Cocycle growth for the Steinberg representation

THÈSE N^{O} 6599 (2016)

PRÉSENTÉE LE 26 FÉVRIER 2016

À LA FACULTÉ DES SCIENCES DE BASE
CHAIRE DE THÉORIE ERGODIQUE ET GÉOMÉTRIQUE DES GROUPES
PROGRAMME DOCTORAL EN MATHÉMATIQUES

ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

POUR L'OBTENTION DU GRADE DE DOCTEUR ÈS SCIENCES

PAR

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ÉCOLE POLYTECHNIQUE
FÉDÉRALE DE LAUSANNE

Suisse
2016

À ma famille.

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2010 *Mathematics Subject Classification.* — 20J06, 22E41, 20E42.

The author was supported by the Advanced Grant 267635 *Rigidity* of the European Research Council attributed to Prof. Nicolas Monod, and by the Swiss Confederation.

COCYCLE GROWTH FOR THE STEINBERG REPRESENTATION

Thibaut Dumont

Abstract. — This thesis investigates the growth of the natural cocycle introduced by Klingler for the Steinberg representation. When possible, we extend the framework of simple algebraic groups over a local field to arbitrary Euclidean buildings. In rank one, the growth of the cocycle is determined to be sublinear. In higher rank, the complexity of the problem leads us to study the geometry of $\widetilde{A}_2$ buildings, where we describe in detail the relative position of three points.

Key words and phrases. — Group theory, cohomology, continuous cohomology, building, Steinberg representation.

Résumé. — Cette thèse étudie la croissance du cocycle naturel pour le module de Steinberg. Nous étendons les travaux de Klingler dans le cas des groups algébriques simples sur un corps local aux immeubles euclidiens lorsque cela est possible. En rang un, la croissance du cocycle de Klingler est sous-linéaire. En rang supérieur la complexité de la question nous entraine dans l'étude de la géométrie des immeubles $\widetilde{A}_2$, où nous décrivons en détail la position relative de trois points.

Mots clefs. — Théorie des groupes, cohomologie, cohomologie continue, immeuble, representation de Steinberg.

ACKNOWLEDGEMENT

I would like to express my deep gratitude to Prof. Nicolas Monod for taking me as his PhD student and for accompanying me during this journey. Working with him has been a real pleasure full of amazement. My admiration is endless.

We thank the jury members, the president, Prof. Kathryn Hess Bellwald, the internal examiner, Prof. Donna Testerman, and the two external examiners, Dr. Yves de Cornulier and Prof. Bruno Klingler, for accepting to be part of it with the duties it implies.

My thanks go to the various people whom I had the opportunity to discuss the present work with. In particular, I warmly thank Prof. Pierre-Emmanuel Caprace who helped me complete some of the arguments of Section 5.2, Kristian K. Olesen for his help with some LaTeX issues, and Guillaume Blanc for taking five minutes to provide the Python script for the proof of Proposition 5.3.13.

To everyone who supported me during my time at EPFL, colleagues, friends, family, I thank you from the bottom of my heart.

Lausanne, February 2, 2016.

CONTENTS

INTRODUCTION

The study of cocycle growth for isometric linear representations is a fine cohomological tool. Bounded cohomology is a good example of a fundamental cohomology theory with a growth condition, see [**Mon06**] for instance. The growth of 1-cocycles for unitary representations of locally compact groups relates to renowned properties such as Kazhdan's Property (T) or Haagerup Property, see [**BHV08**] and [**CTV07**].

The present work investigates the growth of a particular cocycle in the following setting. Let G be a compactly generated locally compact group with an associated word length distance d_S, where S is a compact generating set, and let V be an isometric linear representation of G in a complex Banach space $(V, \|\cdot\|)$. Studying the growth of a G-equivariant n-cocycle $c : G^{n+1} \to V$ amounts to look for possible bounds of

$$\|c(g_0, \ldots, g_n)\|$$

as $(g_0, \ldots, g_n)$ vary in a subset of G^{n+1}, and, preferably, depending on the distances between the variables $g_0, \ldots, g_n \in G$. In particular, bounded cohomology treats with cocycles bounded uniformly on the whole G^{n+1}. In [**CTV07**], the authors look at 1-cocycles of unitary representations that are unbounded, but having some upper bound depending on $d_S(g_0, g_1)$, allowing the bound to only take place outside a compact subset of G^2.

The main motivation for the present research is a problem posed by Monod. In [**MS04**], he and Shalom performed a procedure, called 'quasification', of an unbounded 1-cocycle, yielding a new bounded cohomology class,called the *median class* for groups acting on a tree. The median class has a natural generalization for various groups acting on CAT(0) cube complexes, see [**CFI12**]. In [**Mon06**, Problem P] Monod asks if a similar quasification could be applied to the cocycles defined by Klingler in [**Kli03**]. The first natural step toward an answer is to determine their growth, which is the central objectif of this thesis.

Klingler's cocycle and the Steinberg representation

Let G be the group of F-points of a connected, simply connected, almost F-simple algebraic group over a local field F of characteristic zero. It is a compactly generated totally disconnected locally compact group, which possesses an interesting pre-unitary admissible representation called the Steinberg representation **St**. Klingler built in [**Kli03**] a natural cocycle $\mathrm{vol}_G : G^{n+1} \to \mathbf{St}$ for this G-module generating the continuous cohomology in degree n equal the F-rank of G, *i.e.*

$$\mathrm{H}^n_c(G, \mathbf{St}) = \mathbf{C} \cdot [\mathrm{vol}_G].$$

The aim of our research is to determine the growth of vol_G. In rank $n = 1$, we obtain the following result, (Theorem 4.2.4 and Corollary 4.2.5).

Theorem. — *Let G be $\mathrm{SL}_2(F)$ and d_S be a word metric on G. There exists a constant $C > 0$, depending only on the cardinality of the residue field*[(1)] *of F, such that*

$$\| \mathrm{vol}_G(g_0, g_1)\|_{\mathbf{St}} \leq C \cdot \sqrt{d_S(g_0, g_1)},$$

for all $g_0, g_1 \in G$.

In [**GJ15**], Gournay and Jolissaint independently obtained a finer and more general estimate, which proves our bound to be asymptotically sharp. Nevertheless, we hope that our proof shed light on the combinatorics of Klingler's cocycle. In higher rank, the difficulty is significantly increased.

The definition of the Klingler's cocycle vol_G, as well as the Steinberg representation of G, are closely related to the Bruhat-Tits building of G. The latter is a locally finite irreducible Euclidean building X on which G acts by type preserving automorphism. It can be endowed with a proper CAT(0) metric d for which G acts by isometries. In the preliminary Chapter 1, we recall that the geometries of (X, d) and (G, d_S) are roughly identical, (Proposition 1.1.8), and moreover that the cohomology of G can be fully understood from that of X by considering G-equivariant cocycles on X. In turn, the Bruhat-Tits building X is sufficient to study the cohomology of G and the growth of its cocycles, (Theorem 1.3.9). The second chapter explains the construction of vol_G, but also how the Steinberg representation $\mathbf{St}$ is related to X. Briefly, Klingler [**Kli04**] defined an explicit isomorphism, called the Poisson transform, between $\mathbf{St}$ and a subspace of the square summable functions on the set of chambers of X, (Theorem 2.3.5). The scalar product of the latter opens the way to estimate the norm of Klingler's cocycle vol_G. Again, morally, everything takes place in the Bruhat-Tits building X of G.

This lead us to formulate the question in the general framework of a locally finite irreducible Euclidean building. This is the content of Chapter 3, where we detailed the case of an $\widetilde{A}_2$ building, and of Section 2.2, where Klingler's cocycle is described. (Both can be easily adapted to regular buildings of type $\widetilde{A}_1$, *i.e.* regular trees, as done in Chapter 4.) In this setting, the 2-cocycle of Klingler and its Poisson transform are as follows. Let X be a locally finite Euclidean building of type $\widetilde{A}_2$ and denote ∂X the spherical building attached at infinity. To every triple (x, y, z) of vertices of X, we associate a real function $\mathrm{vol}_X(x, y, z)$ on the set $\Omega := \mathrm{Ch}(X)$ of chamber of ∂X. Given a chamber ξ at infinity of an Euclidean apartment A, the volume $\mathrm{vol}_X(x, y, z)(\xi)$ is defined by taking the oriented volume of the triangle formed by the images of x, y, z under the canonical retraction $\rho_{(A,\xi)}$ onto A centered about ξ. One can think of it as a pull-back of the volume form of A to the entire building X.

As for the Poisson transform, we follow Klingler [**Kli04**], and define a signed measure ν_C attached to each chamber C of X, (Definition 3.4.8). The Poisson transform of $\mathrm{vol}_X(x, y, z)$ is simply the function that integrate the latter against the measures ν_C, *i.e.*

$$\mathcal{P}\mathrm{vol}_X(x, y, z)(C) = \int_\Omega \mathrm{vol}_X(x, y, z) d\nu_C.$$

[(1)]provided it is at least 4,

Hence, determining the growth of Klingler's cocycle is equivalent to estimating

$$\|\mathcal{P}\mathrm{vol}_X(x,y,z)\|^2_{\ell^2(\mathrm{Ch}(X))} = \sum_{C\in\mathrm{Ch}(X)} \left(\int_\Omega \mathrm{vol}_X(x,y,z)d\nu_C\right)^2$$

in terms of the distances between the vertices x, y, z and their relative configuration in the building. In particular, one needs to estimate the value of the Poisson transform of $\mathrm{vol}_X(x, y, z)$ at each given chamber. The key ingredients are sector spheres $S_{m,n}(x)$ and the graph theoretic metric d_1 of the 1-skeleton of X, both introduced in [**CMS94**]. In X, any two vertices sits in a common apartment where they determine a well defined parallelogram (Figure 1, page 37), which consists of their convex hull for the metric d_1. The shape of the former determines two non-negative integers m, n encoding the relative position of the two vertices in X. The set of vertices in this particular position relatively to a vertex x defines the sector sphere $S_{m,n}(x)$ about x. The set Ω of chambers at infinity can be describe as the projective limit of the latter spheres, for (m, n) varying in $\mathbf{N}^2$,

$$\Omega = \varprojlim S_{m,n}(x),$$

which defines a natural compact topology on Ω (Sections 3.1 and 3.2), and a Borel probability measure ν_x, called the visual measure with respect to x, (Section 3.4). The topology is independent of the base point x but the visual measures do. Any two such ν_x, ν_y are always absolutely continuous and have an explicit Radon-Nikodym derivative $\frac{d\nu_x}{d\nu_y}$, (Proposition 3.4.3). A variation of these measures is used to define ν_C. Our next result is an averaging formula over a large sector sphere computing the exact value of $\mathrm{vol}_X(x, y, z)(C)$, (Theorem 3.4.10). It relies on horospherical coordinates $m(x, y, \xi)$, $n(x, y, \xi)$ describing the shift of the two sectors $\mathrm{Sect}_x(\xi)$, $\mathrm{Sect}_y(\xi)$ issuing at x, y respectively and pointing toward ξ, (Lemma 3.2.3). They can be computed by considering the two parallelograms determined by the pairs (x, u) and (y, u) where u is any vertex of the intersection of the two aforementioned sectors, (Lemma 3.2.7).

Theorem. — *Let $\underline{x} = (x, y, z) \in X^3$ be a triple of vertices, $C \in \mathrm{Ch}(X)$, and let x_C be vertex of C of type 0. Then for every natural number $R \in \mathbf{N}$, satisfying*

$$R \geq \max\{d_1(x, y), d_1(x, z), 2d_1(x, x_C)\},$$

we have

$$\mathcal{P}\mathrm{vol}_X(\underline{x})(C) = C_{\mathrm{vol}} \cdot N_{R,R}^{-1} \cdot \sum_{u\in S_{R,R}(x)} \det\begin{pmatrix} m(x,y,u) & m(x,z,u) \\ n(x,y,u) & n(x,z,u)\end{pmatrix} \cdot \frac{d\nu_C}{d\nu_{x_C}}(u) \cdot \frac{d\nu_{x_C}}{d\nu_x}(u),$$

where $N_{R,R}$ is the cardinal of the sphere $S_{R,R}(x)$ and C_{vol} is the constant of Proposition 3.3.3.

The contribution of each term in the above sum is not clear and is the motivation for Chapter 5. Looking at the summand, it seems sufficient to determine how many $u \in S_{R,R}(x)$ are in a given position relatively to the four vertices x, y, z, x_C. But already understanding the relative position of three vertices in an $\widetilde{A}_2$ building is not easy. Ramagge-Robertson-Steger and Lafforgue obtained a description of the convex hull of three points in an $\widetilde{A}_2$ building in the articles [**RRS98**] and [**Laf00**]. In simple terms, either the three parallelograms determined by the pairs of vertices have a common horizontal segment, or there is an equilateral triangle in the 1-skeleton connecting the three convex hulls. In Section 5.2, we give a new proof using the graph theoretic distance of the 1-skeleton, providing unicity of the triangle, (Theorem 5.2.8).

CHAPTER 1

PRELIMINARIES

1.1. Geometry of locally compact groups

This section is a summary of facts, well known to experts, on locally compact groups and their geometry. The second half of the twentieth century witnessed the success of Gromov's theory which considered infinite groups as metric spaces. The study of their geometry and their large scale properties yields interesting characterizations of algebraic properties. Such an approach is part of our field of research: *geometric group theory*. We recall various properties of locally compact groups following notably the book of Cornulier and la Harpe [**CH15**].

Convention 1.1.1. — We adopt the French convention and say that a topological space X is *compact* if it is Hausdorff and if any covering of X by open sets admits a finite subcover. Throughout this thesis topological groups are assumed Hausdorff. Consequently a *locally compact group* is by definition a Hausdorff topological group in which every point has a neighborhood basis consisting of compact subsets. The identity element of a multiplicative group is written 1_G, but we shall often drop the subscript G.

1.1.1. Metric on groups.

— Under mild assumptions a topological group turns out to be metrizable *i.e.* it admits a metric inducing the same topology, such a metric is called *compatible*. However a locally compact group G may admit many metrics that are not necessarily continuous as functions $G \times G \to \mathbf{R}_+$, but still important from the point of view of coarse geometry.

Theorem 1.1.2 (Birkhoff-Kakutani). — [**CH15**, Theorem 2.B.2] *A topological group G is metrizable if and only if it is first countable i.e. each point has a countable neighborhood basis. In this case, there exists a compatible metric d on G that is left-invariant:*

$$d(gx, gy) = d(x, y),$$

for all $x, y, g \in G$. □

In the present thesis we are concerned with *word metrics* on *compactly generated* locally compact groups. We warn the reader that word metrics need not be compatible.

Definition 1.1.3. — Let S be a generating subset of a group G, *i.e.*

$$G = \bigcup_{n \in \mathbf{N}} (S \cup S^{-1})^n.$$

- The *word metric* d_S associated to S is defined as follows. For every $x, y \in G$, let $d_S(x, y)$ be the minimal length of a word $(s_1, \dots, s_n)$ with letters in $S \cup S^{-1}$ such that $x^{-1}y = s_1 s_2 \dots s_n$.
- The *length function* ℓ_S associated to S is defined by $\ell_S(g) = d_S(1_G, g)$ for all $g \in G$, so that $d_S(x, y) = \ell_S(x^{-1}y)$ and therefore d_S is left invariant.
- We say that a topological group G is *compactly generated* if there exists a compact subset S generating G. The exhaustion of G by the compact subsets $(S \cup S^{-1})^n$ shows G to be σ-compact.

Examples 1.1.4. — Let F be a non-discrete locally compact field[(1)], *i.e.* the real numbers $\mathbf{R}$, the p-adic numbers $\mathbf{Q}_p$, the formal Laurent series $\mathbf{F}_p((t))$, or a finite extension of these.

- Among these fields the Archimedean ones, $\mathbf{R}$ and $\mathbf{C}$, are compactly generated as additive groups whereas the non-Archimedean ones are not. Interestingly the compact subset $S = [0, 1]$ generates $(\mathbf{R}, +)$ but the distance d_S is not continuous with respect to the standard topology of $\mathbf{R}$, *i.e.* not compatible. Indeed $d_{[0,1]}(0, 1+\varepsilon) = 2$ for all $\varepsilon > 0$ but $d_{[0,1]}(0, 1) = 1$, [**CH15**, §1.D].
- The matrix groups $\mathrm{GL}_n(F)$ and $\mathrm{SL}_n(F)$ with the subspace topology of $\mathrm{M}_n(F) \cong F^{n\times n}$ are second countable, locally compact, and compactly generated. More generally, so is the group of F-points of a *reductive algebraic group* defined over F, see [**CH15**, Theorem 5.A.12].

Different compact generating sets may certainly yield non-isometric distances on G, but they are equivalent from a coarse point of view. Here are some ways to coarsely distinguish metric spaces.

Definition 1.1.5. — Let $f : X \to Y$ be a map between metric spaces. We say that

- f is *Lipschitz* if there exists a constant $C > 0$ such that
$$d_Y(f(x), f(x')) \leq C \cdot d_X(x, x'),$$
for all $x, x' \in X$.
- f is *bilipschitz* if there exist constants $C, c > 0$ such that
$$c \cdot d_X(x, x') \leq d_Y(f(x), f(x')) \leq C \cdot d_X(x, x'),$$
for all $x, x' \in X$.
- f is a *bilipschitz equivalence* if f is bilipschitz and surjective. In this case, any set theoretic section of f is also bilipschitz and X, Y are said to be *bilipschitz equivalent*.
- f is a *quasi-isometry* if there exist constants $C, c > 0$ and $C', c' \geq 0$ such that
$$c \cdot d_X(x, x') - c' \leq d_Y(f(x), f(x')) \leq C \cdot d_X(x, x') + C',$$
for all $x, x' \in X$, and there is $R > 0$ such that the R-neighborhood of $f(X)$ covers Y. In other words, any point of Y is at a uniform bounded distance from $f(X)$.
- X and Y are *quasi-isometric* if there exist quasi-isometries $f : X \to Y$ and $g : Y \to X$.

It is clear that bilipschitz equivalent metric spaces are quasi-isometric. A classical result of geometric group theory says that all word metrics on a compactly generated locally compact group are bilipschitz equivalent.

[(1)] We consider only commutative fields.

Proposition 1.1.6. — [**CH15**, Proposition 4.B.4] *Let G be a compactly generated locally compact group with two compact generating sets S, S'. Then the identity map $(G, d_S) \to (G, d_{S'})$ is a bilipschitz equivalence.*

Convention 1.1.7 (Local field). — In this thesis, we consider only fields of characteristic zero. Therefore, by a *local field* we mean a locally compact non-Archimedean field of characteristic zero, *i.e.* a finite extension of the field of p-adic numbers $\mathbf{Q}_p$.

The central objects of this thesis are groups like $\mathrm{SL}_n(F)$ of example 1.1.4 called *groups of F-points of a connected simply-connected almost F-simple algebraic group over a local field F*. If G is such a group, we always consider the topology on G induced by that of F. An important feature of these groups is the existence of an associated metric space (X, d) called the *Bruhat-Tits building* on which G acts by isometries. We postpone rigorous descriptions of these objects to Section 1.2 as this is sufficient for now. The Bruhat-Tits building of G is the non-Archimedean equivalent of the symmetric space of a Lie group and as a general philosophy (X, d) mimics G in many ways. One relevant fact to this preliminary chapter is that (X, d) and (G, d_S) are quasi-isometric, S being any compact generating set of G. This can be formulated as follows. For $x_0 \in X$, define the left-invariant map $d_{x_0} : G \times G \to \mathbf{R}_+$ by

$$d_{x_0}(g, h) \coloneqq d(gx_0, hx_0).$$

The stabilizer of x_0 is generally non-trivial, hence we may have $d_{x_0}(g, h) = 0$ even if $g \neq h$. Nevertheless d_{x_0} satisfies all other axioms of a metric and is what we call a *pseudo-metric*. Note that most of the above discussion and Definitions 1.1.5 make sense for pseudo-metrics. This is in fact the point of view of [**CH15**]. The following proposition covers the main example of the present thesis.

