in a locally small cocomplete category $\mathcal{C}$, $\mathcal{A}$-cellular approximation can be made into a functor

$$\mathcal{C}^{[1]} \times \mathrm{Ord} \to \mathcal{C}^{[2]},\ (f,\lambda) \mapsto \big(\iota_\lambda(f), \pi_\lambda(f)\big)$$

such that, for every fixed ordinal λ, $\mathcal{A}$-cellular approximation of length λ is a functorial factorisation on $\mathcal{C}$.

Proof. Given a commutative square in $\mathcal{C}$ and the result upon taking cellular approximation

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "a"'] & Y \arrow[d, "b"] \\ X' \arrow[r, "f'"'] & Y' \end{tikzcd} \qquad \mapsto \qquad \begin{tikzcd} X \arrow[r, "\iota_\lambda(f)"] \arrow[d, "a"'] & C_\lambda(f) \arrow[r, "\pi_\lambda(f)"] \arrow[d, dotted, "\varphi_\lambda"'] & Y \arrow[d, "b"] \\ X' \arrow[r, "\iota_\lambda(f')"'] & C_\lambda(f') \arrow[r, "\pi_\lambda(f')"'] & Y' \end{tikzcd},$$

we need to construct a dotted arrow φ_λ making the diagram commute in a functorial (with respect to f) and natural (with respect to λ) way, which we do by transfinite recursion, starting with

$$(\varphi_0 := a)\colon\ (C_0(f) = X)\ \to\ (X' = C_0(f')).$$

The limit case is easy, as we just need to take colimits, which is clearly functorial with respect to f and natural with respect to λ (as it is defined by a universal property). In the successor case, we get a map $\psi\colon S(\pi_\lambda(f)) \to S(\pi_\lambda(f'))$ by $(i_s, g_s, h_s) \mapsto (i_s, \varphi_\lambda \circ g_s, b \circ h_s)$; i.e.

$$\begin{tikzcd} A_s \arrow[r, "g_s"] \arrow[d, "i_s"'] & C_\lambda(f) \arrow[d, "\pi_\lambda(f)"] \\ B_s \arrow[r, "h_s"'] & Y \end{tikzcd} \qquad \mapsto \qquad \begin{tikzcd} A_s \arrow[r, "g_s"] \arrow[d, "i_s"'] & C_\lambda(f) \arrow[r, "\varphi_\lambda"] \arrow[d, "\pi_\lambda(f)"'] & C_\lambda(f') \arrow[d, "\pi_\lambda(f')"'] \\ B_s \arrow[r, "h_s"'] & Y \arrow[r, "b"'] & Y' \end{tikzcd}.$$

This in turn gives us a morphism of spans as follows

$$\begin{tikzcd} \coprod\limits_{s\in S(\pi_\lambda(w))} B_s \arrow[d, hook] & \coprod\limits_{s\in S(\pi_\lambda(f))} A_s \arrow[l, "\amalg_s i_s"'] \arrow[r, "{[g_s]_s}"] \arrow[d, hook] & C_\lambda(f) \arrow[d, "\varphi_\lambda"] \\ \coprod\limits_{s'\in S(\pi_\lambda(f'))} B_{s'} & \coprod\limits_{s'\in S(\pi_\lambda(f'))} A_{s'} \arrow[l, "\amalg_{s'} i_{s'}"'] \arrow[r, "{[g_{s'}]_{s'}}"] & C_\lambda(f') \end{tikzcd}$$

(where the two unnamed vertical arrows map the s-summand into the ψs-summand by the identity). Taking pushouts, we get $\varphi_{\lambda+1}\colon C_{\lambda+1}(f) \to C_{\lambda+1}(f')$ rendering the square

$$\begin{tikzcd} C_\lambda(f) \arrow[r, "\varphi_\lambda"] \arrow[d] & C_\lambda(f') \arrow[d] \\ C_{\lambda+1}(f) \arrow[r, "\varphi_{\lambda+1}"'] & C_{\lambda+1}(f') \end{tikzcd}$$

commutative (thus proving the naturality with respect to λ). Clearly, this construction is functorial with respect to f because the above assignment of spans is functorial (using the functoriality of φ_λ). $\square$

Cellular approximation is a vastly useful tool in constructing a functorial factorisation to define a model structure on a category $\mathcal{C}$ but not without reservations. The problem is that while we know that in the factorisation $f = \pi_\lambda(f) \circ \iota_\lambda(f)$ that $\iota_\lambda(f) \in \mathrm{Cell}(\mathcal{A})$, we cannot say much about $\pi_\lambda(f)$. Here is where small objects come into play. For convenience let us first prove a small lemma.

(6.4) **Lemma.** Let $\mathcal{A}$ be a class of arrows in a category $\mathcal{C}$, $C \in \mathcal{C}$ and κ a cardinal. Then C is κ-small relative to $\mathrm{Cell}(\mathcal{A})$ iff it is κ-small relative to $\mathcal{A}$-cobase changes, i.e. κ-small relative to

$$\left\{ f\colon X \to Y \;\middle|\; \text{there is a pushout diagram } \begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle j}\downarrow & \ulcorner & \downarrow{\scriptstyle f} \\ B & \longrightarrow & Y \end{array} \text{ with } j \in \mathcal{A} \right\}.$$

Proof. The direction "$\Rightarrow$" is obvious. As for the other direction, consider a κ-directed transfinite sequence $X\colon \lambda \to \mathcal{C}$ of relative $\mathcal{A}$-cell complexes, so that each $X_\beta \to X_{\beta+1}$ is a composition of a transfinite sequence $Y^\beta\colon \mu_\beta \to \mathcal{C}$ of $\mathcal{A}$-pushouts. As in (1.10), we well-order $\nu := \{(\beta,\gamma) \mid \beta < \lambda, \gamma < \mu_\beta\}$ lexicographically and get a transfinite sequence $Y\colon \nu \to \mathcal{C}$ that has the same compositions as X. Because the standard projection $\nu \to \lambda$ is weakly increasing and surjective one easily sees that ν is κ-directed because λ is and so we get a commutative square

$$\begin{tikzcd}
\operatorname{colim} \mathcal{C}(C,X) \arrow[r] \arrow[d, "\cong"'] & \mathcal{C}(C, \operatorname{colim} X) \arrow[d, "\cong"] \\
\operatorname{colim} \mathcal{C}(C,Y) \arrow[r, "\cong"'] & \mathcal{C}(C, \operatorname{colim} Y)
\end{tikzcd} \quad ,$$

where the horizontal arrows are the canonical ones and the vertical arrows are induced by $\lambda \cong \lambda \times \{0\} \hookrightarrow \nu$. They are isomorphisms because $\{(\beta,0)\}_{\beta<\lambda}$ is weakly terminal in ν. □

(6.5) **Theorem. (Small Object Argument)** Let $\mathcal{C}$ be a locally small cocomplete category, κ an infinite cardinal and λ a κ-directed ordinal. If $\mathcal{A}$ is a set of arrows in $\mathcal{C}$, all of whose domains are κ-small relative to $\mathcal{A}$-cobase changes then $\mathcal{A}$-cellular approximation $(\iota_\lambda, \pi_\lambda)$ of length λ is a functorial factorisation with $\iota_\lambda(f) \in \mathrm{Cell}(\mathcal{A})$ and $\pi_\lambda(f) \in \mathcal{A}^{\pitchfork}$ for all f in $\mathcal{C}$. In particular, this makes $(\mathcal{A}\text{-cof}, \mathcal{A}^{\pitchfork})$ a functorial weak factorisation system.

Proof. The only claim not yet proven is that $\pi_\lambda(f) \in \mathcal{A}^{\pitchfork}$ for $f\colon X \to Y$. For this, suppose we are given a commutative square of solid arrows

$$\begin{tikzcd}
 & C_\beta(f) \arrow[dr, dotted] & \\
A \arrow[rr] \arrow[d, "j"'] \arrow[ur, dotted] & & C_\lambda(f) \arrow[d, "\pi_\lambda(f)"] \\
B \arrow[rr] \arrow[urr, dotted] & & Y
\end{tikzcd}$$

with $j \in \mathcal{A}$. By hypothesis and the lemma, we find $\beta < \lambda$ as well as $A \to C_\beta(f)$ with

$$A \to C_\lambda(f) \;=\; A \to C_\beta(f) \to C_\lambda(f), \text{ so that } (j, A \to C_\beta(f), B \to Y) \in S\big(\pi_\beta(f)\big).$$

By construction, we get an arrow from B into the pushout $C_{\beta+1}(f)$ satisfying

$$A \to C_\beta(f) \to C_{\beta+1}(f) \;=\; A \xrightarrow{j} B \to C_{\beta+1}(f) \qquad \text{and}$$

$$B \to Y \;=\; B \to C_{\beta+1}(f) \xrightarrow{\pi_{\beta+1}(f)} Y,$$

so that $B \to C_{\beta+1}(f) \to C_\lambda(f)$ fits into the diagram above. A final word concerning the last claim of the theorem. Clearly $\mathrm{Cell}(\mathcal{A}) \subseteq \mathcal{A}\text{-cof} = {}^{\pitchfork}(\mathcal{A}^{\pitchfork})$ because $\mathcal{A}$-cof is (cellularly) saturated and contains $\mathcal{A}$ (cf. (2.15)) and $\mathcal{A}\text{-cof}^{\pitchfork} = \mathcal{A}^{\pitchfork}$ by (2.3). □

(6.6) **Remark.** As is visible from the proof, we only need the surjectivity part from the definition of a κ-small object.

To apply the small object argument, one usually starts with a set $\mathcal{A}$ of arrows, all of whose domains are small relative to $\mathcal{A}$ -cobase changes and then chooses κ and λ accordingly as follows.

(6.7) **Corollary. (Small Object Argument** [bis]**)** If $\mathcal{A}$ is a set of arrows in a locally small cocomplete category $\mathcal{C}$, all of whose domains are small relative to $\mathcal{A}$-cobase changes then there exists a functorial factorisation (ι, π) with $\iota(f) \in \mathrm{Cell}(\mathcal{A})$ and $\pi(f) \in \mathcal{A}^{\pitchfork}$ for all f in $\mathcal{C}$.

Proof. For $j \in \mathcal{A}$ let us write κ_j for the minimal cardinal such that j is κ_j-small relative to $\mathcal{A}$-cobase changes. These κ_j form a set so that we can define κ to be the supremum of all the κ_j, choose a κ-directed ordinal λ (e.g. $\mathrm{cf}\,\kappa$ or κ^+) and then apply the above theorem. □

Noting that in a locally presentable category, every object is presentable (in particular small), we can use the small object argument to immediately construct an infinity of functorial weak factorisation systems.

(6.8) **Example.** If $\mathcal{C}$ is a locally presentable category then every set of arrows $\mathcal{A}$ in $\mathcal{C}$ defines a functorial factorisation system by the small object argument.

(6.9) **Corollary.** Again for $\mathcal{A}$ a set of arrows in a locally small cocomplete category $\mathcal{C}$, with all domains of arrows in $\mathcal{A}$ small relative to $\mathcal{A}$-cobase changes, we have $\mathcal{A}\text{-cof} = \mathrm{Sat}(\mathcal{A})$. Put differently, every $f \in \mathcal{A}\text{-cof}$ is a strong retract of some $i \in \mathrm{Cell}(\mathcal{A})$.

Proof. Choose a functorial factorisation (ι, π) as in the last corollary, so that $f = \pi(f) \circ \iota(f)$ with $\iota(f) \in \mathrm{Cell}(\mathcal{A})$ and $\pi(f) \in \mathcal{A}^{\pitchfork}$. By our hypothesis, f has the left lifting property with respect to $\pi(f)$ and by the retract argument, f is a strong retract of $\iota(f)$. □

We have already seen that being κ-small relative to $\mathcal{A}$-cobase changes is equivalent to being κ-small relative to $\mathrm{Cell}(\mathcal{A})$ but with our last corollary, we can prove even more. The following proof is almost word by word from Hovey's book [34, Proposition 2.1.16], where it is attributed to Hirschhorn.

(6.10) **Corollary.** Let $\mathcal{C}$ and $\mathcal{A}$ as in the last two corollaries and κ an infinite cardinal. Then a $C \in \mathcal{C}$ is κ-small relative to $\mathcal{A}$-cobase changes iff it is κ-small relative to $\mathcal{A}$-cof.

Proof. The direction "$\Leftarrow$" is trivial because $\mathcal{A}$-cobase changes lie in $\mathrm{Cell}(\mathcal{A}) \subseteq {}^{\pitchfork}(\mathcal{A}^{\pitchfork}) = \mathcal{A}\text{-cof}$ and for the other direction, let $X\colon \lambda \to \mathcal{C}$ be a κ-directed transfinite sequence of arrows in $\mathcal{A}$-cof. We choose a functorial factorisation (ι, π) as in the small object argument and shall use it to construct a transfinite sequence $Y\colon \lambda \to \mathcal{C}$ of arrows in $\mathrm{Cell}(\mathcal{A})$ as well as natural transformations $\sigma\colon X \Rightarrow Y$ and $\rho\colon Y \Rightarrow X$ such that $\rho \circ \sigma = \mathrm{id}_X$, which we do by transfinite recursion (with a dependent choice, cf. (2.16)) starting with $Y_0 := X_0$ and $\rho_0 := \sigma_0 := \mathrm{id}_{X_0}$. If Y_β, ρ_β and σ_β are already defined for some β with $\beta + 1 < \lambda$, we apply (ι, π) to factor the composite $Y_\beta \to X_\beta \to X_{\beta+1}$ into $g_\beta\colon Y_\beta \to Y_{\beta+1}$ and $\rho_{\beta+1}\colon Y_{\beta+1} \to X_{\beta+1}$ with $g_\beta \in \mathrm{Cell}(\mathcal{A})$ and $\rho_{\beta+1} \in \mathcal{A}^{\pitchfork}$. This gives us a commutative diagram

$$\begin{tikzcd}
X_\beta \arrow[r, "\sigma_\beta"] \arrow[dd] & Y_\beta \arrow[rr, "g_\beta"] \arrow[dr, "\rho_\beta"'] & & Y_{\beta+1} \arrow[dd, "\rho_{\beta+1}"] \\
& & X_\beta \arrow[dr] & \\
X_{\beta+1} \arrow[rrr, equal] & & & X_{\beta+1}
\end{tikzcd}$$

and we can choose a lift $\sigma_{\beta+1}\colon X_{\beta+1} \to Y_{\beta+1}$. Finally, if $\beta < \lambda$ is a limit ordinal, we obviously put $Y_\beta := \operatorname{colim}_{\alpha<\beta} Y_\alpha$, $\sigma_\beta := \operatorname{colim}_{\alpha<\beta} \sigma_\alpha$ and $\rho_\alpha := \operatorname{colim}_{\alpha<\beta} \rho_\alpha$. Now ρ and σ induce arrows $r\colon \operatorname{colim} Y \to \operatorname{colim} X$ and $s\colon \operatorname{colim} X \to \operatorname{colim} Y$ such that $r \circ s = \mathrm{id}_{\operatorname{colim} X}$, yielding a commutative diagram whose vertical composites are identities

$$\begin{tikzcd}
\operatorname{colim} \mathcal{C}(C, X) \arrow[r] \arrow[d] & \mathcal{C}(C, \operatorname{colim} X) \arrow[d] \\
\operatorname{colim} \mathcal{C}(C, Y) \arrow[r, "\cong"] \arrow[d] & \mathcal{C}(C, \operatorname{colim} Y) \arrow[d] \\
\operatorname{colim} \mathcal{C}(C, X) \arrow[r] & \mathcal{C}(C, \operatorname{colim} X)
\end{tikzcd} .$$

From this, we readily get an inverse for $\operatorname{colim} \mathcal{C}(C, X) \to \mathcal{C}(C, \operatorname{colim} X)$. $\square$

Chapter 5

MODEL CATEGORIES

In this chapter, we are going to review some results about the interplay between model structures and categorical smallness conditions (such as smallness or presentability of objects). The first section is taken almost verbatim from [34] and is maybe the most well-known. The results about combinatorial model categories can be found in [19] but with lots of details missing; not least due to references to Smith's book about combinatorial model categories, which has never been published. Finally, the account of Smith's theorem given here mostly follows [3] though our presentation differs considerably.

1. Cofibrantly Generated Model Categories

(1.1) **Definition.** A model category $\mathcal{M}$ is called *cofibrantly generated* iff there exist sets $\mathcal{I}$, $\mathcal{J}$ of arrows such that

(a) the domains of arrows in $\mathcal{I}$ and $\mathcal{J}$ are small relative to $\mathcal{I}$- and $\mathcal{J}$-cobase changes respectively;

(b) $\mathcal{I}^{\pitchfork}$ and $\mathcal{J}^{\pitchfork}$ are the classes of acyclic fibrations and fibrations respectively;

The elements of $\mathcal{I}$ are then referred to as *generating cofibrations* and those of $\mathcal{J}$ as *generating acyclic cofibrations.*

(1.2) **Observation.** Given a cofibrantly generated model category as above, the results from the previous section immediately imply

(a) $\mathcal{I}$-cof and $\mathcal{J}$-cof are, respectively, the classes of cofibrations and acyclic cofibrations;

(b) every cofibration is a strong retract of an arrow in $\mathrm{Cell}(\mathcal{I})$, while every acyclic cofibration is a strong retract of an arrow in $\mathrm{Cell}(\mathcal{J})$;

(c) the domains of all arrows in $\mathcal{I}$ and $\mathcal{J}$ are small relative to cofibrations and acyclic cofibrations respectively;

Obviously, we usually don't want to start with a fully-fledged model category and then single out generating (acyclic) cofibrations but rather, we want to start with a subcategory of weak equivalences that we would like to formally invert and then construct a model structure with these weak equivalences by specifying generating (acyclic) cofibrations. Thus the following theorem is the central part of this section.

(1.3) **Theorem. (Recognition Theorem)** Let $\mathcal{M}$ be a bicomplete category, $\mathcal{W}$ a class and $\mathcal{I}$, $\mathcal{J}$ sets of arrows in $\mathcal{M}$. Then there exists a cofibrantly generated model structure on $\mathcal{M}$ having $\mathcal{W}$ as weak equivalences, $\mathcal{I}$ as generating cofibrations and $\mathcal{J}$ as generating acyclic cofibrations iff the following conditions are met:

(a) $\mathcal{W}$ satisfies the 2-out-of-3 axiom and is closed under retracts;

(b) the domains of all arrows in $\mathcal{I}$ and $\mathcal{J}$ are small relative to $\mathcal{I}$- and $\mathcal{J}$-cobase changes respectively;

(c) $\mathcal{I}^{\pitchfork} \subseteq \mathcal{W}$ and $\mathrm{Cell}(\mathcal{J}) \subseteq \mathcal{W}$;

(d) $\mathcal{I}^{\pitchfork} \subseteq \mathcal{J}^{\pitchfork}$ or $\mathcal{J} \subseteq \mathcal{I}\text{-cof}$;

(e) $\mathcal{W} \cap \mathcal{J}^{\pitchfork} \subseteq \mathcal{I}^{\pitchfork}$ or $\mathcal{W} \cap \mathcal{I}\text{-cof} \subseteq \mathcal{J}\text{-cof}$.

Proof. The conditions stated are certainly necessary. Conversely, if they hold, we define a model structure on $\mathcal{M}$ with weak equivalences $\mathcal{W}$ by taking

- $\mathcal{I}\text{-cof} = {}^{\pitchfork}(\mathcal{I}^{\pitchfork})$ as the cofibrations and
- $\mathcal{J}^{\pitchfork}$ as the fibrations.

The first part of (a) is the required 2-out-of-3 axiom and the second part together with the fact that classes of the form $\mathcal{A}^{\pitchfork}$ or ${}^{\pitchfork}\mathcal{A}$ are always closed under retracts gives us the retract axiom. Taking functorial factorisations (ι, π), (ι', π') as provided by the small object argument applied to $\mathcal{I}$ and $\mathcal{J}$ respectively, the factorisation axiom requires

$$\mathcal{I}^{\pitchfork} \subseteq \mathcal{W} \cap \mathcal{J}^{\pitchfork} \qquad \text{and} \qquad \mathrm{Cell}(\mathcal{J}) \subseteq \mathcal{W} \cap \mathcal{I}\text{-cof}.$$

These two inclusions are implied by hypothesis (c) together with both statements of hypothesis (d) (using that $\mathcal{I}$-cof is cellularly saturated). But in fact the two statements in (d) are equivalent because $({}^{\pitchfork}-, -^{\pitchfork})$ is an antitone Galois correspondence.

The only thing left to check is the lifting axiom, which consists exactly of the two inclusions in (e) (where we use that $\mathcal{I}\text{-cof}^{\pitchfork} = \big({}^{\pitchfork}(\mathcal{I}^{\pitchfork})\big)^{\pitchfork} = \mathcal{I}^{\pitchfork}$). Since we only assumed one of them, we need to check that either one implies the other.

Assuming the second one, we let $f \in \mathcal{W} \cap \mathcal{J}^{\pitchfork}$ and factor it as $f = p \circ i$ with $i \in \mathrm{Cell}(\mathcal{I}) \subseteq \mathcal{I}\text{-cof}$ and $p \in \mathcal{I}^{\pitchfork} \cap \mathcal{W}$. By 2-out-of-3, we also have $i \in \mathcal{W}$ and hence $i \in \mathcal{W} \cap \mathcal{I}\text{-cof} \subseteq \mathcal{J}\text{-cof}$ by assumption. It follows that we can solve the lifting problem on the left below and conclude that f is a retract of $p \in \mathcal{I}^{\pitchfork}$ (whence $f \in \mathcal{I}^{\pitchfork}$) as seen on the right.

$$\begin{tikzcd} X \arrow[r, "\mathrm{id}_X"] \arrow[d, "\mathcal{J}\text{-cof}\ni i"'] & X \arrow[d, "f \in \mathcal{J}^{\pitchfork}"] \\ Z \arrow[r, "p"'] \arrow[ur, dotted, "d"] & Y \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} X \arrow[r, "i"] \arrow[d, "f"'] & Z \arrow[r, "d"] \arrow[d, "p"'] & X \arrow[d, "f"'] \\ Y \arrow[r, "\mathrm{id}_Y"'] & Y \arrow[r, "\mathrm{id}_Y"'] & Y \end{tikzcd}$$

The other implication is similar. □

(1.4) **Remark.** The recognition theorem provides several alternative sufficient criteria to get a cofibrantly generated model structure. One important option (used for Smith's theorem) is to check points (a) and (b) and that $\mathcal{I}^{\pitchfork} \subseteq \mathcal{W}$ as well as $\mathcal{J}\text{-cof} = \mathcal{W} \cap \mathcal{I}\text{-cof}$.

2. Combinatorial Model Categories

Cofibrantly generated model categories are ones where the model structure satisfies a certain smallness condition. If in addition, the same holds true for the underlying category, we arrive at the concept of a combinatorial model category.

(2.1) **Definition.** A model category $\mathcal{M}$ is *combinatorial* iff it is cofibrantly generated and locally presentable. More restrictively, for κ a regular cardinal, we are going to call a model category $\mathcal{M}$ *κ-combinatorial* iff it is locally κ-presentable and cofibrantly generated such that the domains as well as the codomains of all arrows in the generating cofibrations and generating acyclic cofibrations are κ-presentable.

(2.2) **Remark.** Since, in a locally presentable category, all objects are presentable (and in particular small), the smallness condition on generating (acyclic) cofibrations of a cofibrantly generated model category is automatically satisfied, so that $\mathcal{M}$ is combinatorial iff

(a) $\mathcal{M}$ is locally presentable;

(b) it has a set of arrows $\mathcal{I}$ such that $\mathcal{I}^{\pitchfork}$ is the class of acyclic fibrations;

(c) it has a set of arrows $\mathcal{J}$ such that $\mathcal{J}^{\pitchfork}$ is the class of fibrations.

Moreover, since (local) κ-presentability implies (local) κ'-presentability for all regular $\kappa' \geqslant \kappa$, $\mathcal{M}$ is combinatorial iff it is κ-combinatorial for some κ.

(2.3) **Nomenclature.** For the sake of conciseness, if $\mathcal{M}$ is any κ-combinatorial model category, let us call *standard factorisations* the two cellular factorisations of length κ with respect to the generating cofibrations and the generating acyclic cofibrations.

(2.4) **Corollary.** In a combinatorial model category, we can always pick generating cofibrations and generating acyclic cofibrations with cofibrant domains. $\square$

The reason to work with locally presentable categories is to be able to reduce certain constructions and arguments to small objects. To do this within a homotopical context, the model structure of a combinatorial category should preserve smallness in some sense. That this is indeed the case is not all that surprising since the whole point of the small object argument was, as the name suggests, to make use of the smallness of objects.

(2.5) **Proposition.** Let $\mathcal{C}$ be a locally κ-presentable category and $\mathcal{A} \subseteq \operatorname{Arr}(\mathcal{C})$ a set of arrows, all of whose domains and codomains are κ-presentable. Then $\mathcal{A}$-cellular factorisation (which is a functor $\mathcal{C}^{[1]} \to \mathcal{C}^{[2]}$) of any length λ is κ-accessible (i.e. preserves κ-filtered colimits).

Proof. First note that κ-presentability in $\mathcal{C}^{[1]}$ and $\mathcal{C}^{[2]}$ is just pointwise presentability, as was shown in (3.6.4). We now proceed by transfinite induction on λ. The base case $\lambda = 0$ is trivial, since the $\mathcal{A}$-cellular approximation of length 0 just factors an arrow $f: A \to B$ as id_A, followed by f. The limit case is also trivial, since there, $\mathcal{A}$-cellular factorisation of length λ is just the colimit of the $\mathcal{A}$-cellular factorisations of strictly smaller length. This leaves the successor case.

So let $\varphi: X \Rightarrow Y$ be a morphism of κ-filtered diagrams $X,\ Y: \mathcal{I} \to \mathcal{C}$ with colimit $f: C \to D$ and let's denote the $\mathcal{A}$-cellular approximation of length λ of a morphism

$$C' \xrightarrow{g} D' \qquad \text{by} \qquad C' \xrightarrow{\iota_\lambda(g)} C_\lambda(g) \xrightarrow{\pi_\lambda(g)} D'.$$

Since $C_{\lambda+1}(g) = C_1(\pi_\lambda(g))$, it suffices to consider the case $\lambda = 1$. We need to show that

$$C \xrightarrow{\iota_1(f)} C_1(f) \xrightarrow{\pi_1(f)} D \qquad \text{is the colimit of} \qquad X \xrightarrow{\iota_1(\varphi)} C_1(\varphi) \xrightarrow{\pi_1(\varphi)} Y.$$

Since colimits are calculated pointwise, we really only need to show that $C_1(f)$ is the colimit of $C_1(\varphi)$. By definition, we need to consider the set $S_0(f) := \coprod_{i\in\mathcal{A}} \mathrm{Hom}_{\mathcal{C}^{[1]}}(i, f)$ consisting of all triples

$$s = (a_s, g_s, h_s) \text{ fitting into a commuting square } \begin{tikzcd} A_s \arrow[r, "g_s"] \arrow[d, "a_s"'] & C \arrow[d, "f"] \\ B_s \arrow[r, "h_s"'] & D \end{tikzcd} \text{ and with } a_s \in \mathcal{A}.$$

With this, $C \to C_1(f)$ is the pushout

$$\begin{tikzcd} \coprod\limits_{s\in S_\beta(f)} A_s \arrow[r, "{[g_s]_s}"] \arrow[d, "\amalg_s a_s"'] & C \arrow[d] \\ \coprod\limits_{s\in S_\beta(f)} B_s \arrow[r] & C_1(f) \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd} ,$$

while $\pi_1(f)\colon C_1(f) \to D$ is the unique arrow induced by the h_s and f (similarly for φ). Since cobase changes of coproducts can be calculated as transfinite compositions, gluing one summand at a time (cf. (4.3.3)), we can assume that $\mathcal{A}$ consists of a single arrow $a\colon A \to B$. We now note that the $S_0(\varphi_I)$ define a diagram $S_0(\varphi)\colon \mathcal{I} \to \mathbf{Sets}$, where a morphism $i\colon I \to J$ in $\mathcal{I}$ induces $(a, g, h) \mapsto (a, Xi \circ g, Yi \circ h)$ or more diagrammatically:

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "a"'] & X_I \arrow[d, "\varphi_I"] \\ B \arrow[r, "h"'] & Y_I \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} A \arrow[r, "g"] \arrow[d, "a"'] & X_I \arrow[r, "Xi"] \arrow[d, "\varphi_I"'] & X_J \arrow[d, "\varphi_J"'] \\ B \arrow[r, "h"'] & Y_I \arrow[r, "Yi"'] & Y_J \end{tikzcd}$$

The κ-presentability of A and B means that $i\colon A \to B$ is κ-presentable in $\mathcal{C}^{[1]}$, which in turn now implies that

$$S_0(f) = \mathrm{Hom}_{\mathcal{C}^{[1]}}(i, f) \cong \mathrm{Hom}_{\mathcal{C}^{[1]}}(i, \operatorname*{colim}_I \varphi_I) \cong \operatorname*{colim}_I \mathrm{Hom}_{\mathcal{C}^{[1]}}(i, \varphi_I) = \operatorname*{colim}_I S_0(\varphi_I).$$

Finally, since $\operatorname{colim}_I A \cdot S_0(\varphi_I) \cong A \cdot S_0(f)$ (the copower functor $A \cdot -$ is left adjoint to $\mathcal{C}(A, -)$) and colimits commute (and $\operatorname{colim}_I C \cong C$ because $\mathcal{I}$ is connected), we get

$$\begin{aligned} C_1(f) = \operatorname{colim}\left(\coprod_{S_0(f)} B \leftarrow \coprod_{S_0(f)} A \to C \right) &\cong \operatorname{colim} \operatorname*{colim}_I \left(\coprod_{S_0(\varphi_I)} B \leftarrow \coprod_{S_0(\varphi_I)} A \to C \right) \\ \cong \operatorname*{colim}_I \operatorname{colim}\left(\coprod_{S_0(\varphi_I)} B \leftarrow \coprod_{S_0(\varphi_I)} A \to C \right) &= \operatorname*{colim}_I C_1(\varphi_I). \end{aligned}$$ □

(2.6) **Corollary.** If $\mathcal{M}$ is a κ-combinatorial model category, the standard factorisations (which are functors $\mathcal{M}^{[1]} \to \mathcal{M}^{[2]}$) preserve κ-filtered colimits. □

As another consequence of the accessibility of the standard factorisations, we can now show that κ-filtered colimits in a κ-combinatorial model category are actually homotopy colimits.

(2.7) **Proposition.** If $\mathcal{M}$ is a κ-combinatorial model category and $\mathcal{I}$ a κ-filtered indexing category then $\operatorname{colim}\colon \mathcal{M}^{\mathcal{I}} \to \mathcal{M}$ preserves (pointwise) fibrations, weak equivalences and acyclic fibrations. In particular, κ-filtered colimits are homotopy colimits.

Proof. Let $\tau\colon X \Rightarrow Y$ be a natural weak equivalence between diagrams $X, Y\colon \mathcal{I} \to \mathcal{M}$ (i.e. every τ_I is a weak equivalence). Since the standard factorisation into an acyclic cofibration followed by a fibration is κ-accessible, we can apply it pointwise to τ to get

$$X_I \overset{\sim}{\rightarrowtail} Z_I \twoheadrightarrow Y_I \qquad \text{functorial in } I \in \mathcal{I}$$

and the colimit of this $X \Rightarrow Z \Rightarrow Y$ is again a factorisation

$$\operatorname*{colim}_{\mathcal{I}} X \overset{\sim}{\rightarrowtail} \operatorname*{colim}_{\mathcal{I}} Z \twoheadrightarrow \operatorname*{colim}_{\mathcal{I}} Y$$

into an acyclic cofibration, followed by a fibration. It suffices to show that $\operatorname{colim} Z \to \operatorname{colim} Y$ is acyclic; i.e. right transverse to every generating cofibration $i\colon A \rightarrowtail B$. But since every τ_I is a weak equivalence, every $Z_I \to Y_I$ is acyclic and a commutative square on the left

$$\begin{tikzcd} A \arrow[r] \arrow[d, tail, "i"'] & \operatorname{colim}_{\mathcal{I}} Z \arrow[d, two heads] \\ B \arrow[r] & \operatorname{colim}_{\mathcal{I}} Y \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} A \arrow[r] \arrow[d, tail, "i"'] & Z_I \arrow[r] \arrow[d, two heads, "\sim"] & \operatorname{colim}_{\mathcal{I}} Z \arrow[d, two heads] \\ B \arrow[r] \arrow[ur, dotted] & Y_I \arrow[r] & \operatorname{colim}_{\mathcal{I}} Y \end{tikzcd}$$

is a morphism in $\mathcal{M}^{[1]}$. There, i is κ-presentable (by (3.6.4)) and so our morphism factors through some $Z_I \to Y_I$, which is an acyclic fibration and we find a lift as on the right.

This last step also proves our claim about the preservation of pointwise (acyclic) fibrations, where we start with a pointwise (acyclic) fibration $Z \Rightarrow Y$ and construct diagonal fillers by the above manner for i a generating (acyclic) cofibration. $\square$

(2.8) **Corollary.** If $\mathcal{M}$ is a combinatorial model category with fibrations $\mathcal{F}$ and weak equivalences $\mathcal{W}$ (viewed as a full subcategories of $\mathcal{M}^{[1]}$), then $\mathcal{F}$, $\mathcal{W}$, $\mathcal{F} \cap \mathcal{W} \hookrightarrow \mathcal{M}^{[1]}$ are all accessibly embedded accessible full subcategories.

Proof. Assuming $\mathcal{M}$ is κ-combinatorial, we have just shown that the three subcategories are κ-accessibly embedded. Using (3.8.17), we shall show that $\mathcal{F}$ and $\mathcal{F} \cap \mathcal{W}$ are closed under κ-pure subobjects. We do this for $\mathcal{F}$, with the case of $\mathcal{F} \cap \mathcal{W}$ being analogous, replacing the generating acyclic cofibrations by generating cofibrations everywhere.

So, given a fibration p, together with a κ-pure subobject $i\colon q \hookrightarrow p$ in $\mathcal{M}^{[1]}$, we need to show that q is right transverse to every generating acyclic cofibration j; i.e. given a commutative square in $\mathcal{M}$

(2.9) $$\begin{tikzcd} J_1 \arrow[r, "f_1"] \arrow[d, tail, "j"', "\sim"] & Q_1 \arrow[d, "q"] \\ J_2 \arrow[r, "f_2"'] & Q_2 \end{tikzcd}$$

with j a generating acyclic cofibration, we need to construct a diagonal filler. For this, we first compose the square with i and find a diagonal filler d as depicted on the left below and then construct the commutative square in $\mathcal{M}^{[1]}$ as depicted on the right.

$$\begin{tikzcd} J_1 \arrow[r, "f_1"] \arrow[d, tail, "j"', "\sim"] & Q_1 \arrow[r, "i_1"] & P_1 \arrow[d, two heads, "p"] \\ J_2 \arrow[r, "f_2"'] \arrow[urr, dotted, "d"] & Q_2 \arrow[r, "i_2"'] & P_2 \end{tikzcd} \qquad \begin{tikzcd} j \arrow[r, "f"] \arrow[d, "{(j,\mathrm{id}_{J_2})}"'] & q \arrow[d, "i"] \\ \mathrm{id}_{J_2} \arrow[r, "{(d, i_2 \circ f_2)}"'] & p \end{tikzcd}$$

Since j and id_{J_2} are κ-presentable in $\mathcal{M}^{[1]}$ (by (3.6.4)), we find an upper diagonal filler $d' = (d'_1, d'_2)\colon \mathrm{id}_{J_2} \to q$ for the square on the right, which satisfies

$$d'_1 \circ j = f_1, \qquad d'_2 \circ \mathrm{id}_{J_2} = f_2 \qquad \text{and} \qquad q \circ d'_1 = d'_2 \circ \mathrm{id}_{J_2} = f_2,$$

so that d'_1 is a diagonal filler for the original square (2.9).

Finally, the weak equivalences are accessible by (3.8.20) since they are the full preimage of the accessibly embedded accessible full subcategory $\mathcal{F} \cap \mathcal{W} \subseteq \mathcal{M}^{[1]}$ under the composite accessible functor

$$\mathcal{M}^{[1]} \to \mathcal{M}^{[2]} \xrightarrow{\delta_0^*} \mathcal{M}^{[1]},$$

where the first one is the standard factorisation into an acyclic cofibration followed by a fibration and δ_0^* maps a pair of arrows $\to\to$ to the second one (which is accessible because it has a right adjoint). □

Another key property of combinatorial model categories is that the standard factorisations preserve presentability above a certain level.

(2.10) **Proposition.** For every κ-combinatorial model category $\mathcal{M}$, there is a regular cardinal $\kappa' \geqslant \kappa$ such that for every regular $\mu \trianglerighteq \kappa'$ (cf. (3.4.2)), the standard factorisations induce $\mathcal{M}_\mu^{[1]} \to \mathcal{M}_\mu^{[2]}$. Put differently, above κ', the standard factorisations preserve the level of presentability.

Proof. The category of κ-presentable objects $\mathcal{M}_\kappa$ is essentially small. Therefore, we find a regular cardinal κ' such that factoring morphisms between κ-presentable objects (using the standard factorisations only produces κ'-presentable ones.

Now given $\mu \trianglerighteq \kappa'$, the Makkai-Paré theorem (3.6.6) tells us that any morphism $f\colon C \to D$ lifts to a morphism $\varphi\colon X \Rightarrow Y$ between κ-filtered and μ-small diagrams, whose colimit is f. Because our standard factorisations are κ-accessible, we can apply them pointwise and get factorisations of diagrams $X \Rightarrow Z \Rightarrow Y$ whose colimits are the standard factorisations of f. By our choice of κ', Z is pointwise κ'-presentable. That is to say, Z is a μ-small diagram of κ'-presentable (whence μ-presentable) objects and so, its colimit is again μ-presentable. □

(2.11) **Scholium.** As seen in the proof, we can take κ' from the above proposition to be the smallest (regular) cardinal such that the standard factorisations map $\mathcal{M}_\kappa^{[1]}$ to $\mathcal{M}_{\kappa'}^{[2]}$ (i.e. such that factoring morphisms between κ-presentable objects only produces κ'-presentable ones).

3. The Interpolating Small Object Argument

An essential ingredient for Smith's theorem, which we are going to prove in the next section, is the "interpolating" version of the small object argument. Its proof follows essentially the same lines as the one of the ordinary small object argument.

(3.1) **Definition.** Given two classes of arrows $\mathcal{A}$, $\mathcal{W}$ in a category $\mathcal{C}$, a third class $\mathcal{B}$ is said to *interpolate* (or to be *interpolating*) from $\mathcal{A}$ to $\mathcal{W}$ iff every morphism $a \to w$ in $\mathcal{C}^{[1]}$ with $a \in \mathcal{A}$ and $w \in \mathcal{W}$ factors through some $b \in \mathcal{B}$. Explicitly, that is to say that for every commutative square as on the left below with $a \in \mathcal{A}$ and $w \in \mathcal{W}$

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "a"'] & A' \arrow[d, "w"] \\ B \arrow[r, "g"'] & B' \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} A \arrow[r] \arrow[d, "a"'] & A'' \arrow[r] \arrow[d, "b"'] & A' \arrow[d, "w"'] \\ B \arrow[r] & B'' \arrow[r] & B' \end{tikzcd}$$

we have a chosen (using the axiom of choice if necessary) commutative rectangle as on the right with $b \in \mathcal{B}$, the composite of the top row being f and the composite of the bottom row being g.

(3.2) **Observation.** If $\mathcal{B}$ interpolates from $\mathcal{A}$ to $\mathcal{W}$, then so does any $\mathcal{B}' \supseteq \mathcal{B}$. On the other hand, if $\mathcal{B}$ interpolates from $\mathcal{A}$ to $\mathcal{W}$ and we have subclasses $\mathcal{A}' \subseteq \mathcal{A}$, $\mathcal{W}' \subseteq \mathcal{W}$, then $\mathcal{B}' := \mathcal{B} \cap \mathcal{W}'$ interpolates from $\mathcal{A}'$ to $\mathcal{W}'$. Finally, if $\mathcal{B}$ interpolates from $\mathcal{A}$ to $\mathcal{W}$ and from $\mathcal{A}'$ to $\mathcal{W}'$, it also interpolates from $\mathcal{A} \cap \mathcal{A}'$ to $\mathcal{W} \cup \mathcal{W}'$.

(3.3) **Example.** If $\mathcal{A}$ is any class of arrows, $\mathcal{W}$ is the class of all isomorphisms and $\mathcal{B}$ contains all identities on the codomains of all arrows in $\mathcal{A}$, then $\mathcal{B}$ interpolates from $\mathcal{A}$ to $\mathcal{B}$. To wit, given any commutative square as on the left, we obtain a commutative diagram as on the right, where the top row's composite is f and $\mathrm{id}_B \in \mathcal{B}$.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "a"'] & A' \arrow[d, "\cong"', "w"] \\ B \arrow[r, "g"'] & B' \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} A \arrow[r, "a"] \arrow[d, "a"'] & B \arrow[r, "g"] \arrow[d, "\mathrm{id}_B"'] & B' \arrow[r, "w^{-1}"] \arrow[d, "\mathrm{id}_{B'}"'] & A' \arrow[d, "w"] \\ B \arrow[r, "\mathrm{id}_B"'] & B \arrow[r, "g"'] & B' \arrow[r, "\mathrm{id}_{B'}"'] & B' \end{tikzcd}$$

With regard to (2.8) above, the following is the most important example of interpolating classes for us.

(3.4) **Example.** If $\mathcal{M}$ is a combinatorial model category with generating cofibrations $\mathcal{I}$ and weak equivalences $\mathcal{W}$, then there is a set of weak equivalences $W \subseteq \mathcal{W}$ that interpolates from $\mathcal{I}$ to $\mathcal{W}$. More generally, if $\mathcal{M}$ is any accessible category and $\mathcal{A}$, $\mathcal{B}$ two classes of morphisms on $\mathcal{M}$ such that

(a) $\mathcal{A}$ is actually a set and

(b) $\mathcal{B}$, viewed as a full subcategory of $\mathcal{M}^{[1]}$, is accessibly embedded and accessible,

then there is a set $B \subseteq \mathcal{B}$ that interpolates from $\mathcal{A}$ to $\mathcal{B}$.

Proof. Since $\mathcal{A}$ is a set, we find some regular κ such that every $a \in \mathcal{A} \subseteq \mathcal{M}^{[1]}$ is κ-presentable. Since $\mathcal{B} \subseteq \mathcal{M}^{[1]}$ is accessible, it is λ-accessible for some regular $\lambda \geqslant \kappa$ and we find a set B of λ-generators. Now, every $b \in \mathcal{B}$ is a λ-filtered colimit (in $\mathcal{M}^{[1]}$) $b \cong \operatorname{colim}_I b_I$ of a diagram in B and because every $a \in \mathcal{A}$ is λ-presentable, every $a \to b$ factors through some b_I. $\square$

(3.5) **Proposition.** If $\mathcal{B}$ interpolates from $\mathcal{A}$ to $\mathcal{W}$, then $\mathcal{B}$ also interpolates from $\mathcal{A}$ to $\mathcal{W}$'s closure under retracts.

Proof. Given any commutative diagram as on the left

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "a"'] & A' \arrow[r, tail, "s"] \arrow[d, "v"'] \arrow[rr, bend left, "\mathrm{id}"] & A'' \arrow[r, two heads, "r"] \arrow[d, "w"'] & A' \arrow[d, "v"'] \\ B \arrow[r, "g"'] & B' \arrow[r, tail, "s'"'] \arrow[rr, bend right, "\mathrm{id}"'] & B'' \arrow[r, two heads, "r'"'] & B' \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "h"] \arrow[d, "a"'] \arrow[rr, bend left, "s\circ f"] & A''' \arrow[r, "k"] \arrow[d, "b"'] & A'' \arrow[d, "w"'] \\ B \arrow[r, "h'"'] \arrow[rr, bend right, "s'\circ g"'] & B''' \arrow[r, "k'"'] & B'' \end{tikzcd}$$

with $a \in \mathcal{A}$, $w \in \mathcal{W}$, we can factorise part of it as depicted on the right, where $b \in \mathcal{B}$. By composing this with the square $(r, r')\colon w \to v$, and noting that r and r' are retractions of s and s', respectively, we get the required interpolation of the square $(f, g)\colon a \to v$. □

(3.6) **Definition.** Let $f\colon X \to Y$ be an arrow in $\mathcal{C}$, $\mathcal{A}$ a set of arrows in $\mathcal{C}$ and $\mathcal{B}$ a class of arrows in $\mathcal{C}$ that interpolates from $\mathcal{A}$ to f. We define the $(\mathcal{A}, \mathcal{B})$-*cellular successor* to be the factorisation

$$X \xrightarrow{\iota_1(f)} C_1(f) \xrightarrow{\pi_1(f)} Y$$

of f obtained as follows. Let $S(f) := \coprod_{i\in\mathcal{A}} \mathrm{Hom}_{\mathcal{C}^{[1]}}(i, f)$ be the set that consists of all triples

$$s = (i_s, g_s, h_s) \quad \text{fitting into a commuting square} \quad \begin{tikzcd} A_s \arrow[r, "g_s"] \arrow[d, "i_s"'] & X \arrow[d, "f"] \\ B_s \arrow[r, "h_s"'] & Y \end{tikzcd} \quad \text{and with } i_s \in \mathcal{A}.$$

Using the chosen interpolations, we factor every such square

$$\begin{tikzcd} A_s \arrow[r, "g_s"] \arrow[d, "i_s"'] & X \arrow[d, "f"] \\ B_s \arrow[r, "h_s"'] & Y \end{tikzcd} \quad \text{as} \quad \begin{tikzcd} A_s \arrow[r, "g'_s"] \arrow[d, "i_s"'] & A'_s \arrow[r, "g''_s"] \arrow[d, "j_s"'] & X \arrow[d, "f"'] \\ B_s \arrow[r, "h'_s"'] & B'_s \arrow[r, "h''_s"'] & Y \end{tikzcd}$$

with $j_s \in \mathcal{B}$ and define $\iota_1(f)\colon X \to C_1(f)$ to be the pushout

$$\begin{tikzcd} \coprod_{s\in S(f)} A'_s \arrow[r, "{[g''_s]_s}"] \arrow[d, "\amalg_s j_s"'] & X \arrow[d, "\iota_1(f)"] \\ \coprod_{s\in S(f)} B'_s \arrow[r] & C_1(f) \arrow[ul, phantom, very near start, "\ulcorner"] \end{tikzcd},$$

while $\pi_1(f)$ is the unique arrow induced by f and $[h''_s]_s$. We take note that, by definition, $\iota_1(f) \in \mathrm{Cell}(\mathcal{B})$.

The problem with this definition is that we generally do not know if $\mathcal{B}$ is again interpolating from $\mathcal{A}$ to $\pi_1(f)$ and we need to find sufficient hypotheses to ensure that we can continue our cellular approximation process.

(3.7) **Definition.** Let $\mathcal{C}$ be a locally small cocomplete category,

- $\mathcal{A}$ a set of arrows in $\mathcal{C}$,
- $\mathcal{W}$ a class of arrows in $\mathcal{C}$ satisfying 2-out-of-3 and such that $\mathcal{W} \cap \mathcal{A}$-cof is cellularly saturated,
- $\mathcal{B} \subseteq \mathcal{W} \cap \mathcal{A}$-cof a class that interpolates from $\mathcal{A}$ to $\mathcal{W}$.

For an arrow $w\colon X \to Y$ in $\mathcal{W}$, we define a transfinite sequence $C(w)\colon \mathrm{Ord} \to \mathcal{C}$ (where Ord is the poset of all ordinals), together with a cocone $\pi(w)\colon C(w) \Rightarrow Y$ recursively by $C_0(w) := X$, $\pi_0(w) = w$ and for λ a limit ordinal necessarily $C_\lambda(w) := \operatorname{colim}_{\alpha<\lambda} C_\alpha(w)$ and $\pi_\lambda(w)$ the unique arrow induced by $\big(\pi_\alpha(w)\big)_{\alpha<\lambda}$. Finally, in the successor case, we define

$$C_{\alpha+1}(w) := C_1\big(\pi_\alpha(w)\big), \quad \pi_{\alpha+1}(w) := \pi_1\big(\pi_\alpha(w)\big) \quad \text{and } C(\alpha < \alpha+1) := \iota_1\big(\pi_\alpha(w)\big).$$

Now for λ any (usually limit) ordinal and $\iota_\lambda(w)\colon X = C_0(w) \to C_\lambda(w)$, the pair $\big(\iota_\lambda(w), \pi_\lambda(w)\big)$ is called an $(\mathcal{A},\mathcal{B})$-*cellular approximation (of length λ)* of w. By construction $\iota_\lambda(w) \in \mathrm{Cell}(\mathcal{B})$ and $\pi_\lambda(w) \circ \iota_\lambda(w) = w$.

(3.8) **Remark.** The cellular saturation of $\mathcal{W} \cap \mathcal{A}$-cof means that this class contains all isomorphisms and is closed under cobase change and transfinite compositions. In particular, $\mathcal{W}$ needs to contain all isomorphisms. In our main application (Smith's theorem), $\mathcal{W}$ is also required to be closed under retracts, so we could even assume that $\mathcal{W} \cap \mathcal{A}$-cof be saturated.

(3.9) **Proposition.** The transfinite sequence $C(w)$ together with $\pi(w)\colon C(w) \Rightarrow Y$ are well-defined.

Proof. We just need to check that we can always form the $(\mathcal{A},\mathcal{B})$-cellular successor. Because $\mathcal{W} \cap \mathcal{A}$-cof is cellularly saturated and $\mathcal{B} \subseteq \mathcal{W} \cap \mathcal{A}$-cof, we also have $\mathrm{Cell}(\mathcal{B}) \subseteq \mathcal{W} \cap \mathcal{A}$-cof. In particular, every $C_0(w) \to C_\alpha(w)$ belongs to $\mathcal{W}$. By 2-out-of-3 then, the same holds true for every $\pi_\alpha(w)\colon C_\alpha(w) \to Y$ and we can form the $(\mathcal{A},\mathcal{B})$-cellular successor. □

The functoriality of the interpolated cellular approximation is very similar to the classical one (4.6.3) and the same proof goes through almost verbatim. The difference is that the interpolated cellular approximation is not defined everywhere.

(3.10) **Proposition.** Under the same hypotheses as in (3.7), we identify $\mathcal{W}$ with the full subcategory of $\mathcal{C}^{[1]}$ defined by it. With this, $(\mathcal{A},\mathcal{B})$-cellular approximation can be made into a functor

$$\mathcal{W} \times \mathrm{Ord} \to \mathcal{C}^{[2]},\ (w,\lambda) \mapsto \big(\iota_\lambda(w), \pi_\lambda(w)\big)$$

such that, for every fixed ordinal λ, $(\mathcal{A},\mathcal{B})$-cellular approximation of length λ is a functorial factorisation.

Proof. Given a commutative square in $\mathcal{C}$ as on the left below with w, $w' \in \mathcal{W}$ as well as the result upon taking interpolated cellular approximation

$$\begin{tikzcd} X \arrow[r, "w"] \arrow[d, "a"'] & Y \arrow[d, "b"] \\ X' \arrow[r, "w'"'] & Y' \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} X \arrow[r, "\iota_\lambda(w)"] \arrow[d, "a"'] & C_\lambda(w) \arrow[r, "\pi_\lambda(w)"] \arrow[d, dotted, "\varphi_\lambda"'] & Y \arrow[d, "b"] \\ X' \arrow[r, "\iota_\lambda(w')"'] & C_\lambda(w') \arrow[r, "\pi_\lambda(w')"'] & Y' \end{tikzcd},$$

we need to construct a dotted arrow φ_λ making the diagram commute in a functorial (with respect to w) and natural (with respect to λ) way, which we do by transfinite recursion, starting with

$$(\varphi_0 := a)\colon\ (C_0(w) = X) \ \to\ (X' = C_0(w')).$$

The limit case is easy, as we just need to take colimits, which is clearly functorial with respect to w and natural with respect to λ (as it is defined by a universal property). In the successor case, we get a map $\psi\colon S(\pi_\lambda(w)) \to S(\pi_\lambda(w'))$ by $(i_s, g_s, h_s) \mapsto (i_s, \varphi_\lambda \circ g_s, b \circ h_s)$; i.e.

$$\begin{tikzcd} A_s \arrow[r, "g_s"] \arrow[d, "i_s"'] & C_\lambda(w) \arrow[d, "\pi_\lambda(w)"] \\ B_s \arrow[r, "h_s"'] & Y \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} A_s \arrow[r, "g_s"] \arrow[d, "i_s"'] & C_\lambda(w) \arrow[r, "\varphi_\lambda"] \arrow[d, "\pi_\lambda(w)"'] & C_\lambda(w') \arrow[d, "\pi_\lambda(w')"'] \\ B_s \arrow[r, "h_s"'] & Y \arrow[r, "b"'] & Y' \end{tikzcd} .$$

This in turn gives us a morphism of spans as follows

$$\begin{tikzcd} \coprod_{s \in S(\pi_\lambda(w))} B_s \arrow[d, hook] & \coprod_{s \in S(\pi_\lambda(w))} A_s \arrow[l, "\amalg_s j_s"'] \arrow[r, "{[g_s]_s}"] \arrow[d, hook] & C_\lambda(w) \arrow[d, "\varphi_\lambda"] \\ \coprod_{s' \in S(\pi_\lambda(w'))} B_{s'} & \coprod_{s' \in S(\pi_\lambda(w'))} A_{s'} \arrow[l, "\amalg_{s'} j_{s'}"'] \arrow[r, "{[g_{s'}]_{s'}}"] & C_\lambda(w') \end{tikzcd}$$

(where the two unnamed vertical arrows map the s-summand into the ψs-summand by the identity). Note that for this step, it is essential to have chosen interpolations, so that j_s at the top is the same as $j_{\psi s}$ at the bottom. Taking pushouts, we get $\varphi_{\lambda+1}\colon C_{\lambda+1}(w) \to C_{\lambda+1}(w')$ rendering the square

$$\begin{tikzcd} C_\lambda(w) \arrow[r, "\varphi_\lambda"] \arrow[d] & C_\lambda(w') \arrow[d] \\ C_{\lambda+1}(w) \arrow[r, "\varphi_{\lambda+1}"'] & C_{\lambda+1}(w') \end{tikzcd}$$

commutative (thus proving the naturality with respect to λ). Clearly, this construction is functorial with respect to w because the above assignment of spans is functorial (using the functoriality of φ_λ). $\square$

Since the following theorem is really only going to be used in the context of presentable categories, the κ-smallness condition can be replaced by κ-presentability and this is not going to affect the generality of the theorem significantly.

(3.11) **Theorem. (Interpolating Small Object Argument)** Under the same hypotheses as in (3.7), if κ is an infinite cardinal such that the domains of all arrows in $\mathcal{A}$ are κ-small relative to $\mathcal{A}$-cobase changes and λ is a κ-directed ordinal then $(\mathcal{A}, \mathcal{B})$-cellular factorisation $(\iota_\lambda, \pi_\lambda)$ of length λ is a functorial factorisation with $\iota_\lambda(w) \in \mathrm{Cell}(\mathcal{B})$ and $\pi_\lambda(w) \in \mathcal{A}^{\pitchfork}$ for all $w \in \mathcal{W}$.

Proof. The only claim we haven't shown yet is that $\pi_\lambda(w) \in \mathcal{A}^{\pitchfork}$ for every $w\colon X \to Y$ in $\mathcal{W}$. For this, suppose we are given a commutative square of solid arrows

$$\begin{tikzcd}
& C_\alpha(w) \arrow[dr, dotted] & \\
A \arrow[ur, dotted] \arrow[rr] \arrow[d, "i"'] & & C_\lambda(w) \arrow[d, "\pi_\lambda(w)"] \\
B \arrow[urr, dotted] \arrow[rr] & & Y
\end{tikzcd}$$

with $i \in \mathcal{A}$. By hypothesis, we find $\alpha < \lambda$ as well as $A \to C_\alpha(w)$ with

$$A \to C_\lambda(w) \;=\; A \to C_\alpha(w) \to C_\lambda(w), \text{ so that } (i, A \to C_\alpha(w), B \to Y) \in S(\pi_\alpha(w)).$$

Interpolating the square, we get

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "i"'] & A' \arrow[r] \arrow[d, "j"'] & C_\alpha(w) \arrow[d, "\pi_\alpha(w)"'] \\
B \arrow[r] & B' \arrow[r] & Y
\end{tikzcd}$$

and by construction of $C_{\alpha+1}(w)$, an arrow from $B' \to C_{\alpha+1}(w)$ satisfying

$$A' \to C_\alpha(w) \to C_{\alpha+1}(w) \;=\; A' \xrightarrow{j} B' \to C_{\alpha+1}(w) \qquad \text{and}$$

$$B' \to Y \;=\; B' \to C_{\alpha+1}(w) \xrightarrow{\pi_{\alpha+1}(w)} Y,$$

so that $B \to B' \to C_{\alpha+1}(w) \to C_\lambda(w)$ fits into the initial diagram. $\square$

(3.12) **Example.** The ordinary small object argument can be recovered from the interpolating version below if we take $\mathcal{B} = \mathcal{A}$, $\mathcal{W} = \operatorname{Arr}\mathcal{C}$ and factor every morphism $(f,g)\colon a \to w$ in $\mathcal{C}^{[1]}$ as the identity followed by (f,g).

(3.13) **Corollary.** Still under the hypotheses of (3.7), if our base category $\mathcal{C}$ is locally presentable and $\mathcal{W}$ closed under retracts, then $\mathcal{B}$-cof $= \operatorname{Sat}(\mathcal{B}) = \mathcal{W} \cap \mathcal{A}$-cof.

Proof. The equality $\mathcal{B}$-cof $= \operatorname{Sat}(\mathcal{B})$ follows from the corollary (4.6.9) to the classical small object argument and the hypothesis that $\mathcal{C}$ be locally presentable. Now, $\mathcal{W} \cap \mathcal{A}$-cof was required to be cellularly saturated but because $\mathcal{W}$ is closed under retracts, it is even saturated, whence $\mathcal{B} \subseteq \mathcal{W} \cap \mathcal{A}$-cof implies $\operatorname{Sat}(\mathcal{B}) \subseteq \mathcal{W} \cap \mathcal{A}$-cof. Conversely, if $f \in \mathcal{W} \cap \mathcal{A}$-cof, we pick a κ-directed ordinal λ (e.g. $\operatorname{cf}(\kappa)$) and factor $f = \pi_\lambda(f) \circ \iota_\lambda(f)$ as in the interpolating small object argument. Since $f \in \mathcal{A}$-cof $= {}^{\pitchfork}(\mathcal{A}^{\pitchfork})$ and $\pi_\lambda(f) \in \mathcal{A}^{\pitchfork}$, the retract argument tells us that f is a (strong) retract of $\iota_\lambda(f) \in \operatorname{Cell}(\mathcal{B})$. $\square$

4. Smith's Theorem

For a locally presentable base category, Smith's theorem is an improvement of the classical recognition theorem (1.3) for cofibrantly generated model categories in the sense that it allows us to do away with the generating acyclic cofibrations.

(4.1) **Theorem. (Smith)** Let $\mathcal{M}$ be a locally presentable category, $\mathcal{W}$ a class and $\mathcal{I}$ a set of arrows in $\mathcal{M}$. Then there exists a combinatorial model structure on $\mathcal{M}$ with $\mathcal{W}$ as weak equivalences and $\mathcal{I}$ as generating cofibrations iff the following conditions are met:

(a) $\mathcal{W}$ satisfies the 2-out-of-3 axiom and is closed under retracts;

(b) $\mathcal{I}^{\pitchfork} \subseteq \mathcal{W}$;

(c) $\mathcal{W} \cap \mathcal{I}$-cof is closed under cobase change and transfinite compositions (hence saturated);

(d) $\mathcal{W} \subseteq \mathcal{M}^{[1]}$ is an accessibly embedded accessible full subcategory.

Proof. By the recognition theorem (1.3) and the remark thereafter, we need to construct a set $\mathcal{J}$ (the generating acyclic cofibrations) of arrows in $\mathcal{M}$ such that $\mathcal{J}$-cof $= \mathcal{W} \cap \mathcal{I}$-cof. By the corollary (3.13) to the interpolating small object argument, it is enough to find a set $\mathcal{J} \subseteq \mathcal{W} \cap \mathcal{I}$-cof that interpolates from $\mathcal{I}$ to $\mathcal{W}$. This is where accessibility comes in.

As we have seen in (3.4) above, our condition (d) implies that there is a set $W \subseteq \mathcal{W}$ that interpolates from $\mathcal{I}$ to $\mathcal{W}$ and hence, it suffices to find a set in $\mathcal{W} \cap \mathcal{I}$-cof, that interpolates from $\mathcal{I}$ to W.

So, let $i\colon A \to B$ in $\mathcal{I}$, $w\colon C \to D$ in W and $(f,g)\colon i \to w$ any morphism in $\mathcal{M}^{[1]}$. Taking the pushout $P := B +_A C$, we use the (ordinary) small object argument for $\mathcal{I}$ to factor the induced morphism $[g,w]\colon P \to D$ as $[g,w] = b \circ a$ with $a \in \mathrm{Cell}(\mathcal{I})$ and $b \in \mathcal{I}^{\pitchfork}$:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "i"'] & C \arrow[d, "i'"] \arrow[rrd, bend left, "w"] & & \\
B \arrow[r, "f'"] \arrow[rrr, bend right, "g"'] & P \arrow[ul, phantom, "\ulcorner", very near start] \arrow[r, "a"] & Q \arrow[r, "b"] & D
\end{tikzcd} .$$

We put $j := a \circ i'$ which lies in $\mathcal{W}$ by 2-out-of-3 (since $b \in \mathcal{W}$ and $b \circ j = w \in \mathcal{W}$) as well as in $\mathcal{I}$-cof (since $a \in \mathrm{Cell}(\mathcal{I}) \subseteq \mathcal{I}$-cof and $i' \in \mathcal{I}$-cof as it is a cobase change of $i \in \mathcal{I} \subseteq \mathcal{I}$-cof). Finally, it interpolates from i to w by $(f, a \circ f')\colon i \to j$ and $(\mathrm{id}_C, b)\colon j \to w$. Letting $\mathcal{J}$ consist of all such j (which is a set as it is indexed by the set-variables $i \in \mathcal{I}$, $w \in \mathcal{W}$ and $(f,g) \in (\mathcal{M}^{[1]})(i,w)$), our proof is complete. $\square$

(4.2) **Remark.** Condition (d) in the theorem can be replaced by the (seemingly weaker) condition that there be a set $W \subseteq \mathcal{W}$ interpolating from $\mathcal{I}$ to $\mathcal{W}$. As we have seen in (3.4), this is always the case for an accessible $\mathcal{W}$. However, combining general stability results such as (3.8.19) and (3.8.20) with the fact that $\mathcal{W} \subseteq \mathcal{M}^{[1]}$ is always accessibly embedded and accessible, accessibility is usually easier to check.

Chapter 6

LEFT BOUSFIELD LOCALISATION

When localising categories (i.e. picking a class of morphisms that we would like to invert), a model structure is what we need to gain some control of the resulting homotopy category. Now when we already have some model structure, we can try to further localise our category; meaning that we might want to add in new weak equivalences. The question now is how we can get a new model structure with the added in weak equivalences. Since fibrations are right transverse to acyclic cofibrations and cofibrations are left transverse to acyclic fibrations, we will certainly not be able to keep the same fibrations and cofibrations. However, we can try to at least keep one of these two classes fixed, leading to the notion of a left and a right Bousfield localisation. As it stands, left Bousfield localisation is used far more often and is better studied than the right one.

1. Simplicial Mapping Spaces

In this section, we are going to recall some basic properties of mapping spaces in a simplicial (that is to say simplicially enriched) model category. This is all well-known and can be found in many different books, back to the originals ones due to Quillen. Recall that $\mathcal{M}$ being a simplicial model category means the following.

(a) $\mathcal{M}$ is enriched over simplicial sets (with the cartesian closed symmetric monoidal structure). That is to say, for every pair of objects A, $B \in \mathcal{M}$, we have a simplicial set (the *mapping space*) $\mathrm{Map}_{\mathcal{M}}(A, B)$ and these come with composition maps

$$\mathrm{Map}_{\mathcal{M}}(A, B) \times \mathrm{Map}_{\mathcal{M}}(B, C) \to \mathrm{Map}_{\mathcal{M}}(A, C)$$

satisfying associativity and where every $\mathrm{Map}_{\mathcal{M}}(A, A)$ has an identity vertex. These then allow us to view "taking the mapping space" as an enriched functor

$$\mathcal{M}^{\mathrm{op}} \times \mathcal{M} \to \mathbf{sSets},\ (A, B) \mapsto \mathrm{Map}_{\mathcal{M}}(A, B)$$

(where the action on mapping spaces is obtained by taking adjuncts of the composition maps). We routinely regard $\mathcal{M}$ as an ordinary category, with Hom-sets given by the vertices $\mathcal{M}(A, B) := \mathrm{Map}_{\mathcal{M}}(A, B)_0$.

(b) $\mathcal{M}$ is powered and copowered (a.k.a. cotensored and tensored). That is to say, we have (enriched) functors

$$\mathbf{sSets}^{\mathrm{op}} \times \mathcal{M} \to \mathcal{M},\ (K, A) \mapsto [K, A],\ \ \mathbf{sSets} \times \mathcal{M} \to \mathcal{M},\ (K, A) \mapsto K \odot A,$$

together with natural isomorphisms of simplicial sets

$$\mathrm{Map}_{\mathcal{M}}(A, [K, B]) \cong \mathrm{Map}_{\mathcal{M}}(K \odot A, B) \cong \mathrm{Map}_{\mathbf{sSets}}(K, \mathrm{Map}_{\mathcal{M}}(A, B)).$$

It suffices to know these functors on objects, as the naturality determines their action on mapping spaces. Taking vertices, we also have such natural isomorphisms between the Hom-sets in the underlying categories, rather than Hom-spaces.

(c) The underlying category (with Hom-sets $\mathcal{M}(A, B) := \mathrm{Map}_{\mathcal{M}}(A, B)_0$) is equipped with a model structure.

(d) These structures are related to each other by the so-called *Quillen SM7-axiom*, which says that for every cofibration $i\colon A \rightarrowtail B$ and every fibration $p\colon X \twoheadrightarrow Y$ in $\mathcal{M}$, the so-called *pullback product* map

$$i \boxtimes p := (i^*, p_*)\colon\ \mathrm{Map}_{\mathcal{M}}(B, X) \to \mathrm{Map}_{\mathcal{M}}(A, X) \times_{\mathrm{Map}_{\mathcal{M}}(A,Y)} \mathrm{Map}_{\mathcal{M}}(B, Y)$$

is a fibration and even acyclic if i or p is acyclic.

(1.1) **Example.** The primordial example for a simplicial model category is the category of simplicial sets with the Quillen model structure, which is cartesian closed (i.e. enriched over itself) by

$$\mathrm{Map}_{\mathbf{sSets}}(K, L)_n = \mathbf{sSets}(\Delta[n] \times K, L).$$

Let us quickly motivate the Quillen SM7-axiom, which might seem very abstract at first sight. In an ordinary (not necessary simplicial) model category $\mathcal{M}$, given two maps $i\colon A \to B$ and $p\colon X \to Y$, the pullback of Hom-sets

$\mathcal{M}(A, X) \times_{\mathcal{M}(A,Y)} \mathcal{M}(B, Y)$ consists of commutative squares
$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r] & Y \end{tikzcd}$$

and the pullback product $i \boxtimes p\colon \mathcal{M}(B, X) \to \mathcal{M}(A, X) \times_{\mathcal{M}(A,Y)} \mathcal{M}(B, Y)$ maps an arrow $d\colon B \to X$ to the commutative square

$$\begin{tikzcd} A \arrow[r, "d\circ i"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r, "p\circ d"'] \arrow[ur, dotted, "d"] & Y \end{tikzcd}.$$

It follows that a commutative square in $\mathcal{M}(A, X) \times_{\mathcal{M}(A,Y)} \mathcal{M}(B, Y)$ has a diagonal filler iff it lies in the image of $i \boxtimes p$ and the fibre is the precisely set of all possible diagonal fillers. With this, the usual lifting axioms for model categories can be restated as $i \boxtimes p$ being surjective if i is a cofibration, p a fibration and at least one of them acyclic. The part of the SM7-axiom where neither i nor p is acyclic is harder to motivate and involves homotopies of commutative squares.

In any case, Quillen's SM7-axiom should be viewed as the correct analogue of the usual lifting axiom for ordinary model categories and does in fact imply it (as discussed above). Moreover, it has the following interesting consequence.

(1.2) **Observation.** Let $\mathcal{M}$ be a simplicial model category, $i\colon A \to B$ and $p\colon X \to Y$. For any vertex $s \in \mathrm{Map}_{\mathcal{M}}(A, X) \times_{\mathrm{Map}_{\mathcal{M}}(A,Y)} \mathrm{Map}_{\mathcal{M}}(B, Y)$ (a commutative square from i to p), we can define the space of diagonal fillers for s as the homotopy fibre of the pullback product

$$i \boxtimes p\colon\ \mathrm{Map}_{\mathcal{M}}(B, X) \to \mathrm{Map}_{\mathcal{M}}(A, X) \times_{\mathrm{Map}_{\mathcal{M}}(A,Y)} \mathrm{Map}_{\mathcal{M}}(B, Y)$$

above s. If i is a cofibration and p a fibration, then, by the SM7-axiom, this pullback product is a Kan fibration and the homotopy fibre agrees with the strict one. If, in addition, i or p is acyclic, the pullback product is even an acyclic Kan fibration and hence, the space of diagonal fillers is contractible.

(1.3) **Proposition.** If $\mathcal{M}$ is a simplicial model category and $A \in \mathcal{M}$ then $* \odot A \cong A$. More generally, $M \odot A \cong M \cdot A$ for every set M (viewed as a discrete simplicial set).

Proof. For $B \in \mathcal{M}$ arbitrary, we easily calculate that

$$\mathcal{M}(* \odot A, B) \cong \mathbf{sSets}(*, \mathrm{Map}_{\mathcal{M}}(A, B)) \cong \mathrm{Map}_{\mathcal{M}}(A, B)_0 = \mathcal{M}(A, B)$$

and the first claim follows from the Yoneda lemma. For the second one, just note that $- \odot A \dashv \mathrm{Map}_{\mathcal{M}}(A, -)$, so that $- \odot A$ preserves colimits. □

Playing around with the adjunctions involved, one can equivalently formulate condition (d) in terms of the copower (a.k.a. tensor) functor, which is what is usually referred to as Quillen's SM7-axiom. Similarly, it can be rephrased in terms of the power functor. Let's be a little more general than necessary.

(1.4) **Definition.** An *adjunction in two-variables* consists of three functors

$$\otimes\colon \mathcal{C} \times \mathcal{D} \to \mathcal{E}, \qquad \mathrm{Hom}_l\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{E} \to \mathcal{D}, \qquad \mathrm{Hom}_r\colon \mathcal{D}^{\mathrm{op}} \times \mathcal{E} \to \mathcal{C}$$

together with natural isomorphisms

$$\mathcal{C}(C, \mathrm{Hom}_r(D, E)) \cong \mathcal{E}(C \otimes D, E) \cong \mathcal{D}(D, \mathrm{Hom}_l(C, E)).$$

It is a *Quillen adjunction in two-variables* iff $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$ are model categories and $\otimes$ is a *(left) Quillen bifunctor*, meaning that it preserves colimits in each variable (which is automatic by adjointness) and satisfies Quillen's SM7 axiom: For every pair of cofibrations $i\colon C \rightarrowtail C'$ in $\mathcal{C}$ and $j\colon D \rightarrowtail D'$ in $\mathcal{D}$, the *pushout product*

$$i \hat{\otimes} j\colon (C \otimes D') +_{C \otimes D} (C' \otimes D) \xrightarrow{[i \otimes \mathrm{id}, \mathrm{id} \otimes j]} C' \otimes D'$$

is again a cofibration and even acyclic if i or j is.

(1.5) **Example.** By definition, the copower, power and mapping space functors for a simplicial model category $\mathcal{M}$ form an adjunction in two variables:

$$\odot\colon \mathbf{sSets} \times \mathcal{M} \to \mathcal{M}, \quad [-,-]\colon \mathbf{sSets}^{\mathrm{op}} \times \mathcal{M} \to \mathcal{M}, \quad \mathrm{Map}_{\mathcal{M}}\colon \mathcal{M}^{\mathrm{op}} \times \mathcal{M} \to \mathbf{sSets}.$$

To see that it is even a Quillen adjunction in two variables, we use the following general observation.

(1.6) **Proposition.** Given an adjunction between two variables

$$\otimes\colon \mathcal{C} \times \mathcal{D} \to \mathcal{E}, \qquad \mathrm{Hom}_l\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{E} \to \mathcal{D}, \qquad \mathrm{Hom}_r\colon \mathcal{D}^{\mathrm{op}} \times \mathcal{E} \to \mathcal{C}$$

with $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$ model categories, the following are equivalent:

(a) For every pair of cofibrations $i\colon C \rightarrowtail C'$ in $\mathcal{C}$ and $j\colon D \rightarrowtail D'$ in $\mathcal{D}$,

$$Q := (C \otimes D') +_{C\otimes D} (C' \otimes D) \longrightarrow C' \otimes D'$$

is again a cofibration and even acyclic if i or j is acyclic.

(b) For every cofibration $i\colon C \rightarrowtail C'$ in $\mathcal{C}$ and every fibration $p\colon E \twoheadrightarrow E'$ in $\mathcal{E}$,

$$\mathrm{Hom}_l(C', E) \longrightarrow \mathrm{Hom}_l(C, E) \times_{\mathrm{Hom}_l(C,E')} \mathrm{Hom}_l(C', E) =: P$$

is a fibration and even acyclic if i or p is acyclic.

(c) For every cofibration $j\colon D \rightarrowtail D'$ in $\mathcal{D}$ and every fibration $p\colon E \twoheadrightarrow E'$ in $\mathcal{E}$,

$$\mathrm{Hom}_r(D', E) \longrightarrow \mathrm{Hom}_r(D, E) \times_{\mathrm{Hom}_r(D,E')} \mathrm{Hom}_r(D', E) =: R$$

is a fibration and even acyclic if j or p is acyclic.

Proof. We show the equivalence of (a) and (b). That of (a) and (c) is analogous. Juggling around with the universal properties of pushouts and pullbacks as well as adjunct morphisms, one can show that for morphisms $i\colon C \to C'$ in $\mathcal{C}$, $j\colon D \to D'$ in $\mathcal{D}$ and $p\colon E \to E'$ in $\mathcal{E}$ (and using the notation from the proposition), there is a one-to-one correspondence between commutative squares (as well as diagonal fillers for them)

$$\begin{tikzcd} Q \arrow[r, "{[f,f']}"] \arrow[d, "{[i_*,j_*]}"'] & E \arrow[d, "p"] \\ C' \otimes D' \arrow[r, "g"'] & E' \end{tikzcd} \quad \leftrightarrow \quad \begin{tikzcd} D \arrow[r, "f'^\sharp"] \arrow[d, "j"'] & \mathrm{Hom}_l(C', E) \arrow[d, "{(i^*,p_*)}"] \\ D' \arrow[r, "{(f^\sharp,g^\sharp)}"'] & P. \end{tikzcd}$$

□

As one can imagine, given an adjunction in two variables, we can always fix one of the variables and get lots of ordinary adjunctions as follows.

