Numerical methods for wave equation in heterogenous media

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In this report we discuss recent developments of numerical methods for the wave equation in a bounded polygonal domain Ω

(1) $$\partial_{tt}u_\varepsilon - \nabla\cdot(a_\varepsilon(x)\nabla u_\varepsilon) = F \text{ in } \Omega\times]0,T[$$

(2) $$u_\varepsilon(x,0) = g^1(x), \partial_t u_\varepsilon(x,0) = g^2(x),\ u_\varepsilon = 0 \text{ on }]0,T[\times\partial\Omega,$$

where $g^1 \in H^1(\Omega)$, $g^2 \in L^2(\Omega)$, $F \in L^2(0,T;L^2(\Omega)$. The family of symmetric tensors satisfy $a_\varepsilon \in (L^\infty(\Omega))^{d\times d}$ and is assumed to be uniformly elliptic and bounded. Here we think of ε as an abstract parameter $0 < \varepsilon << 1$. Furthermore, the derivative of a^ε is assumed to be large and unbounded as $\varepsilon \to 0$ (e.g., $\|a'_\varepsilon\| = \mathcal{O}(\varepsilon^{-1})$). We consider two situations that require different numerical modelling.

Heterogenenous media without scale separation. For the discrete approximation, we pick a piecewise linear finite element space V_h and consider the following problem: find $u_h : [0,T] \to V_h$ such that $\forall v_h \in V_h$ and a.e. $t > 0$

$$\langle \partial_{tt}u_h, v_h\rangle + (a_\varepsilon(x)\nabla u_h(\cdot,t), \nabla v_h)_{L^2(\Omega)} = (F(\cdot,t), v_h)_{L^2(\Omega)},$$

with appropriate discrete initial value. Following the best approximation result of Baker [7] we have ($u^\varepsilon \in C^0(0,T;H^1(\Omega)$ is the solution to the weak form of (1))

$$\|u_\varepsilon - u_h\|_{L^\infty(L^2)} \le C(T)(\|u_\varepsilon - \Pi_h(u_\varepsilon)\|_{L^\infty(L^2)} + \|\partial_t u_\varepsilon - \partial_t\Pi_h(u_\varepsilon)\|_{L^1(L^2)}),$$

where $\Pi_h : H^1_0(\Omega) \to V_h$ is the Ritz-projection on V_h, i.e., the $(a_\varepsilon\nabla\cdot,\nabla\cdot))$-orthogonal projection. An a priori error estimate of the projection error involves the norm of the derivative of a_ε and leads to a rate of convergence that cannot scale better than $C(T)(h/\varepsilon)$ leading to a computational complexity of $\mathcal{O}(h^{-d})$ with $h < \varepsilon$. In what follows, we construct a multiscale space following [11]. We consider a coarse grid V_H and assume that the fine space V_h is obtained by refinement of V_H with $h < \varepsilon << H$. We then consider the decomposition

$$V_h = V_H^{ms} \oplus W_h,$$

where $W_h = Ker(I_H)$ and $I_H : V_h \longrightarrow V_H$ is the L^2 projection. The multiscale space is defined by

$$V_{H,k}^{ms} = \{\Phi_z + Q_{h,k}(\Phi_z), \Phi_z \text{ nodal macro basis fct}\},$$

where $\Phi_z \in V_H$ is a macroscopic basis function and for each $K \in \text{supp}(\Phi_z)$, $Q_{h,k}(\Phi_z) \in W_h(U_k(K))$ is the solution the solution of a localized fine scale elliptic problem in an environment $U(K)$ around K. We can then show [4]

Theorem 1. *Under the regularity assumptions for the wave equation stated above, we have*

$$\|u_\varepsilon - u^{ms}_{H,k}\|_{L^\infty(L^2)} \le C(T)(\|u^\varepsilon - \Pi^{ms}_{H,k}(u^\varepsilon)\|_{L^\infty(L^2)} + \|\partial_t u^\varepsilon - \partial_t\Pi^{ms}_{H,k}(u^\varepsilon)\|_{L^1(L^2)}).$$

Assuming in addition $\partial_t u_\varepsilon \in L^1(H^1)$ *then*

$$\|u_\varepsilon - \Pi^{ms}_{H,k}(u_\varepsilon)\|_{L^\infty(L^2)} \le C(T)H\|u_\varepsilon\|_{L^\infty(H^1)}$$
$$\|\partial_t u_\varepsilon - \partial_t\Pi^{ms}_{H,k}(u_\varepsilon)\|_{L^1(L^2)} \le C(T)H\|\partial_t u_\varepsilon\|_{L^1(H^1)}.$$

One issue in the above estimate that also appears in other multiscale methods developed so far (see the references in [4]) is the boundedness of $\|\partial_t u_\varepsilon\|_{L^1(H^1)}$. A standard a priori error estimates yields $\|\partial_t u_\varepsilon\|_{L^1(H^1)} = \mathcal{O}(\varepsilon^{-1})$. Using a perturbation argument together with $G-$convergence we show in [4] that this term can be bounded and we obtain $\|u_\varepsilon - u_{H,k}^{ms}\|_{L^\infty(L^2)} \leq C(T)(H + r(\varepsilon))$, with C independent of ε and $\lim_{\varepsilon\to 0} r(\varepsilon) = 0$.

