Supplemental Material to "Unconventional dc Transport in Rashba Electron Gases"

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HELICITY EIGENSTATES BASIS

To fix the notation, let us start by giving some details on the helicity eigenstates basis. The helicity operator is defined as $S = [\hat{p} \times \vec{\sigma}]_z$ and its eigenstates, $|\mathbf{k}s\rangle$, with $s = \pm 1$, satisfy the relation $S|\mathbf{k}\pm\rangle = \pm|\mathbf{k}\pm\rangle$. A simple calculation shows in particular that $|\mathbf{k}\pm\rangle$ can be expressed in terms of the standard spin eigenstates, $|\mathbf{k}\uparrow\rangle$ and $|\mathbf{k}\downarrow\rangle$, as $|\mathbf{k}\pm\rangle = (\exp(-i\theta_{\mathbf{k}})|\mathbf{k}\uparrow\rangle \pm i|\mathbf{k}\downarrow\rangle)/\sqrt{2}$ with $\theta_{\mathbf{k}} = \arctan(k_y/k_x)$. Consequently, the matrix $U_{\mathbf{k}}$ which implements the rotation from the spin to the helicity eigenstates basis has the form

$$U_{\mathbf{k}} = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{-i\theta_{\mathbf{k}}} & e^{-i\theta_{\mathbf{k}}} \\ i & -i \end{pmatrix}. \tag{1}$$

In the basis spanned by the states $|\mathbf{k}s\rangle$ the total Hamiltonian (Eq. (3) in main text) can be recast as follows:

$$H = \sum_{\mathbf{k}} c^{\dagger}_{\mathbf{k}} H_k c_{\mathbf{k}} + \sum_{\mathbf{q},\mathbf{k}} V_{\rm imp}(\mathbf{q}) c^{\dagger}_{\mathbf{k}+\mathbf{q}} U^{\dagger}_{\mathbf{k}+\mathbf{q}} U_{\mathbf{k}} c_{\mathbf{k}} \tag{2}$$

where $c_{\mathbf{k}} = (c_{\mathbf{k}+}, c_{\mathbf{k}-})$ and $c^{\dagger}_{\mathbf{k}} = (c^{\dagger}_{\mathbf{k}+}, c^{\dagger}_{\mathbf{k}-})$ are spinor creation and annihilation operators, H_k is the Hamiltonian of the clean Rashba model, *i.e.* $H_k = \mathrm{diag}(E^{+}_k - E_0, E^{-}_k - E_0)$ with $E^{\pm}_k = (k \pm p_0)^2/(2m)$, $p_0 = m\alpha$ and $V_{\rm imp}(\mathbf{q})$ denotes the Fourier transform of the impurity potential, $V_{\rm imp}(\mathbf{q}) = 1/\mathcal{V} \sum_j e^{i\mathbf{q}\cdot\mathbf{R}_j} v_{\rm imp}$. Differently from the main text, in this Supplementary Material where not differently specified we use units $\hbar = e = 1$.

GREEN'S FUNCTION

The Green's function obeys the standard Dyson equation: $G^{-1} = \left(G^0\right)^{-1} - \Sigma$, where G^0 is the Green's function of the Rashba model in the absence of disorder and Σ is the self-energy. Specifically, in the helicity eigenstates basis, G^0 has the form

$$[G^0]_{\alpha\beta} = (i\varepsilon_l - E^{\alpha}_p + E_F)^{-1}\delta_{\alpha\beta}$$

where ε_l is a fermionic Matsubara frequency, and from now on we set the zero of the energy to $-E_0$. Thus, as in the main text, the dominant spin-orbit (DSO) regime is identified by $E_F < E_0$. Within the self-consistent Born approximation (SCBA), $\Sigma = \Sigma(\mathbf{p}, i\varepsilon_n)$ is determined by solving the following equation

$$\Sigma(\mathbf{k}, i\epsilon_n) = \frac{n_i v^2_{\rm imp}}{\mathcal{V}} \sum\nolimits_{\mathbf{p}} U^{\dagger}_{\mathbf{p}} U_{\mathbf{k}} G(\mathbf{p}, i\epsilon_n) U^{\dagger}_{\mathbf{k}} U_{\mathbf{p}} \tag{3}$$

Figure 1: Wigwam diagrams which describe the self-energy within Born approximation. The solid line correspond to the dressed Green function, G, while the crosses indicate averaging over disorder [1, 2].

which corresponds the "wigwam diagram" depicted in Fig. 1 as described *e.g.* in Refs [1, 2]. Note that in the helicity basis to each impurity-scattering vertex, changing the electron momentum from $\mathbf{k}$ to $\mathbf{p}$, one has to associate the spin rotation $U^{\dagger}_{\mathbf{p}} U_{\mathbf{k}}$. Equation (3) admits a momentum- and spin- independent solution. Indeed, assuming $[\Sigma(\mathbf{p}, i\epsilon_n)]_{\alpha\beta} = \Sigma(i\epsilon_n)$, the momentum dependent part on the r.h.s. of this equation averages away. Within SCBA the Green's function $G(\mathbf{p}, i\epsilon_n)$ is thus represented by the following diagonal matrix in the helicity eigenstates basis:

$$[G(\mathbf{p}, i\varepsilon_l)]_{\alpha\beta} = (i\varepsilon_l - E^{\alpha}_p + E_F - \Sigma(i\epsilon_l))^{-1}\delta_{\alpha\beta}. \tag{4}$$