(1.7) **Observation.** Given a two-variable adjunction as above

$$\otimes\colon \mathcal{C} \times \mathcal{D} \to \mathcal{E}, \qquad \mathrm{Hom}_l\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{E} \to \mathcal{D}, \qquad \mathrm{Hom}_r\colon \mathcal{D}^{\mathrm{op}} \times \mathcal{E} \to \mathcal{C}$$

and objects $C \in \mathcal{C}$, $D \in \mathcal{D}$, $E \in \mathcal{E}$, we have

$$C \otimes - \dashv \mathrm{Hom}_l(C, -), \quad - \otimes D \dashv \mathrm{Hom}_r(D, -)$$

and $\mathrm{Hom}_l(-, E)$, $\mathrm{Hom}_r(-, E)$ form a contravariant adjoint pair; i.e.

$$\mathrm{Hom}_l(-, E)^{\mathrm{op}} \dashv \mathrm{Hom}_r(-, E), \quad \mathrm{Hom}_r(-, E)^{\mathrm{op}} \dashv \mathrm{Hom}_l(-, E).$$

In a two-variable Quillen adjunction as above, the functor $\otimes\colon \mathcal{C} \times \mathcal{D} \to \mathcal{E}$ is not quite left Quillen (i.e. it doesn't preserve cofibrations and acyclic cofibrations) and neither is it when we fix one variable. However, it is close enough to have a total left derived functor.

(1.8) **Proposition.** For a two-variable Quillen adjunction as above, if $C \in \mathcal{C}$ and $D \in \mathcal{D}$ are cofibrant then $C \otimes -$ and $- \otimes D$ are left Quillen (whence $\mathrm{Hom}_l(C, -)$ and $\mathrm{Hom}_r(D, -)$ are right Quillen). Similarly, if $E \in \mathcal{E}$ is fibrant, then $\mathrm{Hom}_l(-, E)$ and $\mathrm{Hom}_r(-, E)$ are right Quillen, meaning that they map (acyclic) cofibrations to (acyclic) fibrations.

Proof. We just prove the first statement with the rest being analogous. If $D \in \mathcal{D}$ is cofibrant and $i\colon C \rightarrowtail C'$ an (acyclic) cofibration in $\mathcal{C}$, we apply Quillen's SM7-axiom to i and $\varnothing \rightarrowtail D$ (using that $- \otimes \varnothing$ is constantly $\varnothing$), to get that

$$i \otimes \mathrm{id}_D\colon C \otimes D \cong (C \otimes D) +_{C \otimes \varnothing} C' \otimes \varnothing \to C' \otimes D$$

is a cofibration and even acyclic if i is. □

(1.9) **Proposition.** Given a two-variable Quillen adjunction as above, $\otimes\colon \mathcal{C} \times \mathcal{D} \to \mathcal{E}$ preserves cofibrant objects as well as cofibrations and acyclic cofibrations (hence weak equivalences by Ken Brown's lemma) between them. In particular, it has a total left derived functor $\otimes^{\mathbb{L}}$. Similarly

$$\mathrm{Hom}_l\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{E} \to \mathcal{D} \qquad \text{and} \qquad \mathrm{Hom}_r\colon \mathcal{D}^{\mathrm{op}} \times \mathcal{E} \to \mathcal{C}$$

preserve fibrant objects as well as fibrations and acyclic fibrations (hence weak equivalences) between them. Consequently, they have total right derived functors $\mathbb{R}\,\mathrm{Hom}_l$ and $\mathbb{R}\,\mathrm{Hom}_r$.

Proof. We check the claim for $\otimes$ as the other ones are analogous. Since $\otimes$ preserves colimits in each variable (and in particular initial objects), the preservation of cofibrations between cofibrant objects implies the preservation of cofibrant objects. Now, given cofibrations between cofibrant objects $i\colon C \rightarrowtail C'$ in $\mathcal{C}$ and $j\colon D \rightarrowtail D'$ in $\mathcal{D}$, we just write

$$i \otimes j = (\mathrm{id}_{C'} \otimes j) \circ (i \otimes \mathrm{id}_D),$$

and the claim follows from the previous proposition. □

We can apply these results to the two-variable Quillen adjunction coming from a simplicial model category. For convenience, let us record the following implications that they have, which we are going to use later on.

(1.10) **Proposition.** If $\mathcal{M}$ is a simplicial model category then

(a) $\mathrm{Map}_{\mathcal{M}}(-,-)$ maps colimits in the first variable and limits in the second variable to limits;

(b) every $\mathrm{Map}_{\mathcal{M}}(A, X)$ with A cofibrant and X fibrant is a Kan complex;

(c) for X fibrant, $\mathrm{Map}_{\mathcal{M}}(-, X)$ maps (acyclic) cofibrations to (acyclic) Kan fibrations and consequently weak equivalences between cofibrant objects to weak equivalences of Kan complexes;

(d) dually, for A cofibrant, $\mathrm{Map}_{\mathcal{M}}(A, -)$ maps (acyclic) fibrations to (acyclic) Kan fibrations and consequently weak equivalences between fibrant objects to weak equivalences between Kan complexes;

(e) for every simplicial set K, the power functor $[K, -]\colon \mathcal{M} \to \mathcal{M}$ is right Quillen.

Proof. The only statement that doesn't follow from generalities about two-variable Quillen adjunctions is the last point, where we used that every simplicial set is cofibrant. □

We finish this section by calculating some explicit examples in simplicial model categories involving the copower functor.

(1.11) **Example.** If $\mathcal{M}$ is any simplicial model category and K a contractible simplicial set then $K \odot^{\mathbb{L}} A \simeq A$ for every $A \in \mathcal{M}$. To wit, if A is cofibrant then $- \odot A\colon \mathbf{sSets} \to \mathcal{M}$ is left Quillen and hence, choosing a base point in K, carries the acyclic cofibration $* \to K$ to an acyclic cofibration $A \simeq * \odot A \simeq K \odot A$.

(1.12) **Example.** Let $\mathcal{M}$ be a simplicial model category, $* \in \mathcal{M}$ a cofibrant contractible object (i.e. a cofibrant replacement for the terminal object), $i\colon * \rightarrowtail A$ a cofibration and $j\colon K \rightarrowtail CK$ a monomorphism of simplicial sets with $CK \simeq *$. Then the pushout product of i with j is weakly equivalent to

$$K \odot A \rightarrowtail CK \odot A \simeq A.$$

This is easily seen by forming the pushout defining the pushout product

$$\begin{tikzcd}
K \odot * \arrow[r, tail] \arrow[d, tail] & K \odot A \arrow[d, tail] \arrow[dr, bend left] & \\
CK \odot * \arrow[r, tail] \arrow[rr, bend right] & Q \arrow[r] \arrow[ul, phantom, "\ulcorner"] & CK \odot A,
\end{tikzcd}$$

noting that this is actually a homotopy pushout and using the previous example, yielding $CK \odot * \simeq *$. It now follows that

$$Q \simeq \mathrm{hCof}(K \odot * \to K \odot A) \simeq K \odot \mathrm{hCof}(* \to A) \simeq K \odot A.$$

Finally, for a more interesting example using the simplicial structure for a homotopical (i.e. model categorical) construction, we show how we can do suspensions.

(1.13) **Example.** Let $\mathcal{M}$ be a simplicial model category and $i\colon A \to CA$ a morphism between cofibrant objects in $\mathcal{M}$ with $CA \simeq *$. Then the pushout involved in the pushout product of i with $j\colon \partial\Delta[n] \hookrightarrow \Delta[n]$ is the suspension

$$Q_n := \Delta[n] \odot A +_{\partial\Delta[n]\odot A} \partial\Delta[n] \odot CA \simeq \Sigma^n A,$$

where $\Sigma^n A$ is defined inductively as

$$\Sigma^0 A := A \qquad \text{and} \qquad \Sigma^{n+1}A := \mathrm{hocolim}(* \leftarrow \Sigma^n A \to *).$$

We show this inductively with the case $n = 0$ following from $\partial\Delta[0] \odot -$ being constantly the initial object in $\mathcal{M}$ and $\Delta[n] \odot A \simeq A$ for every n, by the first example above. For the inductive step, we note that $Q_n \simeq \mathrm{hocolim}(A \leftarrow S^{n-1} \odot A \to S^{n-1} \odot CA)$, where $S^{n-1} \simeq \partial\Delta[n]$ is any simplicial model for the $(n-1)$-sphere. Applying the Fubini theorem to

$$\begin{tikzcd}
A \arrow[r, equal] \arrow[d, equal] & A \arrow[r] & CA \\
A & S^{n-1} \odot A \arrow[l] \arrow[r] \arrow[u] \arrow[d] & S^{n-1} \odot CA \arrow[u] \arrow[d] \\
A \arrow[u, equal] \arrow[r, equal] & A \arrow[r] & CA,
\end{tikzcd}$$

and using that $- \odot A$ as well as $- \odot CA$ preserve homotopy pushouts, we get

$$Q_{n+1} \simeq \mathrm{hocolim}(A \leftarrow S^n \odot A \to S^n \odot CA) \simeq \mathrm{hocolim}(CA \leftarrow Q_n \to CA).$$

The claim now follows from the inductive hypothesis $Q_n \simeq \Sigma^n A$.

2. Mapping Spaces and Homotopy Categories

In this section, we establish the folklore fact that for a simplicial model category, Hom-sets in the homotopy category are just path components of the corresponding derived mapping spaces. The author is sure that this result can be found in many textbooks but advises the interested readers to do the proofs themselves, as they are all straightforward.

(2.1) **Proposition.** If $A \in \mathcal{M}$ is cofibrant,

$$A + A \cong \partial\Delta[1] \odot A \overset{j_1 \odot A}{\rightarrowtail} \Delta[1] \odot A \xrightarrow[\sim]{p \odot A} * \cdot A \cong A$$

(where $j_1 \colon \partial\Delta[1] \rightarrowtail \Delta[1]$ and $p \colon \Delta[1] \to *$) is a *good cylinder object* (see [22, Definition 4.2]) for A, called the *standard cylinder*. Explicitly, this means that the first morphism is a cofibration, the second one a weak equivalence and the composite is the fold map $[\mathrm{id}_A, \mathrm{id}_A]$.

Proof. We have already seen in (1.3), that $\partial\Delta[1] \odot A \cong A + A$ and since $- \odot A$ is left Quillen, the claim follows. □

The following lemma holds in any model category (not necessarily simplicial). In the simplicial context, we can combine it with the previous proposition and conclude that for A cofibrant and X fibrant, we can restrict our attention to the cylinder $\Delta[1] \odot A$ of A.

(2.2) **Lemma.** If $f, g \colon A \to X$ are homotopic with A cofibrant and X fibrant then there is a (left) homotopy $f \sim g$ through any good cylinder for A.

Proof. Since $A \in \mathcal{M}$ is cofibrant and $X \in \mathcal{M}$ fibrant, there is some left homotopy $h \colon f \sim_l g$ through a very good cylinder

$$A + A \overset{i}{\rightarrowtail} C \underset{\sim}{\overset{p}{\twoheadrightarrow}} A$$

for A (i.e. there is $h \colon C \to X$ such that $h \circ i = [f, g]$). Now, if

$$A + A \overset{j}{\rightarrowtail} D \xrightarrow[\sim]{w} A$$

is any good cylinder for A, then the following lifting problem has a solution

$$\begin{tikzcd} A + A \arrow[r, tail, "i"] \arrow[d, tail, "j"'] & C \arrow[r, "h"] \arrow[d, two heads, "p", "\sim"'] & X \\ D \arrow[r, "\sim", "w"'] \arrow[ur, dotted, "d"] & A & \end{tikzcd}$$

and we find a left homotopy $h' \colon f \sim_l g$ through D by $h' := h \circ d$. □

(2.3) **Proposition.** If $A \in \mathcal{M}$ is cofibrant and $X \in \mathcal{M}$ fibrant, then

$$\mathcal{M}(A, X) = \mathrm{Map}_{\mathcal{M}}(A, X)_0 \twoheadrightarrow \pi_0 \, \mathrm{Map}_{\mathcal{M}}(A, X)$$

is the quotient map with respect to homotopy.

Proof. Since A is cofibrant and X is fibrant, $\mathrm{Map}_{\mathcal{M}}(A, X)$ is a Kan complex, so that two vertices lie in the same component iff there is an edge between them. Writing

$$A + A \cong \partial\Delta[1] \odot A \xrightarrow{j_1 \odot A} \Delta[1] \odot A \xrightarrow[\sim]{p \odot A} * \cdot A \cong A$$

for the standard cylinder of A, then, by the above, two $f,\ g\colon A \to X$ are homotopic iff there is a left homotopy h between them through $\Delta[1] \odot A$. But by adjointness, morphisms $h\colon \Delta[1] \odot A \to X$ correspond to maps $\Delta[1] \to \mathrm{Map}_{\mathcal{M}}(A, X)$ (i.e. edges $h^\sharp \in \mathrm{Map}_{\mathcal{M}}(A, X)$) and the condition $(j_1 \odot A) \circ h = [f, g]$ corresponds to $h^\sharp$ being an edge from f to g. □

By construction of $\mathrm{Ho}(\mathcal{M})$, the Hom-set $\mathrm{Ho}(\mathcal{M})(A, X)$ for A cofibrant and X fibrant is just $\mathcal{M}(A, X)$ quotiented by the homotopy relation (usually, one even requires A and X to be bifibrant but this is only to have compositions and homotopy inverses). So, for a fixed cofibrant A and a fibrant X, we have a bijection of quotients

$$\mathrm{Ho}(\mathcal{M})(A, X) \cong \pi_0 \,\mathrm{Map}_{\mathcal{M}}(A, X)$$

(i.e. a bijection under $\mathcal{M}(A, X)$). And we can extend this to an isomorphism of categories. For this, we choose cofibrant replacements $(q_A\colon QA \to A)_{A \in \mathcal{M}}$ and fibrant replacements $(r_A\colon A \to RA)_{A \in \mathcal{M}}$ and define the category $\mathrm{Ho}^s(\mathcal{M})$ with the same objects as $\mathcal{C}$ and

$$\mathrm{Ho}^s(\mathcal{M})(A, B) := \pi_0 \mathbb{R}\,\mathrm{Map}_{\mathcal{M}}(A, B) = \pi_0 \,\mathrm{Map}_{\mathcal{M}}(QA, RB).$$

The composition is induced by the composition map of simplicial mapping spaces in $\mathcal{M}$ and the identities are the components of the identities in $\mathcal{M}$. This category comes with an obvious functor $H^s\colon \mathcal{M} \to \mathrm{Ho}^s(\mathcal{M})$, which is the identity on objects and

$$\mathcal{M}(A, X) \xrightarrow{q_A^* r_{A*}} \mathcal{M}(QA, RX) = \mathbb{R}\,\mathrm{Map}(A, X)_0 \twoheadrightarrow \pi_0 \mathbb{R}\,\mathrm{Map}(A, X) = \mathrm{Ho}^s(\mathcal{M})(A, X)$$

on Hom-sets.

(2.4) **Proposition.** The functor $H^s\colon \mathcal{M} \to \mathrm{Ho}^s(\mathcal{M})$ maps weak equivalences to isomorphisms.

Proof. Given a weak equivalence $w\colon A \to B$, by Yoneda, it suffices to check that every

$$w^*\colon\ \mathrm{Ho}^s(\mathcal{M})(B, X) = \pi_0 \mathbb{R}\,\mathrm{Map}_{\mathcal{M}}(B, X) \to \pi_0 \mathbb{R}\,\mathrm{Map}_{\mathcal{M}}(A, X) = \mathrm{Ho}^s(\mathcal{M})(A, X)$$

with $X \in \mathrm{Ho}^s(\mathcal{M}) = \mathcal{M}$ is an isomorphism, which is obvious, since w induces a weak equivalence of derived mapping spaces. □

By the universal property of the homotopy category as the localisation of $\mathcal{M}$ at the weak equivalences, there is a unique induced functor $\bar{H}^s\colon \mathrm{Ho}(\mathcal{M}) \to \mathrm{Ho}^s(\mathcal{M})$ of categories below $\mathcal{M}$ and unsurprisingly, we have the following result.

(2.5) **Proposition.** The functor $\bar{H}^s\colon\ \mathrm{Ho}(\mathcal{M}) \to \mathrm{Ho}^s(\mathcal{M})$ is an isomorphism of categories.

Proof. By definition, $\bar{H}^s$ is bijective on objects and we only need to check that it is fully faithful. Since weak equivalences induce bijections of Hom-sets in both $\mathrm{Ho}(\mathcal{M})$ and $\mathrm{Ho}^s(\mathcal{M})$, it suffices to check this for bifibrant domains and codomains. I.e. for A, X bifibrant, we need to check that

$$\mathrm{Ho}(\mathcal{M})(A, X) \to \pi_0 \mathbb{R}\,\mathrm{Map}_{\mathcal{M}}(A, X) = \pi_0 \,\mathrm{Map}_{\mathcal{M}}(A, X)$$

is a bijection (under $\mathcal{M}(A, X)$), which is trivial since both are quotients of $\mathcal{M}(A, X)$ with respect to the homotopy relation. □

(2.6) **Corollary.** The following are equivalent for a morphism $w\colon A \to B$ in $\mathcal{M}$:

(a) w is a weak equivalence;

(b) every $w^*\colon \mathbb{R}\operatorname{Map}_{\mathcal{M}}(B, X) \to \mathbb{R}\operatorname{Map}_{\mathcal{M}}(A, X)$ with $X \in \mathcal{M}$ is a weak equivalence;

(c) every $\pi_0 w^*\colon \pi_0\mathbb{R}\operatorname{Map}_{\mathcal{M}}(B, X) \to \pi_0\mathbb{R}\operatorname{Map}_{\mathcal{M}}(A, X)$ with $X \in \mathcal{M}$ is a bijection. □

3. Definition and First Properties

In its most general form, left Bousfield localisations of model categories can be defined as follows.

(3.1) **Definition.** A *left Bousfield localisation* $L\mathcal{M}$ of a (simplicial) model category $\mathcal{M}$ is a new model structure on the underlying (simplicial) category $\mathcal{M}$ with the same cofibrations but possibly more weak equivalences. Note that in the simplicial case (which is the usual one), we assume that $\mathcal{M}$ and $L\mathcal{M}$ have the same mapping spaces.

(3.2) **Observation.** Since a left Bousfield localisation $L\mathcal{M}$ of $\mathcal{M}$ has the same cofibrations but more weak equivalences, it has more acyclic cofibrations and hence fewer fibrations. Consequently, the identity functor induces a Quillen adjunction

$$\mathrm{Id}\colon \mathcal{M} \rightleftarrows L\mathcal{M} : \mathrm{Id} \qquad \text{(left adjoint on the left).}$$

Since $L\mathcal{M}$ has fewer fibrations, it also has fewer fibrant objects. Noticing that mapping spaces in $\mathcal{M}$ and $L\mathcal{M}$ agree and since cofibrant replacements in $\mathcal{M}$ and $L\mathcal{M}$ are the same, we can conclude that for every object A

$$\mathbb{R}\operatorname{Map}_{\mathcal{M}}(A, X) = \mathbb{R}\operatorname{Map}_{L\mathcal{M}}(A, X) \qquad \text{if } X \text{ is fibrant in } L\mathcal{M};$$

i.e. the derived mapping spaces agree as well (or rather, they are naturally weakly equivalent).

Usually, when considering left Bousfield localisations, we are working with simplicial model categories. This added simplicial structure of $\mathcal{M}$ already allows us to show that in any left Bousfield localisation, between fibrant objects (in the localisation!), no new weak equivalences are added.

(3.3) **Proposition.** If $L\mathcal{M}$ is any left Bousfield localisation of a simplicial model category $\mathcal{M}$ and X, $Y \in L\mathcal{M}$ fibrant then a morphism $w\colon X \to Y$ is a weak equivalence in $L\mathcal{M}$ iff it is a weak equivalence in $\mathcal{M}$.

Proof. The implication "$\Leftarrow$" is trivial. As for the other direction, we consider the full subcategory $\mathrm{Ho}_{f_L}(\mathcal{M}) \subseteq \mathrm{Ho}(\mathcal{M})$ of objects that are fibrant in $L\mathcal{M}$. If we can show that for every Z, that is fibrant in $L\mathcal{M}$, the induced map

$$w^*\colon\ \mathrm{Ho}_{f_L}(\mathcal{M})(Y, Z) = \mathrm{Ho}(\mathcal{M})(Y, Z) \to \mathrm{Ho}(\mathcal{M})(X, Z) = \mathrm{Ho}_{f_L}(\mathcal{M})(X, Z)$$

is an isomorphism, we are done because then (by the Yoneda lemma), w is an isomorphism in $\mathrm{Ho}_{f_L}(\mathcal{M})$ and hence in $\mathrm{Ho}(\mathcal{M})$ since $\mathrm{Ho}_{f_L}(\mathcal{M}) \subseteq \mathrm{Ho}(\mathcal{M})$ is full. By (2.6),

$$w^*\colon \mathbb{R}\operatorname{Map}_{L\mathcal{M}}(Y, Z) \xrightarrow{\sim} \mathbb{R}\operatorname{Map}_{L\mathcal{M}}(X, Z)$$

is a weak equivalence. But since Z is fibrant in $L\mathcal{M}$, these are also derived mapping spaces in $\mathcal{M}$ by the above observation. In particular, w^* induces an isomorphism on π_0, which are the Hom-sets of $\mathrm{Ho}(\mathcal{M})$ by (2.3). □

As an interesting consequence of this proposition, we can show for simplicial model categories, that being left proper is preserved under left Bousfield localisations.

(3.4) **Proposition.** If $\mathcal{M}$ is a left proper simplicial model category, then any left Bousfield localisation $L\mathcal{M}$ of it is again left proper.

Proof. Given a pushout square on the left

$$\begin{tikzcd} A \arrow[r, tail, "i"] \arrow[d, "w"', "\sim_L"] & C \arrow[d, "v"] \\ B \arrow[r, tail, "j"'] \arrow[ur, phantom, "\ulcorner", very near end] & D \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, tail, "\sim_L"] \arrow[d, "w"', "\sim_L"] & RA \arrow[d, two heads, "\sim"', "w'"] & \\ B \arrow[r, tail, "\sim_L"'] & RB \arrow[r, two heads] & * \end{tikzcd}$$

with w a weak equivalence in $L\mathcal{M}$ and i, j cofibrations, we need to check that v is a weak equivalence in $L\mathcal{M}$, too. For this, we pick a fibrant replacement w' for w in $L\mathcal{M}$ as depicted above. By 2-out-of-3, w' is a weak equivalence in $L\mathcal{M}$ and hence, by the above result, in $\mathcal{M}$. We now form the cubical diagram

$$\begin{tikzcd} & RA \arrow[rr, tail] \arrow[dd, two heads, "\sim"', "w'", near start] & & Q \arrow[dd, dotted, "\sim"] \\ A \arrow[ur, tail, "\sim_L"] \arrow[rr, tail, "i", near end] \arrow[dd, "w"', "\sim_L"] & & C \arrow[ur, tail, "\sim_L"] \arrow[dd, "v", near start] & \\ & RB \arrow[rr, tail] & & R \\ B \arrow[ur, tail, "\sim_L"'] \arrow[rr, tail, "j"'] & & D \arrow[ur, tail, "\sim_L"'] \arrow[ul, phantom, "\ulcorner", very near start] & \end{tikzcd},$$

where the top and bottom faces are pushouts and hence (by the pushout lemma), so is the back face. The two maps $C \to Q$ and $D \to R$ are acyclic cofibrations in $L\mathcal{M}$ since these are stable under cobase change and the induced map $Q \to R$ is a weak equivalence in $\mathcal{M}$ by left properness. Using 2-out-of-3, it follows that v is a weak equivalence in $L\mathcal{M}$, as claimed. □

4. S-Local Objects and Morphisms

Obviously, the general definition of a left Bousfield localisation is not very useful. What we would really like is to be able to add a set S of morphisms to the weak equivalences and still get a model structure that we have some control over. So, for this entire section, let's fix some model category $\mathcal{M}$ that is

- left proper and
- simplicial,

together with a *set* S of *cofibrations* in $\mathcal{M}$ (the set of morphisms that we would like to declare to be weak equivalences). For this section, it is not important that the elements of S be cofibrations but we will make use of it in the next one.

(4.1) **Definition.** An object $X \in \mathcal{M}$ is said to be *S-local* iff

$$s^*\colon \mathbb{R}\operatorname{Map}_{\mathcal{M}}(B, X) \to \mathbb{R}\operatorname{Map}_{\mathcal{M}}(A, X)$$

is a weak equivalence of simplicial sets for every $s\colon A \to B$ in S. With this, a morphism $w\colon A \to B$ in $\mathcal{M}$ is said to be an *S-local weak equivalence* (or just *S-equivalence*) iff

$$w^*\colon \mathbb{R}\operatorname{Map}_{\mathcal{M}}(B, X) \to \mathbb{R}\operatorname{Map}_{\mathcal{M}}(A, X)$$

is a weak equivalence of simplicial sets for every S-local X. If such a w also happens to be a cofibration, we speak of an *S-local acyclic cofibration*.

(4.2) **Remark.** Since we take derived mapping spaces everywhere, the requirement that the maps in S be cofibrations is no restriction as we can always cofibrantly replace them. We could even require their domains to be cofibrant (which is often done) but avoid doing so to be able to prove (6.2) below. Usually in the literature, S-local objects are also required to be fibrant, which we can always impose – again because we take derived mapping spaces everywhere.

(4.3) **Example.** By basic properties of derived mapping spaces, every contractible space is S-local. Similarly, every weak equivalence in $\mathcal{M}$ is an S-local weak equivalence.

Before making a few more comments about the above definition, let us make the following elementary observations, which follow directly from elementary properties of derived mapping spaces and of weak equivalences of simplicial sets.

(4.4) **Observation.**

(a) The class of S-local objects is closed under weak equivalences, retracts and homotopy limits.

(b) The class of S-local weak equivalences is closed under weak equivalences, retracts, homotopy colimits (all in $\mathcal{M}^{[1]}$) and satisfies the 2-out-of-3 axiom.

(c) We can always enlarge S by adding in some S-local weak equivalences without changing the resulting classes of local objects and morphisms.

Let us quickly take a step back and analyse the above definition. From (2.6), we know that we can test weak equivalences by applying derived mapping spaces $\mathbb{R}\operatorname{Map}(-, X)$ (in the left Bousfield localisation but these are just the ones in $\mathcal{M}$ as long as X is fibrant in the localisation). So, if we want the elements of S to become weak equivalences, all that we need to do is to take away enough fibrations such that all fibrant objects left are S-local. As we are usually not so interested in the fibrations or what is taken away, but rather in weak equivalences and what is added, we just assume that we have done so and then (again by (2.6)), the new weak equivalences must be the S-local ones.

With these contemplations in mind, we define the following, seemingly special, case of a left Bousfield localisation. As it turns out in (6.2), every left Bousfield localisation is of this form, as long as $\mathcal{M}$ and the resulting left Bousfield localisation are nice enough.

(4.5) **Definition.** The *left Bousfield localisation* of $\mathcal{M}$ at S (if it exists!) is the model category $L_S\mathcal{M}$ with the same cofibrations $\mathcal{C}$ as $\mathcal{M}$ and where the weak equivalences are the S-local weak equivalences.

The following statement is just an easy extension of (3.3), made possible by the more special left Bousfield localisations that we are considering.

(4.6) **Proposition.** If X, $Y \in \mathcal{M}$ are S-local, then a morphism $w\colon X \to Y$ is an S-local weak equivalence iff it is an ordinary weak equivalence.

Proof. We have already shown the claim for fibrant X, Y in (3.3). For general X, Y, we just fibrantly replace them (in $\mathcal{M}$!) and extend w to get $w'\colon X' \to Y'$ with X', Y' fibrant. Note that since X and Y are S-local and S-local objects are closed under weak equivalences, so are X' and Y'. Now, by 2-out-of-3 for S-local equivalences, w' is one such, which is to say an ordinary weak equivalence. But now, by 2-out-of-3 for ordinary weak equivalences, so is w. □

The result (4.8) below is a key lemma, due to Lurie [37, Lemma A.3.7.1], which shows that we don't need to take derived mapping spaces to check for S-local acyclic cofibrancy and we reproduce his proof here for convenience. For it, we need the following interesting consequence (and at the same time generalisation) of Ken Brown's lemma.

(4.7) **Lemma.** Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor between model categories that maps acyclic cofibrations to weak equivalences. If $w\colon B \to C$ is a weak equivalence in $\mathcal{C}$ and there is a cofibration $i\colon A \rightarrowtail B$ such that $w \circ i$ is a cofibration, too, then Fw is a weak equivalence.

Proof. Recall that a morphism in $A \downarrow \mathcal{C}$ is a fibration/cofibration/weak equivalence iff it is so after applying the standard projection $Q\colon A \downarrow \mathcal{C} \to \mathcal{C}$. Now, since the initial object in $A \downarrow \mathcal{C}$ is id_A, an object $f \in A \downarrow \mathcal{C}$ is cofibrant iff f is a cofibration on $\mathcal{C}$. In particular, $w\colon i \to w \circ i$ can be viewed as a weak equivalence between cofibrant object in $A \downarrow \mathcal{C}$. By hypothesis, the composite functor

$$A \downarrow \mathcal{C} \xrightarrow{Q} \mathcal{C} \xrightarrow{F} \mathcal{D}$$

maps acyclic cofibrations to weak equivalences and thus, by Ken Brown's lemma, Fw is a weak equivalence. □

(4.8) **Lemma.** A cofibration $i\colon A \rightarrowtail B$ is an S-local weak equivalence (i.e. an S-local acyclic cofibration) iff

$$i^*\colon \mathrm{Map}_{\mathcal{M}}(B, X) \to \mathrm{Map}_{\mathcal{M}}(A, X)$$

is an acyclic Kan fibration for every S-local fibrant X.

Proof. Note that, since X is fibrant and i a cofibration, i^* is already a Kan fibration and we only need to show that it is acyclic. That is to say, we need to show that all its strict fibres (which are its homotopy fibres) are contractible. For this, we cofibrantly replace $i\colon A \to B$ by first cofibrantly replacing A by some A' and then factoring the composite $A' \overset{\sim}{\twoheadrightarrow} A \to B$ into a cofibration $i'\colon A' \rightarrowtail B'$ followed by an acyclic fibration $B' \overset{\sim}{\twoheadrightarrow} B$. The map induced by i between (derived) mapping spaces is thus

$$i'^*\colon \mathrm{Map}_{\mathcal{M}}(B', X) \to \mathrm{Map}_{\mathcal{M}}(A', X)$$

and we note that just like for i, this is still a Kan fibration and to show its acyclicity, we just need to show that all its strict fibres are contractible. Put differently, to show the entire lemma, we need to show that i'^* has contractible strict fibres for all S-local X iff i^* has contractible strict fibres for all S-local X. It is a subtle point that we are not claiming this for a fixed S-local X. Now, we first form the following pushout diagram and construct an induced map

$$\begin{tikzcd} A' \arrow[r, tail, "i'"] \arrow[d, two heads, "\sim"'] & B' \arrow[d, "\sim"] \arrow[dr, bend left, two heads, "\sim"] & \\ A \arrow[r, tail, "i''"] \arrow[rr, tail, bend right, "i"'] & P \arrow[r, dotted, "j", "\sim"'] \arrow[ul, phantom, "\ulcorner", very near start] & B, \end{tikzcd}$$

where $B' \to P$ is a weak equivalence by left properness and j is a weak equivalence by 2-out-of-3.

Applying $\mathrm{Map}_{\mathcal{M}}(-, X)$ to this diagram and using that it maps pushouts to pullbacks, we get

$$\begin{tikzcd} \mathrm{Map}_{\mathcal{M}}(B, X) \arrow[r, "j^*", "\sim"'] \arrow[dr, bend right, two heads, "i^*"'] & \mathrm{Map}_{\mathcal{M}}(P, X) \arrow[r] \arrow[d, two heads, "i''^*"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Map}_{\mathcal{M}}(B', X) \arrow[d, two heads, "i'^*"] \\ & \mathrm{Map}_{\mathcal{M}}(A, X) \arrow[r] & \mathrm{Map}_{\mathcal{M}}(A', X), \end{tikzcd}$$

where j^* is a weak equivalence by the generalised Ken Brown lemma (4.7) applied to the functor $\mathrm{Map}_{\mathcal{M}}(-,X)$ (which preserves acyclic cofibrations). As always, this pullback induces an isomorphism between the fibres of i'^* and i''^* but, of course, only over vertices lying in the image of $\mathrm{Map}_{\mathcal{M}}(A,X)$ (i.e. those $A' \to X$ that factor through A).
"$\Rightarrow$": This implication is easy: If all fibres of i'^* are contractible then so are all fibres of i''^* because the above square is a pullback. Using the fibration sequence of homotopy fibres

$$\mathrm{hFib}_p(j^*) \to \mathrm{hFib}_{p\circ i''}(i^*) \to \mathrm{hFib}_{p\circ i''}(i''^*) \simeq \mathrm{hFib}(i'^*)$$

for a vertex $p\colon P \to X$ (where the last fibre is taken over its image in $\mathrm{Map}_{\mathcal{M}}(A',X)$) and the fact that j^* is a weak equivalence, the claim follows.
"$\Leftarrow$": Let $p\colon A' \to X$ be an arbitrary vertex of $\mathrm{Map}(A',X)$, and factor it as $A' \overset{u}{\rightarrowtail} Y \overset{v}{\underset{\sim}{\twoheadrightarrow}} X$. This gives us the following commutative square of mapping spaces, in which the v_* are acyclic Kan fibrations because A' and B' are cofibrant and v an acyclic fibration:

$$\begin{tikzcd}
\mathrm{Map}(B',Y) \arrow[r, two heads, "i'^*"] \arrow[d, two heads, "v_*"', "\sim"] & \mathrm{Map}(A',Y) \arrow[d, two heads, "\sim"', "v_*"] \\
\mathrm{Map}(B',X) \arrow[r, two heads, "i'^*"'] & \mathrm{Map}(A',X).
\end{tikzcd}$$

Since $p = v_*(u)$, it suffices to show that the fibre of i^* above $u \in \mathrm{Map}(A',Y)_0$ is contractible. For this, we form the pushout $Q := A +_{A'} Y$ and take a fibrant replacement Z of it:

$$\begin{tikzcd}
A' \arrow[r, two heads, "\sim"'] \arrow[d, tail, "u"'] & A \arrow[d, tail, "u'"] & & \\
Y \arrow[r, "\sim", "w"'] & Q \arrow[ul, phantom, "\ulcorner", very near start] \arrow[r, tail, "\sim", "k"'] & Z \arrow[r, two heads] & *.
\end{tikzcd}$$

Here, w is a weak equivalence by left properness. Now we get the commutative square of mapping spaces, where the $(k \circ w)_*$ are weak equivalences because A' and B' are cofibrant and $k \circ w\colon Y \xrightarrow{\sim} Z$ is a weak equivalence between fibrant objects

$$\begin{tikzcd}
\mathrm{Map}(B',Y) \arrow[r, two heads, "i'^*"] \arrow[d, "(k\circ w)_*"', "\sim"] & \mathrm{Map}(A',Y) \arrow[d, "\sim"', "(k\circ w)_*"] \\
\mathrm{Map}(B',Z) \arrow[r, two heads, "i'^*"'] & \mathrm{Map}(A',Z).
\end{tikzcd}$$

Again, it now suffices to show that the fibre of i'^* above $k \circ w \circ u = (k \circ w)_*(u) \in \mathrm{Map}(A',Z)_0$ is contractible. This follows from the implication "$\Rightarrow$" applied to the space Z instead of X (which makes sense because S-local spaces are closed under weak equivalences and Z is fibrant) and noting that $k \circ w \circ u$ factors through A. $\square$

5. S-Local Fibrant Replacement

For the entire section, we fix some model category $\mathcal{M}$ that is

- left proper,
- simplicial and
- combinatorial

together with a *set* S of *cofibrations* in $\mathcal{M}$ (the set of morphisms that we would like to declare to be weak equivalences).

On our path to showing that the left Bousfield localisation $L_S\mathcal{M}$ exists and is again a simplicial model model category, let's first show Quillen's SM7 axiom for $L_S\mathcal{M}$.

(5.1) **Lemma.** The S-local objects, S-local weak equivalences and S-local acyclic cofibrations have the following closure properties.

(a) The S-local objects are closed under derived powers; i.e. if X is S-local, then so is $\mathbb{R}[K, X]$ for every simplicial set K.

(b) The fibrant S-local objects are closed under powers; i.e. if X is fibrant and S-local, then so is $[K, X]$,

(c) The S-local weak equivalences are closed under derived copowering with weak equivalences of simplicial sets; i.e. if $s\colon A \to B$ is an S-local weak equivalence then so is $w \odot^{\mathbb{L}} s$ for every weak equivalence of simplicial sets $w\colon K \to L$.

(d) The S-local acyclic cofibrations are closed under copowering with acyclic cofibrations of simplicial sets; i.e. if $s\colon A \rightarrowtail B$ is an S-local acyclic cofibration, then so is $w \odot s$ for every acyclic cofibration of simplicial sets $w\colon K \rightarrowtail L$.

Proof. For all three proofs, we use the notation from the proposition.
Ad (a) & (b): Clearly, (a) is a consequence of (b): If X is not fibrant, we take a fibrant replacement $X \simeq X'$ and use that $\mathbb{R}[K, X] \simeq \mathbb{R}[K, X'] \simeq [K, X']$; so we just need to show (b). First off, $[K, X]$ is fibrant because $[K, -]$ is right Quillen. Now, given $s\colon A \rightarrowtail B$ in S, we first replace it cofibrantly, yielding $s'\colon A' \rightarrowtail B'$ with A', B' cofibrant, so that all mapping spaces in the diagram below are derived (in particular Kan complexes) and s'^* is the morphism between derived mapping spaces induced by s. Now, using adjointness, we have

$$\begin{tikzcd}
\mathrm{Map}_{\mathcal{M}}(B', [K, X]) \arrow[r, "s'^*"] \arrow[d, "\cong"'] & \mathrm{Map}_{\mathcal{M}}(A', [K, X]) \arrow[d, "\cong"] \\
\mathrm{Map}_{\mathbf{sSets}}(K, \mathrm{Map}_{\mathcal{M}}(B', X)) \arrow[r, "\simeq"'] & \mathrm{Map}_{\mathbf{sSets}}(K, \mathrm{Map}_{\mathcal{M}}(A', X)),
\end{tikzcd}$$

where, for the bottom equivalence, we used that s' is an S-local weak equivalence.
Ad (c) & (d): Again, (c) is a consequence of (d) by taking cofibrant replacements and we just need to show (d). But, using (4.8), (d) follows immediately from (b) by adjointness. □

(5.2) **Lemma.** Let $j\colon K \rightarrowtail L$ be any monomorphism of simplicial sets. If $i\colon A \rightarrowtail B$ is an S-local acyclic cofibration, so is the pushout product

$$j \hat{\odot} i\colon\ Q := (K \odot B) +_{K \odot A} (L \odot A) \longrightarrow L \odot B.$$

Proof. The morphism $j \hat{\odot} i$ is a cofibration by Quillen's SM7-axiom for $\mathcal{M}$. To show that it is an S-local weak equivalence, we let X be any S-local fibrant object and apply $\mathrm{Map}_{\mathcal{M}}(-, X)$ (which is enough by (4.8)) to the corresponding pushout square for Q. This gives us a pullback square

$$\begin{tikzcd}
\mathrm{Map}_{\mathcal{M}}(L \odot B, X) \arrow[r, "(j\hat{\odot} i)^*"'] \arrow[rr, bend left] \arrow[dr, bend right] & \mathrm{Map}_{\mathcal{M}}(Q, X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Map}_{\mathcal{M}}(L \odot A, X) \arrow[d] \\
& \mathrm{Map}_{\mathcal{M}}(K \odot B, X) \arrow[r] & \mathrm{Map}_{\mathcal{M}}(K \odot A, X).
\end{tikzcd}$$

By point (d) from the above lemma, the bottom map, as well as the composite along the top are weak equivalences. Because j is a monomorphism (i.e. a cofibration) and **sSets** is right proper, it follows that the top map in the pullback square is also a weak equivalence and hence, by 2-out-of-3, so is $(j\hat{\odot}i)^*$. □

Now, to get a characterisation of the fibrant objects in $L_S\mathcal{M}$, which in turn allows us to construct a fibrant replacement functor, we define a set $S_f \supset S$ of *test functions for fibrancy* as follows. First off, we add a set $\mathcal{J}$ of generating acyclic cofibrations for $\mathcal{M}$ to S. Next, we replace all elements of S cofibrantly, yielding a set S' (whose elements are still S-local acyclic cofibrations by 2-out-of-3) and add in all pushout products

$$j_n\hat{\odot}s'\colon\ Q_{s'} = (\Delta[n]\odot A') +_{\partial\Delta[n]\odot A'} (\partial\Delta[n]\odot B') \longrightarrow \Delta[n]\odot B'$$

with $s'\colon A' \to B'$ in S' and $j_n\colon \partial\Delta[n] \rightarrowtail \Delta[n]$ the boundary inclusion. That is to say,

$$S_f := S \cup \mathcal{J} \cup \{j_n\hat{\odot}s' \mid n \in \mathbb{N},\ s' \in S'\}\,.$$

By the above lemma, $S_f \supset S$ is still a set of S-local acyclic cofibrations and so, passing from S to S_f doesn't change the class of S-local objects or that of S-local weak equivalences.

(5.3) **Proposition.** For an object X of $\mathcal{M}$, the following are equivalent:

(a) X is fibrant in $L_S\mathcal{M}$ (i.e. $i \pitchfork X$ for every S-local acyclic cofibration i);

(b) X is fibrant and every $i^*\colon\ \mathrm{Map}_{\mathcal{M}}(B,X) \to \mathrm{Map}_{\mathcal{M}}(A,X)$ with $i\colon A \to B$ an S-local acyclic cofibration is an acyclic Kan fibration;

(c) $S_f \pitchfork X$ (i.e. $i \pitchfork X$ for every $i \in S_f$);

(d) X is fibrant and every $s'^*\colon\ \mathrm{Map}_{\mathcal{M}}(B',X) \to \mathrm{Map}_{\mathcal{M}}(A',X)$ with $s'\colon A' \to B'$ in S' is an acyclic Kan fibration;

(e) X is fibrant (in $\mathcal{M}$) and S-local.

Proof. "*(a) ⇒ (c), (b) ⇒ (d)*": Trivial.
"*(a) ⇒ (b), (c) ⇒ (d)*": We prove (c) ⇒ (d) and the proof of (a) ⇒ (b) is the same, with the sets S, S_f both replaced by the class of all acyclic cofibrations. Clearly, X is fibrant since S_f contains a set of generating acyclic cofibrations for $\mathcal{M}$. As for the second part, we have already noted in the (dual of the) proof of the different equivalent formulations (1.6) for Quillen's SM7-axiom, that, given $s'\colon A' \to B'$, we have a bijective correspondence between commutative squares (as well as diagonal fillers for them)

$$\begin{tikzcd} Q_{s'} \arrow[r] \arrow[d, "j_n\hat{\odot}s'"'] & X \arrow[d] \\ \Delta[n]\odot B' \arrow[r] & * \end{tikzcd} \quad \leftrightarrow \quad \begin{tikzcd} \partial\Delta[n] \arrow[r] \arrow[d, tail, "j_n"'] & \mathrm{Map}_{\mathcal{M}}(B',X) \arrow[d, "s'^*"] \\ \Delta[n] \arrow[r] & \mathrm{Map}_{\mathcal{M}}(A',X). \end{tikzcd}$$

Since S_f contains all $j_n\hat{\odot}s'$ for $s' \in S'$, the claim follows.
"*(b) ⇒ (a)*": We recall that every acyclic Kan fibration is surjective on vertices (which is just the right lifting condition with respect to $j_0\colon \varnothing \to *$). So, for $i\colon A' \to B'$ an S-local acyclic cofibration, the hypothesis (b) implies that

$$i^*\colon \mathcal{M}(B',X) = \mathrm{Map}_{\mathcal{M}}(B',X)_0 \to \mathrm{Map}_{\mathcal{M}}(A',X)_0 = \mathcal{M}(A',X)$$

is surjective, which is to say $i \pitchfork X$.
"*(d) ⇒ (e)*": By definition of S' and because X is fibrant, the maps in (d) are exactly the induced maps between derived mapping spaces for morphisms in S, so that (e) follows.
"*(e) ⇒ (b)*": This was (4.8) □

Using the above result, we can now use the small object argument (since $\mathcal{M}$ is locally presentable) to get a functorial factorisation $\mathcal{M}^{[1]} \to \mathcal{M}^{[2]}$

$$\left(M \xrightarrow{f} N\right) \quad \mapsto \quad \left(M \xrightarrow{\iota(f)} C(f) \xrightarrow{\pi(f)} N\right)$$

with $\iota(f) \in \mathrm{Cell}(S_f)$ and $\pi(f) \in {S_f}^{\pitchfork}$, which is accessible by (5.2.5). Applying it to a morphism $M \to *$ gives us a factorisation $M \to R_S M \to *$ with $S_f \pitchfork R_S M$; i.e. $R_S M$ is fibrant and S-local. By functoriality of (ι, π), this even extends to an *S-local fibrant replacement functor*

(5.4) $$R_S\colon \mathcal{M}^{[1]} \to \mathcal{M}^{[1]}, \left(M \xrightarrow{f} N\right) \mapsto \left(R_S M \xrightarrow{R_S f} R_S N\right),$$

which (being obtained from the above accessible functorial factorisation) is still accessible.

6. Existence Theorem

Finally, we have all the necessary pieces in place to prove the existence of left Bousfield localisations for sufficiently nice model categories. This theorem is due to Smith but was never published by him. Accounts of it can be found in many places such as Barwick [2] and Lurie [37].

(6.1) **Theorem.** If $\mathcal{M}$ is a left proper, combinatorial simplicial model category and S any set of cofibrations with cofibrant domains in $\mathcal{M}$, the left Bousfield localisation $L_S\mathcal{M}$ of $\mathcal{M}$ at S exists and is again left proper, combinatorial and simplicial (with the same mapping spaces as $\mathcal{M}$).

Proof. We write $\mathcal{W}_S$ for the class of S-local weak equivalences, which satisfies 2-out-of-3 and is closed under retracts. As our set of generating cofibrations, we shall use a set $\mathcal{I}$ of generating cofibrations for $\mathcal{M}$. Clearly then $\mathcal{I}^{\pitchfork} \subseteq \mathcal{W}_S$ because every weak equivalence in $\mathcal{C}$ (and in particular every acyclic fibrations) is an S-local weak equivalence. To apply Smith's theorem, we need to show that $\mathcal{W}_S \cap \mathcal{I}$-cof (which is just the class of S-local acyclic cofibrations) is closed under cobase change and transfinite compositions and that $\mathcal{W}_S \subseteq \mathcal{M}^{[1]}$ is accessibly embedded and accessible.

The first of these claims follows from (4.8): Given a pushout square as below with i an S-local acyclic cofibration, and X any S-local fibrant object

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, tail, "i"', "\sim_S"] & A' \arrow[d, tail, "i'"] \\ B \arrow[r, "f'"'] & B' \arrow[ul, phantom, "\ulcorner"] \end{tikzcd} \qquad \xmapsto{\mathrm{Map}(-,X)} \qquad \begin{tikzcd} \mathrm{Map}_{\mathcal{M}}(B', X) \arrow[r, "f'^*"] \arrow[d, "i'^*"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Map}_{\mathcal{M}}(B, X) \arrow[d, two heads, "i^*", "\sim"'] \\ \mathrm{Map}_{\mathcal{M}}(A', X) \arrow[r, "f^*"'] & \mathrm{Map}_{\mathcal{M}}(A, X), \end{tikzcd}$$

applying $\mathrm{Map}_{\mathcal{M}}(-, X)$ gives us a pullback square as on the right. But i^* is an acyclic Kan fibration by (4.8) and hence, so is i'^*. Similarly, given a transfinite sequence $A\colon \alpha \to \mathcal{M}$ of S-local acyclic cofibrations, applying $\mathrm{Map}_{\mathcal{M}}(-, X)$ for an S-local fibrant X gives a transfinite inverse sequence $\mathrm{Map}_{\mathcal{M}}(A-, X)\colon \alpha^{\mathrm{op}} \to \mathcal{M}$ of acyclic Kan fibrations and these are closed under transfinite "cocomposition".

As a last step towards applying Smith's theorem, we need to check the accessibility of $\mathcal{W}_S \subseteq \mathcal{M}^{[1]}$ (viewed as a full subcategory). For this, we write $\mathcal{W} \subseteq \mathcal{M}^{[1]}$ for the full subcategory of the original weak equivalences of $\mathcal{M}$ and consider the S-local fibrant replacement functor (5.4) $R_S\colon \mathcal{M}^{[1]} \to \mathcal{M}^{[1]}$, which is accessible. It now suffices to show that

$$\mathcal{W}_S = R_S^{-1}(\mathcal{W}) \qquad \text{(the full preimage)}$$

because $\mathcal{W} \subseteq \mathcal{M}^{[1]}$ is accessibly embedded and accessible by (5.2.8) and by (3.8.20), these are stable under full preimages of accessible functors. Now, given any morphism $w\colon M \to N$ in $\mathcal{M}$, by definition of R_S, we have a commutative diagram

$$\begin{tikzcd} M \arrow[r, "i"] \arrow[d, "w"'] & R_S M \arrow[r] \arrow[d, "R_S w"'] & * \arrow[d, equal] \\ N \arrow[r, "j"'] & R_S N \arrow[r] & * \end{tikzcd}$$

with i, j being relative cell complexes of S-local acyclic cofibrations. But we have just shown that these are cellularly saturated and so, i and j are themselves S-local acyclic cofibrations. By 2-out-of-3, it follows that $w \in \mathcal{W}_S$ iff $R_S w \in \mathcal{W}_S$. But $R_S M$ and $R_S N$ are S-local fibrant objects and so, by (4.6), this is equivalent to $R_S w \in \mathcal{W}$.

Concerning the simplicial structure of $L_S\mathcal{M}$, given a cofibration $i\colon A \rightarrowtail B$ and a monomorphism of simplicial sets $j\colon K \rightarrowtail L$, then by Quillen's SM7-axiom for $\mathcal{M}$, the pushout product $j \hat{\odot} i$ is again a cofibration and acyclic if j is. The only remaining case is to show the acyclicity of $j \hat{\odot} i$ for i acyclic, which was (5.2) above. Finally, as we have seen in (3.4), left properness is stable under left Bousfield localisation. □

As a last remark, recall that we initially claimed that every (sufficiently nice) left Bousfield localisation is of the form $L_S\mathcal{M}$. Let us quickly reproduce a proof of this here.

(6.2) **Proposition.** If $\mathcal{M}$ is a left proper, cofibrantly generated simplicial model category and $L\mathcal{M}$ a left Bousfield localisation that is also cofibrantly generated and simplicial (with the same mapping spaces as $\mathcal{M}$), then there is a set S of cofibrations in $\mathcal{M}$ such that $L\mathcal{M} = L_S\mathcal{M}$.

Proof. We show that a set S of generating acyclic cofibrations of $L\mathcal{M}$ has the sought for property. Since the cofibrations in $L\mathcal{M}$ and $L_S\mathcal{M}$ are the same, it suffices to show that they have the same acyclic cofibrations. Writing $\mathcal{A}_L$ and $\mathcal{A}_S$ for the respective classes of acyclic cofibrations in $L\mathcal{M}$ and $L_S\mathcal{M}$, we note that $\mathcal{A}_L = S\text{-cof} \subseteq \mathcal{A}_S$ because $S \subseteq \mathcal{A}_S$. Conversely, given any cofibration $i\colon A \rightarrowtail B$ in $\mathcal{A}_S$ (i.e. an S-local acyclic cofibration), then, by (2.6), we just need to check that

$$i^*\colon \mathbb{R}\,\mathrm{Map}_{\mathcal{M}}(B, X) \to \mathbb{R}\,\mathrm{Map}_{\mathcal{M}}(A, X)$$

is a weak equivalence for every $X \in L\mathcal{M}$ fibrant. But $X \in L\mathcal{M}$ being fibrant implies (again by (2.6)) that every

$$s^*\colon \mathbb{R}\,\mathrm{Map}_{\mathcal{M}}(S, X) \to \mathbb{R}\,\mathrm{Map}_{\mathcal{M}}(R, X)$$

with $s\colon R \to S$ in S is a weak equivalence. That is to say, X is S-local and hence, by definition, every i^* as above with $i \in \mathcal{A}_S$ an S-local acyclic cofibration is a weak equivalence as claimed. □

Part III

Cellular Homotopy Excision

Chapter 7

HOMOTOPICAL PRELIMINARIES

In this chapter, we are going to review some homotopical constructions, introducing our notation along the way. Most of the material in this chapter is not due to us and our main contribution is to offer different perspectives, approaches and proofs as well as sometimes a few more details. The exception is the material in the second section, which we haven't found anywhere in the literature. While at it, let us mention that the material in sections 2 and 3 is only used in our interpretation of Emmanuel Dror Farjoun's proof of the fibre decomposition theorem (9.3). Since we also provide an alternative (and more general) proof, the impatient reader might want to skip them.

1. Basic Constructions and Notations

As already mentioned in the preface, we assume basic familiarity with homotopy theory, including simplicial sets and the homotopy theory of small categories via their nerves. Still, let's start with that in order to establish our notation.

(1.1) **Definition.** For a small category $\mathcal{I}$, its *nerve* is the simplicial set

$$N(\mathcal{I})\colon \mathbf{\Delta}^{\mathrm{op}} \hookrightarrow \mathbf{Cat}^{\mathrm{op}} \xrightarrow{\mathbf{Cat}(-,\mathcal{I})} \mathbf{Sets}.$$

In this way, the nerve defines a functor $N\colon \mathbf{Cat} \to \mathbf{sSets}$, which is fully faithful.

(1.2) **Definition.** Recall that for every small category $\mathcal{I}$, there is the *category of elements functor* (or *Grothendieck construction*) $\int_{\mathcal{I}}\colon \mathbf{Sets}^{\mathcal{I}^{\mathrm{op}}} \to \mathbf{Cat}\downarrow\mathcal{I}$, which is left adjoint to $\mathcal{C} \mapsto \mathrm{Fun}_{\mathcal{I}}(\mathcal{I}\downarrow -, \mathcal{C})$, preserves limits (because its extension $\mathbf{Cat}^{\mathcal{I}^{\mathrm{op}}} \to \mathbf{Cat}\downarrow\mathcal{I}$ has a left adjoint), is fully faithful (because the adjunction's unit is invertible) and is pseudo-natural in $\mathcal{I}$, meaning that for every $F\colon \mathcal{I} \to \mathcal{J}$

$$\begin{tikzcd} \mathbf{Sets}^{\mathcal{J}^{\mathrm{op}}} \arrow[r, "\int_{\mathcal{J}}"] \arrow[d, "F^*"'] & \mathbf{Cat}\downarrow\mathcal{J} \arrow[d, "F^*"] \\ \mathbf{Sets}^{\mathcal{I}^{\mathrm{op}}} \arrow[r, "\int_{\mathcal{I}}"'] & \mathbf{Cat}\downarrow\mathcal{I} \end{tikzcd}$$

(where the right hand vertical arrow is pulling back along F) commutes up to coherent natural isomorphism. Explicitly, for a presheaf $P\colon \mathcal{I}^{\mathrm{op}} \to \mathbf{Sets}$, the category of element $\int_{\mathcal{I}} P$ has objects all (I,x) with $I \in \mathcal{I}$, $x \in PI$ and morphisms $(I,x) \to (J,y)$ all $i\colon I \to J$ in $\mathcal{I}$ such that $y \cdot i = (Pi)y = x$. With this, the projection $\mathrm{Pr}\colon \int_{\mathcal{I}} P \to \mathcal{I}$ is just $(I,x) \mapsto I$, $i \mapsto i$. Alternatively, using the Yoneda lemma, the category of elements $\int_{\mathcal{I}} - \cong \mathbf{y}\downarrow -$ can also be obtained as the comma category below the Yoneda embedding $\mathbf{y}\colon \mathcal{I} \to \mathbf{Sets}^{\mathcal{I}^{\mathrm{op}}}$.

(1.3) **Remark.**

- Using the trivial bundle adjunction $U\colon \mathbf{Cat}\downarrow \mathcal{I} \rightleftarrows \mathbf{Cat} : -\times\mathcal{I}$, we obtain that $\int_{\mathcal{I}}\colon \mathbf{Sets}^{\mathcal{I}^{\mathrm{op}}} \to \mathbf{Cat}$ is left adjoint to $\mathcal{C} \mapsto \mathrm{Fun}_{\mathcal{I}}(\mathcal{I}\downarrow -, \mathcal{C}\times\mathcal{I}) \cong \mathrm{Fun}(\mathcal{I}\downarrow -, \mathcal{C})$.
- When viewed as a functor to **Cat**, $\int_{\mathcal{I}}$ does not preserve limits (because it maps the terminal object to $\mathcal{I}$). It does however preserve pullbacks because these are the same in **Cat** and $\mathbf{Cat}\downarrow\mathcal{I}$. In particular

$$\int_{\mathcal{I}}(P\times Q) \cong \int_{\mathcal{I}}(P\times_{\mathbf{y}[0]} Q) \cong \left(\int_{\mathcal{I}} P\right) \times_{\mathcal{I}} \left(\int_{\mathcal{I}} Q\right),$$

where $\mathbf{y}\colon \mathcal{I} \hookrightarrow \mathbf{Sets}^{\mathcal{I}^{\mathrm{op}}}$ is the Yoneda embedding and $[0]$ is the terminal category.

The main example of the category of elements that we are going to be interested in is the case $\mathcal{I} = \mathbf{\Delta}$, where the category of elements is usually referred to as the *category of simplices*. To avoid unnecessary symbol clutter (especially for symbols as scary as an integral sign), let's make the following convention.

(1.4) **Notation.** If K is the name of a simplicial set, we usually use the same name $\mathcal{K}$ but in a calligraphic font for the associated category of simplices $\int_{\mathbf{\Delta}} K$. In particular, if $\mathcal{C}$ is any category, we write $\mathcal{N}(\mathcal{C})$ for the category of simplices of $\mathcal{C}$'s nerve $N(\mathcal{C})$. For the lack of readily available calligraphic Greek letters, we also write $\mathbf{\Delta}[n]$ for $\int_{\mathbf{\Delta}} \Delta[n]$.

(1.5) **Example.** The category of simplices $\mathbf{\Delta}[0]$ of $\Delta[0]$ is isomorphic to $\mathbf{\Delta}$.

(1.6) **Example.** A simplicial set K can be reconstructed from $\mathcal{K}$ as $K \cong \int^{x\in\mathcal{K}} \Delta[\dim x]$ (every presheaf is a colimit of representables indexed by its category of elements). This is sometimes referred to as the *coYoneda lemma*.

Now that we have treated the basic categorical constructions, let us establish the more homotopical ones. First of all, let us again stress the following convention.

(1.7) **Convention.** We are always working with simplicial sets rather than topological spaces and so, the word "*space*" will be used interchangeably with "*simplicial set*". Also, the category **sSets** of simplicial sets will always be equipped with the *Quillen model structure* (also known as the *Kan model structure*), where

(a) weak equivalences are those maps that induce bijections on connected components as well as isomorphisms between all homotopy groups (of the geometric realisations);

(b) fibrations are Kan fibrations;

(c) cofibrations are the monomorphisms.

This model structure is combinatorial. A set of generating cofibrations is given by all boundary inclusions $\partial\Delta[n] \hookrightarrow \Delta[n]$ with $n\in\mathbb{N}$, where $\partial\Delta[n]$ is the simplicial subset generated by all $\delta^i\colon [n-1] \hookrightarrow [n]$ with $i\in\{0,\dots,n\}$; or more explicitly

$$\partial\Delta[n]_m = \{\xi\colon [m]\to[n] \mid \xi \text{ not surjective}\}.$$

A set of generating acyclic cofibrations is given by all inclusions of horns $\Lambda^k[n] \hookrightarrow \Delta[n]$ with $n\in\mathbb{N}$ and $k\in\{0,\dots,n\}$, where $\Lambda^k[n]$ is the simplicial subset generated by all $\delta^i\colon [n-1]\hookrightarrow[n]$ with $i\in\{0,\dots,n\}$ but $i\neq k$. Again more explicitly

$$\Lambda^k[n]_m = \{\xi\colon [m]\to[n] \mid \operatorname{Im}\xi \neq [n] \text{ and } \operatorname{Im}\xi \neq [n]\setminus\{k\}\}.$$

(1.8) **Definition.** Given $n \in \mathbb{N}$, a space K is *n-connected* iff $\pi_k(X)$ is a singleton for every $k \in \{0, \dots, n\}$. In particular, a 0-connected space (which we will just call *connected*) has to be non-empty. We extend this definition by saying that every space is (-2)-connected and that a space K is (-1)-*connected* iff $K \neq \varnothing$.

This is the same convention as the one used by Goodwillie [30] and a good justification for it (other than just making some results and formulae more concise) in terms of closed classes is given in (8.2.2).

We assume basic familiarity with the notions of homotopy limits and colimits (also, we have given it some treatment from an abstract perspective as derived functors in the first part). However, in contrast to the situation there, where the homotopy (co)limits only exist in the homotopy category, since we have functorial (co)fibrant replacements at our hands for the category of simplicial sets. Still, let us quickly review specifically what "being a homotopy (co)limit" means in a model categorical setting. We treat the homotopy colimit case as the other one is dual.

In the abstract approach via derived functors, given any small category $\mathcal{I}$ and letting $\bar{\mathcal{I}}$ be the category $\mathcal{I}$ with a terminal element $\top$ added, the inclusion $I\colon \mathcal{I} \hookrightarrow \bar{\mathcal{I}}$ gives rise to a left Kan extension functor $I_!\colon \mathbf{sSets}^{\mathcal{I}} \to \mathbf{sSets}^{\bar{\mathcal{I}}}$ (which is just given by taking the colimit at $\top$). With this, a diagram $X\colon \bar{\mathcal{I}} \to \mathbf{sSets}$ is a homotopy colimit diagram iff it lies in the essential image of $\mathbb{L}I_!$. Equivalently, one can require the counit $(\mathbb{L}I_! \circ \operatorname{Ho} I^*)X \to X$ to be a weak equivalence (we should really say an isomorphism since, at this level of generality, it is a morphism in the homotopy category).

For **sSets** (or more generally, any cofibrantly generated model category), we can use the explicit construction of left derived functors via projectively cofibrant replacements; i.e. cofibrant replacements in the projective model structure (cf. section 3). That is to say, we use the fact that

$$\begin{tikzcd} \mathbf{sSets}^{\mathcal{I}} \arrow[r, shift left, "I_!"] \arrow[r, phantom, "\perp"] & \mathbf{sSets}^{\bar{\mathcal{I}}} \arrow[l, shift left, "I^*"] \end{tikzcd}$$

is a Quillen adjunction with respect to the projective model structures. Explicitly, a diagram $X\colon \bar{\mathcal{I}} \to \mathbf{sSets}$ is a homotopy colimit diagram iff any (or equivalently all) cofibrant replacement maps $q\colon Q(I^*X) \to I^*X = X|_{\mathcal{I}}$ in $\mathbf{sSets}^{\mathcal{I}}$ the composite

$$I_!Q(I^*X) \xrightarrow{I_!q} I_!I^*X \to X$$

(where the second arrow is the counit) is a weak equivalence.

(1.9) **Example.** For $\mathcal{I} = \{a \leftarrow b \to c\}$ the indexing category for spans, a diagram $X_a \leftarrow X_b \to X_c$ is projectively cofibrant iff X_b is cofibrant and the two arrows are cofibrations. So, such a diagram can be replaced cofibrantly by first replacing X_b cofibrantly as $QX_b \to X_b$ and then factoring the composite $QX_b \to X_b \to X_a$ into a cofibration, followed by an acyclic fibration. In this way, we obtain a morphism of diagram

$$\begin{tikzcd} QX_a \arrow[d, two heads, "\sim"'] & QX_b \arrow[l, tail] \arrow[r, tail] \arrow[d, two heads, "\sim"'] & QX_c \arrow[d, two heads, "\sim"'] & & \operatorname{hocolim}_{\mathcal{I}} X \arrow[d] \\ X_a & X_b \arrow[l] \arrow[r] & X_c \arrow[r, phantom, "\overset{\mathrm{colim}}{\longmapsto}"] & & \operatorname{colim}_{\mathcal{I}} X \end{tikzcd}$$

and a commutative square $X\colon \bar{\mathcal{I}} \to \mathbf{sSets}$ is a homotopy pushout square iff the composite $\operatorname{hocolim}_{\mathcal{I}} X \to \operatorname{colim}_{\mathcal{I}} X \to X_\top$ is a weak equivalence.

(1.10) **Remark.** One can show that in any model category, weak equivalences between cofibrant objects are stable under pushouts along cofibrations and it follows from the pushout lemma that any pushout square

$$\begin{tikzcd} A \arrow[r, tail] \arrow[d] & B \arrow[d] \\ C \arrow[r, tail] & D \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

with cofibrant objects and $A \rightarrowtail B$ a cofibration is also a homotopy pushout square. Since, in **sSets** all objects are cofibrant, we can forget about the cofibrancy assumption. Dually, while not every simplicial set is fibrant (i.e. a Kan complex), **sSets** is right proper, meaning that weak equivalences are stable under pullbacks along fibrations and it again follows that any pullback square of simplicial sets

$$\begin{tikzcd} A \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, two heads] \\ C \arrow[r] & D \end{tikzcd}$$

with $B \twoheadrightarrow D$ a fibration is also a homotopy pullback.

Many of the usual homotopical constructions can be understood as (a combination of) homotopy limits and homotopy colimits. Let us discuss the most important ones for us. Given any simplicial set A, we define its *suspension* as

$$\Sigma A := \operatorname{hocolim}(* \leftarrow A \rightarrow *) \qquad (\text{where } * := \Delta[0]).$$

Special cases of these are the spheres S^n, which we define as follows. The (-1)-sphere S^{-1} is the empty simplicial set (it will be important later on to distinguish this from the empty set of spaces and so we try to avoid the notation $\varnothing$ for S^{-1}) and $S^{n+1} := \Sigma S^n$.

(1.11) **Notation.** We will sometimes allow ourselves to be sloppy with our notation and just write $*$ for any contractible simplicial set. So, we might say that

$$\begin{tikzcd} A \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & \Sigma A \end{tikzcd}$$

is a homotopy pushout square even though we are working in an unpointed context and there is no natural base point making the square commute. Similarly, we will sometimes just write and speak about identities when we really mean weak equivalences.

More generally, given any map of spaces $f\colon A \to B$, we define its *homotopy cofibre* to be the homotopy pushout

$$\operatorname{Cof}(f) := \operatorname{hocolim}(* \leftarrow A \xrightarrow{f} B).$$

Usually, as long as there is no risk of confusion, we will also just write $B/\!\!/A$ for this homotopy cofibre and it follows that as long as $f\colon A \rightarrowtail B$ is monic, $B/\!\!/A \simeq B/A$ is just the usual strict quotient.

We are also going to need a few homotopical constructions that are inherently (partially) pointed such as loop spaces, wedges, smashes and half-smashes. Given a space X with base-point x, its loop-space (at x) is

$$\Omega_x X := \operatorname{holim}\left(* \xrightarrow{x} X \xleftarrow{x} *\right).$$

As long as the base-point x is clear (or implicitly given), we will simply write $\Omega_* X$. With this, one can show that we have a derived adjunction $\Sigma \dashv \Omega_*$ in an extremely general context; namely for any pointed derivator [31]. As long as X is connected, the homotopy type of $\Omega_x X$ does not depend on x and it is usually safe to just write ΩX. We extend this notation to unconnected (unpointed!) spaces X by defining

$$\Omega X := \begin{cases} \Omega_x X & X \text{ connected} \\ S^{-1} & \text{otherwise,} \end{cases}$$

where $x \in X$ is arbitrary (and so this notation only makes sense in a homotopical context). The reasoning behind this convention is that looping should lower the connectivity of a space.

(1.12) **Warning.** With this convention, some caution is required when dealing with maps and loop spaces. Given a map $f\colon A \to B$, there need not be a map $\Omega f\colon \Omega A \to \Omega B$ because ΩB might be empty. However, Ωf is well-defined (up to homotopy) if B is connected or $B = S^{-1}$. Moreover, one needs to be careful when using the suspension-loop adjunction (which is only sensible in a pointed context anyway).

Again, more generally, given any pointed space Y with base-point y and a map $f\colon X \to Y$, we define its *homotopy fibre* at y to be the homotopy pullback

$$\operatorname{hFib}_y(f) := \operatorname{holim}\left(* \xrightarrow{y} Y \xleftarrow{f} X\right)$$

and again write $\operatorname{hFib}_*(f)$ if the base-point y is clear and even just $\operatorname{hFib}(f)$ for Y connected (so that the homotopy type of $\operatorname{hFib}_*(f)$ does not depend on the base-point). In contrast to loop spaces, we are not going to introduce some unusual convention for a non-connected Y but shall instead work with fibre sets later on.

As for the other pointed constructions that we mentioned, if X and Y are both pointed spaces, their wedge- and smash product are, respectively,

$$X \vee Y := \operatorname{colim}(X \leftarrow * \to Y) \simeq \operatorname{hocolim}(X \leftarrow * \to Y)$$

(where the two maps are base-point inclusions) and

$$X \wedge Y := \operatorname{colim}(* \leftarrow X \vee Y \hookrightarrow X \times Y) \simeq \operatorname{hocolim}(* \leftarrow X \vee Y \hookrightarrow X \times Y)$$

(where $X \vee Y \hookrightarrow X \times Y$ is the inclusion on both X and Y by taking is base-point of the other space as the second coefficient). One easily checks that S^0 (with any one point chosen as its base-point) is a unit for the smash; i.e. $X \wedge S^0 \simeq S^0 \wedge X \simeq \operatorname{hocolim}(* \leftarrow X + * \hookrightarrow X + X) \simeq X$. In case X is pointed but Y is not, we can define the half-smash

$$X \rtimes Y := \operatorname{colim}(* \leftarrow Y \hookrightarrow X \times Y) \simeq \operatorname{hocolim}(* \leftarrow Y \hookrightarrow X \times Y)$$

(where $Y \hookrightarrow X \times Y$ is the inclusion with the first coefficient the base point of X). Similarly for $X \ltimes Y$ if X is unpointed but Y is pointed. The reduced suspension of a pointed space X is the smash product $X \wedge S^1$ (where $S^1 = \Sigma S^0 = \operatorname{hocolim}(* \leftarrow S^0 \to *)$ as above with a base-point coming from a chosen base-point in S^0) and one can show that $\Sigma X \simeq X \wedge S^1$. This can be done purely diagrammatically as follows.

First, using Thomason's theorem [15, Theorem 26.8] (see also (5.2) below), we can write $X \wedge S^1$ as the homotopy colimit of

$$X \wedge S^1 \simeq \operatorname{hocolim}\left(\begin{tikzcd} & X \arrow[dl] \arrow[dr, tail] & \\ * & * \arrow[l] \arrow[r] \arrow[u] \arrow[d] & X \times S^1 \\ & S^1 \arrow[ul] \arrow[ur, tail] & \end{tikzcd}\right).$$

Since all constructions are homotopy invariant, we can assume that X is a Kan complex and we note that by Mather's second cube theorem (7.2), $X \times -$ preserves homotopy colimits (since this is just pulling back along $X \to *$). In particular $X \times S^1 \simeq \operatorname{hocolim}(X \leftarrow X + X \to X)$, where the two arrows are the fold maps. With this, the above diagram can actually be lifted (along hocolim) to a diagram in the category of spans $\mathbf{sSets}^{\{a \leftarrow b \to c\}}$ as

$$\begin{tikzcd} & (X \leftarrow X \to X) \arrow[dl] \arrow[dr] & \\ (* \leftarrow * \to *) & (* \leftarrow * \to *) \arrow[l] \arrow[r] \arrow[u] \arrow[d] & (X \leftarrow X + X \to X) \\ & (* \leftarrow S^0 \to *) \arrow[ul] \arrow[ur] & \end{tikzcd},$$

where the only non-obvious maps are $* \to S^0$, $* \to X + X$ and $X \to X + X$. The first one is just the chosen base-point in S^0 (which we used to make S^1 pointed), while the other two are maps into the summand corresponding to this chosen base-point. By Fubini, we can now take level-wise homotopy colimits first, which can again be calculated using Thomason's theorem (i.e. calculating the central homotopy pushout first). In levels a and c, we just get $*$, while in the middle level b, we get $\operatorname{hocolim}(* \leftarrow X + * \to X + X) \simeq X \wedge S^0 \simeq X$. All in all, after levelwise homotopy colimits, we have $* \leftarrow X \to *$ and it follows that

$$X \wedge S^1 \simeq \Sigma X.$$

A final construction that we are going to need, which also gives a link between pointed and unpointed homotopical constructions is the join. For X, Y two (unpointed) spaces, their join is

$$X * Y := \operatorname{hocolim}(X \leftarrow X \times Y \to Y),$$

where the two maps are the standard projections. Clearly, S^{-1} is a unit for this construction (i.e. $S^{-1} * X \simeq X * S^{-1} \simeq X$ for all X). Moreover, using Thomason's theorem twice, we can show that $X * S^0 \simeq \Sigma X$. To wit:

$$\operatorname{hocolim}\left(X \leftarrow X + X \to S^0\right) \simeq \operatorname{hocolim}\left(\begin{tikzcd} & X \to * \arrow[dl] \\ X & \\ & X \to * \arrow[ul] \end{tikzcd}\right) \simeq \operatorname{hocolim}(* \leftarrow X \to *).$$

As for the link between pointed and unpointed constructions, one can easily show [13, Proposition 4.7], in a purely diagrammatic fashion, that $X * Y \simeq \Sigma(X \wedge Y)$ for X and Y pointed.

In particular, the homotopy type of $\Sigma(X \wedge Y)$ does not depend on base points. This provides an alternative proof that $X * S^0 \simeq \Sigma X$ because

$$X * S^0 \simeq \Sigma(X \wedge S^0) \simeq \Sigma X.$$

2. Non-Degenerate Simplices

In this section, we investigate the role of the non-degenerate simplices of a simplicial set. More specifically, we are going to study the category of non-degenerate simplices, which we are going to put to use later on when studying the fibre decomposition. Let us start off very simple.

(2.1) **Definition.** A simplex $x \in K_n$ of a simplicial set K is *degenerate* iff $x = y \cdot \sigma$ for some surjection $\sigma \neq \mathrm{id}$ in $\mathbf{\Delta}$ and some simplex y. Because every surjection is a composition of generating surjections σ^i, this is equivalent to $x = s_j y$ for some $j \in \{0, \ldots, n-1\}$ and some $y \in K_{n-1}$.

(2.2) **Proposition.** For a simplex $x \in K_n$, the following are equivalent:

(a) x is degenerate;

(b) $x \colon \Delta[n] \to K$ factors through $\Delta[n-1]$;

(c) $x \colon \Delta[n] \to K$ factors through some $\Delta[k]$ with $k < n$.

