Complement to the Paper Stochastic MPC for Controlling the Average Constraint Violation of Periodic Linear Systems with Additive Disturbances

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A. Explicit computation of the SPCI sequence

To present the algorithm to compute the SPCI sequence it is it useful to define the preset operator.

Definition 1: Given a set $\mathcal{M}_{j+1} \subseteq \mathbb{R}^n$ at intra-period $j+1$, the set $\text{Pre}(\mathcal{M}_j)$ is defined as

$$\begin{aligned}\text{Pre}(\mathcal{M}_{j+1}) =: \{x \in \mathbb{R}^n :\ \exists u \in \mathcal{U}_j \mid \\ A_j x + B_j u + w \in \mathcal{M}_{j+1} \quad \forall w \in \mathcal{W}_j \\ g^T_{\sigma_{j+1}}(A_j x + B_j u) \leq q_j(\xi) + h_{\sigma_{j+1}}\}\end{aligned}$$

Algorithm 1 SPCI Computation

1: Initialize the sets $S_j := X^s_j \cup \bar{\mathcal{X}}_j,\ j = 0, 1, \ldots, p-1$ and set $i := 0$
2: Let $h = \text{mod}(i, p)$. Compute $Q(S_h) =: \text{Pre}(S_h) \cap S_{\sigma_{h-1}}$.
3: **if** $i \leq -p$ and $S_{\sigma_{h-1}} = Q(S_h)$ **then** stop. The maximal SPCI sequence has been found.
4: **if** $Q(S_h)$ is empty **then** stop. The maximal SPCI sequence does not exist.
5: Update $S_{h-1} = Q(S_h)$
6: Set $i =: i-1$, and **goto** Step 2.

B. Implicit Parametrization of the SPCI sequence

Consider an MPC problem formulation with prediction horizon N and current time k. The predictions for the control input u_{k+i} are provided by an explicit policy parametrization for all $i \in \mathbb{N}_0^{N-1}$ whereas, for $i \geq N$ a fixed controller is assumed. To give a possible instance, let

$$u_{k+i} = \pi_i(x_k^{k+i}), \quad i \in \mathbb{N}_0^{N-1} \quad \text{(B1)}$$

be the explicit control policy (state-sequence feedback). Further assume the terminal regulator to be a state-feedback controller

$$u_{k+i} = \kappa_f(x_{k+i}), \quad i \geq N.$$

The constraint satisfaction is enforced explicitly for (B1) through constraints on the policy π and implicitly for $i \geq N$ by requiring that the state x_{k+N} lands in a predetermined sequence of invariant sets associated to κ_f. More precisely:

Definition 2: A collection of sets $(\mathcal{X}^f_0, \ldots, \mathcal{X}^f_{p-1})$ is a sequence of terminal periodic invariant sets if it satisfies for each $j = 0, \ldots, p-1$, $\mathcal{X}^f_j \subseteq \mathcal{X}^s_j \cap \bar{\mathcal{X}}_j$ and

$$\begin{aligned}\forall x \in \mathcal{X}^f_j \quad & A_j x + B_j \kappa_f(x) + w \in \mathcal{X}^f_{\sigma_{j+1}} \quad \forall w \in \mathcal{W}_j \\ & g^T_{\sigma_{j+1}}(A_j x + B_j \kappa_f(x)) \leq q_j(\xi) + h_{\sigma_{j+1}} \\ & \kappa_f(x) \in \mathcal{U}_j.\end{aligned}$$

Therefore, at time k, the constraint on the terminal state assumes the form $x_{k+N} \in \mathcal{X}^f_{\sigma_{k+N}}$ which implicitly ensures satisfaction of both (29a) and (29b) for all w_i where $i \geq N$. Regarding the constraint on the policy parametrization for $i \in \mathbb{N}_0^{N-1}$, the constraint (29a) is imposed as it is whereas the second line of (29b) is enforced implicitly as follows. We observe that $x_{k+1} \in \mathcal{S}^{r_k}_{\sigma_{k+1}}$ is guaranteed if $x_{k+i} \in \bar{\mathcal{X}}_{\sigma_{k+i}}$ for $i \in \mathbb{N}_+$ and

$$\mathbf{E}\{l(g^T_{\sigma_{k+r_k+i}} x_{\sigma_{k+r_k+i}} - h_{\sigma_{k+r_k+i}}) \mid x_{k+r_k+i-1}, x_k, v_k\} \leq \xi$$

