Robust H_∞ Controller Design Using Frequency-Domain Data via Convex Optimization

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SUMMARY

A new robust controller design method that satisfies the H_∞ criterion is developed for linear time-invariant single-input single-output (SISO) systems. A data-driven approach is implemented in order to avoid the unmodeled dynamics associated with parametric models. This data-driven method uses fixed order controllers to satisfy the H_∞ criterion in the frequency domain. The necessary and sufficient conditions for the existence of such controllers are presented by a set of convex constraints. These conditions are also extended to systems with frequency-domain uncertainties in polytopic form. It is shown that the upper bound on the weighted infinity norm of the sensitivity function converges monotonically to the optimal value, when the controller order increases. Additionally, the practical issues involved in computing fixed-order rational H_∞ controllers in discrete- or continuous-time by convex optimization techniques are addressed. Copyright © 2015 John Wiley & Sons, Ltd.

1. INTRODUCTION

Data-driven controller design is a very attractive research field within the control community (for a survey, see [1, 2, 3]). In this method, a controller is designed by using either the time-domain or frequency-domain data of a system rather than by using a parametric model of the plant, where the intermediate identification procedure or first principle modeling is not required. Thus, they are expected to perform better than the model based approaches because of the absence of unmodelled dynamics and parametric errors (see [4]).

The majority of the data-driven methods use time-domain data for computing a controller that minimizes a model reference criterion (or more generally, an H_2 control criterion). Model Reference Adaptive Control (MRAC) [5], adaptive switching control [6, 7], Iterative Feedback Tuning (IFT)

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[8], Virtual Reference Feedback Tuning (VRFT) [9] and Iterative Correlation-based Tuning (ICbT) [10] are among the well-known methods using time-domain data.

There are only a few methods that use the frequency-domain data to compute robust controllers to meet some constraints on the stability margins or H_∞ norm of the sensitivity functions. The frequency-domain methods assume that the frequency response of a system is available from an experiment (i.e., in applications where the parametric model is unknown or unavailable). These methods are closely related to time-domain data-driven methods because there is a linear relationship between the time-domain and frequency-domain data (through the Fourier transform). In this paper, the data-driven framework refers to a design method which uses the frequency response of a system in order to compute a robust controller.

A robust fixed-order controller design method using linear programming is proposed in [11]. In this method, the constraints on the gain margin, phase margin and crossover frequency are approximated with linear constraints by using linearly parameterized controllers. The frequency response data are used in [12] to compute the frequency response of a controller that achieves a desired closed-loop pole location. In [13], a complete set of PID controllers is computed that guarantee a gain margin, phase margin and H_∞ performance specification using frequency-domain data. This method is extended to design fixed-order linearly parameterized controllers in [14]. A data-driven synthesis methodology with a fixed structure controller that ensures H_∞ performance is presented in [15]. This method, however, uses the Q parameterization in the frequency domain and solves a non-convex optimization problem to find a local optimum. Another frequency-domain approach is presented in [16] to design reduced order controllers with a guaranteed bounded error on the difference between the desired and achieved magnitude of closed-loop sensitivity functions; this approach also uses a non-convex optimization method. A convex optimization method is used in [17] to compute robust H_∞ controllers for SISO systems represented by their frequency response. An interpretation of this algorithm based on convex-concave optimization for tuning PID controllers is given in [18]. This approach is extended to compute decoupling controllers for multi-input-multi-output (MIMO) systems in [19]. Based on this method, a public domain toolbox for MATLAB is developed which is available in [20].

In this paper, the necessary and sufficient conditions for the existence of robust controllers that guarantee bounded infinity norm on the sensitivity functions are developed. It is shown that these conditions depend only on the frequency response of the plant model and can be represented by convex constraints with respect to the controller parameters. By using fixed-order rational controllers, a convex optimization problem is formulated which produces a solution that ensures H_∞ performance. The results are extended to systems with frequency-domain polytopic uncertainties that are caused by measurement noise or multimodel incertitude. The developed conditions are necessary and sufficient for stable systems and only sufficient for unstable systems with polytopic uncertainties. The main contributions with respect to the work in [17] are: (1) The existence of a multiplier that convexifies the problem is proved (no linearization around a given desired open loop transfer function is performed). (2) Rational controllers are designed instead of linearly parameterized controllers. (3) The convergence of the method to the global optimal solution is proved.

This paper is organized as follows: the system structure and general preliminaries are discussed in Section 2. The problem formulation and main results with regards to satisfying the H_∞ criterion

are addressed in Section 3. The implementation issues associated with the optimization problem are considered in Section 4. The effectiveness of the proposed design scheme is demonstrated with simulation examples and experimental results in Section 5. Finally, the concluding remarks are given in Section 6.

2. PRELIMINARIES

Let $\mathbf{RH}_\infty$ represent the family of all stable, proper, real-rational transfer functions. It is imperative to note that $\mathbf{RH}_\infty$ is closed under multiplication and addition (i.e., if $A(s), B(s) \in \mathbf{RH}_\infty$, then $\{A(s) + B(s), A(s)B(s)\} \in \mathbf{RH}_\infty$). Suppose that a SISO unity feedback control system structure is used where the plant is represented as $G(s) = N(s)M^{-1}(s)$ such that $\{N(s), M(s)\} \in \mathbf{RH}_\infty$. As asserted in [21] and [22], if $N(s)$ and $M(s)$ are coprime, then $G(s) = N(s)M^{-1}(s)$ is called a *coprime factorization* of $G(s)$ over $\mathbf{RH}_\infty$.

The frequency response of such a factorized SISO system is given by:

$$G(j\omega) = N(j\omega)M^{-1}(j\omega), \quad \omega \in \Omega \tag{1}$$

where $\Omega := \mathbf{R} \cup \{\infty\}$ and $N(j\omega), M(j\omega)$ are the frequency responses of bounded analytic functions in the right half plane. It is also assumed that $G(j\infty) = 0$, which implies that $N(j\infty) = 0$ and $M(j\infty) \neq 0$. This representation includes time-delayed systems as well as unstable plants with unbounded infinity norms.

Finding the coprime factors of a given plant is a standard problem in control when the model of the plant is available ([23]). In a data-driven setting, for stable systems, a trivial choice is $N(j\omega) = G(j\omega)$ and $M(j\omega) = 1$. For unstable systems, a stabilizing controller is needed in order to properly formulate $N(j\omega)$ and $M(j\omega)$. In this case, $N(j\omega)$ is the frequency response function between the reference signal and the measured output, while $M(j\omega)$ is the frequency response function between the reference signal and the plant input. Given these definitions, it is evident that $N(j\omega)M^{-1}(j\omega)$ represents the frequency response of the plant model. For notation purposes, the dependence on $j\omega$ will be omitted and will be reiterated when deemed necessary.