By analytical continuation to real frequencies of Eq. (3) (see *e.g.* [2]) we obtain the following self-consistent equations for the scattering rate Γ

$$\Gamma = -\mathrm{Im}[\Sigma^R(0)] = \frac{n_i v^2_{\rm imp}\Gamma}{2\mathcal{V}} \sum_{\mathbf{p}} \left[|g^R_+(p,0)|^2 + |g^R_-(p,0)|^2\right] \tag{5}$$

and the retarded self-energy $\Sigma^R(\omega)$

$$\Sigma^R(\omega) = \frac{n_i v^2_{\rm imp}}{2\mathcal{V}} \sum\nolimits_{\mathbf{p}s} g^R_s(p,\omega)\theta(p_c - p) \tag{6}$$

where $g^R_{\pm}(p,\omega) = \left[\omega - E^{\pm}_p + E_F - \Sigma^R(\omega)\right]^{-1}$ is the Green's function of each chiral eigenstate. To simulate a finite Brillouin zone, in Eq. (6) we introduced an upper momentum cut-off, p_c. The latter is needed, in particular, to regularize the real part of the self-energy, $\mathrm{Re}[\Sigma^R(\omega)]$, which would otherwise diverge logarithmically at the band edge, see *e.g.* Ref.[3]. In these regards, we notice that, contrarily to what happens in standard half-filled systems where $\mathrm{Re}[\Sigma^R(\omega)]$ is approximately ω-independent and it can be absorbed in a redefinition of the Fermi level, in the low-doping regime investigated here $\mathrm{Re}[\Sigma^R(\omega)]$ acquires a non-trivial frequency dependence. We thus need to calculate self-consistently both $\mathrm{Re}[\Sigma^R(\omega)]$ and $\mathrm{Im}[\Sigma^R(\omega)]$. Such self-consistent solution identifies the elastic scattering rate,

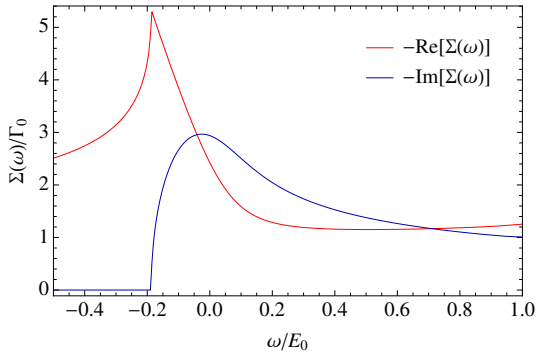

Figure 2: Structure of the real and imaginary part of the self-energy close to the lower band-edge for $E_0 = 40\Gamma_0$.

$\Gamma = -\mathrm{Im}[\Sigma^R(0)]$ and the renormalized density of states (DOS) $N(E) = -\frac{1}{\pi\mathcal{V}}\sum_{\mathbf{p}} \mathrm{Im}[G_R(\mathbf{p}, E)]$. The electronic density at $T = 0$ is given by $n = \int_{-\infty}^{E_F} N(E)dE$. At low doping the impurities lead to a smearing of the van-Hove singularity in the DOS, which reflects in the behavior of the scattering rate Γ, as described in the main text (Fig. 3a). In addition, the presence of impurities gives a shift, Δ_{edge}, of the lower band edge, so that the lower "effective" band edge where $n = 0$ is identified by $\tilde{E}_F = E_F + \Delta_{\mathrm{edge}} = 0$.In Fig.2, as an example we show the structure of the real and imaginary parts of the self-energy as functions of the frequency, for frequencies close to the lower band-edge, located at $E_F = -\Delta_{\mathrm{edge}}$. Notice that these two quantities are connected by Kramers-König relations.

DC CONDUCTIVITY FROM DIAGRAMMATIC PERTURBATION THEORY

Current response function

Within SCBA the static conductivity is given by Eq.(14) of the manuscript, that we report here for convenience

$$\sigma_{dc} = \frac{1}{2\pi}\left(P^{AR}_{xx} - \mathrm{Re}[P^{RR}_{xx}]\right). \tag{7}$$

The derivation of the above equation in the absence of spin-orbit coupling is standard textbooks material (see *e.g.* Refs. [1, 2]) and, since it does not change in the presence of spin-orbit coupling, we do not review it here.