The previous constraint is enforced explicitly along the prediction horizon for a given (x_k, v_k), all $i \in \mathbb{N}_+$ and all the possible trajectories generated by all possible $w_k^{k+r_k+i-2}$.

C. Proof of Theorem 1

(i) At time zero the layer index $r_0 = 1$ and $\beta_0 = \xi$. We need to show that $\mathcal{U}_0(x_0, \chi_0) \neq \emptyset$. But this is guaranteed by the condition $x_0 \in S_0$ and the definition of S_0.

(ii) At time k we assume $r_k = 1$. Hence, feasibility at time k implies that the state will land in $S^1_{\sigma_{k+1}}$ at time $k+1$. To prove feasibility at time $k+1$ one needs to show that the two constraints (29a), (29b) are satisfied for, at least, an admissible input $u_{k+1} \in \mathcal{U}_{j+1}$. Once again, this is ensured by the definition of $S^1_{\sigma_{k+1}}$. Note further that $r_k = 1$ is the only case when the second constraint (29b) is not redundant since $\beta_{k+1} < \bar{\xi}_{\sigma_{k+1}} \Rightarrow r_k = 1$.

For the case $r_k > 1$ we note that feasibility at time k implies that the state process is in $S^{r_k}_{\sigma_{k+1}} = \text{Pre}(S^{r_k-1}_{\sigma_{k+2}}) \bigcap \bar{\mathcal{X}}_{\sigma_{k+2}}$ at the next time iteration. Noticing that $r_{k+1} \geq r_k - 1$ we know that constraint (29a) will be satisfied at time $k+1$ whereas constraint (29b) is redundant, as already underlined.

(iii) Let k be the current time instant and σ_k the correspondent intra-period time. First consider the case $v_k/k \leq \xi$ which can also be written as $\xi k - v_k \geq 0$. From the definition

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This work has received support from the Swiss National Science Foundation under the NRP 70 Energy Turnaround Project (Integration of Intermittent Widespread Energy Sources in Distribution Networks: Storage and Demand Response, grant number 407040 15040/1) and the European Research Council under the European Unions Seventh Framework Programme (FP/2007-2013) / ERC Grant Agreement n. 307608 (BuildNet)

of β_k we have $\beta_k = \xi + \xi k - v_k$. Hence

$$\begin{aligned}\mathbf{E}_k(v_{k+1}) &= \gamma v_k + \mathbf{E}\{l(g^T_{\sigma_{k+1}} x_{k+1} - h_{\sigma_{k+1}} | x_k)\} \\ &\leq \gamma v_k + \gamma(\xi s_k - v_k) + \xi = \xi s_{k+1}\end{aligned}$$

as required.

Consider now a time instant when $v_k/s_k > \xi$ and let τ_k be the first time of return under the threshold ξ. Obviously, whenever $\tau_k < \infty$ the second line of (8) is satisfied. In the case when $\tau_k = \infty$ we can define a new process

$$\eta_i := v_{k+i} - s_{k+i}\xi, \quad i \in \mathbb{N}$$

As a first thing, we show that η_i is a supermartingale

$$\begin{aligned}&\mathbf{E}_{k+i}\{\eta_{i+1} - \eta_i\} \\ &= \mathbf{E}_{k+i}\{(\eta_{i+1} - \eta_i)\} \\ &= \mathbf{E}_{k+i}\{(\gamma - 1)\eta_i + l(g^T_{\sigma_{k+i+1}} x_{k+i+1} - h_{\sigma_{k+i+1}}) - \xi)\} \\ &\leq \underbrace{(\gamma - 1)}_{\leq 0}\underbrace{\eta_i}_{\geq 0} \leq 0\end{aligned}$$

where the penultimate inequality derives from the fact that, when $\tau_k = \infty$, we have $\mathbf{E}\{l(g^T_{\sigma_{k+i+1}} x_{k+i+1} - h_{\sigma_{k+i+1}})\} \leq \xi$. Therefore η_i^τ is a supermartingale process and for Doob's martingale convergence theorem it converges with probability one to some finite random variable η_∞.

