Complement to the Paper Stochastic MPC for Controlling the Average Constraint Violation of Periodic Linear Systems with Additive Disturbances

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A. Explicit computation of the SPCI sequence

To present the algorithm to compute the SPCI sequence it is useful to define the preset operator.

Definition 1: Given a set \( M_{j+1} \subseteq \mathbb{R}^n \) at intra-period \( j+1 \), the set \( \text{Pre}(M_j) \) is defined as
\[
\text{Pre}(M_{j+1}) = \{ x \in \mathbb{R}^n : \exists u \in U_j \mid A_j x + B_j u + w \in M_{j+1}, \forall w \in W_j \}
\]
\[
g^T_{\sigma_{j+1}}(A_j x + B_j u) \leq q_j(\xi) + h_{\sigma_{j+1}}
\]

Algorithm 1 SPCI Computation

1. Initialize the sets \( S_j := X_j^f \cup X_j^i \), \( j = 0,1,\ldots,p-1 \) and set \( h := 0 \)
2. Let \( h = \text{mod}(i, p) \). Compute \( Q(S_h) = \text{Pre}(S_h) \cap S_{\sigma_{h-1}} \).
3. if \( i \leq p \) and \( S_{\sigma_{h-1}} = Q(S_h) \) then stop. The maximal SPCI sequence has been found.
4. if \( Q(S_h) \) is empty then stop. The maximal SPCI sequence does not exist.
5. Update \( S_{h-1} = Q(S_h) \)
6. Set \( i := i - 1 \), and goto Step 2.

B. Implicit Parametrization of the SPCI sequence

Consider an MPC problem formulation with prediction horizon \( N \) and current time \( k \). The predictions for the control input \( u_{k+i} \) are provided by an explicit policy parametrization for all \( i \in \mathbb{N}_0^{N-1} \) whereas, for \( i \geq N \) a fixed controller is assumed. To give a possible instance, let
\[
u_{k+i} = \pi_i(x_{k+i}^k), \quad i \in \mathbb{N}_0^{N-1} \tag{B1}
\]
be the explicit control policy (state-sequence feedback). Further assume the terminal regulator to be a state-feedback controller
\[
u_{k+i} = \kappa_f(x_{k+i}), \quad i \geq N.
\]
The constraint satisfaction is enforced explicitly for (B1) through constraints on the policy \( \pi \) and implicitly for \( i \geq N \) by requiring that the state \( x_{k+N} \) lands in a predetermined sequence of invariant sets associated to \( \kappa_f \). More precisely:

Definition 2: A collection of sets \( (X_0^f, \ldots, X_{p-1}^f) \) is a sequence of terminal periodic invariant sets if it satisfies for each \( j = 0, \ldots, p-1 \), \( X_j^f \subseteq X_j^i \cap X_j^o \) and
\[
\forall x \in X_j^f \quad A_j x + B_j \kappa_f(x) + w \in X_{j+1}^f, \forall w \in W_j
\]
\[
g^T_{\sigma_{j+1}}(A_j x + B_j \kappa_f(x)) \leq q_j(\xi) + h_{\sigma_{j+1}}
\]
\[
\kappa_f(x) \in U_j.
\]

Therefore, at time \( k \), the constraint on the terminal state assumes the form \( x_{k+N} \in X_{\sigma_{N+1}}^f \) which implicitly ensures satisfaction of both (29a) and (29b) for all \( w \) where \( i \geq N \). Regarding the constraint on the policy parametrization for \( i \in \mathbb{N}_0^{N-1} \), the constraint (29a) is imposed as it is whereas the second line of (29b) is enforced implicitly as follows. We observe that \( x_{k+1} \in S_{\sigma_{k+1}}^f \) is guaranteed if \( x_{k+i} \in X_{\sigma_{k+i}}^i \) for \( i \in \mathbb{N}_+ \) and
\[
E\{I(g^T_{\sigma_{k+r_k+i}, x_{\sigma_{k+r_k+i}} - h_{\sigma_{k+r_k+i}}}, x_{k+r_k+i}, x_k, v_k) \leq \xi
\]
The previous constraint is enforced explicitly along the prediction horizon for a given \( (x_k, v_k) \), all \( i \in \mathbb{N}_+ \) and all the possible trajectories generated by all possible \( w^{k+r_k+i-2} \).

C. Proof of Theorem 1

(i) At time zero the layer index \( r_0 \) and \( \beta_0 = \xi \). We need to show that \( U_0(x_0, \chi_0) \neq \emptyset \). But this is guaranteed by the condition \( x_0 \in S_0 \) and the definition of \( S_0 \).

(ii) At time \( k \) we assume \( r_k = 1 \). Hence, feasibility at time \( k \) implies that the state will land in \( S_{\sigma_{k+1}}^1 \) at time \( k+1 \). To prove feasibility at time \( k+1 \) one needs to show that the two constraints (29a), (29b) are satisfied for, at least, an admissible input \( u_{k+1} \in U_{k+1} \). Once again, this is ensured by the definition of \( S_{\sigma_{k+1}}^1 \). Note further that \( r_k = 1 \) is the only case when the second constraint (29b) is not redundant since \( \beta_{k+1} < \xi_{\sigma_{k+1}} \Rightarrow r_k = 1 \).