By applying diagrammatic perturbation theory, one easily sees that the calculation of σ_{dc} implies the summation of all ladder diagrams shown in Fig. 3. This in turn corresponds to calculate the following current-

Figure 3: Ladder diagrams describing the conductivity within Born approximation. Solid lines and empty circles represent respectively $G(\mathbf{p}, i\varepsilon_l)$ and $j_x(\mathbf{p})$.

current response function in Matsubara frequencies:

$$P_{xx}(i\varepsilon_l, i\varepsilon_{l+n}) = \frac{1}{\mathcal{V}}\sum_{\mathbf{p}} \mathrm{Tr}\left\{G(\mathbf{p}, i\varepsilon_l) j_x(\mathbf{p}) G(\mathbf{p}, i\varepsilon_{l+n}) \cdot \right. \\ \left. \cdot \; J_x(\mathbf{p}, i\varepsilon_l, i\varepsilon_{l+n})\right\} \tag{8}$$

where j_x is the bare velocity operator introduced in the main text. We recall that in the helicity basis $j_x = e\mathrm{v}_x$ is represented by the following matrix:

$$\mathrm{v}_x = \frac{p_x}{m}\sigma_0 + \alpha\cos\theta_{\mathbf{p}}\sigma_z + \alpha\sin\theta_{\mathbf{p}}\sigma_y \tag{9}$$

where σ_0 is the 2×2 identity matrix and σ_i with $i = x, y, z$ are the Pauli matrices. The renormalized charge current $J_x(\mathbf{p}, i\varepsilon_l, i\varepsilon_m)$ satisfies the diagrammatic equation shown in Fig. 4 that in helicity space can be written as

$$J_x(\mathbf{k}, i\varepsilon_l, i\varepsilon_m) = j_x(\mathbf{k}) + \frac{n_i v_{\mathrm{imp}}^2}{\mathcal{V}}\sum_{\mathbf{p}} \Big[U^\dagger_{\mathbf{k}} U_{\mathbf{p}} G(\mathbf{p}, i\varepsilon_l)\cdot \\ J_x(\mathbf{p}, i\varepsilon_l, i\varepsilon_m) G(\mathbf{p}, i\varepsilon_m) U^\dagger_{\mathbf{p}} U_{\mathbf{k}}\Big] \tag{10}$$

Before coming to the solution of the above equation, we remark, that, as usual [2], the AR and RR response functions appearing in Eq.(7) correspond respectively to $P_{xx}(0 - i\delta, 0 + i\delta)$ and $P_{xx}(0 + i\delta, 0 + i\delta)$.

As it can be easily verified, by symmetry arguments one finds that J_x has the same matrix structure of the bare current (9), so that we can write:

$$J_x(\mathbf{k}, i\epsilon_l, i\epsilon_m) = \frac{p_x}{m}\sigma_0 + \tilde{\alpha}\cos\theta\sigma_z + \tilde{\alpha}\sin\theta\sigma_y \tag{11}$$

where $\tilde{\alpha}(i\varepsilon_l, i\varepsilon_m)$ satisfies the following self-consistent equation:

$$\tilde{\alpha} = \alpha + \frac{n_i v_{\mathrm{imp}}^2}{2}\sum_{\mathbf{p}} \mathrm{Tr}\big[U^\dagger_{\mathbf{p}}\sigma_y U_{\mathbf{p}} G(\mathbf{p}, i\varepsilon_l)\cdot \\ \cdot J_x(\mathbf{p}, i\varepsilon_l, i\varepsilon_m)\, G(\mathbf{p}, i\varepsilon_m)\big]. \tag{12}$$

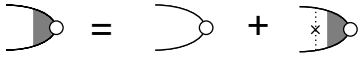

Figure 4: Diagrams describing renormalization of the charge current vertex.

Equation (12) can be solved explicitly to obtain to the following result for the renormalized anomalous vertex:

$$\tilde{\alpha}(i\varepsilon_l, i\varepsilon_m) = \frac{\alpha + \alpha_0(i\varepsilon_l, i\varepsilon_m)}{1 - A(i\varepsilon_l, i\varepsilon_m)} \tag{13}$$

where we introduced the quantities $A(i\varepsilon_l, i\varepsilon_m)$ and $\alpha_0(i\varepsilon_l, i\varepsilon_m)$ given by:

$$A(i\varepsilon_l, i\varepsilon_m) = \frac{n_i v_{\rm imp}^2}{4\mathcal{V}} \sum_{\mathbf{k}ss'} g_s(k, i\varepsilon_l) g_{s'}(k, i\varepsilon_m), \tag{14}$$

$$\alpha_0(i\varepsilon_l, i\varepsilon_m) = \frac{n_i v_{\rm imp}^2}{4\mathcal{V}} \sum_{\mathbf{k}s} \frac{k}{m} s\, g_s(k, i\varepsilon_l) g_s(k, i\varepsilon_m). \tag{15}$$

By replacing Eqs.(9) and (11) in Eq. (8), we arrive at the following expression for the correlation function $P_{xx}(i\varepsilon_l, i\varepsilon_m)$,

$$P_{xx}(i\varepsilon_l, i\varepsilon_m) = P_0(i\varepsilon_l, i\varepsilon_m) + \frac{m\left[(\alpha + \tilde{\alpha})\,\alpha_0 + \alpha\,\tilde{\alpha}\,A\right]}{\Gamma_0}. \tag{16}$$

On the r.h.s. of the above equation the frequency dependences of $\tilde{\alpha}$, α_0 and A, defined in Eqs.(12-15), is implied and we introduced the function $P_0(i\varepsilon_l, i\varepsilon_m)$,

$$P_0(i\varepsilon_l, i\varepsilon_m) = \frac{1}{2\mathcal{V}} \sum_{\mathbf{k}s} \frac{k^2}{m^2} g_s(k, i\varepsilon_l) g_s(k, i\varepsilon_m), \tag{17}$$

that in the absence of Rashba coupling yields the only non-vanishing contribution to the conductivity.