Proof. The implications "(a) $\Rightarrow$ (b) $\Rightarrow$ (c)" are trivial and for "(c) $\Rightarrow$ (a)", assume that $x = y \circ \varphi_*$ (i.e. $x = y \cdot \varphi$) for some $\varphi \colon [n] \to [k]$ with $k < n$ and $y \colon \Delta[k] \to K$. Then $\varphi = \delta \circ \sigma$ for some injection δ and some surjection $\sigma \neq \mathrm{id}$ (because φ cannot be injective), so that $x = (y \cdot \delta) \cdot \sigma$. □

Clearly, every simplicial set is generated by its graded subsets of non-degenerate simplices and every morphism of simplicial sets preserves degenerate simplices (because it is compatible with the degeneracies). It is not true however that every morphism of simplicial sets preserves non-degenerate simplices but this does hold for monic morphisms.

(2.3) **Proposition.** Every monomorphism of simplicial sets $f \colon K \rightarrowtail L$ preserves non-degenerate simplices.

Proof. If $x \in K_n$ is non-degenerate and $f(x) = y \cdot \sigma$ for some surjection σ then we choose a section δ of σ and get

$$f(x \cdot \delta\sigma) = f(x) \cdot \delta\sigma = y \cdot \sigma\delta\sigma = y \cdot \sigma = f(x).$$

Because f is monic (i.e. dimensionwise injective), it follows that $(x \cdot \delta) \cdot \sigma = x$, which is non-degenerate and therefore $\sigma = \mathrm{id}$. □

(2.4) **Definition.** For a simplicial set K, we define the *category of non-degenerate simplices* to be the full subcategory of $\mathcal{K}$ comprising the $\big([n], x\big)$ with $x \in K_n$ non-degenerate. This category will be denoted by $\int_{\Delta}^{*} K$ or $\mathcal{K}^*$.

(2.5) **Observation.** Note that every morphism in a category of non-degenerate simplices is injective. To wit, if $\varphi \colon (m, x) \to (n, y)$ is a morphism of non-degenerate simplices (so that $\varphi \colon [m] \to [n]$ and $x = y \cdot \varphi$), we factor φ as a surjection, followed by an injection, say $\varphi = \delta \circ \sigma$ and since $x = y \cdot \varphi = (y \cdot \delta) \cdot \sigma$ is non-degenerate, σ must be the identity.

The problem with the category of non-degenerate simplices is that this construction is not functorial in any obvious manner. However, it is functorial when restricted to $\mathbf{\Delta} \subseteq \mathbf{sSets}$ or more general non-singular simplicial sets (see below).

(2.6) **Example.** A simplex $\sigma \in \Delta[n]_m$ is just a monotone map $\sigma\colon [m] \to [n]$, which is non-degenerate iff it is injective. So non-degenerate simplices of $\Delta[n]$ are in bijection with non-empty subsets of $\{0, \dots, n\}$ and under this bijection, face maps correspond to inclusions. It follows that $\int_{\mathbf{\Delta}}^{*} \Delta[n]$ is isomorphic to the poset $\mathfrak{P}\{0, \dots, n\} \setminus \{\varnothing\} =: \mathfrak{P}^*[n]$.

In light of this example, it is clear how to make the construction functorial on $\mathbf{\Delta}$. To wit, a morphism $\varphi\colon \Delta[m] \to \Delta[n]$ is just a map $\varphi\colon [m] \to [n]$, which induces $\mathfrak{P}^*[m] \to \mathfrak{P}^*[n]$, $A \mapsto \varphi A$. But we can go a step further and extend this functor to the category of *non-singular* simplicial sets.

(2.7) **Observation.** If K is a simplicial set and $x\colon \Delta[n] \rightarrowtail K$ monic in $\mathbf{sSets}$ (i.e. dimensionwise injective) then $x \in K_n$ is non-degenerate.

Proof. Given a surjection $\sigma\colon [n] \twoheadrightarrow [m]$ and $y \in K_m$ such that $x = y \cdot \sigma$, the triangle

$$\begin{tikzcd} \Delta[n] \arrow[r, tail, "x"] \arrow[d, "\sigma_*"'] & K \\ \Delta[m] \arrow[ur, "y"'] & \end{tikzcd}$$

is commutative and because x is monic, so is σ_*. But any section of σ (which exists), induces a section of σ_*, showing that it is even an isomorphism. Because the Yoneda embedding is fully faithful, σ must be an isomorphism, too, and hence $\sigma = \mathrm{id}$. □

The converse of this observation is not true in general. For example, taking the 2-sphere $S^2 := \Delta[2]/\partial\Delta[2]$ with non-degenerate simplices $*$ and l in dimensions 0 and 2 respectively, l is non-degenerate but $l\colon \Delta[2] \to S^2$ is not monic because all composites $\Delta[0] \to \Delta[2] \to S^2$ are equal.

(2.8) **Definition.** A simplicial set K is *non-singular* iff the converse of the above observation holds. That is to say, if $x \in K_n$ is non-degenerate then $x\colon \Delta[n] \to K$ is monic. Otherwise, the simplicial set K is called *singular*. We write $\mathbf{sSets}_n$ for the full subcategory of $\mathbf{sSets}$ with objects the non-singular simplicial sets.

(2.9) **Example.** All representable simplicial sets $\Delta[n]$ are non-singular. Also, by definition, every simplicial subset of a non-singular simplicial set is non-singular as well.

(2.10) **Example.** If $\mathcal{I}$ is a small skeletal category (i.e. the only isomorphisms are the identities) with only identities as endomorphisms (e.g. a poset) then $N(\mathcal{I})$ is non-singular. First note that being skeletal and having only identity endomorphisms implies that the only split mono- or epimorphisms are the identities, for if $r \circ s = \mathrm{id}$ then also $s \circ r = \mathrm{id}$ as $s \circ r$ is an endomorphism. So, $r = s^{-1}$ and hence $s = r = \mathrm{id}$ because $\mathcal{I}$ is skeletal. Now, by (2.14) below, it suffices to check that given a non-degenerate

$$I_\bullet\colon I_0 \xrightarrow{f_1} I_1 \xrightarrow{f_2} \dots \xrightarrow{f_n} I_n$$

in $N(\mathcal{I})_n$ (i.e. $f_i \neq \mathrm{id}$ for all i), if $d_i I_\bullet = d_j I_\bullet$ then $i = j$. But this is easy because the I_i are pairwise distinct. Indeed, if we had $I_i = I_j$ for some $i < j$ then $I_i \to I_j$ would be an endomorphism (whence the identity) and so, f_{i+1} would be a split monic (thus again an identity).

The problem with the category of non-degenerate simplices of simplicial sets that are singular, which also makes it non-functorial, is that it doesn't capture the boundary relations. Degeneracies on the other hand are not the problem because they are just freely generated by the non-degenerate simplices. To wit, if $x \cdot \sigma = y \cdot \sigma'$ for two non-degenerate simplices x, y and two surjections σ, σ' then $x = y$ and $\sigma = \sigma'$. This is a consequence of the Eilenberg-Zilber lemma (4.7), which will also tell us that the boundary relations for non-singular simplicial sets are determined by their restriction to non-degenerate simplices. Even though we will prove a generalised version of it in (4.7), let us state the usual one here without proof for convenience.

(2.11) **Proposition. (Eilenberg-Zilber Lemma)** For every simplicial set K and every simplex $x \in K_n$ there is a unique surjection $\eta_x \colon [n] \twoheadrightarrow [m]$ (which we call the *Eilenberg-Zilber map* for x) and a unique non-degenerate $x^{\downarrow} \in K_m$ such that $x^{\downarrow} \cdot \eta_x = x$.

(2.12) **Corollary.** A morphism $f \colon K \to L$ of simplicial sets is monic iff it preserves non-degenerate simplices and is monic on them.

Proof. "$\Rightarrow$": By (2.3).
"$\Leftarrow$": Given $x = x^{\downarrow} \cdot \eta_x$, $y = y^{\downarrow} \cdot \eta_y \in K_n$ with $fx = fy$ then $f(x^{\downarrow}) \cdot \eta_x = f(y^{\downarrow}) \cdot \eta_y$. By hypothesis, $f(x^{\downarrow})$ and $f(y^{\downarrow})$ are non-degenerate and so, by Eilenberg-Zilber $f(x^{\downarrow}) = f(y^{\downarrow})$ (whence $x^{\downarrow} = y^{\downarrow}$) and $\eta_x = \eta_y$, proving that $x = y$. $\square$

(2.13) **Example.** For f to be monic, it is not sufficient to be monic on non-degenerate simplices. For example, taking X to be the space obtained from $\Delta[2]$ by identifying two edges and smashing the remaining edge down to a point. The simplicial set X is a model for the 2-sphere and there is an obvious quotient map

$$X \twoheadrightarrow \Delta[2]/\partial\Delta[2],$$

which smashes the remaining non-degenerate 1-simplex down to a point. This quotient map is not monic (as it doesn't preserve non-degeneracy) but it is monic (i.e. dimensionwise injective) on non-degenerate simplices.

(2.14) **Proposition.** For a simplicial set K, the following are equivalent:

(a) K is a non-singular;

(b) for every non-degenerate simplex x and every two injections δ, δ' in $\mathbf{\Delta}$, having $x \cdot \delta = x \cdot \delta'$ implies $\delta = \delta'$;

(c) for every non-degenerate simplex $x \in K_n$ and every two i, $j \in \{0, \ldots, n\}$, having $d_i x = d_j x$ implies $i = j$.

Moreover, if these hold then boundaries of non-degenerate simplices are non-degenerate.

Proof. "*(a) $\Rightarrow$ (b)*": Given $x \in K_n$, δ, $\delta' \colon [m] \rightarrowtail [n]$ as in the claim, $x \colon \Delta[n] \rightarrowtail K$ is monic and because

$$\begin{tikzcd} \Delta[m] \arrow[r, shift left, "\delta_*"] \arrow[r, shift right, "\delta'_*"'] & \Delta[n] \arrow[r, tail, "x"] & K \end{tikzcd}$$

is a cofork, it follows that $\delta_* = \delta'_*$, whence $\delta = \delta'$.

"*(b) ⇔ (c)*": The direction "⇒" is trivial and for the converse, just note that every injection is expressible as a composite of δ^i and proceed by induction.

"*(b) ⇒ (a)*": First, observe that the "Moreover" part follows from (b). To wit, if $x \in K_n$ is non-degenerate and $x \cdot \delta = y \cdot \sigma$ with δ injective, σ surjective and y some simplex, then for any two sections δ', δ'' of σ, we get $x \cdot (\delta\delta') = y = x \cdot (\delta\delta'')$, whence, by hypothesis and injectivity of δ, $\delta' = \delta''$, meaning that σ only has one section and thus $\sigma = \mathrm{id}$. Now, by the previous corollary, $x\colon \Delta[n] \to K$ is monic iff it preserves non-degeneracy (which is the "Moreover" part) and is monic on non-degenerate simplices, which is implied by (b). □

(2.15) **Corollary.** If K is a non-singular simplicial set then its category of non-degenerate simplices $\mathcal{K}^*$ is a poset.

Proof. Follows directly from (b) and the observation (2.5) that all morphisms in $\mathcal{K}^*$ are injections. □

(2.16) **Example.** For a simplicial set to be non-singular, it is not sufficient for boundaries of non-degenerate simplices to be non-degenerate. For example, $S^1 := \Delta[1]/\partial\Delta[1]$ satisfies this property but is singular because, writing $*$ and l for the non-degenerate simplices in dimension 0 and 1 respectively, we have $d_0 l = d_1 l = *$. This motivates the following definition.

(2.17) **Definition.** A simplicial set K is called a *∂-non-singular* iff all boundaries of non-degenerate simplices in K are again non-degenerate. We write $\mathbf{sSets}_{\partial n}$ for the full subcategory of **sSets** with objects all ∂-non-singular simplicial sets.

(2.18) **Example.** Every non-singular simplicial set is also ∂-non-singular but for example $S^1 = \Delta[1]/\partial\Delta[1]$ is a singular ∂-non-singular simplicial set.

(2.19) **Example.** In analogy to (2.10), if $\mathcal{I}$ is a small category whose only split monomorphisms (or equivalently split epimorphisms) are identities then $N(\mathcal{I})$ is ∂-non-singular. Indeed, if

$$I_\bullet\colon I_0 \xrightarrow{f_1} I_1 \xrightarrow{f_2} \dots \xrightarrow{f_n} I_n$$

is non-degenerate (i.e. $f_i \neq \mathrm{id}$ for all i) then none of its boundaries, which are obtained by cutting off and composing some of the f_i, can be degenerate by hypothesis.

The following characterisation of ∂-non-singular simplicial sets (or rather the corollary thereafter) is the reason why we defined them in the first place.

(2.20) **Proposition.** For a simplicial set K, the following are equivalent:

(a) K is ∂-non-singular;

(b) the Eilenberg-Zilber assignment $x \mapsto x^\downarrow$ can be extended to a functor $\mathcal{K} \to \mathcal{K}^*$ in such a way that $\eta_x\colon x \to x^\downarrow$ becomes natural;

(c) the inclusion $\mathcal{K}^* \hookrightarrow \mathcal{K}$ has a left adjoint determined by the object function $x \mapsto x^\downarrow$ and the unit having components $\eta_x\colon x \to x^\downarrow$ the Eilenberg-Zilber maps;

(d) the inclusion $\mathcal{K}^* \hookrightarrow \mathcal{K}$ has a left adjoint.

Proof. "*(a) ⇒ (c)*": It suffices to verify the universal property of η. So given a morphism $\varphi\colon x \to y$ in $\mathcal{K}$ (i.e. $x = y \cdot \varphi$) with y non-degenerate, we must find a unique morphism $\varphi^\flat\colon x^\downarrow \to y$ fitting in to the commutative triangle

$$\begin{tikzcd} x \arrow[r, two heads, "\eta_x"] \arrow[dr, "\varphi"'] & x^\downarrow \arrow[d, dotted, "\varphi^\flat"] \\ & y \end{tikzcd} .$$

For this, we factor φ as $\varphi = \delta \circ \sigma$ with δ monic and σ epi. Now $x^\downarrow \cdot \eta_x = x = y \cdot \varphi = y \cdot \delta \cdot \sigma$ and so, by hypothesis and the Eilenberg-Zilber lemma, $x^\downarrow = y \cdot \delta$ and $\eta_x = \sigma$. With this, $\varphi^\flat := \delta$ makes the triangle commute and is the unique arrow doing so because epi-mono factorisations in $\mathcal{K}$ are unique.

"*(c) ⇒ (b) & (c) ⇒ (d)*": Trivial.

"*(b) ⇒ (a)*": Given any non-degenerate simplex $x \in K_n$ and any injection δ into $[n]$, we get a morphism $\delta\colon x \cdot \delta \to x$ in $\mathcal{K}$ and so, by naturality of η, a commutative square

$$\begin{tikzcd} x \cdot \delta \arrow[r, two heads, "\eta_{x\cdot\delta}"] \arrow[d, tail, "\delta"'] & (x \cdot \delta)^\downarrow \arrow[d, "\delta^\downarrow"] \\ x \arrow[r, equal, "\eta_x"'] & x \end{tikzcd} .$$

Because $\delta = \delta^\downarrow \circ \eta_{x\cdot\delta}$ is injective, so is $\eta_{x\cdot\delta}$ and it follows that $\eta_{x\cdot\delta} = \mathrm{id}$ (since it is also surjective) and $\delta^\downarrow = \delta$. In particular, $x \cdot \delta$ is non-degenerate.

"*(d) ⇒ (c)*": Let us denote the left adjoint by F and the unit by η'. First, note that the inclusion $\mathcal{K}^* \hookrightarrow \mathcal{K}$ is fully faithful and so the counit must be an isomorphism. But $\mathcal{K}$ is skeletal, meaning that the counit is even the identity. In particular, if x is non-degenerate, then $Fx = x$ and $\eta'_x = \mathrm{id}$ (by a triangle identity). Next, for an arbitrary simplex x, we apply F to the Eilenberg-Zilber map $\eta_x\colon x \to x^\downarrow$, yielding $F\eta_x = \eta_x^\flat\colon Fx \to x^\downarrow$, which must be injective because Fx and $x^\downarrow$ are non-degenerate. But $\eta_x = \eta_x^\flat \circ \eta'_x$ is surjective and so $\eta_x^\flat$ must be surjective, too, meaning it is the identity. □

(2.21) **Scholium.** If K is ∂-non-singular, the arrow map of the Eilenberg-Zilber functor $\mathcal{K} \to \mathcal{K}^*$ maps a φ in $\mathcal{K}$ with (unique) epi-mono factorisation $\varphi = \delta \circ \sigma$ to δ.

(2.22) **Corollary.** For every ∂-non-singular K, the inclusion $\mathcal{K}^* \hookrightarrow \mathcal{K}$ is homotopy final (see (4.20)).

Proof. Right adjoint functors are homotopy final. □

3. Projective Model Structures

There are several different approaches to calculating homotopy colimits; the common notion behind all of them being that of a total left derived functor of the colimit functor. The most direct approach is via projective model structures with the caveat of needing a cofibrantly generated base category. In the next section, we shall see an alternative approach via Reedy model structures.

(3.1) **Notation.** Recall that if $\mathcal{M}$ is any category with copowers, $A \in \mathcal{M}$ and X any set, we write $X \cdot A$ for the X-fold copower of A. That is to say $X \cdot A := \coprod_X A$. More generally, if $X\colon \mathcal{I} \to \mathbf{Sets}$ is a diagram of sets and $A\colon \mathcal{J} \to \mathcal{M}$, we write $X \cdot A$ for the pointwise copower. More explicitly

$$X \cdot A\colon \mathcal{I} \times \mathcal{J} \to \mathcal{M},\ (I, J) \mapsto X_I \cdot A_J.$$

A pair of morphism $i\colon I \to I'$ in $\mathcal{I}$ and $j\colon J \to J'$ in $\mathcal{J}$ is mapped to the unique morphism

$$[\mathrm{in}_{(Xi)x} \circ Aj]_{x \in X_I}\colon X_I \cdot A_J \to X_{I'} \cdot A_{J'}$$

making the following square commute for all $x \in X_I$:

$$\begin{tikzcd} A_J \arrow[r, "Aj"] \arrow[d, "\mathrm{in}_x"'] & A_{J'} \arrow[d, "\mathrm{in}_{(Xi)x}"] \\ X_I \cdot A_J \arrow[r, dotted] & X_{I'} \cdot A_{J'}. \end{tikzcd}$$

(3.2) **Theorem.** Let $\mathcal{M}$ be a cofibrantly generated model category with generating cofibrations $\mathcal{I}$ and generating acyclic cofibrations $\mathcal{J}$. Then, for every small indexing category $\mathcal{I}$, the *projective model structure* (with pointwise weak equivalences and fibrations) on $\mathcal{M}^{\mathcal{I}}$ exists and is again cofibrantly generated having

(a) generating cofibrations all $\mathcal{I}(I,-) \cdot A \xrightarrow{\mathcal{I}(I,-)\cdot i} \mathcal{I}(I,-) \cdot B$ with $I \in \mathcal{I}$ and $i\colon A \to B$ in $\mathcal{I}$ a generating cofibration;

(b) generating acyclic cofibrations all $\mathcal{I}(I,-) \cdot A \xrightarrow{\mathcal{I}(I,-)\cdot j} \mathcal{I}(I,-) \cdot B$ with $I \in \mathcal{I}$ and $j\colon A \to B$ in $\mathcal{J}$ a generating acyclic cofibration.

Proof. [33, Theorem 11.6.1]. □

(3.3) **Example.** For $\mathcal{M} = \mathbf{sSets}$ and $\mathcal{I}$ any small indexing category, every representable presheaf $\mathcal{I}(I,-)\colon \mathcal{I} \to \mathbf{Sets}$ can be viewed as a discrete simplicial presheaf. One way to make this formal is that we compose it with $\mathbf{Sets} \hookrightarrow \mathbf{sSets}$, induced by the projection $\mathbf{\Delta} \to \{*\}$. Another way is that the discrete representable presheaf is the (pointwise) copower $\mathcal{I}(I,-)\cdot *$ of a point. With this description, every representable discrete simplicial presheaf is projectively cofibrant.

We will also need the following easy observation that simplicial model categories are stable under taking diagram categories.

(3.4) **Proposition.** If $\mathcal{M}$ is a cofibrantly generated simplicial model category and $\mathcal{I}$ a small indexing category, then $\mathcal{M}^{\mathcal{I}}$ equipped with the projective model structure is again a simplicial model category with

$$\odot\colon \mathbf{sSets} \times \mathcal{M}^{\mathcal{I}} \to \mathcal{M}^{\mathcal{I}}, \qquad [-,-]\colon \mathbf{sSets}^{\mathrm{op}} \times \mathcal{M}^{\mathcal{I}} \to \mathcal{M}^{\mathcal{I}}$$

defined pointwise and mapping spaces

$$\mathrm{Map}_{\mathcal{M}^{\mathcal{I}}}\colon \mathcal{M}^{\mathcal{I}} \times \mathcal{M}^{\mathcal{I}} \to \mathbf{sSets},\ (X, Y) \mapsto \mathrm{Nat}(\Delta[n] \odot X, Y).$$

Proof. [33, Theorem 11.7.3]. □

What is important for us is to be able to recognise *projectively cofibrant* diagrams (i.e. ones that are cofibrant in the projective model structure) for $\mathcal{M} = \mathbf{sSets}$. We have taken the following criterion from [20], where it is left as an exercise for the reader, which is why we shall quickly prove it here.

In what follows, we will consider *simplicial presheaves* $X\colon \mathcal{I} \to \mathbf{sSets}$, which, of course, can also be viewed as functors $X\colon \mathcal{I} \times \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Sets}$ or as functors $X\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{M}^{\mathcal{I}}$ (i.e. a simplicial object in $\mathcal{M}^{\mathcal{I}}$). For $n \in \mathbb{N}$, we write $X_n\colon \mathcal{I} \to \mathcal{M}$ for the evaluation of this latter X at $[n] \in \mathbf{\Delta}^{\mathrm{op}}$.

(3.5) **Definition.** A simplicial presheaf $X\colon \mathcal{I} \to \mathbf{sSets}$ has *free degeneracies* iff there is a family of subpresheaves $(N_n \hookrightarrow X_n)_{n\in\mathbb{N}}$ such that for every $n \in \mathbb{N}$ the map

$$\coprod_{k\in\mathbb{N}} \coprod_{\substack{\sigma\colon [n]\twoheadrightarrow[k] \\ \text{surj.}}} N_\sigma \xrightarrow{[X\sigma]_\sigma} X_n \qquad (\text{with } N_\sigma := N_k \text{ for } \sigma\colon [n] \twoheadrightarrow [k])$$

is an isomorphism (of presheaves). Being subpresheaves explicitly means that $N_{n,I} \subseteq X_{n,I}$ for $I \in \mathcal{I}$ and that every $i_*\colon X_{n,I} \to X_{n,J}$ induced by an $i\colon I \to J$ in $\mathcal{I}$ maps $N_{n,I}$ to $N_{n,J}$.

This definition, while very concise, needs some getting used to. Here is one way to see it: Given any simplicial presheaf $X\colon \mathcal{I} \to \mathbf{sSets}$, the *degenerate part* D_n of X_n consists of the images of all $\sigma^*\colon X_k \hookrightarrow X_n$ induced by proper surjections $\sigma\colon [n] \twoheadrightarrow [k]$ out of $[n]$. Its complement

$$N_n := X_n \setminus \{\mathrm{Im}(\sigma^*\colon X_k \hookrightarrow X_n) \mid k < n,\, \sigma\colon [n] \twoheadrightarrow [k] \text{ surjective}\}$$

is the *non-degenerate part* of X_n. Obviously, we can define this for an arbitrary simplicial presheaf X and always get a decomposition $X_n = N_n + D_n$. Also, the D_n are subpresheaves, meaning that every $i_*\colon X_{n,I} \to X_{n,J}$ induced by an $i\colon I \to J$ in $\mathcal{I}$ maps $D_{n,I} \subseteq X_{n,I}$ to $D_{n,J} \subseteq X_{n,J}$. With this more concrete viewpoint, having free degeneracies means

(a) the images of all non-degenerate parts N_k under all $\sigma^*\colon X_k \hookrightarrow X_n$ induced by proper surjections $\sigma\colon [n] \twoheadrightarrow [k]$ out of $[n]$ are pairwise disjoint;

(b) the non-degenerate parts N_n form subpresheaves $\mathcal{I} \to \mathbf{Sets}$ as well.

Note however, that while, for a simplicial presheaf with free degeneracies, the $N_{n,I}$ are required to be functorial in I, no functoriality is imposed on the index n. We obviously shouldn't because a morphism induced by a proper surjection in $\mathbf{\Delta}$ will map everything (in particular non-degenerates) to the degenerate part. Still, let us make the following observation about the interplay between the N_n and maps in $\mathbf{\Delta}$.

(3.6) **Observation.** Let X be a simplicial presheaf with free degeneracies as above, $\varphi\colon [m] \to [n]$ and $\sigma\colon [n] \twoheadrightarrow [k]$ a surjection. If φ is itself a surjection then $N_\sigma \subseteq X_n$ (which is just the image of N_k under $\sigma^*\colon X_k \to X_n$) is mapped to $N_{\sigma\circ\varphi}$ by definition. For a general φ, we factor $\sigma \circ \varphi$ uniquely as a surjection ε, followed by an injection δ:

$$\begin{tikzcd}
{[m]} \arrow[r, "\varphi"] \arrow[d, two heads, "\varepsilon"'] & {[n]} \arrow[d, two heads, "\sigma"] \\
{[p]} \arrow[r, tail, "\delta"'] & {[k]}
\end{tikzcd}
\qquad \overset{X}{\rightsquigarrow} \qquad
\begin{tikzcd}
X_m & X_n \arrow[l, "\varphi^*"'] & N_\sigma \arrow[l, hook'] \\
X_p \arrow[u, "\varepsilon^*"] & X_k \arrow[l, "\delta^*"] \arrow[u, "\sigma^*"'] & N_k. \arrow[l, hook'] \arrow[u, "\sigma^*"', "\cong"]
\end{tikzcd}$$

Now, we don't know where N_σ is mapped to under φ^* but we do know that δ^* maps every element of N_k to some N_τ with $\tau\colon [p] \twoheadrightarrow [l]$ surjective (potentially different ones for different elements), which in turn is mapped to $N_{\tau\circ\varepsilon} \subseteq X_m$. All in all, we have that

$$\varphi^*(N_\sigma) \subseteq \coprod_{l\in\mathbb{N}} \coprod_{\substack{\tau\colon [p]\twoheadrightarrow[l] \\ \text{surj.}}} N_{\tau\circ\varepsilon} \subseteq \coprod_{l\leqslant k} \coprod_{\substack{\rho\colon [m]\twoheadrightarrow[l] \\ \text{surj.}}} N_\rho$$

(3.7) **Proposition.** With the above notation, if $X\colon \mathcal{I} \to \mathbf{sSets}$ has free degeneracies then X is can be obtained as a colimit of simplicial presheaves

$$\mathrm{Sk}_0 X \hookrightarrow \mathrm{Sk}_1 X \hookrightarrow \ldots \hookrightarrow X,$$

where $\mathrm{Sk}_0 X = N_0 \cdot *$ (i.e. $(\mathrm{Sk}_0 X)_I = N_{0,I}$ is a discrete simplicial set for $I \in \mathcal{I}$) and each transformation $\mathrm{Sk}_{n-1} X \to \mathrm{Sk}_n X$ fitting into a pushout

$$\begin{tikzcd}
N_n \cdot \partial\Delta[n] \arrow[r] \arrow[d, hook] & \mathrm{Sk}_{n-1} X \arrow[d] \\
N_n \cdot \Delta[n] \arrow[r] & \mathrm{Sk}_n X \arrow[ul, phantom, very near start, "\ulcorner"]
\end{tikzcd}$$

(where the right-hand map is the boundary inclusion on each component).

Proof. For $n \in \mathbb{N}$, we define $\mathrm{Sk}_n X \subseteq X$ to be the simplicial presheaf given by

$$(\mathrm{Sk}_n X)_i := \coprod_{k\leqslant n} \coprod_{\substack{\sigma\colon [i]\twoheadrightarrow[k] \\ \text{surj.}}} N_\sigma \subseteq X_i$$

(in particular $(\mathrm{Sk}_n X)_i = X_i$ for all $i \leqslant n$). This is well-defined (i.e. every $\varphi^*\colon X_l \to X_k$ induced by some $\varphi\colon [k] \to [l]$ maps $(\mathrm{Sk}_n X)_l$ to $(\mathrm{Sk}_n X)_k$) by the above observation.

There is an obvious inclusion $\mathrm{Sk}_{n-1} X \to \mathrm{Sk}_n X$ and to see that this fits into a pushout square as required, note that a map $N_n \cdot \Delta[n] \to \mathrm{Sk}_n X$ is nothing but $|N_n|$-many elements in $(\mathrm{Sk}_n X)_n$, for which we can just take N_n itself. Saying that the restriction to $N_n \cdot \partial\Delta[n]$ factors through $\mathrm{Sk}_{n-1} X$ means that all boundaries of the simplices $N_n \subseteq X_n$ lie in $\mathrm{Sk}_{n-1} X$, which is obvious since $(\mathrm{Sk}_{n-1} X)_{n-1} = (\mathrm{Sk}_n X)_{n-1} = X_{n-1}$. Finally, that the square is a pushout can be checked dimensionwise. For this, in dimension i, we need to check that the map $N_n \cdot \Delta[n]_i \to (\mathrm{Sk}_n X)_i$ induces a bijection

$$\big(N_n \cdot \Delta[n]_i\big) \setminus \big(N_n \cdot \partial\Delta[n]_i\big) \cong (\mathrm{Sk}_n X)_i \setminus (\mathrm{Sk}_{n-1} X)_i$$

on complements. And indeed

$$(\mathrm{Sk}_n X)_i = (\mathrm{Sk}_{n-1} X)_i + \coprod_{\substack{\sigma\colon [i]\twoheadrightarrow[n] \\ \text{surj.}}} N_n, \text{ while } \partial\Delta[n]_i = \Big\{ [i] \xrightarrow{\varphi} [n] \Bigm| \varphi \text{ not surjective} \Big\}.$$

□

(3.8) **Corollary.** With the above notation, if $X\colon \mathcal{I} \to \mathbf{sSets}$ is a simplicial presheaf with free degeneracies $N_n \subseteq X_n$ and every $N_n\colon \mathcal{I} \to \mathbf{Sets}$ viewed as a discrete simplicial presheaf is projectively cofibrant, then so is X.

Proof. By (3.4) (for $\mathcal{M} := \mathbf{sSets}$), we know that the pointwise product

$$\times\colon \mathbf{sSets} \times \mathbf{sSets}^{\mathcal{I}} \to \mathbf{sSets}^{\mathcal{I}}$$

is a left Quillen bifunctor. In particular, if $Y\colon \mathcal{I} \to \mathbf{sSets}$ is projectively cofibrant, then every $\partial\Delta[n] \times Y \to \Delta[n] \times Y$ is a cofibration. In our case (using that the product with a discrete simplicial set is just the copower), it follows that every $N_n \cdot \partial\Delta[n] \hookrightarrow N_n \cdot \Delta[n]$ is a cofibration. Therefore, (by the above presentation of $\mathrm{Sk}_{n-1} X \to \mathrm{Sk}_n X$ as a pushout), $\varnothing \to X$ is the transfinite composition

$$\varnothing \rightarrowtail N_0 = \mathrm{Sk}_0 X \rightarrowtail \mathrm{Sk}_1 X \rightarrowtail \mathrm{Sk}_2 X \rightarrowtail \dots$$

of a sequence of cofibrations and hence itself one. □

4. Homotopy Colimits

Maybe the easiest and most straightforward approach to defining the homotopy colimit functor (for a cofibrantly generated base category $\mathcal{M}$ and an indexing category $\mathcal{I}$) is to use the projective model structure on $\mathcal{M}^{\mathcal{I}}$, for which

$$\begin{tikzcd} \mathcal{M}^{\mathcal{I}} \arrow[r, shift left, "\mathrm{colim}"] \arrow[r, phantom, "\perp"] & \mathcal{M} \arrow[l, shift left, "\mathrm{Const}"] \end{tikzcd} \qquad \text{is a Quillen adjunction.}$$

In this section, however, we wish to establish an explicit formula for the homotopy colimit of diagram indexed by (the category of simplices of) a simplicial set. And while one can get explicit formulae using the projective model structure (see [20]), our goal is the formula found in [11], due to Emmanuel Dror Farjoun. Our main contribution here is to put it into a model categorical context by generalising some well-known results about (co)simplicial simplicial sets. A longer systematic treatment (but relying on explicit formulae to reduce to the [co]simplicial case) can be found in [33, Chapter 18].

Our treatment rests on the last one of the following insights [28], [33, Theorem 18.4.11]. We use the notation from section 6.1 for simplicial model categories as well as indices “inj”, “proj” and “Reedy” for the injective, projective and Reedy model structure on diagram categories, respectively.

(4.1) **Theorem.** Let $\mathcal{M}$ be any simplicial model category and $\mathcal{I}$ a small indexing category. Then the functor tensor product (a.k.a. the weighted colimit functor)

$$\odot_{\mathcal{I}}\colon \mathbf{sSets}^{\mathcal{I}^{\mathrm{op}}} \times \mathcal{M}^{\mathcal{I}} \to \mathcal{M},\ (W, X) \mapsto \int^{I\in\mathcal{I}} W_I \odot X_I$$

is a left Quillen bifunctor if it is

(a) viewed as a functor $\left(\mathbf{sSets}^{\mathcal{I}^{\mathrm{op}}}\right)_{\mathrm{inj}} \times \left(\mathcal{M}^{\mathcal{I}}\right)_{\mathrm{proj}} \to \mathcal{M}$ or

(b) as a functor $\left(\mathbf{sSets}^{\mathcal{I}^{\mathrm{op}}}\right)_{\mathrm{proj}} \times \left(\mathcal{M}^{\mathcal{I}}\right)_{\mathrm{inj}} \to \mathcal{M}$.

If, in addition, $\mathcal{I}$ is Reedy then it is also a left Quillen bifunctor if it is

(c) viewed as a functor $\left(\mathbf{sSets}^{\mathcal{I}^{\mathrm{op}}}\right)_{\mathrm{Reedy}} \times \left(\mathcal{M}^{\mathcal{I}}\right)_{\mathrm{Reedy}} \to \mathcal{M}$.

To be able to use this result, we need to combine it with three standard results concerning diagrams indexed by (the categories of elements of) simplicial sets. These are usually stated for diagrams indexed by $\mathbf{\Delta}$ and we give proofs of their generalisations here.

Usually, when learning about Reedy structures, the main example given is that of $\mathbf{\Delta}$. But every simplicial set K has a category of simplices $\mathcal{K} = \int_{\Delta} K$ (with objects all (n, k) where $n \in \mathbb{N}$, $k \in K_n$ and morphisms $(n, k) \to (m, l)$ all $\varphi\colon [n] \to [m]$ in $\mathbf{\Delta}$ such that $k = l \cdot \varphi$) and we can define a Reedy structure on every such $\mathcal{K}$, with $\mathbf{\Delta} \cong \int_{\Delta} \Delta[0]$ being the category of simplices of a point.

To wit, $\mathcal{K} = \int_{\mathbf{\Delta}} K$ is Reedy with degree map $d(n, k) := n$, $\mathcal{K}_+$ consisting of all injections and $\mathcal{K}_-$ of all surjections. As usual, any $\varphi\colon (n, k) \to (m, l)$ in $\mathcal{K}$ factorises (uniquely) into a surjection, followed by an injection

$$\varphi\colon [n] \xrightarrow{\varphi_-} [p] \xrightarrow{\varphi_+} [m], \qquad \text{so that we get} \qquad (n, k) \xrightarrow{\varphi_-} (p, l \cdot \varphi_+) \xrightarrow{\varphi_+} (m, l).$$

(4.2) **Notation.** We often leave out the first component of an object $(n, k) \in \mathcal{K}$ in a category of simplices and view the dimension $\dim k := d(k) := n$ as being an inherent property of the simplex k.

Many important aspects of $\mathbf{\Delta}$ generalise to arbitrary categories of simplices, though there is one important exception.

(4.3) **Observation.** Given any simplicial set K with category of simplices $\mathcal{K} := \int_{\mathbf{\Delta}} K$,

(a) $\mathcal{K}$ is skeletal (i.e. the only only isomorphisms are the identities);

(b) $\mathcal{K}$ has unique (split epi, mono)-factorisations;

(c) every epimorphism in $\mathcal{K}$ is split and these are exactly the surjections;

(d) however, while the monomorphisms are still the injections, not every monomorphism in $\mathcal{K}$ is split; in fact, this is the case if and only if K is discrete.

Proof. Ad (a) & (b): Easy.

Ad (c): Given a surjection $\sigma\colon k \cdot \sigma \to k$ (for k some simplex of K), any retraction, δ of the underlying map σ defines a retraction $\delta\colon k \to k \cdot \sigma$ in $\mathcal{K}$. To see that the epimorphisms are the surjections, assume that we have $\varphi\colon x \cdot \varphi \to x$ with some $i \notin \operatorname{Im} \varphi$. Then, we get a fork

$$k \cdot \varphi \xrightarrow{\varphi} k \underset{\delta^{i+1}}{\overset{\delta^i}{\rightrightarrows}} k \cdot \sigma^i$$

in $\mathcal{K}$ (since σ^i is a common retraction of δ^i and δ^{i+1}), showing that φ is not epi.

Ad (d): The monomorphisms are the injections because given $\varphi\colon k \cdot \varphi \to k$ in $\mathcal{K}$ such that $\varphi(i) = \varphi(j)$ for some i, j, viewing i, j as maps out of $[0]$, we get a cofork

$$k \cdot (\varphi \circ i) = k \cdot (\varphi \circ j) \underset{j}{\overset{i}{\rightrightarrows}} k \cdot \varphi \xrightarrow{\varphi} k$$

in $\mathcal{K}$, which shows that φ is not monic. On the other hand, not every monomorphism needs to be split. For example, if K is not discrete, there is some non-degenerate simplex $k \in K_n$ with $n > 0$ and any injection $\delta\colon [0] \hookrightarrow [n]$ defines a monomorphism $k \cdot \delta \to k$, which cannot have a retraction because k is non-degenerate. $\square$

With the Reedy structure on $\int_{\boldsymbol{\Delta}} K$ defined, the first important observation for our characterisation is the generalisation of the fact [45, Lemma 14.3.7] that every bisimplicial set is Reedy cofibrant. For it, we need the well-known Eilenberg-Zilber lemma for simplicial sets, generalised to diagrams indexed by simplicial sets.

(4.4) **Definition.** Let K be a simplicial set with category of simplices $\mathcal{K} := \int_{\boldsymbol{\Delta}} K$ and $X\colon \mathcal{K}^{\mathrm{op}} \to \mathbf{Sets}$. An element of some X_k is called *degenerate* iff it lies in the image of $\sigma^*\colon X_l \rightarrowtail X_k$ for some proper surjection $\sigma\colon k \twoheadrightarrow l$ (note that σ^* is injective since $\sigma\colon k \twoheadrightarrow l$ is split epi). All other elements are called *non-degenerate*.

More explicitly, given a simplex $k \in K_n$, an element $x \in X_k$ is called *degenerate* iff there is a proper surjection $\sigma\colon [n] \twoheadrightarrow [m]$ in $\boldsymbol{\Delta}$ together with a simplex $l \in K_m$ such that $k = l \cdot \sigma$ (i.e. there is a surjection $\sigma\colon k \twoheadrightarrow l$) and x lies in the image of $\sigma^*\colon X_l \to X_k$.

(4.5) **Example.** In the situation of the definition, if $k \in K_n$ is non-degenerate then every $x \in X_k$ is non-degenerate, since there are no non-trivial l, σ satisfying $k = l \cdot \sigma$.

(4.6) **Example.** If $K = \Delta[0]$ is a point then $\mathcal{K} \cong \boldsymbol{\Delta}$, so that X is an ordinary simplicial set and the above definition is the usual one of (non-)degenerate simplices, since K has exactly one simplex in every dimension.

(4.7) **Proposition. (Eilenberg-Zilber Lemma)** Let K be a simplicial set, $\mathcal{K} := \int_{\boldsymbol{\Delta}} K$ its category of simplices and $X\colon \mathcal{K}^{\mathrm{op}} \to \mathbf{Sets}$. For every simplex $k \in K_n$ and every $x \in X_k$, there is a unique surjection $\eta_x\colon [n] \twoheadrightarrow [m]$, a unique simplex $k_x \in K_m$ and a unique non-degenerate $x^{\downarrow} \in X_{k_x}$ such that

$$k = k_x \cdot \eta_x \text{ (yielding } \eta_x\colon k \twoheadrightarrow k_x) \quad \text{and} \quad x^{\downarrow} \cdot \eta_x := x \text{ (where } \eta_x^*\colon X_{k_x} \to X_k).$$

(4.8) **Remark.** Again, the case of $K = \Delta[0]$ a point is the usual Eilenberg-Zilber lemma and the existence of k_x is trivial since K has exactly one simplex in every dimension. With this example in mind, note that we do not claim that k_x is going to be non-degenerate.

Proof. The existence is by an easy induction on the dimension of k. If $x \in X_k$ then either x is non-degenerate (in which case we have $x^{\downarrow} = x$, $k_x = k$ and $\eta_x = \mathrm{id}$) or it is degenerate, meaning that it lies in the image of some $\sigma^*\colon X_l \to X_k$ with σ surjective and we can apply the inductive hypothesis. For the uniqueness, assume that $k = l \cdot \sigma = l' \cdot \sigma'$ and

$$x = y \cdot \sigma = y' \cdot \sigma'$$

for some $\sigma\colon [n] \twoheadrightarrow [m]$, $\sigma'\colon [n] \twoheadrightarrow [m']$ surjective, $l \in K_m$, $l' \in K_{m'}$ and some $y \in X_l$, $y' \in X_{l'}$ non-degenerate. We pick a section δ of σ and apply it on both sides, yielding

$$l = l' \cdot (\sigma' \circ \delta), \qquad \text{as well as} \qquad y = y' \cdot (\sigma' \circ \delta).$$

Factoring $\sigma' \circ \delta$ into a surjection τ, followed by an injection ε, the non-degeneracy of y tells us that τ is the identity, so that $\sigma' \circ \delta = \varepsilon$ is injective (whence $m \leqslant m'$).

Repeating the argument with the roles of (l, y) and (l', y') reversed and picking a section δ' of σ', we have $l \cdot (\sigma \circ \delta') = l'$, $y \cdot (\sigma \circ \delta') = y'$ and the composite $\sigma \circ \delta'$ is again injective. But then $m = m'$ and $\sigma \circ \delta' = \sigma' \circ \delta$ is the identity, so that $l = l'$, $y = y'$. Since the sections δ, δ' were arbitrary, σ and σ' have the same sections. Put differently, every point in $[m]$ has the same fibres under σ and σ', which means that $\sigma = \sigma'$. □

(4.9) **Remark.** Another way to interpret (or prove) the Eilenberg-Zilber lemma is to say that the category of simplices $\mathcal{K}$ has absolute pushouts of surjections [35].

(4.10) **Proposition.** If K is a simplicial set and $\mathcal{K} := \int_{\mathbf{\Delta}} K$ its category of simplices (equipped with the above Reedy structure) then every diagram $X\colon \mathcal{K}^{\mathrm{op}} \to \mathbf{sSets}$ is Reedy cofibrant, as is every pointed diagram (i.e. one with values in $\mathbf{sSets}_*$).

Proof. Being Reedy cofibrant (in $\mathbf{sSets}$ or $\mathbf{sSets}_*$) means that for every $k \in \mathcal{K}$, the map

$$L_k X = \operatorname*{colim}_{\substack{\sigma\colon k \twoheadrightarrow l \\ \sigma \in \mathcal{K}_-,\, \sigma \neq \mathrm{id}}} X_l \quad \longrightarrow \quad X_k$$

out of the latching space is monic. More precisely, this colimit is taken over the full subcategory of $(\mathcal{K}^{\mathrm{op}})_+ \downarrow k = (\mathcal{K}_-)^{\mathrm{op}} \downarrow k$ where we omit the terminal object $\mathrm{id}\colon k \to k$. We note that this indexing category is connected (the Eilenberg-Zilber map $\eta_k\colon k \twoheadrightarrow k^{\downarrow}$ for the simplex k in K is an initial object) and so the two colimits in $\mathbf{sSets}$ and $\mathbf{sSets}_*$ agree by (2.5.3). Moreover, since, colimits in $\mathbf{sSets}$ are calculated (and monicity checked) pointwise, we can fix a dimension and assume that X is a diagram in $\mathbf{Sets}$.

For two surjections $\sigma\colon k \twoheadrightarrow l$, $\sigma'\colon k \twoheadrightarrow l'$ together with $x \in X_l$, $x' \in X_{l'}$ such that $\sigma^*(x) = \sigma'^*(x') =: y \in X_k$, the uniqueness clause of the Eilenberg-Zilber lemma tells us that

$$k_y = l_x = l'_{x'}, \qquad \eta_y = \eta_x \circ \sigma = \eta_{x'} \circ \sigma' \qquad \text{and} \qquad y^{\downarrow} = x^{\downarrow} = x'^{\downarrow},$$

so that we have a commutative diagram

$$\begin{tikzcd} X_l & X_{k_y} \arrow[l, "\eta_x^*"'] \arrow[r, "\eta_{x'}^*"] \arrow[d, "\eta_y^*"] & X_{l'} \arrow[dl, "\sigma'^*"] \\ & X_k \arrow[from=1-1, "\sigma^*"'] & \end{tikzcd} \qquad \begin{tikzcd} x \arrow[dr, mapsto] & y^{\downarrow} \arrow[l, mapsto] \arrow[r, mapsto] \arrow[d, mapsto] & x' \arrow[dl, mapsto] \\ & y & \end{tikzcd} .$$

But this means that x and x' have the same equivalence class in the colimit and hence, the map $L_k X \to X_k$ is indeed injective. □

As is well-known (see (2.2.7)), the colimit functor is just the weighted colimit with a point-weight (explicitly: $\mathrm{colim}_{\mathcal{I}} X \cong \Delta[0] \odot_{\mathcal{I}} X$, where $\Delta[0]$ is constant) and since we already know that every diagram indexed by a simplicial set is Reedy cofibrant, we only need a Reedy cofibrant replacement of the point to get a homotopy colimit functor. For cosimplicial simplicial sets, such a replacement is given by the Yoneda embedding $\Delta\colon \mathbf{\Delta} \to \mathbf{sSets}$ (see [45, Example 14.3.9]).

In *op.cit.*, the approach taken to show that the Yoneda embedding is Reedy cofibrant is to first show that to be Reedy cofibrant, it is enough for a cosimplicial simplicial set to be *unaugmentable* by showing a cosimplicial analogue of the Eilenberg-Zilber lemma. This approach fails in our general context since not every injection in a category of simplices has a retraction (in contrast to $\mathbf{\Delta}$). However, it is easy enough to show Reedy cofibrancy directly.

(4.11) **Proposition.** If K is a simplicial set and $\mathcal{K} := \int_{\mathbf{\Delta}} K$ then

$$X\colon \mathcal{K} \to \mathbf{\Delta} \hookrightarrow \mathbf{sSets},\ (n,k) \mapsto [n] \mapsto \Delta[n]$$

is a Reedy cofibrant replacement for the constant diagram $\Delta[0]$.

Proof. That X is weakly equivalent to $\Delta[0]$ is obvious since every X_k is contractible. As for Reedy cofibrancy, we need to check that for every simplex $(n,k) \in \mathcal{K}$, the map

$$L_k X = \operatorname*{colim}_{\substack{\delta\colon l \rightarrowtail k \\ \delta \in \mathcal{K}_+,\, \delta \neq \mathrm{id}}} X_l \to X_k$$

is monic. But note that for a non-identity morphism $\delta\colon l \rightarrowtail k$ in $\mathcal{K}_+$, we must have $l = k \cdot \delta$, so that these just correspond to arbitrary injections $\delta\colon [m] \hookrightarrow [n]$ with $m < n$. Moreover, the values of X at k and any $k \cdot \delta$ do not depend on k but only its dimension. All in all, we need to show that

$$\operatorname*{colim}_{\substack{\delta\colon [m]\hookrightarrow[n] \\ \delta \text{ inj}, m<n}} \Delta[m] \to \Delta[n]$$

is monic, which is just the Reedy cofibrancy of the Yoneda embedding. We shall prove this quickly for the sake of completeness. As always, we can check this dimensionwise at some fixed dimension d. Given $\delta\colon [m] \hookrightarrow [n]$ and $\delta'\colon [m'] \hookrightarrow [n]$, if their images are disjoint, no two elements of $X_{k\cdot\delta}$ and $X_{k\cdot\delta'}$ become equivalent in the colimit. Otherwise, we can form their intersection (i.e. their pullback) in $\mathbf{\Delta}$, yielding $\varepsilon\colon [p] \hookrightarrow [m]$, $\varepsilon'\colon [p] \hookrightarrow [m']$. With this, we get a commutative diagram

$$\begin{tikzcd} X_{k\cdot\delta,d} \arrow[dr, bend right, "\delta_*"'] & X_{k\cdot(\delta\cap\delta'),d} \arrow[l, "\varepsilon_*"'] \arrow[r, "\varepsilon'_*"] \arrow[d, "(\delta\cap\delta')_*"'] & X_{k\cdot\delta',d} \arrow[dl, bend left, "\delta'_*"] \\ & X_{k,d} & \end{tikzcd} \quad = \quad \begin{tikzcd} \mathbf{\Delta}(d,m) \arrow[dr, bend right, "\delta_*"'] & \mathbf{\Delta}(d,p) \arrow[l, "\varepsilon_*"'] \arrow[r, "\varepsilon'_*"] \arrow[d, "(\delta\cap\delta')_*"'] & \mathbf{\Delta}(d,m') \arrow[dl, bend left, "\delta'_*"] \\ & \mathbf{\Delta}(d,n) & \end{tikzcd} .$$

Now, if we have some $\varphi\colon [d] \to [m]$ and $\varphi'\colon [d] \to [m']$ such that $\delta \circ \varphi = \delta' \circ \varphi'$, we have an induced map $\psi\colon [d] \to [p]$ into the pullback, meaning that $\varepsilon \circ \psi = \varphi$ and $\varepsilon \circ \psi' = \varphi'$. In particular, φ and φ' represent the same element in the colimit. □

(4.12) **Corollary.** If K is a simplicial set and $\mathcal{K} := \int_{\mathbf{\Delta}} K$ its category of simplices, then, for $\mathcal{M} = \mathbf{sSets}$ or $\mathcal{M} = \mathbf{sSets}_*$, the functor

$$|-|_{\mathcal{K}} : \mathcal{M}^{\mathcal{K}^{\mathrm{op}}} \to \mathcal{M},\ X \mapsto |X|_{\mathcal{K}} := \int^{(n,k)\in\mathcal{K}} \Delta[n] \odot X_{(n,k)}$$

is the homotopy colimit functor (more precisely, it is homotopical and the induced functor between homotopy categories is the left derived functor of $\operatorname{colim}_{\mathcal{K}}$).

Proof. We have already shown that the functor is homotopical. To see that it really is the derived functor of $\operatorname{colim}_{\mathcal{K}}$, we switch to adjoints. Writing $Q\colon \mathcal{K} \to \mathbf{\Delta}$ for the standard projection of the Grothendieck construction and $\Delta\colon \mathbf{\Delta} \hookrightarrow \mathbf{sSets}$ for the Yoneda embedding, we note that the functor from the proposition is just $\Delta \circ Q \odot_{\mathcal{K}} -$, which is left Quillen with respect to the Reedy model structure (since $\Delta \circ Q$ is Reedy cofibrant) and left adjoint to

$$[\Delta \circ Q, -]\colon \mathcal{M} \to \mathcal{M}^{\mathcal{K}^{\mathrm{op}}},\ M \mapsto \Big([(\Delta \circ Q)-, M]\colon (n,k) \mapsto [\Delta[n], M]\Big).$$

This adjunction then induces an adjunction of derived functors. But on fibrant objects M, the functor $[\Delta \circ Q, -]$ (and hence also its right derived functor) is weakly equivalent to the constant diagram functor, since $\Delta[n] \to *$ induces a weak equivalences

$$M \cong [*, M] \xrightarrow{\simeq} [\Delta[n], M].$$

□

(4.13) **Example.** If K is a simplicial set, $\mathcal{K} := \int_{\mathbf{\Delta}} K$ and $X\colon \mathcal{K}^{\mathrm{op}} \to \mathcal{M}$ a constant diagram, then

$$\operatorname*{hocolim}_{\mathcal{K}^{\mathrm{op}}} X \simeq \int^{k\in\mathcal{K}} \Delta[\dim k] \odot X \cong \left(\int^{k\in\mathcal{K}} \Delta[\dim k]\right) \odot X \cong K \odot X$$

by the usual presentation of K as a colimit of representables (sometimes referred to as the *coYoneda lemma*). For $\mathcal{M} = \mathbf{sSets}$ (where the copower $\odot$ is just the usual product) or $\mathcal{M} = \mathbf{sSets}_*$ (where $\odot$ is the half-smash), this means:

$$\operatorname*{hocolim}_{\mathcal{K}^{\mathrm{op}}} \left(\mathcal{K} \xrightarrow{X} \mathbf{sSets}\right) \simeq K \times X \quad \text{and} \quad \operatorname*{hocolim}_{\mathcal{K}^{\mathrm{op}}} \left(\mathcal{K} \xrightarrow{X} \mathbf{sSets}_*\right) \simeq K \ltimes X.$$

Again fixing a simplicial set K with category of simplices $\mathcal{K} := \int_{\mathbf{\Delta}} K$, every diagram $X\colon \mathcal{K}^{\mathrm{op}} \to \mathbf{sSets}$ comes with a unique natural transformation to the constant diagram $*$. By the coYoneda lemma then, we obtain a morphism

$$\operatorname*{hocolim}_{\mathcal{K}^{\mathrm{op}}} X \simeq \int^{k\in\mathcal{K}} \Delta[\dim k] \times X_k \longrightarrow \int^{k\in\mathcal{K}} \Delta[\dim k] \cong K.$$

If X is even a pointed diagram, then the coherent base-points induce a natural transformation $* \Rightarrow X$, which, upon applying $\operatorname{hocolim}_{\mathcal{K}^{\mathrm{op}}}$, gives us a section to the above map.

Alternatively, instead of mapping a diagram $X\colon \mathcal{K}^{\mathrm{op}} \to \mathbf{sSets}$ to a point, we can also map the other component $\Delta[-]$ in the coend to a point, which induces a natural map

$$\operatorname*{hocolim}_{\mathcal{K}^{\mathrm{op}}} X \simeq \int^{k\in\mathcal{K}} \Delta[\dim k] \times X_k \to \int^{k\in\mathcal{K}} * \times X_k \cong \operatorname*{colim}_{\mathcal{K}^{\mathrm{op}}} X.$$

So now we have two canonical morphisms

$$\operatorname*{hocolim}_{\mathcal{K}^{\mathrm{op}}} X \to K, \qquad \operatorname*{hocolim}_{\mathcal{K}^{\mathrm{op}}} X \to \operatorname*{colim}_{\mathcal{K}^{\mathrm{op}}} X$$

coming from two different projection maps and we know that for a contractible diagram X, the former is a weak equivalence but clearly, the latter need not be. For example, taking $X = *$ to be constantly a point, the colimit is discrete but the homotopy colimits is K itself.

Let us quickly investigate the relation between pointed and unpointed homotopy colimits. Assuming X is a pointed diagram, every half-smash $\Delta[\dim k] \ltimes X_k$ (which is pointed) is a quotient of $\Delta[\dim k] \times X_k$. More explicitly, it is the pushout of the span

$$\Delta[\dim k] \times X_k \hookleftarrow \Delta[\dim k] \to *.$$

Here, the map to the left is the inclusion above the base point, which is split monic with the standard projection as a retraction. Taking the coend over $\mathcal{K}$ and using that coends commute with pushouts, we obtain the following.

(4.14) **Corollary.** Given a simplicial set K with $\mathcal{K} := \int_{\mathbf{\Delta}} K$ and a pointed diagram $X\colon \mathcal{K}^{\mathrm{op}} \to \mathbf{sSets}_*$, the base point inclusion $* \Rightarrow X$ induces a cofibration sequence

$$K \hookrightarrow \operatorname*{hocolim}_{\mathcal{K}^{\mathrm{op}}} \left(\mathcal{K}^{\mathrm{op}} \xrightarrow{X} \mathbf{sSets}\right) \longrightarrow \operatorname*{hocolim}_{\mathcal{K}^{\mathrm{op}}}{}_* \left(\mathcal{K}^{\mathrm{op}} \xrightarrow{X} \mathbf{sSets}_*\right).$$

In particular, pointed and unpointed homotopy colimits over a contractible simplicial set agree.

Proof. Applying the coend over $\mathcal{K}$ to the pushout of diagrams indicated above and commuting the two gives the required result. The only necessary observations are that $\int^k \Delta[\dim k] \cong K$ (again the coYoneda lemma), $\int^k * \cong *$ and that the resulting morphism of unpointed homotopy colimits $K \to \operatorname{hocolim}_{\mathcal{K}^{\mathrm{op}}} X$ is still split monic. In particular, the resulting pushout is a homotopy pushout. □

Note that (4.12) above is still not the usual formula [11, Definition 3.9] for a homotopy colimit indexed by a category of simplices. For it, we need an easy observation about functor tensor products over Grothendieck constructions. There is surely a more general statement hidden behind it but the following will suffice.

(4.15) **Lemma.** Let $F\colon \mathcal{I}^{\mathrm{op}} \to \mathbf{Sets}$ be any diagram and $\mathcal{F} := \int_{\mathcal{I}} F$ its category of elements, which comes with a standard projection $Q\colon \mathcal{F} \to \mathcal{I}$. Furthermore, let $\mathcal{M}$ be any simplicially enriched cocomplete category that is powered and copowered. Then any diagram $Y\colon \mathcal{F}^{\mathrm{op}} \to \mathcal{M}$ defines a new diagram $Y'\colon \mathcal{I}^{\mathrm{op}} \to \mathcal{M}$, $I \mapsto \coprod_{x\in FI} Y_{(I,x)}$ and for every $X\colon \mathcal{I} \to \mathbf{sSets}$, there is a natural isomorphism

$$(X \circ Q) \odot_{\mathcal{F}} Y \cong X \odot_{\mathcal{I}} Y'.$$

Proof. Both sides of the claimed isomorphism are special kinds of coends. The diagonal elements for the diagram determining the coend on the left-hand side are

$$(X \circ Q)_{(I,x)} \odot Y_{(I,x)} = X_I \odot Y_{(I,x)},$$

while on the right-hand side, they are

$$X_I \odot Y'_I = X_I \odot \coprod_{x\in FI} Y_{(I,x)} \cong \coprod_{x\in FI} X_I \odot Y_{(I,x)}.$$

It is now straightforward to construct morphisms in both directions and check the necessary compatibilities. □

(4.16) **Proposition.** If K is a simplicial set and $\mathcal{K} := \int_{\mathbf{\Delta}} K$ its category of simplices, then, for $\mathcal{M} = \mathbf{sSets}$ or $\mathcal{M} = \mathbf{sSets}_*$, we have a functor

$$\mathcal{M}^{\mathcal{K}^{\mathrm{op}}} \to \mathcal{M}^{\mathbf{\Delta}^{\mathrm{op}}},\ X \mapsto X' \qquad \text{given by } X'_n := \coprod_{k\in K_n} X_k$$

and with this, the composite

$$\mathcal{M}^{\mathcal{K}^{\mathrm{op}}} \xrightarrow{-'} \mathcal{M}^{\mathbf{\Delta}^{\mathrm{op}}} \xrightarrow{|-|} \mathcal{M},\ X \mapsto |X'| \cong \int^n \coprod_{k\in K_n} (\Delta[n] \odot X_k)$$

is the homotopy colimit functor. □

(4.17) **Example.** For $\mathcal{M} = \mathbf{sSets}$, the copower $\odot$ is just the usual product of simplicial sets, so that in this case, we arrive at the usual formula for the homotopy colimit of a diagram $X\colon \mathcal{K}^{\mathrm{op}} \to \mathbf{sSets}$, indexed by a simplicial set:

$$\operatorname*{hocolim}_{\mathcal{K}^{\mathrm{op}}} X \simeq \left(\coprod_{\substack{n\in\mathbb{N},\\ k\in K_n}} \Delta[n] \times X_k \right) \Big/ \sim\ ,$$

where $\sim$ is the equivalence generated by

$$(\varphi \circ \xi, y) \sim (\xi, y \cdot \varphi) \qquad \text{for } \varphi\colon [m] \to [n],\ \xi \in \Delta[m],\ y \in X_k.$$

(4.18) **Example.** For $\mathcal{M} = \mathbf{sSets}_*$, the coproduct is the wedge, while the copower $\odot$ is the half-smash product:

$$K \odot L = K \ltimes L := (K \times L)/(K + *),$$

for K a simplicial set and L a pointed simplicial set (with base point $*$). With this, we obtain the usual formula for the pointed homotopy colimit of a diagram $X\colon \mathcal{K}^{\mathrm{op}} \to \mathbf{sSets}_*$, indexed by a simplicial set:

$$\operatorname*{hocolim}_{\mathcal{K}^{\mathrm{op}}} X \simeq \left(\bigvee_{\substack{n\in\mathbb{N},\\ k\in K_n}} \Delta[n] \ltimes X_k \right) \Big/ \sim\, ,$$

where $\sim$ is the equivalence generated by

$$[\varphi \circ \xi, y] \sim [\xi, y \cdot \varphi] \qquad \text{for } \varphi\colon [m] \to [n],\ \xi \in \Delta[m],\ y \in X_k.$$

Finally, we need to make a small comment about why every homotopy colimit (not necessarily indexed by a simplicial set) can be reduced to the above case. Given an arbitrary small category $\mathcal{I}$, an obvious candidate for a simplicial set that could replace it as an indexing category is its nerve $N(\mathcal{I})$. It's category of elements $\mathcal{N}(\mathcal{I})$ has as objects all pairs

$$\Big([n], I_\bullet = \Big(I_0 \xrightarrow{i_1} \dots \xrightarrow{i_n} I_n\Big)\Big) \qquad \text{with} \qquad I_\bullet\colon [n] \to \mathcal{I}$$

and as morphisms $([m], I_\bullet) \to ([n], J_\bullet)$ all $\varphi\colon [m] \to [n]$ in $\mathbf{\Delta}$ such that $\varphi^* J_\bullet = J_\bullet \circ \varphi = I_\bullet$; so $I_\bullet$ is obtained from $J_\bullet$ by inserting identities, taking compositions and cutting off at the beginning or the end. For readability's sake, we usually leave out the objects' first components and just identify them with $I_\bullet$.

Importantly for us, the category $\mathcal{N}(\mathcal{I})$ comes with two obvious functors relating it to $\mathcal{I}$, namely the target and source functors:

$$T\colon \mathcal{N}(\mathcal{I}) \to \mathcal{I},\ \Big(I_0 \xrightarrow{i_1} \dots \xrightarrow{i_m} I_m\Big) \mapsto I_m,\ \sigma^j \mapsto \mathrm{id},\ \Big(d_i I_\bullet \xrightarrow{\delta^i} I_\bullet\Big) \mapsto \begin{cases} \mathrm{id}_{I_m} & i < m \\ i_m & i = m \end{cases}$$

(put differently, $T\varphi$ is $I_m = J_{\varphi m} \to J_n$ for $\varphi\colon ([m], I_\bullet) \to ([n], J_\bullet)$),

$$S\colon \mathcal{N}(\mathcal{I})^{\mathrm{op}} \to \mathcal{I},\ \Big(I_0 \xrightarrow{i_1} \dots \xrightarrow{i_m} I_m\Big) \mapsto I_0,\ \sigma^j \mapsto \mathrm{id},\ \Big(d_i I_\bullet \xrightarrow{\delta^i} I_\bullet\Big) \mapsto \begin{cases} \mathrm{id}_{I_0} & i > 0 \\ i_0 & i = 0 \end{cases}$$

(again put differently, $S\varphi$ is $J_0 \to J_{\varphi 0} = I_0$ for $\varphi\colon ([m], I_\bullet) \to ([n], J_\bullet)$ in $\mathcal{N}(\mathcal{I})$). One easily checks that these two functors are strictly natural in $\mathcal{I}$, meaning that they define (strict) natural transformations $T\colon \mathcal{N} \Rightarrow \mathrm{id}_{\mathbf{Cat}}$, $S\colon \mathcal{N}^{\mathrm{op}} \Rightarrow \mathrm{id}_{\mathbf{Cat}}$ (where $\mathcal{N}\colon \mathbf{Cat} \to \mathbf{Cat}, \mathcal{I} \mapsto \mathcal{N}(\mathcal{I})$).

(4.19) **Proposition.** The functor $T\colon \mathcal{N}(\mathcal{I}) \to \mathcal{I}$ is final.

Proof. Take $I \in \mathcal{I}$ and let's check that $I{\downarrow}T$ is (non-empty and) connected. Clearly, the object $I = ([0], I) \in \mathcal{N}(\mathcal{I})$ will play an important role and indeed, for any other $(J_\bullet\colon [n] \to \mathcal{I}, b)$, we have the following diagram in $I \downarrow T$

$$\begin{tikzcd}
 & & I \arrow[dll, "\mathrm{id}"'] \arrow[dl, "b"] \arrow[d, "b"'] \arrow[dr, "b"] & \\
I \arrow[r, "b"'] & J_n & J_n \arrow[l, "\mathrm{id}"] \arrow[r, "\mathrm{id}"'] & J_n \\
I \arrow[r, "\delta^1"'] & \Big(I \xrightarrow{b} J_n\Big) & J_n \arrow[l, "\delta^0"] \arrow[r, "(\delta^0)^n"'] & J_\bullet\ .
\end{tikzcd}$$

$\square$

For the source functor, things are even better. Not only is it final (i.e. all $I \downarrow S$ are [non-empty and] connected) but even homotopy final (i.e. all $I \downarrow S$ are contractible).

(4.20) **Definition.** A functor $F\colon \mathcal{I} \to \mathcal{J}$ between indexing categories is *homotopy final* iff $J \downarrow F$ is contractible for every $J \in \mathcal{J}$. This implies that we have a natural weak equivalence $\operatorname{hocolim}_{\mathcal{I}} \circ F^* \simeq \operatorname{hocolim}_{\mathcal{J}}$ as can be shown in a very general context (such as in any derivator [32]). In fact, this consequence is equivalent to the original condition (as shown in (4.25) below). Dually for *homotopy initial functors*.

(4.21) **Proposition.** The functor $S\colon \mathcal{N}(\mathcal{I})^{\mathrm{op}} \to \mathcal{I}$ is a lax Grothendieck fibration, meaning that for every $I \in \mathcal{I}$ the inclusion functor

$$S^{-1}I \hookrightarrow I \downarrow S,\ J_\bullet \mapsto (J_\bullet, \mathrm{id}_I),\ \varphi \mapsto \varphi$$

has a right adjoint.

Proof. Given $I \in \mathcal{I}$, we map an object $\big([n], J_\bullet, a\colon I \to J_0\big)$ in $I \downarrow S$ to $\big([n+1], I \xrightarrow{a} J_\bullet\big)$ and a morphism $\varphi^{\mathrm{op}}\colon (J_\bullet, a) \to (K_\bullet, b)$ to $(1+\varphi)^{\mathrm{op}}$, where $(1+\varphi)$ is just φ shifted up by one. More formally, $(1+\varphi)0 := 0$ and $(1+\varphi)i := 1 + \varphi(i-1)$ for $i > 0$. The unit of the adjunction (which is a morphism in $\mathcal{N}(\mathcal{I})^{\mathrm{op}}$) is just

$$\eta_{[n],J_\bullet} := (\sigma^0)^{\mathrm{op}}\colon \big([n], J_\bullet\big) \to \big([n+1], I \xrightarrow{\mathrm{id}_I} J_\bullet\big),$$

while the counit (which is again a morphism in $\mathcal{N}(\mathcal{I})^{\mathrm{op}}$) is

$$\varepsilon_{[n],J_\bullet,a} := (\delta^0)^{\mathrm{op}}\colon \big([n+1], I \xrightarrow{a} J_\bullet, \mathrm{id}_I\big) \to \big([n], J_\bullet, a\big).$$

Observing that $1 + \delta^0 = \delta^1$, the triangle identities are easily verified. □

(4.22) **Corollary.** The functor $S\colon \mathcal{N}(\mathcal{I})^{\mathrm{op}} \to \mathcal{I}$ is homotopy final.

Proof. For $I \in \mathcal{I}$, the strict fibre $S^{-1}I$ has $\big([0], I\big)$ as a terminal object. □

With this corollary, combined with the coend formula (4.12) for homotopy colimits, we can prove quite a few things such as the following well-known result.

(4.23) **Proposition.** If $\mathcal{I}$ is an (as always small) indexing category, then $N(\mathcal{I}) \simeq \operatorname{hocolim}_{\mathcal{I}} *$.

Proof. This is just a combination of the last corollary and the coend formula:

$$N(\mathcal{I}) \xleftarrow{\simeq} \operatorname*{hocolim}_{\mathcal{N}(\mathcal{I})^{\mathrm{op}}} * = \operatorname*{hocolim}_{\mathcal{N}(\mathcal{I})^{\mathrm{op}}} (S \circ *) \xrightarrow{\sim} \operatorname*{hocolim}_{\mathcal{I}} *.$$

□

Combining this with the fact (5.2.7) that filtered colimits in **sSets** are homotopy colimits, we get the following result for free, which is usually proved using Kan's Ex^∞-functor.

(4.24) **Corollary.** Filtered categories are contractible.

Proof. For $\mathcal{I}$ any filtered category, we have

$$* \cong \operatorname*{colim}_{\mathcal{I}} * \simeq \operatorname*{hocolim}_{\mathcal{I}} * \simeq N(\mathcal{I}).$$

□

In (2.4.10), we have seen that it suffices to check the finality of a functor against the base category **Sets** and one consequence of the last proposition is the following analogue for homotopy finality.

(4.25) **Corollary.** A functor $F\colon \mathcal{I} \to \mathcal{J}$ is homotopy final iff for every $X\colon \mathcal{J} \to \mathbf{sSets}$, the canonical arrow $\operatorname{hocolim}_{\mathcal{I}}(X \circ F) \to \operatorname{hocolim}_{\mathcal{J}} X$ is a weak equivalence.

Proof. The direction "⇒" was already mentioned above and is shown very generally for a derivator in [32]. The more modest analogue for model categories (which suffices) can for example be found in [45]. As for the other direction, we show the dual claim, for which we consider the derived adjunctions of

$$\mathbf{sSets}^{\mathcal{J}} \underset{F_!}{\overset{F^*}{\underset{\longleftarrow}{\overset{\longrightarrow}{\top}}}} \mathbf{sSets}^{\mathcal{I}} \underset{\mathrm{Const}_{\mathcal{I}}}{\overset{\lim_{\mathcal{I}}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\top}}}} \mathbf{sSets}.$$

By hypothesis, the right derived functors compose, implying that so do the left derived ones, which is to say that $\operatorname{Ho}\mathrm{Const}_{\mathcal{J}} \simeq \mathbb{L}F_! \circ \operatorname{Ho}\mathrm{Const}_{\mathcal{I}}$. For the terminal simplicial set $*$ and any $J \in \mathcal{J}$, this means that

$$* = (\mathrm{Const}_{\mathcal{J}} *)_J \simeq \big(\mathbb{L}F_!(\mathrm{Const}_{\mathcal{I}} *)\big)_J \simeq \operatorname{hocolim}\Big(F \downarrow J \to \mathcal{I} \xrightarrow{*} \mathbf{sSets}\Big) \simeq N(F \downarrow J)$$

and so F is homotopy initial. □

(4.26) **Example.** With this corollary, we will see later in (8.4), that every acyclic Kan fibration $p\colon E \to B$ gives rise to a homotopy final functor $p\colon \mathcal{E}^{\mathrm{op}} \to \mathcal{B}^{\mathrm{op}}$ between the opposite categories of simplices.

5. Comparison Maps and Thomason's Theorem

The term *comparison map* is used for different concepts in different situations (for another usage of the term, see section 10.2). In this section, we mean the map between homotopy colimits, induced by a change of indexing categories. Explicitly, if $F\colon \mathcal{I} \to \mathcal{J}$ is any functor between small categories and $X\colon \mathcal{J} \to \mathbf{sSets}$ (or any [cofibrantly generated] model category), we have a comparison map

$$\operatorname*{hocolim}_{\mathcal{I}} F^*X \to \operatorname*{hocolim}_{\mathcal{J}} X \qquad (\text{where } F^*\colon \mathbf{sSets}^{\mathcal{J}} \to \mathbf{sSets}^{\mathcal{I}},\ X \mapsto X \circ F).$$

For strict colimits (rather than homotopy colimits) the map

$$\operatorname*{colim}_{\mathcal{I}} F^*X \xrightarrow{[\mathrm{in}_{FI}]_I} \operatorname*{colim}_{\mathcal{J}} X$$

is just induced by the colimiting cocone $(F^*X)_I = X_{FI} \xrightarrow{\mathrm{in}_{FI}} \operatorname{colim}_{\mathcal{J}} X$ and in the model categorical setting, we can simply combine this with cofibrant replacements in the respective projective model structures. Explicitly, since F^* is homotopical, we can assume X to be cofibrant. Next, we replace F^*X cofibrantly by $Q(F^*X) \to F^*X$ and obtain

$$\operatorname*{hocolim}_{\mathcal{I}} F^*X = \operatorname*{colim}_{\mathcal{I}} Q(F^*X) \to \operatorname*{colim}_{\mathcal{I}} F^*X \to \operatorname*{colim}_{\mathcal{J}} X = \operatorname*{hocolim}_{\mathcal{J}} X.$$

This approach has two downsides. Firstly, one needs to keep track of the replacements and secondly, we needed X to be cofibrant. Of course, we could first cofibrantly replace it but then we would get a zig-zag, rather than a direct map.

Things look a little better with our explicit model for homotopy colimits. Its downside is that it only works for diagrams indexed by (dual categories of simplices of) spaces. As already mentioned, the general situation can be reduced to this one by precomposing with the source functor $S\colon \mathcal{N}(\mathcal{I})^{\mathrm{op}} \to \mathcal{I}$ (for $\mathcal{I}$ any small category). Explicitly, a map $f\colon K \to L$ of simplicial sets yields a functor between dual categories of simplices $f\colon \mathcal{K}^{\mathrm{op}} \to \mathcal{L}^{\mathrm{op}}$, which in turn gives rise to a map

$$|f^*X|_{\mathcal{K}} = \int^{k\in\mathcal{K}} \Delta[\dim k] \times X_{f(k)} \xrightarrow{[\mathrm{in}_{f(k)}]_k} \int^{l\in\mathcal{L}} \Delta[\dim l] \times X_l = |X|_{\mathcal{L}}\,,$$

induced by all

$$\Delta[\dim k] \times X_{f(k)} = \Delta[\dim f(k)] \xrightarrow{\mathrm{in}_{f(k)}} |X|_{\mathcal{L}}\,.$$

This is a direct map and we don't get any zig-zags because there is no cofibrancy condition involved. We provide yet another (more derivatoresque) way of constructing comparison maps, which, as a natural generalisation, gives us Thomason's theorem.

