Space-dependent source determination in a time-fractional diffusion equation using a local discontinuous Galerkin method

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Abstract This paper is devoted to determining a space-dependent source term in an inverse problem of the time-fractional diffusion equation. We use a method based on a finite difference scheme in time and a local discontinuous Galerkin method (LDG) in space and investigate the numerical stability and convergence of the proposed method. Finally, various numerical examples are used illustrate the effectiveness and accuracy of the method.

Keywords Inverse source problem · fractional diffusion equation · local discontinuous Galerkin method

Mathematics Subject Classification (2000) 65M32 · 65M60 · 35R11

1 Introduction

In recent years, there has been an substantial research on problems involving the fractional order partial differential equations (PDEs), see e.g. [18, 6, 24, 19, 21, 17, 2, 30, 25, 29, 22, 27, 28, 31, 8]. In contrast to the classical diffusion equations, the fractional diffusion equations can be used to describe the anomalous diffusion phenomena such as super-diffusion or sub-diffusion. A time-fractional diffusion equation occurs when replacing the standard time derivative with a time fractional derivative and can be applied in modeling of some problems in porous flows, rheology and mechanical systems, models of a variety of biological processes, control and robotics, transport

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in fusion plasmas, and many other areas of applications. The direct problems corresponding to the time-fractional diffusion equations have been studied extensively in recent years, including uniqueness and existence results[2], some analytical or numerical solutions[13, 7, 31], and numerical methods such as finite element methods or finite difference methods [12, 14].

Here, we focus on an interesting inverse problem defined to the fractional inverse problem pioneered by Murio [18, 16, 17]. After that, many works have been published and we mention some of them in the following. In [2] an inverse problem has been considered to determine the order of the fractional derivative and the diffusion coefficient in a fractional diffusion equation. A uniqueness result has been also obtained. In [15], Liu et al. solved a backward problem for the time-fractional diffusion equation by a quasi-reversibility regularization method. Some Cauchy problems for the time-fractional diffusion equation on a bounded domain and on a strip have been investigated in [35, 36]. A modified kernel method to deal with an inverse fractional diffusion equation has been presented in [19]. Wei et al. [25] have dealt with some space-fractional diffusion equations. Rundell et al. [21] have determined an unknown boundary condition in a fractional diffusion equation. We consider the following time-fractional diffusion equation [27],

$$\begin{cases} D_t^\alpha u = (a(x)u_x)_x + c(x)u + p(t)f(x), & 0 < x < 1, \ 0 < t < T, \\ u(0,t) = k_0(t), & 0 \le t \le T, \\ u(1,t) = k_1(t), & 0 \le t \le T, \\ u(x,0) = \phi(x), & 0 \le x \le 1, \end{cases} \tag{1.1}$$

where the functions a, c, p, k_0, k_1 and ϕ are given and u, f are unknown functions. To solve this inverse source problem we need an additional condition

$$u(x,T) = g(x), \qquad 0 \le x \le 1. \tag{1.2}$$

D_t^α is the Caputo fractional derivative of order α, i.e.

$$D_t^\alpha u(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u(x,s)}{\partial s} \frac{ds}{(t-s)^\alpha}, \quad 0 < \alpha < 1, \tag{1.3}$$

where $\Gamma(\cdot)$ is the Gamma function. Obviously for $\alpha = 1$, $D_t^\alpha u = u_t$. The existence of a unique weak solution for the direct problem (1.1) has been studied in [22] and the only uniqueness theorem for the inverse source problem mentioned above is the following.

Theorem 1.1 *If $p \in C[0,T]$ satisfies $p(t) \ge p_0 > 0$ for all $t \in [0,T]$, $a \in C^1[0,1]$ satisfies $a(x) \ge v > 0$ for all $x \in [0,1]$, and $c \in C[0,1]$ satisfies $c(x) \le 0$ for all $x \in [0,1]$ then the inverse source problem (1.1)-(1.3) has a unique solution (u,f).*

Theorem 1.1 has been proved in [27] where Wei et al. showed that the inverse source problem (1.1)-(1.2) is an ill-posed problem. For $\alpha = 1$, we encounter a classical ill-posed problem which has been studied for example in [9, 23], while for $0 < \alpha < 1$, to the best of our knowledge, only a few works have been carried out. Sakamoto et al. in [22] used additional data $u(x_0,t)$ for some given x_0 in the domain to determine p when f is given and the authors obtained Lipschitz stability for p. Wei et al. in

[29] applied extra data to determine p when $a = 1$, $c = 0$ and f is given. This special case has been also considered in [33] where a local discontinuous Galerkin (LDG) method has been applied. In [35], a Fourier truncation method has been applied. In [26], for solving an inverse source problem with $p = 1$ for one-dimensional case with special coefficients, a Tikhonov regularization method has been provided. In [27], Wei et al. focused on a multi-dimensional problem with variable coefficients in a general bounded domain. To the best of our knowledge, for the inverse source problem with or without fractional operators, there are no results to the DG methods and the development of the DG methods remains limited. Of course, the forward fractional diffusion equation has been solved successfully by some DG methods. For example, Wei et al. [28] have solved the time-fractional diffusion equation using a fully-discrete LDG method and some space-fractional diffusion equations have been solved by Hesthaven et al. [7,31] using a local discontinuous Galerkin method in a semi-discrete setting.

In this paper, we extend the application of the discontinuous Galerkin method to some inverse source problems. We present a local discontinuous Galerkin method for solving the inverse source problem of the time-fractional advection-diffusion equation (1.1)-(1.2). In the proposed method, the time-fractional derivative is discretized by a backward difference scheme and a LDG method is applied for the space variable.

The rest of the paper is organized as follows. At the end of this section, some preliminaries are provided. Section 2 is devoted to the construction of the proposed method for the inverse source problem (1.1)-(1.2) and the development of stability and convergence theorems of the proposed method. Some numerical examples are given in Section 3 to illustrate the accuracy and applicability of the method. Finally, a brief conclusion completes the paper.

First we decompose the domain of the problem into some cells (subintervals). For this purpose, we consider a mesh as $0 = x_{\frac{1}{2}} < x_{\frac{3}{2}} < \cdots < x_{N+\frac{1}{2}} = 1$ with cells $I_j = [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}]$, for $j = 1, \ldots, N$. $\Delta x_j = x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}}$ denotes the cell length and we set $h = \max_{1 \leq j \leq N} \Delta x_j$. We denote by $u^+_{j+\frac{1}{2}}$ and $u^-_{j+\frac{1}{2}}$, respectively the values of u at $x_{j+\frac{1}{2}}$ from the right cell I_{j+1} and from the left cell I_j. The jump of u at cell interfaces is denoted by $[u]_{j+\frac{1}{2}}$, i.e. $u^+_{j+\frac{1}{2}} - u^-_{j+\frac{1}{2}}$. For some integer k, we set

$$V_h^k = \left\{ v \in L^2[0,1] | \, v|_{I_j} \in \mathbb{P}^k(I_j), \, j = 1, \ldots, N \right\},$$

as the space of polynomials of degree up to k in each cell I_j.

We define projections $\mathbf{P}$ and $\mathbf{P}^\pm$ as follows

$$\int_{I_j} (\mathbf{P}\omega(x) - \omega(x))v(x) = 0, \qquad \forall v \in \mathbb{P}^k(I_j), \tag{1.4}$$

$$\int_{I_j} (\mathbf{P}^+\omega(x) - \omega(x))v(x) = 0, \qquad \forall v \in \mathbb{P}^{k-1}(I_j), \tag{1.5}$$

$$\mathbf{P}^+\omega(x^+_{j-\frac{1}{2}}) = \omega(x_{j-\frac{1}{2}}),$$

and

$$\int_{I_j} (\mathbf{P}^- \omega(x) - \omega(x))\upsilon(x) = 0, \qquad \forall \upsilon \in \mathbb{P}^{k-1}(I_j), \qquad (1.6)$$

$$\mathbf{P}^- \omega(x^+_{j+\frac{1}{2}}) = \omega(x_{j+\frac{1}{2}}).$$

These projections satisfy the inequality [5, 32]

$$\| \omega^e \| + h \| \omega^e \|_\infty + h^{\frac{1}{2}} \| \omega^e \|_{\tau_h} \leq C h^{k+1}, \qquad (1.7)$$

where $\omega^e = \mathbf{P}\omega - \omega$ or $\omega^e = \mathbf{P}^{\pm}\omega - \omega$, the positive constant C depends solely on ω and is independent of h, τ_h denotes the set of boundary points of all elements I_j. $\| \cdot \|$ is the $L^2[0,1]$ norm.