Proposition 1.1.8. — [**Abe04**, Theorem 6.6] *Let G be the group of F-points of a reductive group over a local field F. The identity map $(G, d_S) \to (G, d_{x_0})$ is a quasi-isometry for any compact generating set S and any $x_0 \in X$.*

In this setting, let f be a non-negative real valued function on the n-fold product G^n. Suppose that f satisfies the following hypothesis: there exists a real polynomial p in variables x_{ij}, for $1 \leq i, j \leq n$, such that

$$f(g_1, \ldots, g_n) \leq p(d_{x_0}(g_i, g_j)),$$

for all $g_1, \ldots, g_n \in G$. The minimal degree of a polynomial for which the above holds is well defined and Proposition 1.1.8 shows that it does not depend on x_0. By quasi-isometry, d_{x_0} may even be replaced by d_S for any compact generating set of G without changing that degree. We will be more precise later but this describes one way to control the asymptotic behavior of $f(g_1, \ldots, g_n)$ as '$d(g_i, g_j)$ tend to infinity', which in turn does not depend on the metric considered. In practice f is going to be the norm of a *cocycle* $c : G^n \to V$, valued in a normed vector space V:

$$f(g_1, \ldots, g_n) = \|c(g_1, \ldots, g_n)\|.$$

1.1.2. Totally disconnected locally compact groups. — The topological groups under consideration in this thesis are totally disconnected locally compact groups.

Definition 1.1.9. — A locally compact group is *totally disconnected* if its connected components are singletons. We write t.d.l.c. to abbreviate totally disconnected locally compact.

The following is a celebrated theorem of Van Dantzig on the topology of t.d.l.c. groups.

Theorem 1.1.10 (van Dantzig). — [**CH15**, Theorem 2.E.6] *Let G be a t.d.l.c. group, then the set $\mathcal{B}(G)$ of all compact open subgroups of G form a neighborhood basis of the identity 1_G.*

The group $\mathrm{GL}_n(F)$ and its closed subgroups with the F-topology, *i.e.* the subspace topology of $\mathrm{M}_n(F) \cong F^{n\times n}$, are σ-compact, second countable, totally disconnected locally compact groups. In particular the group of F-points of the algebraic groups considered in the present thesis have those properties, see [**CH15**, Example 2.C.12].

1.2. Buildings and groups

This section introduces the main notion of this thesis namely *buildings*. First we briefly recall the various equivalent definitions of simplicial buildings and how to obtain one from the BN-pair of a group. Then we discuss group theoretic consequences of the existence of a BN-pair as well as actions of groups on buildings, *e.g.* strongly transitive actions. We also present the important example of the Bruhat-Tits building associated to an algebraic group over a local field mentioned in the previous section. Such groups have two important related BN-pairs yielding a spherical building and a Euclidean one. The second part of the section covers the geometric realization of the latter as a CAT(0) space which is a property of non-positive curvature of a metric space. Finally we discuss some important consequences of the relation between the two buildings.

1.2.1. Simplicial buildings and Tits systems. — The theory of buildings was introduced by Tits in the middle of the twentieth century and saw a rapid development lead by Tits himself [**Tit74**]. Buildings first appeared in a group theoretic context, for example with the theory of semi-simple Lie groups, where they appear as a consequence of two subgroups sharing interesting axioms and forming what is called a *Tits system* or a *BN-pair*. From there mathematicians extracted some axioms which a simplicial complex must satisfy in order to be called a building. A simple example is the tree associated to $\mathrm{SL}_2(\mathbf{Q}_p)$ which has the particularity to be $(p+1)$-regular, see Serre's book [**Ser77**]. But intuitively, when studying the geometry of regular trees, the fact that the valency is a prime number does not matter, and in general any locally finite tree with no leaf(2) is a building. Later building theory found even more general formulations, such as $\mathbf{R}$-trees, that we do not discuss.

Convention 1.2.1. — In this thesis we consider only locally finite, thick, irreducible buildings.

Classically a *simplicial complex* over a set of vertices V is a non-empty collection Δ of finite subsets of V, called *simplices*, satisfying:

- every singleton $\{v\}$ is a simplex, *i.e.* $\{v\} \in \Delta$, and
- every subset of a simplex is also a simplex.

The cardinal(3) $r \in \mathbf{N}$ of a simplex A is called the *rank of* A, and $r-1$ is called the *dimension* of A. If $A \subsetneq B$ are simplices, the positive difference of their dimensions is called the *codimension* of A in B. Finally a simplex of dimension n is called an *n-simplex* and the set of n-simplices of Δ is denoted $\Delta^{(n)}$.

(2) A leaf is a vertex of valency one, *i.e.* with only one neighbor.
(3) The empty set is considered to be a simplex of rank 0 and dimension -1.

With respect to inclusion, a simplicial complex is a poset with the emptyset $\emptyset$ as a unique minimal element. The poset structure encodes the simplicial one, and a simplicial complex may equivalently be defined as follows, see [**AB08**, Appendix A.1.1]. We follow the chapters 3, 4, 5 and the appendix A of *loc. cit.* for this section.

Definition 1.2.2 (Simplicial complex). — A nonempty poset $(\Delta, \leq)$ is called a *simplicial complex* if it satisfies:

- Any pair $A, B \in \Delta$ have a greatest lower bound.
- For every $A \in \Delta$, the poset $\Delta_{\leq A}$ of elements $\leq A$ is isomorphic to the poset of subsets of $\{1, \ldots, r\}$ for some $r \in \mathbf{N}$.

The unique integer r associated to each $A \in \Delta$ by the second condition plays the role of the rank and the terminology is easily adapted.

This definition is advantageous when defining the Coxeter complex of a Coxeter group or the building associated to a BN-pair. For all notions surrounding Coxeter groups we refer to Bourbaki [**Bou68**, GAL, Chapter IV] or to Abramenko-Brown's book [**AB08**, Chapter 2].

Example 1.2.3. — Let (W, S) be a Coxeter system, *i.e.* $S = \{s_i \mid i \in I\}$ is a finite generating set of a group W admitting a presentation of the form

$$W = \langle S \mid (s_i s_j)^{m(i,j)} = 1, \text{ for all } i, j \in I \rangle,$$

where $m(i,j) \in \mathbf{N} \cup \{\infty\}$ satisfy $m(i,i) = 1$ and $m(i,j) \geq 2$ for all $i \neq j$. For every subset $J \subset I$, the subgroup W_J generated by $S_J = \{s_i \mid i \in J\}$ is called *standard* subgroup of W and its (left) cosets are called *standard cosets*. The set $\Sigma(W, S)$ of all standard cosets with the reverse inclusion is a simplicial complex in the sense of the above definition, called the *Coxeter complex* of (W, S), [**AB08**, Theorem 3.5]. Writing $\leq$ for the reverse inclusion, one has

$$wW_J \leq w'W_{J'} \iff w'W_{J'} \subset wW_J \iff J' \subset J \text{ and } w'W_{J'} = wW_{J'}.$$

The unique standard coset of $W_I = W$ is the minimal element in this case. The Coxeter group W acts on $\Sigma(W, S)$ by poset automorphisms, thus preserving ranks. The stabilizer of a standard coset wW_J for this action is the conjugate wW_Jw^{-1}, hence, morally, a simplex of small rank has a large stabilizer and vice-versa. Interestingly the maximal simplices, *i.e.* cosets of the trivial subgroup, are identified with the elements of the group and, therefore, they all have rank equal to $\mathrm{card}(I) = \mathrm{card}(S)$. Moreover, W acts simply transitively on the set of maximal simplices. The poset $\Sigma(W, S)_{\leq A}$ with $A = \{1_G\}$, consists of the set of standard subgroups and is isomorphic to the poset of subsets of I (or S) for the reverse inclusion.

In Example 1.2.3 the maximal simplices have the same dimension. In a finite dimensional simplicial complex Δ with this property, the maximal simplices are called *chambers* and we denote $\mathrm{Ch}(\Delta)$ the set of chambers. Two chambers are called *adjacent* if they share a codimension 1 face. This defines a graph structure on $\mathrm{Ch}(\Delta)$, called the *chamber graph of* Δ. We endow it with the graph theoretic distance denoted $\mathbf{d}$. In the latter, a finite path is called a *gallery* and a gallery is *minimal* if its length minimizes the distance between its extremities.

Definition 1.2.4 (Chamber complex). — [**AB08**, Appendix A.1.3] A finite dimensional simplicial complex is called a *chamber complex* if all its maximal simplices have the same dimension and if it is *gallery connected*, that is if the gallery graph is connected. In other words, any pair of chambers can be joined by a finite sequence of successively adjacent chambers.

The chamber graph of the Coxeter complex $\Sigma(W,S)$ coincides with the well-known Cayley graph of W with respect to the generating set S. For the definitions of *simplicial subcomplex*, *chamber subcomplex*, *simplicial map* and *chamber map*, we refer to [**AB08**, Appendix A].

Definition 1.2.5. — Let Δ be a chamber complex. A simplex is called a *panel* if it is a codimension 1 face of a chamber. The chamber complex Δ is *thin*, if every panel is contained in exactly two chambers, and Δ is called *thick* if every panel has at least three chambers containing it.

Some authors allow non-thick chamber complex in the definition of a building, so that Coxeter complexes are exactly the thin buildings. We do not follow this convention, the buildings in this thesis are assumed thick.

Definition 1.2.6 (Simplicial building). — A thick chamber complex Δ is called a *building* if there is a family $\mathcal{A}$ of chamber subcomplexes of Δ, the elements of which are called *apartments*, satisfying the following:

(B0) Every apartment is a Coxeter complex.
(B1) Any two simplices $A, B \in \Delta$ are contained in a common apartment.
(B2) For every pair $\Sigma, \Sigma' \in \mathcal{A}$ of apartments both containing simplices A and B, there is an isomorphism $\Sigma \to \Sigma'$ fixing A and B pointwise, *i.e.* fixing all simplices of $\Delta_{\leq A}$ and $\Delta_{\leq B}$.

The axiom (B2) can be replaced by either axioms (B2′) or (B2″), [**AB08**, Chapter 4, §1]:

(B2′) For every pair $\Sigma, \Sigma' \in \mathcal{A}$ of apartments both containing simplices A and C, with C a chamber of Σ, there is an isomorphism $\Sigma \to \Sigma'$ fixing A and C pointwise.
(B2″) For every pair $\Sigma, \Sigma' \in \mathcal{A}$ of apartments both containing a simplex C that is a chamber of Σ, there is an isomorphism $\Sigma \to \Sigma'$ fixing pointwise every simplex in $\Sigma \cap \Sigma'$.

Any collection of subcomplexes satisfying the above axioms is called a *system of apartments of* Δ. There is always a unique maximal system of apartments, called the *complete system of apartments*. Consequently, (B0) implies that a building has unique Coxeter system (W,S) associated to it, [**AB08**, Corollary 4.36], so that the maximal system of apartments consists of all chamber subcomplexes of Δ isomorphic to $\Sigma(W,S)$. Except if stated otherwise, we shall always work with the complete system of apartments.

Another consequence is the existence (and uniqueness) of a canonical retraction $\rho_{C,\Sigma}$ associated to a chamber C and an apartment $\Sigma \in \mathcal{A}$ containing it. For a chamber D, let Σ' be an apartment containing C and D given by (B1). The isomorphism of (B2″) turns out to be unique. We define $\rho_{C,\Sigma}(D)$ to be the image under the unique isomorphism $\Sigma' \to \Sigma$ fixing pointwise $\Sigma \cap \Sigma'$, in particular fixing C pointwise.

Definition 1.2.7 (Canonical retraction). — The map $\rho_{C,\Sigma} : \Delta \to \Sigma$ defined above is called the *canonical retraction onto* Σ *centered at* C. It is the unique chamber map $\Delta \to \Sigma$ sending every apartment containing C isomorphically onto Σ by fixing its intersection with Σ.

The edges of a Cayley graph are usually labeled by the elements of the generating set. A similar coloring can be done in a building by means of a type function, and, as a result, the underlying simplicial structure is entirely determined by the coloring of the edges of the chamber graph. This is a general fact for colorable chamber complexes, see Proposition A.20 in [**AB08**, Appendix A].

Definition 1.2.8 (Type function). — Let Δ be chamber complex of rank n and I be a finite set with n elements. The chamber complex Δ is called *colorable* if it admits a *type function*

$\tau : \Delta \to \Delta_I$, that is a chamber map into the poset Δ_I of all subsets of I (ordered by inclusion). For a simplex $A \in \Delta$, we say that $\tau(A)$ is the *type of* A and that $I \smallsetminus \tau(A)$ is its *cotype.*

In other words, τ assigns to each simplex of Δ a subset of I, and is a simplicial map sending chambers of Δ to the unique chamber of Δ_I, namely I itself. A type function is determined by its value on the vertices of Δ and the type of a simplex is the union of the types of its vertices. A colorable chamber complex has a unique type function up to a bijection of the set of colors [**AB08**, Proposition A.14].

Proposition 1.2.9. — [**AB08**, Proposition 4.6] *A building is colorable, moreover the isomorphism of axiom (B2) is type preserving.* □

The cotype of a panel F, a codimension 1 face of a chamber C, is in this case the type of the unique vertex v of C not in F. In other words, we can label the edges of the chamber graph using the cotypes of panels, similarly to Cayley graphs.

Definition 1.2.10 (i-adjacency and J-residue). — Let Δ be endowed with a type function $\tau : \Delta^{(0)} \to I$. Two adjacent chambers C, C' are called *i-adjacent* if their common panel is of cotype $i \in I$ and we write $C \sim_i C'$. The edges of the chamber graph are therefore labeled[(4)] by the type set I. For $J \subset I$, two chambers are called J-equivalent if there is a path in the chamber graph using only colors of J. Such path is called a *J-gallery*. The J-equivalence classes are called *J-residues*.

In general, this data is called a *chamber system*. It turns out that the chamber system determines the entire simplicial structure of the building. See [**AB08**, Corollary 4.11] and [**AB08**, Proposition A.20] for the following proposition.

Proposition 1.2.11. — *Δ is completely determined by its underlying chamber system. More precisely:*

- *For a simplex $A \in \Delta$, the set $\mathrm{Ch}(\Delta)_{\geq A}$ of chambers having A as a face is a J-residue, where J is the cotype of A.*
- *Every J-residue has the form $\mathrm{Ch}(\Delta)_{\geq A}$ for some simplex A.*
- *For any simplex A, we can recover A from $\mathrm{Ch}(\Delta)_{\geq A}$ via*

$$A = \bigcap_{C \geq A} C$$

- *The poset Δ is isomorphic to the set of residues in the chamber graph, ordered by reverse inclusion.*

A vertex in a building is recovered by knowing the chambers that contain it. Say a vertex x is of type i and that C is a chamber containing x. The vertex x is surrounded by all chambers that can be attained from C using a J-gallery where $J = I \smallsetminus \{i\}$.

Definition 1.2.12 (Link). — Let Δ be a building. The *link* of a vertex x, denote $\mathrm{lk}(x)$, is by definition the set of chambers containing x.

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(4) We shall use ‘colored’ and ‘labeled’ as synonyms.

The link is often defined using the set of simplices that can be joined to x but not containing it. The two definitions yield isomorphic chamber complexes. The link of a vertex x is easily shown to be a building in its own right.

There is another characterization of buildings using W-metrics, proved in [**AB08**, Corollary 5.39]. The idea is that two chambers in a building are contained in various apartments. In each of these isomorphic Coxeter complexes, their relative position is a well defined group element $w \in W$. In an apartment, two chambers C, C' are typically connected by a minimal gallery $\Gamma : C_0 = C, \dots, C_l = C'$, such that $C_j \sim_{i_j} C_{j+1}$ where $i_1, \dots, i_l \in I$ are the types of the gallery. Then the group element $w = s_{i_1} \dots s_{i_l}$ is independent of the choice of gallery connecting C and C' and represents that relative position in the Coxeter complex.

Definition 1.2.13. — The element $w \in W$ defined above is denoted $\delta(C, C')$ and defines the *W-metric* or *Weyl distance* $\delta : \mathrm{Ch}(\Delta) \times \mathrm{Ch}(\Delta) \to W$.

Proposition 1.2.14. — *Let (W, S) be the Coxeter system associated to a building Δ and ℓ be the word length function of (W, S). The W-metric δ satisfies, for all $C, D \in \mathrm{Ch}(\Delta)$, the following conditions:*

(W1) $\delta(C, D) = 1_W$ *if and only if* $C = D$.
(W2) If $\delta(C, D) = w$ and $C' \in \mathrm{Ch}(\Delta)$ satisfies $\delta(C', C) = s$, then $\delta(C', D) = sw$ or w. If, in addition, $\ell(sw) = \ell(w) + 1$, then $\delta(C', D) = sw$.
(W3) If $\delta(C, D) = w$, then for any $s \in S$ there is a chamber $C' \in \mathrm{Ch}(\Delta)$ such that $\delta(C', C) = s$ and $\delta(C', D) = sw$. □

The historical source of buildings comes from Tits systems of classical groups, where a building is associated to a group with two distinguished subgroups B and N. The discussion below follows closely Chapter 6 of [**AB08**], the first section of [**Gar73**] and Bourbaki [**Bou68**, GAL].

Definition 1.2.15 (Tits system). — Let G be a group and B, N be two subgroups of G. We say that (B, N) form a *BN-pair* if, together, they generate G, their intersection $T \coloneqq B \cap N$ is a normal subgroup of N, and the quotient $W \coloneqq N/T$ admits a finite generating set S satisfying:

(BN1) For $s \in S$ and $w \in W$,
$$sBw \subset BswBwB.$$
(BN2) For $s \in S$
$$sBs^{-1} \not\leq B.$$

By abuse, we may speak of the *Tits system* (G, B, N, S). In any case, W is called the *Weyl group* of the BN-pair.

The generating set S is uniquely determined by the BN-pair. Expressions of the form BwB with $w \in W$ are a well defined B double cosets and G has a Bruhat decomposition
$$G = \bigsqcup_{w \in W} BwB.$$
Furthermore (W, S) is a Coxeter system and, as in Example 1.2.3, we write $S = \{s_i \mid i \in I\}$ for some index set I. For every $J \subset I$, the union of double cosets $BW_J B$ form a subgroup, denoted P_J, generated by $\bigsqcup_{i \in J} Bs_i B$.

Definition 1.2.16 (Parabolic subgroup). — Let (G, B, N, S) be a Tits system. A *parabolic subgroup* of G is a subgroup containing a conjugate of B. For $J \subset I$, the subgroups P_J are called the *standard parabolic subgroups.* Among the proper parabolic subgroups, maximal ones are called *maximal parabolic subgroups.*

Lemma 1.2.17. — [**Bou68**, GAL, Chapter 4, §2.6, Theorem 4] *Suppose the intersection of two parabolic subgroups P, Q is parabolic. If $gPg^{-1} \subset Q$, then $g \in Q$ and $P \subset Q$. Consquently, a parabolic subgroup is its own normalizer.* □

Proposition 1.2.18. — [**Bou68**, GAL, Chapter IV, §2.5] *The maps $J \longmapsto W_J \longmapsto P_J$, are poset isomorphisms, for the inclusion relations, from the power set of I to the set of standard subgroups of W, and from the latter onto the set of subgroups of G containing B. Thus a subgroup of G is a parabolic subgroup if and only if it is conjugate to a standard parabolic subgroup.* □

The proposition implies that the parabolic subgroups conjugate to a fixed standard parabolic subgroup P_J correspond bijectively to the left cosets of P_J under the map

$$gP_Jg^{-1} \mapsto gP_J. \tag{1.1}$$

A Coxeter group is trivially endowed with a BN-pair with $B = \{1_W\}$ and $N = W$, the terminology of Example 1.2.3 is coherent with the present. The building associated with a BN-pair is defined similarly to the Coxeter complex of a Coxeter system.

Definition 1.2.19 (Standard coset). — A left coset of a standard parabolic subgroup is called a *standard (parabolic) coset.* The poset of standard cosets endowed with the reverse inclusion is denoted $\Delta(G, B)$ and is called the *building associated to the Tits system*[5] (G, B, N, S).

Thanks to Proposition 1.2.18, $\Delta(G, B)$ is a simplicial complex, but it also comes with a natural action of G by left multiplication on the standard cosets, which corresponds under the map (1.1) to the conjugation action on the parabolic subgroups. Before making the link between Tits systems and buildings, we introduce a strong transitivity property of an action on a building. Since buildings are colorable, there is an obvious notion of a group action by type-preserving automorphisms (of chamber complex).

Definition 1.2.20 (Strongly transitive action). — Let G be a group acting on a building Δ by type-preserving automorphisms and let $\mathcal{A}$ be a G-invariant set of apartments of Δ. We say that G *acts strongly transitively* on Δ with respect to $\mathcal{A}$ if G acts transitively on the set of pairs (C, A) where C is a chamber of an apartment $A \in \mathcal{A}$. We shall often use the complete system of apartments and omit to mention it if clear from the context.