Consider the controller structure, $K = XY^{-1}$, where X and Y are stable transfer functions with bounded infinity norm ($X, Y \in \mathbf{RH}_\infty$). These transfer functions may be discrete- or continuous-time; however, for presentation purposes, the continuous-time transfer functions will be considered. Note that the methods proposed in this work can also be used for computing discrete-time controllers. This will be shown through a simulation example in Section 5.

The objective is to design a controller that meets some constraints on the infinity norm of the weighted sensitivity functions. Some of the sensitivity functions associated with the unity feedback control system structure are given by:

$$S = (1 + GK)^{-1} = MY(NX + MY)^{-1} \tag{2}$$

$$T = GK(1 + GK)^{-1} = NX(NX + MY)^{-1} \tag{3}$$

$$U = K(1 + GK)^{-1} = MX(NX + MY)^{-1} \tag{4}$$

$$V = G(1 + GK)^{-1} = NY(NX + MY)^{-1} \tag{5}$$

An upper bound on the infinity-norm of $H(j\omega) = W_1(j\omega)S(j\omega)$ will be considered, where $W_1(j\omega)$ is the frequency function of a stable system with bounded infinity norm. Therefore, the control objective is to find a stabilizing controller K such that

$$\sup_{\omega \in \Omega} |H(j\omega)| < \gamma \qquad (6)$$

This condition can easily be extended to the other weighted sensitivity functions asserted in equations (2-5).

3. CONVEX PARAMETERIZATION OF ROBUST CONTROLLERS

The main objective is to find a set of convex constraints (with respect to X and Y) to satisfy the constraint in (6). The following lemma will be used in the proof of the main results of this paper:

Lemma 1

Suppose that $H(j\omega) = W_1 MY(NX + MY)^{-1}$ is the frequency response of a bounded analytic function in the right half plane. Then, (6) is met if and only if there exists a stable proper rational transfer function $F(s)$ that satisfies

$$Re\{(NX + MY - \gamma^{-1}|W_1MY|)F(j\omega)\} > 0, \qquad \forall \omega \in \Omega$$

Proof : The basic idea is similar to that of the proof of Theorem 1 in [24]. From Fig. 1, it is clear that (6) is satisfied if and only if the disk of radius $\gamma^{-1}|W_1MY|$ centered at $NX + MY$ does not include the origin for all $\omega \in \Omega$, i.e. $|NX + MY| > \gamma^{-1}|W_1MY|$. This is equivalent to the existence of a line passing through origin that does not intersect the disk. Therefore, at every given frequency, ω, there exists a complex number $f(j\omega)$ that can rotate the disk such that it lays inside the right hand side of the imaginary axis. Hence, we have

$$Re\{(NX + MY - \gamma^{-1}|W_1MY|)f(j\omega)\} > 0 \qquad (7)$$

for all $\omega \in \Omega$. In [24], it is shown that, $f(j\omega)$ can be approximated arbitrarily well by the frequency response of a rational stable transfer function $F(s)$ if and only if

$$Z = (NX + MY - \gamma_0^{-1}|W_1MY|)^{-1} \qquad (8)$$

is analytic in the right half plane for all $\gamma_0 > \gamma$. However, $(NX + MY)^{-1}$ is stable because of the stability of H. On the other hand, by decreasing γ_0 from infinity to γ, the poles of Z move continuously with γ_0. Therefore, Z is not analytic in the right half plane if and only if $Z^{-1}(j\omega) = 0$ for a given frequency, which is not the case because the disk shown in Fig. 1 does not include the origin. ∎
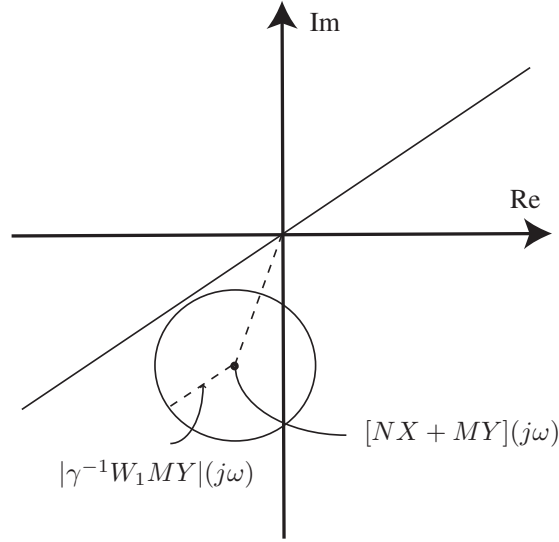
Figure 1. Graphical illustration of nominal performance

3.1. Nominal and robust performance

The set of all controllers that meet the nominal performance condition defined by the weighted norm of sensitivity functions is asserted in the following theorem.

Theorem 1

Given the frequency response model G in (1) and the frequency response of a bounded weighting filter W_1, the following statements are equivalent:

(**a**) There exists a controller K that stabilizes G and

$$\sup_{\omega \in \Omega} |W_1(1+GK)^{-1}| < \gamma \tag{9}$$

(**b**) There exist $X, Y \in \mathbf{RH}_\infty$ with $K = XY^{-1}$, such that

$$\gamma^{-1}|W_1MY|(j\omega) < Re\{[NX+MY](j\omega)\}, \quad \forall \omega \in \Omega \tag{10}$$

Proof : (**b** $\Rightarrow$ **a**) Since $NX + MY$ is analytic in the right half plane and its real part is positive for all $\omega \in \Omega$, it will not encircle the origin when ω travels along the Nyquist contour, so its inverse is stable and therefore K stabilizes G. On the other hand, we have

$$|[NX+MY](j\omega)| \geq Re\{[NX+MY](j\omega)\}, \quad \forall \omega \in \Omega$$

which leads to

$$|W_1MY|(j\omega) < \gamma|NX+MY|(j\omega) \quad \forall \omega \in \Omega$$

and consequently to (9) in Statement (**a**).

(**a** $\Rightarrow$ **b**) Assume that $K = X_0Y_0^{-1}$ satisfies Statement (**a**) but not Statement (**b**). Then, according to Lemma 1 there exists a stable proper rational transfer function $F(s)$, such that

$$Re\{(NX_0 + MY_0 - \gamma^{-1}|W_1MY_0|)F(j\omega)\} > 0 \quad \forall \omega \in \Omega$$

Therefore, there exist $X = X_0 F$ and $Y = Y_0 F$ with $K = XY^{-1} = X_0 Y_0^{-1}$, such that Statement (**b**) holds. ∎

The necessary and sufficient conditions for robust performance of closed-loop systems with disk-type frequency-domain uncertainty can be developed in a similar manner. Suppose that the frequency response of the plant model with some disk additive uncertainty is given as :

$$\tilde{N}(j\omega) = N(j\omega) + |W_n(j\omega)|\delta_n e^{j\theta_n}$$
$$\tilde{M}(j\omega) = M(j\omega) + |W_m(j\omega)|\delta_m e^{j\theta_m}$$

(11)

where $|\delta_n| \leq 1$, $|\delta_m| \leq 1$; $\theta_n, \theta_m \in [0\,,\,2\pi]$; W_n and W_m are computed from the covariance of the estimates for a given confidence interval (see [25]). These types of models can be easily obtained by spectral analysis of measured data.