Once the analytical continuation is performed, Eq. (16) is equivalent to Eqs. (17-18) of the main text. This can be easily seen by inserting in Eq. (16) the explicit expression of A, α_0 and P_0, given above. One then finds that $P_{xx}^{LM}(0,0)$ can be recast as

$$P_{xx}^{LM} = \frac{1}{2\mathcal{V}} \sum_{\mathbf{p}} \Big\{ \Big[\Big(\frac{p^2}{m^2} + \alpha\tilde{\alpha}^{LM} \Big) \left(g_+^L g_+^M + g_-^L g_-^M \right) \Big] + \frac{p(\alpha + \tilde{\alpha})}{m} (g_+^L g_+^M - g_-^L g_-^M) + \alpha\tilde{\alpha}^{LM} (g_+^L g_-^M + g_-^L g_+^M) \Big\}.$$

From the above equation we see P_{xx} is the sum of an inter- and intra-band contribution, *i.e.*

$$P_{xx}^{LM} = \frac{1}{2\mathcal{V}} \sum_{\mathbf{p}} \Big\{ \sum_s \Big[\Big(\frac{p}{m} + s\alpha \Big) \Big(\frac{p}{m} + s\tilde{\alpha}^{LM} \Big) g_s^L g_s^M \Big] + + \alpha\tilde{\alpha}^{LM} \sum_{s \neq s'} g_s^L g_{s'}^M \Big\} \equiv P_{\rm intra}^{LM} + P_{\rm inter}^{LM}. \tag{18}$$

Eventually, since $(\frac{p}{m} + s\alpha)(\frac{p}{m} + s\tilde{\alpha}^{LM}) \equiv \vec{v}_{\mathbf{p}s} \cdot \vec{V}_{\mathbf{p}s}^{LM}$ we arrive at Eq. (18) of the main text.

Analytic approximations in the weak-disorder limit

Starting from the above results in this section we derive approximate analytical expressions for the renormalized vertex and the conductivity. We assume that we are in the weak-disorder limit (WDL), where we can (i) approximate the spectral functions with a delta, *i.e.* set $\mathcal{A}_\pm(p,0) = (\Gamma/\pi)|g_+^R(p,0)|^2 = \delta(E_{\mathbf{p}}^\pm - E_F)$; (ii) neglect the RR contributions.

The WDL approximation for $\tilde{\alpha}^{RA}$ can be derived starting from the analytic continuation of Eqs.(13-15). Performing the angular integral, A^{RA} and α_0^{RA} can be then cast as

$$\alpha_0^{RA}(0,0) = \frac{n_i v_{\rm imp}^2}{8m\pi} \int_0^\infty p^2 \left[|g_+^R(p,0)|^2 - |g_-^R(p,0)|^2 \right] dp, \tag{19}$$

$$A^{RA} = \frac{n_i v_{\rm imp}^2}{8\pi} \int_0^\infty p\, |g_+^R(p,0) + g_-^R(p,0)|^2 dp. \tag{20}$$

The latter equation can be simplified using the self-consistent self-energy equation to obtain:

$$A^{RA}(0,0) = 1/2 + \frac{n_i v_{\rm imp}^2}{4\pi} \int_0^\infty p\, {\rm Re}\big[g_+^R(p,0)\, g_-^R(p,0) \big]\, dp. \tag{21}$$

Approximating the spectral functions with a delta, as stated above, we immediately see that, except in a small density range around $E_F \simeq E_0$, we can neglect the second term and the r.h.s. of Eq.(21), since there is no overlap between the two-chiral bands and we obtain $A^{RA}(0,0) = 1/2$. In the same approximation, setting $\xi_p^\pm = E_p^\pm - E_F$, we can write $\alpha_0^{RA}(0,0)$ as follows

$$\alpha_0^{RA}(0,0) = \frac{n_i v_{\rm imp}^2}{8m\Gamma v_F} \int p^2 \left[\delta(\xi_p^+) - \delta(\xi_p^-) \right] dp\,, \tag{22}$$

which leads to

$$\alpha_0^{RA}(0,0) \simeq -\frac{\Gamma_0}{4m^2 v_F \Gamma} \cdot \begin{cases} (p_-^2 - p_+^2) & E_F > E_0 \\ (p_+^2 + p_-^2) & E_F < E_0 \end{cases} \tag{23}$$

with $\Gamma_0 = n_i v_{\rm imp}^2 m/2$. We recall that here, as in the main text, $p_\pm$ are the momenta on the inner and outer Fermi surface, so that their subscripts refer to the value of the transport helicity η, introduced in the main text in the context of Boltzmann transport. The sign change on the r.h.s. of Eq.(23) is thus due to the fact that for $E_F > E_0$ the two contributions come from the two chiral bands in Eq. (22), while for $E_F < E_0$ only the E_p^- band contributes, with a two-folded Fermi surface. Using the explicit expression $p_\pm$, *i.e.*

$$p_\pm^{E_F > E_0} = m v_F \mp p_0 \quad \text{and} \quad p_\pm^{E_F < E_0} = p_0 \mp m v_F. \tag{24}$$

with $v_F = \sqrt{2E_F/m}$, we eventually obtain:

$$\alpha_0^{RA}(0,0) \simeq \begin{cases} -\alpha & E_F > E_0 \\ -(E_0 + E_F)/p_0 & E_F < E_0 \end{cases} \tag{25}$$