Theorem 1.2.21. — [**AB08**, Theorem 6.65]

(i) Let (G, B, N, S) be a Tits system, then $\Delta(G, B)$ is a building, the G-action on left cosets is strongly transitive and such that B is the stabilizer in G of a chamber (that representing the coset B). Moreover, the subgroup N stabilizes an apartment Σ and acts transitively on its chambers. The system of apartments for which G is strongly transitive is $\mathcal{A} = G\Sigma$.

[5] The building $\Delta(G, B)$ depends not on N. The latter only determines a system of apartments of the building.

(ii) Conversely, suppose a group G acts strongly transitively on a building Δ, with respect to some system of apartments $\mathcal{A}$. Let C be a chamber in an apartment $\Sigma \in \mathcal{A}$. If B denotes the stabilizer of C in G and N the stabilizer of Σ, then (B, N) is a BN-pair in G and Δ is canonically isomorphic to $\Delta(G, B)$.

Strong transitivity has the two following important consequences.

Proposition 1.2.22. — [**AB08**, Proposition 6.6] *Let G be a group acting strongly transitively on a building Δ with respect to a system of apartments $\mathcal{A}$ and let Σ, Σ' be a pair of apartments of $\mathcal{A}$.*

(i) Then every type-preserving automorphism $\phi : \Sigma \to \Sigma'$ is realized by an element $g \in G$, that is $g|_\Sigma = \phi$.
(ii) There is an element $g \in G$ such that $g\Sigma = \Sigma'$ and g fixes $\Sigma \cap \Sigma'$ pointwise.

Corollary 1.2.23. — [**AB08**, Corollary 6.7] *Suppose a group G acts strongly transitively on a building Δ with respect to an apartment system $\mathcal{A}$. Let S be an arbitrary set of simplices of Δ, and denote* $\mathrm{Fix}_G(S)$ *its pointwise fixer, that is the set of all $g \in G$ such that $gA = A$ for all simplices A of S. Then* $\mathrm{Fix}_G(S)$ *acts transitively on the set of apartments in $\mathcal{A}$ containing S.*

1.2.2. CAT(0) geometry of Euclidean buildings. — Recall that a Coxeter group W is called *spherical* if it is finite. In this case W can be realized as a finite reflexion group of a finite dimensional real vector space. *Euclidean* Coxeter groups are those that can be realized as a group of affine reflections stabilizing a locally finite hyperplane arrangement in a finite dimensional real vector space, see [**AB08**, Chapter 10] or [**Bou68**, Chapter V].

Definition 1.2.24. — A building is called *spherical* or *Euclidean* if its underlying Coxeter system is so.

Convention 1.2.25. — In this thesis we consider only irreducible Coxeter systems. Accordingly a Euclidean Coxeter group W_{aff} is the affine reflection group of an irreducible crystallographic root system, so that its Coxeter complex can be realized as the complex of geometric simplices of a Euclidean space, on which W_{aff} acts simply transitively. In particular an affine Coxeter group W_{aff} has an associated finite Weyl group W, which can be taken as the stabilizer in W of a special vertex. See also [**AB08**, §10.1-2].

Let Δ be a Euclidean building with Coxeter group W. Let $|\Sigma(W, S)|$ denote the geometric realization of the Coxeter complex mentioned above with a fixed Euclidean metric d. The geometric realization $X = |\Delta|$ can be endowed with a unique metric inducing d on each of its apartments (in the complete system of apartments), see [**AB08**, Theorem 11.16]. By abuse, we identify a Euclidean building Δ with its geometric realization X which will provoke many abuses of notation.

The pair (X, d) is a CAT(0) metric space, *i.e.* d satisfies an inequality describing the fact that triangles in X are thinner than what they would be in $\mathbf{R}^2$. The reference for CAT(0) geometry is the celebrated book of Bridson and Haefliger [**BH99**].

A *geodesic* in a metric space is a map $\sigma : I \subset \mathbf{R} \to X$ from an interval I that preserves distances. A CAT(0) space X is uniquely geodesic and, if proper, it can be compactified by means of its visual boundary ∂X. By definition ∂X is the set of equivalence classes of geodesic rays $r : \mathbf{R}_+ \to X$ for the relation of being *asymptotic*. Two geodesic rays r, r' are *asymptotic* if

$$\sup_{t \in \mathbf{R}_+} d(r(t), r'(t)) < \infty.$$

The equivalence class of a geodesic ray r is often denoted $r(\infty)$ to emphasis that we think of r as pointing in the direction of $r(\infty)$. For every base point $x \in X$, it is well known that each class $\eta \in \partial X$ has a unique representative starting at x. In other words, points of ∂X correspond bijectively with the geodesic rays issuing at x.

Proposition 1.2.26. — [**AB08**, Chapter 11] *Let X be the geometric realization of a Euclidean building. Then the CAT(0) boundary ∂X has the structure of a spherical building. Its system of apartments is in bijection with the complete system of apartments of X via the map $A \mapsto \partial A$.*

In the previous identification, the chambers at infinity correspond bijectively with the equivalence class of sectors in X, for the relation of containing a common subsector, see [**AB08**, §11.5]. It is well known that for each point $x \in X$ in the Euclidean building, the set of sectors issuing at x is in bijection with the chambers at infinity. Given a chamber at infinity $\xi \in \mathrm{Ch}(X)$, the unique sector at x is denoted $\mathrm{Sect}_x(\xi)$.

At last, we discuss the main example that will be treated in this thesis, namely *Bruhat-Tits buildings*. We refer to the original article of Bruhat and Tits [**BT72**]. This setting is used in [**Kli03**], [**Kli04**], [**BW00**, Chapter X, §2] and [**Bor76**, Chapter II, §3].

Example 1.2.27. — Let G be a connected, simply-connected, almost F-simple algebraic group over a local field F. A consequence of *loc. cit.* is that the group G can be endowed with two Tits systems, with the same subgroup N, (G, B, N, S) and (G, P, N, S_0), for which $\Delta(G, B)$ is a Euclidean building and $\Delta(G, P)$ is a spherical building isomorphic to the building at infinity ∂X of the geometric realization $X = |\Delta(G, B)|$. The corresponding Coxeter groups are denoted W_{aff} and W respectively. We assume their generating sets S and S_0 to satisfy:

$$S = S_0 \cup \{s_0\}, \quad \text{so that} \quad W = \langle S_0 \rangle < W_{\mathrm{aff}} = \langle S \rangle.$$

We moreover index them so that $S = \{s_i \mid i \in I\}$ and $S_0 = \{s_i \mid i \in I_0\}$ with $I_0 = I \smallsetminus \{0\}$.

In this context of a double Tits system, the subgroups containing a conjugate of P are called parabolic subgroups. Another name was needed for the subgroups containing B. On the one hand, P is a generalization of the notion of Borel subgroup of an algebraic groups over an algebraically closed field. On the other, the group B was studied by Iwahori and Matsumoto in [**IM65**]. It was later called an Iwahori subgroup as a portmanteau of 'parabolic' and 'Iwahori'. By drawing the parallel, mathematicians came up with the name of *parahoric subgroups* for the groups containing B, so that B would be a standard minimal parahoric.

The classical example is $G = \mathrm{SL}_n(F)$ over a local field, see [**AB08**, Proposition 11.105] for more details. The Bruhat-Tits buildings X in the main example above are all locally finite: each panel is contained in a finite number of chambers.

Definition 1.2.28. — Let Δ be a building with label set I. Then Δ is *locally finite* if

$$\mathrm{card}(\{C' \in \mathrm{Ch}(\Delta) \mid C' \sim_i C\}) < \infty$$

for all types $i \in I$ and chambers $C \in \mathrm{Ch}(\Delta)$. We say that Δ is *regular* if this cardinal does not depend on C. In this case we denote this cardinal by q_i and call the set $\{q_i \mid i \in I\}$ the *regularity parameters* of X.

If a building Δ admits a strongly transitive action of a group, then it is regular. A more surprising fact is that, except for the type $\widetilde{A}_1$, every locally finite irreducible building is regular, [**Par05**, Theorem 1.7.4]. Moreover if the Coxeter diagram of the building is simply laced the regularity parameters are all equal, $q_i = q_j$ for all $i, j \in I$, [**Par05**, Corollary 1.7.2]. This includes Euclidean buildings of type $\widetilde{A}_n$, $\widetilde{D}_n$, with $n \geq 2$, and $\widetilde{E}_n$, with $n = 6, 7, 8$. In particular, there is $q \in \mathbf{N}$ such that

$$\{C' \in \mathrm{Ch}(\Delta) \mid \delta(C, C') = w\} = q^{\ell(w)},$$

for all $w \in W_{\mathrm{aff}}$ and chambers $C \in \mathrm{Ch}(C)$, in these cases.

1.2.3. Busemann cocycle. — To conclude this section, we present the *Busemann cocycle* of a CAT(0) space and recall some of its elementary features. For more details on the Busemann cocycle, see [**BH99**, Part II, Chapter 8].

Definition 1.2.29 (Busemann cocycle). — Let X be a (proper, complete) CAT(0) space. For every $x, y \in X$, the *Busemann cocycle* is the map $B(x, y) : \partial X \to \mathbf{R}$, defined by

$$B(x, y)(\eta) = \lim_{z \to \eta} d(y, z) - d(x, z),$$

for all $\eta \in \partial X$. The notation is ambiguous but make sense when $\overline{X} = X \sqcup \partial X$ is endowed with the cone topology. A more satisfactory version is perhaps

$$B(x, y)(\eta) = \lim_{t \to \infty} d(y, r(t)) - d(x, r(t)),$$

for any geodesic ray $r : \mathbf{R}_+ \longrightarrow X$ in the equivalence class of η, *i.e.* $r(\infty) = \eta$.

Proposition 1.2.30. — *The Busemann cocycle satisfies:*

- $B(x, x) = 0$,
- $|B(x, y)(\eta)| \leq d(x, y)$,
- $B(x, y) = -B(y, x)$,
- $B(x, y) = B(x, z) - B(y, z)$,
- $B(x, y)(\eta) = B(gx, gy)(g\eta)$

for all $x, y, z \in X$, $\eta \in \partial X$ *and* $g \in \mathrm{Isom}(X)$.

Example 1.2.31. — The Busemann cocycle of $\mathbf{R}^2$ is easy to compute. Let $x, y \in \mathbf{R}^2$, $\eta \in \partial \mathbf{R}^2$ and v be the unit vector such that $t \mapsto r(t) = x + tv$ is the geodesic ray in the class η starting at x. Then

$$\begin{aligned}
B(x, y)(\eta) &= \lim_{t \to \infty} \|y - (x + tv)\| - \|x - (x + tv)\| \\
&= \lim_{t \to \infty} \frac{\|\overrightarrow{xy} - tv\|^2 - \|tv\|^2}{\|\overrightarrow{xy} - tv\| + \|tv\|} \\
&= \lim_{t \to \infty} \frac{\|\overrightarrow{xy}\|^2 - 2t\langle \overrightarrow{xy}, v\rangle + t^2\|v\|^2 - t^2\|v\|^2}{\|\overrightarrow{xy} - tv\| + \|tv\|} \\
&= \lim_{t \to \infty} \frac{\frac{1}{t}\|\overrightarrow{xy}\|^2 - 2\langle \overrightarrow{xy}, v\rangle}{\|\frac{\overrightarrow{xy}}{t} - v\| + \|v\|} \\
&= -\langle \overrightarrow{xy}, v\rangle.
\end{aligned}$$

1.3. Group cohomology

This section contains the many aspects of group cohomology we shall need and some links between them. To begin with, the classical algebraic cohomology of groups was historically introduced in connection with topology and fundamental groups. Later the theory took a more abstract turn in which the bar-resolution yields a fairly simple description of it [**Bro94**]. In the context of topological groups, the bar-resolution can be extended naively in various ways to define *continuous cohomology* and *bounded continuous cohomology.* On the other hand techniques of relative homological algebras produce categorical constructions where powerful tools such as spectral sequences are available. Fortunately, the particular flavor of t.d.l.c. groups simplifies the situation greatly and the naive approach is sufficient, at least at the level of the present work.

Convention 1.3.1. — The vector spaces we consider are over $\mathbf{C}$ and topological vector spaces are implicitly assumed to be locally convex topological vector spaces, (Hausdorff by Convention 1.1.1). By a *G-module* we mean a complex representation (π, V) of a group G. We shall make the standard abuse of omitting either π or V when speaking of the representation. The set of maps between two sets X, V is denoted $\mathscr{F}(X, V)$, and, if X, V are topological spaces, the set of continuous maps is denoted $\mathrm{C}(X, V)$. If $V = \mathbf{C}$ we write $\mathscr{F}(X)$ and $\mathrm{C}(X)$ instead.

1.3.1. Continuous cohomology.

— Let G be a group and V be a G-module. For every $n \in \mathbf{N}$, let $\mathscr{F}^n(G, V)$ be the vector space of functions from the $(n+1)$-fold product G^{n+1} into V endowed with the G-action:

$$(g \cdot f)(x_0, \dots, x_n) = gf(g^{-1}x_0, \dots, g^{-1}x_n), \tag{1.2}$$

for all $g, x_0, \dots, x_n \in G$, where on the right hand side the G-action on V is implicit between the letters g and f. The differential $d^n : \mathscr{F}^n(G, V) \to \mathscr{F}^{n+1}(G, V)$ is given by the classical alternate sum

$$d^n f(x_0, \dots, x_{n+1}) = \sum_{i=0}^{n+1} (-1)^i f(x_0, \dots, \widehat{x}_i, \dots, x_{n+1}), \tag{1.3}$$

where $\widehat{x}_i$ means that we omit the ith variable, is G-equivariant, and satisfies $d^{n+1} \circ d^n = 0$.

Definition 1.3.2. — The *(homogeneous) bar resolution* is the complex $\mathscr{F}^\bullet(G, V)$ defined by :

$$\mathscr{F}^\bullet(G, V) \quad : \quad 0 \longrightarrow \mathscr{F}^0(G, V) \xrightarrow{d^0} \mathscr{F}^1(G, V) \xrightarrow{d^1} \mathscr{F}^2(G, V) \xrightarrow{d^2} \dots \tag{1.4}$$

We speak of the *augmented bar resolution* if we introduce the morphism $\varepsilon : V \to \mathscr{F}(G, V)$ sending v to the constant function $g \mapsto v$ that is

$$0 \longrightarrow V \xrightarrow{\varepsilon} \mathscr{F}^0(G, V) \xrightarrow{d^0} \mathscr{F}^1(G, V) \xrightarrow{d^1} \mathscr{F}^2(G, V) \xrightarrow{d^2} \dots \tag{1.5}$$

The *(abstract) algebraic cohomology of G with coefficient module V* is the graded vector space obtained by taking the cohomology of the G-invariants of the bar resolution:

$$0 \longrightarrow \mathscr{F}^0(G, V)^G \xrightarrow{d^0} \mathscr{F}^1(G, V)^G \xrightarrow{d^1} \mathscr{F}^2(G, V)^G \xrightarrow{d^2} \dots \tag{1.6}$$

In other words,

$$\mathrm{H}^\bullet_{\mathrm{alg}}(G, V) \coloneqq \mathrm{H}(\mathscr{F}^\bullet(G, V)^G). \tag{1.7}$$

The bar resolution is an injective resolution of V and it is well known that the above cohomology does not depend on the choice of such. Suppose G is a topological group and V a topological vector space. One may wish to capture these topological aspects in order to define a notion of 'continuous cohomology'. The naive possibility is to replace $\mathscr{F}^n(G,V)$ by the vector space of continuous functions $\mathrm{C}^n(G,V) \coloneqq \mathrm{C}(G^{n+1},V)$ and proceed exactly as above.

Definition 1.3.3. — Let G be a topological group and V a G-module that is also a topological vector space. The *continuous cohomology of G with coefficient module V* is defined by

$$\mathrm{H}_{\mathrm{c}}^{\bullet}(G,V) \coloneqq \mathrm{H}(\mathrm{C}^{\bullet}(G,V)^G). \tag{1.8}$$

Readily the algebraic cohomology is a particular case of the continuous cohomology by forgetting that G and V carry topologies. Indeed let G be a group and endow it with the discrete topology. We write G_δ the resulting discrete topological group, then

$$\mathrm{H}_{\mathrm{alg}}^{\bullet}(G,V) = \mathrm{H}_{\mathrm{c}}^{\bullet}(G_\delta,V).$$

So far we imposed no restriction on the continuity of the action map $\alpha : G \times V \to V$, which needs to be done for a more categorical formulation. For instance, the authors of [**BW00**, Chapter IX] consider the category $\mathcal{C}_{G,\mathrm{top}}$of *topological G-modules*, *i.e.* G-modules for which α is continuous. For totally disconnected locally compact groups, there is also a notion of *smooth cohomology* and of *continuous smooth cohomology* presented in the book of Borel and Wallach [**BW00**, Chapter X]. Fortunately for us, the various cohomologies coincide in our framwork, see Proposition 1.3.7. Better, for the algebraic groups considered here, the cohomology is computable using the Bruhat-Tits building, see Theorem 1.3.9.

1.3.2. Smooth and admissible representations of t.d.l.c. groups.

— The smooth and admissible representations arise naturally for totally disconnected locally compact groups. In accordance with [**BW00**, Chapter X], t.d.l.c. groups are assumed countable at infinity[(6)] and metrizable for this paragraph, a hypothesis satisfied by the groups with which we are concerned.

Definition 1.3.4 (Smooth representation). — Let (π,V) be a complex representation of a t.d.l.c. group G. A vector in V is called *smooth* if its stabilizer in G is open. The subspace of smooth vectors is denoted V^∞ and is G-invariant. The restriction of π to the smooth vectors is denoted π^∞. The representation (π,V) is called *smooth* if $V = V^\infty$. We also say that V is a *smooth G-module.*

Definition 1.3.5 (Admissible representation). — A smooth representation V is called *admissible* if V^K is finite dimensional for all compact open subgroups K of G.

Since G is t.d.l.c., van Dantzig's Theorem 1.1.10 shows that $v \in V$ being a smooth vector is equivalent to each of the conditions:

- v is fixed by an open subgroup of G.
- v is fixed by a compact open subgroup of G.

(6) In [**CH15**, Remark 2.A.2] it is mentioned that a (Hausdorff) locally compact space X is σ-compact if and only if it is *countable at infinity*, *i.e.* there is a countable exhaustion $\{K_n \mid n \in \mathbf{N}\}$ of X by compact subsets satisfying $K_n \subset \mathrm{int}(K_{n+1})$ for all $n \geq 0$.

Therefore V^∞ is the directed union of its subspaces of the form V^K with K a compact open subgroup of G. Let V_δ be V with the discrete topology, then the smoothness of V is equivalent to the continuity of the action map $G \times V_\delta \to V_\delta$. Indeed one can prove that the latter is continuous if and only if every stabilizer is open. However V_δ is not a topological vector space over $\mathbf{C}$ (for the standard topology of $\mathbf{C}$), hence V_δ is not a topological G-module. On the other hand, V, if smooth, can always be endowed with its finest locally convex topology for which it is a topological G-module according to [**BW00**, Chapter X, §1.3]. If V is moreover admissible, then its finest locally convex topology is that of the strict inductive limit of the finite dimensional subspaces V^K with K ranging among the compact open subgroups. Since G is assumed t.d.l.c. and metrizable, it has a countable basis of identity neighborhoods $\{K_n \mid n \in \mathbf{N}\}$ consisting of compact open subgroups. Therefore the inductive limit is well defined using the sequence $\{V^{K_n} \mid n \in \mathbf{N}\}$.

Definition 1.3.6 (Smooth topological G-module). — Let $V \in \mathcal{C}_{G,\mathrm{top}}$ be a topological G-module. Endow the space V^∞ of smooth vectors of V with the inductive limit topology given by the subspaces V^K, with K compact open subgroup, these being endowed with the subspace topology induced by V. Then V is called a *topological smooth G-module* if $V = V^\infty$ as topological vector spaces. The category of smooth topological G-modules is denoted $\mathcal{C}^\infty_{G,\mathrm{top}}$.

