Numerical methods for wave equation in heterogeneous media

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In this report we discuss recent developments of numerical methods for the wave equation in a bounded polygonal domain Ω

\[ \partial_H u_{\epsilon} - \nabla \cdot (a_{\epsilon}(x) \nabla u_{\epsilon}) = F \text{ in } \Omega \times ]0, T[ \]
\[ u_{\epsilon}(x, 0) = g^1(x), \partial_t u_{\epsilon}(x, 0) = g^2(x), \; u_{\epsilon} = 0 \text{ on } ]0, T[ \times \partial \Omega, \]

where \( g^1 \in H^1(\Omega) \), \( g^2 \in L^2(\Omega) \), \( F \in L^2(0, T; L^2(\Omega)) \). The family of symmetric tensors satisfy \( a_{\epsilon} \in (L^\infty(\Omega))^{d \times d} \) and is assumed to be uniformly elliptic and bounded. Here we think of \( \epsilon \) as an arbitrary parameter \( 0 < \epsilon << 1 \). Furthermore, the derivative of \( a_{\epsilon} \) is assumed to be large and unbounded as \( \epsilon \to 0 \) (e.g., \( \|a_{\epsilon}'\| = O(\epsilon^{-1}) \)).

We consider two situations that require different numerical modelling.

**Heterogeneous media without scale separation.** For the discrete approximation, we pick a piecewise linear finite element space \( V_h \) and consider the following problem: find \( u_h : [0, T] \to V_h \) such that \( \forall v_h \in V_h \) and a.e. \( t > 0 \)

\[ \langle \partial_t u_h, v_h \rangle + \langle a_{\epsilon}(x) \nabla u_h(\cdot, t), \nabla v_h \rangle_{L^2(\Omega)} = \langle F(\cdot, t), v_h \rangle_{L^2(\Omega)}, \]

with appropriate discrete initial value. Following the best approximation result of Baker [7] we have \( u_{\epsilon} \in C^0(0, T; H^1(\Omega)) \) is the solution to the weak form of (1)

\[ \|u_{\epsilon} - u_h\|_{L^\infty(L^2)} \leq C(T)(\|u_{\epsilon} - \Pi_h(u_{\epsilon})\|_{L^\infty(L^2)} + \|\partial_t u_{\epsilon} - \partial_t \Pi_h(u_{\epsilon})\|_{L^1(L^2)}), \]

where \( \Pi_h : H^1_0(\Omega) \to V_h \) is the Ritz-projection on \( V_h \), i.e., the \( (a_{\epsilon} \nabla \cdot, \nabla))\)-orthogonal projection. An a priori error estimate of the projection error involves the norm of the derivative of \( a_{\epsilon} \) and leads to a rate of convergence that cannot scale better than \( C(T)(h/\epsilon) \) leading to a computational complexity of \( O(h^{-d}) \) with \( h < \epsilon \).

We consider a coarse grid \( V_H \) and assume that the fine space \( V_h \) is obtained by refinement of \( V_H \) with \( h < \epsilon << h \). We then consider the decomposition

\[ V_h = V_{H,h} \oplus W_h, \]

where \( W_h = Ker(I_H) \) and \( I_H : V_h \to V_H \) is the \( L^2 \) projection. The multiscale space is defined by

\[ V_{H,h}^\epsilon = \{ \Phi_\epsilon + Q_h,k(\Phi_\epsilon), \Phi_\epsilon \text{ nodal macro basis fct} \}, \]

where \( \Phi_\epsilon \in V_H \) is a macroscopic basis function and for each \( K \in \text{supp}(\Phi_\epsilon) \), \( Q_h,k(\Phi_\epsilon) \in W_h(U_h(K)) \) is the solution of a localized fine scale elliptic problem in an environment \( U(K) \) around \( K \). We can then show [4]

**Theorem 1.** Under the regularity assumptions for the wave equation stated above, we have

\[ \|u_{\epsilon} - v^\epsilon_{H,k}\|_{L^\infty(L^2)} \leq C(T)(\|u_{\epsilon} - \Pi_{H,k}^{ms}(u_{\epsilon})\|_{L^\infty(L^2)} + \|\partial_t u_{\epsilon} - \partial_t \Pi_{H,k}^{ms}(u_{\epsilon})\|_{L^1(L^2)}). \]

Assuming in addition \( \partial_t u_{\epsilon} \in L^1(H^1) \) then

\[ \|u_{\epsilon} - \Pi_{H,k}^{ms}(u_{\epsilon})\|_{L^\infty(L^2)} \leq C(T)H\|u_{\epsilon}\|_{L^\infty(H^1)} \]
\[ \|\partial_t u_{\epsilon} - \partial_t \Pi_{H,k}^{ms}(u_{\epsilon})\|_{L^1(L^2)} \leq C(T)H\|\partial_t u_{\epsilon}\|_{L^1(H^1)}. \]
One issue in the above estimate that also appears in other multiscale methods developed so far (see the references in [4]) is the boundedness of $\|\partial_t u_\varepsilon\|_{L^1(\Omega_t)}$. A standard a priori error estimates yields $\|\partial_t u_\varepsilon\|_{L^1(\Omega_t)} = O(\varepsilon^{-1})$. Using a perturbation argument together with $G-$convergence we show in [4] that this term can be bounded and we obtain $\|u_\varepsilon - u_{H_0}^{ms}\|_{L^\infty(L^2)} \leq C(T) (H + r(\varepsilon))$, with $C$ independent of $\varepsilon$ and $\lim_{\varepsilon \to 0} r(\varepsilon) = 0$.