Here we also used the WDL results for Γ, that can be derived from Eq. (5) in the WDL where $|g_\pm^R(p,0)|^2 =$

$(\pi/\Gamma)\delta(E^{\pm}_{\mathbf{p}s} - E_F)$ as

$$\Gamma^{WDL} = \begin{cases} \Gamma_0 & E_F > E_0 \\ \Gamma_0\sqrt{E_0/E_F} = \Gamma_0 p_0/(m v_F) & E_F < E_0 \end{cases} \tag{26}$$

that coincides with the Boltzmann result from Eq. (6) of the main text. By replacing the result (25) into Eq. (13), along with $A^{RA}(0,0) = 1/2$, we then obtain the estimate of $\tilde{\alpha}^{RA}$ quoted in the main text, i.e.

$$\alpha^{RA}(0,0) \simeq \begin{cases} 0 & E_F > E_0 \\ \alpha(1 - E_F/E_0) & E_F < E_0 \end{cases} \tag{27}$$

Let us now discuss the analytic approximation of the conductivity. As discussed in main text in the WDL we can put

$$\sigma_{dc} \simeq \frac{P^{RA}_{\rm intra}}{2\pi} \quad \text{(WDL)} \tag{28}$$

where the intraband term $P^{RA}_{\rm intra}$ coincides with the first term on the r.h.s. of Eq. (18). By using the result (27) for the anomalous vertex we can rewrite it as follows

$$P^{RA}_{\rm intra} = \frac{\left[p_+^2 + p_-^2\right]}{4m\Gamma} \qquad E_F > E_0 \tag{29}$$

$$P^{RA}_{\rm intra} = \frac{\left[p_+^2 - p_-^2 - m\tilde{\alpha}_{RA}(p_+ - p_-)\right]}{4m\Gamma} \qquad E_F < E_0. \tag{30}$$

Using the expression of $p_\pm$, Γ and $\tilde{\alpha}^{RA}$ derived above, along with the expressions for the particle density in the WDL, i.e.

$$n = \begin{cases} (m/\pi)(E_F + E_0) & E_F > E_0 \\ (p_0^2/\pi)\sqrt{E_F/E_0} = n_0\sqrt{E_F/E_0} & E_F < E_0 \end{cases} \tag{31}$$

Eq.s (29-30) lead to the final expression for the conductivity quoted in Eq.s (1)-(2) of the main text.

To conclude this section we would like to show that the inclusion of vertex corrections is crucial in both regimes. The "bare-bubble" conductivity σ_{bb}, corresponding to the first diagram in Fig. 3, is given by the term P_0 defined in Eq. (17). It can be directly computed from Eq. (18) by replacing $\tilde{\alpha}$ with α, so that the renormalized velocity $\vec{V}_{\mathbf{p}s}$ is replaced by the bare one $\vec{v}_{\mathbf{p}s}$. In the WDL we then easily obtain

$$P^{RA}_{bb} = \frac{\pi}{2\Gamma\mathcal{V}}\sum_{\mathbf{p},s}\vec{v}^{\,2}_{\mathbf{p}s}\delta(E^s_{\mathbf{p}} - E_F) = \frac{v_F}{4\Gamma}(p_+ + p_-) \tag{32}$$

Thus, using Eq.s (24), (26) and (31) into Eq. (28) one easily obtains that (restoring the charge e)

$$\sigma_{\rm bb} = \frac{e^2(n - n_0/2)}{2m\Gamma} = \sigma_{\rm Drude} - \sigma_{n_0}, \quad E_F > E_0 \tag{33}$$

$$\sigma_{\rm bb} = \frac{e^2 n}{4m\Gamma} = \frac{\sigma^{WDL}_{\rm Drude}}{2}, \qquad E_F > E_0 \tag{34}$$

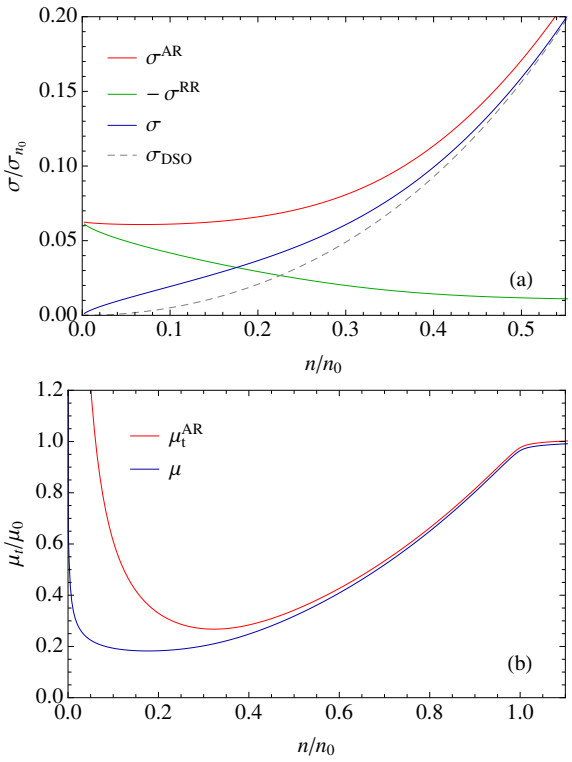

Figure 5: (Color online) (a) Different contributions to the conductivity as a function of the density for $E_0 = 40\Gamma_0$, $\Gamma_0 = 0.5meV$ and $m = 0.7m_e$. (b) Comparison of the total the AR mobilities. Parameters as in panel (a).