By analogy, let $\mathcal{C}_{G,\mathrm{alg}}$ be the category of G-modules, and $\mathcal{C}^\infty_{G,\mathrm{alg}}$ be the category of smooth G-modules in the sense of Definition 1.3.4. The latter category together with $\mathcal{C}_{G,\mathrm{top}}$ and $\mathcal{C}^\infty_{G,\mathrm{top}}$ have *enough invectives*(7), therefore the *derived functors* of the functor taking the G-invariants define a cohomology in each category. We denote them

$$\mathrm{H}^\bullet_{\mathrm{top}}(G,V), \quad \mathrm{H}^\bullet_{\mathrm{top},\infty}(G,V), \quad \text{and} \quad \mathrm{H}^\bullet_{\mathrm{alg},\infty}(G,V), \tag{1.9}$$

whenever V is a G-module in the corresponding category. (In [**BW00**], they are denoted $\mathrm{H}^\bullet_{\mathrm{ct}}$, $\mathrm{H}^\bullet_d$ and $\mathrm{H}^\bullet_e$ respectively.)

Proposition 1.3.7. — *Let V be a G-module that is also a Banach space. Then Definition 1.3.6 and the discussion above it give V^∞ the same topology and all cohomologies of* (1.9) *with coefficient module V^∞ are canonically isomorphic to $\mathrm{H}^\bullet_c(G,V^\infty)$ and to $\mathrm{H}^\bullet_c(G,V)$ as well.*

Proof. — This follows from the results of Chapter IX and X of [**BW00**], notably Proposition 1.6 of Chapter X and its §5. That the topologies are the same follows from the content of [**BW00**, Chapter X, §1.2-1.3]. En route, it is shown that the complex used in (1.8), with the compact-open topology on $\mathrm{C}^n(G,V)$, is an injective resolution of V in the category $\mathcal{C}_{G,\mathrm{top}}$. □

Let G be the group of F-rational points of a connected, simply-connected, almost F-simple algebraic group over a local field F. Let X be the Bruhat-Tits building of G of the previous section. Recall that X is an irreducible locally finite building of Euclidean type, identified with the geometric realization of paragraph 1.2.2 and on which G acts by type-preserving automorphisms. Since X is contractible, the cohomology of G can be related to the cohomology of two complexes defined in terms of X. The first is the complex of G-equivariant simplicial cochains described in [**BW00**, Chapter X, §§1.10-1.12 and §2]. The finite dimensionality of X, among other things, has the following consequences.

(7) see Lemma 1.6 of Chapter IX, Proposition 1.5 of Chapter X, and §5 of the same chapter in [**BW00**].

Theorem 1.3.8. — [**BW00**, Chapter X, Theorem 2.4] *Let G be as above and V be a G-module that is also a Banach space, then* $\mathrm{H}^n_{\mathrm{c}}(G,V)$ *vanishes for all* $n > \mathrm{rank}_F(G)$. *Let B be an Iwahori subgroup, i.e. the stabilizer of a chamber of X. If V^B is finite dimensional, e.g. if V is admissible, then* $\mathrm{H}^n_{\mathrm{c}}(G,V)$ *is finite dimensional for all* $n \in \mathbf{N}$. *If moreover V has no non-zero B-invariant vectors then* $\mathrm{H}^\bullet_{\mathrm{c}}(G,V) = 0$. □

The second complex also uses the action of G on X and that the stabilizers of simplices, *i.e.* the parahoric subgroups, are compact open. Fix a base vertex $x_0 \in X$ with stabilizer K in G, the G-orbit of x_0 is identified with the discrete G-space G/K thanks to K being an open subgroup. Again, let V be a G-module which is also a Banach space, the space $\mathrm{C}((G/K)^{n+1}, V)$ is isomorphic to the space of continuous functions f on G^{n+1} that are K-invariant on the right, that is

$$f(g_0k, \ldots, g_nk) = f(g_0, \ldots, g_n),$$

for all $k \in K$ and $g_0, \ldots, g_n \in G$. We endow it with the G-action given by (1.2) and denote $\mathrm{C}^n(G/K, V)$ the resulting G-submodule of $\mathrm{C}^n(G, V)$.

Theorem 1.3.9. — *If G and V are as above, the inclusion map* $\mathrm{C}^n(G/K,V) \hookrightarrow \mathrm{C}^n(G,V)$ *induces an isomorphism*

$$\mathrm{H}^\bullet_{\mathrm{c}}(G,V) \cong \mathrm{H}^\bullet(\mathrm{C}^\bullet(G/K,V)^G),$$

where the differential map of the right hand side is given by the same formula as (1.3).

Proof. — We may assume V to be a smooth topological G-module thanks to Proposition 1.3.7. Now, Lemma 2.6 of [**BW00**, Chapter X] shows that $\mathrm{C}^n(G/K, V)$, with the compact-open topology, is s-injective in $\mathcal{C}_{G,\mathrm{top}}$. To translate the notation, we refer to Chapter IX, §§1-2 of *loc. cit.* □

Remark 1.3.10. — We can view a cocycle $f \in \mathrm{C}^n(G/K,V)^G$ as a G-equivariant function on a subset of vertices of X, namely on the G-orbit of x_0, satisfying $d^n f = 0$. Clearly, the restriction to a G-orbit of a G-equivariant function $f : X^{n+1} \to V$, defined on the whole building, and satisfying the cocycle identity, gives a cocycle for the above cohomology. The previous theorems says in particular that any cohomology class for the (continuous) bar resolution of G can be obtained in this way.

CHAPTER 2

THE STEINBERG REPRESENTATION AND THE NATURAL COCYCLE OF KLINGLER

This chapter summarizes the two motivational articles [**Kli03**] and [**Kli04**]. Let G be the group of F-rational points of a connected, simply-connected, almost F-simple algebraic group over a local field F. We first give a result of Casselman describing all irreducible admissible coefficient G-modules with non-trivial cohomology. Among them only the Steinberg representation $\mathbf{St}$ is unitarizable and non-trivial. The cohomology of G with coefficient $\mathbf{St}$ vanishes in all degrees except in the rank of G where it is one-dimensional. In [**Kli03**], Klingler constructed a natural cocycle by means of the Bruhat-Tits building of G and produced 'the' non-trivial cohomology class. The method involves only building theoretic tools and can be easily adapted to an arbitrary Euclidean building. The last section investigates the unitarity of $\mathbf{St}$, which was first proved by Casselman and Borel-Serre by different non-explicit methods. Since we want to compute the growth of the norm of the Klingler cocycle, we need a norm as explicit as possible. It is Klingler again who found an explicit isomorphism [**Kli04**], called the *Poisson transform*, between the Steinberg representation $\mathbf{St}$ and the space of smooth square summable harmonic functions on the chambers of the Bruhat-Tits building. In Chapter 4, we shall use it to compute an explicit upper bound to the norm of Klingler's cocycle when the building is a regular tree. However in higher rank the complexity is yet to be overcome.

2.1. Irreducible admissible representations

Definition 2.1.1. — Let X be a totally disconnected locally compact Hausdorff space and V a complex vector space. A function $f : X \to V$ is *locally constant* if every point of X has a neighborhood on which f is constant. The space of V-valued locally constant functions is denoted $\mathrm{C}^\infty(X, V)$ and simply $\mathrm{C}^\infty(X)$ when $V = \mathbf{C}$.

A locally constant function is continuous for all topological vector spaces V.

Example 2.1.2. — Suppose X is the quotient of a t.d.l.c. group G by a closed cocompact subgroup P and consider the left regular representation of G on $\mathscr{F}(X, V)$. Then, according to Definition 1.3.4, a smooth function $f : X \to V$ is K-left-invariant for some compact open subgroup K of G, thus locally constant since Kx is a neighborhood of x. Conversely, a locally constant function f is K-left-invariant for some compact open subgroup. Indeed, the compact open subgroups form a basis of neighborhoods of the identity in G. Thus each point $x \in X$ has a neighborhood of the form Kx, with $K < G$ compact open, on which f is constant. By compactness, we need only finitely many compact open subgroups to cover X; their intersection is a compact open subgroup under which f is left invariant. If V is a topological vector space, we conclude

$$\mathrm{C}^{\infty}(X,V)=\mathscr{F}(X,V)^{\infty}=\mathrm{C}(X,V)^{\infty},$$

the notation is consistent.

Let G be the group of F-rational points of a connected, simply-connected, almost F-simple algebraic group over a local field F. Let $\partial X=\Delta(G,P)$ be the spherical building of G associated to the Tits system (G,P,N,S_0) of the previous chapter. Recall that the parabolic subgroups of G are the conjugates of the standard parabolic subgroups P_J with $J\subset I_0$, *e.g.* $P_\emptyset=P$ and $P_{I_0}=G$. If Q is conjugate to P_J, the integer $\mathrm{prk}(Q)\coloneqq\mathrm{card}(I_0\smallsetminus J)$ is called the *parabolic rank* of Q, (cardinal of the cotype of P_J).

Definition 2.1.3. — Let Q be a parabolic subgroup of G, we define Ind_Q^G, the *induced representation with respect to* Q, to be the left regular representation of G in the space $\mathrm{C}^{\infty}(G/Q)$. The action on a locally constant function $f:G/Q\to\mathbf{C}$ is given by

$$gf(x)=f(g^{-1}x),$$

for all $x\in G/Q$ and $g\in G$.

On the one hand, G/P is compact for the quotient topology, thanks to the Iwasawa decomposition of Proposition 2.3.1, and so is G/Q. On the other hand, G is totally disconnected locally compact, thus Example 2.1.2 applies. Since Ind_Q^G is the smooth induction of an admissible representation, namely of the trivial representation $\mathbf{C}$ of Q, and because G/Q is compact, Ind_Q^G is an admissible representation of G, see [**BW00**, Chapter X, Lemma 1.8]. For $Q\subset Q'$, the surjective map $G/Q\to G/Q'$ induces a G-morphism $\pi_{Q'Q}:\mathrm{Ind}_{Q'}^G\to\mathrm{Ind}_Q^G$ by precomposition.

Definition 2.1.4. — Let V_Q denote the quotient of Ind_Q^G by the submodule generated by the images $\pi_{Q'Q}(\mathrm{Ind}_{Q'}^G)$ where Q' ranges among the parabolic subgroups with $Q\subset Q'$. The module V_P is called the *Steinberg representation* of G and is denoted **St**. Notice that $V_G=\mathrm{Ind}_G^G$ is the space of constant functions on G, *i.e.* the trivial representation.

We can focus on the induced representations with respect to standard parabolic subgroups only. We write $\mathrm{Ind}_J^G\coloneqq\mathrm{Ind}_{P_J}^G$ and $V_J\coloneqq V_{P_J}$. The picture to keep in mind is the following:

$$\begin{array}{cccc}
I_0 & P_{I_0} & \mathrm{Ind}_{I_0}^G=\mathbf{C} & \twoheadrightarrow \mathbf{C}\\
\cup & \cup & \cap & \\
J & P_J & \mathrm{Ind}_J^G & \twoheadrightarrow V_J\\
\cup & \cup & \cap & \\
\emptyset & P_\emptyset=P & \mathrm{Ind}_\emptyset^G & \twoheadrightarrow \mathbf{St}
\end{array}$$

Theorem 2.1.5 (Casselman). — [**BW00**, Chapitre X, Theorem 4.12] *Let V be an irreducible admissible representation of G such that $\mathrm{H}_{\mathrm{c}}^{\bullet}(G,V)\neq 0$. Then V is isomorphic to V_Q for some parabolic subgroup Q. The continuous cohomology $\mathrm{H}_{\mathrm{c}}^{n}(G,V)$ is one-dimensional if $n=\mathrm{prk}(Q)$ and vanishes in all other degrees.* □

Recall that a representation V of G is called *unitarizable* or *pre-unitary*, if V can be endowed with a G-invariant inner product. Interestingly, the representation $\mathbf{St}$ is the only non-trivial unitarizable representation up to isomorphism. The modules V_Q, for $Q \neq G$, and not conjugated to P, are all non-unitarizable, see [**BW00**, Chapter XI, §4]. We discuss the existence of a G-invariant scalar product for $\mathbf{St}$ in Section 2.3.

2.2. Natural cocycle for the Steinberg representation

Let G be the group of F-rational points of a connected, simply-connected, almost F-simple algebraic group over a local field F with Bruhat-Tits building X, let P be the standard minimal parabolic subgroup, and let $\mathbf{St}$ be the Steinberg representation of G. We saw in the previous section that the cohomology with coefficient in the Steinberg module $\mathbf{St}$ is one dimensional in degree the F-rank of G and zero in all other degrees. Any cocycle that is not a coboundary generates the cohomology of G. Klingler [**Kli03**, Theorem 1], built a natural one, $\overline{\mathrm{vol}}_X$, on the Bruhat-Tits building, giving a cocycle $\overline{\mathrm{vol}}_G$ for G, see Remark 1.3.10. The construction starts by building a G-equivariant cocycle vol_X valued in $\mathrm{Ind}_P^G = \mathrm{C}^\infty(G/P)$, whose projection into $\mathbf{St} = V_P$ is $\overline{\mathrm{vol}}_X$, the desired cocycle. If n is the F-rank of G, this is summarized in a commutative diagram of G-maps:

$$\begin{tikzcd}
X^{n+1} \arrow[dr, "\mathrm{vol}_X"] \arrow[ddr, bend right, "\overline{\mathrm{vol}}_X"'] & & G^{n+1} \arrow[ll, "\mathrm{ev}_{x_0}"'] \arrow[dl, "\mathrm{vol}_G"'] \arrow[ddl, bend left, "\overline{\mathrm{vol}}_G"] \\
& \mathrm{C}^\infty(G/P) \arrow[d] & \\
& \mathbf{St} &
\end{tikzcd}$$

Let Ω be the set of chambers of the spherical building ∂X, it corresponds to G/P. The naive definition of $\mathrm{vol}_X(\underline{x})$, for $\underline{x} \in X^{n+1}$, is to consider for every chamber at infinity $\xi \in \Omega$ the retraction $\rho_{(\xi,A)}$ of X onto an apartment A containing a sector of ξ. Then the Euclidean convex hull of the retraction of $\underline{x}$ in A has an oriented volume given by the volume form vol_A of A which is known to satisfy the n-cocycle identity $d^n \mathrm{vol}_A = 0$. This construction can be made G-equivariant by being careful with the choice of orientation of A.

This idea is well defined in any Euclidean building X. We present Klingler's construction of the cocycle in an arbitrary Euclidean building X. The cocycle is equivariant under the full group of type-preserving automorphisms of X. For simplicity we chose to work with a Euclidean building of dimension 2. This is not very restrictive thanks to Tits' classification [**Tit74**], which shows that, in dimension at least 3, all Euclidean buildings arise as the Bruhat-Tits building of some group of algebraic flavor; Klingler already covered this case in [**Kli03**].

2.2.1. The equivariant 2-cocycle vol_X. — In this paragraph, we define the cocycle of Klingler, vol_X, for a Euclidean building X with a type function $\tau : X^{(0)} \to I = \mathbf{Z}/3\mathbf{Z}$, for which the vertices of type 0 are special vertices. The complete system of apartments of X is denote $\mathrm{Apt}(X)$ and consists of all subspaces of X isometric to $\mathbf{R}^2$. We transport the affine structure of $\mathbf{R}^2$ onto each apartment independently of the choice of an isometry with $\mathbf{R}^2$.

Notation 2.2.1. — We use the following notation:

- Ω denotes $\mathrm{Ch}(\partial X)$. We usually denote an element of Ω with the Greek letter ξ.

- $\Omega\mathrm{Apt}(X)$ denotes the set of pairs $(\xi, A) \in \Omega \times \mathrm{Apt}(X)$ such that $\xi \in \mathrm{Ch}(\partial A)$.

We first define orientations.

Definitions 2.2.2. — Let $A \in \mathrm{Apt}(X)$.

- A *frame* in A is an ordered triple (x, y, z) of points in the geometric realization of A such that $\overrightarrow{xy}, \overrightarrow{xz}$ are linearly independent. Equivalently x, y, z are affinely independent.
- An *orientation* o of A is an equivalence class of frames where two frames (x, y, z) and (x', y', z') are equivalent if the linear part of the unique affine map $A \to A$ sending (x, y, z) to (x', y', z') has positive determinant. The set of orientations of A is written $\mathrm{Or}(A)$ and has cardinal 2.

Remark 2.2.3. — In general, for $o, o' \in \mathrm{Or}(A)$, we use the symbol oo' to denote the real number $\delta_{\{o=o'\}} - \delta_{\{o \neq o'\}}$. In other words,

$$oo' = \begin{cases} 1 & \text{if} \quad o = o', \\ -1 & \text{if} \quad o \neq o'. \end{cases}$$

Example 2.2.4. — Let C be a chamber of an apartment $A \in \mathrm{Apt}(X)$. We write $\sigma(C, A)$ the orientation given by the frame (x, y, z) such that x, y, z are the vertices of C with

$$\tau(x, y, z) \coloneqq (\tau(x), \tau(y), \tau(z)) = (0, 1, 2).$$

Given an orientation o of an apartment A, there is a chamber C in A such that $o = \sigma(C, A)$. Indeed two adjacent chambers of A define opposite orientations. So essentially, an orientation of A can be represented by one of its chambers.

Definitions 2.2.5. — A *choice of orientations* of X is by definition a map

$$\sigma : \Omega\mathrm{Apt}(X) \to \bigsqcup_{A \in \mathrm{Apt}(X)} \mathrm{Or}(A)$$

such that $\sigma(\xi, A) \in \mathrm{Or}(A)$.

Let G be the group of all type-preserving automorphisms of X. Recall that G also acts on ∂X by automorphism. Consider the natural actions of G on the following spaces: $\mathrm{Ch}(X)$, Ω, $\mathrm{Apt}(X)$, $\Omega\mathrm{Apt}(X)$. There is an action of G on $\bigsqcup_{A \in \mathrm{Apt}(X)} \mathrm{Or}(A)$ with respect to which $g \in G$ maps an orientation o, given by a frame (x, y, z), of $A \in \mathrm{Apt}(X)$ to the orientation go of gA defined by the frame (gx, gy, gz). If o is the orientation $\sigma(C, A)$ given by a chamber C, then $go = \sigma(gC, gA)$.

Definition 2.2.6. — A choice of orientations σ is called G-*equivariant*, or simply *equivariant*, if $g\sigma(\xi, A) = \sigma(g\xi, gA)$ for all $g \in G$ and $(\xi, A) \in \Omega\mathrm{Apt}(X)$.

Example 2.2.7. — There is an equivariant choice of orientations defined as follows. For $(\xi, A) \in \Omega\mathrm{Apt}(X)$, we define $\sigma(\xi, A) \coloneqq \sigma(C_x(\xi), A)$ where x is any vertex of A of type 0 and $C_x(\xi)$ is the initial chamber of the unique sector issuing at x pointing toward ξ. This is independent of the choice of $x \in A$ of type 0. The G-equivariance of this choice of orientations is clear since G acts on X by type-preserving automorphisms. If G acts strongly transitively on ∂X, which is the case if X is a Bruhat-Tits building, it acts transitively on $\Omega\mathrm{Apt}(X)$. Hence there are only two possible equivariant choices of orientations in this case. At the other extreme, X may have trivial automorphism group. We call σ the *canonical equivariant choice of orientations*.

Definition 2.2.8. — Let o be an orientation of an apartment A of X. We view A as its geometric realization isometric to $\mathbf{R}^2$ endowed with the Lebesgue measure $\mathrm{vol}_A = \mathrm{vol}_{\mathbf{R}^2}$. For every triple (x, y, z) of points in A, *the oriented volume* of (x, y, z) with respect to (A, o) is defined as

$$\mathrm{vol}_{(A,o)}(x, y, z) := \mathrm{vol}_A(\mathrm{conv}(x, y, z)) \left(\delta_{(x,y,z)\in o} - \delta_{(x,y,z)\notin o}\right),$$

where $\delta_{(x,y,z)\in o}$ is 1 if (x, y, z) is a frame yielding the orientation o and 0 else, and $\mathrm{conv}(x, y, z)$ denotes the Euclidean convex hull of $\{x, y, z\}$. We shall write $\underline{x} = (x, y, z)$ and $\mathrm{vol}_{(A,o)}(\underline{x})$.