For this, we need the fact [31, Proposition 1.30] that every combinatorial model category $\mathcal{M}$ (and in particular **sSets**) defines a derivator

$$\mathbb{D}_{\mathcal{M}}\colon \mathcal{I} \mapsto \mathrm{Ho}(\mathcal{M}^{\mathcal{I}}).$$

More generally, this works for any model category as shown by Cisinski [17] and implicitly by Chachólski and Scherer [15]. However, the proof for combinatorial model categories is somewhat easier because one has the injective and projective model structures available. As another result about derivators, we need [31, Proposition 1.24].

(5.1) **Theorem. (Groth)** Every pullback square of small categories

$$\begin{tikzcd} \mathcal{I} \arrow[r, "H"] \arrow[d, "G"'] \arrow[dr, phantom, "\lrcorner \;\; \swArrow_{\mathrm{id}}"] & \mathcal{E} \arrow[d, "P"] \\ \mathcal{J} \arrow[r, "K"'] & \mathcal{B}. \end{tikzcd}$$

with P a Grothendieck opfibration (i.e. arising as the projection of a [covariant] Grothendieck construction) is homotopy exact.

Expanding the definition of a homotopy exact square and applying it to the derviator $\mathbb{D}_{\mathbf{sSets}}$, this means that the square on the left below satisfies the Beck-Chevalley condition (see (1.5.1)); or in other words, that, upon taking mates, we have an isomorphism of functors as in the right-hand suare:

$$\begin{tikzcd} \mathrm{Ho}(\mathbf{sSets}^{\mathcal{E}}) \arrow[d, "\mathrm{Ho}\,H^*"'] \arrow[dr, phantom, "\searrow \mathrm{id}"] & \mathrm{Ho}(\mathbf{sSets}^{\mathcal{B}}) \arrow[l, "\mathrm{Ho}\,P^*"'] \arrow[d, "\mathrm{Ho}\,K^*"] \\ \mathrm{Ho}(\mathbf{sSets}^{\mathcal{I}}) & \mathrm{Ho}(\mathbf{sSets}^{\mathcal{J}}) \arrow[l, "\mathrm{Ho}\,G^*"] \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Ho}(\mathbf{sSets}^{\mathcal{E}}) \arrow[r, "\mathbb{L}P_!"] \arrow[d, "\mathrm{Ho}\,H^*"'] \arrow[dr, phantom, "\nearrow \cong"] & \mathrm{Ho}(\mathbf{sSets}^{\mathcal{B}}) \arrow[d, "\mathrm{Ho}\,K^*"] \\ \mathrm{Ho}(\mathbf{sSets}^{\mathcal{I}}) \arrow[r, "\mathbb{L}G_!"'] & \mathrm{Ho}(\mathbf{sSets}^{\mathcal{J}}). \end{tikzcd}$$

Back to the problem of camparison maps, any functor $F\colon \mathcal{I} \to \mathcal{J}$ can be viewed as a functor $F\colon [1] \to \mathbf{Cat}$ and we can take its Grothendieck construction (also called its *cograph* or *collage*) $\mathrm{Coll}\,F := \int^{[1]} F$, which comes with a Grothendieck opfibration; namely the projection $P\colon\ \mathrm{Coll}\,F \to [1]$. But not only that, it also comes with a target functor

$$T\colon\ \mathrm{Coll}\,F \to \mathcal{J},$$

which is the identity on $\mathcal{J}$ and maps $\mathcal{I}$ to $\mathcal{J}$ via F. Now the comparison map (or rather its lift to **sSets**) $\operatorname{hocolim}_{\mathcal{I}} F^*X \to \operatorname{hocolim}_{\mathcal{J}} X$ for $X\colon \mathcal{J} \to \mathbf{sSets}$ is just the composite

$$\mathrm{Ho}(\mathbf{sSets}^{\mathcal{J}}) \xrightarrow{\mathrm{Ho}\,T^*} \mathrm{Ho}(\mathbf{sSets}^{\mathrm{Coll}\,F}) \xrightarrow{\mathbb{L}P_!} \mathrm{Ho}(\mathbf{sSets}^{[1]})$$

evaluated at X. To see this, we note that Groth's theorem gives us an isomorphism

$$\begin{tikzcd}
\mathrm{Ho}(\mathbf{sSets}^{\mathrm{Coll}\,F}) \arrow[r, "\mathbb{L}P_!"] \arrow[d, "\mathrm{Ho}\,\mathrm{Res}_{\mathcal{I}}"'] \arrow[dr, phantom, "\cong"] & \mathrm{Ho}(\mathbf{sSets}^{[1]}) \arrow[d, "\mathrm{Ho}\,\mathrm{ev}_0"] \\
\mathrm{Ho}(\mathbf{sSets}^{\mathcal{I}}) \arrow[r, "\operatorname{hocolim}_{\mathcal{I}}"'] & \mathrm{Ho}(\mathbf{sSets})
\end{tikzcd}$$

(where $\mathrm{Res}_{\mathcal{I}}$ is the restriction to $\mathcal{I} \subseteq \mathrm{Coll}\,F$ and ev_0 is evaluation at $0 \in [1]$). Using that $F^* = \mathrm{Res}_{\mathcal{I}} \circ T^*$, we have, for every $X\colon \mathcal{J} \to \mathbf{sSets}$, that

$$\operatorname*{hocolim}_{\mathcal{I}} F^*X = (\operatorname*{hocolim}_{\mathcal{I}} \circ \mathrm{Ho}\,\mathrm{Res}_{\mathcal{I}} \circ \mathrm{Ho}\,T^*)X \cong (\mathrm{Ho}\,\mathrm{ev}_0 \circ \mathbb{L}P_! \circ T^*)X,$$

which is the domain of the comparison map constructed above. Similarly for $\mathcal{J}$ instead of $\mathcal{I}$, which gives us the correct codomain.

Now the attentive reader will have noticed that in the above argument, we can replace $\mathrm{Coll}\,F$ by an arbitrary Grothendieck construction, which gives us Thomason's theorem. In this generality, the theorem is really due to Chachólski and Scherer [15], who proved it for an arbitrary model category (in fact, even a little more generally) using their theory of (covariant!) bounded diagrams indexed by spaces (cf. the next section).

(5.2) **Theorem. (Thomason's Theorem)** Given a functor $F\colon \mathcal{I} \to \mathbf{Cat}$ and a diagram $X\colon \int^{\mathcal{I}} F \to \mathbf{sSets}$, there is a natural weak equivalence

$$\operatorname*{hocolim}_{\int^{\mathcal{I}} F} X \simeq \operatorname*{hocolim}_{I \in \mathcal{I}} \operatorname*{hocolim}_{A \in FI} X(I, A).$$

Proof. Writing $P\colon \int^{\mathcal{I}} F \to \mathcal{I}$ for the projection. As always with left Kan extensions (just looking at the right adjoints $P^* \circ \mathrm{Const}_{\int^{\mathcal{I}} F} = \mathrm{Const}_{\mathcal{I}}$), we have a natural isomorphism

$$\operatorname*{hocolim}_{\int^{\mathcal{I}} F} X \simeq \operatorname*{hocolim}_{\mathcal{I}} \mathbb{L}P_! X$$

and all we need to understand $\mathbb{L}P_!X$. Since weak equivalences in diagram categories are pointwise, we can fix $I \in \mathcal{I}$ and proceed as above, so that Groth's theorem applied to the pullback square on the left below gives us an isomorphism as in the right-hand square:

$$\begin{tikzcd}
FI \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner\ \Swarrow \mathrm{id}"] & \int^{\mathcal{I}} F \arrow[d, "P"] \\
\{I\} \arrow[r, hook] & \mathcal{I},
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{Ho}(\mathbf{sSets}^{\int^{\mathcal{I}} F}) \arrow[r, "\mathbb{L}P_!"] \arrow[d, "\mathrm{Ho}\,\mathrm{Res}_{FI}"'] \arrow[dr, phantom, "\cong"] & \mathrm{Ho}(\mathbf{sSets}^{\mathcal{I}}) \arrow[d, "\mathrm{Ho}\,\mathrm{ev}_I"] \\
\mathrm{Ho}(\mathbf{sSets}^{FI}) \arrow[r, "\operatorname{hocolim}_{FI}"'] & \mathrm{Ho}(\mathbf{sSets}).
\end{tikzcd}$$

With this, $\mathrm{Ho}\,\mathrm{ev}_I \circ \mathbb{L}P_! \simeq \operatorname{hocolim}_{FI} X|_{FI}$ and the claim follows. □

6. Bounded Diagrams

As seen at the end of the previous section, when calculating homotopy colimits, we can always assume our indexing category to be of the form $(\int_{\mathbf{\Delta}} K)^{\mathrm{op}}$ for some simplicial set K because, for an arbitrary indexing category $\mathcal{I}$, we can always precompose a diagram indexed by $\mathcal{I}$ with the source functor $S\colon \mathcal{N}(\mathcal{I})^{\mathrm{op}} = (\int_{\mathbf{\Delta}} N(\mathcal{I}))^{\mathrm{op}} \to \mathcal{I}$, which is homotopy final.

In contrast to this, the target functor $T\colon \mathcal{N}(\mathcal{I}) \to \mathcal{I}$ is only final but not necessarily homotopy final, which is sometimes very inconvenient since a category of simplices $\int_{\mathbf{\Delta}} K$ is often much better behaved than its dual with respect to homotopy colimits. For example, homotopy colimits over $\int_{\mathbf{\Delta}} \Delta[n]$ are very easy to calculate since the indexing category has a terminal object, while the dual case is more complicated. Also, as seen in (2.20), if K is ∂-non-singular (e.g. a simplicial complex) and $\mathcal{K} := \int_{\mathbf{\Delta}} K$, the inclusion

$$\mathcal{K}^* \hookrightarrow \mathcal{K}$$

of the non-degenerate simplices has a left adjoint. In particular, it is homotopy final and it suffices to consider the much smaller (and easier to understand) category $\mathcal{K}^*$ for the calculation of homotopy colimits. Unfortunately, when taking duals, the inclusion $(\mathcal{K}^*)^{\mathrm{op}} \hookrightarrow \mathcal{K}^{\mathrm{op}}$ now has a right adjoint instead of a left one.

In this section, we are going to consider *bounded diagrams* indexed by a dual category of simplices $\mathcal{K}^{\mathrm{op}}$. In contrast to this, [15] develops the theory of bounded diagrams indexed by a category of simplices (rather than its dual) and shows how it can be used to construct homotopy colimits in a general model category (and even a bit more general). More precisely, it is shown in *op. cit.*, that for every simplicial set K with $\mathcal{K} := \int_{\mathbf{\Delta}} K$, and every model category $\mathcal{M}$, the full subcategory

$$\mathrm{Fun}^b(\mathcal{K}, \mathcal{M}) \subseteq \mathcal{M}^{\mathcal{K}}$$

of bounded diagrams has a (projective) model structure with pointwise weak equivalences and fibrations. With this, $\mathrm{colim}_{\mathcal{K}}\colon \mathrm{Fun}^b(\mathcal{K}, \mathcal{M}) \to \mathcal{M}$ is left Quillen and thus has a total left derived functor $\mathbb{L}^b \mathrm{colim}$ (which is generally *not* the same as hocolim for $\mathcal{M}^{\mathcal{K}}$). Since, for every indexing category $\mathcal{I}$, the target functor $T\colon \mathcal{N}(\mathcal{I}) \to \mathcal{I}$ is final, the composite

$$\mathcal{M}^{\mathcal{I}} \xrightarrow{T^*} \mathrm{Fun}^b(\mathcal{N}(\mathcal{I}), \mathcal{M}) \xrightarrow{\mathrm{colim}} \mathcal{M}$$

is just $\mathrm{colim}_{\mathcal{I}}$ and they go on to show that $\mathbb{L}^b \mathrm{colim} \circ \mathrm{Ho}\, T^*$ is $\mathrm{hocolim}_{\mathcal{I}}$ (i.e. the total left derived functor of $\mathrm{colim}_{\mathcal{I}}$). So in a sense, the target functor is homotopy final for bounded diagrams. Let us now return to the dual situation.

(6.1) **Definition.** For K a simplicial set with $\mathcal{K} := \int_{\mathbf{\Delta}} K$, a diagram $X\colon \mathcal{K}^{\mathrm{op}} \to \mathbf{sSets}$ is *bounded* iff every surjection $\sigma\colon k \cdot \sigma \to k$ in $\mathcal{K}$ is mapped to an isomorphism $X_\sigma\colon X_k \cong X_{k\cdot\sigma}$.

(6.2) **Remark.** Using the Eilenberg-Zilber lemma, one can show [15, Proposition 10.3] that every bounded diagram is weakly equivalent to a *strongly bounded* one, meaning that it maps surjections to identities. However, we will not need this result here.

(6.3) **Example.** If $K = \Delta[n]$, then the unique non-degenerate simplex $\iota_n \in \Delta[n]_n$ is initial in $\mathbf{\Delta}[n]^{\mathrm{op}}$ (where $\mathbf{\Delta}[n] := \int_{\mathbf{\Delta}} \Delta[n]$) and so, every diagram $X\colon \mathbf{\Delta}[n]^{\mathrm{op}} \to \mathbf{sSets}$ comes with a transformation form the constant diagram $X_{\iota_n} \Rightarrow X$, which is the identity at ι_n. Now, if $n = 0$ (where $\mathbf{\Delta}[0] = \mathbf{\Delta}$) and X is bounded, then (by definition), this is a natural weak equivalence and we conclude that

$$X_{\iota_n} \cong \Delta[0] \times X_{\iota_n} \simeq \operatorname*{hocolim}_{\mathbf{\Delta}[0]^{\mathrm{op}}} X_{\iota_n} \simeq \operatorname*{hocolim}_{\mathbf{\Delta}[0]^{\mathrm{op}}} X \qquad \text{for } X \text{ bounded.}$$

(6.4) **Proposition.** For K a simplicial set with $\mathcal{K} := \int_{\mathbf{\Delta}} K$, a diagram $X\colon \mathcal{K}^{\mathrm{op}} \to \mathbf{sSets}$ is bounded iff it sends all Eilenberg-Zilber maps $\eta_k\colon k \to k^{\downarrow}$ with $k \in K$ to isomorphisms.

Proof. The direction "$\Rightarrow$" is obvious (the Eilenberg-Zilber maps are surjections) and for the converse, we just note that if $\sigma\colon k\cdot\sigma \to k$ is a surjection, then $\eta_k \circ \sigma = \eta_{k\cdot\sigma}$ by the uniqueness of the Eilenberg-Zilber map. □

Now, for K a simplicial set and $\mathcal{K} := \int_{\mathbf{\Delta}} K$, we write $\mathcal{K}^*$ for the full subcategory of non-degenerate simplices and $I\colon \mathcal{K}^* \hookrightarrow \mathcal{K}$ for the inclusion. Recall from (2.20) that if K is ∂-non-singular (e.g. a simplicial complex), the Eilenberg-Zilber maps $\eta_k\colon k \to k^{\downarrow}$ form the unit of an adjunction

$$\begin{tikzcd} \mathcal{K}^* \arrow[r, hook, shift left, "I"] \arrow[r, phantom, "\top"] & \mathcal{K} \arrow[l, shift left, "E"] \end{tikzcd} \qquad \rightsquigarrow \qquad \begin{tikzcd} (\mathcal{K}^*)^{\mathrm{op}} \arrow[r, hook, shift left, "I^{\mathrm{op}}"] \arrow[r, phantom, "\perp"] & \mathcal{K}^{\mathrm{op}}. \arrow[l, shift left, "E^{\mathrm{op}}"] \end{tikzcd}$$

In this situation, the above proposition can be interpreted as follows.

(6.5) **Corollary.** With the same notation as above, if K is ∂-non-singular, then a diagram $X\colon \mathcal{K}^{\mathrm{op}} \to \mathbf{sSets}$ is bounded iff the counit $X\eta\colon XI^{\mathrm{op}}E^{\mathrm{op}} \Rightarrow X$ is an isomorphism. □

(6.6) **Corollary.** Still with the same notation as above, if K is a ∂-non-singular simplicial set and $X\colon \mathcal{K}^{\mathrm{op}} \to \mathbf{sSets}$ bounded, then the comparison map

$$\operatorname*{hocolim}_{(\mathcal{K}^*)^{\mathrm{op}}} X \to \operatorname*{hocolim}_{\mathcal{K}^{\mathrm{op}}} X$$

is a weak equivalence.

Proof. Since $E^{\mathrm{op}}\colon \mathcal{K}^{\mathrm{op}} \to (\mathcal{K}^*)^{\mathrm{op}}$ has a left adjoint, it is homotopy final and thus

$$\operatorname*{hocolim}_{\mathcal{K}^{\mathrm{op}}} X \cong \operatorname*{hocolim}_{\mathcal{K}^{\mathrm{op}}} (E^{\mathrm{op}})^*(X \circ I^{\mathrm{op}}) \simeq \operatorname*{hocolim}_{(\mathcal{K}^*)^{\mathrm{op}}} X.$$

□

For our explicit model $|X|_{\mathcal{K}}$ of the homotopy colimit of a diagram $X\colon \mathcal{K}^{\mathrm{op}} \to \mathbf{sSets}$, we can even be more specific and obtain a stronger result, which will come in handy later on.

(6.7) **Proposition.** Still with the above notation, if K is a ∂-non-singular simplicial set and $X\colon \mathcal{K}^{\mathrm{op}} \to \mathbf{sSets}$ bounded, then the comparison map

$$|X|_{\mathcal{K}^*} = \int^{k^* \in \mathcal{K}^*} \Delta[\dim k^*] \times X_{k^*} \to \int^{k \in \mathcal{K}} \Delta[\dim k] \times X_k = |X|_{\mathcal{K}}$$

is an isomorphism.

Proof. The two adjunctions above induce adjunctions of precomposition functors

$$\begin{tikzcd} \mathbf{sSets}^{\mathcal{K}^*} \arrow[r, shift left, "I^*"] \arrow[r, phantom, "\perp"] & \mathbf{sSets}^{\mathcal{K}}, \arrow[l, shift left, "E^*"] \end{tikzcd} \qquad \begin{tikzcd} \mathbf{sSets}^{(\mathcal{K}^*)^{\mathrm{op}}} \arrow[r, shift left, "(I^{\mathrm{op}})^*"] \arrow[r, phantom, "\top"] & \mathbf{sSets}^{\mathcal{K}^{\mathrm{op}}} \arrow[l, shift left, "(E^{\mathrm{op}})^*"] \end{tikzcd}$$

and therefore

$$I_* \cong E^*, \quad E_! \cong I^*, \quad I^{\mathrm{op}}_! \cong (E^{\mathrm{op}})^*, \quad E^{\mathrm{op}}_* \cong (I^{\mathrm{op}})^*.$$

Using that $(E^{\mathrm{op}})^*(X \circ I^{\mathrm{op}}) \cong X$ (by boundedness), the claim is just a formal calculation using the coend formula for left Kan extensions:

$$\begin{aligned}
|X|_{\mathcal{K}} &\cong |I^{\mathrm{op}}_!(X \circ I^{\mathrm{op}})|_{\mathcal{K}} = \int^{k} \Delta[\dim k] \times \left(\int^{k^*} \mathcal{K}(k,k^*) \cdot X_{k^*}\right) \\
&\cong \int^{k^*}\int^{k} \Delta[\dim k] \times (\mathcal{K}(k,k^*) \cdot X_{k^*}) \cong \int^{k^*}\int^{k} (\Delta[\dim k] \cdot \mathcal{K}(k,k^*)) \times X_{k^*} \\
&\overset{(*)}{\cong} \int^{k^*}\int^{k} (\Delta[\dim k] \cdot \mathcal{K}^*(Ek,k^*)) \times X_{k^*} \cong \int^{k^*} (E_!\Delta[\dim -])_{k^*} \times X_{k^*} \\
&\cong \int^{k^*} (I^*\Delta[\dim -])_{k^*} \times X_{k^*} = \int^{k^*} \Delta[\dim k^*] \times X_{k^*} = |X|_{\mathcal{K}^*}\,,
\end{aligned}$$

where, for $(*)$, we used that $Ek = k^{\downarrow}$ and that every morphism $k \to k^*$ with k^* non-degenerate factors uniquely through $k^{\downarrow}$ (for which one needs K to be ∂-non-singular). □

(6.8) **Remark.** One might think that bounded diagrams $X\colon \mathcal{K}^{\mathrm{op}} \to \mathbf{sSets}$ are exactly the ones obtained by left Kan extension along the inclusion $I^{\mathrm{op}}\colon (\mathcal{K}^*)^{\mathrm{op}} \hookrightarrow \mathcal{K}^{\mathrm{op}}$. In general, this is not true [15, Example 10.10]! However, it is true if K is ∂-non-singular.

7. Mather's Cube Theorems

In its original formulation [41] (for topological spaces rather than simplicial sets), Mather's cube theorems are concerned with cubical diagrams of spaces

$$\begin{tikzcd}
& A \arrow[rr] \arrow[dl] \arrow[dd] & & C \arrow[dl] \arrow[dd] \\
B \arrow[rr, crossing over] \arrow[dd] & & D & \\
& A' \arrow[rr] \arrow[dl] & & C' \arrow[dl] \\
B' \arrow[rr] & & D' \arrow[from=uu, crossing over] & .
\end{tikzcd}$$

A concise version of Mather's cube theorem is the following statement. The direction "⇒" is usually called the first cube theorem, while "⇐" is the second cube theorem.

(7.1) **Theorem. (Mather's Cube Theorems)** Consider a cubical diagram of spaces as above with the left and back face homotopy pullbacks and the bottom face a homotopy pushout. Then the top face is a homotopy pushout iff the front and right face are homotopy pullbacks.

Mather's original proof for topological spaces was quite involved, using explicit calculations of homotopy pullbacks and pushouts. When working in simplicial sets instead, we can even prove more general statements without getting our hands too dirty. Let's first prove the second cube theorem as it is the easier one of the two. The following property is usually referred to as *universality of homotopy colimits* but more often than not, we will also refer to this as Mather's (second) cube theorem.

(7.2) **Theorem. (Mather's Second Cube Theorem)** In **sSets**, homotopy pullbacks preserve homotopy colimits. More precisely, pulling back along a Kan fibration $f\colon K \to L$

$$- \times_L K\colon \mathbf{sSets}\downarrow L \to \mathbf{sSets}\downarrow K$$

preserves homotopy colimits (where the two comma categories inherit a model structure from **sSets** and so [homotopy] colimits in them are calculated in **sSets**).

Proof. Being a presheaf category, **sSets** is locally cartesian closed, meaning that pulling back along f has a right adjoint $\prod_f$ and in particular, $- \times_L K$ preserves colimits. Moreover, since monomorphisms are stable under base change, $- \times_L K$ preserves cofibrations and because **sSets** is right proper (and f a fibration), it is a homotopy functor. In particular, $- \times_L K$ is left Quillen. By preservation of colimits, we obtain, for every indexing category $\mathcal{I}$, a commuting square of functors (up to a canonical isomorphism)

$$\begin{tikzcd} (\mathbf{sSets}\downarrow L)^{\mathcal{I}} \arrow[r, "\mathrm{colim}"] \arrow[d, "(-\times_L K)_*"'] & \mathbf{sSets}\downarrow L \arrow[d, "-\times_L K"] \\ (\mathbf{sSets}\downarrow K)^{\mathcal{I}} \arrow[r, "\mathrm{colim}"'] & \mathbf{sSets}\downarrow K \end{tikzcd} .$$

Equipping the diagram categories with the projective model structures, this is a commuting square of left Quillen functors, the only non-trivial case being $(- \times_L K)_*$, where one just observes that its right adjoint $(\prod_f)_*$ preserves (acyclic) fibrations because those are pointwise. Since left Quillen functors compose (the left derived functor of their composite is the composite of the derived functors), the claim follows from the square's commutativity. □

(7.3) **Corollary.** If K is a Kan complex, then the trivial bundle functor

$$- \times K\colon \mathbf{sSets} \to \mathbf{sSets}\downarrow K$$

preserves homotopy colimits.

Proof. Just take $f\colon K \to *$ in the above theorem. □

(7.4) **Remark.** The fibrancy of f is necessary since in general, $- \times_L K$ is not left Quillen (just take f to be the inclusion $\Delta[0] \hookrightarrow S^1 = \Delta[1]/\partial\Delta[1]$ and consider the diagram $\Delta[0] \hookrightarrow \Delta[1]$ over S^1). Of course, we can always fibrantly replace an arbitrary $f\colon K \to L$ but this involves replacing K as well, so that we land in a different slice category. However, as long as one is only interested in the homotopy type of some space obtained by taking a homotopy pullback of a homotopy colimit, this is irrelevant.

As one particular example of this theorem, we obtain a special case of Puppe's theorem [43], which tells us that taking homotopy colimits commutes with taking homotopy fibres over a fixed base space. Note, however, that if $\tau\colon X \Rightarrow Y$ is a natural transformation of diagrams, it is wrong in general that $\mathrm{hFib}\,\mathrm{hocolim}\,\tau \simeq \mathrm{hocolim}\,\mathrm{hFib}\,\tau$.

(7.5) **Corollary. (Special Puppe Theorem)** Given a diagram $X\colon \mathcal{I} \to$ **sSets** and $\tau\colon X \Rightarrow K$ a transformation to a constant diagram (i.e. X is a diagram in $\mathbf{sSets}\downarrow K$) then

$$\mathrm{hFib}_k\big(\mathrm{hocolim}\,\tau\colon\ \mathrm{hocolim}\,X \to K\big) \simeq \mathrm{hocolim}(\mathrm{hFib}_k\,\tau)$$

for every base-point k of K (where $\mathrm{hFib}_k\,\tau$ is the diagram $I \mapsto \mathrm{hFib}_k(\tau_I\colon XI \to K)$).

Proof. Just take $f\colon PK \to K$ in the theorem to be the standard path space fibration. □

(7.6) **Corollary.** Given a space D and a commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

then

$$\begin{gathered} \mathrm{hFib}_d\big(\mathrm{hocolim}(B \leftarrow A \to C) \to D\big) \simeq \\ \mathrm{hocolim}\big(\mathrm{hFib}_d(B \to D) \leftarrow \mathrm{hFib}_d(A \to D) \to \mathrm{hFib}_d(C \to D)\big). \end{gathered}$$

Proof. In the previous corollary, take $\mathcal{I} = \{\bullet \leftarrow \bullet \rightarrow \bullet\}$ and $K = D$. $\square$

The full Puppe theorem is a generalisation of Mather's first cube theorem and says that given two diagrams $X, Y : \mathcal{I} \to$ **sSets** together with a natural transformation $\tau\colon X \Rightarrow Y$ such that for every $i\colon I \to J$ in $\mathcal{I}$ the square

$$\begin{tikzcd} X_I \arrow[r, "\tau_I"] \arrow[d, "X_i"'] & Y_I \arrow[d, "Y_i"] \\ X_J \arrow[r, "\tau_J"'] & Y_J \end{tikzcd}$$

is a homotopy pullback, then every square

$$\begin{tikzcd} X_I \arrow[r, "\tau_I"] \arrow[d] & Y_I \arrow[d] \\ \operatorname{hocolim}_{\mathcal{I}} X \arrow[r, dotted] & \operatorname{hocolim}_{\mathcal{I}} Y \end{tikzcd}$$

with $I \in \mathcal{I}$ is again a homotopy pullback. However, for the time being, we will only treat the classical case $\mathcal{I} = \{\bullet \leftarrow \bullet \rightarrow \bullet\}$ from Mather's original theorem.

(7.7) **Proposition. (Mather's First Cube Theorem)** Given a cubical diagram

$$\begin{tikzcd} & A \arrow[rr] \arrow[dl] \arrow[dd] & & C \arrow[dl] \arrow[dd] \\ B \arrow[rr] \arrow[dd] & & D \arrow[dd] & \\ & A' \arrow[rr] \arrow[dl] & & C' \arrow[dl] \\ B' \arrow[rr] & & D' & \end{tikzcd}$$

in **sSets** with the top and bottom faces (homotopy) pushouts and the left and back faces (homotopy) pullbacks, the front and right faces are (homotopy) pullbacks, too.

Proof. We can cofibrantly replace $B' \leftarrow A' \rightarrow C'$, and also replace all of $A \to A'$, $B \to B'$, $C \to C'$ by fibrations. With this, we can assume that the back and left face are strict pullbacks, which preserve cofibrations, and that the bottom and top faces are strict pushouts all of whose maps are monic. Since pullbacks in **sSets** are calculated dimensionwise, the claim follows from the lemma below. $\square$

(7.8) **Lemma.** Given a cubical diagram of sets

$$\begin{tikzcd} & A \arrow[rr, hook] \arrow[dl, hook'] \arrow[dd, "\alpha"] & & C \arrow[dl, hook'] \arrow[dd, "\gamma"] \\ B \arrow[rr, hook] \arrow[dd, "\beta"'] & & D \arrow[dd, "\delta"' near end] & \\ & A' \arrow[rr, hook] \arrow[dl, hook'] & & C' \arrow[dl, hook'] \\ B' \arrow[rr, hook] & & D' & \end{tikzcd}$$

with the top and bottom faces being pushouts and the left and back faces being pullbacks, the front and right faces are pullbacks, too.

Proof. We identify

$$D \cong A + (B \setminus A) + (C \setminus A) \qquad \text{and} \quad D' \cong A' + (B' \setminus A') + (C' \setminus A')$$

with the maps $B, C \to D$ and $B', C' \to D'$ then being the obvious inclusions, while $\delta\colon D \to D'$ is induced by $\alpha = \beta|_A = \gamma|_A$, β and γ. With this

$$\begin{aligned} B' \times_{D'} D &= \{(b', d) \in B' \times D \mid b' = \delta d\} \\ &\cong \{(a', a) \in A' \times A \mid a' = \alpha a\} + \{(b', b) \in (B' \setminus A') \times (B \setminus A) \mid b' = \beta b\} \\ &\cong \{(b', b) \in B' \times B \mid b' = \beta b\} \cong B. \end{aligned}$$

Similarly for $C' \times_{D'} D \cong C$. □

8. Some Explicit Formulae

Having the coend formula (4.12) for homotopy colimits allows us to do explicit calculations with homotopy colimits on the point-set level. While we generally try to avoid these (our credo being that we should only use them in homotopical calculations), they sometimes provide quick shortcuts (one might even say "hacks") that allow us to sidestep the technicalities of derived functors.

In this section, using Mather's second cube theorem, we are going to show several such explicit formulae, which we will need later on. We have taken all of these (with the exception of the compatibility with telescopes) from [10], where they are stated without proof, which is why we do them here. All of these proofs are purely formal (if a bit tedious) and involve no homotopy theory whatsoever.

The first formula, in some sense says that if $f\colon L \to K$ is a map of simplicial sets and $X\colon \mathcal{K}^{\mathrm{op}} \to \mathbf{sSets}$, then $\operatorname{hocolim}_{\mathcal{L}^{\mathrm{op}}} f^*X$ is the (strict!) pullback of $\operatorname{hocolim}_{\mathcal{K}^{\mathrm{op}}} X \to K$ along f. Of course, this statement is non-sensical because the homotopy colimit is only defined up to homotopy and we need to pick explicit models.

(8.1) **Proposition.** Given a map of simplicial sets $f\colon L \to K$ and $X\colon \mathcal{K}^{\mathrm{op}} \to \mathbf{sSets}$ then the following is a (strict!) pullback square:

$$\begin{tikzcd} {|f^*X|_{\mathcal{L}}} \arrow[r] \arrow[d] & {|X|_{\mathcal{K}}} \arrow[d] \\ L \arrow[r, "f"'] & K. \end{tikzcd}$$

The downwards arrows are the obvious projections and the top arrow the comparison map induced by the standard inclusions

$$\Delta[\dim l] \times X_{f(l)} = \Delta[\dim f(l)] \times X_{f(l)} \xrightarrow{\mathrm{in}_{f(l)}} \int^k \Delta[\dim k] \times X_k = |X|_{\mathcal{K}}\,.$$

Proof. The claim can be verified dimensionwise and we fix a dimension m for that. Now by Mather's second cube theorem for strict colimits (a.k.a. the universality of colimits), the pullback of a coend along $f\colon L \to K$ is just the coend over the pullback of the corresponding diagram. In our case, if $k \in K$, we form the pullbacks

$$\begin{tikzcd} P_k \times X_k \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near start] & P_k \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near start] & L \arrow[d, "f"] \\ \Delta[\dim k] \times X_k \arrow[r] & \Delta[\dim k] \arrow[r] & K, \end{tikzcd}$$

where, explicitly $P_{k,m} = \{(l,\xi) \in L_m \times \mathbf{\Delta}([m],[\dim k]) \mid f(l) = k\cdot\xi\}$. The map $P_{k\cdot\varphi} \to P_k$ induced by some $\varphi\colon k\cdot\varphi \to k$ in $\mathcal{K}$ is just $(l,\xi) \mapsto (l, \varphi\circ\xi)$. With this, the universality of colimits tells us that $L \times_K |X|_{\mathcal{K}} \cong \int^k P_k \times X_k$ and we only need to establish an isomorphism with $|f^*X|_{\mathcal{L}} = \int^l \Delta[\dim l] \times X_{f(l)}$.

Going one way, for $l \in L$, we map $(\xi, x) \in \Delta[\dim l]_m \times X_{f(l),m}$ (i.e. $\xi\colon [m] \to [\dim l]$) to $f(l) \in K$ and $(l\cdot\xi, \xi, x) \in P_{f(l),m} \times X_{f(l),m}$ (and then compose with the standard inclusion into the coend).

Conversely, given $k \in K$, we map $(l,\xi,x) \in P_{k,m} \times X_{k,m}$ (i.e. $\xi\colon [m] \to [\dim k]$ and $f(l) = k\cdot\xi$; in particular, $\dim l = m$) to $l \in L$ and $(\iota_m, x\cdot\xi) \in \Delta[\dim l]_m \times X_{f(l),m}$, where $\iota_m = \mathrm{id}_{[m]}$ and $x\cdot\xi$ is the image of $x \in X_{k,m}$ under $X\xi\colon X_k \to X_{k\cdot\xi} = X_{f(l)}$ (and then compose with the standard inclusion into the coend). More concisely, we have defined

$$[l,\xi,x] \mapsto [f(l), l\cdot\xi, \xi, x] \qquad \text{and} \qquad [k,l,\xi,x] \mapsto [l, \iota_m, x\cdot\xi].$$

That these two maps are mutually inverse to each other (when passing to the coends) is easily verified. Taking $[l,\xi,x]$ and chasing it through both maps, we get $[l\cdot\xi, \iota_m, x\cdot\xi]$ and the map $\xi\colon l\cdot\xi \to l$ in $\mathcal{L}$ induces

$$\begin{array}{ccccc} \Delta[\dim l] \times X_{f(l)} & \xleftarrow{\xi_*\times\mathrm{id}} & \Delta[m] \times X_{f(l)} & \xrightarrow{\mathrm{id}\times\xi^*} & \Delta[m] \times X_{f(l)\cdot\xi} \\ (\xi, x) & \longleftarrow\!\shortmid & (\iota_m, x) & \longmapsto & (\iota_m, x\cdot\xi), \end{array}$$

so that the two triples are equivalent in the coend. Similarly, if we chase some $[k,l,\xi,x]$ through the two maps, we obtain $[f(l), l, \iota_m, x\cdot\xi]$. Noting that $f(l) = k\cdot\xi$ by definition of P_k, the map $\xi\colon k\cdot\xi \to k$ in $\mathcal{K}$ yields

$$\begin{array}{ccccc} P_k \times X_k & \xleftarrow{\xi_*\times\mathrm{id}} & P_{f(l)} \times X_k & \xrightarrow{\mathrm{id}\times\xi^*} & P_{f(l)} \times X_{f(l)} \\ (l,\xi,x) & \longleftarrow\!\shortmid & (l,\iota_m,x) & \longmapsto & (l,\iota_m,x\cdot\xi) \end{array}$$

and the two triples are again equivalent in the coend. □

As an immediate first corollary, we can just use the pullback lemma to generalise the proposition to arbitrary transformations of diagrams.

(8.2) **Corollary.** Given a map of simplicial sets $f\colon L \to K$ and a natural transformation $\tau\colon X \Rightarrow Y$ of diagrams X, $Y\colon \mathcal{K}^{\mathrm{op}} \to \mathbf{sSets}$, then the maps induced by this transformation together with the comparison maps form a (strict!) pullback square:

$$\begin{array}{ccc} |f^*X|_{\mathcal{L}} & \longrightarrow & |X|_{\mathcal{K}} \\ {\scriptstyle\tau_*}\big\downarrow & & \big\downarrow{\scriptstyle\tau_*} \\ |f^*Y|_{\mathcal{L}} & \xrightarrow[f]{} & |Y|_{\mathcal{K}}\,. \end{array}$$

Proof. Combine the last result for X and Y with the pullback lemma. □

(8.3) **Remark.** One might try to make the two previous statement independent of an explicit choice of model for the homotopy colimit by conjecturing that the squares are actually homotopy pullbacks. This is not true in general. However, in the case of the proposition, it is true if X is a diagram of weak equivalences and for the corollary, if τ is a *transformation by homotopy pullbacks*. This is exactly the content of (the key lemma to prove) Quillen's Theorem B and Puppe's theorem, respectively.

Of course, if the map f is actually a Kan fibration, the square from the proposition is a homotopy pullback and so, in that case, $\operatorname{hocolim}_{\mathcal{L}^{\mathrm{op}}} f^*X = L \times^{\mathbb{R}}_K \operatorname{hocolim}_{\mathcal{K}^{\mathrm{op}}} X$. Using the fact (4.25) that it suffices to check homotopy finality for **sSets**, this leads to the following observation, which we will not need but is still noteworthy.

(8.4) **Corollary.** If $p\colon E \twoheadrightarrow B$ is an acyclic Kan fibration, then the induced functor between opposite categories of simplices $p\colon \mathcal{E}^{\mathrm{op}} \to \mathcal{B}^{\mathrm{op}}$ is homotopy final. □

As another corollary, we obtain that our explicit model for the homotopy colimit (given by the coend formula) preserves cofibrations (i.e. monomorphisms) between the indexing spaces.

(8.5) **Corollary.** If $i\colon L \hookrightarrow K$ is a monomorphism of simplicial sets and $X\colon \mathcal{K}^{\mathrm{op}} \to \mathbf{sSets}$, then the induced map

$$|i^*X|_{\mathcal{L}} \to |X|_{\mathcal{K}}$$

is again a monomorphism.

Proof. Monomorphisms are stable under base changes. □

For our second explicit formula, we take a pushout $P := \operatorname{colim}(L \leftarrow K \to M)$ of simplicial sets. Any category of elements functor (and in particular taking the category of simplices) $\int_{\mathcal{I}}\colon \mathbf{Sets}^{\mathcal{I}} \to \mathbf{Cat}\downarrow\mathcal{I}$ ($\mathcal{I}$ an arbitrary indexing category) preserves colimits because it is left adjoint to $\mathcal{C} \mapsto \operatorname{Fun}_{\mathcal{I}}(\mathcal{I}\downarrow -, \mathcal{C})$. In particular,

$$\mathcal{P}^{\mathrm{op}} = \operatorname{colim}\left(\mathcal{L}^{\mathrm{op}} \leftarrow \mathcal{K}^{\mathrm{op}} \to \mathcal{M}^{\mathrm{op}}\right)$$

is the pushout of the corresponding categories of simplices. As shown in (2.4.8), the Grothendieck construction

$$\mathcal{G} := \int^{\{a\leftarrow b\rightarrow c\}} (\mathcal{L}^{\mathrm{op}} \leftarrow \mathcal{K}^{\mathrm{op}} \to \mathcal{M}^{\mathrm{op}})$$

comes with a final functor $\mathcal{G} \to \mathcal{P}^{\mathrm{op}}$ and using Thomason's theorem (for strict colimits), it follows for every diagram $X\colon \mathcal{P}^{\mathrm{op}} \to \mathbf{sSets}$ (or any other codomain category), that

$$\operatorname*{colim}_{\mathcal{P}^{\mathrm{op}}} X \cong \operatorname{colim}\left(\operatorname*{colim}_{\mathcal{L}^{\mathrm{op}}} X \leftarrow \operatorname*{colim}_{\mathcal{K}^{\mathrm{op}}} X \to \operatorname*{colim}_{\mathcal{M}^{\mathrm{op}}} X\right)$$

(with the obvious abuse of notation). Using Mather's second cube theorem allows us to derive a similar statement for our explicit model of homotopy colimits from the already proven first formula.

(8.6) **Proposition.** Every pushout square of simplicial sets as on the left below and every diagram $X\colon \mathcal{L}^{\mathrm{op}} \to \mathbf{sSets}$ give rise to a (strict!) pushout square as on the right:

$$\begin{tikzcd} K \arrow[r] \arrow[d] \arrow[dr, dotted, "h"] & L \arrow[d, "f"] \\ M \arrow[r, "g"'] & P \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} {|h^*X|_{\mathcal{K}}} \arrow[r] \arrow[d] & {|f^*X|_{\mathcal{L}}} \arrow[d] \\ {|g^*X|_{\mathcal{M}}} \arrow[r] & {|X|_{\mathcal{P}}}\,. \end{tikzcd}$$

Proof. We form the cubical diagram

$$\begin{tikzcd}
& {|h^*X|_{\mathcal{K}}} \arrow[rr] \arrow[dl] \arrow[dd] & & {|f^*X|_{\mathcal{L}}} \arrow[dl] \arrow[dd] \\
{|g^*X|_{\mathcal{M}}} \arrow[rr, crossing over] \arrow[dd] & & {|X|_{\mathcal{P}}} & \\
& K \arrow[rr] \arrow[dl] & & L \arrow[dl] \\
M \arrow[rr] & & P \arrow[from=uu, crossing over] &
\end{tikzcd}$$

(where all the downwards arrows are the canonical projections). The claim now follows from (8.1) in connection with Mather's second cube theorem (7.2). □

As a corollary of the proposition, we get that our explicit (and hence any) homotopy colimit functor preserves "homotopy pushouts for the indexing space". Of course, again, just stated like this, this claim is again non-sensical since $\operatorname{hocolim}_{\mathcal{K}^{\mathrm{op}}}$ is not invariant under weak equivalences of the indexing space K.

(8.7) **Corollary.** Every pushout square of simplicial sets as on the left with $K \rightarrowtail L$ monic and every $X\colon \mathcal{L}^{\mathrm{op}} \to \mathbf{sSets}$ give rise to a homotopy pushout square as on the right:

$$\begin{tikzcd}
K \arrow[r, rightarrowtail] \arrow[d] \arrow[dr, dotted, "h"] & L \arrow[d, "f"] \\
M \arrow[r, "g"'] & P
\end{tikzcd}
\quad \rightsquigarrow \quad
\begin{tikzcd}
\operatorname*{hocolim}_{\mathcal{K}^{\mathrm{op}}} h^*X \arrow[r] \arrow[d] & \operatorname*{hocolim}_{\mathcal{L}^{\mathrm{op}}} f^*X \arrow[d] \\
\operatorname*{hocolim}_{\mathcal{M}^{\mathrm{op}}} g^*X \arrow[r] & \operatorname*{hocolim}_{\mathcal{P}^{\mathrm{op}}} X.
\end{tikzcd}$$

Proof. Picking our usual explicit models for the homotopy colimit, this follows from the last proposition and the fact (8.5) that $|-|_{\mathcal{K}}$ preserves monomorphisms in the indexing space. □

So now that we have checked the compatibility of our explicit homotopy colimits with homotopy pushouts in the indexing space, the next step is obviously to check the compatibility with telescopes. For an arbitrary homotopy colimit functor, given a telescope (i.e. a transfinite sequence indexed by some regular cardinal κ) of spaces

$$K_0 \rightarrowtail K_1 \rightarrowtail \ldots \rightarrowtail K_\alpha \rightarrowtail \ldots \rightarrowtail K,$$

since taking the category of simplices preserves colimits, we obtain a transfinite sequence of categories

$$\mathcal{K}_0^{\mathrm{op}} \rightarrowtail \mathcal{K}_1^{\mathrm{op}} \rightarrowtail \ldots \rightarrowtail \mathcal{K}_\alpha^{\mathrm{op}} \rightarrowtail \ldots \rightarrowtail \mathcal{K}^{\mathrm{op}}.$$

Now, since the Grothendieck construction for a telescope is actually homotopy final in the colimit by (2.4.9), we can use Thomason's theorem to conclude that for every $X\colon \mathcal{K}^{\mathrm{op}} \to \mathbf{sSets}$

$$\operatorname*{hocolim}_{\mathcal{K}} X \simeq \operatorname*{hocolim}_{\alpha<\kappa} \operatorname*{hocolim}_{\mathcal{K}_\alpha^{\mathrm{op}}} X.$$

Using our explicit model, we can prove more.

(8.8) **Proposition.** Let $K\colon \mathcal{I} \to \mathbf{sSets}$ be filtered, $L := \operatorname{colim}_I K_I$ and $\lambda_I\colon K_I \to L$ the colimiting cocone. For every $X\colon \mathcal{L}^{\mathrm{op}} \to \mathbf{sSets}$ and writing $X_I := \lambda_I^* X\colon \mathcal{K}_I^{\mathrm{op}} \to \mathbf{sSets}$, the comparison maps $|X_I|_{\mathcal{K}_I} \to |X|_{\mathcal{L}}$ induce an isomorphism

$$\operatorname*{colim}_{I\in\mathcal{I}} |X_I|_{\mathcal{K}_I} \cong |X|_{\mathcal{L}}\,.$$

Proof. By (8.1) above, we have a pullback of diagrams in $\mathbf{sSets}^{\mathcal{I}}$

$$I \quad \mapsto \quad \begin{tikzcd} |X_I|_{\mathcal{K}_I} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & |X|_{\mathcal{L}} \arrow[d] \\ K_I \arrow[r] & L \end{tikzcd}$$

(where $|X|_{\mathcal{L}}$ and L are constant). Since in **sSets**, pullbacks commute with filtered colimits (i.e. $\operatorname{colim}_{\mathcal{I}} \colon \mathbf{sSets}^{\mathcal{I}} \to \mathbf{sSets}$ preserves pullbacks), the claim follows. $\square$

(8.9) **Corollary.** Let $\mathcal{J} \colon \mathcal{I} \to \mathbf{Cat}$ be filtered, $\mathcal{K} := \operatorname{colim}_I \mathcal{J}_I$ and $\lambda_I \colon \mathcal{J}_I \to \mathcal{K}$ the colimiting cocone. For every $X \colon \mathcal{K} \to \mathbf{sSets}$ and writing $X_I := \lambda_I^* X \colon \mathcal{J}_I \to \mathbf{sSets}$, the comparison maps $\operatorname{hocolim}_{\mathcal{J}_I} X_I \to \operatorname{hocolim}_{\mathcal{K}} X$ induce a weak equivalence

$$\operatorname*{hocolim}_{I \in \mathcal{I}} \operatorname*{hocolim}_{\mathcal{J}_I} X_I \simeq \operatorname*{hocolim}_{\mathcal{K}} X.$$

Proof. We can reduce the situation to indexing spaces by precomposing with the source functors $\mathcal{N}(\mathcal{J}_I)^{\mathrm{op}} \to \mathcal{J}_I$. For this, we need to note that the nerve functor $N \colon \mathbf{Cat} \to \mathbf{sSets}$ preserves filtered colimits (which are calculated pointwise in **sSets**) because

$$N(\operatorname{colim}_I \mathcal{J}_I)_m = \mathbf{Cat}([m], \operatorname{colim}_I \mathcal{J}_I) \cong \operatorname*{colim}_I \mathbf{Cat}([m], \mathcal{J}_I),$$

using that $[m]$ is finitely presentable (even finite). Since the category of simplices preserves all colimits ($\int_{\boldsymbol{\Delta}} - \colon \mathbf{sSets} \to \mathbf{Cat} \downarrow \boldsymbol{\Delta}$ is left adjoint to $\mathcal{C} \mapsto \operatorname{Fun}_{\boldsymbol{\Delta}}(\int_{\boldsymbol{\Delta}} \Delta[-], \mathcal{C})$), as does taking opposites (the 1-functor $-^{\mathrm{op}} \colon \mathbf{Cat} \to \mathbf{Cat}$ is an involution), it follows that $\mathcal{N}(\mathcal{K})^{\mathrm{op}} \cong \operatorname{colim}_I \mathcal{N}(\mathcal{J}_I)^{\mathrm{op}}$. Finally, the claim follows from the above proposition, using the fact (5.2.7) that filtered colimits in **sSets** are actually homotopy colimits. $\square$

As our final result of this section, we are going to show a “local triviality” result for our construction of homotopy colimits, assuming the diagram in question is bounded. For this, let us make a small observation that lots of pushout squares in **Sets** (whence **sSets**) are also pullbacks.

This is probably well-known among category theorists but we stumbled on it during our own studies and, upon checking, weren’t able to find it in the usual references [38; 5]

(8.10) **Lemma.** Given a square of sets

$$\begin{tikzcd} A \arrow[r, tail, "i"] \arrow[d, "f"'] & B \arrow[d, "g"] \\ C \arrow[r, tail, "j"'] & D \end{tikzcd}$$

with $i \colon A \rightarrowtail B$ and $j \colon C \rightarrowtail D$ monic, the following are equivalent:

(a) the square is a pushout;

(b) the square is a pullback (i.e. $A = g^{-1}(C)$) and the map $B \setminus A \cong D \setminus C$ induced by g is bijective.

(c) $g^{-1}(C) \subseteq A$ and the map $B \setminus A \to D \setminus C$ induced by g is bijective;

Since limits, colimits and monicity in **sSets** are determined pointwise, the same is true for simplicial sets (or any presheaf category).

Proof. We identify A and C with their images in B and D, respectively, and view i and j as the corresponding inclusions. By the explicit construction of pushouts and pullbacks in **Sets**, the square's pushout is just $C + (B \setminus A)$, while its pullback is $g^{-1}(C)$.

Ad "(a) ⇒ (b)": The square being a pushout means that we can identify $C + (B \setminus A) \cong D$ with $g\colon B \to D$ becoming the inclusion on $B \setminus A$ and $g\colon A \to C \hookrightarrow D$ on A. With this, obviously $g^{-1}(C) = A$ and $B \setminus A \cong D \setminus C$.

Ad "(b) ⇒ (c)": Trivial.

Ad "(c) ⇒ (a)": For the square to be a pushout, the obvious map

$$[j, g]\colon C + (B \setminus A) \to D$$

needs to be an isomorphism. Since the map is already injective on C, this is equivalent to g inducing a bijection $B \setminus A \cong D \setminus C$. In particular, we must have $g^{-1}(C) \subseteq A$ for such a map to be well-defined. □

As a second observation, given a bounded diagram $X\colon \mathbf{\Delta}[n]^{\mathrm{op}} \to \mathbf{sSets}$, let us investigate the difference between $|X|_{\mathbf{\Delta}[n]} \cong |X|_{\mathbf{\Delta}[n]^*}$ and $|X|_{\partial\mathbf{\Delta}[n]} \cong |X|_{\partial\mathbf{\Delta}[n]^*}$.

(8.11) **Lemma.** Given $X\colon \mathbf{\Delta}[n]^{\mathrm{op}} \to \mathbf{sSets}$ bounded, then $|X|_{\partial\mathbf{\Delta}[n]} \subseteq |X|_{\mathbf{\Delta}[n]}$ by (8.5) and the dimensionwise complement is

$$|X|_{\mathbf{\Delta}[n]} \setminus |X|_{\partial\mathbf{\Delta}[n]} \cong |X|_{\mathbf{\Delta}[n]^*} \setminus |X|_{\partial\mathbf{\Delta}[n]^*} \cong \Delta[n]_{-} \times X_{\iota_n},$$

where $\Delta[n]_{-} := \Delta[n] \setminus \partial\Delta[n]$ consists of all surjections into $[n]$.

Proof. By boundedness of X, we can use (6.7) to replace $\mathbf{\Delta}[n]$ and $\partial\mathbf{\Delta}[n]$ by their respective full subcategories $\mathbf{\Delta}[n]^*$, $\partial\mathbf{\Delta}[n]^*$ of non-degenerate simplices. But then, the only object of $\mathbf{\Delta}[n]^*$ that is not already in $\partial\mathbf{\Delta}[n]^*$ is the unique non-degenerate simplex $\iota_n := \mathrm{id}_{[n]}$ in dimension n, so that every element of $|X|_{\mathbf{\Delta}[n]^*} \setminus |X|_{\partial\mathbf{\Delta}[n]^*}$ is represented by some $[\iota_n, \xi, x]$ with $\xi \in \Delta[n]$ and $x \in X_{\iota_n}$.

Since ι_n is terminal in $\mathbf{\Delta}[n]^*$ and there are no morphisms $\iota_n \to \delta$ for δ a proper injection into $[n]$, the only generating equivalences for the coend $|X|_{\mathbf{\Delta}[n]^*}$ involving such representatives $[\iota_n, \xi, x]$ are of the form

$$[\iota_n, \delta \circ \xi, x] = [\delta, \xi, x \cdot \delta] \qquad \text{for } \delta\colon [m] \rightarrowtail [n] \text{ an arbitrary injection and } \xi \in \Delta[m].$$

To wit, such a generating equivalence originates from the span

$$\Delta[n] \times X_{\iota_n} \xleftarrow{\delta_* \times \mathrm{id}} \Delta[m] \times X_{\iota_n} \xrightarrow{\mathrm{id} \times \delta^*} \Delta[m] \times X_{\delta}.$$

It follows that an element represented by $[\iota_n, \xi, x]$ lies in the complement of $|X|_{\partial\mathbf{\Delta}[n]^*}$ iff ξ does not factor through a proper injection, meaning that it is surjective. Moreover, no two such elements are equivalent in the coend $|X|_{\mathbf{\Delta}[n]^*}$ and the claim follows. □

(8.12) **Proposition.** Every pushout square of simplicial sets, as on the left below and every bounded diagram $X\colon \mathcal{K}^{\mathrm{op}} \to \mathbf{sSets}$ give rise to (strict!) pushouts as on the right

$$\begin{tikzcd}
\partial\Delta[n] \arrow[r] \arrow[d, tail] & L \arrow[d, tail] \\
\Delta[n] \arrow[r] & K \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}
\quad\rightsquigarrow\quad
\begin{tikzcd}
\partial\Delta[n] \times X_{\iota_n} \arrow[r, "\cong"] \arrow[d, tail] & {|X_{\iota_n}|_{\partial\mathbf{\Delta}[n]}} \arrow[r] \arrow[d, tail] & {|X|_{\partial\mathbf{\Delta}[n]}} \arrow[r] \arrow[d, tail] & {|X|_{\mathcal{L}}} \arrow[d, tail] \\
\Delta[n] \times X_{\iota_n} \arrow[r, "\cong"'] & {|X_{\iota_n}|_{\mathbf{\Delta}[n]}} \arrow[r] & {|X|_{\mathbf{\Delta}[n]}} \arrow[r] \arrow[ul, phantom, "\ulcorner", very near start] & {|X|_{\mathcal{K}}} \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd},$$

where ι_n the unique non-degenerate simplex in $\Delta[n]_n$ and $|X_{\iota_n}|_{\mathbf{\Delta}[n]} \to |X|_{\mathbf{\Delta}[n]}$ is induced by the transformation from the constant diagram $X_{\iota_n} \Rightarrow X$ (using that ι_n is initial in $\mathbf{\Delta}[n]^{\mathrm{op}}$).

Proof. Again, using the boundedness of X, we can replace $\mathbf{\Delta}[n]$ and $\partial\mathbf{\Delta}[n]$ by their respective full subcategories $\mathbf{\Delta}[n]^*$, $\partial\mathbf{\Delta}[n]^*$ of non-degenerate simplices by (6.7). We have already seen in proposition (8.6) that the square on the right in the diagram is a pushout and we just need to check that the same is true for the central one. By (8.2), this central square is a pullback and so, using the first lemma (8.10) above, we just need to check that the map

$$f\colon\ |X_{\iota_n}|_{\mathbf{\Delta}[n]^*} \to |X|_{\mathbf{\Delta}[n]^*}\,,\ [\delta,\xi,x] \mapsto [\delta,\xi,x\cdot\delta]$$

induced by the transformation $X_{\iota_n} \Rightarrow X$ yields a bijection of dimensionwise complements

$$|X_{\iota_n}|_{\mathbf{\Delta}[n]^*} \setminus |X_{\iota_n}|_{\partial\mathbf{\Delta}[n]^*} \cong |X|_{\mathbf{\Delta}[n]^*} \setminus |X|_{\partial\mathbf{\Delta}[n]^*}\,.$$

This then follows from the second lemma (8.11) because both sides are $\Delta[n]_- \times X_{\iota_n}$. □

9. Fibre Decomposition

A map of simplicial sets $f\colon A \to B$ is a weak equivalence iff all its homotopy fibres are contractible. When left Bousfield localising at a set of maps M, one might ask if the same is true in $L_M\mathbf{sSets}$. This certainly does not follow from general abstract nonsense because otherwise (by duality), a map of simplicial sets would also be a weak equivalence iff all its homotopy cofibres are contractible, which is not true.

The tool needed to obtain the sought for characterisation of weak equivalences is that of fibrewise localisation. All that follows is inspired by material taken from [26], [10] and [14] though our presentation and methods are slightly different.

(9.1) **Definition.** Given a map $f\colon E \to B$ of simplicial sets and with $\mathcal{B} := \int_{\mathbf{\Delta}} B$, we define a diagram $F_f\colon \mathcal{B} \to \mathbf{sSets}$ by the pullback diagrams

$$\begin{tikzcd} F_f(b) \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & E \arrow[d, "f"] \\ \Delta[\dim b] \arrow[r, "b"'] & B \end{tikzcd}$$

(with the morphism function being defined by the universal property of a pullback).

(9.2) **Remark.** In some of the above references, the diagram F_f is precomposed with the source functor $S\colon \mathcal{N}(\mathcal{B})^{\mathrm{op}} \to \mathcal{B}$ (where $\mathcal{N}(\mathcal{B}) := \int_{\Delta} N(\mathcal{B})$), yielding a *(bounded)* diagram

$$df := F_f \circ S\colon \mathcal{N}(\mathcal{B})^{\mathrm{op}} \to \mathbf{sSets}.$$

It is explicitly given by $df(\varphi_\bullet; b) := F_f(b\cdot\varphi_n \dots \varphi_1)$,

$$df\big((\varphi_2,\dots,\varphi_n;b) \xrightarrow{\delta^0} (\varphi_1,\dots,\varphi_n;b)\big) := F_f(b\cdot\varphi_n\dots\varphi_1) \xrightarrow{F_f\varphi_1} F_f(b\cdot\varphi_n\dots\varphi_2)$$

and $df(\delta^i) = \mathrm{id}$, $df(\sigma^j) = \mathrm{id}$ for all $i \neq 0$ and all j. The reason for this is that in said sources, the homotopy colimit over an arbitrary indexing category $\mathcal{I}$ is defined via first passing to a diagram indexed by $\mathcal{N}(\mathcal{I})^{\mathrm{op}}$ (where $\mathcal{N}(\mathcal{I}) := \int_{\mathbf{\Delta}} N(\mathcal{I})$) and then using the explicit coend formula (4.12). However, as long as we only use abstract properties of the homotopy colimit, there is really no need for that since the source functor S is homotopy final (4.22).

The essential property of the functor F_f from the above definition is that the induced map $\operatorname{hocolim}_{\mathcal{B}} F_f \to \operatorname{colim}_{\mathcal{B}} F_f \cong E$ is a weak equivalence in a compatible way, meaning that the homotopy colimit of the transformation $F_f \Rightarrow *$ is weakly equivalent to $f\colon E \to B$ itself. For the base B a simplicial complex, this was already done by Dror Farjoin in [26] but without the model categorical lingo and we give our interpretation of his proof below. We also provide an alternative one for a general base, which is inspired by the one in [14] but working directly with the projective model structure instead of passing through bounded diagrams.

Recall that if B is a simplicial set with category of simplices $\mathcal{B} := \int_{\mathbf{\Delta}} B$ then, for every diagram $X\colon \mathcal{B}^{\mathrm{op}} \to \mathcal{M}$ (with $\mathcal{M}$ any simplicial model category), and using the coend formula (4.12) to define homotopy colimits, we have a natural morphism

$$\operatorname*{hocolim}_{\mathcal{B}^{\mathrm{op}}} X = \int^{b\in\mathcal{B}} \Delta[\dim b] \odot X_b \to \int^{b\in\mathcal{B}} * \odot X_b = \operatorname*{colim}_{\mathcal{B}^{\mathrm{op}}} X$$

(which is just the unit of the total left derived functor). For a general indexing category $\mathcal{I}$, we use the (homotopy final) source functor $S\colon \mathcal{N}(\mathcal{I})^{\mathrm{op}} \to \mathcal{I}$, where $\mathcal{N}(\mathcal{I}) := \int_{\mathbf{\Delta}} N(\mathcal{I})$, to define $\operatorname{hocolim}_{\mathcal{I}}$ and can use this definition to define a natural morphism $\operatorname{hocolim}_{\mathcal{I}} X \to \operatorname{colim}_{\mathcal{I}} X$ for an $X\colon \mathcal{I} \to \mathcal{M}$ via

$$\begin{tikzcd}
\operatorname*{hocolim}_{\mathcal{N}(\mathcal{I})^{\mathrm{op}}} X \circ S \arrow[r] \arrow[d, equal] & \operatorname*{colim}_{\mathcal{N}(\mathcal{I})^{\mathrm{op}}} X \circ S \arrow[d, "\cong"] \\
\operatorname*{hocolim}_{\mathcal{I}} X \arrow[r, dotted] & \operatorname*{colim}_{\mathcal{I}} X,
\end{tikzcd}$$

where the morphism between strict colimits is induced by S (and is an isomorphism because S is final). Alternatively, this can be done using the projective model structure on the corresponding diagram categories and these definitions would be compatible. To wit, any functor $F\colon \mathcal{I} \to \mathcal{J}$ between indexing categories yields a precomposition functor $F^*\colon \mathcal{M}^{\mathcal{J}} \to \mathcal{M}^{\mathcal{I}}$ and the universal property of colimits gives us a natural transformation

$$\operatorname*{colim}_{\mathcal{I}} \circ F^* \Rightarrow \operatorname*{colim}_{\mathcal{J}}.$$

Taking total left derived functors, we obtain transformations

$$\operatorname*{hocolim}_{\mathcal{I}} \circ \operatorname{Ho} F^* \Rightarrow \mathbb{L}(\operatorname*{colim}_{\mathcal{I}} \circ F^*) \Rightarrow \operatorname*{hocolim}_{\mathcal{J}},$$

where the first one is induced by the units of the derived functors. Since the natural morphism into the strict colimit is nothing but the unit, we get that for every X, the square

$$\begin{tikzcd}
\operatorname*{hocolim}_{\mathcal{I}} X \circ F \arrow[r] \arrow[d] & \operatorname*{colim}_{\mathcal{I}} X \circ F \arrow[d] \\
\operatorname*{hocolim}_{\mathcal{J}} X \arrow[r] & \operatorname*{colim}_{\mathcal{J}} X
\end{tikzcd}$$

commutes up to homotopy (no matter how the derived functors are constructed).

(9.3) **Theorem. (Fibre Decomposition)** For every map $f\colon E \to B$ of simplicial sets and $\mathcal{B} := \int_{\mathbf{\Delta}} B$, the cocones $\big(F_f(b) \to E\big)_b$, $\big(\Delta[\dim b] \to B\big)_b$, together with the natural transformation $\big(F_f(b) \to \Delta[\dim b]\big)_b$ induce a commutative diagram

$$\begin{tikzcd}
\operatorname*{hocolim}_{b\in\mathcal{B}} F_f(b) \arrow[r, "\sim"] \arrow[d] & \operatorname*{colim}_{b\in\mathcal{B}} F_f(b) \arrow[r, "\cong"] \arrow[d] & E \arrow[d, "f"] \\
\operatorname*{hocolim}_{b\in\mathcal{B}} \Delta[\dim b] \arrow[r, "\sim"'] & \operatorname*{colim}_{b\in\mathcal{B}} \Delta[\dim b] \arrow[r, "\cong"'] & B,
\end{tikzcd}$$

where the weak equivalences are the units of the derived functor hocolim.

Proof. Since **sSets** is locally cartesian closed, the functor $E \times_B -\colon \mathbf{sSets}\downarrow B \to \mathbf{sSets}\downarrow E$ given by pulling back along f has a right adjoint and thus preserves colimits, yielding

$$\begin{aligned}
\operatorname*{colim}_{b\in\mathcal{B}} F_f(x) &= \operatorname*{colim}_{b\in\mathcal{B}}\big(E \times_B \Delta[\dim b]\big) \\
&\cong E \times_B \operatorname*{colim}_{b\in\mathcal{B}} \Delta[\dim b] \cong E \times_B B \cong E.
\end{aligned}$$

Note that the bottom weak equivalence in the diagram is a special case of the top weak equivalence, where $f = \mathrm{id}_B$ and so it suffices to show that the map at the top is a weak equivalence for every f, which we state as a lemma. □

(9.4) **Lemma.** In the above situation, the unit map

$$\operatorname*{hocolim}_{b\in\mathcal{B}} F_f(b) \to \operatorname*{colim}_{b\in\mathcal{B}} F_f(b) \cong E$$

is a weak equivalence.

We give two proofs for this lemma. The first one is our interpretation of the original proof [26], due to Dror Farjoun and assumes the base to be a simplicial complex (or more generally a non-singular simplicial set (2.8)). Since we imposed no additional conditions on the map f, one can always replace it by one with such a base. Our second proof works for a general base B. In the following proof, we will freely use the properties (2.14) of non-singular simplicial sets on several occasions.

Proof. (Dror Farjoun) We assume B to be non-singular, so that the inclusion $\mathcal{B}^* \hookrightarrow \mathcal{B}$ of the full subcategory of non-degenerate simplices is homotopy final by (2.22). That is to say, we have a commutative square

$$\begin{tikzcd}
\operatorname*{hocolim}_{b\in\mathcal{B}^*} F_f(b) \arrow[r] \arrow[d, "\sim"'] & \operatorname*{colim}_{b\in\mathcal{B}^*} F_f(b) \arrow[d, "\cong"] \\
\operatorname*{hocolim}_{b\in\mathcal{B}} F_f(b) \arrow[r] & \operatorname*{colim}_{b\in\mathcal{B}} F_f(b)
\end{tikzcd}$$

and it suffices to show that the map at the top is a weak equivalence.

For this, we are going to show that the diagram $F_f\colon \mathcal{B}^* \to \mathbf{sSets}$ is projectively cofibrant by applying the criterion (3.8) using free degeneracies. Fixing $(n, b) \in \mathcal{B}^*$ and a dimension m, a quick consideration leads one to "define" the non-degenerate parts

$$N_m(n, b) \subseteq F_f(n, b)_m = \{(\varphi, e) \mid \varphi\colon [m] \to [n],\, e \in E_m,\, f(e) = b \cdot \varphi\}$$

to consist of all (φ, e) with φ injective. For such an injective φ, it then follows that e must be non-degenerate since b is non-degenerate and B is non-singular. Note that these N_m are actually subpresheaves since the morphisms in $\mathcal{B}^*$ are all injections (see (2.5)). More generally, if $\sigma\colon [m] \twoheadrightarrow [k]$ is surjective, the image $N_\sigma(n,b)$ of $N_k(n,b)$ under the induced map $\sigma^*\colon F_f(n,b)_k \to F_f(n,b)_m$ is

$$N_k(n,b) \cong N_\sigma(n,b) \hookrightarrow F_f(n,b)_m,\ (\delta, e) \mapsto (\delta \circ \sigma, e \cdot \sigma).$$

All these images (with σ ranging over all surjections our of $[m]$) form a partition by Eilenberg-Zilber. Finally, to be able to apply (3.8), we need to check that the discrete simplicial presheaves $N_m\colon \mathcal{B}^* \to \mathbf{Sets} \hookrightarrow \mathbf{sSets}$ are projectively cofibrant.

This is easy since $\mathcal{B}^*$ is a poset by (2.15) and there is a bijective correspondence between injections $\delta\colon [m] \hookrightarrow [n]$ and non-degenerate m-simplices a such that $a \leqslant b$; namely, via $\delta \mapsto b \cdot \delta$ (which is well-defined because boundaries of non-degenerates in B are non-degenerate by non-singularity). Consequently,

$$N_m(n,b) \cong \coprod_{\substack{a \in B_m \\ \text{non-deg.}}} \coprod_{e \in f^{-1}(a)} \mathcal{B}^*\big((m,a),(n,b)\big),$$

meaning that $N_m \cong \coprod_{a,e} \mathcal{B}^*\big((m,a), -\big)$ is a coproduct of representables, which are projectively cofibrant. □

In the above proof, B being non-singular was essential and for a general simplicial set B, we cannot show projective cofibrancy of the diagram in question.

Proof. By the standard skeletal filtration of a simplicial set B, it suffices to check that the class $\mathcal{C}$ of all spaces B for which the lemma is true contains all $\Delta[n]$ and is closed under coproducts, homotopy pushouts and telescopes. The claim clearly holds for $B = \Delta[n]$ since $\int_{\mathbf{\Delta}} \Delta[n]$ has a terminal element (n, id) and the closure of $\mathcal{C}$ under coproducts is trivial.

Now, if $B = \operatorname{colim}(B_a \leftarrow B_b \rightarrowtail B_c)$ with $B_a,\ B_b,\ B_c \in \mathcal{C}$ and $B_b \rightarrowtail B_c$ monic, we pull back $f\colon E \to B$ along the universal maps $i_k\colon B_k \to B$, yielding maps $f_k\colon E_k \to B_k$ and by the pullback lemma $F_{f_k} \cong i_k^* F_f$ is just the restriction of F_f to $\mathcal{B}_k \subseteq \mathcal{B}$.

Next, we use that any category of elements functor (and in particular taking the category of simplices) $\int_{\mathcal{I}}\colon \mathbf{Sets}^{\mathcal{I}} \to \mathbf{Cat} \downarrow \mathcal{I}$ ($\mathcal{I}$ an arbitrary indexing category) preserves colimits because it is left adjoint to $\mathcal{C} \mapsto \mathrm{Fun}_{\mathcal{I}}(\mathcal{I} \downarrow -, \mathcal{C})$. In particular,

$$\mathcal{B} = \operatorname{colim}(\mathcal{B}_a \leftarrow \mathcal{B}_b \to \mathcal{B}_c)$$

is the pushout of the corresponding categories of simplices. Taking a projectively cofibrant replacement F_f^c of $F_f\colon \mathcal{B} \to \mathbf{sSets}$, we need to show that

$$\operatorname*{hocolim}_{\mathcal{B}} F_f = \operatorname*{colim}_{\mathcal{B}} F_f^c \to \operatorname*{colim}_{\mathcal{B}} F_f$$

is a weak equivalence. But we have written $\mathcal{B}$ as a colimit of categories and, as shown in (2.4.8), instead of taking this colimit of categories as our indexing category, we can just as well take the Grothendieck construction

$$\mathcal{B}' := \int^{\{a \leftarrow b \to c\}} (\mathcal{B}_a \leftarrow \mathcal{B}_b \to \mathcal{B}_c), \qquad \text{which comes with a final functor } \mathcal{B}' \to \mathcal{B}.$$

Using Thomason's theorem (for strict colimits), we now need to show that the map between pushouts induced by

$$\begin{tikzcd}
\operatorname*{colim}_{\mathcal{B}_a} F_f^c \arrow[d] & \operatorname*{colim}_{\mathcal{B}_b} F_f^c \arrow[l] \arrow[r] \arrow[d] & \operatorname*{colim}_{\mathcal{B}_c} F_f^c \arrow[d] \\
\operatorname*{colim}_{\mathcal{B}_a} F_f & \operatorname*{colim}_{\mathcal{B}_b} F_f \arrow[l] \arrow[r] & \operatorname*{colim}_{\mathcal{B}_c} F_f
\end{tikzcd}$$

is a weak equivalence. By point (a) in the lemma below, all the restrictions of F_f^c to the $\mathcal{B}_k$ are cofibrant and hence the downward arrows are all weak equivalences by our assumption that $B_k \in \mathcal{C}$. So, it suffices to show that the two pushouts are actually homotopy pushouts. For this, we note that the top map $\operatorname{colim}_{\mathcal{B}_b} F_f^c \to \operatorname{colim}_{\mathcal{B}_c} F_f^c$ is monic by point (b) of the lemma below and that the bottom map $\operatorname{colim}_{\mathcal{B}_b} F_f \to \operatorname{colim}_{\mathcal{B}_c} F_f$ is (isomorphic to) $E_b \to E_c$, which, by the pullback lemma, is the pullback of $B_b \rightarrowtail B_c$ along $E_c \to B_c$ and hence monic, too.

Finally, the case of a telescope $B \cong \operatorname{colim}(B_0 \hookrightarrow B_1 \hookrightarrow \ldots)$ is analogous, using that filtered colimits in **sSets** are actually homotopy colimits (5.2.7). □

(9.5) **Lemma.** Every map $f\colon E \to B$ of simplicial sets induces a functor $f\colon \mathcal{E} \to \mathcal{B}$ between the corresponding categories of simplices, yielding a precomposition functor

$$f^*\colon \mathcal{M}^{\mathcal{B}} \to \mathcal{M}^{\mathcal{E}}$$

for $\mathcal{M}$ any cofibrantly generated model category.

(a) The precomposition functor f^* is both left and right Quillen with respect to the projective model structures.

(b) If $X\colon \mathcal{B} \to \mathcal{M}$ is cofibrant in the projective model structure, then the induced map $\operatorname{colim}_{\mathcal{E}} f^*X \to \operatorname{colim}_{\mathcal{B}} X$ is a cofibration.

Proof. Ad (a): Every precomposition functor between projective model structures on diagram categories is right Quillen. To see that f^* is also left Quillen, we look at its right adjoint f_*, which is right Kan extension along f. Explicitly,

$$(f_*X)_{(n,b)} = \lim\Big((n,b)\downarrow f \to \mathcal{E} \xrightarrow{X} \mathcal{M}\Big),$$

where $b \in B_n$, $X\colon \mathcal{E} \to \mathcal{M}$ and $(n,b)\downarrow f \to \mathcal{E}$ is the standard projection. But note that in $(n,b)\downarrow f$, the full subcategory of all $\big((n,e),\mathrm{id}\big)$ with with $e \in f^{-1}(b) \subseteq E_n$ is initial and discrete. Discreteness is clear because if we have e, $e' \in f^{-1}(b)$ together with $\varphi\colon (n,e) \to (n',e')$ such that $\varphi \circ \mathrm{id} = \mathrm{id}$ then $\varphi = \mathrm{id}$. As for it being initial, given any $e \in E_m$ and $\xi\colon (n,b) \to (m, f(e))$, we must have $b = f(e)\cdot\xi = f(e\cdot\xi)$, meaning that $\xi\colon \big((n,e\cdot\xi),\mathrm{id}\big) \to \big((m,e),\xi\big)$ is a morphism in $(n,b)\downarrow f$. Moreover, if there is any other $e' \in f^{-1}(b)$ together with a morphism $\varphi\colon \big((n,e'),\mathrm{id}\big) \to \big((m,e),\xi\big)$ such that $\varphi \circ \mathrm{id} = \xi$, it immediately follows that $\varphi = \xi$. All in all, the right Kan extension is explicitly given by

$$(f_*X)_{(n,b)} = \lim\Big((n,b)\downarrow f \to \mathcal{E} \xrightarrow{X} \mathcal{M}\Big) = \prod_{e \in f^{-1}(b)} X_e.$$

Since (acyclic) fibrations in $\mathcal{M}$ are closed under products, it follows that f_* is right Quillen since (acyclic) fibrations in the projective model structure are pointwise.