where, as in the main text, $\sigma_{\rm Drude} = e^2 n/(2\Gamma_0 m)$, $\sigma_{n_0} = e^2 n_0/(4\Gamma_0 m)$ and $\sigma^{WDL}_{\rm Drude} = e^2 n/(2\Gamma^{WDL} m)$. We therefore see that the bare-bubble result is inadequate at all densities and chemical potentials. This also shows that even recovering the Drude conductivity at $E_F > E_0$ is a non-trivial result, due to the crucial role of vertex corrections. Indeed, even in the regime $E_F \gtrsim E_0$, where E_F slightly exceeds the Rashba energy, so that two Fermi surfaces are clearly separated, any signature of the Rashba interaction disappears in the dc conductivity, which is given by the usual Drude formula. We also notice that vertex corrections tend to enhance the conductivity with respect to the bare-bubble result. As we shall discuss below, this is the result one usually expects within a Boltzmann picture, where backward and forward scattering processes contribute to the transport scattering time with different weights.

Relevance of the RR contribution

As we mentioned above, our analytical formulae are in principle valid only in the WDL, realized for $\Gamma \ll E_F$.

In this limit we did two approximations, we replaced the spectral functions of the chiral eigenstates with delta functions and neglected the RR part in Eq. (7). As the density decreases in the DSO regime these approximations are not valid anymore and the quantum result for σ_{dc} starts to deviates from the analytical WDL result σ_{DSO}. In this regime the RR current response function becomes as much relevant as the RA one, and it is important to include it in order to reproduce physical results for the dc conductivity. In Figure 5(a) we plot separately the contributions of the RR and AR response function to the conductivity, defined respectively as $\sigma^{AR} = P^{AR}(0,0)/(2\pi)$ and $\sigma^{RR} = -P^{RR}(0,0)/(2\pi)$. As one can see, for our parameters choice, at $n/n_0 \lesssim 0.5$ the RR contribution increases, while the AR one tends to saturates with decreasing density. Neglecting σ_{RR} thus leads to the unphysical result of a finite conductivity for zero density. The vanishing of the conductivity as $n \to 0$ is indeed guaranteed by the cancellation between σ^{AR} and σ^{RR}. This fact is also evident in Fig. 5(b) where along with the total mobility μ_t we also plot the "AR" mobility, μ_t^{AR} defined as $\mu_t^{AR} = \sigma^{AR}/n$. We se that the inclusion of the RR contribution significantly modifies the structure of the mobility curves around the minimum. Eventually we note that the cancellation between RR and AR terms turns out to improve the agreement between σ_{dc} and σ_{DSO} and it enlarges the range where the DSO formula can be used to describe transport.

DC CONDUCTIVITY WITHIN SEMICLASSICAL BOLTZMANN APPROACH

For a detailed discussion of Boltzmann equation we refer the reader to Ref.[4], here we only outline the most significant steps. Note that here we neglect the effect of the off-diagonal terms of the spin-density matrix in the helicity basis at a given $\mathbf{p}$ since as discussed in the main text they give only small corrections to dc charge transport (see e.g. Refs.[5, 6, 9, 10]). Let us call $\rho_{\mathbf{p},s}$ the distribution function for the quasiparticle eigenstates $\varepsilon_{\mathbf{p},s}$, where s denotes in general the band index, which coincides in our case with the chiral index. The time derivative of ρ is determined by the collision integral:

$$\frac{\partial \rho_{\mathbf{p},s}}{\partial t} = -\sum_{\mathbf{p}',s'} Q^{\mathbf{p}s}_{\mathbf{p}'s'}[\rho_{\mathbf{p},s} - \rho_{\mathbf{p}',s'}], \tag{35}$$

where $Q^{\mathbf{p}s}_{\mathbf{p}'s'}$ is the scattering kernel from the state $\varepsilon_{\mathbf{p},s}$ to the state $\varepsilon_{\mathbf{p}',s'}$. In the presence of an electric field $\mathbf{E}$ the l.h.s. of the above equations is given by:

$$\frac{\partial \rho_{\mathbf{p},s}}{\partial t} = -e\mathbf{E}\cdot\frac{\partial \rho_{\mathbf{p},s}}{\partial \mathbf{p}} \simeq -e\mathbf{E}\cdot\vec{v}_{\mathbf{p}s}\frac{\partial \rho^{\rm eq}_{\mathbf{p},s}}{\partial \varepsilon_{\mathbf{p}s}} \tag{36}$$

in the last passage we replaced ρ with its equilibrium value $\rho^{\rm eq}_{\mathbf{p},s} \equiv f(\varepsilon_{\mathbf{p}s})$, since we are interested in the linear response in $\mathbf{E}$. In the relaxation-time approximation we can express the time evolution of ρ via a transport scattering time τ^{tr}_{ps}, so that:

$$\frac{\partial \rho_{\mathbf{p},s}}{\partial t} = -\frac{\rho_{\mathbf{p},s} - \rho^{\rm eq}_{\mathbf{p},s}}{\tau^{tr}_{ps}} \tag{37}$$