Given a pair (A, o), let e_1, e_2 be an orthonormal basis of A giving it the orientation o. The oriented volume of $(x, y, z) \in A^3$ is also computed with the volume form $e_1 \wedge e_2$, that is

$$\mathrm{vol}_{(A,o)}(x, y, z) = \frac{1}{2} e_1 \wedge e_2(\overrightarrow{xy}, \overrightarrow{xz}) = \frac{1}{2} \det \begin{pmatrix} \langle \overrightarrow{xy}, e_1\rangle & \langle \overrightarrow{xy}, e_2\rangle \\ \langle \overrightarrow{xz}, e_1\rangle & \langle \overrightarrow{xz}, e_2\rangle \end{pmatrix}.$$

Proposition 2.2.9. — *Let (A, o) and $\underline{x}$ be as in the previous definition, then*

$$\mathrm{vol}_{(gA,go)}(g\underline{x}) = \mathrm{vol}_{(A,o)}(\underline{x}),$$

for all $g \in G$.

Proof. — The group G acts on X by isometries. Therefore $g \in G$ restricted to A is an isometry onto its image gA. In addition, the orientation o and go are compatible by construction. □

The idea of Klingler cocycle consists of sending a triple $\underline{x} = (x, y, z)$ of points in X onto an apartment by means of retraction and then to compute the oriented volume.

Definition 2.2.10. — For every $(\xi, A) \in \Omega\mathrm{Apt}(X)$, *the canonical retraction onto A centered at* ξ is the type-preserving chamber map $\rho_{(\xi,A)} : X \to A$ defined as follows. For $x \in X$, consider an apartment A' containing it and with $\xi \in \partial A'$. In particular $A \cap A'$ contains a sector in the class ξ, see [**AB08**, Theorem 11.63]. By definition $\rho_{(\xi,A)}(x)$ is the image of x under the unique retraction $\rho_{C,A}$ fixing $A \cap A'$ pointwise, where C is any chamber of that intersection, see axiom (B2$''$) after Definition 1.2.6. For every $\underline{x} = (x, y, z) \in X^3$, we write

$$\rho_{(\xi,A)}(\underline{x}) := (\rho_{(\xi,A)}(x), \rho_{(\xi,A)}(y), \rho_{(\xi,A)}(z)).$$

Remark 2.2.11. — This does not depend on the choice of A'. Given two apartments A, A' having ξ as a chamber at infinity, the maps $\rho_{(\xi,A)}|_{A'}$ and $\rho_{(\xi,A')}|_A$ are mutual inverse isometries. Note also that $\rho_{(g\xi,gA)} = g \circ \rho_{(\xi,A)} \circ g^{-1}$, thus $g(\rho_{(\xi,A)}(\underline{x})) = \rho_{(g\xi,gA)}(g\underline{x})$, for all $g \in G$.

The next theorem enables us to define Klingler's cocycle.

Theorem 2.2.12 (Klingler). — *Let $\sigma : \Omega\mathrm{Apt}(X) \to \bigsqcup_{A\in\mathrm{Apt}(X)} \mathrm{Or}(A)$ be the canonical equivariant choice of orientations of Example 2.2.7. For any triple $\underline{x} = (x, y, z) \in X^3$ and any two pairs $(\xi, A), (\xi, A') \in \Omega\mathrm{Apt}(X)$, one has:*

$$\mathrm{vol}_{(A,\sigma(\xi,A))}(\rho_{(\xi,A)}(\underline{x})) = \mathrm{vol}_{(A',\sigma(\xi,A'))}(\rho_{(\xi,A')}(\underline{x})).$$

The original proof of Klingler [**Kli03**, §3.1.1] is easily adapted to a general Euclidean building (here of dimension 2) admitting a strongly transitive action of a group by type-preserving automorphisms (with respect to the complete system of apartments). This is the case for the Bruhat-Tits building of a connected, simply connected, almost F-simple algebraic group over a local field F.

Proof for strongly transitive actions. — Let P_ξ be the minimal parabolic subgroup of G stabilizing ξ and let N_ξ be the subgroup of all elements fixing pointwise a sector in the class ξ. If $x \in X$, then $\rho_{(\xi,A)}(x)$ is the unique point lying in the intersection of the N_ξ-orbit of x and A. With this characterization, one observes that

$$\rho_{(\xi,A')} \circ \rho_{(\xi,A)} = \rho_{(\xi,A')}.$$

The pointwise stabilizer of the intersection $A \cap A'$ acts transitively on the apartments containing it, see Corollary 1.2.23. Hence there is a $g \in N_\xi$ such that

$$g|_A = \rho_{(\xi,A')}|_A \qquad \text{and} \qquad g^{-1}|_{A'} = \rho_{(\xi,A)}|_{A'}.$$

Obviously, we have $g(\xi, A) = (g\xi, gA) = (\xi, A')$ and, by equivariance, $g\sigma(\xi, A) = \sigma(\xi, A')$. Therefore,

$$\begin{aligned}
\mathrm{vol}_{(A,\sigma(\xi,A))}(\rho_{(\xi,A)}(\underline{x})) &= \mathrm{vol}_{(gA,g\sigma(\xi,A))}(g\rho_{(\xi,A)}(\underline{x})) \\
&= \mathrm{vol}_{(A',\sigma(\xi,A'))}(g|_A \circ \rho_{(\xi,A)}(\underline{x})) \\
&= \mathrm{vol}_{(A',\sigma(\xi,A'))}(\rho_{(\xi,A')} \circ \rho_{(\xi,A)}(\underline{x})) \\
&= \mathrm{vol}_{(A',\sigma(\xi,A'))}(\rho_{(\xi,A')}(\underline{x})),
\end{aligned}$$

as desired. □

Proof in the general case. — Given $x, y, z \in X$ and $\xi \in \Omega$, let A_x, A_y, A_z be apartments containing x, y, z respectively, and ξ as a chamber at infinity. Let A, A' be as in the statement. The five apartments of

$$\mathscr{A} := \{A, A', A_x, A_y, A_z\}$$

have, pairwise, their intersection containing a sector in the class ξ. Therefore, by successive extraction of subsector, we can find a sector S in the global intersection of these five apartments. The canonical choice of orientations of Example 2.2.7 gives A and A' the orientation defined by a single chamber in S. Now for $\Sigma, \Sigma' \in \mathscr{A}$, the retraction $\rho_{(\xi,\Sigma)}|_{\Sigma'}$ is an orientation-preserving isometry fixing S pointwise. Using these isometries the result holds. □

Definition 2.2.13. — If σ denote the canonical equivariant choice of orientations, the quantity in the previous theorem depends only on $\underline{x}$ and ξ. We denote it $\mathrm{vol}_X^\sigma(x, y, z)(\xi)$. This defines a map $\mathrm{vol}_X^\sigma : X^3 \mapsto \mathscr{F}(\Omega)$ called *Klingler's cocycle*, where $\mathscr{F}(\Omega)$ is the space of complex functions on Ω.

Remark 2.2.14. — If $\bar\sigma$ is the opposite of the canonical equivariant choice of orientations then $\mathrm{vol}_X^{\bar\sigma} = -\mathrm{vol}_X^\sigma$. Our purpose being to compute an L^2-norm of this cocycle (see Chapter 2.3), it will be clear that the norm is independent of the choice of σ or $\bar\sigma$. Hence, we shall drop the exponent σ and simply write vol_X.

Theorem 2.2.15 (Klingler, G-equivariance). — *The map* vol_X *is a G-equivariant* 2*-cocycle.*

If X is the Bruhat-Tits building of a connected, simply-connected, almost F-simple algebraic group over a local field F, then the cohomology class of $\overline{\mathrm{vol}}_G$ induced by vol_X is moreover non-trivial in $\mathrm{H}^2_{\mathrm{c}}(G, \mathbf{St})$, see [**Kli03**, §3.1.2] for the proof.

Proof. — The cocycle relation is clear as the volume form is a 2-cocycle in $\mathbf{R}^2$. Let $g \in G$, $\underline{x} \in X^3$, and $\xi \in \Omega$. We need to show $\mathrm{vol}_X(g\underline{x})(g\xi) = \mathrm{vol}_X(\underline{x})(\xi)$. By definition, the right-hand side is

$\mathrm{vol}_{(A,\sigma(\xi,A))}(\rho_{(\xi,A)}(\underline{x}))$ for any A with $(\xi, A) \in \Omega\mathrm{Apt}(X)$. But since $(g\xi, gA) \in \Omega\mathrm{Apt}(X)$, the left-hand side is

$$\begin{aligned}\mathrm{vol}_{(gA,\sigma(g\xi,gA))}(\rho_{(g\xi,gA)}(g\underline{x})) &= \mathrm{vol}_{(gA,\sigma(g\xi,gA))}(g\rho_{(\xi,A)}(\underline{x})) \\ &= \mathrm{vol}_{(gA,g\sigma(\xi,A))}(g\rho_{(\xi,A)}(\underline{x})) \\ &= \mathrm{vol}_{(A,\sigma(\xi,A))}(\rho_{(\xi,A)}(\underline{x})),\end{aligned}$$

where the last equality is because g is an isometry. □

2.3. Unitarizability via the Poisson transform

We come back to the setting of Section 2.1 and suppose that G is the group of F-points of a connected, simply-connected, almost F-simple algebraic group over a local field F. Independently Casselman [**Cas74**] and Borel-Serre [**BS76**, §5.10] showed that the Steinberg representation **St** is unitarizable. The proof of Casselman can be found in his famous unpublished notes [**Cas95**]. Borel-Serre's proof uses a long exact sequence in cohomology which eventually shows **St** to be isomorphic to the space $\mathcal{H}(X)^\infty \cap \mathrm{L}^2(G/B)$ of smooth square integrable harmonic functions on the set of chambers of the Bruhat-Tits building of G. (See below for the definitions.) Knowing this, Klingler [**Kli04**] defined an analogue of the classical Poisson transform for this setting of double Tits systems, which eventually gives an explicit isomorphism with $\mathcal{H}(X)^\infty \cap \mathrm{L}^2(G/B)$, on which G acts continuously by unitary operators.

The buildings X and ∂X are the buildings $\Delta(G, B)$ and $\Delta(G, P)$ associated to the Tits systems (G, B, N, S) and (G, P, N, S_0) respectively. Here P is the minimal standard parabolic subgroup and B is the minimal standard parahoric subgroup. The sets of involutions S_0 and S are indexed with I_0 and $I = I_0 \cup \{0\}$, so that the vertices of type 0 are special, and generate the finite Weyl group W and the affine Weyl group W_{aff} respectively.

Recall that for its F-topology, G is a t.d.l.c. group, B is a compact open subgroup of G and P a cocompact closed subgroup of G. The set of chambers $\mathrm{Ch}(X)$ and $\mathrm{Ch}(\partial X)$ correspond to the set G/B and G/P respectively, see paragraph 1.2.2. With their respective topologies induced from that of G, the quotient G/B is discrete, whereas G/P is compact. Similarly to the paragraph above Theorem 1.3.9, we identify the space of complex valued functions on G/B with the space of continuous functions on G that are B-invariant on the right:

$$\mathrm{C}(G/B) = \{f : G \to \mathbf{C} \mid f(gb) = f(g) \text{ for all } g \in G, b \in B\}.$$

In particular, finitely supported functions on G/B correspond to compactly supported right-B-invariant functions on G, we write $\mathrm{C}_c(G/B)$ the corresponding space. With respect to this identification the Haar measure μ_G of G restricted to $\mathrm{C}_c(G/B)$ corresponds to the counting measure on G/B. Consequently we have G-isomorphisms, for the left regular actions of G,

$$\mathscr{F}(\mathrm{Ch}(X)) \cong \mathrm{C}(G/B) \quad \text{and} \quad \ell^2(\mathrm{Ch}(X)) \cong \mathrm{L}^2(G/B),$$

where $\mathrm{L}^2(G/B)$ is the closure of $\mathrm{C}_c(G/B)$ in $\mathrm{L}^2(G, \mu_G)$.

The aforementioned BN-pairs have their respective Bruhat decompositions

$$G = \bigsqcup_{w \in W_{\mathrm{aff}}} BwB \quad \text{and} \quad G = \bigsqcup_{w \in W} PwP,$$

but are also related by the Iwasawa decomposition.

Proposition 2.3.1 (Iwasawa decomposition). — *Let G, B, P, W be as above, then*

$$G = \bigsqcup_{w \in W} BwP.$$

Proof. — Let x be the (special) vertex of type 0 of the (fundamental) chamber C of X representing B. By Corollary 1.2.23, the stabilizer of C, namely B, acts transitively on the apartments of X containing C, in particular on the set of sectors issuing at x containing C as an initial chamber. This describes the B-orbit of P in $\mathrm{Ch}(\partial X)$. Similarly the stabilizer of a chamber in the link of x acts transitively on the set of sectors having that chamber as initial chamber. Since G is strongly transitive on X, the stabilizer of x, say K, is strongly transitive on $\mathrm{lk}(x)$. But $\mathrm{lk}(x)$ is a building of type W, the chambers wB forming an apartment of $\mathrm{lk}(x)$, namely $N \cap \mathrm{lk}(x)$, for $w \in W$. Hence, the link $\mathrm{lk}(x)$ is decomposed in B-orbits according to the Bruhat decomposition

$$K = \bigsqcup_{w \in W} BwB.$$

Since every chamber at infinity is represented by a unique sector issuing at x, the B-orbits in $\mathrm{lk}(x)$ determine the B-orbits in $G/P = \mathrm{Ch}(\partial X)$. This proves the Iwasawa decomposition, which, by the way, implies $G = KP$. □

Remark 2.3.2. — The Iwasawa decomposition implies that the B-orbits in G/P are

$$\mathcal{O}_w \coloneqq BwP/P,$$

with $w \in W$. The proof shows that $\mathcal{O}_w$ consists of classes of sectors having their initial chamber in the BwB/B, see also the comment in [**Kli04**, §2, Remarques, 1.][(1)]. Note that we only used the strong transitivity of G on the complete system of apartments. This holds generally in this setting, see [**AB08**, Proposition 11.99].

Let ν_w be the unique Borel B-invariant probability measure on $\mathcal{O}_w$. And let ν_B be the signed measure on G/P given by

$$\nu_B = \sum_{w \in W} (-1)^{\ell(w)} \nu_w.$$

Definition 2.3.3 (Poisson tranform). — For $f \in \mathrm{C}(G/P)$, the *Poisson tranform* of f is defined by

$$\mathcal{P}f(g) = \int_{G/P} f(gx) d\nu_B(x),$$

for all $g \in G$.

In measure theoretic terms, the Poisson transform of f is equivalently written

$$\mathcal{P}f(g) = \langle g^{-1} f, \nu_B \rangle = \langle f, g_* \nu_B \rangle, \tag{2.1}$$

where $\langle \cdot, \cdot \rangle$ is the natural pairing of the space of compactly supported functions with the space of measures, $g_* \nu_B$ is the image measure of ν_B under g, and $g^{-1} f$ is the action of g^{-1} on f by left translation. The Poisson transform of f evaluated at $g \in G$ is the integral of f against the measure $g_* \nu_B$. Looking globally at the action of g on X and ∂X, we can think of $g_* \nu_w$ as the unique Borel probability measure on $g\mathcal{O}_w$ invariant under gBg^{-1}. Hence $g_* \nu_B$ is the alternating sum of the $g_* \nu_w$. Intuitively the measure ν_B is defined by the position of the chamber B in the

[(1)] We point out that point 2. of the Remarques is an error without consequence.

building X and $g_*\nu_B$ is defined by that of the chamber gB. This will be made rigorous in Chapter 3, see paragraph 3.4.2.

We are ready to define the space of harmonic functions on X (or rather on the chambers of X).

Definition 2.3.4 (Harmonic function). — A function $\phi : \mathrm{Ch}(X) \to \mathbf{C}$ is *harmonic* if for every chamber $C \in \mathrm{Ch}(X)$ and every type $i \in I$, the average of ϕ over all chambers i-adjacent to C is zero, *i.e.*

$$\sum_{C' \sim_i C} \phi(C') = 0.$$

Recall that C is always i-adjacent to itself for all $i \in I$. The space of harmonic functions is denoted $\mathcal{H}(X)$.

Theorem 2.3.5 (Klingler). — *The Poisson transform* $\mathcal{P}\colon \mathrm{C}(G/P) \to \mathrm{C}(G)$ *induces an isomorphism* $\mathcal{P}\colon \mathbf{St} \to \mathcal{H}(X)^\infty \cap \mathrm{L}^2(G/B)$ *of irreducible admissible representations.*

Sketch of the proof. — The full proof can be found in [**Kli04**]. We sketch some of the ideas. The map is a G-morphism for the left regular action on both sides. For if $g, h \in G$, then

$$\mathcal{P}(hf)(g) = \langle hf, g_*\nu_B\rangle = \langle f, (h^{-1}g)_*\nu_B\rangle = \mathcal{P}f(h^{-1}g) = h(\mathcal{P}f)(g),$$

and linearity is clear. The measure ν_B is B-invariant which means $b_*\nu_B = \nu_B$, for all $b \in B$. Therefore, thanks to (2.1), $\mathcal{P}f$ is B-invariant on the right, *i.e.*

$$\mathcal{P}f(gb) = \mathcal{P}f(g),$$

for $g \in G$ and $b \in B$, so that $\mathcal{P}(\mathrm{C}(G/P))$ sits in $\mathrm{C}(G/B)$. Clearly smooth vectors are sent to smooth vectors because the Poisson transform is a G-morphism. Harmonicity of $\mathcal{P}f$ comes from the measure ν_B. For $i \in I$, let B_i denote the standard parabolic subgroup $B_{\{i\}}$, stabilizing the panel of cotype i of the chamber B (*i.e.* the codimension 1 face opposite to the vertex of type i of the chamber B). The chambers i-adjacent to B correspond to the left cosets of B_i/B. By decomposing the orbits of B_i on $\mathrm{Ch}(\partial X)$ into B-orbits, one shows

$$\sum_{u \in B_i/B} u_*\nu_B = 0,$$

see [**Kli04**, Lemme 6]. This implies

$$\sum_{C \sim_i B} \mathcal{P}f(C) = \sum_{u \in B_i/B} \mathcal{P}f(uB) = \langle f, \sum_{u \in B_i/B} u_*B\rangle = 0.$$

Now for an arbitrary chamber $C' = gB$, the proof follows from the discussion above Definition 2.3.4. Indeed, the action of g yields:

$$\sum_{C \sim_i C'} \mathcal{P}f(C) = \sum_{u \in B_i/B} \mathcal{P}f(guB) = \langle g^{-1}f, \sum_{u \in B_i/B} u_*\nu_B\rangle = 0.$$

To see that the Poisson transform factors through $\mathbf{St}$, we realize $\mathrm{C}^\infty(G/Q)$ as a submodule of $\mathrm{C}^\infty(G/P)$ for every standard parabolic subgroup Q. For $i \in I_0$, it suffices to show that $\mathcal{P}$ vanishes on $\mathrm{C}^\infty(G/P_i)$ where P_i denotes the standard parabolic subgroup $P_{\{i\}}$. A similar argument to that for harmonicity can be applied, [**Kli04**, Lemme 4]. We prove and discuss square summability below. $\square$

In [**Kli04**, §5], the author argues as follows. The Poisson transform evaluated at $1 = 1_G$ can be seen as a linear form on **St**:

$$f \mapsto \mathcal{P}f(1) = \langle f, \nu_B \rangle$$

so that $c(g) = \langle gf, \nu_B \rangle = \mathcal{P}f(g^{-1})$ is a matrix coefficient of **St**. Proposition 2.5.3 of [**Cas95**] states that an admissible irreducible representation has all its matrix coefficients in $\mathrm{L}^2(G, \mu_G)$ if and only if there exists at least one non-zero square integrable matrix coefficient. (Our assumptions on G implies it has finite center.) Consequently the Poisson transform maps **St** into $\mathrm{L}^2(G/B)$ if and only if there exists one $f \in \mathbf{St}$ such that $\mathcal{P}f \in \mathrm{L}^2(G/B)$. Indeed the linear form defined above is non-zero since $\langle \mathbb{1}_{\mathcal{O}_w}, \nu_B \rangle = \langle \mathbb{1}_{\mathcal{O}_w}, \nu_w \rangle = (-1)^{\ell(w)}$, for all $w \in W$. The square integrability is then proved by showing the B-invariant vectors of **St** to have square integrable Poisson transform, [**Kli04**, §5.1]. From the proof, we realized that the only necessary ingredients were the harmonicity of $\mathcal{P}f$ and its smoothness, implied by the B-invariance. Using a Lemma due to Bruhat, one can generalize the proof of Klingler. This was inspired by [**Bro14**, §3] and [**Bor76**, §4].