Ad (b): Let's define the *collage* (or *cograph*) of f to be the Grothendieck construction

$\operatorname{Coll} f := \int^{[1]}(\mathcal{E} \to \mathcal{B})$ on f, viewed as a functor $[1] \to \mathbf{Cat}$. This not only comes with a standard projection $Q\colon \operatorname{Coll} f \to [1]$ (as always) but also with a target functor

$$T\colon \operatorname{Coll} f \to \mathcal{B},\ (0,e) \mapsto f(e),\ (1,b) \mapsto b,\ \varphi \mapsto \varphi$$

(note that this is well-defined on morphisms because an arrow $\varphi\colon (0,e) \to (1,b)$ is just an arrow $\varphi\colon f(e) \to b$ in $\mathcal{B}$). One easily verifies that the induced map $\operatorname{colim}_{\mathcal{E}} f^*X \to \operatorname{colim}_{\mathcal{B}} X$ for a diagram $X\colon \mathcal{B} \to \mathcal{M}$ is obtained as the image of X under the composite

$$\mathcal{M}^{\mathcal{B}} \xrightarrow{T^*} \mathcal{M}^{\operatorname{Coll} f} \xrightarrow{Q_!} \mathcal{M}^{[1]}.$$

Since cofibrant objects in $\mathcal{M}^{[1]}$ (with respect to the projective model structure) are just cofibrations in $\mathcal{M}$ with cofibrant domains and $Q_!$ is clearly left Quillen, it suffices to check that T^* is left Quillen, too. This proof is analogous to the one in (a) above and one finds

$$(T_*X)_b = \lim\left(b \downarrow T \to \operatorname{Coll} f \xrightarrow{X} \mathcal{M}\right) = \prod_{e \in f^{-1}b} X_{(0,e)},$$

which is right Quillen with respect to the projective model structure. □

(9.6) **Remark.** The author thinks that in the proof of point (b), the fact that T^* is left Quillen should be reducible to (a) by showing that $\operatorname{Coll} f$ is the category of simplices of the mapping cylinder of f but a direct verification seems easier.

As a corollary to the above fibre decomposition theorem (or rather the bottom row in the diagram from said theorem), we obtain the following classical result for free.

(9.7) **Corollary.** If $K \in \mathbf{sSets}$ and $\mathcal{K} := \int_{\boldsymbol{\Delta}} K$, then $\operatorname{hocolim}_{\mathcal{K}} * \simeq K$ and $K \simeq N(\mathcal{K})$.

Proof. The first claim is just $\operatorname{hocolim}_{\mathcal{K}} * \xleftarrow{\sim} \operatorname{hocolim}_{x \in \mathcal{K}} \Delta[\dim x] \xrightarrow{\sim} K$. For the second claim, we let $\mathcal{N}(\mathcal{K}) := \int_{\boldsymbol{\Delta}} N(\mathcal{K})$ and consider the source functor $\mathcal{N}(\mathcal{K})^{\mathrm{op}} \to \mathcal{K}$, which is homotopy final (4.22). With this

$$N(\mathcal{K}) \overset{(4.13)}{\simeq} \operatorname*{hocolim}_{N(\mathcal{K})^{\mathrm{op}}} * \simeq \operatorname*{hocolim}_{\mathcal{K}} * \simeq K.$$

□

Chapter 8

CLOSED CLASSES

1. Sets of Spaces

In our study of closed classes and their relation to homotopy excision, we shall often work with sets of spaces (i.e. sets of simplicial sets). For example, if M is any set of spaces, we will consider the closed class $\mathcal{C}(M)$ generated by M (see (2.5)). Classically, $\mathcal{C}(M)$ was only defined for M a single space, which is not really a restriction since one can show that $\mathcal{C}(M) = \mathcal{C}(\bigvee_{A\in M} A)$. However, this approach of just wedging the spaces together has several drawbacks.

First of all, all spaces in M must be required to be non-empty. This seems like an unproblematic restriction but can become a real nuisance. For example, if $f\colon A \to B$ is any map of simplicial sets, we will often have to consider (the closed class generated by) the collection of all homotopy fibres, even in contexts where there might be empty ones.

(1.1) **Notation.** There is usually no risk of confusion between a set of spaces and a single space and, for a space A, we will routinely write $\mathcal{C}(A)$ instead of $\mathcal{C}(\{A\})$. However, there is one notable exception. We will have to clearly distinguish between the empty simplicial set, which we denote by S^{-1}, and the empty collection of spaces because

$\mathcal{C}(S^{-1}) = \mathcal{C}(\{S^{-1}\})$ is the collection of all spaces,

while

$\mathcal{C}(\varnothing)$ is the collection of all contractible spaces.

For this reason, when working with closed classes, the empty simplicial set is *always* denoted by S^{-1} and $\varnothing$ will *always* refer to the empty set of spaces.

Another drawback of always replacing a set of spaces by its wedge is that some homotopical constructions that we might want to apply to every single space (such as loop spaces) will not commute with wedges.

As already indicated above, a natural point at which we need to switch from single spaces to sets of spaces is when considering homotopy fibres. For any map of spaces $f\colon E \to B$ and $b \in B$, we write

$$\mathrm{hFib}_b(f) := \operatorname{holim}\left(* \xrightarrow{b} B \xleftarrow{f} E\right)$$

for the homotopy fibre of f at b. If the base-point is clear from the context (or somehow induced), we sometimes just write $\mathrm{hFib}_*(f)$ instead. As long as B is connected, the homotopy type of this does not depend on the base-point and we allow ourselves to write $\mathrm{hFib}(f)$ in that situation. More generally, if we really want to consider all possible (homotopy types of) homotopy fibres, we define

$$\mathrm{h}\mathfrak{F}\mathrm{ib}(f) := \{\mathrm{hFib}_b(f) \mid b \in B\} \qquad \text{(really } b \in B_0\text{)},$$

where, again, it is enough to take one point b for every component of B. Consequently, as long as B is connected,

$$\mathrm{h}\mathfrak{F}\mathrm{ib}(f) \simeq \{\mathrm{hFib}(f)\}$$

(and we routinely identify this singleton with $\mathrm{hFib}(f)$), where two sets of spaces M, N are said to be *(weakly) equivalent* (and we write $M \simeq N$) iff every $X \in M$ is weakly equivalent to some $Y \in N$ and vice-versa.

(1.2) **Convention.** Whenever we write $\mathrm{hFib}(f)$ for some map f, it is implied that $\mathrm{h}\mathfrak{F}\mathrm{ib}(f)$ is weakly equivalent to a singleton (e.g. if the codomain is connected).

(1.3) **Example.** Obviously $\mathrm{h}\mathfrak{F}\mathrm{ib}(f) = \varnothing$ iff f is the identity $S^{-1} \to S^{-1}$.

(1.4) **Warning.** Given a homotopy pullback square

$$\begin{tikzcd} P \arrow[r, "f"] \arrow[d] & A \arrow[d, "p"] \\ B \arrow[r, "g"'] & C \end{tikzcd}$$

one often says that f and g have "the same homotopy fibre" and it is tempting to conclude that $\mathrm{h}\mathfrak{F}\mathrm{ib}(f) \simeq \mathrm{h}\mathfrak{F}\mathrm{ib}(g)$. But of course, we only have $\mathrm{hFib}_a(f) \simeq \mathrm{hFib}_{p(a)}(f)$ for every $a \in A$ and so, to conclude $\mathrm{h}\mathfrak{F}\mathrm{ib}(f) \simeq \mathrm{h}\mathfrak{F}\mathrm{ib}(g)$, we need that $\pi_0(p)\colon \pi_0(A) \to \pi_0(C)$ is surjective.

Finally, we will often perform homotopical constructions (that are defined for spaces) on sets of spaces and it is always understood that these should be performed elementwise. For example, given a set of spaces M, its suspension is just $\Sigma M := \{\Sigma A \mid A \in M\}$. Similarly given two sets of spaces M, N, their join is $M * N := \{A * B \mid A \in M, B \in N\}$.

2. Characterisation of Closed Classes

In this section, we quickly recall the definition of a closed class as well as equivalent formulations, which are often cited but seldom proved in full detail.

(2.1) **Definition.** A class $\mathcal{C} \neq \varnothing$ of non-empty simplicial sets is *closed* iff it is closed under pointed homotopy colimits. That is to say, if $X\colon \mathcal{I} \to \mathbf{sSets}_*$ is any pointed diagram of simplicial sets with $X_I \in \mathcal{C}$ for all $I \in \mathbf{I}$ and a simplicial set $Y \simeq \mathrm{hocolim}_{\mathbf{I}} X$ is a homotopy colimit of X then $Y \in \mathcal{C}$. In particular, $\mathcal{C}$ is required to be closed under weak equivalences. For convenience, if $\mathcal{C}$ contains S^{-1}, we define that $\mathcal{C}$ is closed iff it is the class of all simplicial sets.

(2.2) **Example.**

(a) Connectivity is expressible in terms of closed classes. To wit, for $n \in \mathbb{N} \cup \{-2, -1\}$, the closed class $\mathcal{C}(S^n)$ is the class of all n-connected spaces [26].

(b) If $n \in \mathbb{N}_{>1}$ and p any prime then $\mathcal{C}\big(M(\mathbb{Z}/p\mathbb{Z}, n)\big)$ consists of all $(n-1)$-connected spaces A such that $\pi_n(A)$ is generated by elements of order p and every $\pi_k(A)$ with $k > n$ is a p-group [6].

(c) Letting M be the set of all countable simplicial sets A with $\widetilde{H}(A;\mathbb{Z}) = 0$, the closed class $\mathcal{C}(M)$ consists of all spaces $X \neq \varnothing$ such that $\widetilde{H}(X;\mathbb{Z}) = 0$ [26].

Owing to the explicit formula for the calculation of homotopy colimits, obtained in section 7.4, it suffices to check the closedness of a class of simplicial sets under certain pointed homotopy colimits. More explicitly, combining (7.4.22) with (7.4.17), we have the following characterisation of a closed class.

(2.3) **Proposition.** A class $\mathcal{C} \neq \varnothing$ of non-empty simplicial sets is closed iff it is closed under weak equivalences, wedges and geometric realisations of bisimplicial sets. □

As always for geometric realisations, we have a skeletal filtration, where each succesive skeletion is obtained from the previous one by a homotopy pushout (since one of the morphisms in the defining span is monic). We could have alternatively done this directly for the Reedy structure on a category of simplices [46].

(2.4) **Proposition.** A class $\mathcal{C} \neq \varnothing$ of non-empty simplicial sets is closed iff it is closed under weak equivalences, wedges, homotopy pushouts and telescopes (meaning that for a diagram $X\colon \omega \to \mathbf{sSets}$, having $X_n \in \mathcal{C}$ for every n implies $\operatorname{colim}_\omega X \in \mathcal{C}$). □

(2.5) **Definition.** Given any set M of simplicial sets, we write $\mathcal{C}(M)$ for the smallest closed class containing it and the elements of $\mathcal{C}(M)$ are called *M-cellular*.

(2.6) **Observation.** While not every closed class $\mathcal{C}$ needs to be of the form $\mathcal{C}(M)$, still

$$\mathcal{C} = \bigcup_{A \in \mathcal{C}} \mathcal{C}(A) \qquad \text{for every closed class } \mathcal{C}.$$

We will mainly be concerned with the closed classes $\bar{\mathcal{C}}(A)$ that are also closed under extensions by fibrations and in section 5, we will see how they are characterised by left Bousfield localisations. As it turns out [33, Theorems 5.1.5 & 5.1.6], a similar characterisation is possible for the closed classes $\mathcal{C}(M)$. One can show, that if all spaces in M are non-empty then, the class $\mathcal{C}(M)$ coincides with the class of cofibrant objects in a right Bousfield localisation $R_M\mathbf{sSets}_*$ of $\mathbf{sSets}_*$.

(2.7) **Notation.** If $\mathcal{C}$ is a closed class and $\mathcal{I}$ any indexing category, we write $X\colon \mathcal{I} \to \mathcal{C}$ for a diagram $X\colon \mathcal{I} \to \mathbf{sSets}$ such that $X_I \in \mathcal{C}$ for every $I \in \mathcal{I}$. However, $\operatorname{hocolim}_{\mathcal{I}} X$ still needs to be understood as the homotopy colimit of X in $\mathbf{sSets}$ (and not the full subcategory corresponding to $\mathcal{C}$).

3. Properties of Closed Classes

In this section, we are going to collect some often-used elementary results about properties of closed classes. This is purely for convenience, so that we don't have to give references every time we use them later on.

(3.1) **Observation.** We have $\Delta[0] \in \mathcal{C}$ for every closed class $\mathcal{C}$ since this is the empty pointed homotopy colimit.

(3.2) **Proposition.** Every closed class $\mathcal{C}$ is closed under homotopy retracts; i.e. given $K \in \mathcal{C}$ and $i\colon A \to K$, $r\colon K \to A$ such that $r \circ i$ is weak equivalence then $A \in \mathcal{C}$.

Proof. Because **sSets** is an $\aleph_0$-combinatorial model category, filtered colimits are even homotopy colimits by (5.2.7). In particular

$$\operatorname{hocolim}\left(K \xrightarrow{ir} K \xrightarrow{ir} \dots\right) \simeq \operatorname{colim}\left(K \xrightarrow{ir} K \xrightarrow{ir} \dots\right) \cong \operatorname{colim}\left(A \xrightarrow{ri} A \xrightarrow{ri} \dots\right).$$

But this colimit is weakly equivalent to A because each π_n preserves filtered colimits and ri is a weak equivalence. □

For our next statement, we recall that for every simplicial set K (with category of simplices $\mathcal{K}$) and every pointed diagram $X\colon \mathcal{K}^{\mathrm{op}} \to \mathbf{sSets}_*$, the unpointed homotopy colimit $\operatorname{hocolim} X$ and the pointed homotopy colimit $\operatorname{hocolim}_* X$ fit into a cofibration sequence

$$K \to \operatorname*{hocolim}_{\mathcal{K}^{\mathrm{op}}} X \to \operatorname*{hocolim}_{\mathcal{K}^{\mathrm{op}}}{}_* X$$

as seen in (7.4.14). Combining this with the fact that for $\mathcal{I}$ any indexing category, the source functor $S\colon \mathcal{N}(\mathcal{I})^{\mathrm{op}} \to \mathcal{I}$ is homotopy final, we get a cofibration sequence

$$N(\mathcal{I}) \simeq \operatorname*{hocolim}_{\mathcal{I}} * \to \operatorname*{hocolim}_{\mathcal{I}} X \to \operatorname*{hocolim}_{\mathcal{I}}{}_* X$$

and in particular, if $\mathcal{I}$ is contractible, the pointed and unpointed homotopy colimit are weakly equivalent. With this, the following proposition follows. One is now tempted to say that closed classes are closed under (unpointed) homotopy colimits indexed by contractible categories. However, in the above cofibre sequence, the diagram X still needs to be pointed.

One of the insights of Chachólski in [10] is that one can in fact remove the restriction of the diagram needing to be pointed. Since, in *op. cit.* all spaces are assumed to be connected, we sketch the proof of this fact to make sure that one can do away with this restriction. For this, given a closed class $\mathcal{C}$ we define

$$D(\mathcal{C}) := \left\{K \in \mathbf{sSets} \;\middle|\; \operatorname*{hocolim}_{\mathcal{K}^{\mathrm{op}}} X \in \mathcal{C} \text{ for all } X\colon \mathcal{K}^{\mathrm{op}} \to \mathcal{C}\right\}.$$

Before sketching the necessary steps (with the proofs taken directly from *op cit.*), recall that for $n \in \mathbb{N}$ and $k \in \{0, \dots, n\}$, the k^{th} horn $\Lambda^k[n] \subseteq \Delta[n]$ is the simplicial subset generated by all $\delta^i\colon [n-1] \hookrightarrow [n]$ except $i = k$. More explicitly,

$$\Lambda^k[n]_m = \{\varphi\colon [m] \to [n] \mid \operatorname{Im}\varphi \neq [n] \text{ and } \operatorname{Im}\varphi \neq [n] \setminus \{k\}\}.$$

(3.3) **Lemma.** For every closed class $\mathcal{C}$:

(a) $\Delta[n] \in D(\mathcal{C})$ for all $n \in \mathbb{N}$;

(b) $D(\mathcal{C})$ is closed under homotopy pushouts: Given simplicial sets $L \hookleftarrow K \to M$ in $D(\mathcal{C})$ with $K \hookrightarrow L$ monic then also $\operatorname{colim}(L \hookleftarrow K \to M) \in D(\mathcal{C})$;

(c) $\Lambda^k[n] \in D(\mathcal{C})$ for all $n \in \mathbb{N}$, $k \in \{0, \dots, n\}$;

(d) if $f\colon K \to L$ is a weak equivalence and $K \in D(\mathcal{C})$ then also $L \in D(\mathcal{C})$.

Proof. The case where $\mathcal{C}$ consists of all simplicial sets is trivial and so, we assume $\varnothing \notin \mathcal{C}$.
Ad (a): The category of simplices $\mathbf{\Delta}[n] := \int_{\mathbf{\Delta}} \Delta[n]$ has a terminal object, namely $(n, \mathrm{id}_{[n]})$. Consequently, $\mathbf{\Delta}[n]^{\mathrm{op}}$ has an initial object and every diagram $X\colon \mathbf{\Delta}[n]^{\mathrm{op}} \to \mathcal{C}$ can be pointed.
Ad (b): Follows directly from (7.8.7).
Ad (c): The horn $\Lambda^k[n]$ is just the boundary $\partial\Delta[n]$ with its k^{th} face removed. Consequently, walking around the vertex k opposite to the removed face, $\Lambda^k[n]$ can be obtained by successively gluing n-many faces $\cong \Delta[n-1]$ together along a common subface $\cong \Delta[n-2]$, yielding a simplicial set X and finally identifying two subfaces $\cong \Delta[n-2]$ (from the first and the last face attached), which is just taking a colimit of the form

$$\Lambda^k[n] \cong \operatorname{colim}\big(\Delta[n-2] \leftarrow \Delta[n-2] \vee \Delta[n-2] \hookrightarrow X\big).$$

The claim now follows from (a) and (b).
Ad (d): If the map f is even an acyclic Kan fibration, the claim follows from (7.8.1) and the fact that acyclic Kan fibrations are stable under pullback. Otherwise, we use the small object argument (i.e. cellular approximation) with respect to the generating acyclic cofibrations $\{\Lambda^k[n] \hookrightarrow \Delta[n]\}_{n,k}$ to factor f into an acyclic cofibration $i\colon K \hookrightarrow M$, followed by an acyclic Kan fibration $p\colon M \twoheadrightarrow L$. By construction i is a transfinite composition

$$K = K_0 \hookrightarrow K_1 \hookrightarrow \ldots \hookrightarrow K_\alpha \hookrightarrow \ldots \hookrightarrow K_\kappa =: M$$

for some regular cardinal κ and every $K_{\alpha+1}$ is a pushout of the form $\Delta[n] \hookleftarrow \Lambda^k[n] \to K_\alpha$ (so that $K_\alpha \in D(\mathcal{C})$ implies $K_{\alpha+1} \in D(\mathcal{C})$ by the previous points). Now, either using (7.8.8) or the fact (2.4.9) that the Grothendieck construction for a telescope is actually homotopy final in the colimit in connection with Thomason's theorem, we conclude that

$$\operatorname*{hocolim}_{\mathcal{K}_\beta^{\mathrm{op}}} X \simeq \operatorname*{hocolim}_{\alpha<\beta} \operatorname*{hocolim}_{\mathcal{K}_\alpha^{\mathrm{op}}} X$$

for every $X\colon \mathcal{M}^{\mathrm{op}} \to \mathbf{sSets}$ and every limit ordinal $\beta \leqslant \kappa$ (and in particular for $K_\kappa = M$). Since $K = K_0 \in D(\mathcal{C})$ and we have already treated the successor case, it follows that $M \in D(\mathcal{C})$ since $\mathcal{C}$ is closed under telescopes. Finally, $L \in D(\mathcal{C})$ by our initial comment in the proof of this point. □

(3.4) **Proposition.** If $\mathcal{C}$ is a closed class, $\mathcal{I}$ a contractible indexing category and $X\colon \mathcal{I} \to \mathcal{C}$ then $\operatorname{hocolim}_{\mathcal{I}} X \in \mathcal{C}$.

Proof. We can assume that $\mathcal{I} = \mathcal{K}^{\mathrm{op}}$ is the dual category of simplices of a simplicial set K since the source functor $\mathcal{N}(\mathcal{I})^{\mathrm{op}} \to \mathcal{I}$ is homotopy final. Now any base point inclusion $* \to K$ is a weak equivalence and the claim follows from point (d) of the lemma. □

(3.5) **Remark.** In the original paper [10], Chachólski goes quite a bit further and in fact shows that $D(\mathcal{C}) \subseteq \mathcal{C}$ is itself a closed class, closed under extensions by fibrations (see (5.1) below). He also derives that this is another characterisation of closed classes. To wit, a collection of spaces $\mathcal{C}$ is a closed class iff it is closed under unpointed homotopy colimits indexed by contractible categories.

(3.6) **Corollary.** If K is a simplicial set with category of simplices $\mathcal{K}$, $\mathcal{C}$ is a closed class and $X\colon \mathcal{K}^{\mathrm{op}} \to \mathcal{C}$ then, for every vertex $k \in K$

$$\mathrm{hFib}_k\left(\operatorname*{hocolim}_{\mathcal{K}^{\mathrm{op}}} X \to K\right) \in \mathcal{C}$$

(where the map is the canonical one induced by $X \Rightarrow *$). More generally, if $\mathcal{I}$ is any indexing category and $X\colon \mathcal{I} \to \mathcal{C}$ then, for every $I \in \mathcal{I}$

$$\mathrm{hFib}_I \left(\operatorname*{hocolim}_{\mathcal{I}} X \to \operatorname*{hocolim}_{\mathcal{I}} * \xrightarrow{\simeq} \operatorname*{hocolim}_{\mathcal{N}(\mathcal{I})^{\mathrm{op}}} * \simeq N(\mathcal{I}) \right) \in \mathcal{C}.$$

Proof. We replace $k\colon \Delta[0] \to K$ by a fibration $p\colon PK \twoheadrightarrow K$ and use (7.8.1), by which

$$\mathrm{hFib}_k \left(\operatorname*{hocolim}_{\mathcal{K}^{\mathrm{op}}} X \to K \right) \simeq PK \times_K \operatorname*{hocolim}_{\mathcal{K}^{\mathrm{op}}} X \simeq \operatorname*{hocolim}_{(\mathcal{P}K)^{\mathrm{op}}} p^* X,$$

which is a homotopy colimit of a diagram in $\mathcal{C}$, indexed by a contractible category. □

4. Dror Farjoun's Theorem

A central result to bootstrap the entire calculus of cellular inequalities is Dror Farjoun's theorem. In its simplest form [26, Proposition 1.D.2] it reads as in (4.1). Even though, we are only going to introduce M-equivalences and periodisations in a later section, we feel like it fits better at this place and we are going to state a more general form for closed classes due to Chachólski later in the section.

This result can be understood as a generalisation of (3.6), which considers the homotopy fibres of maps $\mathrm{hocolim}_{\mathcal{K}^{\mathrm{op}}} X \to \mathrm{hocolim}_{\mathcal{K}^{\mathrm{op}}} * \simeq K$, induced by $X \Rightarrow *$. What Dror Farjoun's theorem does now is to replace the terminal diagram $*$ by an arbitrary one.

(4.1) **Theorem.** Let M be any set of spaces and $\varphi\colon E \Rightarrow B$ a natural transformation of diagrams E, $B\colon \mathcal{I} \to$ **sSets**. If every $\varphi_I\colon E_I \to B_I$ with $I \in \mathcal{I}$ is an M-equivalence then so is $\varphi_*\colon\ \mathrm{hocolim}_{\mathcal{I}} E \to \mathrm{hocolim}_{\mathcal{I}} B$.

Proof. Being an M-equivalence means being a weak equivalence in L_M**sSets** and since the identity functor id: **sSets** $\to L_M$**sSets** is left Quillen, it preserves homotopy colimits. Now, a weak equivalence φ between diagrams in L_M**sSets** is obviously mapped to a weak equivalence in L_M**sSets**. □

Combining this with the fact (6.3) that M-equivalences are exactly those maps whose homotopy fibres are killed by M, we get the following version of this theorem.

(4.2) **Corollary.** Let M be any set of spaces and $\varphi\colon E \Rightarrow B$ a natural transformation of diagrams E, $B\colon \mathcal{I} \to$ **sSets**. If $\mathrm{h}\mathfrak{F}\mathrm{ib}(\varphi_I) > M$ for all $I \in \mathcal{I}$ then also

$$\mathrm{h}\mathfrak{F}\mathrm{ib} \left(\operatorname*{hocolim}_{\mathcal{I}} E \xrightarrow{\varphi_*} \operatorname*{hocolim}_{\mathcal{I}} B \right) > M.$$

□

Next, we are going to repeat Chachólski's proof of the generalised Dror Farjoun theorem. The reason for this is that, in the original paper [10], all spaces were assumed to be connected, which is an unnecessary restriction that be need to remove for later purposes.

(4.3) **Lemma.** Given a closed class $\mathcal{C}$ and a natural transformation $\tau\colon X \Rightarrow Y$ of diagrams X, $Y\colon \mathcal{I} \to \mathcal{C}$, together with a map $f\colon\ \mathrm{hocolim}_{\mathcal{I}} X \to A$ having $A \in \mathcal{C}$, then also

$$\mathrm{hocolim} \left(\operatorname*{hocolim}_{\mathcal{I}} Y \xleftarrow{\tau_*} \operatorname*{hocolim}_{\mathcal{I}} X \xrightarrow{f} A \right) \in \mathcal{C}.$$

Proof. By Thomason's theorem, the homotopy pushout from the claim can be obtained as a single homotopy colimit over the Grothendieck construction of

$$\{a \leftarrow b \rightarrow c\} \rightarrow \mathbf{Cat}, \quad (a \leftarrow b \rightarrow c) \mapsto \left(\mathcal{I} \xleftarrow{\mathrm{Id}} \mathcal{I} \rightarrow *\right)$$

and by (3.4), it suffices to check that the resulting indexing category is contractible. But this is simple because the inclusion

$$\int^{\{b\to c\}} (\mathcal{I} \rightarrow *) \hookrightarrow \int^{\{a\leftarrow b\to c\}} (\mathcal{I} \leftarrow \mathcal{I} \rightarrow *)$$

has a right adjoint being the identity on $\{b\}\times\mathcal{I}$ and $\{(c,*)\}$, while mapping an $(a,I) \in \{a\}\times\mathcal{I}$ to (b,I). In particular, the (nerves of the) two categories are weakly equivalence and the left-hand one has a terminal object. □

Combining this lemma with the fibre decomposition (7.9.3), we obtain the following geometric version of the lemma.

(4.4) **Theorem. (Bundle Theorem)** Let $\mathcal{C}$ be a closed class, $f\colon E \to E'$ a morphism of spaces over a fixed base B and $g\colon E \to K$:

$$\begin{tikzcd} K & E \arrow[l, "g"'] \arrow[r, "f"] \arrow[dr, "p"'] & E' \arrow[d, "p'"] \\ & & B \end{tikzcd}$$

If $K \in \mathcal{C}$ and $\mathrm{h}\mathfrak{F}\mathrm{ib}(p)$, $\mathrm{h}\mathfrak{F}\mathrm{ib}(p') \subseteq \mathcal{C}$, then also $\operatorname{hocolim}(K \xleftarrow{g} E \xrightarrow{f} E') \in \mathcal{C}$.

Proof. We by fibrant replacement, we can assume that p and p' are Kan fibrations. To wit, we first factor p' as an acyclic cofibration, followed by a fibration $p'\colon E' \overset{\sim}{\rightarrowtail} \bar{E}' \twoheadrightarrow B$. Next, we factor $E \to E' \to \bar{E}'$ into an acyclic cofibration, followed by a fibration, yielding $E \overset{\sim}{\rightarrowtail} \bar{E} \twoheadrightarrow \bar{E}'$. And finally, we replace K fibrantly by $K \overset{\sim}{\rightarrowtail} \bar{K}$ and choose a lift as in the following diagram (which exists because $\bar{K}$ is fibrant)

$$\begin{tikzcd} K \arrow[d, tail, "\sim"'] & E \arrow[l, "g"'] \arrow[r, "f"] \arrow[d, tail, "\sim"'] & E' \arrow[d, tail, "\sim"'] \\ \bar{K} & \bar{E} \arrow[l, dotted] \arrow[r, two heads] \arrow[dr, two heads] & \bar{E}' \arrow[d, two heads] \\ & & B \end{tikzcd}$$

Now the homotopy pushouts of the two rows agree and we can indeed assume that p and p' are Kan fibrations. With this, the diagrams F_p, $F_{p'}\colon \mathcal{B}^{\mathrm{op}} \to \mathbf{sSets}$ of the fibre decomposition (7.9.3) consist, respectively, of the homotopy fibres of p and p' and the map f lifts to a morphism of diagrams $F_p \Rightarrow F_{p'}$, so that the diagram from the theorem becomes

$$\begin{tikzcd} K \arrow[d, equal] & \operatorname*{hocolim}_{\mathcal{B}^{\mathrm{op}}} F_p \arrow[l] \arrow[r] \arrow[d, "\sim"'] & \operatorname*{hocolim}_{\mathcal{B}^{\mathrm{op}}} F_{p'} \arrow[d, "\sim"'] \\ K & E \arrow[l, "g"'] \arrow[r, "f"] \arrow[dr, "p"'] & E' \arrow[d, "p'"] \\ & & B \end{tikzcd}$$

The claim now follows directly from the previous lemma. □

The bundle theorem has all kinds of special cases and consequences, which we use all the time. The following list of immediate corollaries is taken directly from [10].

(4.5) **Corollary.** Let $\mathcal{C}$ be closed class.

(a) If $p'\colon E' \to B$ is a map of simplicial sets that has a section $s\colon B \to E'$ and such that $K \in \mathcal{C}$ as well as $\mathrm{h}\mathfrak{F}\mathrm{ib}(p') \subseteq \mathcal{C}$, then

$$\operatorname{hocolim}\left(K \xleftarrow{g} B \xrightarrow{s} E'\right) \in \mathcal{C} \qquad \text{for every map } g\colon B \to K.$$

(b) Given two spaces $p\colon E \to B$, $p'\colon E' \to B$ over a fixed base B and a morphism $f\colon E \to E'$ of spaces over B, if $\mathrm{h}\mathfrak{F}\mathrm{ib}(p)$, $\mathrm{h}\mathfrak{F}\mathrm{ib}(p') \subseteq \mathcal{C}$ and $E \in \mathcal{C}$ then also $E' \in \mathcal{C}$.

(c) Closed classes are closed under split extensions: If $p\colon E \to B$ is a map of simplicial sets with a section, $\mathrm{h}\mathfrak{F}\mathrm{ib}(p) \subseteq \mathcal{C}$ and $B \in \mathcal{C}$, then also $E \in \mathcal{C}$.

(d) Closed classes contain base spaces: If $p\colon E \to B$ is a map of simplicial sets such that $\mathrm{h}\mathfrak{F}\mathrm{ib}(p) \subseteq \mathcal{C}$ and $E \in \mathcal{C}$ then also $B \in \mathcal{C}$.

(e) If $p\colon E \to B$ is a map of simplicial sets with B connected and $\mathrm{hFib}(p)$, $\Omega B \in \mathcal{C}$, then also $E \in \mathcal{C}$.

Proof. Ad (a): Take $E := B$, $p := \mathrm{id}_B$ and $f := s$ in the theorem.
Ad (b): Take $K := E$ and $g := \mathrm{id}_E$ in the theorem.
Ad (c): Follows from (a) for $K := B$ and $g := \mathrm{id}_B$.
Ad (d): Follows from (b) for $E' := B$, $p' := \mathrm{id}_B$ and $f := p$.
Ad (e): Apply (d) to the fibre sequence $\Omega B \to \mathrm{hFib}(p) \to E$. Note that the connectivity of B is necessary for this statement to be useful because if B is not connected, $b \in B$ and p hits some other connected component, then some homotopy fibres of $\mathrm{hFib}_b(p) \to E$ are empty and we would need $\mathcal{C}$ to be trivial. □

When looking at the details of the fibre decomposition, one might be tempted to think that one could replace the homotopy fibres in the above theorem and all of its corollaries by strict fibres and one would be half-right! However, the corresponding result would not be very useful, since, first off, in the strict case, we not only need to consider fibres but all possible pullbacks along $b\colon \Delta[n] \to B$ (not necessarily $n = 0$). Also, since we need to take all "fibres" into account, as soon as our maps are not surjective, the closed class $\mathcal{C}$ must be trivial since at least one fibre is going to be empty.

More importantly, having the strict pullbacks in a closed class is actually a stronger condition than having the homotopy fibres in it, as the following observation shows.

(4.6) **Proposition.** Given a closed class $\mathcal{C}$ and a map $f\colon E \to B$ of simplicial sets, if

$$F_f(b) := \lim\left(E \xrightarrow{f} B \xleftarrow{b} \Delta[\dim b]\right) \in \mathcal{C}$$

for every simplex $b \in B$ then also $\mathrm{h}\mathfrak{F}\mathrm{ib}(f) \subseteq \mathcal{C}$.

Proof. By the fibre decomposition theorem (7.9.3), the map f is weakly equivalent to the canonical projection $\operatorname{hocolim}_{\mathcal{B}^{\mathrm{op}}} F_f \to B$ and by hypothesis, F_f is a diagram in $\mathcal{C}$. The claim now follows from (3.6). □

As so often, following [10], we are going to prove Dror Farjoun's theorem by induction over the indexing space, to reduce it to the cases of homotopy pushouts and telescopes. For this, let us state the homotopy pushout case as a lemma.

(4.7) **Lemma.** Consider a closed class $\mathcal{C}$ together with a commutative diagram of simplicial sets as on the left with its right square a pullback, i, j cofibrations

$$\begin{tikzcd} E_1 \arrow[d, "p_1"'] & E_2 \arrow[l, "g"'] \arrow[r, tail, "j"] \arrow[d, "p_2"'] \arrow[dr, phantom, "\lrcorner", very near start] & E_3 \arrow[d, "p_3"'] \\ B_1 & B_2 \arrow[l, "f"] \arrow[r, tail, "i"'] & B_3 \end{tikzcd} \qquad \text{and pushouts} \qquad \begin{tikzcd} E \arrow[d, "p"'] \\ B. \end{tikzcd}$$

If $\mathrm{h}\mathfrak{F}\mathrm{ib}(p_1)$, $\mathrm{h}\mathfrak{F}\mathrm{ib}(p_2)$, $\mathrm{h}\mathfrak{F}\mathrm{ib}(p_3) \subseteq \mathcal{C}$, then also $\mathrm{h}\mathfrak{F}\mathrm{ib}(p\colon E \to B) \subseteq \mathcal{C}$.

Proof. By fibrant replacement, we can assume that all the p_k are Kan fibrations and by (4.6) above, it suffices to check that

$$F_p(b) := \lim\left(E \xrightarrow{p} B \xleftarrow{b} \Delta[\dim b]\right) \in \mathcal{C}$$

for every simplex $b \in B = \operatorname{colim}(B_1 \leftarrow B_2 \to B_3)$. Fixing such a simplex $b \in B$, it lies in the image of B_1 or B_3. In the case of B_1, we note that by (7.8.10) together with the pullback hypothesis from the claim, the square

$$\begin{tikzcd} E_1 \arrow[r, tail] \arrow[d, two heads, "p_1"'] & E \arrow[d, "p"] \\ B_1 \arrow[r, tail] & B \end{tikzcd}$$

is a pullback and so (since $b\colon \Delta[\dim b] \to B$ factors through B_1) $F_p(b) \simeq \mathrm{hFib}_b(p_1) \in \mathcal{C}$. The other case, where b lies in the image of B_3 is a little more complicated. For the sake of simplicity, we rename $b\colon \Delta[\dim b] \to B_3$ to $h\colon A \to B_3$ (in fact, the following construction works for an arbitrary such map) and form pullbacks

$$\begin{tikzcd} K \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "h"] \\ B_2 \arrow[r, tail] & B_3, \end{tikzcd} \quad \begin{tikzcd} F_3 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E_3 \arrow[d, two heads, "p_3"] \\ A \arrow[r, "h"'] & B_3, \end{tikzcd} \quad \begin{tikzcd} F_2 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E_2 \arrow[d, two heads, "p_2"] \\ K \arrow[r] & B_2, \end{tikzcd} \quad \begin{tikzcd} F_1 \arrow[rr] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & & E_1 \arrow[d, two heads, "p_1"] \\ K \arrow[r] & B_2 \arrow[r, "f"'] & B_1. \end{tikzcd}$$

Explicitly, making the usual identifications of subobjects with their images, we can identify

$$K \cong h^{-1}(B_2), \quad F_3 \cong \{(a, e_3) \in A \times E_3 \mid h(a) = p_3(e_3)\}$$

$$F_2 \cong \left\{(a, e_2) \in h^{-1}(B_2) \times E_2 \;\middle|\; h(a) = p_2(e_2)\right\},$$

$$F_1 \cong \left\{(a, e_1) \in h^{-1}(B_2) \times E_1 \;\middle|\; (f \circ h)(a) = p_1(e_1)\right\}.$$

Now, since, by definition, $E \cong (E_1 + E_3)/(g(e_2) \sim e_2)$ and writing $[b_2]$ for the image of some $b_2 \in B_2$ in the pushout B, the pullback

$$\lim\left(E \xrightarrow{p} B \xleftarrow{h} A\right) \cong \{(e, a) \in E \times A \mid [h(a)] = p(e)\}$$

is isomorphic to the pushout

$$\operatorname{colim}(F_1 \leftarrow F_2 \rightarrowtail F_3) \cong \frac{F_1 + F_3}{(a, g(e_2)) \sim (a, e_1)}.$$

In our case, where $h\colon A \to B_3$ is $b\colon \Delta[\dim b] \to B_3$, we note that $F_3 \simeq \mathrm{hFib}_b(p_3) \in \mathcal{C}$ and that $\mathrm{h}\mathfrak{F}\mathrm{ib}(F_k \to K) \subseteq \mathrm{h}\mathfrak{F}\mathrm{ib}(p_k)$ for $k \in \{1, 2\}$. Consequently, the diagram $F_1 \leftarrow F_2 \to F_3$ satisfies the hypotheses of the Bundle Theorem and we conclude that its colimit (which is a homotopy colimit since $F_2 \rightarrowtail F_3$ is a cofibration) $F_p(b)$ lies in $\mathcal{C}$ as well. □

With this lemma, we can finally prove Chachólski's generalisation of Dror Farjoun's theorem. By abuse of language, we will also refer to this generalisation as "Dror Farjoun's theorem".

(4.8) **Theorem. (Dror Farjoun)** Let $\mathcal{C}$ be a closed class and $\varphi\colon E \Rightarrow B$ a natural transformation of diagrams $E, B\colon \mathcal{I} \to \mathbf{sSets}$. If $\mathrm{h}\mathfrak{F}\mathrm{ib}(\varphi_I\colon E_I \to B_I) \subseteq \mathcal{C}$ for every $I \in \mathcal{I}$ then also

$$\mathrm{h}\mathfrak{F}\mathrm{ib}\left(\operatorname*{hocolim}_{\mathcal{I}} E \xrightarrow{\varphi_*} \operatorname*{hocolim}_{\mathcal{I}} B\right) \subseteq \mathcal{C}.$$

Proof. As always, by precomposing with the homotopy final source functor $S\colon \mathcal{N}(\mathcal{I})^{\mathrm{op}} \to \mathcal{I}$, we can assume that $\mathcal{I} = \mathcal{K}^{\mathrm{op}}$, where $\mathcal{K} = \int_{\mathbf{\Delta}} K$ for some simplicial set K and that E, B are bounded. By skeletal decomposition, it suffices to check that the class of all spaces K for which the claim holds contains $\Delta[0]$ and is closed under coproducts, attaching n-simplices and telescopes. The case $K = \Delta[0]$ follows easily from (7.6.3) and the closure under coproducts is trivial.

Now, assume that $K = \operatorname{colim}(L \leftarrow \partial\Delta[n] \hookrightarrow \Delta[n])$ is obtained from a space L for which the claim holds by attaching an n-simplex. Using our explicit model $|-|_{\mathcal{K}}$ for the homotopy colimit and writing $\iota_n \in \Delta[n]_n$ for the unique non-degenerate simplex, our explicit formula (7.8.12) tells us that we have

$$\begin{array}{ccc} |E|_{\mathcal{K}} & \cong & \operatorname{colim}\Big(|E|_{\mathcal{L}} \longleftarrow \partial\Delta[n] \times E_{\iota_n} \rightarrowtail \Delta[n] \times E_{\iota_n}\Big) \\ \downarrow{\scriptstyle \varphi_*} & & \quad\ \downarrow{\scriptstyle \varphi_*} \qquad\qquad \downarrow{\scriptstyle \varphi_*} \qquad\qquad \downarrow{\scriptstyle \varphi_*} \\ |B|_{\mathcal{K}} & \cong & \operatorname{colim}\Big(|B|_{\mathcal{L}} \longleftarrow \partial\Delta[n] \times B_{\iota_n} \rightarrowtail \Delta[n] \times B_{\iota_n}\Big) \end{array}.$$

By our assumption for L and the hypotheses from the claim, the three homotopy fibres of the morphism between spans lie in $\mathcal{C}$ and hence, by the lemma above, so do the homotopy fibres of the map between pushouts.

Finally, for telescopes, if $K = \operatorname{colim}_{\alpha<\kappa} K_\alpha$ for some transfinite sequence $(K_\alpha)_{\alpha<\kappa}$ with κ a regular cardinal and the claim true for every K_α, then, by (7.8.8), the map $|E|_{\mathcal{K}} \to |B|_{\mathcal{K}}$ can be obtained as the colimit

$$\begin{array}{ccc} |E|_{\mathcal{K}} & \cong & \operatorname{colim}_\kappa\Big(|E|_{\mathcal{K}_0} \longrightarrow |E|_{\mathcal{K}_1} \longrightarrow \dots\Big) \\ \downarrow & & \quad\ \downarrow \qquad\qquad \downarrow \\ |B|_{\mathcal{K}} & \cong & \operatorname{colim}_\kappa\Big(|B|_{\mathcal{K}_0} \longrightarrow |X|_{\mathcal{K}_1} \longrightarrow \dots\Big) \end{array}$$

(where $\mathcal{K}_\alpha := \int_{\mathbf{\Delta}} K_\alpha$). Again, we can assume every $|E|_{\mathcal{K}_\alpha} \to |B|_{\mathcal{K}_\alpha}$ to be a fibration, which won't change the homotopy type of the (filtered!) colimits since, by (5.2.7), these are actually homotopy colimits. Now, since $\Delta[0]$ is finitely presentable, every vertex $b\colon \Delta[0] \to |B|_{\mathcal{K}}$ factors through some $|B|_{\mathcal{K}_\alpha}$ and we get a sequence of cospans

$$\beta \mapsto \left(\Delta[0] \xrightarrow{b} |B|_{\mathcal{K}_\beta} \leftarrow |E|_{\mathcal{K}_\beta}\right).$$

Taking pullbacks and using that finite limits in **sSets** commute with filtered colimits, we find

$$\mathrm{hFib}_b(|E|_{\mathcal{K}} \to |B|_{\mathcal{K}}) \simeq \operatorname*{colim}_{\alpha \leqslant \beta < \kappa} \mathrm{hFib}_b(|E|_{\mathcal{K}_\beta} \to |B|_{\mathcal{K}_\beta}),$$

which lies in $\mathcal{C}$ by hypothesis and the closure of $\mathcal{C}$ under telescopes. □

One very important corollary of Dror Farjoun's theorem, which we are going to use time and again is the following theorem due to Chachólski.

(4.9) **Theorem. (Chachólski)** Given a closed class $\mathcal{C}$ and a homotopy pushout square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & B \arrow[d, "j"] \\ C \arrow[r, "g"'] & D, \end{tikzcd}$$

if $\mathrm{h}\mathfrak{F}\mathrm{ib}(f) \subseteq \mathcal{C}$, then also $\mathrm{h}\mathfrak{F}\mathrm{ib}(g) \subseteq \mathcal{C}$.

Proof. We can express g as an induced map between homotopy pushouts

$$\begin{tikzcd} C \arrow[dd, "g"'] & \simeq & \mathrm{hocolim}\,\big(C & A \arrow[l, "i"'] \arrow[r, "\mathrm{id}"] & A\big) \\ & & & & \\ D & \simeq & \mathrm{hocolim}\,\big(C \arrow[from=uu, "\mathrm{id}"'] & A \arrow[from=uu, "\mathrm{id}"'] \arrow[l, "i"] \arrow[r, "f"'] & B\big) \arrow[from=uu, "i"'] \end{tikzcd}$$

and the claim follows immediately from Dror Farjoun's theorem. □

5. Closed Classes and Fibrations

(5.1) **Definition.** We say that a closed class $\mathcal{C}$ is *closed under (extensions by) fibrations* iff for every fibre sequence $F \to E \to B$ (with respect to an arbitrary base-point in B), having $F, B \in \mathcal{C}$ implies $E \in \mathcal{C}$. With this, given any set of simplicial set M, we write $\bar{\mathcal{C}}(M)$ for the smallest closed class that contains M and is closed under extensions by fibrations. This is called the *Bousfield class* of M and its elements are called the *M-acyclic* spaces.

(5.2) **Remark.** It may seem strange that we only require $F \in \mathcal{C}$ for a single homotopy fibre in the above definition. However, since we also require $B \in \mathcal{C}$, the above definition is actually equivalent to the condition that for every $f\colon E \to B$, if $B \in \mathcal{C}$ and $\mathrm{h}\mathfrak{F}\mathrm{ib}(f) \subseteq \mathcal{C}$, then also $E \in \mathcal{C}$. Indeed, if B is connected or $B = S^{-1}$ the two conditions are obviously equivalent. If $B \neq S^{-1}$ is not connected, then $\mathcal{C}$ contains at least all non-empty spaces since $B \in \mathcal{C}$, so that the case $E \neq S^{-1}$ is trivial. Finally, if $E = S^{-1}$ then $F = S^{-1}$ for every homotopy fibre F of $E \to B$ and so, if $F \in \mathcal{C}$, then also $E \in \mathcal{C}$.

(5.3) **Definition.** Given any set M of simplicial sets, we can form the left Bousfield localisation $L_M\mathbf{sSets} := L_S\mathbf{sSets}$ at the set $S := \{A \hookrightarrow CA \mid A \in M\}$ (where every CA is contractible) and write $P_M\colon \mathbf{sSets} \to \mathbf{sSets}$ for a fibrant replacement functor in $L_M\mathbf{sSets}$ (an explicit construction is given in section 6.5). This functor is called *M-nullification* or *M-periodisation.*

(5.4) **Example.** If M is any set of simplicial sets then the class $\mathcal{C}$ of all simplicial sets that become contractible after passing to the left Bousfield localisation $L_M\mathbf{sSets}$ is a closed class, closed under fibrations. Indeed, the identity functor $\mathbf{sSets} \to L_M\mathbf{sSets}$ is left Quillen and hence preserves homotopy colimits. But a pointed homotopy colimit of contractible objects is contractible, so that $\mathcal{C}$ is closed. Finally, if $F \to E \to B$ is a fibre sequence (in $\mathbf{sSets}$) with F, $B \in \mathcal{C}$, we use that the identity functor the other way $L_M\mathbf{sSets} \to \mathbf{sSets}$ is right Quillen, and in particular preserves homotopy fibres. But then, F is also the homotopy fibre of $E \to B$ in $L_M\mathbf{sSets}$ and since B is contractible there, this means that E and F are weakly equivalent (i.e. S-locally) and in particular, E is contractible in $L_M\mathbf{sSets}$.

The characterisation (5.6) of Bousfield classes below can be found in [10, Corollary 17.3] though there, it is only stated for M being a singleton and the proof involves A-cellular approximation functors CW_A, which ours does not. To prove it, we need a small lemma, where we, unfortunately, need to look more closely at the construction of the periodisation functor P_M done in 6.5.

Explicitly, the functor P_M is obtained by the small object argument with respect to a set S_f of maps, which is a union of three sets:

(a) the original set $S = \{A \rightarrowtail CA \mid A \in M\}$;

(b) a set of generating acyclic cofibrations for $\mathbf{sSets}$, such as $\left\{\Lambda^k[n] \rightarrowtail \Delta[n] \;\middle|\; n,\, k\right\}$;

(c) the set of pushout products

$$\left\{\Delta[n] \times A +_{\partial\Delta[n]\times A} \partial\Delta[n] \times CA \rightarrowtail \Delta[n] \times CA \;\middle|\; A \in M,\, n \in \mathbb{N}\right\}.$$

What is important for us is that we understand the homotopy type of the pushout products in the last set. In fact, it is just $\Sigma^n A \to *$ as shown in (6.1.13).

Finally, for an ordinal α, we write $P_{M,\alpha}\colon \mathbf{sSets} \to \mathbf{sSets}$ for the α^{th} stage in the construction of P_M. That is to say, it is S_f-cellular approximation of length α, with the functor P_M itself being $P_{M,\kappa}$ for a sufficiently large regular cardinal κ.

(5.5) **Lemma.** For every set M of simplicial sets and every $B \in \mathbf{sSets}$

(a) $\mathrm{h}\mathfrak{F}\mathrm{ib}\big(P_{M,\alpha}(B) \to P_{M,\alpha+1}(B)\big) \subseteq \mathcal{C}(M)$ for all α;

(b) $\mathrm{h}\mathfrak{F}\mathrm{ib}\big(B \to P_{M,\alpha}(B)\big) \subseteq \bar{\mathcal{C}}(M)$ for all α.

Proof. Ad (a): The successor $P_{M,\alpha+1}(B)$ is defined as the pushout

$$\begin{tikzcd}
\coprod_{s,f} A_s \arrow[r, "{[f]_{s,f}}"] \arrow[d, tail, "\amalg_{s,f} s"'] & P_{M,\alpha}(B) \arrow[d, tail] \\
\coprod_{s,f} B_s \arrow[r] & P_{M,\alpha+1}(B) \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd},$$

where $s\colon A_s \to B_s$ ranges over all elements of S_f and, for a fixed such s, f ranges over all morphisms $A_s \to P_{M,\alpha}(B)$. By Chachólski's theorem then

$$\mathrm{h}\mathfrak{F}\mathrm{ib}(P_{M,\alpha}(B) \to P_{M,\alpha+1}(B)) \gg \bigcup_{s\in S_f} \mathrm{h}\mathfrak{F}\mathrm{ib}(s).$$

But the homotopy fibres of morphisms in S_f are either some $A \in M$ (for $A \rightarrowtail CA$) or contractible (for the generating acyclic cofibrations) or some $\Sigma^n A$ with $A \in M$ (for the pushout products). Since all of these lie in $\mathcal{C}(M)$, the claim follows.
Ad (b): The case $\alpha = 0$ is trivial. In the successor case, we form the the fibre sequence associated to the composable pair of maps

$$B \xrightarrow{f} P_{M,\alpha}(B) \xrightarrow{g} P_{M,\alpha+1}(B),$$

which, together with point (a), shows that $\mathrm{hFib}_{g(x)}(B \to P_{M,\alpha+1}(B))$ for every base point $x \in P_{M,\alpha}(B)$. If there is a base point $y \in P_{M,\alpha+1}(B)$, whose component is not hit by g, then $\mathrm{h}\mathfrak{F}\mathrm{ib}(P_{M,\alpha}(B) \to P_{M,\alpha+1}(B))$ contains S^{-1} and hence $\mathcal{C}(M)$ is the class of all spaces by (a), making the claim trivial.

Finally, for α a limit ordinal, we use that in **sSets** filtered colimits are homotopy colimits and commute with homotopy fibres. More explicitly, we can again assume that all components of $P_{M,\alpha}(B)$ are hit because $* \in$ **sSets** is finitely presentable and so, every $* \to P_{M,\alpha}(B)$ factors through some $P_{M,\beta}(B)$ with $\beta < \alpha$. Hence, if not all components of $P_{M,\alpha}(B)$ are hit then this is already the case for some $P_{M,\beta}(B)$ with $\beta < \alpha$ and it follows from the inductive hypothesis that $\bar{\mathcal{C}}(M)$ is the class of all spaces.

Now, the strict fibre of the composite $B \to P_{M,\alpha}(B)$ above a base point coming from B is given by the pullback of colimits

$$\operatorname*{colim}_{\beta<\alpha} B \longrightarrow \operatorname*{colim}_{\beta<\alpha} P_{M,\alpha}(B) \longleftarrow \operatorname*{colim}_{\beta<\alpha} *,$$

where the two outer diagrams are constant. By right properness of **sSets**, it suffices to replace $* \Rightarrow P_{M,-}(B)$ by a pointwise fibration, to get the homotopy fibres and pullbacks commute with filtered colimits, meaing that

$$\mathrm{hFib}(B \to P_{M,\alpha}(B)) \simeq \operatorname*{colim}_{\beta<\alpha} \mathrm{hFib}(B \to P_{M,\beta}(B))$$

(still everything above a base point coming from b) and the claim follows from the inductive hypothesis and the closure of $\bar{\mathcal{C}}(M)$ under telescopes. □

(5.6) **Proposition.** For every set M of simplicial sets, the following classes are the same:

(a) $\bar{\mathcal{C}}(M)$;

(b) the class of all simplicial sets that become contractible after passing to L_M**sSets**;

(c) $\bar{\mathcal{C}}'(M) := \{B \in \mathbf{sSets} \mid P_M B \simeq *\}$.

Proof. That the last two classes agree follows from (6.3.3), using that $P_M B$ is just a fibrant replacement of B in L_M**sSets**. Concerning the other equality, by definition, $M \in \bar{\mathcal{C}}'(M)$, which is a closed class, closed under fibrations (as seen in example (5.4) above), whence $\bar{\mathcal{C}}(M) \subseteq \bar{\mathcal{C}}'(M)$. For the reverse inclusion, we take $B \in \bar{\mathcal{C}}'(M)$ and apply the above lemma to $P_M B = P_{M,\kappa}(B)$ for a sufficiently large regular cardinal κ, by which then

$$\mathrm{h}\mathfrak{F}\mathrm{ib}(B \to P_M B) \simeq \mathrm{h}\mathfrak{F}\mathrm{ib}(B \to *) \simeq \{B\} \subseteq \bar{\mathcal{C}}(M).$$

□

(5.7) **Corollary.** For every set M of simplicial sets, given a space X, any fibre F of the M-local fibrant replacement $r\colon X \to P_M X$ is M-acyclic (i.e. $P_M F \simeq *$ or equivalently $F \in \bar{\mathcal{C}}(M)$).

Proof. By point (b) of the lemma (5.5) above, applied to a sufficiently large regular cardinal, we get that $F \in \bar{\mathcal{C}}(M)$, which, by the proposition, just means that $P_M F \simeq *$. □

(5.8) **Remark.** For a general left Bousfield localisation (with respect to any set of cofibrations S), the conclusion of this corollary does not hold. In fact, one can show that this is a distinguishing property of periodisation functors! That is to say, a set S of cofibrations is of the form $S = \{A \rightarrowtail CA \mid A \in M\}$ for some set of spaces M iff $R_S\,\mathrm{h}\mathfrak{F}\mathrm{ib}(X \to R_S X) \simeq *$ for every space X.

6. Fibrewise Localisation

A map of simplicial sets is a weak equivalence iff all its homotopy fibres are contractible and we ask if the same is true in L_S**sSets** for a set of maps S. For this, we are going to use the fibre decomposition (7.9.3) to construct the so-called *fibrewise localisation*, which will give us the required result.

For this entire section, we fix a set S of cofibrations of simplicial sets and a functorial fibrant replacement functor R_S for the left Bousfield localisation L_S**sSets**. If not specified differently, all homotopical constructions (in particular homotopy colimits and homotopy fibres) are to be understood in **sSets**. Recall from (6.3.3) that a map $K \to L$ of simplicial sets is an S-equivalence (i.e. a weak equivalence in L_S**sSets**) iff its image under the functorial S-local fibrant replacement $R_S K \to R_S L$ is a weak equivalence (in **sSets**). In particular, since every weak equivalence is also an S-equivalence, it follows that R_S preserves weak equivalences.

The following construction is taken almost verbatim from Dror Farjoun's book [26], where it is done for more general coaugmented functors (with the same proof).

(6.1) **Theorem. (Fibrewise Localisation)** To every Kan fibration $p\colon E \twoheadrightarrow B$, we can functorially assign a map of spaces over B

$$\begin{tikzcd} E \arrow[rr] \arrow[dr, two heads, "p"'] & & \bar{E} \arrow[dl, "\bar{p}"] \\ & B & \end{tikzcd}$$

such that

(a) the map $E \to \bar{E}$ is an S-equivalence and

(b) $\mathrm{hFib}_b(\bar{p}) \simeq R_S\,\mathrm{hFib}_b(p)$ for every base point $b \in B$.

Proof. Using the fibre decomposition (7.9.3), we have a commutative diagram

$$\begin{tikzcd} \tilde{E} := \mathrm{hocolim}_{b\in\mathcal{B}}\, R_S F_p(b) \arrow[d] & E' := \mathrm{hocolim}_{b\in\mathcal{B}}\, F_p(b) \arrow[l, "\sim_S"'] \arrow[r, "\sim"] \arrow[d] & E \arrow[d, two heads, "p"] \\ B'' := \mathrm{hocolim}_{b\in\mathcal{B}}\, R_S\Delta[\dim b] & B' := \mathrm{hocolim}_{b\in\mathcal{B}}\, \Delta[\dim b] \arrow[l, "\sim"] \arrow[r, "w", "\sim"'] & B, \end{tikzcd}$$

where the leftwards pointing arrows are induced by the (pointwise) S-local fibrant replacement of the corresponding diagrams. The bottom arrow pointing to the left is a weak equivalence because every contractible space is S-local and in particular, every $\Delta[n] \to R_S\Delta[n]$ is a weak equivalence (it is an S-equivalence between S-local fibrant objects). Rather than a weak equivalence, the top arrow pointing to the left is an S-equivalence because it is induced by an S-equivalence of diagrams and $\mathrm{id}\colon \mathbf{sSets} \to L_S\mathbf{sSets}$ preserves homotopy colimits (being left Quillen).

Our next goal is to get a diagram over a constant base B, for which we first take the homotopy pullback in the left-hand square above. Explicitly, we factor $\tilde{E} \to B''$ into an acyclic cofibration, followed by a fibration, say $\tilde{E} \overset{\sim}{\rightarrowtail} \tilde{E}' \twoheadrightarrow B''$ and form the homotopy pullback $\tilde{E}'' := \operatorname{colim}(\tilde{E}' \twoheadrightarrow B'' \leftarrow B')$, which fits into the diagram

$$\begin{tikzcd}
\tilde{E} \arrow[d, tail, "\sim"'] & E' \arrow[l, "\sim_S"'] \arrow[r, "\sim"] \arrow[d, "\sim_S"] & E \arrow[dd, two heads, "p"] \\
\tilde{E}' \arrow[d, two heads] & \tilde{E}'' \arrow[l, "\sim"'] \arrow[d, two heads] & \\
B'' & B' \arrow[l, "\sim"] \arrow[r, "w", "\sim"'] & B.
\end{tikzcd}$$

Finally, we simply take the homotopy pushout of the diagram $\tilde{E}'' \leftarrow E' \to E$ over B ($\tilde{E}''$ is a space over B by composing $\tilde{E}'' \twoheadrightarrow B'$ with w). Explicitly, we factor $E' \to \tilde{E}''$ into a cofibration (which is an S-equivalence by 2-out-of-3), followed by an acyclic fibration, yielding $E' \overset{\sim_S}{\rightarrowtail} \tilde{E}''' \overset{\sim}{\twoheadrightarrow} \tilde{E}''$. With this, we get a commutative diagram

$$\begin{tikzcd}
E' \arrow[r, "\sim"] \arrow[d, tail, "\sim_S"'] & E \arrow[d, tail, "\sim_S"] \arrow[ddr, bend left, two heads, "p"] & \\
\tilde{E}''' \arrow[r, "\sim"'] \arrow[d, two heads, "\sim"'] \arrow[ur, phantom, "\ulcorner"] & \bar{E} \arrow[dr, dotted] & \\
\tilde{E}'' \arrow[r, two heads] & B' \arrow[r, "\sim"'] & B
\end{tikzcd}$$

Since $\bar{E} \to B$ is connected to $\tilde{E} \to B''$ by a zig-zag of weak equivalences, the two maps have the same homotopy fibres. But since p is a fibration, $F_p\colon \mathcal{B} \to \mathbf{sSets}$ (and hence R_SF_p since R_S preserves weak equivalences) is a diagram of weak equivalences. Using Quillen's theorem B [26, p.186], it follows that the homotopy fibre of $\tilde{E} \to B''$ above (a point corresponding to the component of) $b \in B$ is weakly equivalent to $R_SF_p(b)$. □

Before proving that we can check S-equivalences by looking at their homotopy fibres, we need a small lemma to compare homotopy fibres in $L_S\mathbf{sSets}$ with those in $\mathbf{sSets}$. For this, let's introduce the notation hFib_b^S for the homotopy fibre in $L_S\mathbf{sSets}$ above a point b (while hFib_b still refers to the usual one in $\mathbf{sSets}$).

(6.2) **Lemma.** Let $f\colon E \to B$ be a map of simplicial sets.

(a) If E and B are S-local and fibrant, then the homotopy fibres in $L_S\mathbf{sSets}$ and $\mathbf{sSets}$ agree (i.e. they are weakly equivalent in $\mathbf{sSets}$).

(b) Writing $r\colon B \to R_SB$ for the S-local fibrant replacement and letting $b \in B$ be an arbitrary base point,

$$\mathrm{hFib}_b^S(f) \simeq \mathrm{hFib}_{r(b)}(R_Sf) \simeq R_S\,\mathrm{hFib}_b(f).$$

(c) In case where S is of the form $S = \{A \rightarrowtail CA \mid A \in M\}$ for some set of spaces M (so that $R_S = P_M$ is M-periodisation), we also have

$$\mathrm{h}\mathfrak{F}\mathrm{ib}(P_M f) \simeq P_M\, \mathrm{h}\mathfrak{F}\mathrm{ib}(f).$$

Proof. Ad (a): In $L_S\mathbf{sSets}$, we factor $b\colon * \to B$ into an S-acyclic cofibration $i\colon * \sim_S PB$, followed by a fibration $p\colon PB \to B$. Since B is S-local and fibrant (i.e. fibrant in $L_S\mathbf{sSets}$), so is PB and since S-equivalences between S-local fibrant objects are weak equivalences, $* \simeq PB$. Since $PB \twoheadrightarrow B$ is a fibration in $L_S\mathbf{sSets}$, it is also a Kan fibration and so, the pullback of f along it is both (weakly equivalent to) the homotopy fibre in $L_S\mathbf{sSets}$ and $\mathbf{sSets}$. Note that since $L_S\mathbf{sSets}$ need not be right proper, we really have to require E to be S-local and fibrant.

Ad (b): Since $f \simeq_S R_S f$, the two maps have the same homotopy fibres in $L_S\mathbf{sSets}$. More explicitly,

$$\mathrm{hFib}^S_b(f) \simeq \mathrm{hFib}^S_{r(b)}(R_S f) \simeq \mathrm{hFib}_{r(b)}(R_S f),$$

where the first weak equivalence stems from the fact that homotopy fibres can always be assumed to be fibrant (and so S-equivalences between them are weak equivalences) and the second weak equivalence is by point (a). For the other claim, we assume that f is a Kan fibration and form the fibrewise localisation $\bar{f}$ as above. We then get a commutative diagram (where the top row consists of the corresponding homotopy fibres)

$$\begin{tikzcd}
\mathrm{hFib}_b(f) \arrow[r, "\sim_S"] \arrow[d] & R_S\, \mathrm{hFib}_b(f) \arrow[r, "\sim_S"] \arrow[d] & \mathrm{hFib}^S_b(\bar{f}) \arrow[d] \\
E \arrow[r, "\sim_S"] \arrow[d, two heads, "f"'] & \bar{E} \arrow[r, "\sim_S"] \arrow[d, "\bar{f}"] & R_S\bar{E} \arrow[d, "R_S\bar{f}"] \\
B \arrow[r, equal] & B \arrow[r, "\sim_S"] & R_S B.
\end{tikzcd}$$

Since $f \simeq_S \bar{f}$ (as shown by the diagram), $\mathrm{hFib}^S_b(f) \simeq_S \mathrm{hFib}^S_b(\bar{f})$ is an S-equivalence between S-local fibrant objects, it is actually a weak equivalence. By the same argument, $R_S\, \mathrm{hFib}_b(f) \simeq_S \mathrm{hFib}^S_b(\bar{f})$ is actually a weak equivalence and the claim follows.

Ad (c): The case where $S^{-1} \in M$ is trivial because then S^{-1} is the only S-local fibrant object and hence $P_M X = S^{-1}$ for every X. So let's assume that $S^{-1} \notin M$. From (5.7), we know that $\mathrm{h}\mathfrak{F}\mathrm{ib}(B \to P_M B) \subseteq \bar{\mathcal{C}}(M)$ and in particular (since $S^{-1} \notin \bar{\mathcal{C}}(M)$), all homotopy fibres of $B \to P_M B$ are non-empty. But this means that all components of $P_M B$ are hit and the claim follows from point (b). □

(6.3) **Proposition.** If a map of simplicial sets $f\colon E \to B$ is an S-equivalence then all its homotopy fibres $\mathrm{hFib}_b(f)$ with $b \in B$ become contractible in $L_S\mathbf{sSets}$ (i.e. $R_S\, \mathrm{hFib}_b(f) \simeq *$). In case S is of the form $S = \{A \rightarrowtail CA \mid A \in M\}$ for some set of spaces M (so that $R_S = P_M$ is M-periodisation), the converse is also true.

Proof. Writing $r\colon B \to R_S B$ for an S-local fibrant replacement, if f is an S-equivalence then $R_S f$ is a weak equivalence and so

$$R_S\, \mathrm{hFib}_b(f) \simeq \mathrm{hFib}_{r(b)}(R_S f) \simeq *.$$

In case $R_S = P_M$ is a periodisation functor, we even have

$$\mathrm{h}\mathfrak{F}\mathrm{ib}(P_M f) \simeq P_M\, \mathrm{h}\mathfrak{F}\mathrm{ib}(f)$$

and so, $P_M f$ is a weak equivalence (i.e. f is an M-equivalence) iff $P_M\, \mathrm{hFib}_b(f) \simeq *$ for all base-points $b \in B$. □

7. Cellular and Acyclic Inequalities

A significant amount of the strength of the approach to (unstable) homotopy theory through closed classes and Bousfield localisations stems from calculus of the cellular and acyclic inequalities, which we are going to introduce in this section. We are then going to state a few useful standard results. Since we have already established the more involved results (such as Dror Farjoun's theorem) necessary, it should be easy enough to follow the proofs of these standard results present in the current literature.

(7.1) **Definition.** Recall that given any set M of simplicial sets, $\mathcal{C}(M)$ is the smallest closed class containing it, while $\bar{\mathcal{C}}(M)$ is the smallest closed class, closed under extensions by fibrations containing M. Now, for N any other set of simplicial sets, we write

(a) $N \gg M$ for $\mathcal{C}(N) \subseteq \mathcal{C}(M)$ (or equivalently $N \subseteq \mathcal{C}(M)$) and then say that M *constructs* N (or, as before, that [every element of] N is M-cellular);

(b) $N > M$ for $\bar{\mathcal{C}}(N) \subseteq \bar{\mathcal{C}}(M)$ (or equivalently $N \subseteq \bar{\mathcal{C}}(M)$) and then say that M *kills* N (or, as before, that [every element of] N is M-acyclic).

(7.2) **Observation.** By definition, cellular and acyclic inequalities are transitive, meaning that if $N \gg M \gg L$ then also $N \gg L$ (and similarly for $>$). Also note that, again by definition, $N \gg M$ implies $N > M$.

(7.3) **Example.** Recall that $\mathcal{C}(S^{-1}) = \bar{\mathcal{C}}(S^{-1})$ is the class of all spaces, so that $X \gg S^{-1}$ and $X > S^{-1}$ are true for every X. Slightly more restrictive, if $A \neq S^{-1}$ is non-connected then $\mathcal{C}(A) = \bar{\mathcal{C}}(A)$ consists of all non-empty spaces. Consequently, the inequality $X \gg A$ (or $X > A$) just means that $X \neq S^{-1}$.

(7.4) **Example.** More generally, recall that $\mathcal{C}(S^{n+1}) = \bar{\mathcal{C}}(S^{n+1})$ is the class of all n-connected spaces, so that $X \gg S^{n+1}$ (or $X > S^{n+1}$) just means that X is n-connected.

(7.5) **Example.** In these terms, Dror Farjoun's theorem (4.8) says that given a natural transformation $\varphi\colon E \Rightarrow B$ of diagrams $E, B\colon \mathcal{I} \to \mathbf{sSets}$, then

$$\mathrm{h}\mathfrak{F}\mathrm{ib}\left(\operatorname*{hocolim}_{\mathcal{I}} E \xrightarrow{\varphi_*} \operatorname*{hocolim}_{\mathcal{I}} B\right) > \bigcup_{I \in \mathcal{I}} \mathrm{h}\mathfrak{F}\mathrm{ib}(\varphi_I\colon E_I \to B_I).$$

(7.6) **Example.** Similarly, Chachólski's theorem (4.9) is that given a homotopy pushout

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[d] \\ C \arrow[r, "g"'] & D, \end{tikzcd}$$

we have $\mathrm{h}\mathfrak{F}\mathrm{ib}(g) > \mathrm{h}\mathfrak{F}\mathrm{ib}(f)$.

(7.7) **Example.** In terms of acyclic inequalities, the closure of $\bar{\mathcal{C}}(M)$ under extensions by fibrations just means that for every fibre sequence $F \to E \to B$, we have $E > \{F, B\}$. Alternatively, as remarked in (5.2), it can also be expressed as $E > \mathrm{h}\mathfrak{F}\mathrm{ib}(f) \cup \{B\}$.

An oft-used trick when studying cellular or acyclic inequalities is that every composable pair of maps $f\colon A \to B$ and $g\colon B \to C$ and every base-point $b \in B$ give rise to a fibre sequence

$$\mathrm{hFib}_b(f) \to \mathrm{hFib}_{g(b)}(g \circ f) \to \mathrm{hFib}_{g(b)}(g).$$

Consequently, we have $\mathrm{hFib}_{g(b)}(g \circ f) > \{\mathrm{hFib}_b(f), \mathrm{hFib}_{g(b)}(g)\}$. Alternatively, we can avoid choosing base-points and then have the following result.

(7.8) **Proposition.** Given a pair of composable maps $A \xrightarrow{f} B \xrightarrow{g} C$, then

(a) $\mathrm{h}\mathfrak{F}\mathrm{ib}(g \circ f) > \mathrm{h}\mathfrak{F}\mathrm{ib}(f) \cup \mathrm{h}\mathfrak{F}\mathrm{ib}(g)$ and

(b) $\mathrm{h}\mathfrak{F}\mathrm{ib}(f) > \mathrm{h}\mathfrak{F}\mathrm{ib}(g \circ f) \cup \Omega\, \mathrm{h}\mathfrak{F}\mathrm{ib}(g)$.

Proof. Ad (a): If $g\colon B \to C$ hits all components of C, the claim follows easily from the pointed version. Otherwise, we have $S^{-1} \in \mathrm{h}\mathfrak{F}\mathrm{ib}(g)$ and the inequality is trivial.
Ad (b): This is even simpler because we have a fibration sequence

$$\Omega_* \,\mathrm{hFib}_{g(b)}(g) \to \mathrm{hFib}_b(f) \to \mathrm{hFib}_{g(b)}(g \circ f)$$

for every $b \in B$ and since $\Omega_* X \gg \Omega X$ for every pointed X (by definition of Ω)

$$\mathrm{h}\mathfrak{F}\mathrm{ib}(f) > \Big\{\Omega\, \mathrm{hFib}_{g(b)}(g) \;\Big|\; b \in B\Big\} \cup \Big\{\mathrm{hFib}_{g(b)}(g \circ f) \;\Big|\; b \in B\Big\}.$$ □

As a special case of the fibre sequence associated to a composable pair of maps, consider a map $s\colon A \to B$ with a retraction $r\colon B \to A$. The associated fibre sequence with respect to $b \in B$ is then

$$\mathrm{hFib}_b(s) \to * \xrightarrow{s_*} \mathrm{hFib}_{r(b)}(r).$$

Beware that s need not be pointed and the induced map $s_*\colon * \to \mathrm{hFib}_{r(b)}(r)$ need not be the base-point inclusion. Still, as long as $\mathrm{hFib}_{r(b)}(r)$ is connected, the above fibre sequence just says that $\mathrm{hFib}_b(s) \simeq \Omega_* \,\mathrm{hFib}_{r(b)}(r)$. If we want to remove the base-point and work with sets of spaces instead, we obtain the following result.

(7.9) **Proposition.** Given $s\colon A \to B$ with a retraction $r\colon B \to A$ then

(a) $\mathrm{h}\mathfrak{F}\mathrm{ib}(r) \gg \mathrm{h}\mathfrak{F}\mathrm{ib}(s)$ and (b) $\mathrm{h}\mathfrak{F}\mathrm{ib}(s) \gg \Omega\, \mathrm{h}\mathfrak{F}\mathrm{ib}(r)$.

Moreover, if $\pi_0(r)$ is injective, then

(c) every homotopy fibre of r is connected and $\mathrm{h}\mathfrak{F}\mathrm{ib}(s) \simeq \Omega\, \mathrm{h}\mathfrak{F}\mathrm{ib}(r)$.

Proof. First note that $\pi_0(r) \circ \pi_0(s) = \mathrm{id}$ and in particular, $\pi_0(r)$ is surjective, so that

$$\mathrm{h}\mathfrak{F}\mathrm{ib}(r) \simeq \Big\{\mathrm{hFib}_{r(b)}(r) \;\Big|\; b \in B\Big\}.$$

Ad (c): Having sections, $\pi_1(r)$ and $\pi_0(r)$ are surjective and fit into a long exact sequence

$$\pi_1(B,b) \xrightarrow{\pi_1(r)} \pi_1(A, r(b)) \xrightarrow{0} \pi_0(\mathrm{hFib}_{r(b)}(r)) \to \pi_0(B) \xrightarrow{\pi_0(r)} \pi_0(A)$$

for every base-point $b \in B$. In particular, if $\pi_0(r)$ is injective, it follows that $\mathrm{hFib}_{r(b)}(r)$ is connected, and so $\mathrm{hFib}_b(s) \simeq \Omega_* \,\mathrm{hFib}_{r(b)}(r) \simeq \Omega\, \mathrm{hFib}_{r(b)}(r)$.

Ad (a): If $A' \subseteq A$ is a component, we get $A' \overset{s'}{\rightarrowtail} r^{-1}(A') \xrightarrow{r'} A'$ induced by s and r. Since $r^{-1}(A')$ is just a union of some components of B, $\mathrm{h}\mathfrak{F}\mathrm{ib}(s') \subseteq \mathrm{h}\mathfrak{F}\mathrm{ib}(s)$ and we get all possible homotopy fibres of r by running through all components A'. Showing the claim for each component A' individually, we can assume that A is connected. Moreover, we can also assume that B is connected because otherwise $S^{-1} \in \mathrm{h}\mathfrak{F}\mathrm{ib}(s)$ and the claim is trivial. With this, $\pi_0(r)$ is injective and (c) gives $\mathrm{hFib}(s) \simeq \Omega\,\mathrm{hFib}(r) \ll \mathrm{hFib}(r)$.