If we consider the nominal performance as defined in (6), the robust performance condition given by [22]:

$$|W_1 \tilde{M} Y| < \gamma |\tilde{M} Y + \tilde{N} X| \quad \forall \omega \in \Omega$$

(12)

$$|W_1 M + W_1|W_m|\delta_m e^{j\theta_m} Y| < \gamma|\tilde{M} Y + \tilde{N} X + |W_m|\delta_m e^{j\theta_m} + |W_n|\delta_n e^{j\theta_n}| \quad \forall \omega \in \Omega$$

(13)

becomes:

$$\sup_{\omega \in \Omega} \frac{|W_1 M Y| + |W_1 W_m Y|}{|NX + MY| - |W_n X| - |W_m Y|} < \gamma$$

(14)

Equivalently, at any $\omega \in \Omega$, a disk of radius

$$r(\omega) = \gamma^{-1}|W_1 M Y| + \gamma^{-1}|W_1 W_m Y| + |W_n X| + |W_m Y|$$

(15)

centered at $[NX + MY](j\omega)$ should not include the origin. This can be presented as a set of convex constraints with respect to X and Y as follows:

$$r(\omega) < Re\{[NX + MY](j\omega)\}, \quad \forall \omega \in \Omega$$

(16)

3.2. Multimodel and frequency-domain polytopic uncertainty

Let the frequency-domain polytopic uncertainty be defined as:

$$G(\lambda, j\omega) = N(\lambda, j\omega)M^{-1}(\lambda, j\omega)$$

(17)

where

$$N(\lambda, j\omega) = \sum_{i=1}^{m} \lambda_i N_i(j\omega) \qquad ; \qquad M(\lambda, j\omega) = \sum_{i=1}^{m} \lambda_i M_i(j\omega)$$

$\lambda_i \geq 0$, $\sum_{i=1}^{q} \lambda_i = 1$ and λ is the convex hull of λ_i's. This uncertainty should not be confound with the parametric polytopic uncertainty, which is defined in the parameter space in model-based robust control approaches.

It is clear that the following constraints

$$\gamma^{-1}|W_1 M_i Y| < Re\{[N_i X + M_i Y](j\omega)\}, \quad \forall \omega \in \Omega \quad , \quad \text{for} \quad i = 1, \ldots, m$$

(18)

are necessary and sufficient conditions for robust performance of the closed-loop system with multimodel uncertainty. However, it can be shown that there are only sufficient for frequency-domain polytopic uncertainty. It suffices to compute the convex combination of the constraints in (18) as

$$\gamma^{-1} \sum_{i=1}^{m} \lambda_i |W_1 M_i Y| < Re \left\{ \sum_{i=1}^{m} \lambda_i [N_i X + M_i Y] \right\}, \forall \omega \in \Omega$$

and for $i = 1, \ldots, m$. Noting that:

$$\left| \sum_{i=1}^{m} \lambda_i W_1 M_i Y \right| \leq \sum_{i=1}^{m} \lambda_i |W_1 M_i Y| \tag{19}$$

we obtain:

$$\gamma^{-1} |W_1 M(\lambda) Y| < Re\{N(\lambda) X + M(\lambda) Y\}(j\omega)\}, \quad \forall \omega \in \Omega$$

Then, according to Theorem 1, the upper bound for the weighted sensitivity function is satisfied for all λ.

Although the constraints for polytopic uncertainty are only sufficient, the necessary and sufficient conditions can be developed for some class of models and some sensitivity functions. The following theorem represents the results for systems that have polytopic uncertainty only in N.

Theorem 2

Consider the model given in (17) with $N(\lambda, j\omega) = \sum_{i=1}^{m} \lambda_i N_i(j\omega)$ and $M(\lambda, j\omega) = M(j\omega)$. Then, the following statements are equivalent:

(a) Controller K stabilizes $G(\lambda) = N(\lambda) M^{-1}$ and

$$\sup_{\omega \in \Omega} |W_1 (1 + G(\lambda) K)^{-1}| < \gamma$$

(b) There exist $X, Y \in \mathbf{RH}_\infty$ such that $K = XY^{-1}$, and

$$\gamma^{-1} |W_1 MY(j\omega)| < Re\{[N_i X + MY](j\omega)\}, \quad \forall \omega \in \Omega, \quad \text{for} \quad i = 1, \ldots, m \tag{20}$$

Proof : (**b** $\Rightarrow$ **a**) The convex combination of the constraints in (20) leads to

$$\gamma^{-1} |W_1 MY(j\omega)| < Re\{[N(\lambda) X + MY](j\omega)\} \tag{21}$$

for all $\omega \in \Omega$ and for all λ. So Statement (**a**) can be concluded using the result of Theorem 1.

(**a** $\Rightarrow$ **b**) Suppose that (**a**) is satisfied with the controller $K = X_0 Y_0^{-1}$. Therefore, all disks of the same radius, $\gamma^{-1} |W_1 M Y_0|$, centered inside a polygon with m vertices, $N_i X_0 + MY_0$, do not include the origin. This represents a convex set, which is the convex hull of the m disks. Therefore, there exists a line that passes through the origin and does not intersect this convex set. As a result, similar to the proof of Lemma 1, there exists a stable transfer function $F(s)$ such that:

$$Re\{[N_i X_0 + MY_0 - \gamma^{-1} |W_1 MY_0|] F(j\omega)\} > 0, \quad \forall \omega \in \Omega, \quad \text{for} \quad i = 1, \ldots, m$$

Hence $X = X_0 F$ and $Y = Y_0 F$ satisfies the inequalities in Statement (**b**). ∎
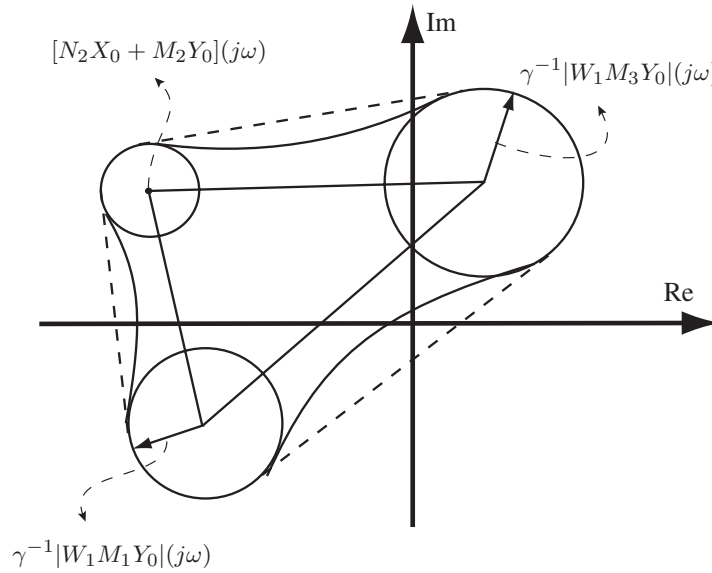
Figure 2. Illustration of the constraints for polytopic uncertainty with 3 vertices