Proposition 2.3.6. — *Suppose that the Bruhat-Tits building of G has regularity parameter $q = q_i$ for all $i \in I$ and let $\phi : \mathrm{Ch}(X) \to \mathbf{C}$ be a harmonic function. If ϕ is smooth, i.e. invariant under the left regular action of a compact open subgroup of G, then $\phi \in \ell^2(\mathrm{Ch}(X))$.*

Example 2.3.7. — Suppose X is a regular tree of valency $q+1$, for $q \in \mathbf{N}$, *i.e.* a locally finite regular Euclidean building of type $\widetilde{A}_1$. In this case, the classical harmonicity of a function ϕ, defined on the edges of X, coincides with our definition. The main ingredient for the proof of Proposition 2.3.6 is the following. Suppose ϕ is a harmonic function with the following property. There is an edge C_0 of X such that, for every $d \in \mathbf{N}$, ϕ is constant on the set of edges at distance d from C_0. Suppose D is at distance $d \in \mathbf{N}$ from C_0, and that C is adjacent to D but at distance $d+1$ from C_0. Then ϕ being harmonic implies

$$-\phi(D) = \sum_{\substack{C' \sim D \\ d(C', C_0) = d+1}} \phi(C') = q \cdot \phi(C),$$

whence $\phi(C) = -\frac{1}{q}\phi(D)$. The intuition is that, away from C, ϕ is spreading uniformly on the neighboring edges. Now ϕ is easily shown square integrable on the set of edges by grouping them in each sphere about C_0.

$$\sum_C |\phi(C)|^2 = \sum_{d \in \mathbf{N}} \sum_{d(C,C_0)=d} |\phi(C)|^2 = \sum_{d \in \mathbf{N}} q^{-2d} |\phi(C_0)|^2 \sum_{d(C,C_0)=d} 1 = 2|\phi(C_0)|^2 \sum_{d \in \mathbf{N}} q^{-d}.$$

In fact, it suffices that the uniform spreading phenomenon occurs outside a large ball about C_0.

This idea can be adapted to a Bruhat-Tits building X, using the combinatorial distance $\mathbf{d}$ on the set of chambers. Bruhat's Lemma guarantees the uniform spreading phenomenon to occur outside a large ball provided ϕ is smooth.

Lemma 2.3.8 (Bruhat). — [**Bor76**, Lemma 4.1] *Let U be a compact open subgroup of G and denote C_0 the fundamental chamber of X corresponding to B. There is a constant $d_0 > 0$ with the following property: given a chamber C such that $\mathbf{d}(C, C_0) > d_0$, there exists a chamber D adjacent to C, say $D \sim_i C$ satisfying the two following conditions:*

(i) $\mathbf{d}(D, C_0) = \mathbf{d}(C, C_0) - 1$.

(ii) the U-orbit of C contains the set of chambers i-adjacent but not equal to D.

On the proof. — The last section of [**Bor76**] contains a detailed proof. We simply note that if $U = B$ is the stabilizer of C_0, Klingler showed that we can take $d_0 = 1$. More precisely, for a chamber C at W-distance $w = \delta(C, C_0)$ from C_0, the B-orbit of C is the set of all chambers at W-distance w of C_0, namely BwB/B. In particular, his argument applies to Euclidean buildings admitting a strongly transitive group of type-preserving automorphism. Bruhat's Lemma, however, relies on the fact that X is a Bruhat-Tits building of a group of algebraic type and has no clear generalization. □

Proof of Proposition 2.3.6. — Let $\phi : \mathrm{Ch}(X) \to \mathbf{C}$ be a harmonic function invariant on the left under the action of a compact open subgroup U. With the notation of Lemma 2.3.8, we show that

$$\sum_{\mathbf{d}(C,C_0)\geq d_0} \phi(C)^2 < \infty,$$

where the sum is taken over the chambers of X at distance at least d_0 from C_0. The result follows since the balls for $\mathbf{d}$ are finite (by locally finiteness of X). We claim that for every such chamber C there exist a chamber D_C and a gallery

$$\Gamma_C : D_0 = D_C, \ldots, D_l = C,$$

of length $l \in \mathbf{N}$ such that:

(i) $\mathbf{d}(D_0, C_0) = d_0$,
(ii) $\mathbf{d}(D_{i-1}, C_0) = \mathbf{d}(D_i, C_0) - 1$, for all $i = 1, \ldots, l$, and
(iii) the U-orbit of D_i contains all chambers not equal to D_{i-1} and having $D_i \cap D_{i-1}$ as a panel, for all $i = 1, \ldots, l$.

In particular we have $l = \mathbf{d}(C, C_0) - d_0$ and Γ_C is minimal. To prove the claim, set $l \coloneqq \mathbf{d}(C, C_0) - d_0$ and $D_l \coloneqq C$, and apply Bruhat's Lemma to C to obtain a chamber D_{l-1} at distance $\mathbf{d}(D_{l-1}, C_0) = \mathbf{d}(C, C_0) - 1$ and satisfying (iii). By successive applications of Bruhat's Lemma we obtain the desired gallery. The regularity assumptions on X implies the following regularity on the chambers. For $w \in W_{\mathrm{aff}}$, the sphere of type w centered at C_0 has cardinal

$$\mathrm{card}(\{C' \in \mathrm{Ch}(X) \mid \delta(C', C_0) = w\}) = q^{\ell(w)}.$$

In particular, each panels is contained in exactly $q+1$ chambers regardless of its type. The function ϕ being U-invariant and harmonic, we have, using (iii),

$$\phi(D_i) = -\frac{1}{q}\phi(D_{i-1}),$$

for all $i = 1, \ldots, l$. Thus,

$$|\phi(C)| = q^{-l}|\phi(D_C)|, \tag{2.2}$$

with $l = \mathbf{d}(C, C_0) - d_0$. Recall that in general $\ell(\delta(C, C')) = \mathbf{d}(C, C')$, so that

$$\mathrm{card}(\{C' \in \mathrm{Ch}(X) \mid \mathbf{d}(C', C_0) = d\}) = q^d S(d),$$

where $S(d)$ denotes the number of group elements of W_{aff} of length $d \in \mathbf{N}$. The function S, called the *growth function* of W_{aff}, is known to be bounded by a polynomial. All together we have:

$$\begin{aligned}
\sum_{\mathbf{d}(C,C_0)\geq d_0} \phi(C)^2 &= \sum_{d\geq d_0} \sum_{\mathbf{d}(C,C_0)=d} \phi(C)^2 \\
&= \sum_{d\geq d_0} q^{-2(d-d_0)} \phi(D_C)^2 \\
&\leq \max\{\phi(D)^2 \mid \mathbf{d}(D,C_0)=d_0\} \sum_{d\geq d_0} q^{-2(d-d_0)} q^d S(d) \\
&= \max\{\phi(D)^2 \mid \mathbf{d}(D,C_0)=d_0\} \sum_{d\geq d_0} q^{-d+2d_0} S(d) \\
&= \max\{\phi(D)^2 \mid \mathbf{d}(D,C_0)=d_0\} \cdot q^{d_0} \cdot \sum_{l\geq 0} q^{-l} S(d_0+l).
\end{aligned}$$

The last series is absolutely convergent since S can be bounded by a polynomial. □

Comment 2.3.9. — Suppose vol_X is the 2-cocycle of Klingler defined on the Bruhat-Tits building X. The G-equivariance of vol_X implies that $\mathrm{vol}_X(x,y,z)$ is invariant under the left regular action of the compact open subgroup $K = \mathrm{Stab}_G(x) \cap \mathrm{Stab}_G(y) \cap \mathrm{Stab}_G(z)$. Hence so is the Poisson transform $\mathcal{P}\mathrm{vol}_X(x,y,z)$. Therefore the previous proof, with $U = K$, yields a bound for the norm of the latter. We can backtrack, in [**Bor76**], the origin of the constant d_0 which depends on K and on the diameter of its fix point set. However, even if d_0 were to grow linearly with the distances between the points x, y, z, the above estimate has an exponential factor q^{d_0}. The bound is hopeless.

CHAPTER 3

BUILDING THEORETIC FORMULATION

The construction of Klingler's natural cocycle relies mostly on the geometry of the building: use of retractions with respect to chambers at infinity and Euclidean volume in the apartments. It is fairly natural to extend the construction to an arbitrary Euclidean (irreducible) building X, a more general setting than Bruhat-Tits buildings associated to groups over a local field (at least when the rank is less than 3). The Steinberg module, however, is not defined, but we may consider the space of harmonic ℓ^2-functions on the set of chambers of X instead. We also need a notion of Poisson transform and, in particular, suitable measures replacing ν_B of Section 2.3. For this we introduce the so-called *visual measure* associated to a chamber and show that it generalizes indeed $g_*\nu_B$, maybe up to a sign. However harmonicity is not clear in this context, and similarly square summability has a priori no reason to occur, since the notion of smooth functions on the chambers is not available. Of course if Bruhat's Lemma 2.3.8 is available then the proof of Proposition 2.3.6 may be adapted to ensure square integrability of harmonic functions.

The success of harmonic analysis and spherical functions on locally finite regular trees, (see for instance the book [**FTN91**] of Figà-Talamanca and Nebbia), lead to similar studies in abstract Euclidean buildings. Starting with $\widetilde{A}_2$ buildings [**CMS94**], the harmonic analysts later developed tools in Euclidean buildings of type $\widetilde{A}_n$, *e.g.* [**Car01**], followed by various simultaneous generalizations including [**Par06**] to general Euclidean buildings. For simplicity we focus on the $\widetilde{A}_n$ case, which includes the Bruhat-Tits building of $\mathrm{SL}_{n+1}(F)$ over a local field F. A similar treatment for other types of buildings seems reasonably possible as some of the methods used here are also available, see [**Par06**]. As [**Car01**] testifies, the notations and the combinatorics of $\widetilde{A}_n$ buildings are cumbersome if not painful. We admittedly choose to expose here only the case of an $\widetilde{A}_2$ building for the sake of clarity. A full exposition of the (regular) $\widetilde{A}_1$ case, *i.e.* the case of regular tree, is done in Chapter 4, but it should be obvious how to translate the present chapter to regular trees. There, we determine the growth of the norm of Klingler's 1-cocycle. For $n = 2$, the results of Chapter 5 on the geometry of $\widetilde{A}_2$-buildings point in the right direction.

A treatment of higher rank may be done in future research. A point to keep in mind is that the Weyl (Coxeter) group of a root system of type A_n is the symmetric group on $n+1$ letters, thus of cardinal $(n+1)!$, whereas the number of roots is $n(n+1)$. These two natural numbers coincide if and only if $n \leq 2$. Thus one should be careful when generalizing the present chapter.

Two ingredients we call *sector coordinates* and the corresponding *sector spheres* allow us to describe the set $\mathrm{Ch}(\partial X)$ of chambers at infinity as a (topological) projective limit of the sector spheres. This idea is mentioned and used in [**CMS94**]. From there Klingler's cocycle is locally constant with respect to this topology and is determined by its projection on any sphere of sufficiently large radius.

The projective limit construction yields natural *visual measures* as a projective limit of the counting measure on the sector spheres, which are of course finite since the building X is assumed locally finite. Then we define the visual measure with respect to a chamber C, needed for the Poisson transform. It is done via the visual measure at a special vertex x of the given chamber, by replacing the orbits of the stabilizer of C by a local condition in the link of x. The latter indeed generalizes the measures $g_*\nu_B$ of Klingler, see the proof of the Iwasawa decomposition, Proposition 2.3.1.

Setting 3.0.1. — *Let X be a (thick) locally finite Euclidean building of type $\widetilde{A}_2$. As usual, we identify it with its natural CAT(0)-geometric realizations and shall therefore make some abuse of notation. Recall that in this setting X is chamber regular, i.e. there is an integer $q \geq 3$ such that every panel of X is contained in exactly $q+1$ chambers, see Section 1.2.2 and* [**Par06**, §1.7].

3.1. The boundary Ω

The goal of this chapter is to present some technical tools that will allow us to understand better the combinatorics of $\mathrm{vol}_X(\underline{x})$. They will prove themselves useful to derive a formula for the Poisson transform of the latter, see Theorem 3.4.10. In the process we give various descriptions of Ω, one of which induces a compact topology on it. Finally a necessary tool will be a family of Borel measures on Ω, called *visual measures*.

The CAT(0) metric on X, or rather on its geometric realization, is denoted d and assumed to be normalized so that the sides of a chamber have length 1, that is $d(x, y) = 1$ for all pairs of distinct vertices x, y of a chamber. This is very specific of the fact that X is of type $\widetilde{A}_2$, the Coxeter complex of which is the tesselation of $\mathbf{R}^2$ by equilateral triangles.

3.1.1. Sector coordinates and sector distance. — The spherical building ∂X is of type A_2. In the literature, the Greek letter ξ often refers to a point in the visual boundary of a CAT(0) space. Here we use it for chambers at infinity, *i.e.* $\xi \in \Omega$. However the vertices of ξ shall be denoted by ξ_1 and ξ_2 and a generic point by $\eta \in \partial X$.

Notation 3.1.1. — Given a point $x \in X$, a chamber at infinity $\xi \in \Omega$, and a point at infinity $\eta \in \partial X$, we use the following notation:

- The unique geodesic ray issuing from x in the class of η will be denoted by $r_x^\eta : \mathbf{R}_+ \to X$ meaning $r_x^\eta(0) = x$ and $r_x^\eta(\infty) = \eta$. By abuse, we think of a geodesic and its image in X as a single object and hence use the notation $r_x^\eta = [x, \eta[$.
- The unique geodesic between two points $x, y \in X$ is denoted by $[x, y]$ and is characterized by

$$[x, y] = \{p \in X \mid d(x, y) = d(x, p) + d(p, y)\}.$$

- Let $\overline{\mathrm{Sect}}_x(\xi)$ denote the closure in X of $\mathrm{Sect}_x(\xi)$, the unique sector at x pointing in the direction of ξ. If ξ_1, ξ_2 are the vertices of ξ then

$$\overline{\mathrm{Sect}}_x(\xi) = \mathrm{Sect}_x(\xi) \cup [x, \xi_1[\cup [x, \xi_2[.$$

We shall use a convenient labeling (type function) of ∂X. More precisely, we define below two maps $(-)_1, (-)_2 : \Omega \to (\partial X)^{(0)}$ and a type function (or a labeling) of ∂X again denoted τ so that $\tau(\xi_i) = i$ for $i = 1, 2$.

Definition 3.1.2 (Panel of a sector). — Let $x \in X$ and $\xi \in \Omega$.

- For $i = 1, 2$, let ξ_i be the vertex of ξ such that $r_x^{\xi_i}(1)$ has type $\tau(x) + i \mod 3$.
- The geodesic ray $[x, \xi_1[$ is called the *right panel* of $\mathrm{Sect}_x(\xi)$ and $[x, \xi_2[$ is called the *left panel* of $\mathrm{Sect}_x(\xi)$.
- We may also say that $[x, \xi_i[$ is the *panel of type* i of $\mathrm{Sect}_x(\xi)$.

In the previous definition, the labels of ξ_1, ξ_2 depend on the vertex x. To be very careful we should have called them $\xi_{x,1}, \xi_{x,2}$. The next proposition shows this to be independent of x and thus defines a type function $\tau : (\partial X)^{(0)} \to (\mathbf{Z}/3\mathbf{Z})^*$ with $\tau(\xi_i) = i$. Therefore, the vertices of a chamber at infinity $\xi \in \Omega$ shall always be labelled so that $\tau(\xi_i) = i$ as above.

Proposition 3.1.3. — *Let $\xi \in \Omega$, and x, y be vertices of X. Then $\xi_{x,i} = \xi_{y,i}$, for $i = 1, 2$.*

Proof. — A geodesic ray r is convex and isometric to a subset of $\mathbf{R}^2$, thus it is contained in some apartment A. Assume r starts at some vertex x and stays in the 1-skeleton $X^{(1)}$ (equivalently $r(0) \in X^{(0)}$ and $r(\infty) \in (\partial X)^{(0)}$). Then r will go through vertices, the types of which will cyclically appear as either

$$\dots, 0, 1, 2, 0, 1, 2, \dots \quad \text{or} \quad \dots, 0, 2, 1, 0, 2, 1, \dots,$$

as explained in [**RRS98**, §1.2]. In A, two parallel geodesic rays staying in $X^{(1)}$ and issuing at different vertices x, y will witness the same cycles of types, maybe shifted by $\tau(x) - \tau(y)$. For the general case, let A_x and A_y be apartments of X containing $\mathrm{Sect}_x(\xi)$ and $\mathrm{Sect}_y(\xi)$ respectively. If the two sectors are contained in a common apartment, *i.e.* if $A_x = A_y$ is possible, they form two pairs of parallel panels and we are in the above situation. If not, let $\mathrm{Sect}_u(\xi)$ be a common subsector, that is

$$\mathrm{Sect}_u(\xi) \subset \mathrm{Sect}_x(\xi) \cap \mathrm{Sect}_y(\xi) \subset A_x \cap A_y.$$

We can apply the previous case to $\mathrm{Sect}_u(\xi)$ and $\mathrm{Sect}_x(\xi)$ which are contained in the apartment A_x. We conclude by doing the same with $\mathrm{Sect}_u(\xi)$ and $\mathrm{Sect}_y(\xi)$ in A_y. □

Given a sector $\mathrm{Sect}_x(\xi)$ in an apartment A, we shall use the affine structure of the latter to assign coordinates to the vertices in the closure of the sector (independently of the choice of an apartment A). Let x' and x'' be the vertices of $C_x(\xi)$, the initial chamber of $\mathrm{Sect}_x(\xi)$, with $\tau(x') = \tau(x) + 1 \mod 3$ and $\tau(x'') = \tau(x) + 2 \mod 3$. In other words, $x' = r_x^{\xi_1}(1)$ and $x'' = r_x^{\xi_2}(1)$. We consider the linearly independent vectors

$$v_1 = \overrightarrow{xx'} \quad \text{and} \quad v_2 = \overrightarrow{xx''}$$

sitting inside A. Any vertex $u \in \overline{\mathrm{Sect}_x}(\xi)$ is written as an affine combination $u = x + mv_1 + nv_2$ with non-negative integer coefficients $m, n \in \mathbf{N}$. This is independent of the apartment A containing $\mathrm{Sect}_x(\xi)$. (For if u is contained in another sector $\mathrm{Sect}_x(\xi')$ and m', n' are the corresponding integers, one sees that $m' = m$ and $n' = n$.) Thus the coordinates do not depend on a particular sector issuing at x containing u. The degenerate situation of a vertex sitting on a panel common to two sectors should be kept in mind.

Definitions 3.1.4 (Sector coordinates). — Let x, u be arbitrary vertices in X.

- The (right and left) *sector coordinates* of u with respect to x are the non-negative integers (m,n) describe above. We denote them $(m_x(u), n_x(u))$ or $(m_x^1(u), m_x^2(u))$ and use each notation according to the use of 'left' and 'right' or of the types $i = 1, 2$.
- We call $m_x(u)$ the *right sector coordinate* of u with respect to x or the *sector coordinate of type* 1.
- Similarly, $n_x(u)$ is the *left sector coordinate* or the *sector coordinate of type* 2.
- The *sector distance* between x and u is $d_1(x,u) = m_x(u) + n_x(u)$.

See [**Car01**] for equivalent notions in $\widetilde{A}_n$ buildings.

Remark 3.1.5. — Clearly, given a sector $\mathrm{Sect}_x(\xi)$, the sector coordinates give a bijection from the points of $\overline{\mathrm{Sect}}_x(\xi)$ onto $\mathbf{N}^2$, the inverse of which sends $(m,n) \in \mathbf{N}^2$ to the unique point of $\overline{\mathrm{Sect}}_x(\xi)$ with coordinates (m,n) with respect to x.

Proposition 3.1.6. — *Let x, u be vertices in X. Then,*

(i) $(m_u(x), n_u(x)) = (n_x(u), m_x(u))$, *hence* $d_1(x,u) = d_1(u,x)$.
(ii) $d(x,u)^2 = m_x(u)^2 + m_x(u)n_x(u) + n_x(u)^2$.
(iii) $d_1(x,u)^2 = d(x,u)^2 + m_x(u)n_x(u)$.
(iv) $d(x,u) \leq d_1(x,u)$.
(v) d_1 *is the graph theoretic distance in the* 1*-skeleton of* X.