2 The LDG scheme

In this section, we describe a numerical scheme for the solution of problem (1.1)-(1.2). For some positive integer M, let $\Delta t = T/M$ be the time mesh size and $t_n = n\Delta t$, $n = 0, \ldots, M$ be the mesh points. The time fractional derivative (1.3) is approximated by a simple quadrature formula known as the L1 rule [13],

$$D_t^\alpha u(x, t_n) = \frac{(\Delta t)^{1-\alpha}}{\Gamma(2-\alpha)} \sum_{i=0}^{n-1} b_i \frac{u(x, t_{n-i}) - u(x, t_{n-i-1})}{\Delta t} + \gamma^n(x),$$

where $b_i = (i+1)^{1-\alpha} - i^{1-\alpha}$ and γ^n is the truncation error with the estimate [14]

$$\|\gamma^n\| \leq C(\Delta t)^{2-\alpha}.$$

Here C is a constant depending on u, α and T. It is easy to check that $b_i > 0$ for each i, $1 = b_0 > b_1 > \cdots$ and $b_n \to 0$ as $n \to \infty$.

We rewrite (1.1) as a first-order system

$$q = u_x, \qquad D_t^\alpha u(x, t) - (a(x)q)_x - c(x)u = f(x)p(t), \qquad u(x, T) = g(x). \quad (2.1)$$

Let $u_h^n, q_h^n, f_h \in V_h^k$ be the approximation of $u(\cdot, t_n), q(\cdot, t_n), f(\cdot)$ respectively, and $p^n = p(t_n)$. We define a local discontinuous Galerkin scheme as follows: find $u_h^n, q_h^n \in V_h^k$, such that for all test functions $v, w, y \in V_h^k$,

$$\begin{cases} \int_\Omega u_h^n v \, dx + \beta \left(\int_\Omega (a q_h^n) v_x \, dx - \sum_{j=1}^N ((\hat{a}\hat{q}_h^n v^-)_{j+\frac{1}{2}} - (\hat{a}\hat{q}_h^n v^+)_{j-\frac{1}{2}}) \right) - \\ \qquad \beta \int_\Omega c u_h^n v \, dx = \beta p^n \int_\Omega f_h v \, dx + \sum_{i=1}^{n-1} (b_{i-1} - b_i) \int_\Omega u_h^{n-i} v \, dx + b_{n-1} \int_\Omega u_h^0 v \, dx, \\ \int_\Omega q_h^n w \, dx + \int_\Omega u_h^n w_x \, dx - \sum_{j=1}^N ((\hat{u}_h^n w^-)_{j+\frac{1}{2}} - (\hat{u}_h^n w^+)_{j-\frac{1}{2}}) = 0, \\ \int_\Omega u_h^M y \, dx = \int_\Omega g y \, dx, \end{cases}$$

$$(2.2)$$

where $\Omega = [0,1]$ and $\beta = (\Delta t)^\alpha \Gamma(2-\alpha)$. The "hat" terms in (2.2) are the "numerical fluxes", which are single valued functions defined on the edges. These are selected based on different guiding principles for different PDEs to ensure stability of the scheme. We take the following simple choice non-unique

$$\hat{u}_h^n = (u_h^n)^-, \qquad \hat{q}_h^n = (q_h^n)^+,$$

or

$$\hat{u}_h^n = (u_h^n)^+, \qquad \hat{q}_h^n = (q_h^n)^-.$$

Following to [3,4], $\hat{u}_h^n$ and $\hat{q}_h^n$ must be taken from opposite sides. For convenience and without loss of generality, we consider the case $a(x) = 1$ and $c(x) = 0$. Using the first equation in (1.1) and equation (1.2), we have

$$f(x) = \frac{1}{p^M} \left(D_t^\alpha u \Big|_{t=T} - g''(x) \right). \tag{2.3}$$

Lemma 2.1 *If u_t for all $t \in [0,T]$ is bounded and g is sufficiently smooth with bounded derivatives, then f will be bounded.*

Proof At first, if $|u_t| \le c$ then we have

$$\left| D_t^\alpha u(\cdot,T) \right| = \left| \frac{1}{\Gamma(1-\alpha)} \int_0^T \frac{\partial u(\cdot,s)}{\partial s} \frac{ds}{(T-s)^\alpha} \right| \le \frac{c}{\Gamma(1-\alpha)} \left| \int_0^T \frac{ds}{(T-s)^\alpha} \right| \tag{2.4}$$
$$= \frac{c\,T^{(1-\alpha)}}{\Gamma(1-\alpha)} = C_1,$$

where C_1 is a constant that depends on u, α and T. From Theorem 1.1 and Eq. (2.4)

$$|f(\cdot)| = \left| \frac{1}{p^M} \left(D_t^\alpha u(\cdot,T) - g''(\cdot) \right) \right| \le \frac{1}{p_0} | \left(C_1 - g''(\cdot) \right) | \le C_2, \tag{2.5}$$

where C_2 is a constant depending on u, g, α and T.

$\square$

Stability of the scheme (2.2) is given in the following result.

Theorem 2.1 *If u_t for all $t \in [0,T]$ is bounded and g is sufficiently smooth with bounded derivatives, then for periodic or compactly supported boundary conditions, the fully-discrete LDG scheme (2.2) is unconditionally stable, and*

$$\|u_h^n\| \le \|u_h^0\| + \kappa, \qquad n = 1, \ldots, M, \tag{2.6}$$

where κ is a constant depending on f.

Proof We prove this theorem using induction. When $n = 1$, (2.2) is

$$\int_\Omega u_h^1 v dx + \beta \left(\int_\Omega q_h^1 v_x dx - \sum_{j=1}^N ((\hat{q}_h^1 v^-)_{j+\frac{1}{2}} - (\hat{q}_h^1 v^+)_{j-\frac{1}{2}}) \right) + \int_\Omega q_h^1 w dx + \int_\Omega u_h^1 w_x dx$$
$$- \sum_{j=1}^N ((\hat{u}_h^1 w^-)_{j+\frac{1}{2}} - (\hat{u}_h^1 w^+)_{j-\frac{1}{2}}) = \beta p^1 \int_\Omega f_h v dx + \int_\Omega u_h^0 v dx.$$

Taking test functions $v = u_h^1$ and $w = \beta q_h^1$, for periodic or compactly supported boundary conditions, we obtain

$$\beta \left(\int_\Omega q_h^1 v_x dx - \sum_{j=1}^N ((\hat{q}_h^1 v^-)_{j+\frac{1}{2}} - (\hat{q}_h^1 v^+)_{j-\frac{1}{2}}) \right) - \sum_{j=1}^N ((\hat{u}_h^1 w^-)_{j+\frac{1}{2}} - (\hat{u}_h^1 w^+)_{j-\frac{1}{2}})$$

$$+ \int_\Omega u_h^1 w_x dx = \beta \int_\Omega q_h^1 (u_h^1)_x dx - \beta \sum_{j=1}^N \left(((q_h^1)^+ (u_h^1)^-)_{j+\frac{1}{2}} - ((q_h^1)^+ (u_h^1)^+)_{j-\frac{1}{2}} \right)$$

$$+ \beta \int_\Omega u_h^1 (q_h^1)_x dx - \beta \sum_{j=1}^N \left(((u_h^1)^- (q_h^1)^-)_{j+\frac{1}{2}} - ((u_h^1)^- (q_h^1)^+)_{j-\frac{1}{2}} \right) = 0.$$