**Heterogenous media with scale separation.** Using $G-$convergence, one can show there exists a subsequence of solution of (1) that converges weakly* in $L^\infty(H^1_0)$ to a homogenized function $u_0$ solution of

$$\partial_t u_0 - \nabla \cdot (a_0(x)\nabla u_0) = F \text{ in } \Omega \times [0,T],$$

$$u_0(x,0) = g^1(x), \partial_t u_0(x,0) = g^2(x), u_0 = 0 \text{ on } [0,T] \times \partial \Omega,$$

where $a_0$ is again uniformly elliptic and bounded but independent of the small scale $\varepsilon$ [8]. For periodic (or locally periodic) coefficients, $a_0$ is obtained from $d$ solutions $\chi^1, \ldots, \chi^d$ of so-called cell problems (localized elliptic problems).

**Finite element heterogeneous multiscale method.** We pick a standard macroscopic finite element space $V_H$ and define a sampling domain $K_\delta$ (of size $\delta$ comparable to $\varepsilon$) within each macro element $K$. We consider then the following problem: find $u_H : [0,T] \to V_H$ such that

$$(\partial_t u_H, v_H) + B_H(u_H, v_H) = (F, v_H) \quad \forall v_H \in V_H,$$

with appropriate projection of the true initial conditions, where $B_H(u^H, v^H) = \sum_{K \in T_H} \frac{|K|}{|K_\delta|} \int_{K_\delta} a^H(x) \nabla u^H_K \cdot \nabla v^H_K \, dx$ and $u^H_K$ (respectively $v^H_K$) are solutions of a micro problem in a localized sampling domain $K_\delta \subset K$ with $\delta \approx \varepsilon$. A generalized version of the above method is shown to converge in [1] towards the homogenized solution $u^0$. However, with increasing time, due to dispersive effects, the true solution, $u^*$, deviates from the classical homogenized limit $u^H$ [13, 10]. In [5] the solutions of the following family of effective equations

$$(\partial_t \tilde{u}, v_H) + \tilde{B}(\tilde{u}, v_H) = (\tilde{F}, v_H) \quad \forall v_H \in V_H,$$

is shown to capture the dispersive effects over time $O(\varepsilon^{-2})$. This generalizes results in [10, 9]. In the above equation, we assume $x \mapsto \tilde{u}(t,x)$ is $\Omega$ periodic.

**Theorem 2.** Under appropriate regularity assumptions of the data, for any $\mu = (\chi)$ and any real numbers $\tilde{b}^\theta, \tilde{a}^2$ such that $\tilde{b}^\theta = b^\theta + (\chi)^2$, $\tilde{b}^0 = b^0$, $\tilde{a}^2 = a^0(\chi)^2$, the solution of the effective equation

$$\partial_t \tilde{u} - a^0 \partial_{xx} \tilde{u} + \varepsilon^2 (\tilde{a}^2 \partial_{xxxx} \tilde{u} - \tilde{b}^0 \partial_{txx} \tilde{u}) = F,$$

is an effective equation that satisfies $\|u^* - \tilde{u}\|_{L^\infty(0, \varepsilon^{-2}T; L^2(\Omega))} \leq C\varepsilon$, over longtime $\varepsilon^{-2}T$, where $C$ is independent of $\varepsilon$.

Next following [3], we consider the FE-HMM-L method obtained from (3) by replacing $(\partial_t u_H, v_H)$ with $(\partial_t u_H, v_H)_Q$ where

$$(u_H, v_H)_Q = (v_H, w_H) + \sum_{K \in T_H} \frac{|K|}{|K_\delta|} \int_{K_\delta} (u^H_K - u_H)(v^H_K - v_H) \, dx,$$
where $u_k^h$ (respectively $v_k^h$) are the micro functions already in (3). A fully discrete a priori error analysis over long-time has been obtained for the FE-HMM-L in [5].

**Theorem 3.** Under suitable regularity assumptions we have

$$||u^e - u_H||_{L^\infty(0,\varepsilon^{-2};L^2(\Omega))} \leq C \left( \varepsilon + (h/\varepsilon^2)^2 + H/\varepsilon \right),$$

where $C$ is independent of $\varepsilon$. Generalization to higher order macro elements and higher order estimates are also derived (the term $H/\varepsilon$ can be replaced by $H^\ell/\varepsilon$). As error estimates for classical resolved FEM yields a bound of the type $C(h/\varepsilon^3)$, the FE-HMM-L achieves significant reduction in the computational complexity (the size of the linear system of ODEs scale as $O((tol\cdot \varepsilon)^{-1})$ for a resolved FEM while it only scale as $O((tol\cdot \varepsilon)^{-\ell})$ for the FE-HMM-L). A seneralization to multidimensional wave problems has been obtained in [6].

**Acknowledgements.** This research has been partially supported by the Swiss National Foundation.

**REFERENCES**