Proof. — The point (i) is elementary. Let $B = (v_1, v_2)$ be the basis of an apartment A containing x, u constructed as in the paragraph above Definition 3.1.4. The Gramm matrix of B is given by

$$G_B = \begin{pmatrix} \langle v_1, v_1\rangle & \langle v_1, v_2\rangle \\ \langle v_2, v_1\rangle & \langle v_2, v_2\rangle \end{pmatrix} = \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}.$$

Let $v^t = (m_x(u), n_x(u))$. On the one hand,

$$d(x,u)^2 = \|\overrightarrow{xu}\|^2 = v^t G_B v = m_x(u)^2 + m_x(u)n_x(u) + n_x(u)^2,$$

hence (ii). On the other hand,

$$d_1(x,u)^2 = (m_x(u) + n_x(u))^2 = m_x(u)^2 + 2m_x(u)n_x(u) + n_x(u)^2 = d(x,u)^2 + m_x(u)n_x(u),$$

proving (iii). Finally, (iv) follows from (ii), (iii), and the inequality:

$$(m_x(u) + n_x(u))^2 \leq 2(m_x(u)^2 + n_x(u)^2).$$

The last statement certainly holds in an apartment A. On the other hand, the retraction of X onto an apartment can only decrease the graph theoretic distance. □

Definition 3.1.7 (Vertex convex hull). — The *vertex convex hull* of two vertices $x, u \in X^{(0)}$, denoted $\mathrm{Conv}^{(0)}(x,u)$, is by definition the set of vertices t aligned with x, u for the graph theoretic metric d_1 on the 1-skeleton $X^{(1)}$:

$$d_1(x,u) = d_1(x,t) + d_1(t,u).$$

In the case of two vertices sitting on the panel of some sector, the vertex convex hull consists of all vertices of the geodesic segment $[x,y]$. If x, u are not on a common wall, $\mathrm{Conv}^{(0)}(x,u)$ is the set of vertices of $\mathrm{Conv}(x,u)$ the smallest chamber subcomplex of X containing $\{x,u\}$. It is the parallelogram pictured in Figure 1.

Remark 3.1.8. — This convex hull does not depend on a particular sector issuing at x containing u, or rather on a particular apartment containing x and u.

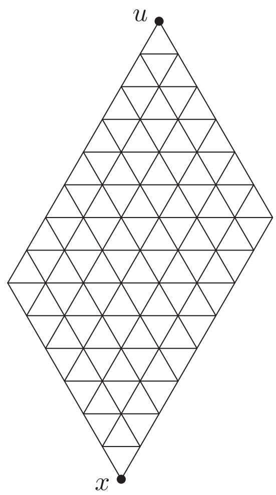

FIGURE 1. $\mathrm{Conv}^{(0)}(x,u)$ consists of the vertices of this parallelogram.

3.1.2. The boundary Ω as a projective limit. — The purpose of this section is to prove a convenient characterization of Ω as a projective limit of sector spheres.

Definition 3.1.9 (Sector sphere). — Let x be a vertex of X. For every $(m,n) \in \mathbf{N}^2$, the *sector sphere of type* (m,n) *centered at* x is defined by

$$S_{m,n}(x) = \{u \in X^{(0)} \mid (m_x(u), n_x(u)) = (m,n)\}.$$

Remark 3.1.10. — The symmetry of the sector coordinates of Proposition 3.1.6 implies that $u \in S_{m,n}(x)$ if and only if $x \in S_{n,m}(u)$.

The local finiteness of X ensures that each sector sphere has finite cardinal and the latter depends not on its center.

Proposition 3.1.11 (Vertex regularity of X). — *The cardinal $N_{m,n} = |S_{m,n}(x)|$ is independent of $x \in X^{(0)}$ and is given by:*

$$N_{0,0} = 1, \quad N_{m,0} = N_{0,m} = (q^2+q+1)q^{2(m-1)}, \quad \textit{and} \quad N_{m,n} = (q^2+q+1)(q+1)q^{2(m+n)-3}.$$

Proof. — A helpful description of the link is given in Paragraph 3.4.2. For the complete proof, see [**CMS94**, p. 218]. □

We fix a basis vertex $x \in X^{(0)}$ for the remainder of the section. The characterization of Ω as a (topological) projective limit will eventually be shown to be independent of x.

Definition 3.1.12 (Projections). — Given $(m,n),(m',n') \in \mathbf{N}^2$ satisfying $m' \leq m$ and $n' \leq n$, we define the *projection* $p^{x,m,n}_{m',n'}$ to be the map

$$p^{x,m,n}_{m',n'} : S_{m,n}(x) \longrightarrow S_{m',n'}(x)$$

that sends $u \in S_{m,n}(x)$ to the vertex in $\mathrm{Conv}^{(0)}(x,u)$ having (m',n') as sector coordinates with respect to x.

Lemma 3.1.13. — *Endow* $\mathbf{N}^2$ *with the product order:*

$$(m',n') \leq (m,n) \quad \text{if and only if} \quad m' \leq m \ \text{ and } \ n' \leq n.$$

Then the projections $\mathscr{S}(x) = \{p^{x,m,n}_{m',n'} \mid (m',n') \leq (m,n) \in \mathbf{N}^2\}$ *form a projective system over* $(\mathbf{N}^2, \leq)$*, that is*

(i) $p^{x,m,n}_{m,n}$ *is the identity map on* $S_{m,n}(x)$ *for all* $(m,n) \in \mathbf{N}^2$*, and*

(ii) $p^{x,m',n'}_{m'',n''} \circ p^{x,m,n}_{m',n'} = p^{x,m,n}_{m'',n''}$*, whenever* $(m'',n'') \leq (m',n') \leq (m,n)$. □

Definition 3.1.14 (Projective limit). — We denote $\varprojlim \mathscr{S}(x)$ the *projective limit* of $\mathscr{S}(x)$. It consists of the subspace of the product

$$\varprojlim \mathscr{S}(x) \subset \prod_{(m,n)\in\mathbf{N}^2} S_{m,n}(x)$$

of those elements compatible with the projections. More precisely, $f \in \varprojlim \mathscr{S}(x)$ is by definition a map $f : \mathbf{N}^2 \to X$ such that $f(m,n) \in S_{m,n}(x)$ for all $(m,n) \in \mathbf{N}^2$ and

$$p^{x,m,n}_{m',n'}(f(m,n)) = f(m',n'), \tag{3.1}$$

whenever $(m',n') \leq (m,n)$.

Let $p^x_{m,n} : \varprojlim \mathscr{S}(x) \to S_{m,n}(x)$ denote the projection on the (m,n)-th coordinate. For every $(m',n') \leq (m,n)$, the following diagram commutes:

$$\begin{tikzcd}
\varprojlim \mathscr{S}(x) \arrow[d, "p^x_{m,n}"'] \arrow[dd, bend left=60, "p^x_{m',n'}"] \\
S_{m,n}(x) \arrow[d, "p^{x,m,n}_{m',n'}"'] \\
S_{m',n'}(x)
\end{tikzcd}$$

In other words, $p^{x,m,n}_{m',n'} \circ p^x_{m,n} = p^x_{m',n'}$.

Notation 3.1.15. — Thanks to the compatibility of the projections and in order to simplify the notations, we shall often drop the upper indices m,n in $p^{x,m,n}_{m',n'}$. We can think of $p^x_{m',n'}$ as a projection of the union of the larger spheres $S_{m,n}(x)$, with $(m',n') \leq (m,n)$, together with the 'celestial sphere' $\varprojlim \mathscr{S}(x)$, onto $S_{m',n'}(x)$.

The sector spheres $S_{m,n}(x)$ are endowed with the discrete topology. The product of the latter endowed with the product topology is a compact space thanks to Tykhonov's theorem. Endow $\varprojlim \mathscr{S}(x)$ with the subspace topology. Since the projective limit is closed in the product, it is compact as well. The maps $p^x_{m,n} : \varprojlim \mathscr{S}(x) \to S_{m,n}(x)$ are continuous by the very definitions of the product topology and the subspace topology. Indeed, if $u \in S_{m,n}(x)$, then

$$\begin{aligned}
(p^x_{m,n})^{-1}(\{u\}) &= \left\{ f \in \varprojlim \mathscr{S}(x) \mid p^x_{m,n}(f) = u \right\} \\
&= \left\{ f \in \varprojlim \mathscr{S}(x) \mid f(m,n) = u \right\} \\
&= \varprojlim \mathscr{S}(x) \cap \left(\{u\} \times \prod_{(m',n')\neq(m,n)} S_{m',n'}(x) \right),
\end{aligned}$$

The projective limit $\varprojlim \mathscr{S}(x)$ is universal in the sense of the following proposition.

Proposition 3.1.16 (Universal property). — *Let Y be a topological space and $\varphi_{m,n} : Y \to S_{m,n}(x)$ for every $(m,n) \in \mathbf{N}^2$ be continuous maps compatible with the projections $p^{x,m,n}_{m',n'} : S_{m,n}(x) \to S_{m',n'}(x)$, that is*

$$p^{x,m,n}_{m',n'} \circ \varphi_{m,n} = \varphi_{m',n'},$$

for all $(m',n') \leq (m,n)$. Then there is a unique continuous map $f : Y \to \varprojlim \mathscr{S}(x)$ making the following diagram commute:

$$\begin{tikzcd}
& Y \arrow[d, "\exists! f"] \arrow[ddl, bend right, "\varphi_{m,n}"'] \arrow[ddr, bend left, "\varphi_{m',n'}"] & \\
& \varprojlim \mathscr{S}(x) \arrow[dl, "p^x_{m,n}"'] \arrow[dr, "p^x_{m',n'}"] & \\
S_{m,n}(x) \arrow[rr, "p^x_{m',n'}"] & & S_{m',n'}(x)
\end{tikzcd}$$

In other words, $p^x_{m,n} \circ f = \varphi_{m,n}$, for all $(m,n) \in \mathbf{N}^2$. One deduces that $\varprojlim \mathscr{S}(x)$ is the unique terminal object of the system $\mathscr{S}(x)$. □

Recall that there is a bijection $\mathrm{Sect}_x : \Omega \to \mathrm{Sect}_x(\Omega)$ sending $\xi \in \Omega$ to the unique sector issuing at x in the class of ξ, namely $\mathrm{Sect}_x(\xi)$. We now describe a topology on $\mathrm{Sect}_x(\Omega)$, the set of sectors issuing at x.

Definition 3.1.17. — Let $u \in X^{(0)}$. We define,

- $\Omega_x(u) = \{\xi \in \Omega \mid u \in \overline{\mathrm{Sect}}_x(\xi)\}$, so that,
- $\mathrm{Sect}_x(\Omega_x(u)) = \{$sectors at x containing u in their closure$\}$.

Proposition 3.1.18. — *The collections $\{\Omega_x(u) \mid u \in X^{(0)}\}$ and $\{\mathrm{Sect}_x(\Omega_x(u)) \mid u \in X^{(0)}\}$ form topological bases on Ω and $\mathrm{Sect}_x(\Omega)$ respectively, with respect to which Sect_x is a homeomorphism. Moreover the resulting topology is Hausdorff.*

Proof. — The statements for Ω and $\mathrm{Sect}_x(\Omega)$ are equivalent thanks to the bijection Sect_x; the homeomorphism follows easily. It is clear that

$$\bigcup_{u \in X^{(0)}} \Omega_x(u) = \Omega.$$

But in fact, for every $(m,n) \in \mathbf{N}^2$, we have

$$\bigsqcup_{u \in S_{m,n}(x)} \Omega_x(u) = \Omega,$$

and the corresponding statement is true in $\mathrm{Sect}_x(\Omega)$.

We show that for any ξ in the intersection $\Omega_x(u) \cap \Omega_x(v)$, there is $\Omega_x(w)$ in the intersection that contains ξ. Let $\xi \in \Omega_x(u) \cap \Omega_x(v)$ for some $u, v \in X^{(0)}$. In particular $u, v \in \overline{\mathrm{Sect}}_x(\xi)$. Let $w \in \mathrm{Sect}_x(\xi)$ be such that $u, v \in \mathrm{Conv}^{(0)}(x, w)$. This is possible as soon as $w \in \mathrm{Sect}_x(\xi)$ has coordinates $m_x(w), n_x(w)$ sufficiently large, say greater than the sum of those of u and v. Hence, $\xi \in \Omega_x(w)$ and clearly any sector at x containing w contains u and v, proving $\xi \in \Omega_x(w) \subset \Omega_x(u) \cap \Omega_x(v)$.

For the separation axiom, let $\xi, \xi' \in \Omega$ be distinct. We look for two disjoint open subsets $U, U' \subset \Omega$ with $\xi \in U$ and $\xi' \in U'$. Since $\mathrm{Sect}_x(\xi) \neq \mathrm{Sect}_x(\xi')$, there exists $(m,n) \in \mathbf{N}^2$ such that the respective vertices u, u' having sector coordinates (m, n) are distinct. Therefore, $\Omega_x(u)$ and $\Omega_x(u')$ are the desired open subsets. □

Scholia 3.1.19. — *Let x be a vertex of X.*

(i) For every $(m, n) \in \mathbf{N}^2$,

$$\bigsqcup_{u \in S_{m,n}(x)} \Omega_x(u) = \Omega.$$

(ii) For every $u, v \in X^{(0)}$, if $u = p^x_{m,n}(v)$ for some $(m, n) \in \mathbf{N}^2$, then

$$\Omega_x(v) \subset \Omega_x(u).$$

(iii) The condition $u \in \mathrm{Conv}^{(0)}(x, v)$ is equivalent to $u = p^x_{m,n}(v)$ for some $(m, n) \in \mathbf{N}^2$. □

Notation 3.1.20. — We denote $\mathcal{T}_{\Omega,x}$ the topology on Ω of the previous proposition and $\mathcal{T}_{\mathrm{Sect},x} \coloneqq \mathrm{Sect}_x(\mathcal{T}_{\Omega,x})$ that on $\mathrm{Sect}_x(\Omega)$.

Remark 3.1.21. — The topology really is defined on $\mathrm{Sect}_x(\Omega)$ and then transported to Ω via the bijection. The situation has been complicated for notational purposes.

Let $\mathcal{T}_{\lim,x}$ denote the compact topology on $\varprojlim \mathscr{S}(x)$ defined in the paragraph above Proposition 3.1.16. The next proposition assembles the pieces together as it gives an explicit homeomorphism between $(\varprojlim \mathscr{S}(x), \mathcal{T}_{\lim,x})$ and $(\mathrm{Sect}_x(\Omega), \mathcal{T}_{\mathrm{Sect},x})$, hence they are homeomorphic to $(\Omega, \mathcal{T}_{\Omega,x})$ as well.

Notation 3.1.22. — For every sector $\mathrm{Sect}_x(\xi)$ issuing at x and every $(m, n) \in \mathbf{N}^2$, let $f^\xi_x(m, n)$ be the unique vertex in $\overline{\mathrm{Sect}}_x(\xi)$ with coordinates (m, n), that is

$$\{f^\xi_x(m, n)\} = \overline{\mathrm{Sect}}_x(\xi) \cap S_{m,n}(x).$$

This defines a map $f_x : \Omega \to \prod S_{m,n}(x)$ sending ξ to f^ξ_x. Define further $\overline{f}_x : \mathrm{Sect}_x(\Omega) \to \prod S_{m,n}(x)$ by the commutative diagram:

$$\begin{tikzcd} \Omega \arrow[rr, "\mathrm{Sect}_x"] \arrow[dr, "f_x"'] & & \mathrm{Sect}_x(\Omega) \arrow[dl, "\overline{f}_x"] \\ & \prod S_{m,n}(x) & \end{tikzcd}$$

By definition, we have $\overline{f}_x = f_x \circ \mathrm{Sect}_x^{-1}$.

Proposition 3.1.23. — *The maps $f_x : \Omega \to \varprojlim \mathscr{S}(x)$ and $\overline{f}_x : \mathrm{Sect}_x(\Omega) \to \varprojlim \mathscr{S}(x)$ are homeomorphisms. We have the following commutative diagram of homeomorphisms:*

$$\begin{tikzcd} \Omega \arrow[rr, "\mathrm{Sect}_x"] \arrow[dr, "f_x"'] & & \mathrm{Sect}_x(\Omega) \arrow[dl, "\overline{f}_x"] \\ & \varprojlim \mathscr{S}(x) & \end{tikzcd}$$

Proof. — The proposition first states that f_x maps Ω into $\varprojlim \mathscr{S}(x)$ and does so surjectively. We need to show that f^ξ_x satisfies equation (3.1), for all $\xi \in \Omega$. Let $(m', n') \leq (m, n) \in \mathbf{N}^2$ and set $u \coloneqq f^\xi_x(m, n)$. Since the vertex convex hull $\mathrm{Conv}^{(0)}(x, u)$ is contained in $\overline{\mathrm{Sect}}_x(\xi)$, it is clear that $p^x_{m',n'}(u) \in \overline{\mathrm{Sect}}_x(\xi)$. But the latter is $f^\xi_x(m', n')$ by definition of f^ξ_x, hence f^ξ_x satisfies (3.1). Surjectivity is proven by noticing that, for a given $f \in \varprojlim \mathscr{S}(x)$, the set $\{f(m, n) \mid (m, n) \in \mathbf{N}^2\}$

is exactly the set of all vertices of some closed sector issuing at x. This is proved in Lemma 3.1.24 below. Injectivity follows from:

$$\begin{aligned}\xi = \xi' &\iff \mathrm{Sect}_x(\xi) = \mathrm{Sect}_x(\xi'),\\ &\iff \overline{\mathrm{Sect}}_x(\xi) = \overline{\mathrm{Sect}}_x(\xi'),\\ &\iff \overline{\mathrm{Sect}}_x(\xi) \cap X^{(0)} = \overline{\mathrm{Sect}}_x(\xi') \cap X^{(0)},\\ &\iff f_x^{\xi} = f_x^{\xi'},\end{aligned}$$

for all $\xi, \xi' \in \Omega$, where the last equivalence also uses the lemma.

Let $u \in S_{m,n}(x)$ and $\xi \in \Omega$; we have the equivalences:

$$\begin{aligned}\xi \in \Omega_x(u) &\iff \overline{\mathrm{Sect}}_x(\xi) \cap S_{m,n}(x) = \{u\},\\ &\iff f_x^{\xi}(m,n) = u,\\ &\iff f_x^{\xi} \in (p^x_{m,n})^{-1}(\{u\}) = \varprojlim \mathscr{S}(x) \cap \left(\{u\} \times \prod_{(m',n')\neq(m,n)} S_{m',n'}(x)\right).\end{aligned}$$

In other word $f_x(\Omega_x(u)) = (p^x_{m,n})^{-1}(\{u\})$, thus f_x is a homeomorphism and so is $\overline{f}_x = f_x \circ \mathrm{Sect}_x$ by composition of homeomorphisms. □

Lemma 3.1.24. — *Let $f \in \prod S_{m,n}(x)$; then the following are equivalent:*

(i) $f \in \varprojlim \mathscr{S}(x)$,

(ii) $\mathrm{Conv}^{(0)}(x, f(n,n)) \subset f(\mathbf{N}^2)$ *for all* $n \in \mathbf{N}$,

(iii) $\bigcup_{n\in\mathbf{N}} \mathrm{Conv}^{(0)}(x, f(n,n)) = f(\mathbf{N}^2)$,

(iv) $f(\mathbf{N}^2)$ *is the set of vertices of a closed sector issuing at* x,

(v) $f = f_x^{\xi}$, *for some* $\xi \in \Omega$.

Proof. — The equivalence between (iv) and (v) is trivial but worth including in the statement. Some obvious remarks concerning any $f \in \prod S_{m,n}(x)$:

- $f(m,n) \in S_{m,n}(x)$ for all $(m,n) \in \mathbf{N}^2$,
- $f(m,n) = f(m',n') \iff (m,n) = (m',n')$, *i.e.* f is injective.
- For every $n \in \mathbf{N}$, the convex hull $\mathrm{Conv}^{(0)}(x, f(n,n))$ contains exactly one point of each sphere $S_{m',n'}(x)$ with $m', n' \leq n$.
- If $f \in \varprojlim \mathscr{S}(x)$ and $m', n' \leq n$, the unique point in $S_{m',n'}(x) \cap \mathrm{Conv}^{(0)}(x, f(n,n))$ is
$$p^x_{m',n'}(f(n,n)) = f(m',n').$$

The implications (i) $\Longrightarrow$ (ii) $\Longleftrightarrow$ (iii) follow from the remarks.