By combining Eqs. (36)-(37) we then have:

$$\rho_{\mathbf{p},s} = \rho^{\rm eq}_{\mathbf{p},s} + e\mathbf{E}\cdot\vec{v}_{\mathbf{p}s}\tau^{tr}_{ps}\frac{\partial \rho^{\rm eq}_{\mathbf{p},s}}{\partial \varepsilon_{\mathbf{p}s}}. \tag{38}$$

For a field in the x direction the current can then be written as:

$$j_x = -e\sum_{\mathbf{p}s} v^x_{\mathbf{p}s}\rho_{\mathbf{p},s} = e^2E_x\sum_{\mathbf{p}s}(v^x_{\mathbf{p}s})^2\tau^{tr}_{ps}\left(-\frac{\partial \rho^{\rm eq}_{\mathbf{p},s}}{\partial \varepsilon_{\mathbf{p}s}}\right) \tag{39}$$

where we used the fact that there is no current in the equilibrium state. At $T = 0$ we can put $\partial\rho^{\rm eq}_{\mathbf{p},s}/\partial\varepsilon_{\mathbf{p}s} = \partial f(\varepsilon_{\mathbf{p}s})/\partial\varepsilon_{\mathbf{p}s} = \delta(\varepsilon_{\mathbf{p}s}-\mu)$. Thus, by identifying $\varepsilon_{\mathbf{p}s}-\mu \equiv E^s_{\mathbf{p}} - E_F$ we arrive at Eq.(8) of the main text

$$\sigma_{dc} = \frac{e^2}{2}\sum_{\mathbf{p}s}|\vec{v}_{\mathbf{p}s}|^2\tau^{tr}_{ps}\delta(E_{\mathbf{p}s} - E_F) \tag{40}$$

A set of equations for the transport scattering times can be derived by substitution of Eq. (36) into Eq. (35), once that one uses the *Ansatz* (37). By doing so, since the scattering kernel Q conserves the energy, and the equilibrium function $\rho^{\rm eq}_{\mathbf{p},s}$ does not depend on the chiral index but only on the energy, one is left with:

$$\begin{aligned}\vec{v}_{\mathbf{p}s} &= \sum_{\mathbf{p}'s'} Q^{\mathbf{p}s}_{\mathbf{p}'s'}\left[\tau^{tr}_{ps}\vec{v}_{\mathbf{p}s} - \tau^{tr}_{p's'}\vec{v}_{\mathbf{p}'s'}\right] = \\ &= \frac{\tau^{tr}_{ps}}{\tau\left(E^s_p\right)}\vec{v}_{\mathbf{p}s} - \sum_{\mathbf{p}'s'} Q^{\mathbf{p}s}_{\mathbf{p}'s'}\tau^{tr}_{p's'}\vec{v}_{\mathbf{p}'s'},\end{aligned} \tag{41}$$

where we introduced the quasiparticle scattering time

$$\frac{1}{\tau\left(E^s_p\right)} = \sum_{\mathbf{p}'s'} Q^{\mathbf{p}s}_{\mathbf{p}'s'}. \tag{42}$$

By using the fact that only the component of $\vec{v}_{\mathbf{p}'s'}$ in the direction of $\vec{v}_{\mathbf{p}s}$ survives after momentum integration, Eq. (41) finally reduces to Eq.(9) of the main text

$$\frac{\tau^{tr}_{ps}}{\tau\left(E^s_p\right)} = 1 + \sum_{\mathbf{p}'s'} Q^{\mathbf{p}s}_{\mathbf{p}'s'}\tau^{tr}_{p's'}\frac{\vec{v}_{\mathbf{p}'s'}\cdot\hat{v}_{\mathbf{p}s}}{|\vec{v}_{\mathbf{p}s}|} \tag{43}$$

Notice that Eq. (43) differs from the one proposed *e.g.* in Ref. [7], where the band-dependence of the transport scattering times on the r.h.s. of Eq. (43) has been overlooked, leading to decoupled equations for the τ^{tr}_{ps}. Here instead the set of coupled equations (43) is analogous to the self-consistence equations (10) introduced above for the renormalized current in the quantum language. This

analogy can be exploited further by the identification of the renormalized Boltzmann current as

$$\mathbf{J}^B_{\mathbf{p}s} = e\vec{v}_{\mathbf{p}s}\frac{\tau^{tr}_{ps}}{\tau\left(E^s_p\right)} \tag{44}$$

As already discussed in Ref. [8] for the case $E_F > E_0$, both the quantum and the Boltzmann approaches lead to the same renormalized currents. This result also holds in the DSO regime $E_F < E_0$, as one can see from the explicit solution for the τ's derived below.