Ad (b): We can assume that r is injective (whence bijective) on π_0 because otherwise, $\mathrm{h}\mathfrak{F}\mathrm{ib}(r)$ contains some non-connected space and hence $S^{-1} \in \Omega\,\mathrm{h}\mathfrak{F}\mathrm{ib}(r)$, making the claim trivial. But in this case, point (c) even gives us a weak equivalence, rather than a cellular inequality. □

(7.10) **Example.** As the easy example $s\colon * \rightarrowtail S^0$ some base-point and $r\colon S^0 \to *$, we cannot expect more than the cellular inequality (b) above in the general case because here, we have

$$\mathrm{h}\mathfrak{F}\mathrm{ib}(s) = \{S^{-1}, *\} \qquad \text{while} \qquad \mathrm{h}\mathfrak{F}\mathrm{ib}(r) = \{S^0\}.$$

Consequently, $\Omega\,\mathrm{h}\mathfrak{F}\mathrm{ib}(r) = \{S^{-1}\}$, which is not weakly equivalent to $\mathrm{h}\mathfrak{F}\mathrm{ib}(s)$.

Chapter 9

HOMOTOPY EXCISION FOR SQUARES

In this chapter, we are going to show an "acyclic" version of the classical Blakers-Massey theorem [4] with generalisations due to Brown and Loday [9], Elis and Steiner [24] and with its most well-known formulation maybe being the one due to Goodwillie [30]. What follows is essentially just a slightly modified version of our article [16].

1. Total Fibres, Pushout Fibres and the James Map

In this section, we are going to introduce some basics that we will need for the remainder of this chapter. For this, let's fix a commutative square of spaces

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "h"] \\ C \arrow[r, "k"'] & D. \end{tikzcd}$$

(1.1) **Definition.** Given such a square with homotopy pullback $P := \mathrm{holim}(B \to D \leftarrow C)$ and pushout $Q := \mathrm{hocolim}(B \leftarrow A \to C)$, the square's *total fibre set* is the homotopy fibre set

$$\mathfrak{T} := \mathrm{h}\mathfrak{F}\mathrm{ib}(A \to P)$$

of the comparison map with the homotopy pullback, while the *pushout fibre set* is the homotopy fibre set

$$\mathfrak{R} := \mathrm{h}\mathfrak{F}\mathrm{ib}(Q \to D)$$

of the comparison map with the homotopy pushout. If $\mathfrak{T}$ happens to consist only of a single (homotopy type of a) space (e.g. if P is connected), we simply speak of the square's *total fibre* and denote it by a non-fraktur symbol T. Similarly for a single *pushout fibre*.

There is a classical result (sometimes called the *Cohen-Moore-Neisendorfer Lemma*), which allows one to calculate the total fibre of a square as the homotopy fibre of the induced map between horizontal (or vertical) homotopy fibres. However, when the spaces in question are not connected, one needs to take more care.

(1.2) **Lemma.** Given a commutative square of spaces

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "h"] \\ C \arrow[r, "k"'] & D \end{tikzcd}$$

together with base-points $b \in B$, $c \in C$ such that $h(b) = k(c) =: d$ (e.g. both coming from a base-point in A), then

$$\mathrm{hFib}_*(A \to \mathrm{holim}(B \to D \leftarrow C)) \simeq \mathrm{hFib}_*(\mathrm{hFib}_b(f) \to \mathrm{hFib}_d(k))$$

(all with respect to the base-points induced by b and c).

Proof. This follows from Fubini, applied to the commutative diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "h"'] & * \arrow[l, "b"'] \arrow[d] \\ C \arrow[r, "k"] & D & * \arrow[l, "d"'] \\ C \arrow[u, "\mathrm{id}"] \arrow[r, "\mathrm{id}"] & C \arrow[u, "k"'] & * \arrow[l, "c"'] \arrow[u] \end{tikzcd} .$$

□

In the unpointed context, when trying to establish a weak equivalence between homotopy fibre sets, some additional hypotheses (such as the following one) are needed.

(1.3) **Lemma.** If, in the above square, B and D are connected, the total fibre set is weakly equivalent to the homotopy fibre set of the induced map $\mathrm{hFib}(A \to B) \to \mathrm{hFib}(C \to D)$.

Proof. Writing $P := \mathrm{holim}(B \to D \leftarrow C)$ for the homotopy pullback, we observe that $\mathrm{hFib}(P \to B)$ and $\mathrm{hFib}(C \to D)$ are weakly equivalent and the induced square:

$$\begin{tikzcd} \mathrm{hFib}(A \to B) \arrow[r] \arrow[d] & A \arrow[d] \\ \mathrm{hFib}(C \to D) \arrow[r] & P \end{tikzcd}$$

is a homotopy pullback. The map $\mathrm{hFib}(C \to D) \to P$ induces a surjection on the sets of components and every choice of a base-point in $\mathrm{hFib}(C \to D)$ determines a base point in P. Consequently, this new square's vertical homotopy fibres over these base points are weakly equivalent, which proves the lemma. □

Given any space A, the James map $A \to \Omega\Sigma A$ is simply the unit of the suspension-loop adjunction. Of course, this is only well-defined (up to homotopy) for $A \neq S^{-1}$ but with our convention about loops, this extends trivially to the case $A = S^{-1}$.

There is another way of producing such a map $A \to \Omega\Sigma A$. Namely, we can first construct the (unreduced) suspension ΣA as the homotopy pushout

$$\begin{tikzcd} A \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & \Sigma A \end{tikzcd}$$

and then take the corresponding homotopy pullback, which yields $\Omega\Sigma A$. Again, this is compatible with our convention about loop spaces because if $A = S^{-1}$ then the homotopy pullback of $* \to \Sigma A \leftarrow *$ is going to be S^{-1} again. If $A \neq S^{-1}$ then ΣA is connected and the homotopy pullback is the usual loop space. We are now going to show (in a derivatoresque manner) that these two maps are actually the same.

(1.4) **Lemma.** Consider the homotopy pushout square

$$\begin{tikzcd} A \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & \Sigma A \end{tikzcd}$$

and the homotopy pull-back $\Omega\Sigma A$. In this case, the comparison map $\eta\colon A \to \Omega\Sigma A$ is the James map; i.e. the unit of the suspension-loop adjunction.

Proof. Following [31, Proposition 3.17], the (derived) adjunction $\Sigma \dashv \Omega$ can be constructed as follows. Writing $\mathbf{sSets}_*$ for the category of pointed spaces, $\ulcorner = \{\{0\} \leftarrow \varnothing \rightarrow \{1\}\}$, $\lrcorner = \{\{0\} \rightarrow \{0,1\} \leftarrow \{1\}\}$ and $\square = \mathfrak{P}\{0,1\}$, the inclusions

$$\varnothing\colon * \hookrightarrow \ulcorner, \quad \{0,1\}\colon * \hookrightarrow \lrcorner, \quad i_\ulcorner\colon \ulcorner \hookrightarrow \square, \quad i_\lrcorner\colon \lrcorner \hookrightarrow \square$$

induce adjunctions

$$\mathbf{sSets}_* \underset{\varnothing_*}{\overset{\varnothing^*}{\leftrightarrows}} \mathbf{sSets}_*^\ulcorner \underset{i_\ulcorner{}^*}{\overset{i_{\ulcorner !}}{\rightleftarrows}} \mathbf{sSets}_*^\square \underset{i_{\lrcorner *}}{\overset{i_\lrcorner{}^*}{\rightleftarrows}} \mathbf{sSets}_*^\lrcorner \underset{\{0,1\}^*}{\overset{\{0,1\}_!}{\leftrightarrows}} \mathbf{sSets}_*,$$

where $-^*$ denotes precomposition, $-_*$ right Kan extension, and $-_!$ left Kan extension. Note that $\varnothing_*\colon A \mapsto (* \leftarrow A \rightarrow *)$, $\{0,1\}_!\colon A \mapsto (* \rightarrow A \leftarrow *)$ and $i_{\ulcorner !}$, $i_{\lrcorner *}$ are given by respectively completing pushout and pull-back diagrams to squares. All these adjunctions can be derived (the first and the last one even consist of homotopy functors) and when replacing $\mathrm{Ho}(\mathbf{sSets}_*^\ulcorner)$, $\mathrm{Ho}(\mathbf{sSets}_*^\square)$ and $\mathrm{Ho}(\mathbf{sSets}_*^\lrcorner)$ by the corresponding full subcategories that have contractible objects at the corners $\{0\}$ and $\{1\}$ (which we indicate by adding an asterisk to the category's name), we get

$$\Sigma\colon \mathrm{Ho}(\mathbf{sSets}_*) \simeq \mathrm{Ho}(\mathbf{sSets}_*^\ulcorner)^* \underset{i_\ulcorner{}^*}{\overset{i_{\ulcorner !}}{\rightleftarrows}} \mathrm{Ho}(\mathbf{sSets}_*^\square)^* \underset{i_{\lrcorner *}}{\overset{i_\lrcorner{}^*}{\rightleftarrows}} \mathrm{Ho}(\mathbf{sSets}_*^\lrcorner)^* \simeq \mathrm{Ho}(\mathbf{sSets}_*) : \Omega.$$

Now, because $i_\ulcorner$ is fully faithful, the unit of $i_{\ulcorner !} \dashv i_\ulcorner{}^*$ is an isomorphism (cf. [31, Proposition 1.20]) and hence the composite adjunction's unit is just the unit of $i_\lrcorner{}^* \dashv i_{\lrcorner *}$ (composed with some isomorphisms), which is exactly the comparison from the claim. □

(1.5) **Lemma.** For any X, $\mathrm{h}\mathfrak{F}\mathrm{ib}(\eta\colon X \to \Omega\Sigma X) > \Omega X * \Omega X$.

Proof. The case where X is connected has been proved in [12, Theorem 7.2]. If X is not connected, then the lemma is vacuously true by our convention that then $\Omega X = S^{-1}$. □

2. Main Theorem and Examples

Even though we are going to prove a more general version of the acyclic Blakers-Massey theorem for arbitrary commutative squares, the most important case is that of a homotopy pushout.

(7.1) **Theorem.** Given a homotopy pushout square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

with total fibre set $\mathfrak{T}$, then

$$\mathfrak{T} > \Omega\, \mathrm{h}\mathfrak{F}\mathrm{ib}(f) * \Omega\, \mathrm{h}\mathfrak{F}\mathrm{ib}(g).$$

We delay the proof until a later section and first would like to show how the classical connectivity statement can be derived from our version and give a few examples.

(2.2) **Corollary.** In the same situation as in the theorem, if $\mathrm{h}\mathfrak{Fib}(f)$ is n-connected and $\mathrm{h}\mathfrak{Fib}(g)$ is m-connected for some m, $n \geqslant -1$, then the total fibre set $\mathfrak{T}$ is $(m+n)$-connected.

Proof. The connectivity assumptions can be reformulated as inequalities $\mathrm{h}\mathfrak{Fib}(f) > S^{n+1}$ and $\mathrm{h}\mathfrak{Fib}(g) > S^{m+1}$. The claim now follows from $\Omega S^{n+1} * \Omega S^{m+1} > S^{n+m+1}$ and the fact that the join construction preserves inequalities.

For the rest of this section, we give some examples illustrating particular cases of the acyclic Blakers-Massey theorem for squares.

(2.3) **Example.** Let A and B be connected spaces. For some chosen base-points, the collapse maps $A \to A \vee B \to B$ fit into a homotopy pushout square

$$\begin{tikzcd} A \vee B \arrow[r] \arrow[d] & B \arrow[d] \\ A \arrow[r] & * \end{tikzcd} .$$

This is a typical homotopy pushout with the terminal vertex contractible. The total fibre T of this square is the homotopy fibre of the inclusion $A \vee B \hookrightarrow A \times B$ and by Puppe's theorem, this is the join $\Omega A * \Omega B$. The same theorem allows us to identify $\mathrm{hFib}(A \vee B \to A) \simeq B \rtimes \Omega A$ and $\mathrm{hFib}(A \vee B \to B) \simeq A \rtimes \Omega B$. Thus, in this case, the total fibre $\Omega A * \Omega B$ is a retract of the join

$$\Omega(B \rtimes \Omega A) * \Omega(A \rtimes \Omega B) \simeq \Omega\,\mathrm{hFib}(A \vee B \to A) * \Omega\,\mathrm{hFib}(A \vee B \to B).$$

The inequality $T > \Omega\,\mathrm{hFib}(A \vee B \to A) * \Omega\,\mathrm{hFib}(A \vee B \to B)$ guaranteed by our theorem above is much weaker.

(2.4) **Example.** In the previous example, the total fibre of a homotopy pushout square was first expressed using the homotopy fibres of the horizontal and vertical maps to the terminal vertex. This is not to be expected in general as shown by the following example. Let us choose an integral homology equivalence $X \to Y$, for example the one described by Whitehead in [49, Example IV.7.3], where $Y = S^1$ and X is obtained from $S^1 \vee S^2$ by attaching a single 3-cell via the attaching map

$$S^2 \xrightarrow{(2,-1)} S^2 \vee S^2 \hookrightarrow \bigvee_{-\infty}^{\infty} S^2 \simeq \widetilde{S^1 \vee S^2} \to S^1 \vee S^2.$$

Here the first map has degree 2 on the first sphere and degree -1 on the second one, while the second map is the inclusion on the zeroth and first summands of the infinite wedge. Finally, the last map is the universal cover. We then consider the homotopy pushout square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & S^1 \arrow[d] \\ * \arrow[r] & * \end{tikzcd} ,$$

where f is the first Postnikov section. The join of the loops of the homotopy fibres of $* \to *$ and $S^1 \to *$ is contractible, but the total fibre T is the universal cover $\tilde{X}$ of X, which is not contractible.

But, of course, the inequality of our main theorem still holds because $\mathrm{hFib}(f) = \tilde{X}$ and $\mathrm{hFib}(g) = X$. As $\Omega\,\mathrm{hFib}(g)$ is not connected, $\bar{\mathcal{C}}(\Omega\,\mathrm{hFib}(f) * \Omega\,\mathrm{hFib}(g)) = \bar{\mathcal{C}}(\Sigma\Omega\tilde{X})$ and so $T = \tilde{X}$ is killed by $\Omega\,\mathrm{hFib}(f) * \Omega\,\mathrm{hFib}(g)$ (see [12, Corollary 3.5 (2)]).

In our last example, we illustrate both the necessity to deal with spaces that are not connected and the importance of considering all homotopy fibres at once. It also confirms the usefulness of our convention $\Omega S^0 := S^{-1}$ to be able to deal with the borderline cases.

(2.5) **Example.** Let $f\colon S^1 + S^1 \to * + S^1$ be the disjoint union of the collapse map and the identity, while $g\colon S^1 + S^1 \to S^1$ is the fold map (the identity on both copies of S^1). Consider the homotopy pushout

$$\begin{tikzcd} S^1 + S^1 \arrow[r, "f"] \arrow[d, "g"'] & * + S^1 \arrow[d] \\ S^1 \arrow[r] & * \end{tikzcd}$$

The homotopy pullback P is the disjoint union $(S^1 \times *) + (S^1 \times S^1)$ and the total fibre set $\mathfrak{T}$ consists of a contractible space and ΩS^1.

The homotopy fibre set $\mathrm{h}\mathfrak{F}\mathrm{ib}(f)$ in this example is equivalent to $\{S^1, *\}$, whereas $\mathrm{h}\mathfrak{F}\mathrm{ib}(g)$ is equivalent to $\{S^0\}$. Thus the join $\Omega\,\mathrm{h}\mathfrak{F}\mathrm{ib}(f) * \Omega\,\mathrm{h}\mathfrak{F}\mathrm{ib}(g)$ is equivalent to the set $\{\Omega S^1, *\}$. Since $\{\Omega S^1, *\} > S^0$, our acyclic Blakers-Massey theorem tells us that the total fibre set is killed by S^0; i.e. every space in the total fibre set is non-empty.

3. Reduction to Fake Wedges

Homotopy pushout diagrams in which the terminal vertex is contractible are easier to handle because the homotopy pullback one needs to form in order to compute the total fibre is simply a product. The aim of this section is to reduce the Blakers-Massey theorem to this situation, which Klein and Peter call a *fake wedge* in [36].

(3.1) **Proposition.** Given a homotopy pushout square as on the left below with D connected, then the spaces $\mathrm{hFib}(h \circ f)$, $\mathrm{hFib}(h)$ and $\mathrm{hFib}(k)$ fit into a homotopy pushout as on the right

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "h"] \\ C \arrow[r, "k"'] & D \end{tikzcd} \qquad \begin{tikzcd} \mathrm{hFib}(h \circ f) \arrow[r, "f'"] \arrow[d, "g'"'] & \mathrm{hFib}(h) \arrow[d] \\ \mathrm{hFib}(k) \arrow[r] & * \end{tikzcd}$$

with the following properties:

- $\mathrm{h}\mathfrak{F}\mathrm{ib}(f')$ is equivalent to $\mathrm{h}\mathfrak{F}\mathrm{ib}(f)$;
- $\mathrm{h}\mathfrak{F}\mathrm{ib}(g')$ is equivalent to $\mathrm{h}\mathfrak{F}\mathrm{ib}(g)$;
- the total fibre sets of the two squares are weakly equivalent.

Proof. Choose a fibration $P \twoheadrightarrow D$ with P contractible and pull back the entire pushout square along this map to form the following commutative cube:

$$\begin{tikzcd}
& \mathrm{hFib}(h) \arrow[rr, two heads] \arrow[dd] & & C \arrow[dd, "h"] \\
\mathrm{hFib}(h \circ f) \arrow[ur, "g'"] \arrow[rr, two heads, crossing over] \arrow[dd, "f'"'] & & A \arrow[ur, "g"'] & \\
& P \arrow[rr, two heads] & & D \\
\mathrm{hFib}(k) \arrow[ur] \arrow[rr, two heads] & & B \arrow[ur, "k"'] \arrow[from=uu, crossing over, "f"'] &
\end{tikzcd}.$$

According to Mather's Second Cube Theorem, the face in this cube containing f' and g' is a homotopy pushout. Since P is contractible, the square:

$$\begin{tikzcd}
\mathrm{hFib}(hf) \arrow[r, "f'"] \arrow[d, "g'"'] & \mathrm{hFib}(h) \arrow[d] \\
\mathrm{hFib}(k) \arrow[r] & *
\end{tikzcd}$$

is a homotopy pushout. As the map $P \twoheadrightarrow D$ induces an epimorphism on π_0, so do the maps $\mathrm{hFib}(h) \twoheadrightarrow C$ and $\mathrm{hFib}(k) \twoheadrightarrow B$. This implies that the set $\mathrm{h}\mathfrak{F}\mathrm{ib}(f)$ is equivalent to $\mathrm{h}\mathfrak{F}\mathrm{ib}(f')$ and $\mathrm{h}\mathfrak{F}\mathrm{ib}(g)$ is equivalent to $\mathrm{h}\mathfrak{F}\mathrm{ib}(g')$. Exactly the same argument gives an equivalence between the total fibre sets. □

4. Cofibres

In this section, we treat the special case of a homotopy pushout given by the cofibre of a map. To wit, given a map $A \to X$ to a connected space X, we give an estimate for the total fibre set $\mathfrak{T}$ of

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & X \arrow[d] \\
* \arrow[r] & X /\!\!/ A
\end{tikzcd}$$

(4.1) **Proposition.** Given a map $A \to X$ to a connected X with homotopy fibre F and letting $\mathfrak{T}$ be the total homotopy fibre set of the above homotopy pushout defined by the map's cofibre, then

$$\mathfrak{T} > \{\Omega F * \Omega F,\, \Omega(F * \Omega X)\}.$$

Proof. If F is not connected then $\Omega F * \Omega F = S^{-1}$ by our convention and the claim is trivial. So we assume that F is connected. This implies that A is also connected, and then so is $X /\!\!/ A$. Hence a choice of a base point in F turns this situation into a pointed one. The total fibre set $\mathfrak{T}$ is then the homotopy fibre of the induced map $\alpha\colon F \to \Omega(X /\!\!/ A)$, which factors through $\eta\colon F \to \Omega\Sigma F$ as $\alpha = (\Omega\alpha^\sharp) \circ \eta$, where $\alpha^\sharp\colon \Sigma F \to X /\!\!/ A$ is the adjunct of α. Using (8.7.8), we then obtain $\mathfrak{T} = \mathrm{h}\mathfrak{F}\mathrm{ib}(\alpha) = \mathrm{h}\mathfrak{F}\mathrm{ib}((\Omega\alpha^\sharp) \circ \eta) > \mathrm{h}\mathfrak{F}\mathrm{ib}(\Omega\alpha^\sharp) \cup \mathrm{h}\mathfrak{F}\mathrm{ib}(\eta)$.

According to (1.5), $\mathrm{h}\mathfrak{Fib}(\eta) > \Omega F * \Omega F$. The adjunct map $\alpha^\sharp$ fits into the following commutative diagram, where all the squares are homotopy pushouts:

$$\begin{tikzcd} A \arrow[r] \arrow[d] & A/F \arrow[r] \arrow[d] & X \arrow[d] \\ CA \arrow[r] & \Sigma F \arrow[r, "\alpha^\sharp"] & X/A. \end{tikzcd}$$

$\mathrm{hFib}(\alpha^\sharp) > \mathrm{hFib}(A/F \to X) \simeq F * \Omega X$ which yields $\mathrm{hFib}(\Omega\alpha^\sharp) > \Omega(F * \Omega X)$. These two relations give the desired inequality. □

5. A Rough Estimate

In this section, we obtain a first, rather rough estimate for the total fibre. By combining this weak estimate with our results for cofibration sequences, we will be able to prove our main theorem. Throughout this section, let us fix a homotopy pushout square of the form

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d] \\ C \arrow[r] & * \end{tikzcd} .$$

In the case where B is connected, we write $F \to A$ for the homotopy fibre map of f over the unique component of B. Similarly, if C is connected, we write $G \to A$ for the homotopy fibre map of g over the unique component of C. By definition, the total fibre set $\mathfrak{T}$ of the above square is the homotopy fibre set of the map $(f, g)\colon A \to B \times C$. By Lemma (1.3), when B is connected, this total fibre set can alternatively be described as the homotopy fibre set of the composite $\alpha\colon F \to C$ of the homotopy fibre map $F \to A$ and g.

(5.1) **Lemma.** If B and C are connected, then the homotopy cofibre $C /\!\!/ F$ of the map $\alpha\colon F \to C$ is killed by $F * \Omega B$. In particular, $C /\!\!/ F$ is 2-connected if F is 1-connected.

Proof. We have a homotopy pushout square

$$\begin{tikzcd} A /\!\!/ F \arrow[r] \arrow[d] & B \arrow[d] \\ C /\!\!/ F \arrow[r] & * \end{tikzcd} .$$

Therefore, we infer from Chachólski's theorem (8.7.6) that $C /\!\!/ F$ is killed by the homotopy fibre $\mathrm{hFib}(A /\!\!/ F \to B) \simeq F * \Omega B$. Finally, if F is 1-connected and B is connected, this join is 2-connected. □

Here is our "rough estimate". The roughness of this acyclic inequality comes from the fact that it only involves one of the fibres. As we know from the classical version of the Blakers-Massey Theorem, the connectivity of the total fibre should be related to the sum of the connectivities of both fibres.

(5.2) **Proposition.** If B and $\mathfrak{T}$ are connected, then $\mathfrak{T} > \Sigma\Omega F$.

Proof. If ΩF is not connected, then it is either empty or contains S^0 as a retract. In the first case, $\Sigma\Omega F = S^0$ and $\mathfrak{T} > \Sigma\Omega F$ is clear because every space in $\mathfrak{T}$ is connected and hence non-empty. If S^0 is a retract of ΩF, then S^1 is a retract of $\Sigma\Omega F$ and hence $\mathfrak{T} > \Sigma\Omega F$ follows from the assumption that all the spaces in $\mathfrak{T}$ are connected.

Let us assume that ΩF is connected and that, therefore, F is 1-connected. This implies that A is connected. Moreover, according to Chachólski's theorem (8.7.6), $C > F$ and so C is 1-connected as well. By the above lemma (5.1), we also know that $C /\!\!/ F > S^3$. Now, the total fibre set $\mathfrak{T}$ consists of a single space T, which is equivalent to the homotopy fibre of the map $\alpha\colon F \to C$. This map fits into the following commutative diagram:

$$\begin{tikzcd}
 & & C \arrow[rrr, "\mathrm{id}"] \arrow[ddd] & & & \ast \arrow[ddd] \\
F \arrow[urr, "\alpha"] \arrow[dr, "\beta"] \arrow[rrr, "\mathrm{id}"] \arrow[ddd] & & & F \arrow[urr] \arrow[dr, "\eta"] \arrow[ddd] & & \\
 & H \arrow[uur] \arrow[rrr, "\gamma"] \arrow[ddl] & & & \Omega\Sigma F \arrow[uur] \arrow[ddl] & \\
 & & C /\!\!/ F \arrow[rrr] & & & \Sigma F \\
CF \arrow[urr] \arrow[rrr] & & & CF \arrow[urr] & &
\end{tikzcd},$$

where $F \to CF$ is a cone above F;

- the faces $\begin{tikzcd} F \arrow[r, "\alpha"] \arrow[d] & C \arrow[d] \\ CF \arrow[r] & C /\!\!/ F \end{tikzcd}$ and $\begin{tikzcd} F \arrow[r] \arrow[d] & \ast \arrow[d] \\ CF \arrow[r] & \Sigma F \end{tikzcd}$ are homotopy pushouts;
- the squares $\begin{tikzcd} H \arrow[r] \arrow[d] & C \arrow[d] \\ CF \arrow[r] & C /\!\!/ F \end{tikzcd}$ and $\begin{tikzcd} \Omega\Sigma F \arrow[r] \arrow[d] & \ast \arrow[d] \\ CF \arrow[r] & \Sigma F \end{tikzcd}$ are homotopy pull-backs.

In this way, we expressed $\alpha\colon F \to C$ as a composition of $\beta\colon F \to H$ and $H \to C$, which gives:

$$T = \mathrm{hFib}(\alpha\colon F \to C) > \mathrm{hFib}(\beta\colon F \to H) \cup \mathrm{hFib}(H \to C).$$

To prove the proposition, it is now enough to show that both fibre sets $\mathrm{hFib}(H \to C)$ and $\mathrm{hFib}(\beta\colon F \to H)$ are killed by $\Sigma\Omega F$. Starting with $\mathrm{hFib}(H \to C)$, we note that we have the following sequence of relations:

$$\mathrm{hFib}(H \to C) \overset{(a)}{\simeq} \Omega(C /\!\!/ F) \overset{(b)}{>} \Omega(F * \Omega B) \overset{(c)}{>} \Omega\Sigma F \overset{(d)}{>} F \overset{(e)}{>} \Sigma\Omega F,$$

where the weak equivalence (a) is a consequence of the fact that the relevant square is a homotopy pull-back; the inequality (b) follows from lemma (5.1) above; connectedness of B gives (c); Chachólski's theorem (8.7.6) gives (d); and, finally, (e) is true for any non-empty F (see for example [12, Corollary 3.5]).

It remains to show that $\mathrm{hFib}(\beta\colon F \to H) > \Sigma\Omega F$. The space H is the homotopy fibre of the cofibre map $C \to C /\!\!/ F$ and hence $H > F$ by Chachólski's theorem (8.7.6).

Thus H is also 1-connected and consequently, $\mathrm{hFib}(\gamma\colon H \to \Omega\Sigma F)$ is connected. According to the diagram above, the composition of $\beta\colon F \to H$ and $\gamma\colon H \to \Omega\Sigma F$ is the James map $\eta\colon F \to \Omega\Sigma F$. The fibres of these three maps therefore fit into a fibration sequence

$$\mathrm{hFib}(\beta\colon F \to H) \to \mathrm{hFib}(\eta\colon F \to \Omega\Sigma F) \to \mathrm{hFib}(\gamma\colon H \to \Omega\Sigma F).$$

We have just argued that the base in this fibration is connected. As F is 1-connected, so is the total space in this fibration. We can therefore form a new fibration sequence

$$\Omega\,\mathrm{hFib}(\gamma\colon H \to \Omega\Sigma F) \to \mathrm{hFib}(\beta\colon F \to H) \to \mathrm{hFib}(\eta\colon F \to \Omega\Sigma F).$$

By lemma (1.5), $\mathrm{hFib}(\eta\colon F \to \Omega\Sigma F) > \Omega F * \Omega F > \Sigma\Omega F$ (using that ΩF is connected). With this, once we show $\Omega\,\mathrm{hFib}(\gamma\colon H \to \Omega\Sigma F) > \Sigma\Omega F$, the desired inequality follows. To do so, we note that $\mathrm{hFib}(\gamma\colon H \to \Omega\Sigma F)$ is the total fibre of the homotopy pushout square

$$\begin{tikzcd} C \arrow[r] \arrow[d] & * \arrow[d] \\ C /\!\!/ F \arrow[r] & \Sigma F. \end{tikzcd}$$

By Proposition (4.1), this fibre is killed by $\{\Omega H * \Omega H, \Omega(H * \Omega(C /\!\!/ F))\}$. Recalling that $H > F > S^2$ and $C /\!\!/ F > S^3$ (see Lemma (5.1)), these inequalities imply

$$\mathrm{hFib}(\gamma\colon H \to \Omega\Sigma F) > \{\Omega F * \Omega F,\ \Omega(F * S^2)\} > \{\Sigma^2\Omega F,\ \Omega\Sigma^3 F\}.$$

Since $\Omega\Sigma^3 F > \Sigma^2\Omega F$, we obtain $\mathrm{hFib}(\gamma\colon H \to \Omega\Sigma F) > \Sigma^2\Omega F$. By looping this inequality, we finally get $\Omega\,\mathrm{hFib}(\gamma\colon H \to \Omega\Sigma F) > \Omega\Sigma^2\Omega F > \Sigma\Omega F$. □

Compared to the classical Blakers-Massey theorem, the previous result might seem too strong. This is because our claim at the beginning of this section – that we would use only one fibre – was not entirely honest. We have used the fibre G implicitly in assuming that B is connected (implying that so is G), which allowed us to pick up a suspension for the inequality $\mathfrak{T} > \Sigma\Omega F$. For a non-connected B, one can only establish $\mathfrak{T} > \Omega\,\mathrm{h}\mathfrak{F}\mathrm{ib}(f)$, as the following example shows.

(5.3) **Example.** Let $n \geqslant 0$, $x\colon * \to S^n$ a base-point and consider the following homotopy pushout square

$$\begin{tikzcd} S^n + * \arrow[r] \arrow[d] & * + * \arrow[d] \\ S^n \arrow[r] & * \end{tikzcd} ,$$

where the vertical map on the left is given by the identity on S^n and x on $*$, while the top horizontal map is the coproduct of the unique maps into $*$. Thus the homotopy fibre set $\mathfrak{F}$ of the top horizontal map is equivalent to $\{S^n, *\}$. The total fibre set $\mathfrak{T}$ of this square, however, is equivalent to $\{\Omega S^n, *\}$. Therefore, in this case, it is not true that $\mathfrak{T} > \Sigma\Omega\mathfrak{F}$, even though, for $n > 1$, every total fibre in $\mathfrak{T}$ is connected.

6. Connectivity of the Total Fibre

Before we proceed to the proof of the acyclic Blakers-Massey theorem, we first need to establish a relationship between the connectivity of the fibers of the maps in a homotopy pushout square and the connectivity of its total fiber in order to be able to use our rough estimate (5.2).

(6.1) **Proposition.** Given a homotopy pushout square the spaces

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "h"] \\ C \arrow[r, "k"'] & D \end{tikzcd}$$

with B, C, D, $F := \mathrm{hFib}(f)$ and $G = \mathrm{hFib}(g)$ connected, then the total fiber set $\mathfrak{T}$ of this square consists of a single space that is again connected.

Proof. The connectivity assumptions imply that the homotopy pullback of the diagram $C \to D \leftarrow B$ is connected and hence, the total fiber set $\mathfrak{T}$ consists of one space T. Using proposition (3.1), we assume, without loss of generality, that D is contractible. Now, the maps $f_*\colon \pi_1(A) \to \pi_1(B)$ and $g_*\colon \pi_1(A) \to \pi_1(C)$ are surjective by connectedness of F and G. Using the long exact homotopy sequence for $T \to A \to B \times C$, we need to show that $(f_*, g_*)\colon \pi_1(A) \to \pi_1(B) \times \pi_1(C)$ is surjective. But $\pi_1(B) *_{\pi_1(A)} \pi_1(C)$ is the trivial group by the Seifert-van Kampen Theorem and the claim follows from the following lemma. □

(6.2) **Lemma.** Given a pushout diagram in the category of groups

$$\begin{tikzcd} G \arrow[r, two heads, "\varphi"] \arrow[d, two heads, "\psi"'] & H \arrow[d] \\ K \arrow[r] \arrow[ur, phantom, "\ulcorner"] & 1 \end{tikzcd}$$

with φ and ψ surjective, the homomorphism $(\varphi, \psi)\colon G \to H \times K$ is surjective, too.

Proof. Writing $M = \operatorname{Ker}\varphi$, $N = \operatorname{Ker}\psi$ and identifying $H \cong G/M$, $K \cong G/N$, we can reformulate the hypothesis $H *_G K \cong G/(M \triangledown N) \cong 1$ as $M \triangledown N = G$, where $M \triangledown N$ is the normal closure of $M \cup N$ in G. But M and N are normal subgroups and so $G = M \triangledown N = MN$. By the second isomorphism theorem then $G/M = (MN)/M \cong N/(M \cap N)$ and likewise $G/N \cong M/(M \cap N)$, so that both $\varphi\colon N \to G/M \cong H$ and $\psi\colon M \to G/N \cong K$ are surjective. Finally, this implies the surjectivity of $(\varphi, \psi)\colon G \to H \times K$ because if $(h, k) \in H \times K$, we find $n \in N$, $m \in M$ such that $\varphi(n) = h$, $\psi(m) = k$ and thus $(\varphi, \psi)(nm) = (h, 1)(1, k) = (h, k)$. □

7. Proof of the Main Theorem

Finally, we are ready to attack to proof of the acyclic Blakers-Massey theorem (for homotopy pushout squares). Recall that it was the following.

(7.1) **Theorem.** Given a homotopy pushout square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

with total fibre set $\mathfrak{T}$, then

$$\mathfrak{T} > \Omega\, \mathrm{h}\mathfrak{F}\mathrm{ib}(f) * \Omega\, \mathrm{h}\mathfrak{F}\mathrm{ib}(g).$$

We divide the proof of this theorem into several parts. The first part consists of reducing the proof to the easier situation when the homotopy fibres of f and g are connected.

Reduction to connected fibres: If D is empty, then so are A, B, C and the claim is trivially true. We assume therefore that D is non-empty. For $D_0 \subseteq D$ a connected component, we define $B_0 \subseteq B$, $C_0 \subseteq C$ and $A_0 \subseteq A$ of D_0 along h, k and $h \circ f = k \circ h$, respectively. In this manner, we obtain a homotopy pushout square:

$$\begin{tikzcd} A_0 \arrow[r, "f_0"] \arrow[d, "g_0"'] & B_0 \arrow[d, "h_0"] \\ C_0 \arrow[r, "k_0"'] & D_0 \end{tikzcd} ,$$

where the maps are all just restrictions of the original ones. Now, the fibers of the maps in the original square are simply the sums of those of the corresponding restricted maps taken over all $D_0 \in \pi_0(D)$ and the same is true for total fibers. Therefore, the claimed acyclic inequality holds if and only if the same inequality holds for all the restricted squares (with $D_0 \in \pi_0(D)$). We can therefore assume that D connected.

A further reduction can then be obtained from (3.1), which states that the general case for a connected D follows from the case $D = *$. So let us make this assumption $D = *$. This implies that both $\pi_0(f)\colon \pi_0(A) \to \pi_0(B)$ and $\pi_0(g)\colon \pi_0(A) \to \pi_0(C)$ are surjective.

If both sets $\mathrm{h}\mathfrak{F}\mathrm{ib}(f)$ and $\mathrm{h}\mathfrak{F}\mathrm{ib}(g)$ contain a non-connected space, then according to our convention, the acyclic classes $\bar{\mathcal{C}}(\Omega\, \mathrm{h}\mathfrak{F}\mathrm{ib}(f))$ and $\bar{\mathcal{C}}(\Omega\, \mathrm{h}\mathfrak{F}\mathrm{ib}(g))$ consist of all spaces and hence so does $\bar{\mathcal{C}}(\Omega\, \mathrm{h}\mathfrak{F}\mathrm{ib}(f) * \Omega\, \mathrm{h}\mathfrak{F}\mathrm{ib}(g))$. It is then clear that the total fibre set belongs to this acyclic class, which means that $\mathfrak{T} > \Omega\, \mathrm{h}\mathfrak{F}\mathrm{ib}(f) * \Omega\, \mathrm{h}\mathfrak{F}\mathrm{ib}(g)$.

We can then assume that $\mathrm{h}\mathfrak{F}\mathrm{ib}(f)$ consists of connected spaces only, which implies that $\pi_0(f)$ is a bijection and so – as an easy $H_0(-, \mathbb{Z})$ calculation shows – the space C has to be connected because $D = *$ is connected. Thus, for any total fibre $T_0 \in \mathfrak{T}$, there is a $F_0 \in \mathrm{h}\mathfrak{F}\mathrm{ib}(f)$ fitting into a fibration sequence $T_0 \to F_0 \to C$, which implies that $\mathrm{hFib}(T_0 \to F_0) \simeq \Omega C$. Furthermore, by Chachólski's theorem (8.7.6), $C > \mathrm{h}\mathfrak{F}\mathrm{ib}(f)$, which implies $\Omega C > \Omega\, \mathrm{h}\mathfrak{F}\mathrm{ib}(f)$. Now, if $\mathrm{hFib}(g)$ is not connected then $M = M * \Omega\, \mathrm{hFib}(g)$ for any set of spaces M and from the acyclic inequality for composable maps (8.7.8), we obtain

$$T_0 > \{\Omega C, F_0\} > \Omega\, \mathrm{h}\mathfrak{F}\mathrm{ib}(f) \cup \{F_0\} > \Omega\, \mathrm{h}\mathfrak{F}\mathrm{ib}(f) > \Omega\, \mathrm{h}\mathfrak{F}\mathrm{ib}(f) * \Omega\, \mathrm{hFib}(g).$$

Because $T_0 \in \mathfrak{T}$ was arbitrary, we get the desired inequality.

Connected fibres: The remaining case is when both $\mathrm{h}\mathfrak{F}\mathrm{ib}(f)$ and $\mathrm{h}\mathfrak{F}\mathrm{ib}(g)$ consist of connected spaces. As above, this implies that B and C are connected, so that $\mathrm{h}\mathfrak{F}\mathrm{ib}(f)$ and $\mathrm{h}\mathfrak{F}\mathrm{ib}(g)$ are weakly equivalent singletons and, as before, we write $F \to A$ for the homotopy fibre map of f and $G \to A$ for the homotopy fibre map of g. Now, by (6.1) the total fibre set $\mathfrak{T}$ consists also of a single connected space T and, lastly, A has to be connected too.

To get an estimate for T, which is the homotopy fibre of $\alpha\colon F \to C$ by Lemma (1.3), we consider the following commutative diagram (compare with the proof of (5.2)),

$$\begin{tikzcd} F \arrow[rr, "\alpha"] \arrow[dr, "\beta"'] \arrow[dd] & & C \arrow[dd] \\ & H \arrow[ur] \arrow[dl] & \\ CF \arrow[rr] & & C/\!\!/F \end{tikzcd}$$

where CF is a cone for F, the outside square is a homotopy pushout and the inside one is a homotopy pullback. Since T is the homotopy fibre of the composite $F \to H \to C$,

$$T > \mathrm{h}\mathfrak{F}\mathrm{ib}(\beta\colon F \to H) \cup \mathrm{h}\mathfrak{F}\mathrm{ib}(H \to C).$$

The homotopy fibre set $\mathrm{h}\mathfrak{F}\mathrm{ib}(\beta)$ is the total fibre set of the outside homotopy pushout square. Since C is connected, $\mathrm{h}\mathfrak{F}\mathrm{ib}(\beta) > \{\Omega T * \Omega T, \Omega(T * \Omega C)\}$ (see Proposition (4.1)). We can then use the rough estimates from (5.2) (with respect to both F and G) and the fact that $C > F$ to conclude that

$$\mathrm{h}\mathfrak{F}\mathrm{ib}(\beta) > \{\Omega\Sigma\Omega F * \Omega\Sigma\Omega G, \Omega(\Sigma\Omega G * \Omega F)\} > \Omega F * \Omega G,$$

where we used that $\Omega\Sigma X > X$ for any space X (by Chachólski's theorem (8.7.6)). Finally, since $\mathrm{hFib}(H \to C) = \Omega(C/F)$ and $C/F > F * \Omega B$ by Lemma (5.1), we get

$$\mathrm{hFib}(H \to C) > \Omega(F * \Omega B) > \Omega(F * \Omega G) > \Omega F * \Omega G.$$

8. Arbitrary Squares

From the case of a homotopy pushout square, which we just proved in the previous section, we easily deduce the following, more general statement for an arbitrary square.

(8.1) **Theorem.** Given a commutative square:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d] \\
C \arrow[r] & D
\end{tikzcd}$$

with total fibre set $\mathfrak{T}$ and pushout fibre set $\mathfrak{R}$ (see section 1). Then

$$\mathfrak{T} > \Omega\,\mathrm{h}\mathfrak{F}\mathrm{ib}(f) * \Omega\,\mathrm{h}\mathfrak{F}\mathrm{ib}(g) \cup \Omega\mathfrak{R}.$$

Proof. Writing $\mu\colon Q \to D$ for the comparison map between the homotopy pushout and the terminal object of the commutative square and replacing the relevant maps cofibrations and fibrations, if necessary, the square from our claim fits into a commutative diagram

$$\begin{tikzcd}
 & & B \arrow[drrr, "id"] \arrow[dddd] & & & \\
A \arrow[urr, "f"] \arrow[drrr, "id"] \arrow[dr, "\beta"] \arrow[dddd, "g"'] & & & & & B \arrow[dddd] \\
 & P_2 \arrow[uur] \arrow[drrr, "\gamma"] \arrow[dddl, two heads] & & A \arrow[urr, "f"] \arrow[dr, "\alpha"] \arrow[dddd, "g"] & & \\
 & & & & P_1 \arrow[uur] \arrow[dddl, two heads] & \\
 & & Q \arrow[drrr, "\mu"] & & & \\
C \arrow[urr] \arrow[drrr, "id"'] & & & & & D \\
 & & & C \arrow[urr] & &
\end{tikzcd}$$

,

where

- the face $\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d] \\ C \arrow[r] & Q \end{tikzcd}$ is a homotopy pushoute, while

- the squares $\begin{tikzcd} P_1 \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$ and $\begin{tikzcd} P_2 \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & Q \end{tikzcd}$ are homotopy pullbacks.

By definition, $\mathfrak{T} = \mathrm{h}\mathfrak{F}\mathrm{ib}(\alpha\colon A \to P_1)$ and $\alpha\colon A \to P_1$ factors as the composition of $\beta\colon A \to P_2$ with $\gamma\colon P_2 \to P_1$. We therefore get the inequality

$$\mathfrak{T} = \mathrm{h}\mathfrak{F}\mathrm{ib}(\alpha\colon A \to P_1) > \mathrm{h}\mathfrak{F}\mathrm{ib}(\beta\colon A \to P_2) \cup \mathrm{h}\mathfrak{F}\mathrm{ib}(\gamma\colon P_2 \to P_1).$$

According to our acyclic Blakers-Massey theorem for homotopy pushout squares (7.1), we have $\mathrm{h}\mathfrak{F}\mathrm{ib}(\beta\colon A \to P_2) > \Omega\,\mathrm{h}\mathfrak{F}\mathrm{ib}(f) * \Omega\,\mathrm{h}\mathfrak{F}\mathrm{ib}(g)$ and since

$$\mathrm{h}\mathfrak{F}\mathrm{ib}(\gamma\colon P_2 \to P_1) \subseteq \Omega\,\mathrm{h}\mathfrak{F}\mathrm{ib}(\mu\colon Q \to D) = \Omega\mathfrak{R},$$

we get $\mathrm{h}\mathfrak{F}\mathrm{ib}(\gamma\colon P_2 \to P_1) > \Omega\mathfrak{R}$ and the claim follows. □

9. Pullbacks and Suspensions

A nice application of our version of Blakers-Massey for squares is that we are able to compare the suspension of a pullback to the pullback of the suspensions as follows.

(9.1) **Proposition.** Given two homotopy pullbacks

$$\begin{tikzcd} P \arrow[r, "g'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "f"] \\ C \arrow[r, "g"'] & D \end{tikzcd} \qquad \begin{tikzcd} P_\Sigma \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Sigma B \arrow[d] \\ \Sigma C \arrow[r] & \Sigma D \end{tikzcd}$$

with comparison map $p\colon \Sigma P \to P_\Sigma$, we have

$$\mathrm{h}\mathfrak{F}\mathrm{ib}\left(\Sigma P \xrightarrow{p} P_\Sigma\right) > F * G, \qquad \text{where } F := \mathrm{h}\mathfrak{F}\mathrm{ib}(f),\ G := \mathrm{h}\mathfrak{F}\mathrm{ib}(g).$$

Proof. Let's write $Q := \mathrm{hocolim}(B \leftarrow P \to C)$, which comes with a comparison map $h\colon Q \to D$, whose fibre is $\mathrm{h}\mathfrak{F}\mathrm{ib}(h) \simeq F * G$ by Puppe's theorem (7.7.6). Since pushouts commute with suspensions, Blakers-Massey, tells us that

$$\begin{aligned} \mathrm{h}\mathfrak{F}\mathrm{ib}(\Sigma P \to P_\Sigma) &> \{\Omega\,\mathrm{h}\mathfrak{F}\mathrm{ib}(\Sigma f') * \Omega\,\mathrm{h}\mathfrak{F}\mathrm{ib}(\Sigma g'), \Omega\,\mathrm{h}\mathfrak{F}\mathrm{ib}(\Sigma h)\} \\ &> \{\Omega\Sigma\,\mathrm{h}\mathfrak{F}\mathrm{ib}(f') * \Omega\Sigma\,\mathrm{h}\mathfrak{F}\mathrm{ib}(g'), \Omega\Sigma\,\mathrm{h}\mathfrak{F}\mathrm{ib}(h)\} \\ &> \{\mathrm{h}\mathfrak{F}\mathrm{ib}(f') * \mathrm{h}\mathfrak{F}\mathrm{ib}(g'), \mathrm{h}\mathfrak{F}\mathrm{ib}(h)\} \simeq F * G. \end{aligned}$$

□

(9.2) **Example.** Let's fix a simplicial set Y and consider

$$\begin{tikzcd} \Omega Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & * \arrow[d] \\ * \arrow[r] & Y, \end{tikzcd} \qquad \begin{tikzcd} \Omega\Sigma Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & * \arrow[d] \\ * \arrow[r] & \Sigma Y. \end{tikzcd}$$

By our proposition then

$$\mathrm{h}\mathfrak{F}\mathrm{ib}(\Sigma\Omega Y \to \Omega\Sigma Y) > \Omega Y * \Omega Y.$$

The same result can be obtained using [12, Section 7]. There, it is shown that for any simplicial set Y, the fibres of the counit and unit map satisfy

$$\mathrm{h}\mathfrak{F}\mathrm{ib}(\Sigma\Omega Y \to Y) \simeq \Omega Y * \Omega Y \qquad \text{and} \qquad \mathrm{h}\mathfrak{F}\mathrm{ib}(Y \to \Omega\Sigma Y) \gg \Omega Y * \Omega Y.$$

Now, one can simply use the fibre sequence associated to $\Sigma\Omega Y \to Y \to \Omega\Sigma Y$.

(9.3) **Example.** Similarly, let's consider $f\colon A \to B$ and

$$\begin{tikzcd} \mathrm{h}\mathfrak{F}\mathrm{ib}(c) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "c"] \\ * \arrow[r] & B /\!\!/ A\,, \end{tikzcd} \qquad \begin{tikzcd} \mathrm{h}\mathfrak{F}\mathrm{ib}(\Sigma c) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Sigma B \arrow[d, "\Sigma c"] \\ * \arrow[r] & \Sigma(B /\!\!/ A)\,. \end{tikzcd}$$

By the proposition, we have

$$\mathrm{h}\mathfrak{F}\mathrm{ib}\big(\Sigma\,\mathrm{h}\mathfrak{F}\mathrm{ib}(c) \to \mathrm{h}\mathfrak{F}\mathrm{ib}(\Sigma c)\big) > \Omega(B /\!\!/ A) * \mathrm{h}\mathfrak{F}\mathrm{ib}(c).$$

Combining this with $B /\!\!/ A \gg \Sigma\,\mathrm{h}\mathfrak{F}\mathrm{ib}(f)$ [11, Proposition 10.5] and $\mathrm{h}\mathfrak{F}\mathrm{ib}(c) \gg A$ [12, Proposition 8.1], we obtain

$$\mathrm{h}\mathfrak{F}\mathrm{ib}\big(\Sigma\,\mathrm{h}\mathfrak{F}\mathrm{ib}(c) \to \mathrm{h}\mathfrak{F}\mathrm{ib}(\Sigma c)\big) > \Omega\Sigma\,\mathrm{h}\mathfrak{F}\mathrm{ib}(f) * A.$$

In this last example, it is not essential that we start with a pushout (that is, a cofibre $B \to B /\!\!/ A$) and we can more generally derive the following from the proposition.

(9.4) **Corollary.** Given any map $f\colon E \to B$, we have

$$\mathrm{h}\mathfrak{F}\mathrm{ib}\big(\Sigma\,\mathrm{h}\mathfrak{F}\mathrm{ib}(f) \to \mathrm{h}\mathfrak{F}\mathrm{ib}(\Sigma f)\big) > \Omega B * \mathrm{h}\mathfrak{F}\mathrm{ib}(f).$$

□

Chapter 10

CUBICAL DIAGRAMS

As a preliminary to our acyclic homotopy excision theorem for cubes (which we will show in the next chapter), we need to introduce cubical diagrams and – as a main ingredient of our proof – show the so-called *Web Trick* (sometimes also referred to as *Thomason Magic*).

1. Setup and Notation

In the following chapter, our main objects of study are cubical diagrams; meaning diagrams of the form $\mathfrak{P}\{1,\ldots,n\} \to \mathbf{sSets}$. Usually, the ones we are interested in are those that arise as higher-dimensional homotopy pushout diagrams from a diagram of the form

$$\mathfrak{P}\{1,\ldots,n\} \setminus \{1,\ldots,n\} \to \mathbf{sSets}.$$

However, a lot of things that we do work in a more general context and we fix some notation for this.

(1.1) **Notation.** Given a poset P with a bottom element $\perp$, we write $\bar{P} := P + \{\top\}$ for the poset P with a top element added and $P' := \bar{P} \setminus \{\perp\}$ for the poset P with its bottom element exchanged for a top one.

(1.2) **Notation.** Given a poset P as above and a diagram $X\colon P \to \mathbf{sSets}$, we write $\bar{X}$ for its left Kan extension along $P \hookrightarrow \bar{P}$ (which is obtained by completing the colimit). We then write $X' := \bar{X}|_{P'}$ for its restriction to $P' \subseteq \bar{P}$. Similarly, if $Y\colon P' \to \mathbf{sSets}$ is a diagram indexed by P', we write $\bar{Y}$ for its right Kan extension along $P' \hookrightarrow \bar{P}$ (which is obtained by completing the limit).

Still given a poset P with a bottom element $\perp$ as well as a simplicial set A, we will sometimes have to consider the diagram $G_A\colon P \to \mathbf{sSets}$ obtained from A by taking the right Kan extension along $\{\perp\} \hookrightarrow P$. Let us make this more explicit.

(1.3) **Definition.** Given a poset P with a bottom element $\perp$ and a simplicial set A, we define the diagram $G_A\colon P \to \mathbf{sSets}$, which maps $\perp$ to A and everything else to the terminal simplicial set $*$. This construction is clearly functorial and defines a functor $G\colon \mathbf{sSets} \to \mathbf{sSets}^P$, which is right adjoint to $\mathrm{Ev}_\perp$. While we are at it, let us fix the notation

$$\Sigma^P A := \operatorname*{hocolim}_P G_A,$$

which (assuming a choice of functorial homotopy colimits; e.g. by using a functorial cofibrant replacement in the projective model structure) defines a functor $\Sigma^P\colon \mathbf{sSets} \to \mathbf{sSets}$.

(1.4) **Example.** If $P = \ulcorner^2 = \{\{1\} \leftarrow \varnothing \rightarrow \{2\}\}$ then G_A is the span $* \leftarrow A \rightarrow *$ and thus $\Sigma^P A \simeq \Sigma A$ is the usual suspension. More generally, if $P = \mathfrak{P}\{1,\dots,n\} \setminus \{1,\dots,n\} = \ulcorner^n$ then $\Sigma^P A \simeq \Sigma^{n-1} A$ is the $(n-1)$-fold suspension, as is shown in (5.7) below.

(1.5) **Example.** Obviously, $\Sigma^P A$ need not be a suspension of A. For example, if we take $P = \{\varnothing, \{1\}, \{2\}, \{3\}\}$ ordered by inclusion, then $\Sigma^P A \simeq \Sigma A \vee \Sigma A$.

More generally, instead of building a diagram out of a single object and always map into the terminal space, we can build a diagram starting from some fixed map.

(1.6) **Definition.** Given a poset P with a bottom element $\perp$ as well as a map of simplicial sets $f\colon A \to B$, we define the diagram $G_f\colon P \to \mathbf{sSets}$, which maps $\perp$ to A, everything else to B, $\perp < p$ to f and every other morphism to the identity. Just like above, we write

$$\Sigma^P f := \operatorname*{hocolim}_P G_f.$$

Obviously, G_A is a special case of this definition (namely $G_A = G_{A\to\{*\}}$). However, contrary to G_A, the newly defined G_f is not a Kan extension in any obvious way. Rather, it is precomposition of $f\colon [1] \to \mathbf{sSets}$ by the quotient map

$$P \twoheadrightarrow P/(P \setminus \{\perp\}) \cong [1],$$

so that we obtain a functor

$$G\colon \mathbf{sSets}^{[1]} \to \mathbf{sSets}^P,\ f \mapsto G_f,$$

which sends a morphism $(g,h)\colon f \to f'$ to the natural transformation $G_f \Rightarrow G_{f'}$ that is g at $\perp$ and h everywhere else. In particular, for $f\colon A \to B$, if we consider $(f, \mathrm{id}_B)\colon f \Rightarrow \mathrm{id}_B$, we obtain the following observation.

(1.7) **Observation.** Every such diagram G_f with $f\colon A \to B$ comes with a canonical natural transformation $G_f \Rightarrow \Delta_B$, which is f at $\perp$ and the identity everywhere else.

Taking homotopy colimits of the canonical natural transformation $G_f \Rightarrow \Delta_B$ and noting that $\operatorname{hocolim} \Delta_B \simeq B$ (because P is contractible), we get an induced map $\Sigma^P f \to B$, whose homotopy fibre we will need in later sections. Informally, one could say that $\Sigma^P f$ is a fibrewise Σ^P-construction.

(1.8) **Proposition.** For every poset P with a bottom element, $f\colon A \to B$ and $b \in B$, $\mathrm{hFib}_b(\Sigma^P f \to B) \simeq \Sigma^P \mathrm{hFib}_b(f)$. Consequently, $\mathrm{h}\mathfrak{F}\mathrm{ib}(\Sigma^P f \to B) \simeq \Sigma^P \mathrm{h}\mathfrak{F}\mathrm{ib}(f)$.

Proof. By the special Puppe theorem (7.7.5)

$$\begin{aligned}
\mathrm{hFib}_b(\Sigma^P f \to B) &= \mathrm{hFib}_b(\operatorname{hocolim}_P G_f \to B)\\
&\simeq \operatorname*{hocolim}_{p\in P} \mathrm{hFib}_b(G_f(p) \to B)\\
&\simeq \operatorname*{hocolim}_{p\in P} G_{\mathrm{hFib}_b(f)}\\
&= \Sigma^P \mathrm{hFib}_b(f),
\end{aligned}$$

where in the second to last step, we used that $G_f(p) \to B$ is the identity everywhere except at $\perp$, where it is f. □

(1.9) **Example.** For $P = \ulcorner^2$ and $f\colon A \to B$, then G_f is $B \xleftarrow{f} A \xrightarrow{f} B$ and, as already mentioned in the proof, by Puppe's theorem $\mathrm{h}\mathfrak{F}\mathrm{ib}(\operatorname{hocolim} G_f \to B) \simeq \Sigma\, \mathrm{h}\mathfrak{F}\mathrm{ib}(f)$.

2. Comparison Maps

Given a diagram $B \leftarrow A \rightarrow C$ in any bicomplete category $\mathcal{C}$ with pushout D, we can consider the pullback P of $B \rightarrow D \leftarrow C$ and obtain a canonical comparison map $A \rightarrow P$ making the obvious diagram commute. If we work in the homotopical context and replace strict (co)limits by homotopy (co)limits, it is not immediately obvious how to obtain this comparison map.

Classically, the unique map into the limit, induced by a cone (which we are also going to call a "comparison map") is part of the definition of a limit. If, instead of a single limit, we want to consider the entire limit functor (for some fixed indexing category $\mathcal{I}$), a cone above a diagram $X : \mathcal{I} \rightarrow \mathcal{C}$ with vertex C is nothing but a natural transformation $\gamma : \Delta C \Rightarrow X$ and the comparison map $C \rightarrow \lim_{\mathcal{I}} X$ induced by it is just the image of γ under the natural bijection

$$\mathrm{Nat}(\Delta C, X) \cong \mathcal{C}(C, \lim_{\mathcal{I}} X) \qquad \text{coming from} \qquad \begin{tikzcd} \mathcal{C} \arrow[r, shift left, "\Delta"] \arrow[r, phantom, "\perp"] & \mathcal{C}^{\mathcal{I}} \arrow[l, shift left, "\lim_{\mathcal{I}}"] \end{tikzcd} .$$

Put differently, writing η for the adjunction's unit, it is $\lim_{\mathcal{I}} \gamma \circ \eta_C$.

In the derived setting, this viewpoint is not helpful because the derived unit is a morphism in $\mathrm{Ho}\,\mathcal{C}$, rather than $\mathcal{C}$. As long as $\mathcal{C}$ is a model category (which will always be the case for us), we can simply use the explicit construction of the homotopy limit by means of fibrant replacements. To wit, if we again have a cone $\gamma : \Delta C \Rightarrow X$, we choose fibrant replacements

$$\begin{tikzcd} \Delta C \arrow[r, "\gamma"] \arrow[d, tail, "r_{\Delta C}"', "\sim"] & X \arrow[d, tail, "\sim"', "r_X"] \\ R\Delta C \arrow[r, dotted, "R\gamma"'] & RX \end{tikzcd}$$

together with an extension $R\gamma$ to these replacements. Now, the derived unit's component at $C \in \mathrm{Ho}\,\mathcal{C}$ is (the image in $\mathrm{Ho}\,\mathcal{C}$ of) the comparison map

$$\dot{\eta}_C : C \xrightarrow{\eta_C} \lim \Delta C \xrightarrow{\lim r_{\Delta C}} \lim R\Delta C = \mathrm{holim}\, \Delta C.$$

induced by the cone $r_{\Delta C}$. Consequently, the comparison map $C \rightarrow \mathrm{holim}\, X$, which is obtained as $\mathrm{holim}\, \gamma \circ \dot{\eta}_C$, is the image in $\mathrm{Ho}\,\mathcal{C}$ of the comparison map

$$C \xrightarrow{\eta_C} \lim \Delta C \xrightarrow{\lim r_{\Delta C}} \lim R\Delta C \xrightarrow{\lim R\gamma} \lim RX$$

induced by the cone $R\gamma \circ r_{\Delta C}$ or alternatively (by commutativity of the above square), it is

$$C \xrightarrow{\eta_C} \lim \Delta C \xrightarrow{\lim \gamma} \lim X \xrightarrow{\lim r_X} \lim RX.$$

Note that this latter description is just the underived comparison map composed with the counit $\lim X \rightarrow \mathrm{holim}\, X$ of the right Kan extension $\mathrm{holim} = \mathbb{R} \lim$. In any case, the derived comparison map (which lives in $\mathrm{Ho}\,\mathcal{C}$) lifts to a morphism in $\mathcal{C}$.

Even though we now know (for model categories) that we can lift comparison maps to morphisms in the model category itself (rather than the homotopy category), this depends on an explicit construction of derived functors by replacements, which means that we always have to keep track of them. There is another approach, better suited for many situations concerning homotopy (co)limits or more generally Kan extensions, which is working with the derivator defined by a model category.

First off, viewing a cone as a natural transformation (i.e. a morphism in a diagram category) is not well-suited for the derived setting because morphisms in homotopy categories are complicated. Instead, we augment our indexing category $\mathcal{I}$ with an initial object $\perp$ and obtain

$$\underline{\mathcal{I}} := \mathcal{I} + \{\perp\}, \qquad \text{which comes with an inclusion } B\colon \mathcal{I} \hookrightarrow \underline{\mathcal{I}}.$$

(2.1) **Remark.** Note that this new indexing category $\underline{\mathcal{I}}$ is just the (contravariant) Grothendieck construction for the functor

$$[1]^{\mathrm{op}} \to \mathbf{Cat},\ 0 \mapsto \{\perp\},\ 1 \mapsto \mathcal{I}.$$

With this, a cone with vertex C above a diagram $X\colon \mathcal{I} \to \mathcal{C}$ is the same as an augmented diagram $Y\colon \underline{\mathcal{I}} \to \mathcal{C}$ such that $Y_\perp = C$ and $Y|_{\mathcal{I}} = B^*Y = X$. Now, note that right Kan extension along B maps a diagram X to a diagram $B_*X\colon \underline{\mathcal{I}} \to \mathcal{C}$, which agrees with X on $\mathcal{I}$ (because B is fully faithful) and has

$$(B_*X)_\perp = \lim\left(\perp \downarrow \mathcal{I} \xrightarrow{\cong} \mathcal{I} \xrightarrow{X} \mathcal{C}\right) = \lim_{\mathcal{I}} X.$$

Instead of a direct calculation, we can more abstractly observe that $\mathcal{C}^-\colon \mathbf{CAT}^{\mathrm{op}} \to \mathbf{CAT}$ is a (meta-)2-functor and hence maps the adjunction

$$\begin{tikzcd} \{\perp\} \arrow[r, hook, shift left=1.5, "I_\perp"] \arrow[r, phantom, "\perp"] & \underline{\mathcal{I}} \arrow[l, two heads, shift left=1.5, "P"] \end{tikzcd} \qquad \text{to an adjunction} \qquad \begin{tikzcd} \mathcal{C} \arrow[r, shift right=1.5, "P^*=\Delta"'] \arrow[r, phantom, "\top"] & \mathcal{C}^{\underline{\mathcal{I}}} \arrow[l, shift right=1.5, "I_\perp^*=\mathrm{ev}_\perp"'] \end{tikzcd}$$

(this is really just saying that taking the limit over an indexing category with an initial object is just evaluation at that object). With this, we get composite adjunctions

$$\begin{tikzcd} \mathcal{C} \arrow[r, shift left=1.5, "\Delta"] \arrow[r, phantom, "\perp"] & \mathcal{C}^{\underline{\mathcal{I}}} \arrow[l, shift left=1.5, "\mathrm{ev}_\perp"] \arrow[r, shift left=1.5, "B^*"] \arrow[r, phantom, "\perp"] & \mathcal{C}^{\mathcal{I}}, \arrow[l, shift left=1.5, "B_*"] \end{tikzcd}$$

which again proves that B_*X is just the completion of the diagram X by the corresponding limit but with the advantage of implying the same for the derived setting.

(2.2) **Observation.** If $\mathcal{C}$ is a model category then the restriction functor $B^*\colon \mathcal{C}^{\underline{\mathcal{I}}} \to \mathcal{C}^{\mathcal{I}}$ is both left and right Quillen with respect to the injective model structures (with pointwise weak equivalences and cofibrations), provided these exist (e.g. for $\mathcal{C}$ combinatorial).

Proof. The functor B^* is clearly left Quillen, for it preserves pointwise weak equivalences and cofibrations. To see that it is also right Quillen, note that its left adjoint just extends a diagram $X\colon \mathcal{I} \to \mathcal{C}$ to $\underline{\mathcal{I}}$ by an initial object:

$$(B_!X)_\perp = \operatorname{colim}\Big(\underbrace{B \downarrow \perp}_{\varnothing} \to \mathcal{I} \xrightarrow{X} \mathcal{C}\Big) = \varnothing.$$

In particular, $B_!$ is left Quillen. □

Now the comparison map $C = Y_\perp \to \lim_{\mathcal{I}} X = \lim_{\mathcal{I}} Y|_{\mathcal{I}}$ could again be obtained as the $\perp$-component of the unit $Y \to B_*B^*Y$ but we wouldn't gain anything that way because this would still be a morphism in $\operatorname{Ho}\mathcal{C}$.

Instead, we are going to construct the comparison map as an object in $\mathrm{Ho}(\mathcal{C}^{[1]})$. To this end, we define $\mathcal{J}$ to be the full subcategory of $\underline{\mathcal{I}} \times [1]$, consisting of all objects except $(\perp, 1)$, which comes with three inclusion functors

$$\underline{\mathcal{I}} \cong \underline{\mathcal{I}} \times \{0\} \xhookrightarrow{I_t} \mathcal{J}, \qquad \mathcal{I} \cong \mathcal{I} \times \{1\} \xhookrightarrow{I_b} \mathcal{J} \qquad \text{and} \qquad \mathcal{J} \xhookrightarrow{J} \mathcal{I} \times [1].$$

Now, the left Kan extension along the top-inclusion functor I_t is just the *doubling* of a diagram $X\colon \underline{\mathcal{I}} \to \mathcal{C}$. Indeed, $I_{t!}X$ agrees with X on $\underline{\mathcal{I}} \times \{0\}$ because I_t is fully faithful and has

$$\left(I_{t!}X\right)_{I,1} = \operatorname{colim}\left(I_t \downarrow (I,1) \to \underline{\mathcal{I}} \xrightarrow{X} \mathcal{C}\right) = X_I$$

because (I, id_I) is terminal in $I_t \downarrow (I,1)$.

(2.3) **Claim.** Given a cone $X\colon \underline{\mathcal{I}} \to \mathcal{C}$, the comparison map $X_\perp \to \lim_{\mathcal{I}} B^*X$ can be obtained by first doubling X to $\mathcal{J}$ and then evaluating the right Kan extension along J at $\perp$. More concisely, it is just $\mathrm{ev}_\perp J_* I_{t!} X$.

Let us first check that (homotopy) right Kan extension along J_* does indeed just complete the bottom layer of a $\mathcal{J}$-indexed diagram by the (homotopy) limit. More precisely, let us show that the following diagram of functors commutes up to natural isomorphism

(2.4)
$$\begin{tikzcd}
\mathcal{C}^{\underline{\mathcal{I}}} \arrow[r, "I_{t!}"] \arrow[d, "B^*"'] & \mathcal{C}^{\mathcal{J}} \arrow[r, "J_*"] \arrow[dl, "I_b^*"] & \mathcal{C}^{\underline{\mathcal{I}}\times[1]} \arrow[d, "I_1^*"] \\
\mathcal{C}^{\mathcal{I}} \arrow[rr, "B_*"'] & & \mathcal{C}^{\underline{\mathcal{I}}}
\end{tikzcd},$$

where I_1^* is the functor induced by $\{1\} \hookrightarrow [1]$ (i.e. restriction to the $(\underline{\mathcal{I}} \times \{1\})$-layer). This could be done by a direct calculation but would then require additional arguments for the derived setting. We shall do all in one step by taking left adjoints everywhere.

(2.5) **Remark.** The commutativity of the triangle in the above diagram is immediate from the explicit description of $I_{t!}$. Moreover, since all functors in it are homotopic, the same is true in the derived setting. Since we are going to need it later, let's switch to left adjoints anyway.

The only functor, where it is not immediately obvious that it has a left adjoint, is $I_{t!}$. There, we note that I_t has a retraction $R\colon \mathcal{J} \to \underline{\mathcal{I}}$, which is just the projection to the first factor: $R(I,n) = I$. Even better, $I_t \dashv R$ (i.e. $\underline{\mathcal{I}}$ is a coreflective subcategory of $\mathcal{J}$) with unit $\mathrm{Id}_{\underline{\mathcal{I}}}$ and counit $\varepsilon\colon I_t \circ R \Rightarrow \mathrm{Id}_{\mathcal{J}}$ given by

$$\varepsilon_{(I,0)} := \mathrm{id}_{(I,0)} \qquad \text{and} \qquad \varepsilon_{(I,1)} := (\mathrm{id}_I, 0 \leqslant 1).$$

By general abstract nonsense about Kan extensions (or direct verification), it follows that $I_{t!} \cong R^*$ and hence $I_{t!}$ has a further left adjoint $I_t^? := R_!$, which is easily explicited as

$$(I_t^? X)_I = \begin{cases} X_{I,1} & I \neq \perp \\ X_{\perp,0} & I = \perp \end{cases}$$

with the obvious functions on morphisms. In particular, this functor is left Quillen (and hence $I_{t!}$ is both left and right Quillen) with respect to the injective model structure (assuming $\mathcal{C}$ is a model category and the injective model structures exist).

As already mentioned, commutativity of the triangle in (2.4) is obvious (one can switch to left adjoints if one wants to). To check the commutativity of the trapezoid (after taking left adjoints), we observe that both

$$\left(I_{1!}X\right)_{I,0} = \operatorname{colim}\Big(\underbrace{I_1 \downarrow (I,0)}_{\varnothing} \to \underline{\mathcal{I}} \xrightarrow{X} \mathcal{C}\Big) = \varnothing \qquad \text{and}$$

$$\left(I_{b!}X\right)_{I,0} = \operatorname{colim}\Big(\underbrace{I_b \downarrow (I,0)}_{\varnothing} \to \mathcal{I} \xrightarrow{X} \mathcal{C}\Big) = \varnothing$$

are extensions of diagrams by initial objects. With this, commutativity upon switching to left adjoints is easily verified. So, now that we know that $I_{t!}$ is just doubling and J_* is completion by a (homotopy) limit, we can make the following definition.

(2.6) **Definition.** With the same notation as above, given a diagram $X\colon \underline{\mathcal{I}} \to \mathcal{C}$, the induced *comparison map* is the image of X under

$$\mathcal{C}^{\underline{\mathcal{I}}} \xrightarrow{I_{t!}} \mathcal{C}^{\mathcal{J}} \xrightarrow{J_*} \mathcal{C}^{\underline{\mathcal{I}}\times[1]} \cong \left(\mathcal{C}^{[1]}\right)^{\underline{\mathcal{I}}} \xrightarrow{\mathrm{ev}_\perp} \mathcal{C}^{[1]}$$

(or rather the right derived version thereof). Dually for (homotopy) colimits, which, by abuse of language, is also going to be referred to as the *comparison map*.

(2.7) **Remark.** One way to directly characterise the comparison map functor, is to say that the projection functor $Q\colon \underline{\mathcal{I}} \to [1]$ (coming from the construction of $\underline{\mathcal{I}}$ as a Grothendieck construction on [1]) induces

$$Q^*\colon \mathcal{C}^{[1]} \to \mathcal{C}^{\underline{\mathcal{I}}},$$

which maps $A \to B$ to a diagram that sends $\perp$ to A and everything else to B. Now, the comparison map functor is just the right adjoint Q_*. This is easily verified by taking left adjoints of all functors involved in the definition and checking that their composite is indeed Q^*.

(2.8) **Proposition.** If $X\colon \underline{\mathcal{I}} \to \mathcal{C}$ is a limit diagram then the comparison map is an isomorphism. Similarly in the derived context.

Proof. Recall that a diagram is a limit diagram iff it lies in the essential image of $B_*\colon \mathcal{C}^{\mathcal{I}} \to \mathcal{C}^{\underline{\mathcal{I}}}$. So, we need to show that the composite

$$\mathcal{C}^{\mathcal{I}} \xrightarrow{B_*} \mathcal{C}^{\underline{\mathcal{I}}} \xrightarrow{I_{t!}} \mathcal{C}^{\mathcal{J}} \xrightarrow{J_*} \mathcal{C}^{\underline{\mathcal{I}}\times[1]} \xrightarrow{\mathrm{ev}_\perp} \mathcal{C}^{[1]} \qquad \text{(or rather } \mathcal{C}^{\mathcal{I}} \xrightarrow{B_*} \mathcal{C}^{\underline{\mathcal{I}}} \xrightarrow{Q_*} \mathcal{C}^{[1]}\text{)}$$

is the same as (i.e. isomorphic to)

$$\mathcal{C}^{\mathcal{I}} \xrightarrow{\lim} \mathcal{C} \xrightarrow{\Delta} \mathcal{C}^{[1]}, \qquad \text{(which maps a diagram to its limit's identity morphism).}$$

Unsurprisingly, we do so by taking left adjoints everywhere (which will also imply the corresponding result for the derived functors). In both cases, the composite left adjoint maps a $Y_0 \to Y_1$ to the constant diagram Y_1. For the first composite, all left adjoints have already been discussed above and the claim is easily checked, while for the second composite, we just note that the left adjoint to $\Delta\colon \mathcal{C} \to \mathcal{C}^{[1]}$ is $\mathrm{ev}_1 = \operatorname{colim}$. □

3. Cubical Diagrams

In what follows, finite semilattices will play a central role. We usually prefer to work with join-semilattices; however, one could just as well work with meet-semilattices since the category of join-semilattices is isomorphic to that of meet-semilattices (using the opposite order). Our usage of the term "semilattice" includes finite (co)completeness as a category and it is understood that every semilattice has a bottom element (and since we are in the finite case, also a top, which is just the join of all elements).

(3.1) **Definition.** An *n-dimensional cube* (or just *n-cube*) is a poset that is a free semilattice on n generators (i.e. isomorphic to $\mathfrak{P}\{1, \ldots, n\}$). A *cubical diagram (of dimension) n* (or, by abuse of language, again just an *n-cube*) is a diagram $X\colon P \to \mathbf{sSets}$ indexed by an n-cube P.

(3.2) **Notation.** Given a cubical diagram $X\colon P \to \mathbf{sSets}$ with P generated (under joins) by $g_1, \ldots, g_n$, we write $X_{g_{i_1}, \ldots, g_{i_k}} := X_{g_{i_1} \vee \ldots \vee g_{i_k}}$ for $i_1, \ldots, i_k \in \{1, \ldots, n\}$. In particular, if $P = \mathfrak{P}\{1, \ldots, n\}$, we simply write $X_{i_1, \ldots, i_k}$ instead of $X_{\{i_1, \ldots, i_k\}}$.