Remark: Theorem 2 considers only the plant model with polytopic uncertainty in N. This represents the class of stable systems that may have some fixed poles on the imaginary axis. The theorem also holds for unstable systems with no uncertainty in M. A polytopic uncertainty in M will change the radius of the disks centered at $N_i X_0 + M_i Y_0$, such that the whole set of the disks will not be necessarily convex. Figure 2 shows a case in which the set of the disks is not convex but is inside the convex hull of the disks. This is always true because of the constraint in (19). In the special case shown in Fig. 2, we observe that the set of disks does not include the origin but the convex hull does. Similarly, Statement (**b**) in Theorem 2 is a sufficient condition for satisfying an upper bound on the weighted sensitivity functions T or V, since the radius of the disks, at each frequency, will not be constant for the whole polygon. However, it will be necessary and sufficient for an upper bound on the weighted sensitivity function U in (4).

3.3. Parametric uncertainty

The approach proposed in this paper requires only the frequency response of a model to design a robust controller. However, if a parametric model is available, the approach can be still used by computing the frequency response of the model. It is well known that the interval deterministic parametric uncertainty cannot be converted to the ellipsoid uncertainty in the frequency-domain. In a data-driven framework, for an identified parametric model using noisy data, the parametric uncertainties have stochastic bounds and can be transferred to the frequency-domain in a stochastic sense.

In a data-driven approach, a parametric model of the plant is identified together with its parametric uncertainty using the classical prediction error methods (see [25]). The parametric uncertainty is characterized by an ellipsoid in the parameter space and can be computed using the asymptotic covariance matrix of the parameters for a given probability level. Thanks to the invariance property

of the Maximum Likelihood Estimators, any function of the estimated parameters will converge to a normal distribution with a covariance matrix that can be computed based on the derivative of the function with respect to the parameters and its covariance matrix. In the complex plane, this parametric uncertainty is represented by an ellipse at each frequency that is well approximated with an m-side polygon ($m > 2$) of minimum area that circumscribes each ellipse. In this manner, the parametric uncertainty can be taken into account using the frequency-domain polytopic uncertainty with almost no conservatism.

Suppose that a stable parametric model $\hat{G}(\theta)$ is identified from a set of noisy data and the covariance matrix of the parameters, $\text{cov}(\theta)$, is computed. Then, the frequency response of the identified model can be computed and its real and imaginary parts put in a vector as:

$$\hat{G}_v(\omega) = \begin{bmatrix} R_e\{\hat{G}(j\omega)\} & I_m\{\hat{G}(j\omega)\} \end{bmatrix}^T \tag{22}$$

This vector has a joint normal distribution with the covariance $C_G(\omega)$ that can be estimated from $\text{cov}(\theta)$ using a linear approximation as follows:

$$C_G(\omega) = \left(\frac{\partial \hat{G}_v(\omega)}{\partial \theta} \right) \text{cov}(\theta) \left(\frac{\partial \hat{G}_v(\omega)}{\partial \theta} \right)^T \tag{23}$$

Note that $\theta \in \mathbb{R}^n$, $\text{cov}(\theta) \in \mathbb{R}^{n \times n}$ and $C_G(\omega) \in \mathbb{R}^{2 \times 2}$. Then, the true frequency response will belong to the following ellipse in the complex plane with a probability of $1 - \alpha$:

$$\begin{bmatrix} x - R_e\{\hat{G}\} \\ y - I_m\{\hat{G}\} \end{bmatrix}^T C_G^{-1}(\omega) \begin{bmatrix} x - R_e\{\hat{G}\} \\ y - I_m\{\hat{G}\} \end{bmatrix} \leq \mathcal{X}_2^2(\alpha) \tag{24}$$

where $\mathcal{X}_2^2$ is the chi-square distribution with two degrees of freedom. For a confidence interval of 0.95 ($\alpha = 0.05$), we have $\mathcal{X}_2^2(0.05) = 5.99$. Since the uncertainty set is an ellipse, the disk uncertainty in (11) cannot be used to model it without conservatism. However, the ellipse can be represented by an n_q-sided polygon with minimum area that circumscribes it, and is approximated by frequency-domain polytopic uncertainty as:

$$G(\lambda) = \sum_{k=1}^{n_q} \lambda_i \hat{G}_k(j\omega) \tag{25}$$

where

$$\hat{G}_k(j\omega) = \hat{G}(j\omega) + \begin{bmatrix} 1 & j \end{bmatrix} \sqrt{5.99 C_G(\omega)} \begin{bmatrix} \frac{\cos(2\pi k/n_w)}{\cos(\pi/n_q)} \\ \frac{\sin(2\pi k/n_w)}{\cos(\pi/n_q)} \end{bmatrix} \tag{26}$$

The last vector in (26) represents the k-th coordinate of a vertex of a polygon circumscribing the unit circle while the matrix $\sqrt{5.99 C_G(\omega)}$ designates the size and direction of the uncertainty (for 0.95 probability).

4. FIXED-ORDER CONTROLLER DESIGN

The minimization of the H_∞ norm becomes an optimization problem that can be solved as follows:

$$\min_{X,Y} \quad \gamma$$

subject to

(27)

$$|W_1 MY(j\omega)| < \gamma Re\{[NX + MY](j\omega)\}, \forall \omega \in \Omega$$

In general, this optimization problem is not convex. However, by linearly parameterizing the controllers X and Y, it becomes a quasi-convex optimization problem and can be solved by using a bisection algorithm to obtain the optimal solution for γ. Within a given tolerance, the bisection algorithm ensures the convergence to the global optimum solution. There are several practical and implementation issues in this optimization problem that will be addressed in this section.

4.1. Controller parameterization

A linear parameterization of X and Y keeps the constraints in (27) convex. As a result, $X(s)$ and $Y(s)$ are linearly parameterized as $X(s) = \rho_x^T \phi(s)$ and $Y(s) = \rho_y^T \phi(s)$, where $\rho_x^T = [\rho_{x_0}, \ldots, \rho_{x_n}]$ and $\rho_y^T = [1, \rho_{y_1}, \ldots, \rho_{y_n}]$ are the vectors of the controller parameters and

$$\phi^T(s) = [1, \phi_1(s) \cdots, \phi_n(s)] \tag{28}$$

is a vector of stable orthogonal basis functions. A simple choice is the Laguerre basis functions given by

$$\phi_i(s) = \frac{\sqrt{2\xi}(s - \xi)^{i-1}}{(s + \xi)^i}$$

with $\xi > 0$ and $i = 1, \cdots, n$. These basis functions have only one parameter to be selected (ξ). The effect of the Laguerre parameter on the control performance for low order controllers is illustrated in a simulation example in the next section.