Then by using (2.3) and Lemma 2.1 we get

$$\| u_h^1 \|^2 + \beta \| q_h^1 \|^2 = \int_\Omega u_h^0 u_h^1 dx + \beta p^1 \int_\Omega f_h u_h^1 dx = \int_\Omega \left(u_h^0 + \beta p^1 f_h \right) u_h^1 dx$$

$$\leq \frac{1}{2} \left(\left\| u_h^0 + \beta p^1 f_h \right\|^2 + \| u_h^1 \|^2 \right)$$

$$\leq \frac{1}{2} \left(\left(\| u_h^0 \| + \beta p^1 \| f_h \| \right)^2 + \| u_h^1 \|^2 \right)$$

$$\leq \frac{1}{2} \left(\left(\| u_h^0 \| + \kappa \right)^2 + \| u_h^1 \|^2 \right),$$

Therefore

$$\| u_h^1 \| \leq \| u_h^0 \| + \kappa. \tag{2.7}$$

Now for the following inequality to hold

$$\| u_h^m \| \leq \| u_h^0 \| + \kappa, \qquad m = 1, 2, \cdots, l, \tag{2.8}$$

we must prove that $\| u_h^{l+1} \| \leq \| u_h^0 \| + \kappa$. Let $n = l+1$ and take the test functions $v = u_h^{l+1}$ and $w = \beta u_h^{l+1}$ in scheme (2.2), and using

$$\sum_{i=1}^l (b_{i-1} - b_i) + b_l = 1,$$

to obtain

$$\| u_h^{l+1} \|^2 + \beta \| q_h^{l+1} \|^2 = b_l \int_\Omega u_h^0 u_h^{l+1} dx + \beta p^{l+1} \int_\Omega f_h u_h^{l+1} dx$$

$$+ \sum_{i=1}^l (b_{i-1} - b_i) \int_\Omega u_h^{l+1-i} u_h^{l+1} dx = b_l \int_\Omega \left(u_h^0 + \frac{\beta p^{l+1}}{b_l} f_h \right) u_h^{l+1} dx$$

$$+ \sum_{i=1}^l (b_{i-1} - b_i) \int_\Omega u_h^{l+1-i} u_h^{l+1} dx \leq b_l \left(\| u_h^0 \| + \kappa \right) \| u_h^{l+1} \|$$

$$+ \sum_{i=1}^l (b_{i-1} - b_i) \left(\| u_h^0 \| + \kappa \right) \| u_h^{l+1} \| = \left(\| u_h^0 \| + \kappa \right) \| u_h^{l+1} \|$$

$$\leq \frac{1}{2} \left(\left(\| u_h^0 \| + \kappa \right)^2 + \| u_h^{l+1} \|^2 \right),$$

and then

$$\| u_h^{l+1} \| \leq \| u_h^0 \| + \kappa.$$

□

Convergence of the scheme is given as.

Theorem 2.2 *Let $u(\cdot, t_n)$ be the exact solution of (1.1), u_h^n be the numerical solution of the fully discrete LDG scheme (2.2), g be sufficiently smooth with bounded derivatives and u_t for all $t \in [0, T]$ be bounded. Then there holds the following error estimate*

$$\|u(\cdot, t_n) - u_h^n\| \leq C(h^{k+1} + (\Delta t)^2 + (\Delta t)^{\frac{\alpha}{2}} h^{k+\frac{1}{2}}) + \tilde{C}(\Delta t)^{\frac{\alpha}{2}} h^{k+1},$$

where C is a constant depending on u, α, T and $\tilde{C}$ on f, p.

Proof We denote

$$
\begin{aligned}
e_u^n &= u(x, t_n) - u_h^n = \mathbf{P}^- e_u^n - (\mathbf{P}^- u(x, t_n) - u(x, t_n)), \\
e_q^n &= q(x, t_n) - q_h^n = \mathbf{P} e_q^n - (\mathbf{P} q(x, t_n) - q(x, t_n)).
\end{aligned}
\tag{2.9}
$$

It is easy to verify that the exact solution of (2.2) satisfies

$$
\begin{aligned}
\int_\Omega u(x, t_n) v \, dx &+ \beta \left(\int_\Omega q(x, t_n) v_x \, dx - \sum_{j=1}^N ((q(x, t_n) v^-)_{j+\frac{1}{2}} - (q(x, t_n) v^+)_{j-\frac{1}{2}}) \right) \\
&+ \int_\Omega u(x, t_n) w_x \, dx - \sum_{j=1}^N ((u(x, t_n) w^-)_{j+\frac{1}{2}} - (u(x, t_n) w^+)_{j-\frac{1}{2}}) \\
&- \sum_{i=1}^{n-1} (b_{i-1} - b_i) \int_\Omega u(x, t_{n-i}) v \, dx - b_{n-1} \int_\Omega u(x, t_0) v \, dx + \int_\Omega q(x, t_n) w \, dx \\
&+ \beta \int_\Omega \gamma^n(x) v \, dx - \beta p^n \int_\Omega f(x) v \, dx = 0.
\end{aligned}
\tag{2.10}
$$

Subtracting equation (2.2) from (2.10), we obtain the error equation

$$
\begin{aligned}
\int_\Omega e_u^n v \, dx &+ \beta \left(\int_\Omega e_q^n v_x \, dx - \sum_{j=1}^N (((e_q^n)^+ v^-)_{j+\frac{1}{2}} - ((e_q^n)^+ v^+)_{j-\frac{1}{2}}) \right) + \int_\Omega e_q^n w \, dx \\
&+ \int_\Omega e_u^n w_x \, dx - \sum_{j=1}^N (((e_u^n)^- w^-)_{j+\frac{1}{2}} - ((e_u^n)^- w^+)_{j-\frac{1}{2}}) - b_{n-1} \int_\Omega e_u^0 v \, dx \\
&- \sum_{i=1}^{n-1} (b_{i-1} - b_i) \int_\Omega e_u^{n-i} v \, dx + \beta \int_\Omega \gamma^n(x) v \, dx - \beta p^n \int_\Omega (f(x) - f_h) v \, dx = 0,
\end{aligned}
$$

Using equation (2.9), the error equation (2.10) can be written

$$
\int_\Omega \mathbf{P}^- e_u^n v dx + \beta \left(\int_\Omega \mathbf{P} e_q^n v_x dx - \sum_{j=1}^N \left(((\mathbf{P} e_q^n)^+ v^-)_{j+\frac{1}{2}} - ((\mathbf{P} e_q^n)^+ v^+)_{j-\frac{1}{2}} \right) \right)
$$

$$
+ \int_\Omega \mathbf{P} e_q^n w dx + \int_\Omega \mathbf{P}^- e_u^n w_x dx - \sum_{j=1}^N \left(((\mathbf{P}^- e_u^n)^- w^-)_{j+\frac{1}{2}} - ((\mathbf{P}^- e_u^n)^- w^+)_{j-\frac{1}{2}} \right)
$$

$$
- \beta p^n \int_\Omega (f(x) - f_h) v dx = b_{n-1} \int_\Omega \mathbf{P}^- e_u^0 v dx + \sum_{i=1}^{n-1} (b_{i-1} - b_i) \int_\Omega \mathbf{P}^- e_u^{n-i} v dx
$$

$$
- \beta \int_\Omega \gamma^n(x) v dx + \int_\Omega (\mathbf{P}^- u(x,t_n) - u(x,t_n)) v dx + \beta \left(\int_\Omega (\mathbf{P} q(x,t_n) - q(x,t_n)) v_x dx \right.
$$

$$
- \sum_{j=1}^N \left(((\mathbf{P} q(x,t_n) - q(x,t_n))^+ v^-)_{j+\frac{1}{2}} - ((\mathbf{P} q(x,t_n) - q(x,t_n))^+ v^+)_{j-\frac{1}{2}} \right) \Big)
$$

$$
+ \int_\Omega (\mathbf{P} q(x,t_n) - q(x,t_n)) w dx + \int_\Omega (\mathbf{P}^- u(x,t_n) - u(x,t_n)) w_x dx
$$