(3.3) **Example.**

- Given a finite set M, the free semilattice on M is just (isomorphic to) $\mathfrak{P}(M)$, ordered by inclusion. So, we can always assume an n-cube to be of the form $\mathfrak{P}(M)$. This is the classical definition of a cube but unsuitable for our purposes because later on, we will find ourselves in situations where there is no natural way to identify a cube with some $\mathfrak{P}(M)$.
- Another, more geometrical, description is that a cube is a product of (categorical) intervals. That is to say, it is a category isomorphic to some $[1]^n$.

(3.4) **Remark.**

- We used the slightly awkward formulation "a poset that is a free semilattice" to stress the point that morphisms between cubes are arbitrary functors rather than just (co)limit preserving ones.
- Since $\mathfrak{P}(M)$ is self-dual, it is both (isomorphic to) the free join-semilattice and the free meet-semilattice on M (though in the latter case, it is generated by all $M \setminus \{m\}$ with $m \in M$).
- Since every free semilattice P is generated (as a category) by all arrows $a \leqslant a \vee m$ with $m \in M$, $m \nleqslant a$ (i.e. the order relation on P is just the reflexive and transitive closure of these), we usually won't bother indicating any other arrows in our diagrams.

(3.5) **Notation.** Writing $\langle n \rangle := \{1, \ldots, n\}$, we denote the *standard n-cube* $\mathfrak{P}\langle n \rangle$ by $\square^n$. We will also have to consider the indexing poset for n-dimensional pushouts $\ulcorner^n := \square^n \setminus \{\langle n \rangle\}$, which is just an n-cube with its top element removed. Dually, the indexing set for n-dimensional pullbacks is $\lrcorner^n := \square^n \setminus \{\varnothing\}$, which is just an n-cube with its bottom element removed.

(3.6) **Notation.** Since we will have to consider edges of the form $\langle n \rangle \setminus \{k\} \to \langle n \rangle$ into the terminal vertex of $\square^n$, let us define

$$\hat{k} := \neg k := \langle n \rangle \setminus \{k\} = \{1, \ldots, k-1, k+1, \ldots, n\}$$

(assuming n is clear from the context).

Let us take a second to note that cubes (being posets isomorphic to some finite powerset) are distributive lattices and, in fact, even Boolean algebras. The elements of a cube corresponding to singletons in a powerset (i.e. its generators under join) are called its *atoms*.

(3.7) **Remark.** As is well-known, every finite Boolean algebra is isomorphic to a powerset (so the terms "cube", "finite powerset", "finite free semilattice" and "finite Boolean algebra" are all equivalent). In particular they are all *atomistic*, meaning that every element is a join of atoms.

(3.8) **Definition.** Given a cube P and $a \leqslant b \in P$, we define the *face* ∂_a^b of P to be the subposet

$$\partial_a^b := \{p \in P \mid a \leqslant p \leqslant b\}.$$

Similarly, if $X\colon P \to \mathbf{sSets}$ is a cubical diagram, we define the *face* $\partial_a^b X := X|_{\partial_a^b}$.

(3.9) **Observation.** Let's write G for the generating set of P. Given $a \leqslant b$, b is of the form $b = a \vee g_1 \vee \ldots \vee g_n$ for some (necessarily unique up to permutation) pairwise distinct $g_1, \cdots, g_n \in G$. It follows that ∂_a^b is an n-cube by

$$\mathfrak{P}\langle n\rangle \cong \partial_a^b,\ I \mapsto a \vee \bigvee_{i\in I} g_i.$$

The dimension of ∂_a^b can either be obtained by looking at generators (as we just did) or by counting the number of elements of ∂_a^b.

(3.10) **Notation.** Given $a = g_1 \vee \ldots \vee g_k$ and $b = a \vee g_{k+1} \vee \cdots \vee g_{k+n}$, we write

$$\partial_{g_1,\ldots,g_k}^{g_1,\ldots,g_n} := \partial_a^b.$$

In particular, $\partial^b := \partial_\perp^b$ (the empty join is $\perp$) but since $\partial_a^\perp$ is silly (because then $a = \perp$ and the face is just $\perp$ itself), we define the special case $\partial_a := \partial_a^\top$.

(3.11) **Example.** Using a standard cube $\square^n$, then, in our notation, the edges $\varnothing \to \{k\}$ out of the initial vertex are denoted by ∂^k while the edges $\hat{k} \to \langle n\rangle$ into the terminal vertex are $\partial_{\hat{k}}$ (or $\partial_{\neg k}$). On the other hand ∂_k and $\partial^{\neg k}$ are $(n-1)$-dimensional faces.

(3.12) **Example.** Given a cubical diagram $X\colon \mathfrak{P}(M) \to \mathbf{sSets}$ and subsets $L \subseteq N \subseteq M$, the $|N \setminus L|$-dimensional face $\partial_L^N X$ of X is classically defined [30] to be the composite

$$\mathfrak{P}(N \setminus L) \hookrightarrow \mathfrak{P}(N) \subseteq \mathfrak{P}(M) \xrightarrow{X} \mathbf{sSets}, \quad S \mapsto X(S \cup L),$$

where the first inclusion is given by $S \mapsto S \cup L$. This already foreshadows complications arising from restricting our attention to $\mathfrak{P}(M)$ as indexing posets.

(3.13) **Notation.** Though we do not generally restrict our indexing posets to power sets, it is sometimes useful to have an ordering of the generators. Given $X\colon \square^n \to \mathbf{sSets}$ (or $X\colon \ulcorner^n \to \mathbf{sSets}$), we will suggestively write

$$X_{\mathrm{top}} := \partial^{\langle n-1\rangle} X = X|_{\square^{n-1}} \qquad \text{and} \qquad X_{\mathrm{bot}} := \partial_n X$$

for its top and bottom $(n-1)$-face, respectively.

(3.14) **Example.** A cube over $M = \{1, 2, 3\}$ is a commutative diagram of spaces

$$\begin{tikzcd}
 & X_\varnothing \arrow[rr] \arrow[dl] \arrow[dd] & & X_1 \arrow[dl] \arrow[dd] \\
X_2 \arrow[rr] \arrow[dd] & & X_{1,2} \arrow[dd] & \\
 & X_3 \arrow[rr] \arrow[dl] & & X_{1,3} \arrow[dl] \\
X_{2,3} \arrow[rr] & & X_{1,2,3} &
\end{tikzcd}$$

(where, again, we left out all the diagonals). The face $\partial^1 X$ is just $X_\varnothing \to X_1$, $\partial^{1,2} X$ is the top square, $\partial_{1,2} X$ is $X_{1,2} \to X_{1,2,3}$, $\partial_1 X$ is the right-hand square etc.

(3.15) **Observation.** Given two faces ∂_a^b and $\partial_{a'}^{b'}$, we have $\partial_{a'}^{b'} \subseteq \partial_a^b$ iff both $b' \leqslant b$ and $a \leqslant a'$. Consequently, the largest common subface (if any) of two faces ∂_a^b and $\partial_{a'}^{b'}$ is $\partial_{a \vee a'}^{b \wedge b'}$. This is only well-defined iff $a' \leqslant b$ and $a \leqslant b'$. Otherwise, the two faces do not share a common subface.

Let us record for later that every $(n-1)$-face ∂_a^b of an n-dimensional cube P (say generated by $g_1, \cdots, g_n$) has a unique *opposite face*, characterised by being the only one that doesn't intersect it (necessarily in a $(n-2)$-dimensional subface). Indeed, ∂_a^b is either of the form $b = \top$ and $a = g_i$ or $b = \neg g_i = g_1 \vee \ldots \widehat{g_i} \vee \ldots g_n$ and $a = \bot$ for some $i \in \{1, \ldots, n\}$. With this, ∂_{g_i} and $\partial^{\neg g_i}$ are opposite.

(3.16) **Definition.** For a cube generated by $g_1, \cdots, g_n$, faces of the form $\partial^{\neg g_i}$ (i.e. the $(n-1)$-dimensonal faces that contain $\bot$) are called *initial*, while those of the form ∂_{g_i} (i.e. the $(n-1)$-dimensional faces that contain $\top$) are called *terminal*.

4. Grothendieck Construction for Posets

(4.1) **Observation.** For any set I (viewed as a discrete category), there is a natural equivalence of categories

$$\widehat{I} = \mathbf{Sets}^I \simeq \mathbf{Sets} \downarrow I, \quad (X_i)_{i \in I} \mapsto \coprod_{i \in I} X_i, \quad (p^{-1} i)_{i \in I} \leftarrow\!\shortmid p$$

between I-indexed families of sets and sets over I.

Now instead of a set I and an I-indexed family (which is just a functor $I \to \mathbf{Sets}$), we can take any small category $\mathcal{I}$ and perform an analogous construction for a functor $\mathcal{I} \to \mathbf{Cat}$.

(4.2) **Definition.** Given a small category $\mathcal{I}$ the *Grothendieck construction* on $F \colon \mathcal{I} \to \mathbf{Cat}$ is the category $\int^{\mathcal{I}} F$, with objects all pairs (I, C) where $I \in \mathcal{I}$, $C \in FI$ and

$$\mathrm{Hom}_{\int^{\mathcal{I}} F}((I, C), (J, D)) := \{(i, f) \mid i \colon I \to J \text{ in } \mathcal{I},\ f \colon (Fi)C \to D \text{ in } FJ\}$$

for two such objects (I, C), $(J, D) \in \int^{\mathcal{I}} F$. The composite of $(I, C) \xrightarrow{(i,f)} (J, D) \xrightarrow{(j,g)} (K, E)$ is just

$$\left(I \xrightarrow{i} J \xrightarrow{j} K, (Fj)(Fi)C \xrightarrow{(Fj)f} (Fj)D \xrightarrow{g} E\right),$$

so that the identity on (I, C) is $(\mathrm{id}_I, \mathrm{id}_C)$.

(4.3) **Proposition.** If $\mathcal{I}$ and every FI with $I \in \mathcal{I}$ is even a poset then so is $P := \int^{\mathcal{I}} F$.

Proof. Clearly, P is a preorder (i.e. there is at most one morphism between any two objects) by definition of its Hom-sets. To see that it is even a poset, assume that we have isomorphic objects $(I, C) \leqslant (J, D) \leqslant (I, C)$. By definition then $I \leqslant J \leqslant I$ (whence $I = J$) and

$$F(I \leqslant J)C \leqslant D \qquad \text{as well as} \qquad F(J \leqslant I)D \leqslant C.$$

But $I \leqslant J$ and $J \leqslant I$ are actually identities and so $C \leqslant D \leqslant C$. $\square$

(4.4) **Example.** Consider the functor

$$F\colon \ulcorner^2 \to \mathbf{Cat},\ (\{2\} \leftarrow \varnothing \to \{1\}) \mapsto \left(\ulcorner^2 \xleftarrow{\mathrm{id}} \ulcorner^2 \to \{\varnothing\}\right),$$

whose Grothendieck construction is

$$\begin{tikzcd}
(\{2\},\{1\}) & (\varnothing,\{1\}) \arrow[l] \arrow[dr] & \\
(\{2\},\varnothing) \arrow[u] \arrow[d] & (\varnothing,\varnothing) \arrow[l] \arrow[r] \arrow[u] \arrow[d] & (\{1\},\varnothing) \\
(\{2\},\{2\}) & (\varnothing,\{2\}) \arrow[l] \arrow[ur] &
\end{tikzcd}
\quad \cong \quad
\begin{tikzcd}
\{1,3\} & \{1\} \arrow[l] \arrow[dr] & \\
\{3\} \arrow[u] \arrow[d] & \varnothing \arrow[l] \arrow[r] \arrow[u] \arrow[d] & \{1,2\} \\
\{2,3\} & \{2\} \arrow[l] \arrow[ur] &
\end{tikzcd}
\quad = \quad \ulcorner^3.$$

As already observed, the category of posets is closed under Grothendieck constructions. Still, in order to generalise the previous example, let us record some easy special properties of the Grothendieck construction occurring there.

(4.5) **Observation.** Let P be a poset and $F\colon P \to \mathbf{Cat}$ a functor such that every Fp with $p \in P$ is again a poset and all morphisms $F(p \leqslant p')\colon Fp \to Fp'$ are identities or morphisms into the terminal poset $\{*\}$. Then $\int^P F$ is again a poset and given (p, q), $(p', q') \in \int^P F$, we have $(p, q) \leqslant (p', q')$ iff firstly $p \leqslant p'$ and secondly $Fp' = \{*\}$ or $Fp = Fp'$ with $q \leqslant q'$.

(4.6) **Remark.** Of course, there is also a dual version of the Grothendieck construction, which assigns to a functor $F\colon \mathcal{I}^{\mathrm{op}} \to \mathbf{Cat}$ the category $\int_{\mathcal{I}} F$, which again has as objects all pairs (I, C) with $I \in \mathcal{I}$, $C \in FI$ but

$$\mathrm{Hom}_{\int_{\mathcal{I}} F}\big((I, C), (J, D)\big) := \{(i, f) \mid i\colon I \to J \text{ in } \mathcal{I},\ f\colon C \to (Fi)D \text{ in } FI\}$$

with the composition of $(I, C) \xrightarrow{(i,f)} (J, D) \xrightarrow{(j,g)} (K, E)$ being

$$\left(I \xrightarrow{i} J \xrightarrow{j} K, C \xrightarrow{f} (Fi)D \xrightarrow{(Fi)g} (Fi)(Fj)E\right).$$

Now, given a functor $F\colon \mathcal{I} \to \mathbf{Cat}$, we get a second functor $F^{\mathrm{op}}\colon \mathcal{I} \to \mathbf{Cat}$ by composing F with the dualisation functor $-^{\mathrm{op}}\colon \mathbf{Cat} \to \mathbf{Cat}$. One can easily verify that then

$$\left(\int^{\mathcal{I}} F\right)^{\mathrm{op}} = \int_{\mathcal{I}^{\mathrm{op}}} F^{\mathrm{op}}.$$

5. Building Higher Dimensional Cubes

The goal of this section is to see two ways of how an $(n+1)$-cube $\square^{n+1}$ can be built from lower dimensional cubes using the Grothendieck construction. The most obvious way to do this, while not very useful, is to just interpret the fact that $\square^{n+1} \cong \square^n \times [1]$ as a Grothendieck construction by saying that

$$\square^{n+1} \cong \int^{[1]} \mathrm{Const}_{\square^n}.$$

Since this is just a complicated way of expressing the categorical product, we shall not do it. Another obvious identification, which we, again, shall not use, is

$$\square^{n+1} \cong \int^{[1]} (\ulcorner^{n+1} \to \{*\}).$$

However, combining these two identifications gives us something useful. To wit, we define a functor

$$F \colon \square^2 \to \mathbf{Cat}, \quad \begin{tikzcd} \varnothing \arrow[r] \arrow[d] & \{1\} \arrow[d] \\ \{2\} \arrow[r] & \{1,2\} \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} \ulcorner^n \arrow[r] \arrow[d, "\mathrm{Id}"'] & \{\varnothing\} \arrow[d] \\ \ulcorner^n \arrow[r] & \{\varnothing\} \end{tikzcd}.$$

Clearly, $\int^{\square^2} F \cong \square^{n+1}$ with the top $\ulcorner^n \to \{\varnothing\}$ corresponding to the top face and the bottom $\ulcorner^n \to \{\varnothing\}$ corresponding to the bottom face. Since such an identification depends on the choice of two such opposite faces, we get a total of $n+1$ possible identifications.

(5.1) **Notation.** For readability's sake, we will omit the curly brackets from singletons in the Grothendieck construction that follow. So for example, we are going to write "$(1,2)$" instead of "$(\{1\},\{2\})$".

(5.2) **Definition.** Given $k \in \langle n+1\rangle$, and letting $\delta^k \colon [n] \to [n+1]$ be the usual k^{th} coface map, we write

$$\Gamma_k \colon \int^{\square^2} F \cong \square^{n+1}, \qquad \begin{array}{rcl} (\varnothing, S) & \mapsto & \delta^k S \\ (1, \varnothing) & \mapsto & \langle n+1\rangle \setminus \{k\} \\ (2, S) & \mapsto & \delta^k S \cup \{k\} \\ (\{1,2\}, \varnothing) & \mapsto & \langle n+1\rangle \end{array}$$

for the identification corresponding to the two opposite faces ∂_k and $\partial^{\neg k}$. The name "Γ" firstly stands for *Grothendieck* and secondly (as a symbol) resembles the $\ulcorner^n$ used in the construction.

The default such identification for us (when it is not explicitly specified) is going to be Γ_{n+1} because there, the isomorphism looks particularly simple as δ^{n+1} has no effect and can thus be omitted.

By restricting F, we also get $\int^{\ulcorner^2} F \cong \ulcorner^{n+1}$. Dually, we can construct $\square^{n+1}$ as a contravariant Grothendieck construction; namely that of

$$G \colon (\square^2)^{\mathrm{op}} \to \mathbf{Cat}, \quad \begin{tikzcd} \varnothing \arrow[r] \arrow[d] & \{1\} \arrow[d] \\ \{2\} \arrow[r] & \{1,2\} \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} \{\varnothing\} & \lrcorner^n \arrow[l] \\ \{\varnothing\} \arrow[u] & \lrcorner^n \arrow[l] \arrow[u, "\mathrm{Id}"'] \end{tikzcd}$$

(note that upon forming the Grothendieck construction $\int_{\square^2} G$, the direction of all arrows in the right-hand square is again reversed). Let us also fix a notation for these identifications

(5.3) **Definition.** Given $k \in \langle n+1 \rangle$, and again letting $\delta^k \colon [n] \to [n+1]$ be the k^{th} coface map, we write

$$\Gamma^k \colon \int_{\square^2} G \cong \square^{n+1}, \qquad \begin{array}{rcl} (\varnothing,\varnothing) & \mapsto & \varnothing \\ (1,S) & \mapsto & \delta^k S \\ (2,\varnothing) & \mapsto & \{k\} \\ (\{1,2\},S) & \mapsto & \delta^k S \cup \{k\} \end{array}$$

for the identification corresponding to the two opposite faces ∂_k and $\partial^{\neg k}$.

Again, as in the dual case, we get $\int_{\lrcorner^2} G \cong \lrcorner^{n+1}$ by restriction. In particular, using Thomason's theorem, cubical pushouts and pullbacks can be calculated inductively as follows.

(5.4) **Proposition.** Given $k \in \langle n \rangle$ and a diagram $X \colon \lrcorner^n \to \mathbf{sSets}$, its homotopy pullback can be calculated as

$$\operatorname*{holim}_{\lrcorner^n} X \simeq \operatorname{holim}\left(X_k \to \operatorname*{holim}_{\lrcorner^{n-1}}(X|_{\partial_k}) \leftarrow \operatorname*{holim}_{\lrcorner^{n-1}}(X|_{\partial^{\neg k}}) \right).$$

Dually for homotopy pushouts.

Proof. This is a direct application of Thomason's theorem (7.5.2). $\square$

Since we are going to need it later on, let us introduce some notation for the homotopy pullback cube obtained from a given cubical diagram by using Thomason's theorem in connection with Γ_k.

(5.5) **Definition.** Given $n \in \mathbb{N}_{\geqslant 3}$ and $k \in \langle n \rangle$, we define $\mathcal{T}^k := \square^n \setminus \{\varnothing, \{k\}\}$. With this then, for $X \colon \square^n \to \mathbf{sSets}$, we write

$$\mathbb{R}_k X := \mathbb{R}\operatorname{Kan}(X|_{\mathcal{T}^k}) \colon \square^n \to \mathbf{sSets}.$$

(5.6) **Remark.** By Thomason's theorem, every $\mathbb{R}_k X$ is a homotopy pullback cube and it comes with a canonical transformation $X \Rightarrow \mathbb{R}_k X$, namely the derived unit. If X happens to already be a homotopy pullback, then by the above proposition, the square

$$\begin{tikzcd} X_\varnothing \arrow[r] \arrow[d] & X_k \arrow[d] \\ (\mathbb{R}_k X)_\varnothing \arrow[r] & (\mathbb{R}_k X)_k \end{tikzcd}$$

defined by the $\varnothing$- and the k-component of the derived unit is a homotopy pullback.

(5.7) **Example.** Given a diagram $X \colon \ulcorner^n \to \mathbf{sSets}$ that has $X_S = *$ for every $S \neq \varnothing$ (i.e. $X = G_{X_\varnothing}$ in terms of (1.3)), its homotopy colimit is $\operatorname{hocolim} X \simeq \Sigma^{n-1} X_\varnothing$. The cases $n = 1$ and $n = 2$ are easy. For bigger n, we just note that

$$\begin{aligned} \operatorname*{hocolim}_{\ulcorner^n} X &\simeq \operatorname{hocolim}\big(X_{\{1,\ldots,n-1\}} \leftarrow \operatorname{hocolim}(X_{\text{top}}|_{\ulcorner^{n-1}}) \to \operatorname{hocolim}(X_{\text{bot}})\big) \\ &\simeq \operatorname{hocolim}(* \leftarrow \Sigma^{n-2} X_\varnothing \to *) \simeq \Sigma^{n-1} X_\varnothing. \end{aligned}$$

by induction and because X_{bot} is constantly $*$.

(5.8) **Example.** Given simplicial sets $A_1, \dots, A_n$ and $X \colon \ulcorner^n \to \mathbf{sSets}$ defined as

$$X_S := \prod_{k \notin S} A_k$$

with all maps being standard projections. Then $\operatorname{hocolim} X \simeq \ast_{k=1}^n A_k$. The case $n = 1$ is trivial and for $n = 2$, this is the definition of $A_1 \ast A_2$. For bigger n, we can assume all A_k to be Kan complexes. By Mather's Cube Theorem (7.7.3), taking products with Kan complexes preserves homotopy colimits and with Thomason's theorem (7.5.2), we calculate

$$\begin{aligned}\operatorname*{hocolim}_{\ulcorner^n} X &\simeq \operatorname{hocolim}(X_{\{1,\dots,n-1\}} \leftarrow \operatorname{hocolim}(X_{\mathrm{top}}|_{\ulcorner^{n-1}}) \to \operatorname{hocolim}(X_{\mathrm{bot}})) \\ &\simeq \operatorname{hocolim}\left(A_n \leftarrow A_n \times \mathop{\ast}_{k=1}^{n-1} A_k \to \mathop{\ast}_{k=1}^{n-1} A_k\right) \simeq \mathop{\ast}_{k=1}^{n} A_k.\end{aligned}$$

For another extremely useful method of constructing higher dimensional cubes, we note that the diagram F of posets in the Grothendieck construction for $\square^{n+1}$ contains two copies of $\ulcorner^n$. We can now apply the same Grothendieck construction to these and obtain

$$\square^{n+1} \quad\cong\quad \int^{\square^2} \left(\begin{tikzcd} \ulcorner^n \arrow[r] \arrow[d] & \{\varnothing\} \arrow[d] \\ \ulcorner^n \arrow[r] & \{\varnothing\} \end{tikzcd}\right) \quad\cong\quad \int^{\square^3} \left(\begin{tikzcd}[row sep=small, column sep=small] & \ulcorner^{n-1} \arrow[rr] \arrow[dl] \arrow[dd] & & \{\varnothing\} \arrow[dl] \arrow[dd] \\ \ulcorner^{n-1} \arrow[rr, crossing over] \arrow[dd] & & \{\varnothing\} & \\ & \ulcorner^{n-1} \arrow[rr] \arrow[dl] & & \{\varnothing\} \arrow[dl] \\ \ulcorner^{n-1} \arrow[rr] & & \{\varnothing\} \arrow[from=uu, crossing over] & \end{tikzcd}\right).$$

We can now just keep going and get $\square^{n+1}$ as a Grothendieck construction of a diagram containing only points and $\ulcorner^k$ for an arbitrary $k \in \{1, \dots, n+1\}$. The most interesting case for us, however, is going to be $k = 2$. Instead of going through the described inductive process, we are going to establish the isomorphism directly. For this, let us recycle some functor names. We now let

$$F \colon \square^n \to \mathbf{Cat},\ S \mapsto \begin{cases} \ulcorner^2 & n \notin S \\ \{\varnothing\} & n \in S \end{cases}$$

(with all arrows being either identities or the unique maps into a point). Note that the observation (4.5) applies to this functor.

(5.9) **Lemma.** The Grothendieck construction $\int^{\square^n} F$ is a join-semilattice generated by

$(\varnothing, 1)$, $(\varnothing, 2)$, $(1, \varnothing)$, $(2, \varnothing)$, $\dots$, $(n-1, \varnothing)$.

Proof. Clearly, $(\varnothing, \varnothing)$ is a bottom element. Now, given (S, T), $(S', T') \in \int^{\square^n} F$, to get the join $(S, T) \vee (S', T')$, we observe that its first component must certainly contain $S \cup S'$ and distinguish several cases. If $n \in S \cup S'$ then $(S, T) \vee (S', T') = (S \cup S', \varnothing)$. If $n \notin S \cup S'$, we again need to distinguish two cases. If $T = \varnothing$ or $T' = \varnothing$ or $T = T'$ then $(S, T) \vee (S', T') = (S \cup S', T \cup T')$ and otherwise $(S, T) \vee (S', T') = (S \cup S' \cup \{n\}, \varnothing)$ (in particular $(\varnothing, 1) \vee (\varnothing, 2) = (n, \varnothing)$). So $\int^{\square^n} F$ is a join-semilattice. It is generated by the claimed elements because given $(S, T) \in \int^{\square^n} F$, if $T = \varnothing$ then

$$(S, T) = \bigvee_{i \in S} (i, \varnothing) \qquad \text{(where we write $(n, \varnothing)$ as $(\varnothing, 1) \vee (\varnothing, 2)$ in case $n \in S$).}$$

On the other hand, if $T \neq \varnothing$ (so that $n \notin S$) then

$$(S, T) = (\varnothing, T) \vee \bigvee_{i \in S} (i, \varnothing).$$

$\square$

(5.10) **Proposition.** There are isomorphisms $\square^{n+1} \cong \int^{\square^n} F$.

Proof. Because $\square^{n+1}$ is the free join-semilattice on $\langle n+1\rangle$, the map

$$1, 2, \ldots, n+1 \quad \mapsto \quad (\varnothing, 1), (\varnothing, 2), (1, \varnothing), (2, \varnothing), \ldots, (n-1, \varnothing)$$

(any bijection will do) can be extended to a morphism of join-semilattices $\square^{n+1} \to \int^{\square^n} F$. This morphism is certainly surjective because it hits all the generators. To see that it is even bijective (and thus an isomorphism of join-semilattices), we just need to check that the two posets involved have the same number of elements. Obviously, $|\square^{n+1}| = 2^{n+1}$ and on the other hand, every $S \in \square^n$ gives us three elements in $\int^{\square^n} F$ if $n \notin S$ (i.e. $S \in \mathfrak{P}\langle n-1\rangle$) and one element otherwise (i.e. $S = T \cup \{n\}$ for $T \in \mathfrak{P}\langle n-1\rangle$). So, all in all,

$$\left|\int^{\square^n} F\right| = 2^{n-1} \cdot 3 + 2^{n-1} \cdot 1 = 2^{n-1} \cdot 4 = 2^{n+1}.$$

$\square$

(5.11) **Corollary.** There are isomorphisms $\ulcorner^{n+1} \cong \int^{\ulcorner^n} F$.

Proof. The two posets are respectively obtained from $\square^{n+1}$ and $\int^{\square^n} F$ by removing the top element. $\square$

From the proof of the proposition, we know that the identification $\square^{n+1} \cong \int^{\square^n} F$ depends on a bijection with $1, \ldots, n+1$. Consequently, there are $(n+1)!$ such identifications but let's look a little closer.

By definition, F is constantly $\ulcorner^2$ on the face $\partial^{\neg n} \subseteq \square^n$ and constantly $\{\varnothing\}$ on ∂_n. Now, many identifications $\square^{n+1} \cong \int^{\square^n} F$ arise from one another by some symmetry of $\square^{n-1}$ applied to both of these opposite faces and since F is constant on them, two such identifications are the same for all intents and purposes. When identifying such isomorphisms with each other, every equivalence class has $(n-1)!$ members.

What really makes a difference is with which face $\partial_n \times \{\varnothing\} \subseteq \int^{\square^n} F$ is identified. Note that such a face is necessarily of the form $\partial_{k,l}$ as it has to contain $\langle n+1\rangle$. To establish our notation, let us pick the following representing family of identifications.

(5.12) **Definition.** Given $k, l \in \langle n+1\rangle$ with $k < l$ (i.e. an $(n-2)$-face $\partial_{k,l}$) and writing $\delta^{k,l} := \delta^k \circ \delta^l \colon [n-1] \to [n+1]$ for the monotone injection that avoids k and l, we define

$$\Gamma_{k,l} \colon \int^{\square^n} F \cong \square^{n+1}$$

to be the isomorphism determined by

$$\Gamma_{k,l}(\varnothing, 1) := k, \quad \Gamma_{k,l}(\varnothing, 2) := l \quad \text{and} \quad \Gamma_{k,l}(i, \varnothing) := \delta^{k,l} i \text{ for } i \in \langle n-1\rangle.$$

Obviously, we can dualise the above proposition and its corollary. This is particularly useful because the cubes $\square^n$ are self-dual, yielding

$$\square^{n+1} \cong (\square^{n+1})^{\mathrm{op}} \cong \left(\int^{\square^n} F\right)^{\mathrm{op}} = \int_{(\square^n)^{\mathrm{op}}} F^{\mathrm{op}}.$$

Upon also identifying $(\square^n)^{\mathrm{op}} \cong \square^n$ appearing as the indexing category in the right-hand Grothendieck construction, we obtain the following result.

(5.13) **Proposition.** There are isomorphisms $\square^{n+1} \cong \int_{\square^n} G$, where

$$G\colon (\square^n)^{\mathrm{op}} \to \mathbf{Cat},\ S \mapsto \begin{cases} \lrcorner^2 & n \in S \\ \{\varnothing\} & n \notin S \end{cases}$$

(with all arrows being either identities or the unique maps into a point). $\square$

Just like in the dual case above, we are really only interested in such identifications up to symmetry of $\square^{n-1}$. Let us also fix some notation for a representing family of such identifications.

(5.14) **Definition.** Given $k, l \in \langle n+1 \rangle$ with $k < l$ (i.e. an $(n-2)$-face $\partial^{\neg\{k,l\}}$) and writing $\delta^{k,l} := \delta^k \circ \delta^l\colon [n-1] \to [n+1]$ for the monotone injection that avoids k and l, we define

$$\Gamma^{k,l}\colon \int_{\square^n} G \cong \square^{n+1}$$

to be the isomorphism determined by

$$\Gamma^{k,l}(n,1) := k, \quad \Gamma^{k,l}(n,2) := l \quad \text{and} \quad \Gamma^{k,l}(i,\varnothing) := \delta^{k,l} i \text{ for } i \in \langle n-1 \rangle.$$

(this determines the isomorphism since these elements generate $\int_{\square^n} G$ as a join-semi-lattice).

The most important aspect of the Grothendieck construction $\int^{\square^n} F$ that makes it a worthwhile model for $\square^{n+1}$ is the existence of distinguished 2-faces as on the left below for $S \subseteq \langle n \rangle$ with $n \notin S$.

(5.15)
$$\begin{tikzcd} (S,\varnothing) \arrow[r] \arrow[d] & (S,1) \arrow[d] \\ (S,2) \arrow[r] & (S \cup \{n\}, \varnothing) \end{tikzcd} \qquad \begin{tikzcd} (S,\varnothing) \arrow[r] \arrow[d] & (S \cup \{n\},1) \arrow[d] \\ (S \cup \{n\},2) \arrow[r] & (S \cup \{n\}, \{1,2\}) \end{tikzcd}$$

These are indeed 2-faces, namely $\partial^{(S\cup\{n\},\varnothing)}_{(S,\varnothing)}$. Let us call these the *Thomason faces* of $\int^{\square^n} F$. Dually, the *Thomason faces* in $\int_{\square^n} G$ are those 2-faces as on the right above, again with $n \notin S$.

(5.16) **Remark.** While not important for our applications, let's see what the other 2-faces of $\int^{\square^n} F$ are, as the Thomason faces only account for 2^{n-1} of the $E_{2,n+1} = n(n+1)/2 \cdot 2^{n-1}$ many. All other 2-faces are obtained from 2-faces of $\square^n$ as follows.

(a) Every 2-face ∂^T_S with $n \notin T$ yields 3 different 2-faces in $\square^{n+1}$ by adding $\varnothing$, 1 or 2 as a second coefficient.

(b) Every 2-face ∂^T_S with $n \notin S$ but $n \in T$ yields 2 different 2-faces in $\square^{n+1}$ by adding $\varnothing$ as a second coefficient to $S \cup \{n\}$, T and 1 or 2 as a second coefficient to its two other vertices ($\varnothing$ is not allowed here, since we have $(S,\varnothing) \leqslant (S,1) \leqslant (S \cup \{n\},\varnothing)$).

(c) Every 2-faces ∂^T_S with $n \in S$ yields a 2-face in $\square^{n+1}$ by adding $\varnothing$ as a second coefficient.

These, together with the faces described above, give us all 2-faces of $\square^{n+1}$. Indeed, from (a), we get $3 \cdot E_{2,n-1}$ faces; from (b), we get $2 \cdot E_{1,n-1}$; and from (c), we get $E_{2,n-1}$. So, all in all,

$$\begin{aligned}4E_{2,n-1} + 2E_{1,n-1} &= 4\frac{(n-1)n}{2}\,2^{n-3} + (n-1)2^{n-1} = \frac{(n-1)n}{2}\,2^{n-1} + (n-1)2^{n-1} \\ &= \frac{(n-1)(n+2)}{2}\,2^{n-1} = \frac{n^2+n-2}{2}\,2^{n-1}.\end{aligned}$$

Adding in the 2^{n-1} 2-faces constructed above, we obtain all $E_{2,n+1} = (n^2+n)/2 \cdot 2^{n-1}$ faces.

(5.17) **Definition.** A set of 2-faces in $\square^n$ is called *Thomason* iff it is the image of the Thomason faces under some isomorphism $\int^{\square^{n-1}} F \cong \square^n$ (without loss of generality, some $\Gamma_{k,l}$).

Let us record the following elementary characterisations as well as one elementary property of Thomason sets.

(5.18) **Proposition.** Let $n \in \mathbb{N}$, $n > 3$ and $\mathfrak{T}$ a set of 2-faces in $\square^n$.

(a) $\mathfrak{T}$ is Thomason iff it is the image of the Thomason 2-faces under some identification $\int_{\square^{n-1}} G \cong \square^n$ (without loss of generality, some $\Gamma^{k,l}$).

(b) $\mathfrak{T}$ is Thomason iff there are distinct $k, l \in \langle n \rangle$ such that

$$\mathfrak{T} = \mathfrak{T}_{k,l} := \left\{ \partial_S^{S\cup\{k,l\}} \;\middle|\; S \subset \langle n \rangle,\ k, l \notin S \right\}.$$

(c) Every 2-face of $\square^n$ is contained in exactly one Thomason set.

Proof. Ad (a): The two Thomason faces (5.15) have the same image under $\Gamma_{k,l}$ and $\Gamma^{k,l}$ respectively.
Ad (b): The terminal vertices of all Thomason faces of $\int^{\square^{n-1}} F$ are precisely the vertices in $\partial_{n-1} \times \{\varnothing\}$ and, as was already observed, any identification $\int^{\square^{n-1}} F \cong \square^n$ maps $\partial_{n-1} \times \{\varnothing\}$ to some $\partial_{k,l}$. Conversely, every $\partial_{k,l}$ determines $(n-2)!$ possible identifications mapping $\partial_{n-1} \times \{\varnothing\}$ to $\partial_{k,l}$. With this, the claim follows.
Ad (c): Any 2-face is of the form $\partial_S^{S\cup\{k,l\}}$ for some distinct $k, l \notin S$. Consequently, the unique Thomason set that contains it is $\mathfrak{T}_{k,l}$. □

The usefulness of Thomason sets plainly stems from the fact that we can calculate the homotopy colimit of a diagram $X \colon \ulcorner^n \to \mathbf{sSets}$ by choosing a Thomason set $\mathfrak{T}_{k,l}$ and replacing every span

$$X_{S\cup\{k\}} \leftarrow X_S \to X_{S\cup\{l\}}$$

in $\mathfrak{T}_{k,l}$ by its homotopy pushout. Using the comparison maps to the $X_{S\cup\{k,l\}}$ then determines a diagram $\ulcorner^{n-1} \to \mathbf{sSets}$, whose homotopy colimit is weakly equivalent to that of X. Similarly for homotopy limits.

Of course, this method boils down to the same thing as the one using the identifications Γ_k from the beginning of this section. The only difference is that here, we construct a $\ulcorner^{n-1}$-indexed diagram by calculating some 2-dimensional homotopy pushouts, whereas with the first approach, we form a $\ulcorner^2$-indexed diagram by calculating some $(n-1)$-dimensional pushouts.

Since we are going to need it later, let's establish some notation for the pullback cubes obtained from a given cube $X \colon \square^n \to \mathbf{sSets}$ by using a certain Thomason set.

(5.19) **Definition.** For $n \in \mathbb{N}_{\geqslant 3}$ and two distinct $k, l \in \langle n \rangle$, we define $\mathfrak{I}^{k,l} \subset \square^n$ to be the subposet of all $S \in \square^n$ with $k \in S$ or $l \in S$. These are exactly the initial vertices of all the Thomason faces in $\mathfrak{T}_{k,l}$. Put differently, $\mathfrak{I}^{k,l}$ is obtained by restricting

$$\Gamma^{k,l} \colon \int_{\square^{n-1}} G \cong \square^n \qquad \text{to } \partial_n \subset \square^{n-1}.$$

Now, given a cubical diagram $X \colon \square^n \to \mathbf{sSets}$ and distinct $k, l \in \langle n \rangle$, we define

$$\mathbb{R}_{k,l} X := \mathbb{R}\,\mathrm{Kan}\,(X|_{\mathfrak{I}^{k,l}}) : \square^n \to \mathbf{sSets},$$

to be the cube obtained by replacing all Thomason faces of X in $\mathfrak{T}_{k,l}$ by homotopy pullbacks. Dually for $\mathfrak{I}_{k,l}$ and $\mathbb{L}_{k,l} X$, which is obtained by replacing all Thomason faces belonging to $\mathfrak{T}_{k,l}$ by homotopy pushouts.

(5.20) **Remark.** By Thomason's theorem, every $\mathbb{R}_{k,l} X$ is a homotopy pullback cube. Moreover, it comes with a canonical transformation $X \Rightarrow \mathbb{R}_{k,l} X$, namely the derived unit.

6. Higher Pushouts and Pullbacks

As is well-known, if we put the projective model structure on the category of spans $\mathbf{sSets}^{\ulcorner^2}$, the cofibrant diagrams (which is sufficient for their pushout to be a homotopy pushout) $X \colon \ulcorner^2 \to \mathbf{sSets}$ are those where $X_\varnothing$ is cofibrant and $X_\varnothing \to X_1$, $X_\varnothing \to X_2$ are both cofibrations. Dually for the injective model structure.

The projective and injective model structure on $\mathbf{sSets}^{\ulcorner^2}$ exist either because $\mathbf{sSets}$ is a combinatorial model category (see [37, Proposition A.2.8.2]) or because $\ulcorner^2$ is "very small" (see [22]). Alternatively (and this is really what lies behind the "very small"-argument), they can be seen as special instances of Reedy model structures, which is what we are going to do in higher dimensions and quickly describe how to obtain (co)fibrant replacements.

(6.1) **Definition.** The *standard* (or *projective*) *Reedy structure* on $\ulcorner^n$ has degree function

$$d(S) := |S|\,.$$

With this, $\ulcorner^n$ is actually a direct category (see [34]) and the induced Reedy model structure is the projective one. Dually, the *standard* (or *injective*) *Reedy structure* on $\lrcorner^n$ has the degree function

$$d(S) := n - |S|\,,$$

so that $\lrcorner^n$ is inverse and the induced Reedy model structure is the injective one.

Working in the case of $\ulcorner^n$ equipped with the projective Reedy structure (the case of $\lrcorner^n$ being dual), the latching object at an object $S \in \ulcorner^n$ for some diagram X is just

$$L_S X = \operatorname*{colim}_{T \subsetneq S} X_T = \operatorname*{colim}_{\ulcorner^{|S|}} X|_{\partial^S_\varnothing \setminus \{S\}}$$

the (strict!) pushout of X restricted to the face $\partial^S_\varnothing$ with the terminal vertex S removed.

Now, for a diagram $X \colon \ulcorner^n \to \mathbf{sSets}$ to be cofibrant, we need every induced map $L_S X \to X_S$ with $S \in \ulcorner^n$ to be a cofibration. For $S = \varnothing$, this just means that $X_\varnothing$ must be cofibrant and for $S = \{i\}$, we get that $X_\varnothing \to X_i$ must be a cofibration. So, for $n = 2$ we can cofibrantly replace a diagram X by first taking a cofibrant replacement $Q_\varnothing$ of $X_\varnothing$ and then factoring the composite maps

$$Q_\varnothing \xrightarrow{\sim} X_\varnothing \to X_i \qquad \text{as} \qquad Q_\varnothing \rightarrowtail Q_i \xrightarrow{\sim} X_i \qquad (\text{for } i \in \{1,2\}).$$

For a general n, we first replace $X_\varnothing$ (say by a $Q_\varnothing$) as well as all $X_\varnothing \to X_i$ as above (say by Q_i). We then replace the rest of the X_S by a recursion on $|S|$. To wit, having already replaced all X_T (say by Q_T) with $|T| \leqslant |S|-1$, we can calculate L_SQ and then replace the composite map

$$L_SQ \to L_SX \to X_S \qquad \text{by} \qquad L_SQ \rightarrowtail Q_S \xrightarrow{\sim} X_S.$$

(6.2) **Observation.** If $X\colon \ulcorner^n \to \mathbf{sSets}$ is cofibrant (so that its colimit is a homotopy colimit), the same is true for every restriction of X to a face.

7. Homotopy Pullbacks and Closed Classes

Dror Farjoun's theorem (8.4.8) gives us an extremely powerful cellular inequality for the fibre of an induced map between homotopy colimits. In this section, we are going to look at the dual situation and would like to have an estimate for the Bousfield class of the fibre of an induced map between homotopy limits. Of course, since fibres and homotopy limits commute, we can just forget about the "fibre" part and directly focus on the Bousfield class of a homotopy limit.

We are confident that there is some more general cellular statement hidden behind this, which does not restrict the form of the indexing category. However, since we are only going to be interested in the cubical case, let us restrict our attention to cubes.

(7.1) **Proposition.** If $X\colon \square^n \to \mathbf{sSets}$ is a homotopy pullback cube then

$$X_\varnothing > \left\{ \Omega_*^{|S|-1} X_S \;\middle|\; S \in \lrcorner^n \right\} > \left\{ \Omega^{|S|-1} X_S \;\middle|\; S \in \lrcorner^n \right\}.$$

for every base-point in $X_\varnothing$ and with all X_S having the induced base-points.

Proof. The second acyclic inequality is trivial because, by definition, $\Omega_* A > \Omega A$ for every pointed space A. So let us assume we are in the pointed context. In the degenerate case $n=1$, being a homotopy pullback just means being a weak equivalence and the result is trivial. For $n=2$, we start with a homotopy pullback square

$$\begin{tikzcd} P \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

and writing $F := \mathrm{hFib}_*(P \to B) \simeq \mathrm{hFib}_*(C \to D)$, we have

$$P > \{F, B\} \qquad \text{while} \qquad F > \{C, \Omega_* D\}.$$

For the general case, we proceed by induction. To wit, using Thomason's theorem, we have a homotopy pullback square

$$\begin{tikzcd} X_\varnothing \arrow[r] \arrow[d] & \operatorname{holim}_{\lrcorner^{n-1}} \partial^{\neg n} X \arrow[d] \\ X_n \arrow[r] & \operatorname{holim}_{\lrcorner^{n-1}} \partial_n X. \end{tikzcd}$$

From the dimension 2 case, we know that

$$X_\varnothing > \{X_n, \operatorname{holim} \partial^{\neg n} X, \Omega_* \operatorname{holim} \partial_n X\}$$

and by the inductive hypothesis,

$$\operatorname{holim} \partial^{\neg n} X > \left\{ \Omega_*^{|S|-1} X_S \,\middle|\, n \notin S,\, S \neq \varnothing \right\},$$

while

$$\operatorname{holim} \partial_n X > \left\{ \Omega_*^{|S|-2} X_S \,\middle|\, n \in S,\, S \neq \{n\} \right\}$$

(the exponent $|S| - 2$ comes from the identification $\square^{n-1} \cong \partial_n$, $S \mapsto S \cup \{n\}$). □

(7.2) **Example.** Given a natural transformation $X \Rightarrow Y$ between homotopy pullbacks $X, Y \colon \square^n \to \mathbf{sSets}$, then

$$\operatorname{hFib}_*(X_\varnothing \to Y_\varnothing) \simeq \operatorname*{holim}_{S \in \lrcorner^n} \operatorname{hFib}_*(X_S \to Y_S) > \left\{ \Omega^{|S|-1} \operatorname{hFib}_*(X_S \to Y_S) \,\middle|\, S \in \lrcorner^n \right\}$$

(all with respect to an arbitrary base-point coming from $X_\varnothing$).

8. Strong Homotopy Colimits

(8.1) **Definition.** Given a poset P, and $p \in P$, we write $\downarrow\! p := \{q \in P \mid q < p\}$. With this, a diagram $X \colon P \to \mathbf{sSets}$ is a *strong homotopy colimit diagram* iff for every $p \in P$ such that $p = \bigvee_{q \in \downarrow p} q$, the induced map

$$\operatorname*{hocolim}_{\downarrow p}(X|_{\downarrow p}) \to X_p$$

is a weak equivalence. As the most important special case, a cubical diagram $X \colon \square^n \to \mathbf{sSets}$ that is a strong homotopy colimit will be called a *strong homotopy pushout (cube).* Dually for a *strong homotopy limit* and a *strong homotopy pullback (cube).*

(8.2) **Remark.** Obviously, we could more generally define strong homotopy colimits indexed by an arbitrary category $\mathcal{I}$ by letting $\downarrow\! I$ (with $I \in \mathcal{I}$) be the full subcategory of $\mathcal{I} \downarrow I$ consisting of all objects except the terminal one (I, id_I) and interpreting $\bigvee \downarrow\! I$ as the colimit of the canonical projection $\downarrow\! I \to \mathcal{I}$.

Note that if P is of the form $P = P' + \{\top\}$, where $\top$ is a top element in P and P' does not have a top element, then a strong homotopy colimit diagram $X \colon P \to \mathbf{sSets}$ is in particular a homotopy colimit diagram (i.e. $\operatorname{hocolim}_{P'} X \to X_\top$ is a weak equivalence). However, we explicitly want to allow diagrams without a terminal vertex, such as in the following example.

(8.3) **Example.** Taking P to be

$$\begin{tikzcd} \bullet \arrow[d] & \bullet \arrow[l] \arrow[r] \arrow[d] & \bullet \arrow[d] \\ \bullet & \bullet \arrow[l] \arrow[r] & \bullet \end{tikzcd},$$

a P-indexed diagram is a strong homotopy colimit iff the two squares are homotopy pushouts.

(8.4) **Example.** For $P = \square^n$, being a strong homotopy pushout $X\colon \square^n \to \mathbf{sSets}$ just means that all faces (including the entire cube) have to be homotopy pushouts.

Since Mather's second cube theorem is the essential ingredient in our study of closed classes, let us quickly record that it is also valid for strong homotopy colimits.

(8.5) **Theorem. (Mather's Second Cube Theorem, Strong Version)** In **sSets**, homotopy pullbacks preserve strong homotopy colimits. More precisely, pulling back along a Kan fibration $f\colon K \to L$

$$f^* := (- \times_L K)\colon \mathbf{sSets}\downarrow L \to \mathbf{sSets}\downarrow K$$

preserves strong homotopy colimits (where the two comma categories inherit a model structure from **sSets** and so strong homotopy colimits in them are calculated in **sSets**).

Proof. Given a poset P, $p \in P$ such that $p = \bigvee_{q \in \downarrow p} q$ and $X\colon P \to \mathbf{sSets}$ a strong homotopy colimit diagram, we get weak equivalences

$$\operatorname*{hocolim}_{\downarrow p}(f^* X|_{\downarrow p}) \simeq f^* \operatorname*{hocolim}_{\downarrow p}(X|_{\downarrow p}) \simeq f^* X\left(\bigvee\nolimits_{q\in\downarrow p} q\right) \simeq f^* X_p,$$

where the first one is by the ordinary second cube theorem and the second one by f being a Kan fibration (so that pulling back along f preserves weak equivalences). $\square$

(8.6) **Corollary. (Special Puppe Theorem, Strong Version)** Given a poset P and a strong homotopy colimit diagram $X\colon P \to \mathbf{sSets}$ together with a transformation $\tau\colon X \Rightarrow K$ to a constant diagram (i.e. X is a diagram in $\mathbf{sSets}\downarrow K$) then

$$\mathrm{hFib}_k\, \tau\colon P \to \mathbf{sSets},\ p \mapsto \mathrm{hFib}_k(\tau_p\colon X_p \to K)$$

is again a strong homotopy colimit for every base-point k of K.

Proof. Just take $f\colon PK \to K$ in the theorem to be the standard path space fibration. $\square$

The classical definition of a strong homotopy pushout cube requires all its 2-faces to be homotopy pushouts and seems weaker than our definition. Thanks to the work done in section 5 we can show that the two conditions are actually equivalent.

(8.7) **Proposition.** An n-cube $X\colon \square^n \to \mathbf{sSets}$ is a strong homotopy pushout iff all its 2-faces are homotopy pushouts.

Proof. The direction "$\Rightarrow$" is obvious. We prove the converse by induction on n, where the case $n = 2$ is trivial. Now given $X\colon \square^{n+1} \to \mathbf{sSets}$ with all 2-faces homotopy pushouts, the inductive hypothesis tells us that all of X's proper faces are homotopy pushouts. Moreover, upon identifying $\ulcorner^{n+1} \cong \int^{\ulcorner^n} F$ as in (5.11), Thomason's theorem yields

$$\operatorname{hocolim} X|_{\ulcorner^{n+1}} \simeq \operatorname*{hocolim}_{S\in\ulcorner^n} \operatorname*{hocolim}_{T\in FS} X(S,T).$$

But FS is either a point (if $n \in S$) or $\ulcorner^2$ (if $n \notin S$), so that

$$\operatorname*{hocolim}_{T\in FS} X(S,T) \simeq X\left(\bigvee_{T\in FS}(S,T)\right) = X(S\cup\{n\}, \varnothing)$$

by our hypothesis. All in all

$$\operatorname{hocolim} X|_{\ulcorner^{n+1}} = \operatorname*{hocolim}_{S\in\ulcorner^n} X(S\cup\{n\},\varnothing).$$

But this is just the homotopy colimit of an n-dimensional subcube of X and the inductive hypothesis gives us that

$$\operatorname{hocolim} X|_{\ulcorner^{n+1}} \simeq X\left(\bigvee_{S\in\ulcorner^n} X(S\cup\{n\},\varnothing)\right) = X(\langle n\rangle,\varnothing).$$

□

9. Strong Homotopy Pushouts and Pullbacks

Given a cubical diagram $X\colon \square^n \to \mathbf{sSets}$, we can simply forget its terminal vertex $X_{\langle n\rangle}$ and take the homotopy pushout of the restricted diagram $X|_{\ulcorner^n}$. If, instead of an ordinary homotopy pushout, we want to take the strong homotopy pushout (rendering every face a homotopy pushout), we have to forget all values of X except $X_\varnothing$ and the X_i with $i\in\langle n\rangle$. To construct these strong homotopy pushouts and reason about them, let us fix the notation

$${}^{\triangleleft}\langle n\rangle := \{\varnothing,\{1\},\dots,\{n\}\}\subseteq \mathfrak{P}\langle n\rangle$$

for the categorical cone on the discrete category $\langle n\rangle$ obtained by adding an initial object to it. Dually, we have the categorical cone

$$\langle n\rangle^{\triangleright} := \{\hat{1},\dots,\hat{n},\langle n\rangle\}\subseteq \mathfrak{P}\langle n\rangle \qquad (\text{where } \hat{k} := \langle n\rangle\setminus\{k\})$$

on the discrete category containing all $\{1,\dots,\hat{k},\dots,n\}$, obtained by adding a terminal object.

For a diagram $X\colon {}^{\triangleleft}\langle n\rangle \to \mathbf{sSets}$, we can now construct its strong homotopy pushout as the homotopy left Kan extension along the inclusion $I\colon {}^{\triangleleft}\langle n\rangle \hookrightarrow \square^n$.

(9.1) **Proposition.** Given a diagram $X\colon {}^{\triangleleft}\langle n\rangle \to \mathbf{sSets}$, then the extended diagram $\mathbb{L}I_!X\colon \square^n\to\mathbf{sSets}$ is a strong homotopy colimit, which is weakly equivalent to X on ${}^{\triangleleft}\langle n\rangle$.

Proof. That $\mathbb{L}I_!X$ agrees with X on ${}^{\triangleleft}\langle n\rangle$ is due to I being fully faithful. Now, given $\mathfrak{S}\subseteq\square^n$ with join $T := \bigvee\mathfrak{S}$ we need to check that the induced map

$$\operatorname{hocolim}\big((\mathbb{L}I_!X)|_{\downarrow\mathfrak{S}}\big) \to (\mathbb{L}I_!X)_T$$

is a weak equivalence. For this, we note that the homotopy colimit on the left is nothing but the homotopy left Kan extension of $(\mathbb{L}I_!X)|_{\downarrow\mathfrak{S}}$ along the inclusion $J\colon \downarrow\mathfrak{S}\hookrightarrow \overline{\downarrow\mathfrak{S}} = \downarrow\mathfrak{S}\cup\{T\}$ evaluated at T. So, we need to show that

$$\Big((\mathbb{L}J_!)\big((\mathbb{L}I_!X)|_{\downarrow\mathfrak{S}}\big)\Big)_T \to (\mathbb{L}I_!X)_T$$

is a weak equivalence. Here, we simply use the fact that left Kan extensions as well as left derived functors compose. More precisely, by Dubuc's theorem [38, Exercise X.3.4.3], the left Kan extension $I_!$ can be split up into a composition of two left Kan extensions along

$${}^{\triangleleft}\langle n\rangle \xhookrightarrow{I'} \downarrow\mathfrak{S} \xhookrightarrow{I''} \square^n, \qquad \text{i.e.} \qquad I_! \cong I''_!\circ I'_!$$

(in our context, this also follows immediately from looking at the right adjoints). Since everything is fully faithful, we then get

$$(I_!X)|_{\downarrow X} = (I'')^*I_!X \cong (I'')^*I''_!I'_!X \cong I'_!X$$

(and the same in the derived setting). Finally, writing $K\colon {}^{\triangleleft}\langle n\rangle \hookrightarrow \overline{\downarrow\mathfrak{S}}$, we get

$$\Big((\mathbb{L}J_!)\big((\mathbb{L}I_!X)|_{\downarrow\mathfrak{S}}\big)\Big)_T \simeq \big((\mathbb{L}J_!)(\mathbb{L}I'_!)X\big)_T \simeq (\mathbb{L}K_!X)_T \simeq (\mathbb{L}I_!X)_T,$$

where the second equivalence is again by Dubuc's theorem and the last one by the same argument as above, using that evaluation at T is just restriction to $\{T\}$. $\square$

(9.2) **Example.** Given a diagram $X\colon \langle n\rangle^{\triangleright} \to \mathbf{sSets}$ with strong homotopy pullback $P\colon \square^n \to \mathbf{sSets}$ and $X_{\langle n\rangle} = *$ then

$$P_S \simeq \prod_{k\notin S} X_{\hat{k}}$$

with the maps $P_S \to P_T$ induced by the $S \subseteq T$ corresponding to the standard projections. Indeed, the injective model structure on $\mathbf{sSets}^{\langle n\rangle^{\triangleright}}$ is also the one induced by the inverse Reedy structure on $\langle n\rangle^{\triangleright}$ where all $\hat{k}$ have degree 1 and $\langle n\rangle$ has degree 0. It is then easily checked that X is fibrant iff every $X_{\hat{k}}$ is a Kan complex. Replacing the $X_{\hat{k}}$ if necessary, we calculate P_S for $S \in \square^n$ as

$$\lim\Big(S\downarrow\langle n\rangle^{\triangleright} \to \langle n\rangle^{\triangleright} \xrightarrow{X} \mathbf{sSets}\Big) = \lim\Big(\Big\{\hat{k}\ \Big|\ k\notin S\Big\} \cup \{\langle n\rangle\} \xrightarrow{X} \mathbf{sSets}\Big) = \prod_{k\notin S} X_{\hat{k}}.$$

The construction of a strong homotopy pushout cube as a homotopy left Kan extension shows that these satisfy the following stronger condition: For every subset S, the induced map

$$\operatorname*{hocolim}_{\downarrow S}(X|_{\downarrow S}) \to X\Big(\bigvee S\Big)$$

is a weak equivalence, where

$$\downarrow S := \{p \mid \exists s \in S\colon p \leqslant s\}\,.$$

But even with this stronger requirement, we are not sure if this is the "correct" notion of a strong homotopy colimit diagram because, even with this stronger condition, a strong homotopy colimit diagram need not be a homotopy colimit diagram (for pathological cases).

With this consideration in mind, a better condition (at least for a poset P with a bottom element $\perp$) would be to require a strong homotopy colimit diagram $X\colon P \to \mathbf{sSets}$ to be weakly equivalent to a left homotopy Kan extension along the inclusion $A \hookrightarrow P$ of all *atoms* (plus the bottom element)

$$A := \{a \in A \mid \text{there is no } b \in A \text{ such that } \perp < b < a\}\,.$$

While this definition seems better conceptually, there is no obvious way to extend it to arbitrary indexing categories.

10. Puppe's Theorem

In this section, we are going to start from a strong homotopy pullback $X\colon \square^n \to \mathbf{sSets}$ and study the comparison map $Q \to X_{\langle n\rangle}$, where Q is the (ordinary) homotopy pushout of $X|_{\ulcorner^n}$. Because Puppe's theorem (7.7.5) (or rather, its strong analogue (8.6)) allows us to easily calculate the fibre of this comparison map, we will refer to this new result as Puppe's theorem, as well.

Before stating the theorem, note that if $X\colon \square^n \to \mathbf{sSets}$ is a strong homotopy pullback, lots of fibres can be identified with each other. In fact, given $S \subseteq T$ in $\square^n$, we note that the two squares in

$$\begin{tikzcd} X_\varnothing \arrow[r] \arrow[d] & X_S \arrow[r] \arrow[d] & X_{(\langle n\rangle\setminus T)\cup S} \arrow[d] \\ X_{T\setminus S} \arrow[r] & X_T \arrow[r] & X_{\langle n\rangle} \end{tikzcd}$$

are homotopy pullbacks, so that the induced maps

$$\mathrm{hFib}(X_\varnothing \to X_{T\setminus S}) \to \mathrm{hFib}(X_S \to X_T) \to \mathrm{hFib}\left(X_{(\langle n\rangle\setminus T)\cup S} \to X_{\langle n\rangle}\right)$$

are both weak equivalences. Put differently it suffices to know the fibres of all maps into the terminal vertex $X_{\langle n\rangle}$ (or out of the initial vertex $X_\varnothing$ but the former seems more natural as we are talking about strong homotopy pullbacks). Finally, since pullbacks commute with taking fibres, and every X_S is the strong homotopy pullback of all $X_{\hat\imath} \to X_{\langle n\rangle}$ with $i \in S$, it is even enough to know all fibres

$$F_{\hat\imath} := \mathrm{hFib}\big(X_{\hat\imath} \to X_{\langle n\rangle}\big)$$

(which can be identified with the $F_i := \mathrm{hFib}(X_\varnothing \to X_i)$ by the above argument). Let us make this relation more precise.

(10.1) **Lemma.** Let $X\colon \square^n \to \mathbf{sSets}$ be a strong homotopy pullback diagram with fibres $F_{\hat\imath} := \mathrm{hFib}_x\big(X_{\hat\imath} \to X_{\langle n\rangle}\big)$ above some base-point $x \in X_{\langle n\rangle}$. Then

$$\mathrm{hFib}_x\big(X_S \to X_{\langle n\rangle}\big) \simeq \prod_{i\notin S} F_{\hat\imath}$$

for all $S \in \square^n$ with the maps $\mathrm{hFib}_x\big(X_S \to X_{\langle n\rangle}\big) \to \mathrm{hFib}_x\big(X_T \to X_{\langle n\rangle}\big)$ induced by the $S \subseteq T$ corresponding to the standard projections.

Proof. This is just the strong Puppe theorem (8.6), combined with example (9.2). □

(10.2) **Theorem. (Puppe)** For $X\colon \square^n \to \mathbf{sSets}$ a strong homotopy pullback, the homotopy fibre of the comparison map $q\colon \operatorname{hocolim} X|_{\ulcorner^n} \to X_{\langle n\rangle}$ above $x \in X_{\langle n\rangle}$ is

$$\mathrm{hFib}_x\, q \simeq \mathop{\circledast}_{i=1}^{n} F_{\hat\imath} \qquad \text{where } F_{\hat\imath} := \mathrm{hFib}_x(X_{\hat\imath} \to X_{\langle n\rangle}).$$

In particular then, $\mathrm{h}\mathfrak{F}\mathrm{ib}(q) \simeq \circledast_{i=1}^{n} \mathrm{h}\mathfrak{F}\mathrm{ib}\big(X_{\hat\imath} \to X_{\langle n\rangle}\big)$.

Proof. By the strong Puppe theorem (8.6), we know that

$$\begin{aligned}\mathrm{hFib}_x\left(\operatorname*{hocolim}_{S\in\ulcorner^n} X_S \xrightarrow{q} X_{\langle n\rangle}\right) &\simeq \operatorname*{hocolim}_{S\in\ulcorner^n} \mathrm{hFib}_x\big(X_S \to X_{\langle n\rangle}\big) \\ &\simeq \operatorname*{hocolim}_{S\in\ulcorner^n} \prod_{i\notin S} F_{\hat\imath} \overset{(5.8)}{\simeq} \mathop{\circledast}_{i=1}^{n} F_{\hat\imath}.\end{aligned}$$

□

(10.3) **Corollary.** Let $A\colon \square^n \to \mathbf{sSets}$ be a strong homotopy pushout with fibres

$$\mathfrak{F}_i := \mathrm{h}\mathfrak{F}\mathrm{ib}(A_\varnothing \to A_i) \qquad \text{and} \qquad \mathfrak{F}_{\hat\imath} := \mathrm{h}\mathfrak{F}\mathrm{ib}(A_{\hat\imath} \to A_{\langle n\rangle})$$

If $X\colon \square^n \to \mathbf{sSets}$ is the strong homotopy pullback obtained from all $A_{\hat\imath} \to A_{\langle n\rangle}$ then

$$\mathrm{h}\mathfrak{F}\mathrm{ib}\left(\operatorname{hocolim} X|_{\ulcorner^n} \xrightarrow{q} A_{\langle n\rangle}\right) \simeq \mathop{\circledast}_{i=1}^{n} \mathfrak{F}_{\hat\imath} \gg \mathop{\circledast}_{i=1}^{n} \mathfrak{F}_i$$

Proof. This is just the last theorem, combined with Chachólski's theorem applied to the homotopy pushouts

$$\begin{array}{ccc} A_\varnothing & \longrightarrow & A_{\hat{i}} \\ \downarrow & & \downarrow \\ A_i & \longrightarrow & A_{\langle n \rangle} \end{array} .$$

□

11. Quotients of Posets

In this section, we are going to study quotients of posets, which we will have to consider in the next section. More precisely, we are asking when the quotient of a poset by a subset (which a priori is only a preorder) is again a poset. This is purely out of convenience and the quotient in the next section is concrete enough to allow for a direct verification. However, the general case is easy and interesting enough to justify doing it.

Given a preorder P together with a subset $Q \subseteq P$, we can define a new preorder P/Q by collapsing Q down to a point as follows.

(11.1) **Definition.** Given a preorder P, together with a subset $Q \subseteq P$, the preorder P/Q has P/Q as its underlying set and

$$[x] \leqslant [y] \qquad \text{iff} \qquad x \leqslant y \text{ or there are } q,\, q' \in Q \text{ such that } x \leqslant q \text{ and } q' \leqslant y,$$

which is well-defined and easily checked to be reflexive and transitive. Moreover, with this definition, the quotient map $P \to P/Q$ is a morphism of preorders.

We will usually be in the situation where P is a poset. However, for a general $Q \subseteq P$, the quotient P/Q need not be a poset and one has to further identify isomorphic objects.

(11.2) **Example.** Taking $P := [2] = \{0 < 1 < 2\}$ and $Q := \{0, 2\}$, the quotient P/Q is the groupoidal interval with two isomorphic objects.

Instead of taking the preorder quotient and then further identifying isomorphic objects to obtain the posetal quotient, we can just as well replace Q first by some bigger subset that guarantees the quotient to be a poset.

(11.3) **Definition.** A subset Q of a poset P is called *convex* iff whenever we have $p \leqslant x \leqslant q$ in P with p, $q \in Q$ then also $x \in Q$. Clearly, P is convex and an intersection of convex subsets is again convex, so that every subset $Q \subseteq P$ has a *convex closure*.

(11.4) **Observation.** Instead of abstractly defining the convex closure $\bar{Q}$ of a set Q as the smallest convex set containing it, we can be more explicit:

$$\bar{Q} = \{x \in X \mid \text{there are } p,\, q \in Q \text{ with } p \leqslant x \leqslant q\}\,.$$

This set is convex because if we have $x \leqslant z \leqslant y$ with x, $y \in \bar{Q}$ then there are p, p', q, $q' \in Q$ such that $p \leqslant x \leqslant p'$ and $q \leqslant y \leqslant q'$, so that $p \leqslant x \leqslant z \leqslant y \leqslant q'$ and thus $z \in \bar{Q}$. Also, clearly $Q \subseteq \bar{Q}$ and $\bar{Q}$ must be contained in every convex subset that contains Q.

(11.5) **Proposition.** A subset $Q \subseteq P$ of a poset P is convex iff P/Q is again a poset.

Proof. "$\Rightarrow$": Let Q be convex and $[x] \leqslant [y] \leqslant [x]$ in P/Q, which is to say that $x \leqslant y \leqslant x$ (and thus $x = y$) or that $y \leqslant x$ and there are

$$q,\, q' \in Q \qquad \text{such that} \qquad x \leqslant q \text{ and } q' \leqslant y,$$

in which case $q' \leqslant y \leqslant x \leqslant q$, so that $x, y \in Q$ by convexity and in particular $[x] = [y] \in P/Q$. Similarly for the case where $x \leqslant y$. Finally, there might be

$$q,\, q',\, q'',\, q''' \in Q \qquad \text{such that} \qquad x \leqslant q,\, q' \leqslant y \leqslant q'',\, q''' \leqslant x.$$

Again, by convexity of Q, it follows that $x, y \in Q$ and in particular $[x] = [y]$ in P/Q.
"$\Leftarrow$": If $p \leqslant x \leqslant q$ with $p, q \in Q$ then $[p] \leqslant [x] \leqslant [q]$ in P/Q. But $[p] = [q]$ and P/Q is a poset, so that even $[p] = [x] = [q]$, meaning $x \in Q$. □

(11.6) **Proposition.** Let P be a poset and $Q \subseteq P$ with convex closure $\bar{Q}$. Writing $P/\!\!/Q$ for the poset associated to P/Q (obtained by identifying isomorphic objects), we have an isomorphism of posets

$$P/\bar{Q} \cong P/\!\!/Q,\ [x] \leftrightarrow [x].$$

Proof. For better readability, we denote the equivalence class of $x \in P$ in P/Q, $P/\bar{Q}$ and $P/\!\!/Q$ respectively by $[x]$, $\bar{x}$ and $[\![x]\!]$. We first check that the map from the proposition is a well-defined bijection. For this, we note that $[\![x]\!] = [\![y]\!]$ iff $[x] \leqslant [y] \leqslant [x]$ for which there are four possibilities. We can have $x \leqslant y \leqslant x$ (in which case $x = y$) or that there are $q, q' \in Q$ with $q' \leqslant y \leqslant x \leqslant q$ or $q' \leqslant x \leqslant y \leqslant q$ (either way $x, y \in \bar{Q}$) or, finally, that there are q, q', q'', $q''' \in Q$ with $x \leqslant q$, $q' \leqslant y \leqslant q''$, $q''' \leqslant x$ (where again $x, y \in \bar{Q}$). All in all, we have $[\![x]\!] = [\![y]\!]$ iff $x = y$ or $x, y \in \bar{Q}$, which is to say $\bar{x} = \bar{y}$.

So our map is a well-defined bijection and we need to check that $\bar{x} \leqslant \bar{y}$ iff $[\![x]\!] \leqslant [\![y]\!]$. In one direction, we have $[\![x]\!] \leqslant [\![y]\!]$ iff $[x] \leqslant [y]$ (because $P/\!\!/Q$ is obtained from P/Q by identifying isomorphic objects), which is the case iff $x \leqslant y$ or there are $q, q' \in Q$ with $x \leqslant q$ and $q' \leqslant y$. But $Q \subseteq \bar{Q}$ and so, this implies $\bar{x} \leqslant \bar{y}$. Conversely, for $\bar{x} \leqslant \bar{y}$ we must have $x \leqslant y$ or there are $\bar{p}, \bar{q} \in \bar{Q}$ with $x \leqslant \bar{p}$, $y \leqslant \bar{q}$, which means that there are $p, p', q, q' \in Q$ with

$$x \leqslant \bar{p},\, p \leqslant \bar{p} \leqslant p' \qquad \text{and} \qquad \bar{q} \leqslant y,\, q \leqslant \bar{q} \leqslant q'.$$

But then $x \leqslant p'$ and $q \leqslant y$, so that $[x] \leqslant [y]$ and thus $[\![x]\!] \leqslant [\![y]\!]$. □

12. Webs and Layers

In the following section, we generalise a very useful trick, which we learned from [13], where it is used to show the suspended square case (11.2.1). The idea is to start with a span $B \leftarrow A \rightarrow C$ and extend it by zig-zags at the end, yielding

$$* \leftarrow B \rightarrow B \leftarrow A \rightarrow C \leftarrow C \rightarrow *.$$

By viewing this as a Grothendieck construction in two different ways an applying Thomason's theorem (7.5.2), one then shows that the homotopy colimit does not change but can use the extended diagram for some cellular arguments that wouldn't have been possible with the original span. Since this is the motivating example and all techniques are just generalisations of the arguments outlined in [13], we are going to keep track of this more tractable case throughout this section.

(12.1) **Definition.** Given a poset P with a bottom element $\bot$, we define its *web* $\mathrm{Web}(P)$ to be the poset obtained as the Grothendieck construction

$$\mathrm{Web}(P) := \int^P W \qquad \text{where} \qquad W\colon P \to \mathbf{Cat},\, p \mapsto \begin{cases} \{\varnothing\} & p = \bot \\ \ulcorner^2 & p \neq \bot \end{cases}$$

with $W(\bot \leqslant x)\colon \{\varnothing\} \hookrightarrow \ulcorner^2$ being the element $\{1\}$ for $x \neq \bot$ and all other morphisms being identities. For later use, we also define a smaller version of the *web*

$$\mathrm{web}(P) := \int^P w \qquad \text{where} \qquad w\colon P \to \mathbf{Cat},\, p \mapsto \begin{cases} \{\varnothing\} & p = \bot \\ \square^1 = \{\varnothing \hookrightarrow \{1\}\} & p \neq \bot \end{cases}$$

again with $w(\bot \leqslant x)\colon \{\varnothing\} \hookrightarrow \square^1$ being the element $\{1\}$ for $x \neq \bot$ and all other morphisms being identities.

(12.2) **Remark.** Note that if we let $C\colon P \to \mathbf{Cat}$ send $\bot$ to $\varnothing$ and everything else to $\lrcorner^1 = \{1\}$ (so C is constantly a singleton) then the canonical projection $\int^P C \to P$ is an isomorphism and the pointwise inclusions define natural transformations $C \Rightarrow w \Rightarrow W$, which then in turn give us canonical inclusions $P \cong \int^P C \hookrightarrow \mathrm{web}(P) \hookrightarrow \mathrm{Web}(P)$. On the other hand, we also have natural transformations in the other direction $W \Rightarrow w \Rightarrow C$, where the first one's component at $x \neq \bot$ maps $\{2\}$ to $\varnothing$ and every other element of $\ulcorner^2$ to itself. These then assemble to canonical retractions $\mathrm{Web}(P) \to \mathrm{web}(P) \to \int^P C \cong P$ of the above inclusions. Note that $\mathrm{web}(P) \to P$ and the composite $\mathrm{Web}(P) \to P$ are just the natural projections, which exist for every Grothendieck construction.

(12.3) **Example.** As already mentioned, the motivational example for the web construction is the case where we take $P = \ulcorner^2 = \{\bullet \leftarrow \bullet \rightarrow \bullet\}$, treated in [13]. Here, its web looks as follows:

$$\mathrm{Web}(P) = \{\bullet \leftarrow \bullet \rightarrow \bullet \leftarrow \bullet \rightarrow \bullet \leftarrow \bullet \rightarrow \bullet\}, \quad \mathrm{web}(P) = \{\bullet \rightarrow \bullet \leftarrow \bullet \rightarrow \bullet \leftarrow \bullet\}$$

Since this is the motivational example, maybe the name *zig-zag* would have been appropriate if not a bit uninspired.

(12.4) **Example.** The term "web" stems from the example $P = \ulcorner^3$, where $\mathrm{Web}(P)$ is of the following form (omitting some composites)

$$\begin{tikzcd}
 & & & (\{2,3\},2) & & & \\
(\{3\},2) \arrow[urrr] \arrow[dddddd] & & & (\{2,3\},\varnothing) \arrow[u] \arrow[d] & & & (\{2\},2) \arrow[ulll] \arrow[dddddd] \\
 & (\{3\},\varnothing) \arrow[ul] \arrow[urr] \arrow[dr] \arrow[dddd] & & (\{2,3\},1) & & (\{2\},\varnothing) \arrow[ull] \arrow[ur] \arrow[dl] \arrow[dddd] & \\
 & & (\{3\},1) \arrow[ur] \arrow[dd] & & (\{2\},1) \arrow[ul] \arrow[dd] & & \\
 & & & (\varnothing,\varnothing) \arrow[ul] \arrow[uu] \arrow[ur] \arrow[dl] \arrow[dr] \arrow[dd] & & & \\
 & & (\{1,3\},1) & & (\{1,2\},1) & & \\
 & (\{1,3\},\varnothing) \arrow[ur] \arrow[dl] & & (\{1\},1) \arrow[ul] \arrow[ur] & & (\{1,2\},\varnothing) \arrow[ul] \arrow[dr] & \\
(\{1,3\},2) & & & (\{1\},\varnothing) \arrow[ull] \arrow[u] \arrow[urr] \arrow[d] & & & (\{1,2\},2) \\
 & & & (\{1\},2) \arrow[ulll] \arrow[urrr] & & &
\end{tikzcd},$$

while $\mathrm{web}(P)$ looks the same but with the outer layer removed.