4.2. Convergence to the optimal solution

In this sub-section, we will show that the optimal solution (γ_n) to the optimization problem in (27) (for a linear parameterization of X and Y by the orthogonal basis functions of order n) will converge to the least upper bound of the infinity norm of the weighted sensitivity function when n goes to infinity. The following Lemma is required to prove this convergence:

Lemma 2
([26]) Let $X_n^*(s)$ be the projection of $X_o(s) \in \mathbf{RH}_\infty$ into the subspace spanned by the orthogonal basis functions $\phi(s)$ in (28). Then

$$\lim_{n \to \infty} \|X_o - X_n^*\|_\infty = 0$$

Theorem 3

Suppose that the controller $K_o(s)$ achieves the optimal H_∞ performance for the plant model $G = NM^{-1}$ such that

$$\gamma_o = \inf_K \sup_\omega |W_1(1 + GK)| = \sup_\omega |W_1(1 + GK_o)|$$

Suppose also that γ_n is the optimal solution of the convex optimization problem in (27) when X and Y are parameterized by an n dimensional orthogonal basis function. Then γ_n converges monotonically from above to γ_o when $n \to \infty$.

Proof : According to Theorem 1, there exist $X_o(s), Y_o(s) \in \mathbf{RH}_\infty$ such that $K_o(s) = X_o(s)Y_o^{-1}(s)$ and

$$\gamma_o = \sup_{\omega \in \Omega} \left| \frac{W_1 M Y_o}{N X_o + M Y_o} \right| \tag{29}$$

Take X_n^* and Y_n^* as the projections of X_o and Y_o into the subspace spanned by an n dimensional orthogonal basis functions and define

$$\gamma_n^* = \sup_{\omega \in \Omega} \left| \frac{W_1 M Y_n^*}{N X_n^* + M Y_n^*} \right| \tag{30}$$

We assume that γ_n^* is bounded, i.e., $N X_n^* + M Y_n^*$ has no zero on the imaginary axis. This can be proved if n is large enough using contradiction and based on the fact that $\|N X_o + M Y_o\|_\infty > \epsilon > 0$. Assume that $j\omega^*$ is a zero of $N X_n^* + M Y_n^*$. Therefore, at $\omega = \omega^*$, one has:

$$|N X_o + M Y_o| = |N(X_o - X_n^*) + M(Y_o - Y_n^*)| > \epsilon \tag{31}$$

However, $|N(X_o - X_n^*) + M(Y_o - Y_n^*)|$ can be made arbitrarily small by increasing n, which shows that for large but finite n, $N X_n^* + M Y_n^*$ will not have a zero on the imaginary axis.

Now, let us compute

$$|\gamma_n^* - \gamma_o| \leq \sup_{\omega \in \Omega} \left| \frac{W_1 M Y_n^*}{N X_n^* + M Y_n^*} - \frac{W_1 M Y_o}{N X_o + M Y_o} \right| \tag{32}$$

$$\leq \sup_{\omega \in \Omega} \left| \frac{W_1 M N [X_n^*(Y_o - Y_n^*) - Y_n^*(X_o - X_n^*)]}{(N X_n^* + M Y_n^*)(N X_o + M Y_o)} \right|$$

Moreover, according to Lemma 2:

$$\lim_{n \to \infty} \|X_o - X_n^*\|_\infty = 0, \quad \lim_{n \to \infty} \|Y_o - Y_n^*\|_\infty = 0$$

Therefore, since all frequency functions in (32) are bounded and the denominator has no zero on the imaginary axis

$$\lim_{n \to \infty} |\gamma_n^* - \gamma_o| = 0 \tag{33}$$

On the other hand, γ_n is always less than or equal to γ_n^* and greater than the optimal solution γ_o. Thus γ_n converges from above to γ_o and this convergence is monotonic because the basis functions of order n are a subset of those of order $n + 1$, which ensures that $\gamma_{n+1} \leq \gamma_n$. ∎

4.3. Finite number of constraints

The constraints in (27) should be satisfied for all $\omega \in \Omega$, which is an infinite set. This problem is known as semi-infinite programming (SIP) problem and there exist different methods to solve it. A very simple and practical solution to this problem is to choose a finite set of frequencies $\Omega_p = \{\omega_1, \omega_2, \cdots, \omega_p\}$ and satisfy the constraints for this set. In this manner, the optimization problem is converted to a semi-definite programming (SDP) problem which can be solved efficiently with solvers that are readily available.

Another solution is to use a randomized approach where the constraints are satisfied for a finite set of randomly chosen frequencies. In this approach, a bound on the violation probability of the constraints can be derived and approaches zero when the number of samples goes to infinity (see [27] and [28]). It should be mentioned that in a data-driven framework, the frequency domain uncertainties are given by some stochastic bounds. Therefore, even if the constraints are met for all ω, the stability, robustness and performance are guaranteed within a probability level. As a result, the use of randomized methods to solve the robust optimization problem in (27) is fully compatible with the uncertainty description of the frequency-domain model of the proposed approach.

4.4. Solution by linear programming

The convex constraints in (27) are equivalent to the following linear constraints:

$$Re\{[NX + MY](j\omega) - \gamma^{-1}e^{j\theta}W_1MY(j\omega)\} > 0, \tag{34}$$

$\forall \omega \in \Omega$ and $\forall \theta \in [0, 2\pi[$. In fact, $\gamma^{-1}e^{j\theta}W_1MY(j\omega)$ represents the circle in Fig. 1. Note that $e^{j\theta}$ can be very well approximated by a polygon of $q > 2$ vertices with least area that circumscribes it. By gridding ω and bounding the circle $e^{j\theta}$, a finite set of linear constraints can be obtained as:

$$Re\left\{[NX + MY](j\omega_i) - \gamma^{-1}\frac{e^{j2\pi k/q}}{\cos(\pi/q)}W_1MY(j\omega_i)\right\} > 0 \tag{35}$$

for $i = 1, \ldots, p$ and $k = 1, \ldots, q$. Therefore, the convex constraints in (27) can be replaced by $p \times q$ linear constraints.

5. CASE STUDIES

5.1. Case 1: Multimodel uncertainty

In this example, a simulation is carried out to compare the traditional μ-synthesis method and the proposed approach for a set of unstable models. The controlled plants are taken from an example in the robust control toolbox of MATLAB. The nominal plant model is a first-order unstable system

$G_0(s) = 2(s-2)^{-1}$, and the family of perturbed plants are variations of $G_0(s)$ as follows,

$$G_1(s) = G_0(s) \cdot (0.06s + 1)^{-1}$$
$$G_2(s) = G_0(s) \cdot e^{-0.04s}$$
$$G_3(s) = G_0(s) \cdot 50^2(s^2 + 10s + 50^2)^{-1}$$
$$G_5(s) = 2.4(s - 2.2)^{-1}$$
$$G_4(s) = G_0(s) \cdot 70^2(s^2 + 28s + 70^2)^{-1}$$
$$G_6(s) = 1.6(s - 1.8)^{-1}$$

Remark: It is imperative to note that these models are simply used to obtain the frequency response functions of the perturbed plants. The actual controller synthesis does not rely on these parametric models.