$$
- \sum_{j=1}^N \left(((\mathbf{P}^- u(x,t_n) - u(x,t_n))^- w^-)_{j+\frac{1}{2}} - ((\mathbf{P}^- u(x,t_n) - u(x,t_n))^- w^+)_{j-\frac{1}{2}} \right)
$$

$$
- b_{n-1} \int_\Omega (\mathbf{P}^- u(x,t_0) - u(x,t_0)) v dx - \sum_{i=1}^{n-1} (b_{i-1} - b_i) \int_\Omega (\mathbf{P}^- u(x,t_{n-i}) - u(x,t_{n-i})) v dx.
$$

$$(2.11)$$

Taking test functions $v = u_h^n$ and $w = \beta q_h^n$, for periodic or compactly supported boundary conditions, we obtain

$$
\int_\Omega (\mathbf{P}^- e_u^n)^2 dx + \int_\Omega (\mathbf{P} e_q^n)^2 dx - \beta p^n \int_\Omega (f(x) - f_h) \mathbf{P}^- e_u^n dx
$$

$$
= b_{n-1} \int_\Omega \mathbf{P}^- e_u^0 \mathbf{P}^- e_u^n dx + \sum_{i=1}^{n-1} (b_{i-1} - b_i) \int_\Omega \mathbf{P}^- e_u^{n-i} \mathbf{P}^- e_u^n dx
$$

$$
- \beta \int_\Omega \gamma^n(x) \mathbf{P}^- e_u^n dx + \beta \sum_{j=1}^N \left(((\mathbf{P} q(x,t_n) - q(x,t_n))^+ \left[\mathbf{P}^- e_u^n \right])_{j-\frac{1}{2}} \right.
$$

$$
+ \int_\Omega (\mathbf{P}^- u(x,t_n) - u(x,t_n)) \mathbf{P}^- e_u^n dx - b_{n-1} \int_\Omega (\mathbf{P}^- u(x,t_0) - u(x,t_0)) \mathbf{P}^- e_u^n dx
$$

$$
- \sum_{i=1}^{n-1} (b_{i-1} - b_i) \int_\Omega (\mathbf{P}^- u(x,t_{n-i}) - u(x,t_{n-i})) \mathbf{P}^- e_u^n dx.
$$

$$(2.12)$$

For $n = 1$, we have

$$
\int_\Omega (\mathbf{P}^- e_u^1)^2 dx + \beta \int_\Omega (\mathbf{P} e_q^1)^2 dx - \beta p^1 \int_\Omega (f(x) - f_h) \mathbf{P}^- e_u^1 dx = \int_\Omega \mathbf{P}^- e_u^0 \mathbf{P}^- e_u^1 dx
$$

$$
- \beta \int_\Omega \gamma^1(x) \mathbf{P}^- e_u^1 dx + \beta \sum_{j=1}^N \left(((\mathbf{P} q(x,t_1) - q(x,t_1))^+ \left[\mathbf{P}^- e_u^1 \right])_{j-\frac{1}{2}} \right.
$$

$$
+ \int_\Omega (\mathbf{P}^- u(x,t_1) - u(x,t_1)) \mathbf{P}^- e_u^1 dx - \int_\Omega (\mathbf{P}^- u(x,t_0) - u(x,t_0)) \mathbf{P}^- e_u^1 dx.
$$

Recalling that

$$
\| P^- e_u^0 \| \leq C h^{k+1}, \quad \| P^- e_u^1 \| \leq C h^{k+1}, \quad \| f(x) - f_h \| \leq c h^{k+1} \| f \|, \quad ab \leq \varepsilon a^2 + \frac{1}{4\varepsilon} b^2,
$$

we can write

$$
\begin{aligned}
\| \mathbf{P}^- e_u^1 \|^2 + \beta \| \mathbf{P} e_q^1 \|^2 \leq {} & (\| \mathbf{P}^- e_u^0 \| + \beta \| \gamma^1(x) \| + \| \mathbf{P}^- u(x,t_1) - u(x,t_1) \| \\
& + \| \mathbf{P}^- u(x,t_0) - u(x,t_0) \|) \| \mathbf{P}^- e_u^1 \| + \beta p^1 \| f(x) - f_h \| \| \| \mathbf{P}^- e_u^1 \| \| \| \\
& + \frac{\beta}{4} \sum_{j=1}^N \left((\mathbf{P} q(x,t_1) - q(x,t_1))^+ \right)^2_{j-\frac{1}{2}} + \varepsilon \beta \sum_{j=1}^N \left[\mathbf{P}^- e_u^1 \right]_{j-\frac{1}{2}} \\
\leq {} & C(h^{k+1} + (\Delta t)^2 + (\Delta t)^{\frac{\alpha}{2}} h^{k+\frac{1}{2}})^2 + \varepsilon \| \mathbf{P}^- e_u^1 \|^2 \\
& + c(\Delta t)^\alpha p^1 h^{2k+2} \| f \| + \varepsilon \beta \sum_{j=1}^N \left[\mathbf{P}^- e_u^1 \right]_{j-\frac{1}{2}}.
\end{aligned}
$$

If we choose ε very small and use Lemma 2.1, we obtain

$$
\| \mathbf{P}^- e_u^1 \|^2 + \beta \| \mathbf{P} e_q^1 \|^2 \leq C(h^{k+1} + (\Delta t)^2 + (\Delta t)^{\frac{\alpha}{2}} h^{k+\frac{1}{2}})^2 + \tilde{C}(\Delta t)^\alpha h^{2k+2},
$$

where $\tilde{C}$ is a constant independent of h, and

$$
\| \mathbf{P}^- e_u^m \|_\Omega \leq C(h^{k+1} + (\Delta t)^2 + (\Delta t)^\alpha h^{k+\frac{1}{2}}) + \tilde{C}(\Delta t)^{\frac{\alpha}{2}} h^{k+1}, \quad m = 1, 2, \ldots, K.
$$

Using Eq. (2.12) and choosing a small enough ε, we obtain

$$
\| \mathbf{P}^- e_u^{K+1} \| \leq C(h^{k+1} + (\Delta t)^2 + (\Delta t)^{\frac{\alpha}{2}} h^{k+\frac{1}{2}}) + \tilde{C}(\Delta t)^{\frac{\alpha}{2}} h^{k+1}.
$$

$\square$

3 Implementation details

Here, we aim to describe our method somewhat more. We set

$$
\mathbb{P}^k(I_j) = \mathrm{Span}\{\phi_1, \ldots, \phi_k\},
$$

with

$$
\phi_1(x) = 1, \quad \phi_2(x) = \left(2 \frac{x - \bar{x}_j}{\Delta x_j} \right), \quad \ldots, \quad \phi_k(x) = \left(2 \frac{x - \bar{x}_j}{\Delta x_j} \right)^k,
$$

and $\bar{x}_j = \frac{1}{2}(x_{j+\frac{1}{2}} + x_{j-\frac{1}{2}})$ is the midpoint of the interval I_j. The global basis functions $\Phi_1, \ldots, \Phi_{N_p}$ for the space V_h^k are obtained from the local basis functions by extending them by zero, i.e.

$$
\Phi_{(j-1)k+i}(x) = \begin{cases} \phi_i(x), & x \in I_j, \\ 0, & x \notin I_j, \end{cases}
$$

for $i = 1, \ldots, k$ and $j = 1, \ldots, N$.