(12.5) **Observation.** For a poset P with a bottom element, the subset $\mathrm{web}(P) \subseteq \mathrm{Web}(P)$ is convex (see (11.3)) and in fact even downwards closed because, given

$$(p, M) \leqslant (q, N) \qquad \text{with } (q, N) \in \mathrm{web}(P),$$

we must have $N = \varnothing$ or $N = \{1\}$ and consequently also $M = \varnothing$ or $M = \{1\}$, since $M \subseteq N$. This shows that also $(p, M) \in \mathrm{web}(P)$. Therefore, the quotient $\mathrm{Web}(P)/\mathrm{web}(P)$ is again a poset by (11.5) and even better

$$P \cong \mathrm{Web}(P)/\mathrm{web}(P),\ p \mapsto \begin{cases} [\mathrm{web}(P)] & p = \perp \\ (p, 2) & p \neq \perp. \end{cases}$$

There are many ways to extend diagrams $X\colon P \to \mathbf{sSets}$ indexed by some poset P to diagrams indexed by $\mathrm{Web}(P)$. The most important one for us is to send the outermost part of the web to the terminal object. This extension to the outer layer is nothing but the right Kan extension

$$\mathbf{sSets}^{\mathrm{web}(P)} \hookrightarrow \mathbf{sSets}^{\mathrm{Web}(P)}$$

along the canonical inclusion $\mathrm{web}(P) \hookrightarrow \mathrm{Web}(P)$. For the extension of a diagram $P \to \mathbf{sSets}$ to $\mathrm{web}(P)$ we can do two different things. We can either precompose with the canonical projection $\mathrm{web}(P) \to P$ (i.e. extend the diagram by identities) or again take the right Kan extension along $P \hookrightarrow \mathrm{web}(P)$. Luckily, these two constructions agree up to isomorphism.

(12.6) **Proposition.** For P a poset with a bottom element $\perp$ and $X\colon P \to \mathbf{sSets}$, its right Kan extension $\tilde{X}$ along $P \hookrightarrow \mathrm{web}(P)$ is isomorphic to $X \circ \pi$, where $\pi\colon\ \mathrm{web}(P) \to P$ is the canonical projection.

Proof. Since the inclusion $P \hookrightarrow \mathrm{web}(P)$ is fully faithful, so is the right Kan extension functor and therefore $\tilde{X}|_P \cong X$. For $(p, \varnothing) \notin P \subseteq \mathrm{web}(P)$, we have

$$\widetilde{X}(p, \varnothing) \cong \lim \Big((p, \varnothing) \downarrow P \to P \xrightarrow{X} \mathbf{sSets} \Big) .$$

But $(p, \varnothing) \downarrow P$, which consists of all $\big(q, \{1\}\big)$ with $p \leqslant q$, has an initial object; namely $\big(p, \{1\}\big)$ and so the above limit is (isomorphic to) X_p. $\square$

(12.7) **Corollary.** For every poset P with a bottom element, the right Kan extension functor along the canonical inclusion $P \hookrightarrow \mathrm{Web}(P)$,

$$\mathbf{sSets}^P \hookrightarrow \mathbf{sSets}^{\mathrm{Web}(P)},\ X \mapsto \tilde{X},$$

(is isomorphic to the functor which) has $\tilde{X}(p, 2) = *$ for $p \in P \setminus \{\perp\}$ and $\tilde{X}(p, S) = X_p$ for $S \neq \{2\}$ (with $\tilde{X}(p, S) \to \tilde{X}(q, T)$ being $X_p \to X_q$ or the unique map to $*$). $\square$

(12.8) **Example.** For a span $B \leftarrow A \to C$, which is just a diagram indexed by $\ulcorner^2$, we get the initial example

$$* \leftarrow B \to B \leftarrow A \to C \leftarrow C \to *.$$

as its (right Kan) extension to $\mathrm{Web}(\ulcorner^2)$.

Just like in (1.3), given a poset P with a bottom element $\perp$ as well as a simplicial set K, let us write $G_K\colon P \to \mathbf{sSets}$ for the diagram that maps $\perp$ to K and everything else to $*$. As this is just the right Kan extension of the diagram K along $\{\perp\} \hookrightarrow P$, we note that if we extend further to $\mathrm{Web}(P)$, we again get that $\tilde{G}_K$ is K on $(\perp, \varnothing)$ and $*$ everywhere else. In particular, there is a canonical natural transformation $\tilde{X} \Rightarrow \tilde{G}_{X_\perp}$ for every $X\colon P \to \mathbf{sSets}$, which is the identity at $(\perp, \varnothing)$ and the unique map to $*$ everywhere else.

(12.9) **Proposition.** Let P be a poset with a bottom element $\perp$, $X\colon P \to \mathbf{sSets}$ any diagram and $\tilde{X}\colon \mathrm{Web}(P) \to \mathbf{sSets}$ its extension to $\mathrm{Web}(P)$. Then the canonical maps

$$\operatorname*{hocolim}_{\mathrm{Web}(P)} \tilde{X} \to \underbrace{\operatorname*{hocolim}_{\mathrm{Web}(P)} \tilde{G}_{X_\perp}}_{\Sigma^{\mathrm{Web}(P)} X_\perp} \to \underbrace{\operatorname*{hocolim}_{P} G_{X_\perp}}_{\Sigma^P X_\perp}$$

(where the first one is induced by $\tilde{X} \Rightarrow \tilde{G}_{X_\perp}$ and the second one by $\mathrm{Web}(P) \to P$ as well as the identification $\tilde{G}_{X_\perp}|_P \cong G_{X_\perp}$) are both weak equivalences.

Proof. Let us first consider the composite map. Writing $W\colon P \to \mathbf{Cat}$ for the functor we used to construct $\mathrm{Web}(P) = \int^P W$, Thomason's theorem tells us that

$$\operatorname*{hocolim}_{\mathrm{Web}(P)} \tilde{X} \simeq \operatorname*{hocolim}_{p \in P} \operatorname*{hocolim}_{S \in W_p} \tilde{X}(p, S).$$

But for $p = \perp$, we have $W_\perp = \{\varnothing\}$ and so $\mathrm{hocolim}_{S \in W_\perp} \tilde{X}(\perp, S) = \tilde{X}(\perp, \varnothing) = X_\perp$ while for $p \neq \perp$, we have $W_p = \ulcorner^2$, whence

$$\begin{aligned}\operatorname*{hocolim}_{S \in W_p} \tilde{X}(p, S) &= \mathrm{hocolim}\big(X(p, \{2\}) \leftarrow X(p, \varnothing) \to X(p, \{1\})\big) \\ &= \mathrm{hocolim}(* \leftarrow X_p \to X_p) \simeq *.\end{aligned}$$

With this, $\mathrm{hocolim}_{\mathrm{Web}(P)} \tilde{X} \simeq \mathrm{hocolim}_P\, G_{X_\perp}$. Replacing $\tilde{X}$ by $\tilde{G}_{X_\perp}$ in the above argument shows that the second map from the proposition is a weak equivalence and by 2-out-of-3, so is the first one. $\square$

(12.10) **Example.** The example of a span $B \leftarrow A \to C$ exemplifies the proof really well. In order to calculate the homotopy colimit of the extended diagram

$$\boxed{* \leftarrow B \to B} \leftarrow A \to \boxed{C \leftarrow C \to *},$$

we can use Thomason's theorem to first the homotopy pushouts of the two appendages, framed above. In both cases, we get a point and then calculate

$$\mathrm{hocolim}(* \leftarrow A \to *) \simeq \Sigma A.$$

To use our web construction in analysing homotopy colimits, we will have to replace the central part of a given diagram as follows. Let $X\colon \mathrm{Web}(P) \to \mathbf{sSets}$, $Y\colon P \to \mathbf{sSets}$ and $\tau\colon X|_P \Rightarrow Y$. We define a new diagram $X_\tau\colon \mathrm{Web}(P) \to \mathbf{sSets}$ (or, by abuse of notation, X_Y if the transformation is clear) as being Y on P and X on $\mathrm{Web}(P) \setminus P$, using τ to connect the two. More explicitly, on objects, X_τ is given by

$$X_\tau(\perp, \varnothing) := Y_\perp,\ X_\tau(p, \{1\}) := Y_p \quad \text{and} \quad X_\tau(p, S) := X(p, S) \text{ for } p \neq \perp,\ S \neq \{1\}.$$

On $P \subseteq \mathrm{Web}(P)$, the arrow function of X_τ is just that of Y and on $\mathrm{Web}(P) \setminus P$, it is that of X; the only remaining case being arrows of the form $(p, \varnothing) \to (p, \{1\})$, which we define to be mapped to

$$X(p, \varnothing) \to X(p, \{1\}) \xrightarrow{\tau_p} Y_p.$$

This completely determines a well-defined arrow function, for if we have $p \leqslant q$ in P then $(p, \varnothing) \leqslant (q, \{1\})$ is mapped to $X_{(p,\varnothing)} \to Y_q$ obtained from the commutative diagram

$$\begin{tikzcd} X(p,\varnothing) \arrow[r] & X(p,\{1\}) \arrow[r, "\tau_p"] \arrow[d] & Y_p \arrow[d] \\ & X(q,\{1\}) \arrow[r, "\tau_q"'] & Y_q \end{tikzcd} .$$

(12.11) **Remark.** If our diagram X is a right Kan extension of a diagram $P \to \mathbf{sSets}$ then the arrow function of X_τ evaluated at $(p, \varnothing) \to \big(p, \{1\}\big)$ is just τ_p.

Note that each diagram X_τ comes with a canonical natural transformation $X \Rightarrow X_\tau$, which is τ on P and the identity on $\mathrm{Web}(P) \setminus P$. The naturality of this transformation follows immediately from the definition of X_τ's arrow function.

Similarly, again starting with a diagram $X\colon \mathrm{Web}(P) \to \mathbf{sSets}$, $Y\colon \mathrm{web}(P) \to \mathbf{sSets}$ and $\tau\colon Y \Rightarrow X|_{\mathrm{web}(P)}$, we define $X^\tau\colon \mathrm{Web}(P) \to \mathbf{sSets}$ (or X^Y by abuse of notation) as being Y on $\mathrm{web}(P)$ and X on $\mathrm{Web}(P) \setminus \mathrm{web}(P)$. Again, the arrow function of X^τ is just that of Y on $\mathrm{web}(P)$, that of X on $\mathrm{Web}(P) \setminus \mathrm{web}(P)$ and maps an arrow $(p, \varnothing) \to \big(p, \{2\}\big)$ to

$$Y(p,\varnothing) \xrightarrow{\tau_{(p,\varnothing)}} X(p,\varnothing) \to X\big(p,\{2\}\big).$$

Just like in the above case, this determines a well-defined arrow function and the new diagram X^τ comes with a canonical natural transformation $X^\tau \Rightarrow X$, which is τ on $\mathrm{web}(P)$ and the identity on the rest of $\mathrm{Web}(P)$.

What is important for us is that there is an alternative way to construct the web, associated to a poset. Instead of starting with the poset and adding appendages to every non-bottom element, we can instead first construct the outer layer and then add in the central copy of the original poset.

(12.12) **Definition.** Let P be a poset with a bottom element and let $P' := \big(P \setminus \{\bot\}\big) + \{\top\}$ be the poset obtained from P by exchanging the bottom for a top element. We define the *layering* $\mathrm{Lay}(P)$ of P to be the following subposet of the Grothendieck construction for the constant functor $[1] = \{0 < 1\}$:

$$\mathrm{Lay}(P) := \left(\int^{P'} [1]\right) \setminus \{(\top, 1)\}.$$

The way to visualise this is that $\mathrm{Lay}(P)$ consists of two layers of $P \setminus \{\bot\}$ with a top element added to the zeroth layer. Finally, we define the following central blowup functor

$$B\colon \mathrm{Lay}(P) \to \mathbf{Cat},\ (p,i) \mapsto \begin{cases} \{p\} & p \neq \top \\ P & p = \top \end{cases}$$

(here's where the bottom element comes in again) with $B\big((p,0) < (\top,0)\big) := (\{p\} \hookrightarrow P)$ and all identity morphisms otherwise.

(12.13) **Proposition.** Given a poset P with a bottom element $\bot$, we have a natural isomorphism $\mathrm{Web}(P) \cong \int^{\mathrm{Lay}(P)} B$ given by

$$(p,\varnothing) \mapsto (p,0,p), \qquad (p,1) \mapsto (\top,0,p), \qquad (p,2) \mapsto (p,1,p).$$

Proof. Straightforward direct verification. □

(12.14) **Example.** Taking $P = \ulcorner^2$ we get $P' = \{1 \to \top \leftarrow 2\}$, so that

$$\mathrm{Lay}(P)\colon \quad \begin{tikzcd} (1,0) \arrow[r] \arrow[d] & (\top,0) & (2,0) \arrow[l] \arrow[d] \\ (1,1) & & (2,1) \end{tikzcd}$$

and then

$$\textstyle\int^{\mathrm{Lay}(P)} B\colon \quad \begin{tikzcd} (1,0,1) \arrow[r] \arrow[d] & (\top,0,1) & (\top,0,\varnothing) \arrow[l] \arrow[r] & (\top,0,2) & (2,0,2) \arrow[l] \arrow[d] \\ (1,1,1) & & & & (2,1,2) \end{tikzcd}$$

which is clearly isomorphic to

$$\mathrm{Web}(P)\colon \qquad \bullet \leftarrow \bullet \rightarrow \bullet \leftarrow \bullet \rightarrow \bullet \leftarrow \bullet \rightarrow \bullet$$

We can use this alternative description of $\mathrm{Web}(P)$ to describe a particular homotopy colimit indexed by it. Explicitly, starting from $X\colon P \to \mathbf{sSets}$, we are going to calculate the homotopy colimit of $\tilde{X}_\lambda$ where $\lambda\colon X \Rightarrow \operatorname{hocolim} X$ is the colimiting cocone.

(12.15) **Proposition.** Let P be a poset with a bottom element, $X\colon P \to \mathbf{sSets}$ and $\lambda\colon X \Rightarrow \operatorname{hocolim} X$ the colimiting cocone. Then the canonical transformation $\tilde{X} \Rightarrow \tilde{X}_\lambda$ induces a weak equivalence between homotopy colimits.

Proof. By the above identification $\mathrm{Web}(P) \cong \int^{\mathrm{Lay}(P)} B$ and Thomason's theorem

$$\begin{aligned} \operatorname{hocolim} \tilde{X} &\simeq \operatorname*{hocolim}_{(p,i)\in\mathrm{Lay}(P)} \operatorname*{hocolim}_{q\in B(p,i)} \tilde{X}(p,i,q) \\ &\simeq \operatorname*{hocolim}_{(p,i)\in\mathrm{Lay}(P)} H(p,i), \end{aligned}$$

where $H(p,i) := \tilde{X}(p,i)$ for $p \neq \top$ (i.e. $H(p,0) = X_p$, $H(p,1) = *$) and $H(\top,0) := \operatorname{hocolim} X$, where the maps $H(p,0) = X_p \to H(\top,0) = \operatorname{hocolim} X$ are the universal cocone's components. Similarly, we have

$$\begin{aligned} \operatorname{hocolim} \tilde{X}_\lambda &\simeq \operatorname*{hocolim}_{(p,i)\in\mathrm{Lay}(P)} \operatorname*{hocolim}_{q\in B(p,i)} \tilde{X}_\lambda(p,i,q) \\ &\simeq \operatorname*{hocolim}_{(p,i)\in\mathrm{Lay}(P)} H'(p,i), \end{aligned}$$

where again $H'(p,0) = X_p$, $H'(p,1) = *$ and $H'(\top,0) = \operatorname{hocolim} \Delta_{\operatorname{hocolim} X}$ is the homotopy colimit of the diagram $P \to \mathbf{sSets}$ that is constantly $\operatorname{hocolim} X$. But P is contractible (it has a bottom element), so that $\operatorname{hocolim} \Delta_{\operatorname{hocolim} X} \simeq \operatorname{hocolim} X$ with all induced maps $X_p \to \operatorname{hocolim} X$ again being the universal cocone's components. $\square$

(12.16) **Example.** For a span $B \leftarrow A \to C$ with homotopy pushout D, the proposition says that the transformation

$$\begin{tikzcd} * \arrow[d] & B \arrow[l] \arrow[r] \arrow[d] & B \arrow[d] & A \arrow[l] \arrow[r] \arrow[d] & C \arrow[d] & C \arrow[l] \arrow[r] \arrow[d] & * \arrow[d] \\ * & B \arrow[l] \arrow[r] & D & D \arrow[l] \arrow[r] & D & C \arrow[l] \arrow[r] & * \end{tikzcd}$$

induces a weak equivalence between homotopy colimits (which are then weakly equivalent to ΣA as seen in (12.10)). Using the layer-description of the web, we can first take the central homotopy pushout and the resulting transformation is a natural weak equivalence.

As always for the Grothendieck construction, the unique transformation $B \Rightarrow \Delta_{\{*\}}$ to the terminal $\mathrm{Lay}(P)$-diagram induces the canonical projection $\int^{\mathrm{Lay}(P)} B \twoheadrightarrow \mathrm{Lay}(P)$. If we now identify $\mathrm{Web}(P) \cong \int^{\mathrm{Lay}(P)} B$ as in (12.13) above, then this canonical projection becomes the quotient map

$$\mathrm{Web}(P) \twoheadrightarrow \mathrm{Web}(P)/P.$$

The following proof of the web-trick for a general poset P is rather technical but the idea behind it is simple enough. We suggest first going through the case $P = \ulcorner^2$ as outlined in (12.19) below, which can be very instructional.

(12.17) **Proposition. (Web-Trick)** Let P be a poset with bottom element and $\tau\colon X \Rightarrow Y$ a morphism between diagrams $X, Y\colon P \to \mathbf{sSets}$ such that every component τ_p with $p \neq \perp$ is a weak equivalence. Then

(a) $\mathrm{h}\mathfrak{Fib}\big(\mathrm{hocolim}_P\, \tau\colon\ \mathrm{hocolim}_P\, X \to \mathrm{hocolim}_P\, Y\big) \gg \Sigma^P\, \mathrm{h}\mathfrak{Fib}(\tau_\perp\colon X_\perp \to Y_\perp)$;

(b) $\mathrm{h}\mathfrak{Fib}(\Sigma^P \tau_\perp\colon \Sigma^P X_\perp \to \Sigma^P Y_\perp) \gg \mathrm{h}\mathfrak{Fib}\big(\mathrm{hocolim}_P\, \tau\colon\ \mathrm{hocolim}_P\, X \to \mathrm{hocolim}_P\, Y\big)$.

Proof. First off, we can assume that all components τ_p with $p \neq \perp$ are even identities. In fact, we can define a new diagram $X'\colon P \to \mathbf{sSets}$, which has $X'_\perp := X_\perp$ but $X'_p := Y_p$ for $p \neq \perp$ with

$$X'(\perp < p)\colon X_\perp \xrightarrow{\tau_\perp} Y_\perp \to Y_p.$$

We then clearly have natural transformations $X \Rightarrow X' \Rightarrow Y$, where the first one is the natural weak equivalence with components $\mathrm{id}_{X_\perp}$ at $\perp$ and τ_p at $p \neq \perp$, while the second one is $\tau_\perp$ at $\perp$ and the identity everywhere else.
Ad (a): We extend the diagram Y to $\mathrm{Web}(P)$ by precomposing with the quotient map $\pi\colon \mathrm{Web}(P) \to \mathrm{Web}(P)/\,\mathrm{web}(P) \cong P$ (cf. (12.5)), which collapses $\mathrm{web}(P)$ to $\perp$. Noticing that all appendages $(p, 1) \leftarrow (p, \varnothing) \rightarrow (p, 2)$ are mapped to spans of the form $Y_\perp \leftarrow Y_\perp \to Y_p$, whose homotopy pushout is Y_p, we conclude that

$$\operatorname*{hocolim}_{\mathrm{Web}(P)} \pi^* Y \simeq \operatorname*{hocolim}_{P} Y$$

by Thomason's theorem (7.5.2). To bring X back into play, we consider the canonical transformation $\tau_{\perp *}\colon G_{\tau_\perp} \Rightarrow \Delta_{Y_\perp}$ of $\mathrm{web}(P)$-indexed diagrams, where $G_{\tau_\perp}$ (as defined in (1.6)) is just constantly $Y_\perp$ except at $\perp$, where it is $X_\perp$. We use this transformation $\tau_{\perp *}$ to replace the $\mathrm{web}(P)$-part of $\pi^* Y$ by $G_{\tau_\perp}$. Now, again by Thomason's theorem the homotopy colimit of the resulting diagram $(\pi^* Y)^{\tau_{\perp *}}\colon \mathrm{Web}(P) \to \mathbf{sSets}$ is (equivalent to) that of X. To wit, the central vertex $(\perp, \varnothing)$ is mapped to $X_\perp$, while an appendage $(p, 1) \leftarrow (p, \varnothing) \rightarrow (p, 2)$ is mapped to $Y_\perp \leftarrow Y_\perp \to X_p$ (recalling that $X_p = Y_p$), whose homotopy pushout is X_p. All in all, the map

$$\operatorname*{hocolim}_{P} \tau\colon\ \operatorname*{hocolim}_{P} X \to \operatorname*{hocolim}_{P} Y$$

can be obtained alternatively as the hocolim of the canonical transformation

$$(\pi^* Y)^{\tau_{\perp *}} \Rightarrow \pi^* Y \qquad \text{induced by } \tau_\perp.$$

We can now use the alternative description of $\mathrm{Web}(P)$ to better understand this map. In fact, the homotopy colimits of the two diagrams $(\pi^* Y)^{\tau_{\perp *}}$ and $\pi^* Y$ can be calculated as

$$\operatorname*{hocolim}_{\mathrm{Web}(P)} (\pi^* Y)^{\tau_{\perp *}} \simeq \operatorname*{hocolim}_{(p,S) \in \mathrm{Lay}(P)}\ \operatorname*{hocolim}_{T \in B(p,S)} (\pi^* Y)^{\tau_{\perp *}}(p, S, T).$$

But for $p \neq \top$ in P' and $S \in [1]$, we have $B(p,S) = \{p\}$ with $(\pi^*Y)^{\tau_\perp *}(p,S,T) = Y_\perp$ for $S = 0$ and X_p otherwise, while $B(\top,0) = P$ with $(\pi^*Y)^{\tau_\perp *}(\top,0,-)$ being the diagram $G_{\tau_\perp}\colon P \to \mathbf{sSets}$, whose homotopy colimit we denote by $\Sigma^P\tau_\perp$. Similarly for π^*Y except that there, $(\pi^*Y)(\top,0,-) = \Delta_{Y_\perp}$ is constant and its homotopy colimit is equivalent to $Y_\perp$ (P is contractible). All in all, taking the homotopy colimit of both these central parts, we get two new diagrams X', $Y'\colon \mathrm{Web}(P)/P \to \mathbf{sSets}$ together with a transformation $\tau'\colon X' \Rightarrow Y'$ which is the identity everywhere except at the equivalence class of P, where it is the map $\Sigma^P\tau_\perp \to Y$ induced by the canonical transformation $G_{\tau_\perp} \Rightarrow \Delta_{Y_\perp}$. By (1.8), the homotopy fibre set of this map is equivalent to $\Sigma^P\,\mathrm{h}\mathfrak{F}\mathrm{ib}(f)$ and applying Dror Farjoun's theorem (8.4.8), the claim finally follows.

Ad (b): As seen in (12.9), the map $\Sigma^P\tau_\perp$ can be obtained by taking the homotopy colimit of $\tilde\tau\colon \tilde X \to \tilde Y$. Again using the alternative description of $\mathrm{Web}(P)$, we can take the homotopy colimit of the central copy of P first and $\tilde\tau$ induces a transformation $\tau'\colon X' \to Y'$ between $\mathrm{Web}(P)/P$-indexed diagrams. The components of τ' are all identities except at the equivalence class of P, where it is (equivalent to) $\mathrm{hocolim}_P\,\tau$. The claim then follows from Dror Farjoun's theorem (8.4.8). □

One particular case of interest for us is when the indexing category is a cube (or rather the indexing category for higher-dimensional pushouts).

(12.18) **Corollary.** Given a transformation $\tau\colon X \Rightarrow Y$ of diagrams $X, Y\colon \ulcorner^n \to \mathbf{sSets}$ such that every component τ_S with $S \neq \varnothing$ is a weak equivalence, then

(a) $\mathrm{h}\mathfrak{F}\mathrm{ib}\big(\mathrm{hocolim}\,\tau\colon\ \mathrm{hocolim}\,X \to \mathrm{hocolim}\,Y\big) \gg \Sigma^{n-1}\,\mathrm{h}\mathfrak{F}\mathrm{ib}(\tau_\varnothing\colon X_\varnothing \to Y_\varnothing)$;

(b) $\mathrm{h}\mathfrak{F}\mathrm{ib}(\Sigma^{n-1}\tau_\varnothing\colon \Sigma^{n-1}X_\varnothing \to \Sigma^{n-1}Y_\varnothing) \gg \mathrm{h}\mathfrak{F}\mathrm{ib}\big(\mathrm{hocolim}\,\tau\colon\ \mathrm{hocolim}\,X \to \mathrm{hocolim}\,Y\big)$.

Proof. This is just the Web-Trick together with the fact (1.4) that $\Sigma^{\ulcorner^n} = \Sigma^{n-1}$. □

(12.19) **Example.** Again, let us consider the more readily visualisable case of the web-trick for $P = \ulcorner^2$ a span. So given a transformation

$$\begin{tikzcd} B \arrow[d, equal] & A \arrow[l] \arrow[r] \arrow[d, "f"'] & C \arrow[d, equal] \\ B & A' \arrow[l] \arrow[r] & C \end{tikzcd} \qquad \text{with hocolim} \qquad \begin{tikzcd} P \arrow[d, "f_*"] \\ Q \end{tikzcd},$$

we need to show that $\mathrm{h}\mathfrak{F}\mathrm{ib}(\Sigma f) \gg \mathrm{h}\mathfrak{F}\mathrm{ib}(f_*) \gg \Sigma\,\mathrm{h}\mathfrak{F}\mathrm{ib}(f)$. The way it is done in the above proof is to note that f_* can alternatively be obtained as the map between homotopy colimits induced by

$$\begin{tikzcd} B \arrow[d, equal] & A' \arrow[l] \arrow[r] \arrow[d, equal] & A' \arrow[d, equal] & A \arrow[l] \arrow[r] \arrow[d, "f"'] & A' \arrow[d, equal] & A' \arrow[l] \arrow[r] \arrow[d, equal] & C \arrow[d, equal] \\ B & A' \arrow[l] \arrow[r] & A' & A' \arrow[l] \arrow[r] & A' & A' \arrow[l] \arrow[r] & C \end{tikzcd},$$

which can easily been seen by taking homotopy pushouts of the outer appendages first. But instead of taking the outer homotopy pushouts first, we can alternatively take the central ones. From (1.9), we know that the induced map between the central pushouts has $\Sigma\,\mathrm{h}\mathfrak{F}\mathrm{ib}(f)$ as its homotopy fibre set. Dror Farjoun's theorem then tells us that $\mathrm{h}\mathfrak{F}\mathrm{ib}(f_*) \gg \Sigma\,\mathrm{h}\mathfrak{F}\mathrm{ib}(f)$.

For the other cellular inequality $\mathrm{h}\mathfrak{F}\mathrm{ib}(\Sigma f) \gg \mathrm{h}\mathfrak{F}\mathrm{ib}(f_*)$, we write Σf as the map between homotopy colimits induced by

$$\begin{tikzcd}
* \arrow[d, equal] & B \arrow[l] \arrow[r] \arrow[d, equal] & B \arrow[d, equal] & A \arrow[l] \arrow[r] \arrow[d, "f"'] & C \arrow[d, equal] & C \arrow[l] \arrow[r] \arrow[d, equal] & * \arrow[d, equal] \\
* & B \arrow[l] \arrow[r] & B & A' \arrow[l] \arrow[r] & C & C \arrow[l] \arrow[r] & *
\end{tikzcd},$$

which does indeed yield Σf, again by first taking homotopy pushouts of the appendages. Again taking the homotopy pushouts of the central spans first, we can alternatively obtain Σf as the map between homotopy colimits induced by

$$\begin{tikzcd}
* \arrow[d, equal] & B \arrow[l] \arrow[r] \arrow[d, equal] & P \arrow[d, "f_*"'] & C \arrow[l] \arrow[r] \arrow[d, equal] & * \arrow[d, equal] \\
* & B \arrow[l] \arrow[r] & Q & C \arrow[l] \arrow[r] & *
\end{tikzcd}$$

and $\mathrm{h}\mathfrak{F}\mathrm{ib}(\Sigma f) \gg \mathrm{h}\mathfrak{F}\mathrm{ib}(f_*)$ again follows from Dror Farjoun's theorem.

13. Serre's Theorem

In this section, we are going to consider a homotopy pullback square of simplicial sets as on the left below.

$$\begin{tikzcd}
P \arrow[r, "f"] \arrow[d, "g"'] & X \arrow[d] \\
Y \arrow[r] & Z
\end{tikzcd}
\qquad
\begin{tikzcd}
P \arrow[r, "f"] \arrow[d, "g"'] & X \arrow[d] \\
Y \arrow[r] & Q
\end{tikzcd}$$

Assuming $P \neq \varnothing$ any choice of a base point in P makes this a diagram of pointed simplicial sets. We are then going to investigate what we can say about the fibres when fitting this homotopy pullback square into larger diagrams (or rather transformations between diagrams) that mainly contain identity maps. To this end, we will have to consider the homotopy pushout as on the right above. Puppe's theorem allows us to identify the fibre of the comparison map $Q \to Z$ as

$$\mathrm{hFib}_*(Q \to Z) \simeq F * G \qquad \text{where } F := \mathrm{hFib}_* f \text{ and } G := \mathrm{hFib}_* g.$$

Before proceeding to the theorem, a quick comment on its name. We refer to it as *Serre's theorem* after [13] because it is a generalisation of a theorem due to Serre [49, Theorem VII.6.1], stated in terms of connectivity and homology (rather than general closed classes).

(13.1) **Theorem. (Serre)** Given a (transformation of) diagram(s) of simplicial sets

$$\begin{tikzcd}
B \arrow[d, equal] & P \arrow[l] \arrow[r, "f"] \arrow[d, "g"'] & X \arrow[d] \\
B & Y \arrow[l] \arrow[r] & Z
\end{tikzcd}
\qquad \text{with (unpointed!) homotopy colimit} \qquad
\begin{tikzcd}
H_1 \arrow[d] \\
H_2
\end{tikzcd}$$

and the right square a homotopy pullback, then $\mathrm{hFib}_*(H_1 \to H_2) \gg F * G$ for every base point of P (and all other spaces equipped with the induced base points).

Proof. The key observation is that, if we replace Z by $Q := \operatorname{hocolim}(Y \xleftarrow{g} P \xrightarrow{f} X)$ in the diagram from the proposition, then the induced map between homotopy colimits is going to be a weak equivalence. Indeed, by Fubini, we have

$$\begin{tikzcd}
B & P \arrow[l] \arrow[r] & X & & H_1 \\
B \arrow[u, equal] \arrow[d, equal] & P \arrow[l] \arrow[r, equal] \arrow[u, equal] \arrow[d] & P \arrow[u] \arrow[d] & & B \arrow[u] \arrow[d, equal] \\
B & Y \arrow[l] \arrow[r, equal] & Y & & B \\
& & & & \\
B & Y \arrow[l] \arrow[r] & Q & & H_1
\end{tikzcd}$$

(where all the induced identities should really be weak equivalences). Now, with this, we extend the diagram from the theorem by inserting the homotopy pushout in the middle, leading to

$$\begin{tikzcd}
B \arrow[d, equal] & P \arrow[l] \arrow[r] \arrow[d] & X \arrow[d] & & H_1 \arrow[d, equal] \\
B \arrow[d, equal] & Y \arrow[l] \arrow[r] \arrow[d, equal] & Q \arrow[d] & \overset{\text{hocolim}}{\rightsquigarrow} & H_1 \arrow[d] \\
B & Y \arrow[l] \arrow[r] & Z & & H_2\,.
\end{tikzcd}$$

Finally, Dror Farjoun's theorem together with Puppe's theorem give

$$\operatorname{hFib}_*(H_1 \to H_2) \gg \operatorname{hFib}_*(Q \to Z) \simeq F * G. \qquad \square$$

(13.2) **Corollary.** For every pointed map of simplicial sets $f\colon X \to Y$ with Y connected,

$$\operatorname{hCof}(f) \gg \Sigma\operatorname{hFib}_*(f).$$

In particular, if Y is connected, then $\operatorname{hCof}(f) \gg \Sigma\operatorname{hFib}(f)$.

Proof. Apply Serre's theorem to

$$\begin{tikzcd}
* \arrow[d, equal] & \operatorname{hFib}_*(f) \arrow[l] \arrow[r] \arrow[d] & * \arrow[d] & & \Sigma\operatorname{hFib}_*(f) \arrow[d] \\
* & X \arrow[l] \arrow[r, "f"'] & Y & \overset{\text{hocolim}}{\rightsquigarrow} & \operatorname{hCof}(f),
\end{tikzcd}$$

which tells us that the fibre of the induced map $\Sigma\operatorname{hFib}_*(f) \to \operatorname{hCof}(f)$ is killed by

$$\operatorname{hFib}_*(f) * \operatorname{hFib}_*\big(\operatorname{hFib}_*(f) \to X\big) \simeq \operatorname{hFib}_*(f) * \Omega_* Y.$$

Since Y is connected (i.e. $Y \gg S^1$), we know that $\Omega_* Y \gg S^0$ and therefore

$$\operatorname{hFib}_*(f) * \Omega_* Y \gg \operatorname{hFib}_*(f) * S^0 \simeq \Sigma\operatorname{hFib}_*(f). \qquad \square$$

For our treatment of the higher-dimensional Blakers-Massey theorem, it will be necessary to generalise Serre's theorem to higher-dimensional cubes (or rather transformations of higher-dimensional pushout diagrams). We will do so in two ways. Firstly, we are going to replace the indexing category $\ulcorner^2$ by a general $\ulcorner^n$ and secondly, we are going to replace the central homotopy pullback by a general strong homotopy pullback of arbitrary dimension.

(13.3) **Lemma.** Let $\tau\colon X \Rightarrow Y$ be a transformation of diagrams $X, Y\colon \ulcorner^n \to \mathbf{sSets}$ and $m \in \mathbb{N}_{\leqslant n}$ such that

$$\tau|_{\square^{m-1}}\colon X|_{\square^{m-1}} \Rightarrow Y|_{\square^{m-1}} \qquad \begin{array}{c}\text{viewed as a diagram } \square^m \to \mathbf{sSets} \\ \text{is a homotopy pushout}\end{array}$$

and such that τ is a weak equivalence outside of $\square^{m-1}$. Then $\tau_*\colon \operatorname{hocolim} X \to \operatorname{hocolim} Y$ is a weak equivalence.

Proof. If $n < 2$, the claim is trivial. The case $n = 2$ was the "key observation" in the proof of Serre's theorem and the general case is by induction on n. We use Thomason's theorem and write the (induced map between) homotopy colimits as

$$\begin{array}{ccccccc}
\operatorname{hocolim} X & \simeq & \operatorname{hocolim}\Big(\operatorname{hocolim}_{\ulcorner^{n-1}} X|_{\partial_n} & \longleftarrow & \operatorname{hocolim}_{\ulcorner^{n-1}} X|_{\partial^{\neg n}} & \longrightarrow & X_{\neg n}\Big) \\
\vdots & & \downarrow \simeq & & \downarrow & & \downarrow \\
\operatorname{hocolim} Y & \simeq & \operatorname{hocolim}\Big(\operatorname{hocolim}_{\ulcorner^{n-1}} Y|_{\partial_n} & \longleftarrow & \operatorname{hocolim}_{\ulcorner^{n-1}} Y|_{\partial^{\neg n}} & \longrightarrow & Y_{\neg n}\Big)
\end{array},$$

where the solid arrow on the left is a weak equivalences by the claim's hypotheses. Now, if $m < n$, then the other two solid arrows are weak equivalences, too. For the right one, this is just by the claim's hypotheses, while for the middle one, it is by the inductive hypothesis. Finally, if $m = n$, then the other two solid arrows need not be weak equivalences but by the claim's hypotheses and Thomason's theorem, the right-hand square in the diagram is a pushout and the claim follows from the case $n = 2$. $\square$

(13.4) **Theorem. (Serre** [bis]**)** Let $\tau\colon X \Rightarrow Z$ be a natural transformation of diagrams $X, Z\colon \ulcorner^n \to \mathbf{sSets}$ and $m \in \mathbb{N}_{\leqslant n}$ such that

$$\tau|_{\square^{m-1}}\colon X|_{\square^{m-1}} \Rightarrow Z|_{\square^{m-1}} \qquad \begin{array}{c}\text{viewed as a diagram } \square^m \to \mathbf{sSets} \\ \text{is a strong homotopy pullback}\end{array}$$

and such that τ is a weak equivalence everywhere else. Furthermore, let us fix any base point in $X_\varnothing$ and denote the fibres of the strong homotopy pullback $\tau|_{\square^{m-1}}$ by $F_1, \dots, F_m$; i.e.

$$F_k := \operatorname{hFib}_*(X_\varnothing \to X_k) \text{ for } k < m \qquad \text{and} \qquad F_m := \operatorname{hFib}_*(X_\varnothing \to Z_\varnothing).$$

Then, $\operatorname{hFib}_*(\operatorname{hocolim} X \to \operatorname{hocolim} Z) \gg \Sigma^{n-m}(F_1 * \dots * F_m)$.

Proof. The case $n = 2$ was Serre's theorem above. For a general n, we proceed similarly to the proof of Serre's theorem. First, we factor τ as $X \Rightarrow Y \Rightarrow Z$, where Y agrees with Z everywhere except at $Y_{\langle m-1\rangle}$, which is such that

$$\tau|_{\square^{m-1}}\colon X|_{\square^{m-1}} \Rightarrow Y|_{\square^{m-1}} \qquad \begin{array}{c}\text{viewed as a diagram } \square^m \to \mathbf{sSets} \\ \text{is a homotopy pushout.}\end{array}$$

More formally, τ is an object of $\mathbf{sSets}^{\ulcorner^n \times [1]}$ and the morphism of posets

$$Q\colon \quad P := (\ulcorner^n \times [2]) \setminus \{(1, \langle m-1\rangle)\} \quad \to \quad \ulcorner^n \times [1]$$

induced by $\delta^1\colon [2] \to [1]$ gives

$$Q^*\colon \mathbf{sSets}^{\ulcorner^n \times [1]} \to \mathbf{sSets}^P,$$

which, pictorially, just fits a copy of Z between X and Z with its $\langle m-1\rangle$-vertex removed. Finally, homotopy left Kan extension along the inclusion $I\colon P \hookrightarrow \ulcorner^n \times [2]$ let's us construct the diagram Y described above by mapping τ through the composite

$$\mathrm{Ho}\left(\mathbf{sSets}^{\ulcorner^n\times[1]}\right) \xrightarrow{\mathrm{Ho}\,Q^*} \mathrm{Ho}\left(\mathbf{sSets}^{P}\right) \xrightarrow{\mathbb{L}I_!} \mathrm{Ho}\left(\mathbf{sSets}^{\ulcorner^n\times[2]}\right).$$

By the lemma above, the induced morphism $\operatorname{hocolim} X \to \operatorname{hocolim} Y$ is a weak equivalence and so it suffices to show that

$$\mathrm{hFib}_*(\operatorname{hocolim} Y \to \operatorname{hocolim} Z) \gg \Sigma^{n-m}(F_1 * \ldots * F_m).$$

Since $Y_{\langle m-1\rangle} \to Z_{\langle m-1\rangle}$ is just the comparison map for the strong homotopy pullback $X|_{\square^{m-1}} \Rightarrow Z|_{\square^{m-1}}$, Puppe's theorem (10.2) allows us to identify its fibre as being $F_1 * \ldots * F_m$. So, what we are really going to show is that

$$\mathrm{hFib}_*(\operatorname{hocolim} Y \to \operatorname{hocolim} Z) \gg \Sigma^{n-m}\,\mathrm{hFib}_*\left(Y_{\langle m-1\rangle} \to Z_{\langle m-1\rangle}\right).$$

The idea here is to repeatedly use Thomason's theorem to move this comparison map into a central location of some subcube and then apply the Web Trick (12.18). Identifying $\ulcorner^n \cong \int^{\ulcorner^2}(\ulcorner^{n-1} \leftarrow \ulcorner^{n-1} \to \{*\})$ via Γ^1 as in (5.3), we use Thomason's theorem to write the (induced map between) homotopy colimits as

$$\begin{tikzcd}
\operatorname{hocolim} Y \arrow[d, dotted] & \simeq & \operatorname{hocolim}\Big(\operatorname{hocolim}_{\ulcorner^{n-1}} Y|_{\partial_1} \arrow[d] & \operatorname{hocolim}_{\ulcorner^{n-1}} Y|_{\partial^{\neg 1}} \arrow[l] \arrow[r] \arrow[d, "\simeq"'] & Y_{\neg 1}\Big) \arrow[d, "\simeq"'] \\
\operatorname{hocolim} Z & \simeq & \operatorname{hocolim}\Big(\operatorname{hocolim}_{\ulcorner^{n-1}} Z|_{\partial_1} & \operatorname{hocolim}_{\ulcorner^{n-1}} Z|_{\partial^{\neg 1}} \arrow[l] \arrow[r] & Z_{\neg 1}\Big).
\end{tikzcd}$$

Observing that Y and Z agree everywhere except at $\langle m-1\rangle$, the two solid arrows on the right are weak equivalences and using Dror-Farjoun's theorem, it suffices to show that

$$\mathrm{hFib}_*\Big(\operatorname*{hocolim}_{\ulcorner^{n-1}} Y|_{\partial_1} \to \operatorname*{hocolim}_{\ulcorner^{n-1}} Z|_{\partial_1}\Big) \gg \Sigma^{n-m}\,\mathrm{hFib}_*\left(Y_{\langle m-1\rangle} \to Z_{\langle m-1\rangle}\right).$$

For this, we simply repeat the above, identify $\ulcorner^{n-1} \simeq \int^{\ulcorner^2}(\ulcorner^{n-2} \leftarrow \ulcorner^{n-2} \to \{*\})$ via Γ^1 and use Thomason's theorem again etc. We do this $m-1$ times and noting that $(Y|_{\partial_1})|_{\partial_1} = Y|_{\partial_{1,2}} = Y|_{\partial_{\langle 2\rangle}}$ (and similarly for $\neg 1$, Z and higher order restrictions), we end up needing to show that

$$\mathrm{hFib}_*\Big(\operatorname*{hocolim}_{\ulcorner^{n-m+1}} Y|_{\partial_{\langle m-1\rangle}} \to \operatorname*{hocolim}_{\ulcorner^{n-m+1}} Z|_{\partial_{\langle m-1\rangle}}\Big) \gg \Sigma^{n-m}\,\mathrm{hFib}_*\left(Y_{\langle m-1\rangle} \to Z_{\langle m-1\rangle}\right).$$

But now $\langle m-1\rangle$ is the initial vertex of $\partial_{\langle m-1\rangle}$ and the claimed cellular inequality is exactly what we get from the Web Trick (12.18). $\square$

Chapter 11

HOMOTOPY EXCISION FOR CUBES

1. Suspended Comparison Maps

In chapter 9, we showed a cellular analogue of the Blakers-Massey theorem for squares by directly considering the fiber of the corresponding comparison map. By first suspending the comparison map, we can use the web-trick (10.12.18) to reduce the problem to the dual situation, where we have Puppe's theorem (10.10.2) to further facilitate things.

To wit, we start with a strong homotopy pushout $A\colon \square^n \to \mathbf{sSets}$ and wish to understand the homotopy fibre of some suspension of the comparison map $q\colon A_\varnothing \to P_\varnothing$, where $P\colon \square^n \to \mathbf{sSets}$ agrees with A everywhere except at $\varnothing$, where it is the homotopy pullback of $A|_{\lrcorner^n}$; i.e.

$$P := \mathbb{R}\,\mathrm{Kan}\big(A|_{\lrcorner^n}\big)\colon \square^n \to \mathbf{sSets}.$$

From the web-trick (10.12.18), we know that

$$\mathrm{h}\mathfrak{F}\mathrm{ib}(\Sigma^{n-1}q) \gg \mathrm{h}\mathfrak{F}\mathrm{ib}\left(A_{\langle n\rangle} \xrightarrow{s} \operatorname*{hocolim}_{\ulcorner^n} P\right)$$

and so it suffices to understand this fibre set. Restricting the obvious natural transformations $A \Rightarrow P \Rightarrow \mathrm{Const}_{A_{\langle n\rangle}}$ to $\ulcorner^n$ and taking homotopy colimits, we get that

$$s\colon A_{\langle n\rangle} \to \operatorname*{hocolim}_{\ulcorner^n} P \qquad \text{has a retraction} \qquad r\colon \operatorname*{hocolim}_{\ulcorner^n} P \to A_{\langle n\rangle}.$$

By (8.7.9) then $\mathrm{h}\mathfrak{F}\mathrm{ib}(s) \gg \Omega\,\mathrm{h}\mathfrak{F}\mathrm{ib}(r)$ and so, instead of studying $\mathrm{h}\mathfrak{F}\mathrm{ib}(s)$, we might just as well study $\mathrm{h}\mathfrak{F}\mathrm{ib}(r)$. For this, let $S\colon \square^n \to \mathbf{sSets}$ be the strong homotopy pullback of A (cf. section 10.9). That is to say, we let $\langle n\rangle^{\triangleright} \subset \square^n$ be the subposet consisting of all $\hat{k} \to \langle n\rangle$ and then

$$S := \mathbb{R}\,\mathrm{Kan}(A|_{\langle n\rangle^{\triangleright}})\colon \square^n \to \mathbf{sSets}.$$

Now, the inclusion of posets $\langle n\rangle^{\triangleright} \hookrightarrow \lrcorner^n$ yields a natural transformation $P \Rightarrow S$ and upon composing with the obvious transformation $S \Rightarrow \mathrm{Const}_{A_{\langle n\rangle}}$, restricting to $\ulcorner^n$ and taking homotopy colimits, we get a factorisation

$$\operatorname*{hocolim}_{\ulcorner^n} P \xrightarrow{p} \operatorname*{hocolim}_{\ulcorner^n} S \xrightarrow{p'} A_{\langle n\rangle} \qquad \text{of} \qquad r\colon \operatorname*{hocolim}_{\ulcorner^n} P \to A_{\langle n\rangle}.$$

The acyclic inequality (8.7.8) associated to this composable pair is

$$\mathrm{h}\mathfrak{F}\mathrm{ib}(p' \circ p) > \mathrm{h}\mathfrak{F}\mathrm{ib}(p) \cup \mathrm{h}\mathfrak{F}\mathrm{ib}(p'). \tag{1.1}$$

Finally, Puppe's theorem (10.10.2) together with Chachólski's theorem (8.7.6) yield

$$\mathrm{h}\mathfrak{F}\mathrm{ib}(p') \simeq \mathop{\circledast}_{k=1}^{n} \mathrm{h}\mathfrak{F}\mathrm{ib}\big(A_{\hat{k}} \to A_{\langle n\rangle}\big) \gg \mathop{\circledast}_{k=1}^{n} \mathfrak{F}(A_\varnothing \to A_k).$$

Let us summarise the above discussion in the following proposition.

(1.2) **Proposition.** Let $A\colon \square^n \to \mathbf{sSets}$ be a strong homotopy pushout with corresponding homotopy pullback P and strong homotopy pullback S. Writing

- $q\colon A_\varnothing \to P_\varnothing$ for the comparison map and
- $p\colon \operatorname{hocolim}_{\ulcorner^n} P \to \operatorname{hocolim}_{\ulcorner^n} S$ for the map induced by the canonical transformation $P \Rightarrow S$ (itself induced by $\langle n\rangle^{\triangleright} \hookrightarrow \lrcorner^n$),

then we have a cellular inequality

$$\mathrm{h}\mathfrak{F}\mathrm{ib}(\Sigma^{n-1}q) > \Omega\left(\mathrm{h}\mathfrak{F}\mathrm{ib}(p) \cup \mathop{\circledast}_{k=1}^{n} \mathrm{h}\mathfrak{F}\mathrm{ib}(A_\varnothing \to A_k)\right).$$

□

2. Suspended Square Case

With the last proposition in place, we can immediately treat the suspended square case, which was already done in [13].

(2.1) **Theorem.** If $A\colon \square^2 \to \mathbf{sSets}$ is a homotopy pushout square with comparison map $q\colon A_\varnothing \to \operatorname{holim}(A_1 \to A_{1,2} \leftarrow A_2)$, then

$$\mathrm{h}\mathfrak{F}\mathrm{ib}(\Sigma q) > \Omega\big(\mathrm{h}\mathfrak{F}\mathrm{ib}(A_\varnothing \to A_1) * \mathrm{h}\mathfrak{F}\mathrm{ib}(A_\varnothing \to A_2)\big)$$

and consequently

$$\mathrm{h}\mathfrak{F}\mathrm{ib}(\Sigma q) > \mathrm{h}\mathfrak{F}\mathrm{ib}(A_\varnothing \to A_1) \wedge \mathrm{h}\mathfrak{F}\mathrm{ib}(A_\varnothing \to A_2)$$

for any choice of base-points (assuming none of the fibres are empty).

Proof. In the square case here, the ordinary homotopy pullback of $A_1 \to A_{1,2} \leftarrow A_2$ is already strong and so, (1.2) exactly says that

$$\mathrm{h}\mathfrak{F}\mathrm{ib}(\Sigma q) > \Omega\big(\mathrm{h}\mathfrak{F}\mathrm{ib}(A_\varnothing \to A_1) * \mathrm{h}\mathfrak{F}\mathrm{ib}(A_\varnothing \to A_2)\big)$$

For the pointed case, we just use that $\Omega(X * Y) \simeq \Omega\Sigma(X \wedge Y) \gg X \wedge Y$ for any pointed spaces X, Y. □

(2.2) **Remark.** We can actually do better and even get cellular inequalities in the above theorem, rather than just acyclic ones. Going through the last section again, the point where we went from a cellular inequality to an acyclic one was (1.1). In our square case, however, $P = S$ and so, the map p in (1.1) is a weak equivalence, implying that even $\mathrm{h}\mathfrak{F}\mathrm{ib}(p' \circ p) \simeq \mathrm{h}\mathfrak{F}\mathrm{ib}(p')$.

3. Cubical Case

As we saw in section 1, given a strong homotopy pushout $A\colon \square^3 \to \mathbf{sSets}$ with corresponding homotopy pullback $P\colon \square^3 \to \mathbf{sSets}$ (which comes with a comparison map $q\colon A_\varnothing \to P_\varnothing$) and strong homotopy pullback $S\colon \square^3 \to \mathbf{sSets}$, if we would like to understand (the Bousfield class of) $\mathrm{h}\mathfrak{F}\mathrm{ib}(\Sigma^2 q)$, we need to understand $\mathrm{h}\mathfrak{F}\mathrm{ib}(p)$, where $p\colon \operatorname{hocolim}_{\ulcorner^3} P \to \operatorname{hocolim}_{\ulcorner^3} S$ is the comparison map. For this, we construct a sequence of homotopy pullback cubes

$$P \simeq P^{(0)} \Rightarrow P^{(1)} \Rightarrow P^{(2)} \Rightarrow P^{(3)} \simeq S,$$

where each transformation induces a map between homotopy colimits over $\ulcorner^3$ that we can understand. These three cubes correspond to the three possible ways of calculating the homotopy pullback using $\mathcal{T}^1$, $\mathcal{T}^2$ and $\mathcal{T}^3$ (see (10.5.5)). In this way, we subsequently replace all possible 2-faces by homotopy pullbacks and therefore eventually arrive at a strong homotopy pullback. Unfortunately, we are only able to deal with the connected case but we are confident that with some additional effort, our strategy can be generalised.

There is a fair amount of bookkeeping involved in the following proof and to facilitate it, we collect some of the fibres occurring in a cube into a set and establish cellular inequalities involving those.

(3.1) **Definition.** Given a cubical diagram $X\colon \square^n \to \mathbf{sSets}$ such that all X_M with $M \neq \varnothing$ are connected and $k \in \langle n \rangle$, we define

$$\mathfrak{F}_k^X := \Big\{ \mathrm{hFib}\Big(X_M \to X_{M\cup\{k\}}\Big) \;\Big|\; M \subseteq \langle n \rangle \setminus \{k\} \Big\}$$

(where we omit the top index "X" if it is clear from the context). We call this the *collective* k^{th} *(homotopy) fibre* of X.

(3.2) **Example.** If X is a strong pullback cube then all fibres in $\mathfrak{F}_k$ are weakly equivalent to each other. In that case, we shall usually work with $F_k = \mathrm{hFib}(X_\varnothing \to X_k)$ or $F_{\hat{k}} = \mathrm{hFib}(X_{\hat{k}} \to X_{\langle n \rangle})$, which both represent the fibres in $\mathfrak{F}_k$.

(3.3) **Example.** If X is a strong pushout cube, the fibres in $\mathfrak{F}_k$ are no longer equivalent but from Chachólski's theorem (8.7.6), we know that at least $\mathfrak{F}_k \gg F_k$ (and obviously $F_k \gg \mathfrak{F}_k$), so that, from a cellular viewpoint, $\mathfrak{F}_k$ still collapses to a single fibre.

There are several links between strong and weak cellular inequalities. There are "crude" results like that $A \gg B$ implies $A > B$ and that $A > \Sigma B$ implies $A \gg B$. But things get more subtle, when working with fibre sequences. What follows is one of these subtle results, which apparently allows one to pick up a suspension when working with weak cellular inequalities.

(3.4) **Lemma.** Given a fibre sequence $F \to E \to B$ where $F > A$ and $E > \Sigma A$ for some simplicial set A, then $B > \Sigma A$.

Proof. We form the cofibre sequence

$$E \to B \to B /\!\!/ E.$$

Here $B /\!\!/ E \gg \Sigma F$ by [11, Proposition 10.5] and $\mathrm{h}\mathfrak{F}\mathrm{ib}(B \to B /\!\!/ E) \gg E$ by [12, Theorem 8.1], from which the claim follows. $\square$

(3.5) **Proposition.** Let $A\colon \square^3 \to \mathbf{sSets}$ be a strong homotopy pushout cube of connected spaces and write $F_k := \mathrm{hFib}(A_\varnothing \to A_k)$ for its homotopy fibres. If $P\colon \square^n \to \mathbf{sSets}$ is the corresponding homotopy pullback, then $\mathfrak{F}_k^P > \Sigma\Omega F_k$ for all $k \in \langle 3 \rangle$ as long as all fibres F_1, F_2 and F_3 is connected.

Proof. Without loss of generality, let's assume $k = 1$ and that F_2 is connected. The collective fibres $\mathfrak{F}_1^A$ and $\mathfrak{F}_1^P$ are almost the same, with the former being killed by F_1. The only fibre in $\mathfrak{F}_1^P$ that is not in $\mathfrak{F}_1^A$ is $\mathrm{hFib}(P_\varnothing \to A_1)$. Picking any base-point on $A_\varnothing$ makes everything (and in particular $P_\varnothing$) pointed. We now consider the fibre sequence

$$\mathrm{hFib}(A_\varnothing \to P_\varnothing) \;\longrightarrow\; \underbrace{\mathrm{hFib}(A_\varnothing \to A_1)}_{F_1} \;\longrightarrow\; \mathrm{hFib}(P_\varnothing \to A_1)$$

associated to $A_\varnothing \to P_\varnothing \to A_1$. Because $F_1 \gg \Sigma\Omega F_1$ (see [11, Corollary 10.6]) and using the lemma above, it suffices to show that $\mathrm{hFib}(A_\varnothing \to P_\varnothing) > \Omega F_1$. For this, we recall that $P_\varnothing$ fits into a homotopy pullback square (of solid arrows)

$$\begin{tikzcd}
A_\varnothing \arrow[r, dotted] & P_\varnothing \arrow[r, "f"] \arrow[d] & \mathrm{holim}(A_1 \to A_{1,2} \leftarrow A_2) \arrow[d, "p"] \\
& A_3 \arrow[r, "g"'] & \mathrm{holim}(A_{1,3} \to A_{1,2,3} \leftarrow A_{2,3}).
\end{tikzcd}$$

The homotopy fibre of p above any base-point of $\mathrm{holim}(A_{1,3} \to A_{1,2,3} \leftarrow A_{2,3})$ is just the homotopy pullback

$$G := \mathrm{holim}\big(\mathrm{hFib}(A_1 \to A_{1,3}) \to \mathrm{hFib}(A_{1,2} \to A_{1,2,3}) \leftarrow \mathrm{hFib}(A_2 \to A_{2,3})\big).$$

Since A is a strong homotopy pushout, all these fibres are killed by F_3, which is connected, and hence connected themselves. The fibre sequence associated to the homotopy pullback G is

$$\Omega\,\mathrm{hFib}(A_{1,2} \to A_{1,2,3}) \to G \to \mathrm{hFib}(A_1 \to A_{1,3}) \times \mathrm{hFib}(A_2 \to A_{2,3})$$

and since both the base and the fibre are killed by ΩF_3, it follows that also $G > \Omega F_3 > S_0$, so that $\pi_0(G) \neq \varnothing$. Since the base-point was arbitrary, it follows that p hits all components of the base and hence $\mathrm{h\mathfrak{F}ib}(g) \simeq \mathrm{h\mathfrak{F}ib}(f)$. From the unsuspended square case, we know that

$$\mathrm{h\mathfrak{F}ib}(g) > \Omega\,\mathrm{hFib}(A_3 \to A_{1,3}) * \Omega\,\mathrm{hFib}(A_3 \to A_{2,3}) > \Omega F_1 * \Omega F_2 > \Sigma\Omega F_1,$$

where, for the last inequality, we used that F_2 is connected. By the same argument, we also have that

$$\mathrm{h\mathfrak{F}ib}\big(A_\varnothing \to \mathrm{holim}(A_1 \to A_{1,2} \leftarrow A_2)\big) > \Omega F_1 * \Omega F_2 > \Sigma\Omega F_1.$$

We now consider the acyclic inequality (8.7.8) associated to the top composite in the last diagram, which is

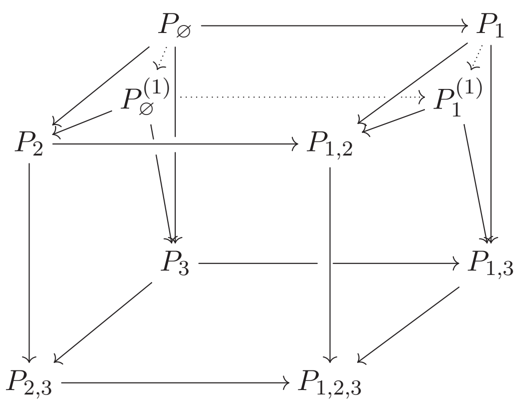
$$\mathrm{hFib}(A_\varnothing \to P_\varnothing) > \mathrm{h\mathfrak{F}ib}\big(A_\varnothing \to \mathrm{holim}(A_1 \to A_{1,2} \leftarrow A_2)\big) \cup \mathrm{h\mathfrak{F}ib}(f)$$

and finally get $\mathrm{hFib}(A_\varnothing \to P_\varnothing) > \Omega\Sigma\Omega F_1 > \Omega F_1$. □

As already mentioned, we start with a homotopy pullback $P\colon \square^3 \to \mathbf{sSets}$ and use the subposets $\mathcal{T}^1$, $\mathcal{T}^2$ and $\mathcal{T}^3$ of $\square^3$ to build the corresponding strong pullback in several steps. More explicitly, using the notation of (10.5.5):

(3.6) $\quad P^{(1)} := \mathbb{R}_1 P, \qquad P^{(2)} := \mathbb{R}_2 P^{(1)} \qquad \text{and} \qquad P^{(3)} := \mathbb{R}_3 P^{(2)}.$

Pictorially, we are going to do the following (using $\mathcal{T}^1$ here):



By definition, $P^{(1)}_{\varnothing}$ and $P^{(1)}_1$ are the homotopy pullbacks of the corresponding faces. Moreover, by Thomason's theorem and using the hypothesis that P is a homotopy pullback cube, the square formed by $P_\varnothing$, P_1, $P^{(1)}_{\varnothing}$ and $P^{(1)}_1$ is also a homotopy pullback. Since, when passing to $P^{(2)}$, the vertex at $\varnothing$ and 2 change, we need to make sure that the left face stays a homotopy pullback. Since everything is symmetric, we might just as well check that in the cubical diagram above, if the back and front faces are homotopy pullbacks, so is the square containing $P^{(1)}_{\varnothing}$, $P^{(1)}_1$, P_3 and $P_{1,3}$. This follows from Thomason's theorem, by which

$$\begin{tikzcd}
P^{(1)}_{\varnothing} \arrow[r] \arrow[d] & \operatorname{holim}\big(P_3 \to P_{1,3} \leftarrow P^{(1)}_1\big) \arrow[d] \\
P_2 \arrow[r] & \operatorname{holim}\big(P_{2,3} \to P_{1,2,3} \leftarrow P_{1,2}\big)
\end{tikzcd}$$

is a homotopy pullback and the bottom map is a weak equivalence by assumption. Similarly when passing from $P^{(2)}$ to $P^{(3)}$, where then, the cube has become the strong homotopy pullback obtained from all the $P_{\hat{k}} \to P_{\langle 3\rangle}$ because all 2-faces of $P^{(3)}$ are homotopy pullbacks.

(3.7) **Proposition.** Given a homotopy pullback $P\colon \square^3 \to \mathbf{sSets}$ and letting $P' := \mathbb{R}_k P$ for some $k \in \langle 3\rangle$. If all P_M and P'_M with $M \neq \varnothing$ are connected, then $\mathfrak{F}^{P'}_l \subseteq \mathfrak{F}^{P}_l$ (up to weak equivalences) for all $l \in \langle 3\rangle$, whence $\mathfrak{F}^{P'}_l \gg \mathfrak{F}^{P}_l$.

Proof. Without loss of generality, let's assume $k = 1$ (i.e. $P' = P^{(1)}$). The only vertices, where P and $P^{(1)}$ differ are those at $\varnothing$ and 1 and so, the only fibres that change are those of the morphism associated to $\varnothing \to 1$ (in $\mathfrak{F}_1$) as well as those associated to $\varnothing \to \{l\}$ and $1 \to \{1, l\}$ (in $\mathfrak{F}_l$) for $1 \neq l$. The first one is easy because

$$\operatorname{hFib}\left(P^{(1)}_{\varnothing} \to P^{(1)}_1\right) \simeq \operatorname{hFib}(P_\varnothing \to P_1)$$

(because the corresponding square is a homotopy pullback) and therefore $\mathfrak{F}^{P'}_1 \simeq \mathfrak{F}^{P}_1$. As for the other ones, we just use that $P^{(1)}_{\varnothing}$ and $P^{(1)}_1$ are obtained as the pullbacks of the left and right faces in the cube above. So, for example, taking $l = 2$,

$$\operatorname{hFib}\left(P^{(1)}_{\varnothing} \to P_2\right) \simeq \operatorname{hFib}(P_3 \to P_{2,3}), \quad \operatorname{hFib}\left(P^{(1)}_1 \to P_{1,2}\right) \simeq \operatorname{hFib}(P_{1,3} \to P_{1,2,3}).$$

So $\mathfrak{F}^{P'}_2$ comprises just these two fibres and is hence contained in $\mathfrak{F}^{P}_2$. □

(3.8) **Proposition.** Let $P\colon \square^3 \to \mathbf{sSets}$ be a homotopy pullback and $P' := \mathbb{R}_k P$ for some $k \in \langle 3\rangle$. If all P_M and P'_M with $M \neq \varnothing$ are connected, then the canonical transformation $P \Rightarrow P'$ induces a map

$$\operatorname*{hocolim}_{\ulcorner^3} P \to \operatorname*{hocolim}_{\ulcorner^3} P',$$

whose homotopy fibre F satisfies $F \gg \Sigma\big(\mathfrak{F}^{P}_k * \operatorname{hFib}(P_k \to P'_k)\big)$.

(3.9) **Remark.** Implicit in our proposition is that $\operatorname{hocolim}_{\ulcorner^3} P'$ is connected as well, which follows from $\{\text{connected spaces}\} = \mathcal{C}(S^1)$ being a closed class and in particular closed under cubical homotopy pushouts.

Proof. Without loss of generality, let's treat the case $k = 1$ (i.e. $P' = P^{(1)}$). Note that $P \Rightarrow P'$ consists of identities everywhere except at $\varnothing$ and 1, where we have the homotopy pullback square

$$\begin{tikzcd} P_\varnothing \arrow[r] \arrow[d] & P_1 \arrow[d] \\ P^{(1)}_\varnothing \arrow[r] & P^{(1)}_1 \end{tikzcd} .$$

By Serre's theorem (10.13.4), we get that the homotopy fibre F of the induced map between homotopy colimits satisfies

$$F \gg \Sigma\Big(\underbrace{\mathrm{hFib}(P_\varnothing \to P_1)}_{\in \mathfrak{F}_1^P} * \mathrm{hFib}\Big(P_1 \to P_1^{(1)}\Big)\Big)$$

□

(3.10) **Corollary.** Let $P\colon \square^3 \to \mathbf{sSets}$ be a homotopy pullback with corresponding strong homotopy pullback $S\colon \square^3 \to \mathbf{sSets}$. If all P_M and S_M with $M \neq \varnothing$ are connected, then the homotopy fibre of the canonical map

$$p\colon \operatorname*{hocolim}_{\ulcorner^3} P \to \operatorname*{hocolim}_{\ulcorner^3} S$$

(induced by the inclusion $\langle 3\rangle^{\triangleright} \hookrightarrow \lrcorner^3$) satisfies

$$\mathrm{hFib}(p) \gg \Sigma\left\{\mathfrak{F}_k^P * \mathrm{hFib}(P_k \to S_k) \;\middle|\; k \in \langle 3\rangle\right\}.$$

Proof. We construct S in several steps, as outlined in (3.6) and note that $P_1^{(1)} \simeq S_1$, $P_2^{(2)} \simeq S_2$ and $P_3^{(3)} \simeq S_3$, while $P_2^{(1)} \simeq P_2$ and $P_3^{(2)} \simeq P_3^{(1)} \simeq P_3$. Now, writing

$$p_k\colon \operatorname*{hocolim}_{\ulcorner^3} P^{(k-1)} \to \operatorname*{hocolim}_{\ulcorner^3} P^{(k)} \qquad (\text{where } P^{(0)} := P)$$

for the canonical maps, we have $p = p_3 \circ p_2 \circ p_1$, so that

$$\mathrm{hFib}(p) > \{\mathrm{hFib}(p_k) \mid k \in \langle 3\rangle\}$$

(using the fibre sequences associated to composable pairs of arrows). By (3.8), we have

$$\mathrm{hFib}(p_1) \gg \Sigma\left(\mathfrak{F}_1^P * \mathrm{hFib}(P_1 \to S_1)\right),$$

$$\mathrm{hFib}(p_2) \gg \Sigma\left(\mathfrak{F}_2^{P^{(1)}} * \mathrm{hFib}(P_2 \to S_2)\right),$$

$$\mathrm{hFib}(p_3) \gg \Sigma\left(\mathfrak{F}_3^{P^{(2)}} * \mathrm{hFib}(P_3 \to S_3)\right).$$

Using (3.7), we can now replace $\mathfrak{F}_2^{P^{(1)}}$ and $\mathfrak{F}_3^{P^{(2)}}$ by $\mathfrak{F}_2^P$ and $\mathfrak{F}_3^P$, respectively. □

Finally, by combining this corollary with (3.5) and the unsuspended square case, we obtain the following weak version of the 3-dimensional Blakers-Massey theorem.

(3.11) **Theorem.** Let $A\colon \square^3 \to \mathbf{sSets}$ be a strong homotopy pushout of connected spaces, with homotopy fibres $F_k := \mathrm{hFib}(A_\varnothing \to A_k)$ and comparison map $q\colon A_\varnothing \to \operatorname{holim}_{\lrcorner^3} A$. As long as the homotopy fibres F_1, F_2 and F_3 are again connected,

$$\mathrm{hFib}(\Sigma^2 q) > \Sigma(\Omega F_1 * \Omega F_2 * \Omega F_3).$$

Proof. Let $P\colon \square^3 \to \mathbf{sSets}$ be the corresponding homotopy pullback and $S\colon \square^3 \to \mathbf{sSets}$ the strong homotopy pullback of all the $A_{\hat{k}} \to A_{\langle 3\rangle}$. We first note that all P_M and S_M with $M \neq \varnothing$ are connected by the acyclic Blakers-Massey theorem for squares. To wit, when passing from P to $P^{(1)}$, the only new space (except for the initial vertex) is $P_1^{(1)}$, where

$$\mathrm{h}\mathfrak{F}\mathrm{ib}\left(P_1 \to P_1^{(1)}\right) > \Omega\,\mathrm{hFib}(P_1 \to P_{1,2}) * \Omega\,\mathrm{hFib}(P_1 \to P_{1,3}) > \Omega F_2 * \Omega F_3 > S^1,$$

where the first acyclic inequality is the acyclic Blakers-Massey theorem for squares (9.7.1), the second one is Chachólski's theorem (8.7.6) and the last one follows from the hypothesis $F_k > S^1$ for all k. We conclude that $P_1 \to P_1^{(1)}$ has no empty homotopy fibres, meaning that all components are hit and since $P_1 = A_1$ is connected, so is $P_1^{(1)}$. Similarly when passing to $P^{(2)}$ and $P^{(3)} = S$.

With the connectivities established, we can use the previous results from this section to establish the claimed acyclic inequality. As always, we have the comparison map

$$p\colon \operatorname*{hocolim}_{\ulcorner^3} P \to \operatorname*{hocolim}_{\ulcorner^3} S$$

and by (1.2), it suffices to show that $\mathrm{hFib}(p) > \Sigma^2(\Omega F_1 * \Omega F_2 * \Omega F_3)$ because then

$$\mathrm{hFib}(\Sigma^2 q) \gg \Omega\,\mathrm{hFib}(p) > \Omega\Sigma^2(\Omega F_1 * \Omega F_2 * \Omega F_3) > \Sigma(\Omega F_1 * \Omega F_2 * \Omega F_3).$$

From the above corollary (and noting that $P_k = A_k$), we already know that

$$\mathrm{hFib}(p) \gg \Sigma\left\{\mathfrak{F}_k^P * \mathrm{hFib}(A_k \to S_k) \;\middle|\; k \in \langle 3\rangle\right\}$$

and by (3.5), we then have

$$\mathrm{hFib}(p) > \Sigma\left\{\Sigma\Omega F_k * \mathrm{hFib}(A_k \to S_k) \mid k \in \langle 3\rangle\right\}.$$

But by definition $S_1 = \operatorname{holim}(A_{1,2} \to A_{1,2,3} \leftarrow A_{1,3})$, so that, by the unsuspended Blakers-Massey theorem for squares and Chachólski's theorem,

$$\mathrm{hFib}(A_1 \to S_1) > \Omega\,\mathrm{hFib}(A_1 \to A_{1,2}) * \Omega\,\mathrm{hFib}(A_1 \to A_{1,3}) > \Omega F_2 * \Omega F_3.$$

Similarly for the other two homotopy fibres, so that, all in all,

$$\mathrm{hFib}(p) > \Sigma^2(\Omega F_1 * \Omega F_2 * \Omega F_3).$$

$\square$

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Languages

German:	Mother tongue
English:	Fluent
French:	Fluent
Swedish:	Conversational level

Education

November 2015 - April 2016:	Visiting researcher at the Kungliga Tekniska Högskolan Stockholm (SE), collaborating with Prof. Dr. Wojciech Chachòlski
October 2011 - :	PhD in mathematics specialising in homotopy theory at the École Polytechnique Fédérale de Lausanne (CH) under the supervision of Prof. Dr. Jérôme Scherer and Prof. Dr. Kathryn Hess-Bellwald
August 2008 - April 2011:	MSc in mathematics at the University of Basel (CH) with specialisation modules: homological algebra and group cohomology (both at the ETH Zurich [CH]), algebraic geometry and linear algebraic groups

Employment

October 2011 - :	Assistant in mathematics at the École Polytechnique Fédérale de Lausanne (CH)
August 2007 - June 2011:	Teaching assistant in mathematics at the University of Basel (CH)
April 2007 - September 2007:	Teaching assistant in computer science at the University of Basel (CH)

Teaching experience

October 2011 - :	Exercises: Point-set Topology, Algebraic Topology, Linear Algebra and Set Theory
Autumn 2008:	Lecture: Supplementary category theory course (with a special focus on groupoids) to a group theory lecture given by Giordano Favi at the university of Basel
August 2007 - June 2011:	Exercises: Linear Algebra, Calculus, Real Analysis, Functional Analysis, Group Theory, Topology
April 2007 - September 2007:	Exercises: Algorithms and Data Structures

Publications & Preprints

Chachólski, W., Scherer, J., Werndli, K.; *Homotopy Excision and Cellularity*; August 2014; http://arxiv.org/abs/1408.3252 — to appear in the *Annales de l'Institut Fourier*

Werndli, K., Zaganidis, D.; *Real-Valued Persistence Diagrams*; in preparation

Werndli, K.; *Double Homotopy (Co)limits for Relative Categories*; in preparation

Awards & Grants

November 2014:	Swiss NSF Doc.Mobility grant for a half-year visit at the Kungliga Tekniska Högskolan Stockholm (SE)
September 2014:	Birkhäuser Price for best expository talk awarded for the talk *Cellularity and Homotopy Excision.*

Other Professional Experiences & Abilities

Organiser of the 2015 Young Topologists' Meeting in Lausanne (CH)

Organiser of the 2013 Young Topologists' Meeting in Lausanne (CH)

Programming experience in Haskell, C/C++, Java, Python, Matlab, Bash, TeX and Assembler

Doctoral Thesis

"Cellular Homotopy Excision" at the École Polytechnique Fédérale de Lausanne under the supervision of Dr. Jérôme Scherer and Prof. Dr. Kathryn Hess Bellwald, strengthening the well-known classical homotopy excision theorem using Bousfield classes.

Conferences & Schools

September 2015:	Workshop in Category Theory and Algebraic Topology, Louvain-la-Neuve (BE), Talk *A Cellular Version of Blakers-Massey*
July 2015:	Young Topologists' Meeting, Lausanne (CH)
April 2015:	AC Building Bridges between Algebra and Topology, Barcelona (ES)
February 2015:	Winter School: Novel Mathematical Approaches to Solving Biological Problems, Kandersteg (CH)
September 2014:	Summer School: Category Theory and Algebraic Topology, Lausanne (CH)
September 2014:	Graduate Colloquium of the Swiss Doctoral Program, Talk *Cellularity and Homotopy Excision*, Basel (CH)
July 2014:	Young Topologists' Meeting, Copenhagen (DK)
January 2014:	Introductory Workshop: Algebraic Topology, MSRI, Berkeley (US)
January 2014:	Connections for Women: Algebraic Topology, MSRI, Berkeley (US)
September 2013:	Barcelona Topology Workshop, Barcelona (ES)
September 2013:	Summer School: Category Theory and Algebraic Topology, Louvain-la-Neuve (BE)
July 2013:	Young Topologists' Meeting, Lausanne (CH)
June 2013:	Conference in Honour of George Maltsiniotis, Paris (FR)
February 2013:	Graduate Colloquium of the Swiss Doctoral Program, Bern (CH)
August 2012:	Arolla Conference on Algebraic Topology, Arolla (CH)
July 2012:	Young Topologists' Meeting, Copenhagen (DK)
April 2012:	Modern Perspectives in Homotopy Theory, Swansea (GB)
September 2011:	Graduate Colloquium of the Swiss Doctoral Program, Basel (CH)
December 2009:	Trends in Essential Dimension, Basel (CH)
May - June 2009:	PIMS Algebra Summer School, Alberta (CA)