Compared with the nominal plant, G_1 has an extra lag, G_2 has an additional time delay, G_3 and G_4 have high frequency resonance mode, G_5 and G_6 have pole and gain migrations. The control task is to design a linear controller to simultaneously stabilize this family of unstable plants and minimize the infinity norm of the weighted sensitivity functions, i.e.:

$$\min \gamma$$

$$\|W_1 S_i\|_\infty < \gamma \quad , \quad \|W_2 T_i\|_\infty < \gamma \quad , \quad \text{for} \quad i = 0, \ldots, 6$$

where
$$W_1(s) = \frac{0.33s + 4.248}{s + 0.008496} \quad ; \quad W_2(s) = \frac{0.1975s^2 + 0.6284s + 1}{7.901e^{-5}s^2 + 0.2514s + 400}$$

The μ-synthesis method from the MATLAB robust control toolbox is used to solve this problem. The multimodel uncertainty is approximated with a fourth-order uncertainty weighting filter and a 18th-order controller is designed that achieves a performance of $\gamma_o = 1.0248$. Comparable performance is achieved after reducing the controller order to 6.

Continuous-time Laguerre basis functions of order 5 with $\xi = 20$ and an integrator are used for the controller parameterization. A high frequency pole at 100 is used for constructing N_i and M_i for the models. For example, for $G_6(s) = N_6(s)M_6^{-1}(s)$:

$$N_6(s) = \frac{1.6}{s + 100}, \qquad M_6(s) = \frac{s - 1.8}{s + 100}$$

The frequency response of the model is computed at $N = 200$ logarithmically spaced frequency points between 10^{-3} and 10^4 rad/s. The linearized constraints in (35) are used with a polygon of $q = 25$ vertices for over bounding $e^{j\theta}$. Solving the optimization problem leads to the following controller:

$$K(s) = \frac{0.26773(s+1)(s+2348)(s^2 + 19.82s + 131.3)(s^2 + 28.5s + 3510)}{s(s + 7.759)(s^2 + 27.77s + 556.5)(s^2 + 94.5s + 12440)}$$

which leads to the step disturbance response depicted in Fig. 3. The resulting performance obtained from the proposed optimization problem is $\gamma_o = 0.8852$. This is much smaller than that of the μ-synthesis method; in the proposed approach, there is no conservatism in modeling the multimodel uncertainties. It should be mentioned that in the μ-synthesis approach the time delay in $G_2(s)$ is
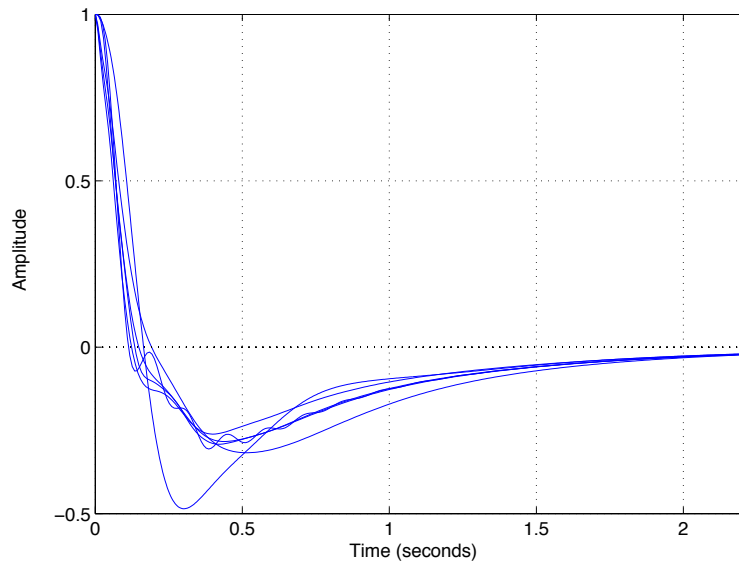
Figure 3. Step responses of the family of closed-loop systems

approximated with a first-order Pade function, while the time-delay is taken into account with no approximation in the proposed approach.

5.2. Case 2: Convergence to optimal performance

Consider a discrete-time SISO system as follows:

$$G(z) = \frac{z - 0.186}{z^3 - 1.116z^2 + 0.465z - 0.093}, \tag{36}$$

The goal is to design a controller with an integrator that minimizes $\|W_1 S\|_\infty$, where

$$W_1(z) = \frac{0.4902(z^2 - 1.0431z + 0.3263)}{(z - 1)(z - 0.282)}. \tag{37}$$

For discrete-time controller synthesis, the controller is parameterized by discrete-time Laguerre basis functions as follows:

$$K(z) = X(z)Y^{-1}(z) \quad ; \quad X(z) = \rho_x^T \phi(z), \, Y(z) = \rho_y^T \phi(z)$$

where n is the controller order, $\phi^T(z) = [1, \phi_1(z), \ldots, \phi_n(z)]$ with

$$\phi_i(z) = \frac{\sqrt{1 - a^2}}{z - a} \left(\frac{1 - az}{z - a} \right)^{i-1}$$

and $-1 < a < 1$. It will be shown that by increasing the controller order, the side effect of the selection of the parameter of Laguerre function, a, is reduced.
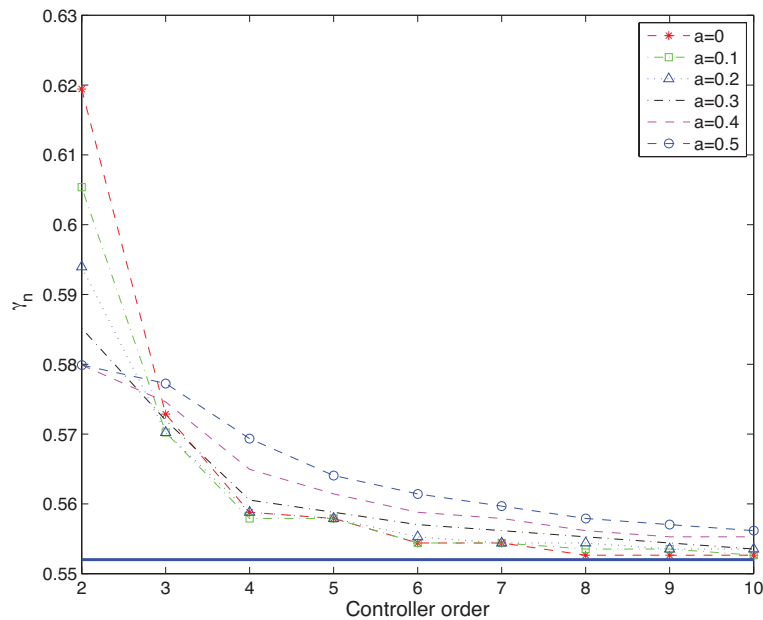
Figure 4. γ_n versus the controller order with different Laguerre parameter

In this example, 50 equally spaced frequency points between 0 and π are chosen. In order to have an integrator in the controller and to avoid unboundedness of W_1 at $\omega = 0$, the basis functions for $Y(z)$ are multiplied by $(z-1)/z$. Since the system is stable, we choose $N(j\omega) = G(j\omega)$ and $M(j\omega) = 1$. The convex constraints are linearized by approximating $e^{j\theta}$ with a polygon of $q = 50$ vertices.