We set

$$
u_h^n(x) = u_h(x, n\Delta t) = \sum_{i=1}^{N_p} \delta_i^n \Phi_i(x), \qquad q_h^n(x) = q_h(x, n\Delta t) = \sum_{i=1}^{N_p} \gamma_i^n \Phi_i(x),
$$

where N_p is the total number of basis functions and

$$
F = \left(\int_\Omega f_h(x) \Phi_1(x) dx \quad \cdots \quad \int_\Omega f_h(x) \Phi_{N_p}(x) dx \right)^T,
$$

$$G = \left(\int_{\Omega} g(x)\Phi_1(x)dx \cdots \int_{\Omega} g(x)\Phi_{N_p}(x)dx \right)^T.$$

Setting $\delta^n = (\delta_1^n \cdots \delta_{N_p}^n)^T$ and $\gamma^n = (\gamma_1^n \cdots \gamma_{N_p}^n)^T$, scheme (2.2) leads to the following iteration scheme

$$\begin{cases} K_{11}\delta^n + K_{12}\gamma^n = \beta p^n F + \sum_{i=1}^{n-1}(b_{i-1} - b_i)K_{22}\delta^{n-i}, \\ K_{21}\delta^n + K_{22}\gamma^n = 0, \\ K_{22}\delta^M = G, \end{cases} \tag{3.1}$$

where $n = 1,\ldots,M$ and

$$(K_{11})_{lr} = \int_{\Omega} \Phi_l(x)\Phi_r(x)dx - \beta \int_{\Omega} c(x)\Phi_l(x)\Phi_r(x)dx,$$

$$(K_{12})_{lr} = \beta \int_{\Omega} a(x)\Phi_l(x)(\Phi_r(x))_x dx \\ - \beta \sum_{j=1}^{N} \left(a(x_{j+\frac{1}{2}}^-)\Phi_l(x_{j+\frac{1}{2}}^-)\Phi_r(x_{j+\frac{1}{2}}^-) - a(x_{j-\frac{1}{2}}^-)\Phi_l(x_{j-\frac{1}{2}}^-)\Phi_r(x_{j-\frac{1}{2}}^+) \right),$$

$$(K_{21})_{lr} = \int_{\Omega} \Phi_l(x)(\Phi_r(x))_x dx - \sum_{j=1}^{N} \left(\Phi_l(x_{j+\frac{1}{2}}^+)\Phi_r(x_{j+\frac{1}{2}}^-) - \Phi_l(x_{j-\frac{1}{2}}^+)\Phi_r(x_{j-\frac{1}{2}}^-) \right),$$

$$(K_{22})_{lr} = \int_{\Omega} \Phi_l(x)\Phi_r(x)dx.$$

For solving the direct problem, we have

$$M\begin{pmatrix} \delta^n \\ \gamma^n \end{pmatrix} = \begin{pmatrix} 0 \\ \beta p^n F + \sum_{i=1}^{n-1}(b_{i-1} - b_i)K_{22}\delta^{n-i} \end{pmatrix}, \quad M = \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix}, \tag{3.2}$$

where matrix K_{22} is nonsingular and block diagonal which every block is a $k \times k$ (k is degree of basis polynomials) matrix and if $1 - \beta c(x) \geq 0$ then K_{11} is positive definite and clearly nonsingular.

For solving the inverse problem, using the first and second equations in (3.1), we have

$$K\delta^n = \beta p^n F + \sum_{i=1}^{n-1}(b_{i-1} - b_i)K_{22}\delta^{n-i},$$

where $K = K_{11} - K_{12}K_{22}^{-1}K_{21}$. Provided M is invertible, the positive definiteness of K_{11} and K_{22} guarantees that K is invertible [1]. Furthermore,

$$\begin{cases} \delta^1 = K^{-1}\beta p^1 F = A_1 F, & n = 1, \\ \delta^2 = K^{-1}(\beta p^2 F + (b_0 - b_1)K_{22}\alpha^1) = A_2 F, & n = 2, \\ \vdots & \\ \delta^{M-1} = A_{M-1}F, & n = M-1. \end{cases} \tag{3.3}$$

For $n = M$, using numerical fluxes $\hat{u}_h^M = g(x)$ and $\hat{q}_h^M = g'(x)$, by defining the following vectors

$$(G_1)_l = \sum_{j=1}^{N} (g(x_{j+\frac{1}{2}}^-)\Phi_l(x_{j+\frac{1}{2}}^-) - g(x_{j-\frac{1}{2}}^-)\Phi_l(x_{j-\frac{1}{2}}^+)),$$

$$(G_2)_l = \sum_{j=1}^{N} (g'(x_{j+\frac{1}{2}}^+)\Phi_l(x_{j+\frac{1}{2}}^-) - g'(x_{j-\frac{1}{2}}^+)\Phi_l(x_{j-\frac{1}{2}}^+)),$$

we obtain

$$\begin{cases} K_{11}\delta^M + \beta\bar{K}_{12}\gamma^M - \beta G_1 = \beta p^M F + \sum_{i=1}^{M-1}(b_{i-1} - b_i)K_1\delta^{M-i}, \\ \bar{K}_{21}\delta^M + K_{22}\gamma^M - G_2 = 0, \\ K_{22}\delta^M = G, \end{cases} \qquad (3.4)$$

where

$$(\bar{K}_{12})_{lr} = \int_\Omega a(x)\Phi_l(x)(\Phi_r(x))_x dx, \quad (\bar{K}_{21})_{lr} = \int_\Omega \Phi_l(x)(\Phi_r(x))_x dx.$$

Finally, we have

$$A_M F = K_M G - \beta G_1 + \beta\bar{K}_{12}K_{22}^{-1}G_2, \quad K_M = (K_{11} - \beta\bar{K}_{12}K_{22}^{-1}\bar{K}_{21})K_{22}^{-1}. \qquad (3.5)$$

Solving the above linear system, we obtain F and by applying a suitable integration rule we recover f_h as an approximation of the source term. Through (3.3), we find the coefficients of u_h at each time step and obtain

$$u_h^n(x) = u_h(x, n\Delta t) = \sum_{i=1}^{N_p} \delta_i^n \Phi_i(x).$$

4 Numerical examples

In this section, we present some numerical results to investigate the effectiveness and stability of the proposed method. The maximum time is $T = 1$ unless otherwise specified. The space and time step is $h = 1/N$ and $\Delta t = 1/M$, respectively. For noisy data, we use

$$g^\delta(x_i) = g(x_i) + \varepsilon g(x_i)(2\text{rand}(i) - 1),$$

where $g(x_i)$ is the exact data, rand(i) is an uniformly distributed random number in $[0, 1]$ and the magnitude ε indicates a relative noise level. The corresponding noise level is calculated $\delta = \|g^\delta - g\|$. To measure the accuracy of the numerical solutions, we compute the approximate L^2 error as

$$e(f, \varepsilon) = \|f - f^\delta\|,$$

and the approximate relative error in the L^2-norm as

$$e_r(f, \varepsilon) = \|f - f^\delta\|/\|f\|.$$

Table 4.1 Accuracy test for Example 4.1 for different α

	n	L^2 error	CO	L^∞ error	CO
$\alpha = 0.3$	5	0.005563	-	0.007862	-
	10	0.000722	2.945792363366441	0.000976	3.009943310861208
	15	0.000215	2.987648225558304	0.000305	2.868682869489701
	20	0.000091	2.988641295174632	0.000127	3.045461549557176
$\alpha = 0.5$	5	0.005390	-	0.007618	-
	10	0.000717	2.910240248937963	0.000969	2.974843717148259
	15	0.000215	2.970509147116684	0.000304	2.859030006042589
	20	0.000091	2.988641295174632	0.000127	3.034045908765933
$\alpha = 0.7$	5	0.004832	-	0.006828	-
	10	0.000697	2.793389893456656	0.000942	2.857664093414955
	15	0.000212	2.935392250252534	0.000300	2.822000653173150
	20	0.000090	2.978206452142942	0.000216	3.015483586833959
$\alpha = 1$	10	0.000574	-	0.000777	-
	20	0.000085	2.755515990537986	0.000120	2.694880192799192
	30	0.000026	2.921471403534026	0.000037	2.901788111010409
	40	0.000011	2.990110777122867	0.000016	2.914082143735104

To verify the convergence rate, we use the following definition

$$CO = \log_2 \frac{e(f, 2\varepsilon)}{e(f, \varepsilon)}.$$

Example 4.1 Consider the inverse source problem of the time-fractional diffusion equation (1.1) with the exact solution $u(x,t) = t^2 \sin(2\pi x)$ and $f(x) = \sin(2\pi x), p(t) = (2\pi t)^2 + \frac{2t^{2-\alpha}}{\Gamma(3-\alpha)}$. We take piecewise P^2 polynomials as the basis functions. Setting Δt very small, we show in Table 4.1 that the order of convergence of the proposed method for several α is about three as we expected. In Fig. 1, we show the errors in L^2-norm and L^∞-norm for piecewise $P^k, k = 1, 2, 3$ polynomials for $\alpha = 0.5$.