To conclude the proof we need to demonstrate that, in the case $\tau_t = \infty$, we have $v_{k+i}/s_{k+i} \to \xi$. To this aim two cases need to be considered.

When $\gamma = 1$, $s_{k+i} = k + i \to \infty$ as $i \to \infty$ and

$$\begin{aligned}\lim_{i\to\infty} \frac{v_{k+i}}{k+i} - \xi &= \lim_{i\to\infty} \frac{v_{k+i} - (k+i)\xi}{k+i} \\ &= \lim_{i\to\infty} \frac{\eta_i}{k+i} = \lim_{i\to\infty} \frac{\eta_\infty}{k+i} = 0\end{aligned}$$

For the case $\gamma \in [0, 1)$ it is sufficient to prove that $\eta_i \to 0$. To this aim, given the convergence of η we just need to prove

$$\mathbb{P}\left(\bigcup_{t=1}^{\infty} \left\{\inf_{i\geq 0}(\eta_i) > 1/t\right\}\right) = 0$$

and, exploiting Boole's inequality, a sufficient condition for this to hold is

$$\mathbb{P}(\tau(t) = \infty) = 0 \quad \forall t \in \mathbb{N}_+$$

with $\tau(t) := \inf\{i \geq 0 \,|\, \eta_i \leq 1/t\}$. To show this we over-bound the trajectories of η_i by a random walk with a drift. Thanks to the assumptions on the loss function $l(\cdot)$, it is possible to write

$$\begin{aligned}&l(g^T_{\sigma_{k+i+1}} x_{k+i+1} - h_{\sigma_{k+i+1}}) \\ &\qquad \leq l(q_{\sigma_{k+i}}(\xi) + g^T_{\sigma_{k+i}} w_{\sigma_{k+i}})\end{aligned}$$

Therefore

$$\begin{aligned}\eta_{i+1} &= \gamma\eta_i + l(g^T_{\sigma_{k+i+1}} x_{k+i+1} - h_{\sigma_{k+i+1}}) \\ &\leq \gamma\eta_i + l(q_{\sigma_{k+i}}(\xi) + g^T_{\sigma_{k+i}} w_{\sigma_{k+i}}) - \xi\end{aligned}$$

which means that the trajectories of η_i are bounded by the AR(1) process of the form

$$X_{i+1} = \gamma X_i + z_i \quad X_0 = \eta_0 > 0$$

with $z_i := l(q_{\sigma_{k+i}}(\xi) + g^T_{\sigma_{k+i}} w_{\sigma_{k+i}}) - \xi$ that is an i.i.d innovation with non-positive mean. Moreover, if $\tau(t) = \infty$ we have $\eta_i > 1/t$ which means that the trajectories of X_i (and hence η_i) are over bounded by the random walk with a drift

$$Y_{i+1} = Y_i + z_i - (1 - \gamma)/t, \quad Y_0 = X_0 = \eta_0 > 0$$

The drift of this random walk, $\mathbf{E}\{z_i\} - (1-\gamma)/t$, is strictly negative and bounded away from zero since $\mathbf{E}\{z_i\} \leq 0$ and $(1 - \gamma/t) > 0$; this implies that the expected return time below $1/t$ is finite and, as a result, $\mathbf{E}\{\tau(t)\} < \infty$ which implies $\mathbb{P}(\tau(t) = \infty) = 0$. This finishes the proof.