The standard H_∞ control method in the Robust Control toolbox of MATLAB leads to the optimal value of $\gamma_o = 0.552$ with a 6th order controller. Fig. 4 shows the optimal value, γ_n, for different choice of the parameter a in Laguerre basis function and different controller order n. It can be observed that the optimal solution converges monotonically and is independent of the value of a. The best results are obtained for $a = 0$, which almost achieves γ_o for an 8th-order controller.

5.3. Case 3: Flexible Transmission System

In this example, the experimental data are used to compute a robust controller with respect to frequency-domain uncertainty. An electro-mechanical flexible transmission system which consists of three disks connected by elastic belts is considered. The first disk is coupled to a servo motor which is derived by a current amplifier. The position of the third disk is measured with an incremental encoder and controlled by a proportional controller. The input of the system is the reference position for the third disk (see Fig. 5). This system is excited by a PRBS signal with a sampling period of $T_s = 40$ms and the data length is 765. Figure 6 shows the experimental data that are used to identify a frequency domain model using spa command in Identification toolbox of MATLAB. The Nyquist diagram of this spectral model together with the uncertainty disks of 0.95 probability are given in Fig. 7. The uncertainty disks are approximated by a polygon of $m = 20$ vertices and the goal is to design a stabilizing controller that minimizes γ where $\|W_1 S\|_\infty < \gamma$,
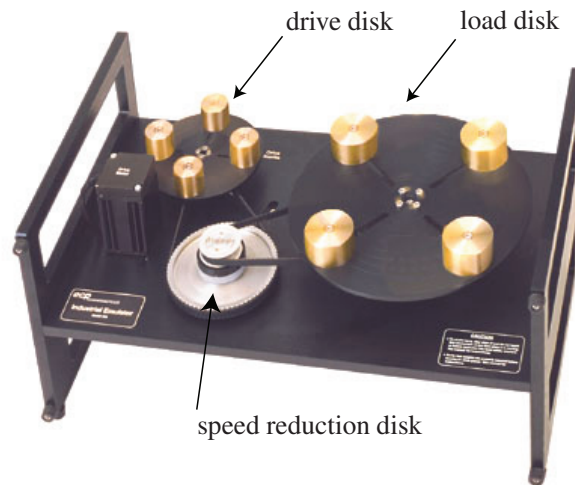
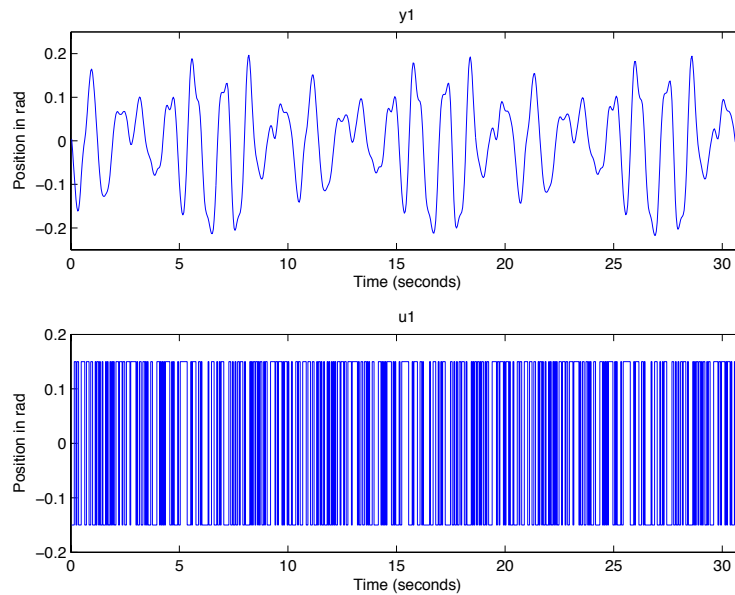Figure 5. Flexible transmission system

Figure 6. Experimental identification data.

with

$$W_1(z) = \frac{z - 0.96}{z - 1}$$

In the proposed method, discrete-time Laguerre basis functions of order 4 with $a = 0$ (FIR filter) are considered for X and Y. The resulting controller is

$$K(z) = \frac{20.3(z^2 - 1.88z + 0.92)(z^2 - 1.278z + 0.6057)}{(z + 0.72)(z - 1)(z^2 + 0.209z + 0.563)},$$

which achieves an optimal performance of $\gamma = 2.12$. Figure 8 shows the magnitude of the Bode diagram of the sensitivity function for the nominal model. It can be observed that the sensitivity
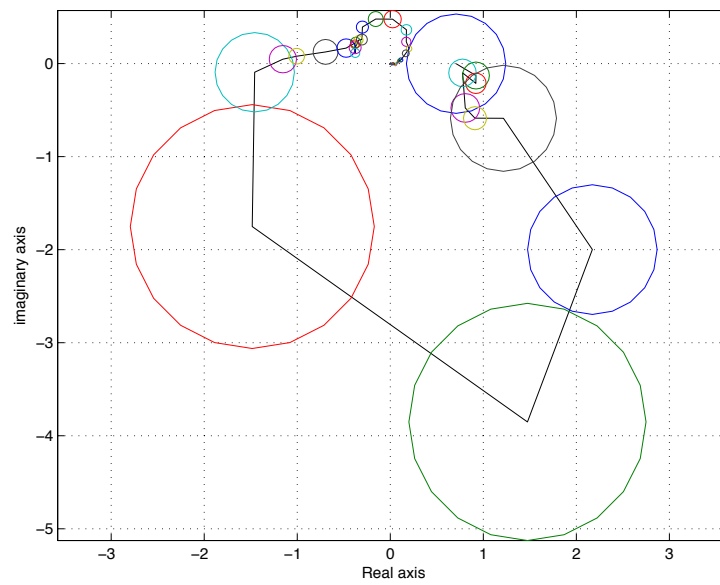
Figure 7. Nyquist diagram of the spectral model together with uncertainty disks
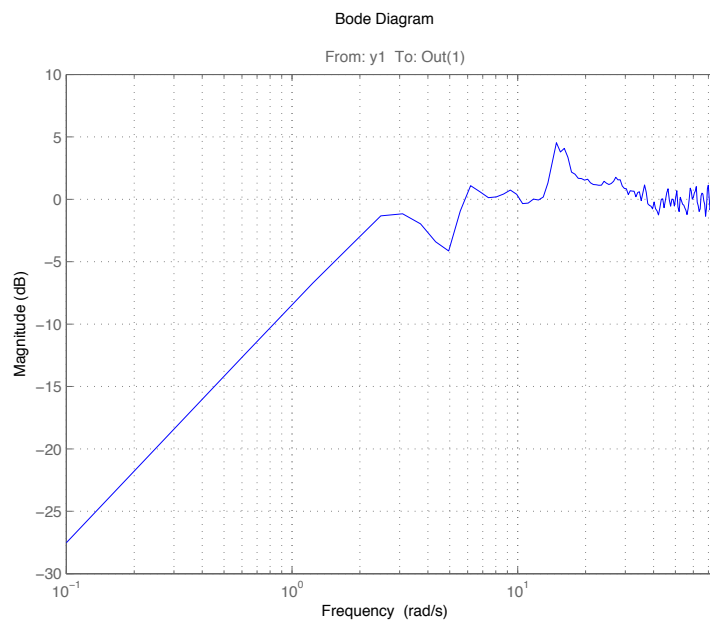