In order to investigate the impact of the "inverse crime", we consider the following two cases for solving the proposed problem. We consider $k = 1$ for both cases.
Case 1: We use $g(x) = \sin(2\pi x)$ and solve the inverse problem using the proposed LDG method. Errors of the method are shown in Table 4.2.
Case 2: We first solve a direct problem using an LDG method to obtain the input data g and solve the inverse problem using the proposed method. We use a finer grid to solve the forward problem, i.e. we take $N = 100$ (the grid points for the space interval) and $M = 200$ (the grid points for the time interval) and choose $N = 50$ and $M = 200$ for solving the inverse problem. Errors of the method are shown in Table 4.3.
It can be seen that for both cases the numerical solutions obtained by the LDG method are not sensitive with respect to the perturbation in the initial data and we do not observe the impact of the "inverse crime".

Example 4.2 Consider the inverse source problem of the time-fractional diffusion equation (1.1) with $a(x) = x^2 + 1$ and $c(x) = -(x+1)$. Take a source function $p(t) = e^{-t}$ and $f(x) = (x(1-x))^\alpha \sin(5\pi x)$. Since the exact solution of this problem is not
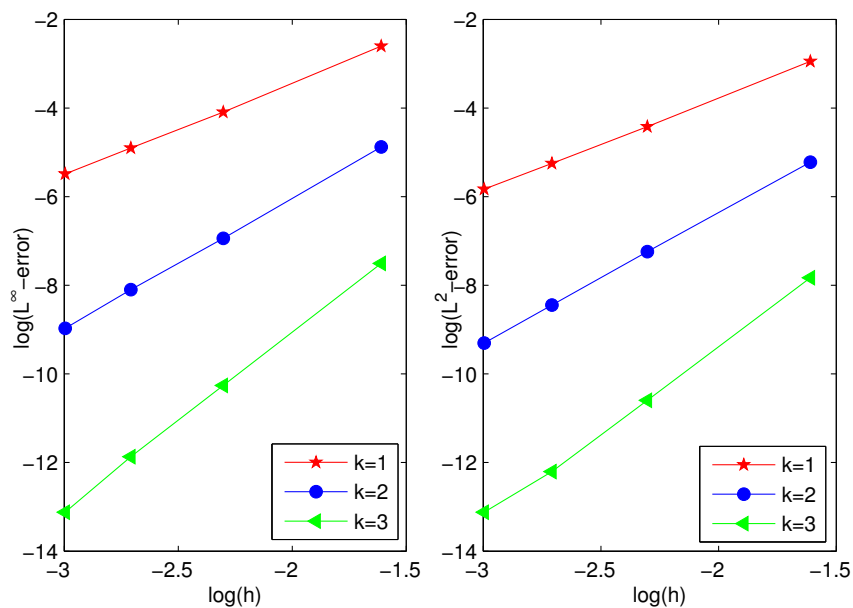
Fig. 4.1 log(error) as a function of log(h) for $\alpha = 0.5$ when using piecewise $P^k, k = 1, 2, 3$ polynomials for Example 4.1

Table 4.2 Errors of the LDG method (case 1) for example 1 for different α with $\varepsilon = 0.01$ (left) and for different ε with $\alpha = 0.6$ (right).

α	$e(f, 0.01)$	$e_r(f, 0.01)$
0.05	1.00×10^{-5}	9.88×10^{-6}
0.1	1.60×10^{-5}	1.59×10^{-5}
0.3	8.00×10^{-5}	7.96×10^{-5}
0.7	8.22×10^{-4}	8.22×10^{-4}
0.9	2.27×10^{-3}	2.26×10^{-3}
0.95	2.90×10^{-3}	2.90×10^{-3}

ε	$e(f, \varepsilon)$	$e_r(f, \varepsilon)$
0.0005	0.000471	0.000470948
0.001	0.000471	0.000471018
0.002	0.000471	0.000471459
0.004	0.000472	0.000472316
0.008	0.000479	0.000479103
0.016	0.000491	0.000491375
0.032	0.000556	0.000555792
0.064	0.000770	0.000770034

Table 4.3 Errors of the LDG method (case 2) for example 1 for different α with $\varepsilon = 0.01$ (left) and for different ε with $\alpha = 0.6$ (right).

α	$e(f, 0.01)$	$e_r(f, 0.01)$
0.05	5.20×10^{-5}	5.18×10^{-5}
0.1	1.26×10^{-4}	1.26×10^{-4}
0.3	7.66×10^{-4}	7.66×10^{-4}
0.7	6.55×10^{-3}	6.55×10^{-3}
0.9	1.42×10^{-2}	1.42×10^{-2}
0.95	1.67×10^{-2}	1.67×10^{-2}

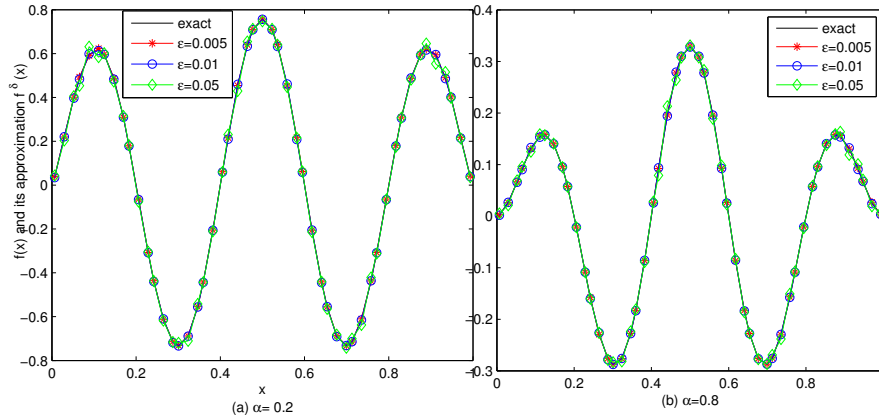| ε	$e(f, \varepsilon)$	$e_r(f, \varepsilon)$
0.0005	0.004143	0.004142571
0.001	0.004142	0.004142483
0.002	0.004142	0.004141573
0.004	0.004141	0.004141178
0.008	0.004146	0.004146441
0.016	0.004153	0.004152914
0.032	0.004166	0.004166003
0.064	0.004209	0.004208976

Fig. 4.2 The exact and approximate f for Example 4.2

Table 4.4 Numerical results of Example 4.2 for different α with $\varepsilon = 0.01$

α	$e(f, 0.01)$ of [27]	$e(f, 0.01)$	$e_r(f, 0.01)$ of [27]	$e_r(f, 0.01)$
0.05	0.0198	0.005997	0.0308	0.005381
0.1	0.0180	0.003882	0.0306	0.003822
0.3	0.0132	0.003702	0.0323	0.005222
0.7	0.0079	0.002768	0.0377	0.007661
0.9	0.0060	0.001415	0.0393	0.005400
0.95	0.0055	0.001237	0.0396	0.005110