Heterogenous media with scale separation. Using $G-$convergence, one can show there exists a subsequence of solution of (1) that converges weakly* in $L^\infty(H_0^1)$ to a homogenized function u_0 solution of

$$\begin{aligned}&\partial_{tt}u_0 - \nabla\cdot(a_0(x)\nabla u_0) = F \text{ in } \Omega\times]0,T[\\ &u_0(x,0) = g^1(x), \partial_t u_0(x,0) = g^2(x),\ u_0 = 0 \text{ on }]0,T[\times\partial\Omega,\end{aligned}$$

where a^0 is again uniformly elliptic and bounded but independent of the small scale ε [8]. For periodic (or locally periodic) coefficients, a^0 is obtained from d solutions $\chi^1,\ldots,\chi^d$ of so-called cell problems (localized elliptic problems).

Finite element heterogeneous multiscale method. We pick a standard macroscopic finite element space V_H and define a sampling domain K_δ (of size δ comparable to ε) within each macro element K. We consider then the following problem: find $u_H : [0,T] \to V_H$ such that

$$(\partial_{tt}u_H, v_H) + B_H(u_H, v_H) = (F, v_H) \qquad \forall v_H \in V_H, \tag{3}$$

with appropriate projection of the true initial conditions, where $B_H(u^H, v^H) = \sum_{K\in\mathcal{T}_H} \frac{|K|}{|K_\delta|}\int_{K_\delta} a^\varepsilon(x)\nabla u_K^h\cdot\nabla v_K^h dx$ and u_K^h (respectively v_K^h) are solutions of a micro problem in a localized sampling domain $K_\delta \subset K$ with $\delta \simeq \varepsilon$. A generalized version of the above method is shown to converge in [1] towards the homogenized solution u^0. However, with increasing time, due to dispersive effects, the true solution, u^ε, deviates from the classical homogenized limit u^0 [13, 10]. In [5] the solutions of the following family of effective equations

$$\partial_{tt}\tilde{u} = a^0\partial_{xx}\tilde{u} - \varepsilon^2\big(\tilde{a}^2\partial_{xxxx}\tilde{u} - \tilde{b}^0\partial_{xx}\partial_{tt}\tilde{u}\big) \quad \text{in } (0, T^\varepsilon]\times\Omega, \tag{4}$$

is shown to capture the dispersive effects over time $\mathcal{O}(\varepsilon^{-2})$. This generalizes results in [10, 9]. In the above equation, we assume $x \mapsto \tilde{u}(t,x)$ is Ω periodic.

Theorem 2. Under appropriate regularity assumptions of the data, for any $\mu = \langle\chi\rangle$ and any real numbers $\tilde{b}^0, \tilde{a}^2$ such that $\tilde{b}^0 = b^0 + \langle\chi\rangle^2$, $b^0 = \langle(\chi - \langle\chi\rangle)^2\rangle$, $\tilde{a}^2 = a^0\langle\chi\rangle^2$ the solution of the effective equation

$$\partial_{tt}\tilde{u} - a^0\partial_{xx}\tilde{u} + \varepsilon^2(\tilde{a}^2\partial_{xxxx}\tilde{u} - \tilde{b}^0\partial_{ttxx}\tilde{u}) = F,$$

is an effective equation that satisfies $\|u^\varepsilon - \tilde{u}\|_{L^\infty(0,\varepsilon^{-2}T;L^2(\Omega))} \leq C\varepsilon$, over longtime $\varepsilon^{-2}T$, where C is independent of ε.

Next following [3], we consider the FE-HMM-L method obtained from (3) by replacing $(\partial_{tt}u^H, v^H)$ with $(\partial_{tt}u^H, v^H)_Q$ where

$$(u_H, v_H)_Q = (v_H, w_H) + \sum_{K\in\mathcal{T}_H}\frac{|K|}{|K_\delta|}\int_{K_\delta}(u_K^h - u_H)(v_K^h - v_H)dx,$$

where u_K^h (respectively v_K^h) are the micro functions already in (3). A fully discrete a prior error analysis over long-time has been obtained for the FE-HMM-L in [5].

Theorem 3. Under suitable regularity assumptions we have

$$\|u^\varepsilon - u_H\|_{L^\infty(0,\varepsilon^{-2}T;L^2(\Omega))} \leq C\left(\varepsilon + \left(h/\varepsilon^2\right)^2 + H/\varepsilon\right),$$

where C is independent of ε. Generalization to higher order macro elements and higher order estimates are also derived (the term H/ε can be replaced by H^ℓ/ε). As error estimates for classical resolved FEM yields a bound of the type $C(h/\varepsilon^3)$, the FE-HMM-L achieves significant reduction in the computational complexity (the size of the linear system of ODEs scale as $\mathcal{O}((tol\cdot\varepsilon^3)^{-1})$ for a resolved FEM while it only scale as $\mathcal{O}((tol\cdot\varepsilon)^{-\ell})$ for the FE-HMM-L). A seneralization to multi-dimensional wave problems has been obtained in [6].

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