Figure 8. Magnitude Bode diagram of the sensitivity function

function is small at low frequencies and its maximum value is less than 5db which guarantees a good stability margin.

6. CONCLUSION

A robust controller design method for LTI-SISO systems based on frequency-domain data is proposed. In comparison with the classical H_∞ controller design methods, the following features can be highlighted:

- The frequency response of the plant is the only requisite for controller synthesis where no parametric model is required
- Pure input/output time delay is considered with no approximation.
- Frequency-domain uncertainty is taken into account with reduced conservatism.
- Parametric uncertainty in identified models with noisy data can be considered in a stochastic sense with reduced conservatism.
- Fixed-order controllers can be designed in a convex optimization problem that considers a finite amount of constraints in the frequency domain.

It is shown that the choice of the basis functions affects the optimization results for low-order controllers. The optimal choice of the basis function and the extension to multivariable systems are considered for future research works.

REFERENCES

1. Bazanella AS, Campestrini L, Eckhard D. *Data-driven Controller Design: The H_2 Approach*. Springer, 2012.
2. Hou ZS, Wang Z. From model-based control to data-driven control: Survey, classification and perspective. *Information Sciences* 2013; **235**:3–35.
3. Yin S, Li X, Gao H, Kaynak O. Data-based techniques focused on modern industry: an overview. *Industrial Electronics, IEEE Transactions on* 2015; **62**(1):657–667.
4. Formentin S, van Heusden K, Karimi A. A comparison of model-based and data-driven controller tuning. *International Journal of Adaptive Control and Signal Processing* 2014; **28**(10):882–897.
5. Landau ID, Lozano R, M'Saad M, Karimi A. *Adaptive Control: Algorithms, Analysis and Applications*. Springer-Verlag: London, 2011.
6. Stefanovic M, Safonov MG. *Safe adaptive control: Data-driven stability analysis and robust synthesis*, vol. 405. Springer, 2011.
7. Battistelli G, Hespanha JP, Mosca E, Tesi P. Model-free adaptive switching control of time-varying plants. *Automatic Control, IEEE Transactions on* 2013; **58**(5):1208–1220.
8. Hjalmarsson H, Gevers M, Gunnarsson S, Lequin O. Iterative feedback tuning: Theory and application. *IEEE Control Systems Magazine* 1998; :26–41.
9. Campi MC, Lecchini A, Savaresi SM. Virtual reference feedback tuning: A direct method for the design of feedback controllers. *Automatica* 2002; **38**:1337–1346.
10. Karimi A, Mišković L, Bonvin D. Iterative correlation-based controller tuning. *Int. Journal of Adaptive Control and Signal Processing* 2004; **18**(8):645–664.
11. Karimi A, Kunze M, Longchamp R. Robust controller design by linear programming with application to a double-axis positioning system. *Control Engineering Practice* February 2007; **15**(2):197–208.
12. Hoogendijk R, Den Hamer AJ, Angelis G, van de Molengraft R, Steinbuch M. Frequency response data based optimal control using the data based symmetric root locus. *IEEE Int. Conference on Control Applications*, Yokohama, Japan, 2010; 257–262.
13. Keel LH, Bhattacharyya SP. Controller synthesis free of analytical models: Three term controllers. *IEEE Trans. on Automatic Control* July 2008; **53**(6):1353–1369.
14. Parastvand H, Khosrowjerdi MJ. Controller synthesis free of analytical model: fixed-order controllers. *Int. Journal of Systems Science* 2014; (ahead-of-print).
15. Den Hamer AJ, Weiland S, Steinbuch M. Model-free norm-based fixed structure controller synthesis. *48th IEEE Conference on Decision and Control*, Shanghai, China, 2009; 4030–4035.

16. Khadraoui S, Nounou H, Nounou M, Datta A, Bhattacharyya S. A measurement-based approach for designing reduced-order controllers with guaranteed bounded error. *Int. Journal of Control* 2013; **86**(9):1586–1596.

17. Karimi A, Galdos G. Fixed-order H_∞ controller design for nonparametric models by convex optimization. *Automatica* 2010; **46**(8):1388–1394.

18. Hast M, Aström K, Bernhardsson B, Boyd S. Pid design by convex-concave optimization. *Proceedings European Control Conference*, Citeseer, 2013; 4460–4465.

19. Galdos G, Karimi A, Longchamp R. H_∞ controller design for spectral MIMO models by convex optimization. *Journal of Process Control* 2010; **20**(10):1175 – 1182.

20. Karimi A. Frequency-domain robust control toolbox. *52nd IEEE Conference in Decision and Control*, 2013; 3744 – 3749.

21. Zhou K, Doyle JC. *Essentials of robust control*. Prentice-Hall: N.Y., 1998.

22. Doyle CJ, Francis BA, Tannenbaum AR. *Feedback Control Theory*. Mc Millan: New York, 1992.

23. Zhou K. *Essentials of Robust Control*. Prentice Hall: New Jersey, 1998.

24. Rantzer A, Megretski A. Convex parameterization of robustly stabilizing controllers. *IEEE Trans. on Automatic Control* September 1994; **39**(9):1802–1808.

25. Ljung L. *System Identification - Theory for the User*. second edn., Prentice Hall: NJ, USA, 1999.

26. Akcay H, Ninness B. Orthonormal basis functions for continuous-time systems and l_p convergence. *Mathematics of Control, Signals, and Systems* 1999; **12**:295–305.

27. Calafiore G, Campi MC. The scenario approach to robust control design. *IEEE Trans. on Automatic Control* May 2006; **51**(5):742–753.

28. Alamo T, Tempo R, Luque A. On the sample complexity of probabilistic analysis and design methods. *Perspectives in Mathematical System Theory, Control, and Signal Processing*. Springer, 2010; 39–55.