Table 4.5 Numerical results of Example 4.2 for different ε with $\alpha = 0.6$

ε	$e(f, \varepsilon)$ of [27]	$e(f, \varepsilon)$	$e_r(f, \varepsilon)$ of [27]	$e_r(f, \varepsilon)$	Order of [27]	Order
0.0005	0.0024	0.000121	0.0096	0.000285	-	-
0.001	0.0031	0.000249	0.0128	0.000585	0.4	1.0
0.002	0.0043	0.000657	0.0176	0.001544	0.5	1.4
0.004	0.0060	0.000974	0.0243	0.002288	0.5	0.6
0.008	0.0082	0.001943	0.0332	0.004564	0.4	1.0
0.016	0.0110	0.004485	0.0448	0.010533	0.4	1.2
0.032	0.0146	0.010353	0.0606	0.024314	0.4	1.2
0.064	0.0207	0.020734	0.0841	0.048695	0.5	1.0

accessible, we first solve a direct problem using a suitable LDG method to obtain the input data g and solve the inverse problem. The numerical results for various noise levels $\varepsilon = 0.005, 0.01, 0.05$ for $\alpha = 0.2, 0.8$ are shown in Fig. 2. In Table 4.4, we compare the numerical errors for the method proposed in [27] with the proposed LDG method for different α with a fixed $\varepsilon = 0.01$. In Table 4.5, we compare the numerical errors and convergence orders for the proposed method in [27] with the proposed LDG method for different ε with a fixed $\alpha = 0.6$. Without applying any regularization methods, our results are better than reported in [27].
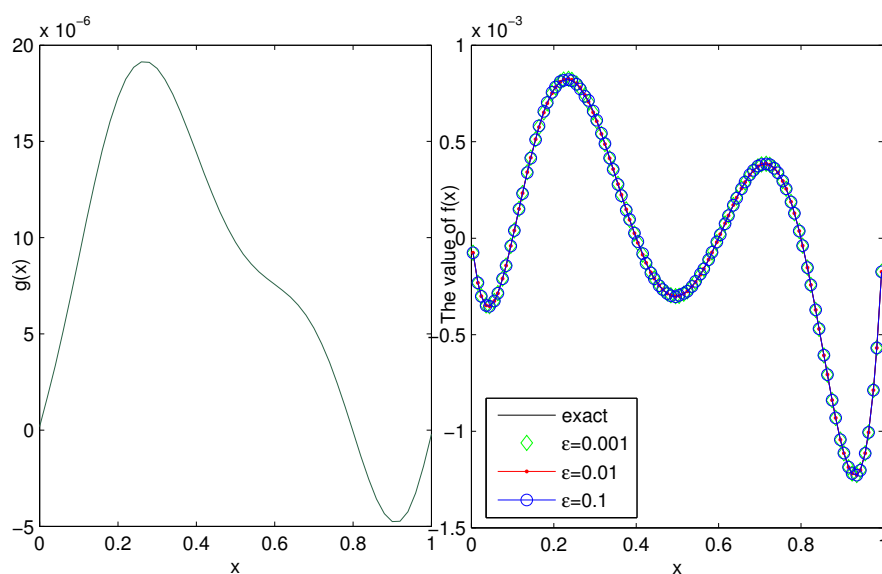
Fig. 4.3 Results of Example 4.3 with $\alpha = 0.95$

Example 4.3 Take $f(x) = x(x-0.1)(x-0.4)(x-0.6)(x-0.8)(x-1)$. Since the exact solution of this problem is not accessible, we first solve a direct problem using a suitable LDG method to obtain the input data g and solve the inverse problem. Numerical results for $\alpha = 0.95$ with various noise levels $\varepsilon = 0.1\%, 1\%, 10\%$ are presented in Fig. 3 in which $e_r(f, \varepsilon) = 0.000178099, 0.000560097, 0.004863713$. Without applying any regularization methods, our results are in good agreement with the results of [34].

Example 4.4 Consider a continuous piecewise smooth function, namely

$$f(x) = \begin{cases} 2x, & x \in [0, 0.5], \\ -2x+2, & x \in (0.5, 1]. \end{cases}$$

Numerical results for $\alpha = 0.95$ with various noise levels $\varepsilon = 0.1\%, 1\%, 10\%$ are presented in Fig. 4 in which $e_r(f, \varepsilon) = 0.000175441, 0.002526905, 0.019711478$. Without applying any regularization methods, our results are in good agreement with the results of [34].

Example 4.5 This example involves reconstructing a discontinuous function given as

$$f(x) = \begin{cases} 0, & x \in [0, 0.2], \\ 1, & x \in (0.2, 0.4], \\ 0, & x \in (0.4, 0.6], \\ -1, & x \in (0.6, 0.8], \\ 0, & x \in (0.8, 1]. \end{cases}$$
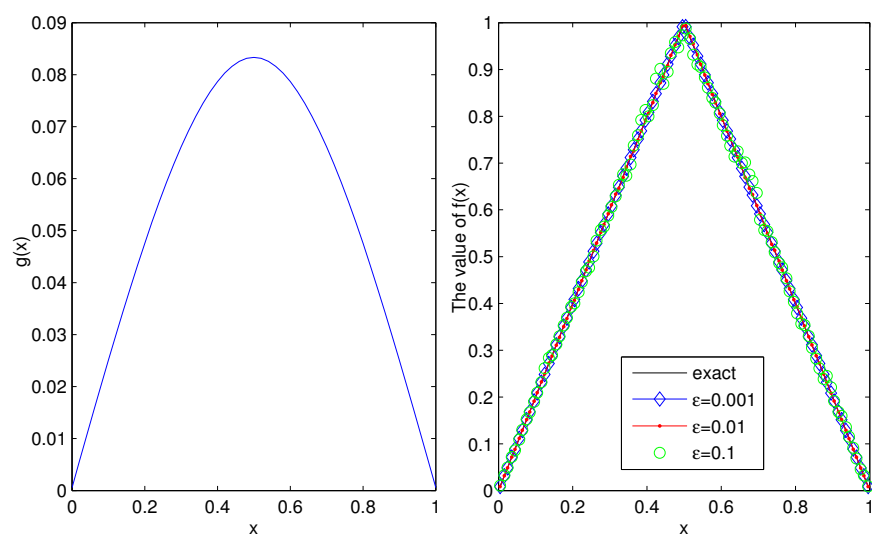
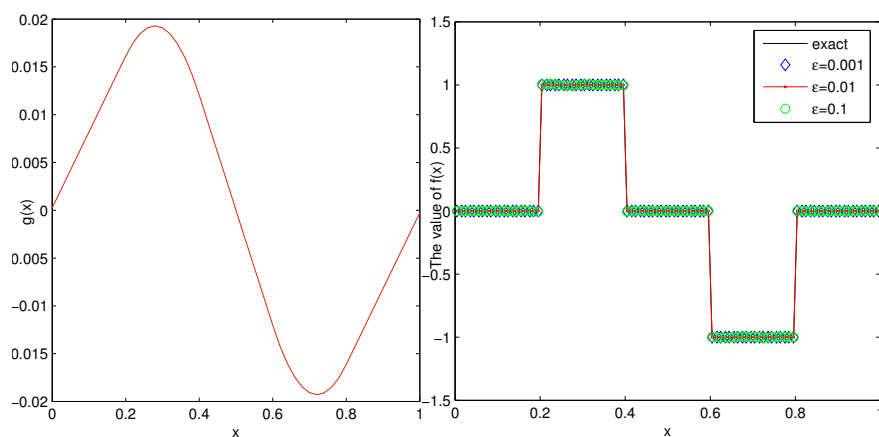**Fig. 4.4** Results of Example 4.4 with $\alpha = 0.95$

Fig. 4.5 Results of Example 4.5 with $\alpha = 0.95$.

Numerical results for $\alpha = 0.95$ with various noise levels $\varepsilon = 0.1\%, 1\%, 10\%$ are presented in Fig. 5 in which $e_r(f, \varepsilon) = 0.000048381, 0.000456007, 0.004768183$. Without applying any regularization methods, our results are in good agreement with the results of [34].

5 Appendix

We seek to explain why our proposed method need not regularization method. Consider the time-fractional diffusion equation (1.1), with $a(x) = 1$ and $c(x) = 0$ (as we consider in examples 4.1 and 4.3-4.5). Let us decompose the domain of the problem into cells of equal length h and choose for local basis functions of $\mathbb{P}^1(I_j)$ the monomial basis functions [20]. Solving (3.4), we have

$$K_{11} = K_{22} = \text{diag}(h, \frac{h}{3}, \ldots, h, \frac{h}{3}), \qquad \bar{K}_{12} = \bar{K}_{21} = \text{diag}(Z, \ldots, Z), \qquad Z = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix},$$

therefore $\bar{K}_{12} K_{22}^{-1} \bar{K}_{21} K_{22}^{-1} = (0)_{2N \times 2N}$, $K_m = I$ and

$$A_M F = G - \beta G_1 + \beta \bar{K}_{12} K_{22}^{-1} G_2. \qquad (5.1)$$

A_M has a complex structure and we are not able to find a closed form for it. In Table 5.1, we report $\| A_M^{-1} \|_2$ for different values of α. Obviously $\| A_M^{-1} \|_2$ has a reasonable size and therefore in examples 4.1 and 4.3-4.5 numerical solutions are not sensitive with respect to the perturbation in the initial data.