For the case \( r_k > 1 \) we note that feasibility at time \( k \) implies that the state process is in \( S_{\sigma_{k+1}}^1 = \text{Pre}(S_{\sigma_{k+2}}^1) \cap X_{\sigma_{k+2}} \) at the next time iteration. Noticing that \( r_{k+1} \geq r_k - 1 \) we know that constraint (29a) will be satisfied at time \( k+1 \) whereas constraint (29b) is redundant, as already underlined.

(iii) Let \( k \) be the current time instant and \( \sigma_k \) the corresponding intra-period time. First consider the case \( v_k/k \leq \xi \) which can also be written as \( \xi k - v_k \geq 0 \). From the definition...
of $\beta_k$ we have $\beta_k = \xi + \xi k - v_k$. Hence
\[
E_{k}(v_{k+1}) = \gamma v_k + E\{l(g^T_{\sigma_{k+1}}x_{k+1} - h_{\sigma_{k+1}}|x_k)\} \\
\leq \gamma v_k + \gamma(\xi s_k - v_k) + \xi = \xi s_{k+1}
\]
as required.

Consider now a time instant when $v_k/s_k > \xi$ and let $\tau_k$ be the first time of return under the threshold $\xi$. Obviously, whenever $\tau_k < \infty$ the second line of (8) is satisfied. In the case when $\tau_k = \infty$ we can define a new process
\[
\eta_i := v_{k+i} - s_{k+i} \xi, \quad i \in \mathbb{N}
\]
As a first thing, we show that $\eta_i$ is a supermartingale
\[
E_{k+i}\{\eta_{i+1} - \eta_i\} \\
= E_{k+i}\{(\eta_{i+1} - \eta_i)\} \\
= E_{k+i}\{(\gamma - 1)\eta_i + l(g^T_{\sigma_{k+i+1}}x_{k+i+1} - h_{\sigma_{k+i+1}})\} \\
\leq (\gamma - 1) \eta_i \leq 0
\]
where the penultimate inequality derives from the fact that, when $\tau_k = \infty$, we have $E\{l(g^T_{\sigma_{k+i+1}}x_{k+i+1} - h_{\sigma_{k+i+1}})\} \leq \xi$. Therefore $\eta^*_i$ is a supermartingale process and for Doob’s martingale convergence theorem it converges with probability one to some finite random variable $\eta_\infty$.

To conclude the proof we need to demonstrate that, in the case $\tau_\infty = \infty$, we have $v_{k+i}/s_{k+i} \rightarrow \xi$. To this aim two cases need to be considered.

When $\gamma = 1$, $s_{k+i} = k + i \rightarrow \infty$ as $i \rightarrow \infty$ and
\[
\lim_{i \rightarrow \infty} \frac{v_{k+i}}{k + i} - \xi = \lim_{i \rightarrow \infty} \frac{v_{k+i} - (k + i)\xi}{k + i} = \lim_{i \rightarrow \infty} \frac{\eta_i}{k + i} = \lim_{i \rightarrow \infty} \frac{\eta_\infty}{k + i} = 0
\]

For the case $\gamma \in [0, 1)$ it is sufficient to prove that $\eta_i \rightarrow 0$. To this aim, given the convergence of $\eta$ we just need to prove
\[
P\left(\bigcup_{i=1}^{\infty} \left\{ \inf_{i \geq 0} (\eta_i) > 1/t \right\} \right) = 0
\]
and, exploiting Boole’s inequality, a sufficient condition for this to hold is
\[
P(\tau(t) = \infty) = 0 \quad \forall t \in \mathbb{N}_+
\]
with $\tau(t) := \inf\{i \geq 0 | \eta_i \leq 1/t\}$. To show this we over-bound the trajectories of $\eta_i$ by a random walk with a drift. Thanks to the assumptions on the loss function $l(\cdot)$, it is possible to write
\[
l(g^T_{\sigma_{k+i}}x_{k+i+1} - h_{\sigma_{k+i+1}}) \\
\leq l(q_{\sigma_{k+i}}(\xi) + g^T_{\sigma_{k+i}}w_{\sigma_{k+i}})
\]
Therefore
\[
\eta_{i+1} = \gamma \eta_i + l(g^T_{\sigma_{k+i+1}}x_{k+i+1} - h_{\sigma_{k+i+1}}) \\
\leq \gamma \eta_i + l(q_{\sigma_{k+i}}(\xi) + g^T_{\sigma_{k+i}}w_{\sigma_{k+i}}) - \xi
\]
which means that the trajectories of $\eta_i$ are bounded by the AR(1) process of the form
\[
X_{i+1} = \gamma X_i + z_i \quad X_0 = \eta_0 > 0
\]