Collision integral

By using Fermi Golden Rule, the scattering rate from the state $|\mathbf{p}, s\rangle$ to the state $|\mathbf{p}', s\rangle$ can be written as:

$$Q^{\mathbf{p}s}_{\mathbf{p}'s'} = \frac{2\pi}{V}|\langle \mathbf{p}s|V_{\rm imp}|\mathbf{p}'s'\rangle|^2\delta(\varepsilon_{\mathbf{p}s} - \varepsilon_{\mathbf{p}'s'}) \tag{45}$$

Using the explicit expression of the helicity eigenstates in plane waves, we can rewrite the above equation as follows:

$$Q^{\mathbf{p}s}_{\mathbf{p}'s'} = \frac{2\pi}{\mathcal{V}^2}\left|\int d\mathbf{r}\, e^{i(\mathbf{p}-\mathbf{p}')\mathbf{r}}V_{\rm imp}(\mathbf{r})W^{\mathbf{p}\mathbf{p}'}_{s's}\right|^2\delta(E^s_{\mathbf{p}} - E^{s'}_{\mathbf{p}'}) \tag{46}$$

Here the matrix $\hat{W}^{\mathbf{p}\mathbf{p}'} = U^\dagger_{\mathbf{p}'}U_{\mathbf{p}}$ comes from the scalar product of the helicity eigenvectors. Within our approximations, (self-averaging delta-correlated disorder and Born scattering), we can write $\langle V_{\rm imp}(\mathbf{r})V_{\rm imp}(\mathbf{r}')\rangle \simeq n_i v^2_{\rm imp}\delta(\mathbf{r}-\mathbf{r}')$ and we can recast the above equation as

$$Q^{\mathbf{p}s}_{\mathbf{p}'s'} = \frac{2\pi}{\mathcal{V}}n_i v^2_{\rm imp}\left|W^{\mathbf{p}\mathbf{p}'}_{s's}\right|^2\delta(E^s_{\mathbf{p}} - E^{s'}_{\mathbf{p}'}). \tag{47}$$

where

$$|W^{\mathbf{p}\mathbf{p}'}_{s's}|^2 = \frac{1+{\rm sign}(ss')\cos(\theta_{\mathbf{p}}-\theta_{\mathbf{p}'})}{2},$$

so that one recovers Eq. (5) of the main text.

Solution of Boltzmann equations

Using the explicit expression of $Q^{\mathbf{p}'s'}_{\mathbf{p}s}$, Eq.(43) reads

$$\frac{\tau^{tr}_{ps}}{\tau\left(E^s_p\right)} = 1 + \frac{\pi n_i v^2_{\rm imp}}{\mathcal{V}}\sum_{\mathbf{p}'s'}(1+ss'\hat{p}\cdot\hat{p}')\frac{\vec{v}_{\mathbf{p}'s'}\cdot\hat{v}_{\mathbf{p}s}}{|\vec{v}_{\mathbf{p}s}|}\cdot \cdot\delta(E^s_{\mathbf{p}} - E^{s'}_{\mathbf{p}'})\tau^{tr}_{p's'}. \tag{48}$$

Now recalling that $\vec{v}_{\mathbf{p}s} = v_{\mathbf{p}s}\hat{p}$ and that $|\vec{v}_{\mathbf{p}s}| = v_F = \sqrt{2mE_F}$ for $E^s_{\mathbf{p}} = E_F$, for states at the Fermi level we can rewrite the above equation as:

$$\frac{\tau^{tr}_{ps}}{\tau} = 1 + \frac{\pi n_i v^2_{\rm imp}}{\mathcal{V}}\sum_{\mathbf{p}'s'}(\hat{p}\cdot\hat{p}')^2\eta_{ps}\eta_{p's'}\delta(E_F - E^{s'}_{\mathbf{p}'})\tau^{tr}_{p's'} \tag{49}$$

where we set $\tau(E_F) = \tau$ and $\eta_{ps} = s(\hat{v}_{\mathbf{p}s}\cdot\hat{p}) = \pm 1$. Performing the angular integral $\mathbf{p}'$ and changing variables from p, s to E, η we eventually recover equation (10) of the main text:

$$\frac{\tau^{\rm tr}_\eta}{\tau} = 1 + \frac{1}{4\tau_0 m v_F}\sum_{\eta'}\eta\eta' p_{\eta'}\tau^{\rm tr}_{\eta'}, \tag{50}$$

where τ_0 denotes as usual the quasiparticle scattering time in the absence of spin-orbit, $\tau_0 = 1/(mn_i v^2_{\rm imp})$, $p_\eta = p_\eta(E_F)$ indicate the two Fermi momenta introduced in Eqs.(24) and we set $\tau^{\rm tr}_\eta = \tau^{\rm tr}_\eta(E_F)$.

To solve this equation it is useful to note that $\tau/\tau_0 = mv_F/\bar{p}_F$ where $\bar{p}_F = 1/2\sum_\eta p_\eta$ that allows us to recast Eq. 50 as:

$$\bar{\tau}^{\rm tr}_\eta = 1 + \frac{1}{4}\sum_{\eta'}\eta\eta'\bar{p}'_\eta\bar{\tau}^{\rm tr}_{\eta'}. \tag{51}$$

with $\bar{\tau}^{\rm tr}_\eta = \tau^{\rm tr}_\eta/\tau$ and $\bar{p}_\eta = p_\eta/\bar{p}_F$. As one can easily check, the solution of this equation reads: $\bar{\tau}^{\rm tr}_\eta = \bar{p}_\eta$ that coincides with Eq. (12) of the main text.

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