Table 5.1 $\| A_M^{-1} \|_2$ for different values of α

α	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$\| A_M^{-1} \|_2$	2.2842	4.6703	8.6683	15.039	24.861	39.783	62.628	98.208	154.09

In table 5.2, we show L^2-norm of matrices G, $A_M^{-1} G$ and A_M^{-1} for example 4.2. Obviously, $\| A_M^{-1} G^\delta \|_2$ is small and $\| A_M^{-1} \|_2$ has a reasonable size. Therefore numerical solutions are not sensitive with respect to the perturbation in the initial data. Therefore, we need not any regularization method.

Table 5.2 Norm 2 of some matrices for example 4.2 for different α with $\varepsilon = 0.01$ (left) and for different ε with $\alpha = 0.6$ (right).

α	$\|G\|_2$	$\|A_M^{-1}G\|_2$	$\|A_M^{-1}\|_2$
0.05	1.9846×10^{-4}	0.40753	5.8379
0.1	1.8721×10^{-4}	0.29199	11.371
0.3	1.4698×10^{-4}	0.07701	64.902
0.7	8.0296×10^{-5}	0.00501	436.65
0.9	5.7313×10^{-5}	0.00129	965.3
0.95	5.2690×10^{-5}	0.00092	1169.8

ε	$\|G\|_2$	$\|A_M^{-1}G\|_2$	$\|A_M^{-1}\|_2$
0.0005	9.4606×10^{-5}	0.0099594	289.24
0.001	9.4606×10^{-5}	0.0099593	289.24
0.002	9.4608×10^{-5}	0.0099596	289.24
0.004	9.4606×10^{-5}	0.0099593	289.24
0.008	9.4604×10^{-5}	0.0099597	289.24
0.016	9.4601×10^{-5}	0.0099590	289.24
0.032	9.4612×10^{-5}	0.0099586	289.24
0.064	9.4609×10^{-5}	0.0099596	289.24

The sequel of this appendix is devoted to investigating the discrete Picard condition [10] in order to show that our proposed method need not any regularization. Here we just investigate Example 4.2. Similar results obtain for Examples 4.3-4.5. We
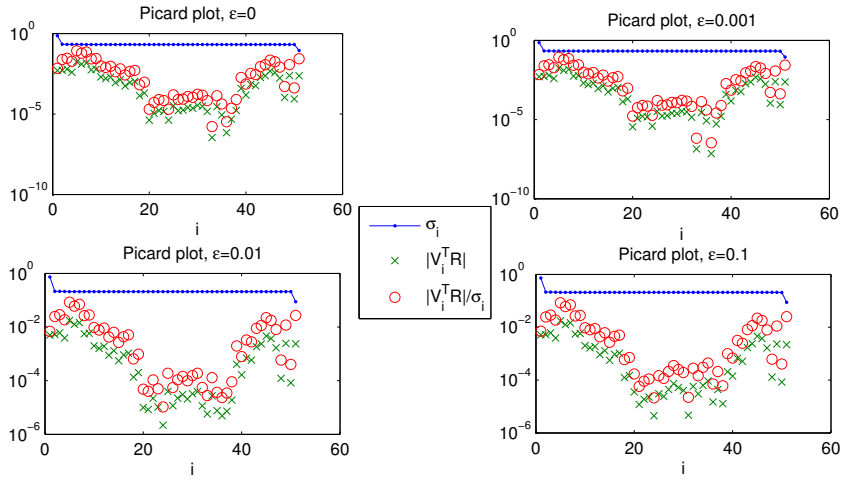
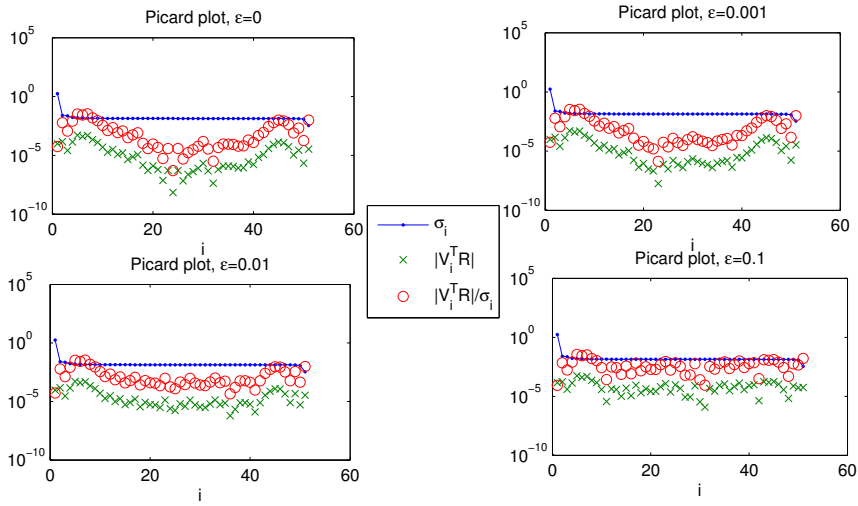**Fig. 5.1** Picard plot for Example 4.2 ($\alpha = 0.1$).

Fig. 5.2 Picard plot for Example 4.2 ($\alpha = 0.6$).

use the MATLAB codes developed by Hansen [11] to prepare Picard plots for both the unperturbed and the perturbed data with various noise levels $\varepsilon = 0.1\%, 1\%, 10\%$ which presented in Figures 5.1-5.3 for $\alpha = 0.1, 0.6, 0.95$, respectively. In all cases, the Fourier coefficients $V_i^T R$ decay to zero faster than the σ_i. Here, (σ_i, V_i)'s are the pair of singular values and corresponding (left) singular vectors of matrix A_M and R is the right-hand side of the linear system which will be solved, i.e. $A_M F = R := K_M G - \beta G_1 + \beta \bar{K}_{12} K_{22}^{-1} G_2$. Therefore, according to the discrete Picard condition, our method need not any regularization.
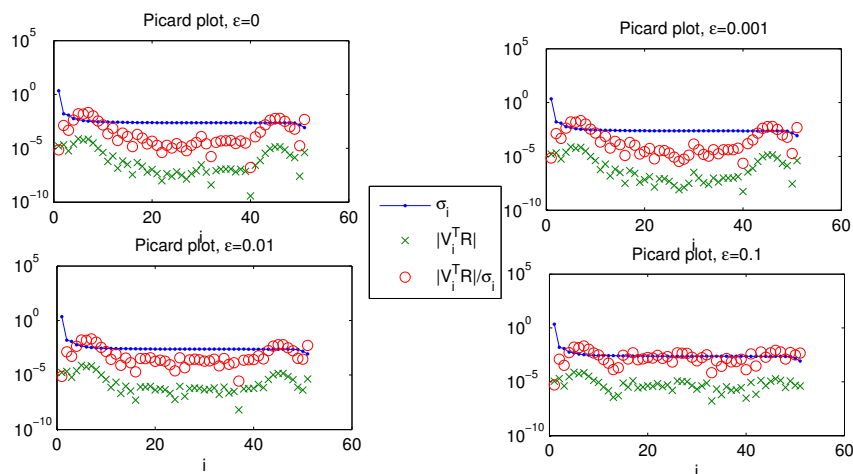
Fig. 5.3 Picard plot for Example 4.2 ($\alpha = 0.95$).

6 Conclusion

In this paper, we have developed a local discontinuous Galerkin finite element method for solving an inverse source problem for the time-fractional diffusion equation. Applying this method without using any regularization, we obtain stable and accurate numerical approximations to the space-dependent source term using an additional condition. The numerical stability and convergence of the method have been investigated and theoretically proven. Several examples with smooth or none-smooth data are given and the numerical results demonstrate the reliability and efficiency of the proposed method.

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