For (i) $\Longleftarrow$ (ii) one needs to show

$$p^x_{m',n'}(f(m,n)) = f(m',n'),$$

whenever $(m',n') \leq (m,n)$. This is equivalent to showing $f(m',n')$ is the unique point in $\mathrm{Conv}^{(0)}(x, f(m,n)) \cap S_{m',n'}(x)$. Set $N := m+n$. Since $\mathrm{Conv}^{(0)}(x, f(N,N)) \subset f(\mathbf{N}^2)$ by hypothesis, we have $f(m',n'), f(m,n) \in \mathrm{Conv}^{(0)}(x, f(N,N))$ and the result follows.

For (iii) $\Longleftarrow$ (iv) we recall that a closed sector is convex. The converse (iii) $\Longrightarrow$ (iv) is more challenging. Under the hypothesis (iii), $f(n,n) \in \mathrm{Conv}^{(0)}(x, f(N,N))$ for all $n \leq N$, in which case the geodesic $\sigma_N = [x, f(N)]$ is an extension of $\sigma_n = [x, f(n)]$. Let σ be the limit geodesic ray, see the remark below, and A be an apartment containing σ. Since σ goes through each point of the diagonal of f, namely $\{f(n,n) \mid n \in \mathbf{N}\} \subset \sigma(\mathbf{R})$, we have $\mathrm{Conv}^{(0)}(x, f(n,n)) \subset A$. But the assertion is clear in an apartment. $\square$

Remark 3.1.25. — The limit of $(\sigma_n)_{n\in\mathbf{N}}$ exists in the CAT(0)-compactification $\overline{X} = X \sqcup \partial X$ endowed with the cone topology [**BH99**, Chapter II.8]. The latter can be described as follows. Identify $\overline{X}$ with the set $R_x(X)$ of all maps $r : \mathbf{R}_+ \to X$ with $r(0) = x$ being either a geodesic ray, or a geodesic for a finite time then a constant map. The identification is $r \mapsto r(\infty)$, associating to r its end point. The cone topology can be defined on $R_x(X)$ and transported to a topology on $\overline{X}$ which does not depend on the choice of $x \in X$. The cone topology is in fact metrizable. Indeed, the following formula defines a compatible metric on $R_x(X)$ for which $(\sigma_n)_{n\in\mathbf{N}}$ is easily shown to be a Cauchy sequence:

$$D(r, r') = \int_{\mathbf{R}_+} d(r(t), r'(t))e^{-t}dt.$$

Remark 3.1.26. — If X is the Bruhat-Tits building of a connected, simply connected, almost F-simple algebraic group over a local field F, we could use a result of [**BT72**, Proposition 2.8.3]. They showed that an increasing union of subsets each contained in an apartment is itself contained in an apartment. However our proof works for any locally finite $\widetilde{A}_2$ building. Parkinson has a general building theoretic proof in his thesis [**Par05**, Appendix B.2].

Proposition 3.1.27 (On the topology $\mathcal{T}_{\Omega,x}$). — *The subsets $\Omega_x(u)$ are closed for all $u \in X^{(0)}$, hence compact. Thus $(\Omega, \mathcal{T}_{\Omega,x})$ is a totally disconnected space. In summary, it is a profinite space.*

Proof. — This is clear from the partition $\bigsqcup_{u\in S_{m,n}(x)} \Omega_x(u) = \Omega$, with $(m,n) \in \mathbf{N}^2$. $\square$

3.2. Topological independence of the base point

In the previous section, we defined a topology $\mathcal{T}_{\Omega,x}$ on Ω depending on a base point $x \in X^{(0)}$ and gave homeomorphisms:

$$(\Omega, \mathcal{T}_{\Omega,x}) \simeq (\varprojlim \mathscr{S}(x), \mathcal{T}_{\lim,x}) \simeq (\mathrm{Sect}_x(\Omega), \mathcal{T}_{\mathrm{Sect},x}).$$

The goal of the current section is to prove that the construction is independent of x.

Proposition 3.2.1. — *Let $x, y \in X^{(0)}$ be two vertices. For every $u \in X^{(0)}$ and $\xi \in \Omega_x(u)$, there is $v \in X^{(0)}$ satisfying*

$$\xi \in \Omega_y(v) \subset \Omega_x(u).$$

Hence, $\mathcal{T}_{\Omega,x} = \mathcal{T}_{\Omega,y}$.

We shall therefore drop the subscripts x or y. To establish the proposition we recall some notations and some technical results of [**CMS94**].

Notation 3.2.2. — We denote $p^x_{m,n}(\xi)$ the unique point of $S_{m,n}(x) \cap \overline{\mathrm{Sect}}_x(\xi)$ for all $(m,n) \in \mathbf{N}^2$ and $\xi \in \Omega$. This is redundant with Notation 3.1.22 where we denoted this point $f^\xi_x(m,n)$. We wish to keep only the present notation as f_x was introduced for Proposition 3.1.23, only. In other words, thanks to the homeomorphisms above, we use the same notation for $p^x_{m,n}$ and $p^x_{m,n} \circ f_x$, that is

$$p^x_{m,n}(\xi) \coloneqq p^x_{m,n}(f^\xi_x) = f^\xi_x(m,n),$$

for all $(m, n) \in \mathbf{N}^2$ and $\xi \in \Omega$. As before, the following diagram commutes,

$$\begin{tikzcd}
\Omega \arrow[rr, "f_x", "\simeq"'] \arrow[dr, "p^x_{m,n}"'] & & \varprojlim \mathscr{S}(x) \arrow[dl, "p^x_{m,n}"] \\
& S_{m,n}(x) &
\end{tikzcd}$$

We can now forget about the f_x notation and simply remember that $p^x_{m,n}$ is the projection onto $S_{m,n}(x)$, in every possible sense.

Lemma 3.2.3. — [**CMS94**, Lemma 2.1] *Let $\xi \in \Omega$ and $x, y \in X^{(0)}$. There are integers $m(x, y, \xi)$ and $n(x, y, \xi) \in \mathbf{Z}$, and a natural number $M = M(x, y, \xi) \in \mathbf{N}$ such that*

$$p^x_{m,n}(\xi) = p^y_{m+m(x,y,\xi),n+n(x,y,\xi)}(\xi), \tag{3.2}$$

or equivalently

$$f^\xi_x(m, n) = f^\xi_y(m + m(x, y, \xi), n + n(x, y, \xi)),$$

for all $m, n \geq M$.

Proof. — The idea is to take any point u in the intersection of $\mathrm{Sect}_x(\xi) \cap \mathrm{Sect}_y(\xi)$ and to look at the sector $\mathrm{Sect}_u(\xi)$ issuing at u. It is a subsector of both sectors. Suppose u has coordinates (m, n) in $\mathrm{Sect}_x(\xi)$ and (k, l) in $\mathrm{Sect}_y(\xi)$. Then a vertex v of $\mathrm{Sect}_u(\xi)$ with coordinates (i, j) satisfies :

$$v = p^x_{m+i,n+j}(\xi) = p^y_{k+i,l+j}(\xi).$$

In other words $p^u_{i,j}(\xi) = p^x_{m+i,n+j}(\xi) = p^y_{k+i,l+j}(\xi)$ for all $i, j \geq 0$. Thus the pair $(k-m, l-n) \in \mathbf{Z}^2$ does not depend on the choice of u and we set $(m(x, y, \xi), n(x, y, \xi)) \coloneqq (k-m, l-n)$. The constant M is chosen so that $m, n \geq M$ implies $p^x_{m,n}(\xi)$ is in $\mathrm{Sect}_y(\xi)$. (For example, take M to be the sum of the coordinates of u with respect to x.) $\square$

Corollary 3.2.4. — *Let $\xi \in \Omega$ and $x, y \in X^{(0)}$. It is immediate from the definitions that*

$$m(x, y, \xi) = -m(y, x, \xi), \quad n(x, y, \xi) = -n(y, x, \xi) \quad \text{and} \quad m(x, x, \xi) = n(x, x, \xi) = 0.$$

Remark 3.2.5. — It is worth understanding $m(x, u, \xi)$ and $n(x, u, \xi)$ for a vertex $u \in \overline{\mathrm{Sect}}_x(\xi)$. The hypothesis translates as

$$p^x_{m_x(u),n_x(u)}(\xi) = u = p^u_{0,0}(\xi) \quad \text{and} \quad p^x_{m+m_x(u),n+n_x(u)}(\xi) = p^u_{m,n}(\xi),$$

for all $(m, n) \in \mathbf{N}^2$. Thus (3.2) implies $m(x, u, \xi) = -m_x(u)$ and $n(x, u, \xi) = -n_x(u)$.

In Lemma 3.2.3, one may use the bound $M = d_1(x, y)$, which is uniform in $\xi \in \Omega$, thanks to the following result.

Lemma 3.2.6. — [**CMS94**, Corollary 2.3] *Let $x, y \in X^{(0)}$ and $\xi \in \Omega$, then $p^x_{m,n}(\xi) \in \mathrm{Sect}_x(\xi) \cap \mathrm{Sect}_y(\xi)$, for all $m, n \geq d_1(x, y)$.* $\square$

The next lemma follows immediatly.

Lemma 3.2.7. — [**CMS94**, Lemma 2.4] *Let x, y, u be vertices in X with $m_x(u), n_x(u) \geq d_1(x, y)$. Then,*

$$\Omega_x(u) \subset \Omega_y(u).$$

Moreover, for every $\xi \in \Omega_x(u)$,

$$m(x, y, \xi) = m_y(u) - m_x(u) = -m(y, x, \xi) \quad \text{and} \quad n(x, y, \xi) = n_y(u) - n_x(u) = -n(y, x, \xi),$$

and in light of Remark 3.2.5,

$$m(x, y, \xi) = m(x, u, \xi) - m(y, u, \xi) \quad \text{and} \quad n(x, y, \xi) = n(x, u, \xi) - n(y, u, \xi).$$

□

Corollary 3.2.8. — *Let x, y, u be vertices in X with, $m_x(u), n_x(u), m_y(u), n_y(u) \geq d_1(x, y)$. Then,*

$$\Omega_x(u) = \Omega_y(u).$$

In general $m_x(u), n_x(u) \geq 2d_1(x, y)$ implies $m_y(u), n_y(u) \geq d_1(x, y)$.

Proof of Corollary 3.2.8. — The first statement is clear from Lemma 3.2.7. We show

$$m_x(u), n_x(u) \geq 2d_1(x, y) \implies m_y(u), n_y(u) \geq d_1(x, y).$$

Indeed, write $d \coloneqq d_1(x, y)$ and let $\xi \in \Omega_x(u)$, then $p_1 \coloneqq p^x_{d,d}(\xi) \in \mathrm{Sect}_x(\xi) \cap \mathrm{Sect}_y(\xi)$ by Lemma 3.2.6. Therefore $p_1 = p^y_{i,j}(\xi)$ for some $i, j \in \mathbf{N}$. Set $p_2 \coloneqq p^x_{2d,2d}(\xi) = p^y_{i+d,j+d}(\xi)$, the hypothesis on u implies

$$(p_1 = p^x_{d,d}(u) = p^y_{i,j}(u) \quad \text{and}) \quad p_2 = p^x_{2d,2d}(u) = p^y_{i+d,j+d}(u),$$

from which we deduce that $m_y(u), n_y(u) \geq d$. □

Corollary 3.2.9 (1-cocycle relation). — *Let x, y, z be vertices in X and $\xi \in \Omega$. Then,*

$$m(x, y, \xi) = m(x, z, \xi) - m(y, z, \xi) \quad \text{and} \quad n(x, y, \xi) = n(x, z, \xi) - n(y, z, \xi).$$

Proof. — Let $m, n \geq \max\{d_1(s, t) \mid s, t \in \{x, y, z\}\}$ and $u \coloneqq p^x_{m,n}(\xi)$. By Lemma 3.2.6, u is in the intersection $\mathrm{Sect}_x(\xi) \cap \mathrm{Sect}_y(\xi) \cap \mathrm{Sect}_z(\xi)$. The result is an easy computation using three times Lemma 3.2.7. □

Proof of Proposition 3.2.1. — Let $x, y \in X^{(0)}$ be two base vertices and set $d \coloneqq d_1(x, y)$. Let $u \in X^{(0)}$ and $\xi \in \Omega_x(u)$. Set $v \coloneqq p^x_{m_x(u)+2d, n_x(u)+2d}(\xi)$. Since both $m_x(v), n_x(v)$ are greater than or equal to $2d_1(x, y)$, we have $\Omega_y(v) = \Omega_x(v) \subset \Omega_x(u)$, thanks to Corollary 3.2.8 and Scholia 3.1.19. Finally ξ sits in $\Omega_x(v)$ (by definition). □

3.3. Factorisation of vol_X through large spheres

In order to understand the possible values of the Poisson transform of $\mathrm{vol}_X(\underline{x})$, we first investigate the possible values of $\mathrm{vol}_X(\underline{x})$ on Ω.

Notation 3.3.1. — Whenever a function $f : \Omega \to \mathbf{C}$ is constant on each $\Omega_x(u)$ for u varying in a sector sphere $S_{m,n}(x)$ centered at a vertex x, we denote $f(u)$ the value of $f(\xi)$ for $\xi \in \Omega_x(u)$. The function f is said to *factor through* $S_{m,n}(x)$ and we have a commutative diagram:

$$\begin{tikzcd}
\Omega \arrow[d, "p^x_{m,n}"'] \arrow[dr, "f"] & \\
S_{m,n}(x) \arrow[r, "f"'] & \mathbf{C}
\end{tikzcd}$$

That is $f \circ p^x_{m,n} = f$. We use the same convention for maps $f : \Omega \to Z$ ranging in a set Z. A function factorizing through a sphere is locally constant. Conversly, since the topology of Ω is profinite, a locally constant function on Ω always factor through a sector sphere $S_{m,n}(x)$ provided m, n are large enough.

Example 3.3.2. — Let $x, y \in X^{(0)}$ and $m, n \geq d_1(x, y)$. Then the functions $\xi \mapsto m(x, y, \xi)$, $\xi \mapsto n(x, y, \xi)$ factor through $S_{m,n}(x)$. This is clear from Lemma 3.2.7. As a consequence, the Radon-Nikodym derivative given by

$$\frac{d\nu_x}{d\nu_y}(\xi) = q^{2(m(x,y,\xi)+n(x,y,\xi))},$$

factors through the same sphere, see Section 3.4.

Let $\underline{x}$ be a triple of vertices of X. We shall achieve the two following goals in parallel:

- Prove that the factorisation of $\mathrm{vol}_X(\underline{x})$ through a sphere $S_{m,n}(x)$ occurs as soon as m, n are greater than the diameter of $\underline{x}$ with respect to the metric d_1 and provided x is one of the vertices of the triple $\underline{x}$. (We shall write $x \in \underline{x}$.)
- Compute the value of $\mathrm{vol}_X(\underline{x})$ on $\Omega_x(u)$, for $u \in S_{m,n}(x)$, and show it to depend only on the integers $m(x, y, \xi), n(x, y, \xi)$ for $x, y \in \underline{x}$ and $\xi \in \Omega_x(u)$.

Proposition 3.3.3. — *There is a constant $C_{\mathrm{vol}} \in \mathbf{R}$, namely $\frac{1}{4}$, depending only on the building X (or rather only on its type, $\widetilde{A}_2$ here), such that*

$$\mathrm{vol}_X(x, y, z)(\xi) = C_{\mathrm{vol}} \cdot \det \begin{pmatrix} m(x, y, \xi) & m(x, z, \xi) \\ n(x, y, \xi) & n(x, z, \xi) \end{pmatrix},$$

for all $x, y, z \in X^{(0)}$ and $\xi \in \Omega$.

The proof uses the following lemmas. First, we compute the volume of a triangle sitting in a sector using the sector coordinates of the vertices. This will be useful to compute the volume after having retracted the building via $\rho_{(A,\xi)}$.

Lemma 3.3.4. — *There is a constant $C_1 \in \mathbf{R}$, namely $\frac{1}{4}$, such that for every $\xi \in \Omega$ and every triple of vertices x, y, z contained in a closed sector $\overline{\mathrm{Sect}}_u(\xi)$ issuing at some $u \in X^{(0)}$, we have*

$$\mathrm{vol}_X(x, y, z)(\xi) = C_1 \cdot \det \begin{pmatrix} m(y, z, \xi) & m(z, x, \xi) \\ n(y, x, \xi) & n(z, x, \xi) \end{pmatrix}.$$

Proof. — Here everything is really happening in $\mathbf{R}^2$. Assume, without loss of generality, that u is of type 0. Let u', u'' denote the vertices of $C_u(\xi)$ so that $\tau(u, u', u'') = (0, 1, 2)$, and set $v_1 = \overrightarrow{uu'}$, $v_2 = \overrightarrow{uu''}$, and $\mathcal{B} = (v_1, v_2)$. The sector coordinates of x, y, z with respect to u are the coefficients of $\overrightarrow{ux}$, $\overrightarrow{uy}$, $\overrightarrow{uz}$ expressed in the base $\mathcal{B}$. We write

$$[\overrightarrow{ux}]_{\mathcal{B}} = \begin{pmatrix} m_u(x) \\ n_u(x) \end{pmatrix} \qquad [\overrightarrow{uy}]_{\mathcal{B}} = \begin{pmatrix} m_u(y) \\ n_u(y) \end{pmatrix} \qquad [\overrightarrow{uz}]_{\mathcal{B}} = \begin{pmatrix} m_u(z) \\ n_u(z) \end{pmatrix},$$

hence

$$[\overrightarrow{xy}]_{\mathcal{B}} = \begin{pmatrix} m_u(y) - m_u(x) \\ n_u(y) - n_u(x) \end{pmatrix} \quad [\overrightarrow{xz}]_{\mathcal{B}} = \begin{pmatrix} m_u(z) - m_u(x) \\ n_u(z) - n_u(x) \end{pmatrix} \quad [\overrightarrow{yz}]_{\mathcal{B}} = \begin{pmatrix} m_u(z) - m_u(y) \\ n_u(z) - n_u(y) \end{pmatrix}.$$

Fix an apartment A containing $\mathrm{Sect}_u(\xi)$ and an orthonormal basis $\mathcal{C} = (e_1, e_2)$ of A so that $v_1 = e_1$ and $v_2 = \cos(\pi/3)e_1 + \sin(\pi/3)e_2 = \sqrt{3}/2e_1 + 1/2e_2$. The orientation of A given by (e_1, e_2) is $\sigma(A, \xi)$ (which corresponds to the orientation of the frame (u, u', u''), *i.e.* that of the base $\mathcal{B} = (v_1, v_2)$,) since the matrix

$$M = [\mathrm{id}_A]_{\mathcal{C}\mathcal{B}} = \begin{pmatrix} 1 & \sqrt{3}/2 \\ 0 & 1/2 \end{pmatrix}$$

has positive determinant. Our choices yield

$$\mathrm{vol}_X(x,y,z)(\xi) = \mathrm{vol}_{A,\sigma(A,\xi)}(x,y,z) = \frac{1}{2} e_1 \wedge e_2(\overrightarrow{xy}, \overrightarrow{xz}) = \frac{1}{2} \det \begin{pmatrix} \langle \overrightarrow{xy}, e_1 \rangle & \langle \overrightarrow{xy}, e_2 \rangle \\ \langle \overrightarrow{xz}, e_1 \rangle & \langle \overrightarrow{xz}, e_2 \rangle \end{pmatrix}.$$

The factor 1/2 comes from the dimension of A and shows the dependency on the building. More generally in $\mathbf{R}^n$ the volume of the n-tetrahedron formed by a basis is $\frac{1}{n!}$ times the volume of the n-parallelotope generated by the same basis. Since $\mathcal{C} = (e_1, e_2)$ is orthonormal, the matrix on the right hand side is the transpose of the matrix whose columns are $[\overrightarrow{xy}]_{\mathcal{C}}, [\overrightarrow{xz}]_{\mathcal{C}}$. Therefore,

$$\begin{aligned}
\mathrm{vol}_X(x,y,z)(\xi) &= \frac{1}{2} \det \left([\overrightarrow{xy}]_{\mathcal{C}} \mid [\overrightarrow{xz}]_{\mathcal{C}} \right) \\
&= \frac{1}{2} \det \left(M[\overrightarrow{xy}]_{\mathcal{B}} \mid M[\overrightarrow{xz}]_{\mathcal{B}} \right) \\
&= \frac{1}{2} \det(M) \det \left([\overrightarrow{xy}]_{\mathcal{B}} \mid [\overrightarrow{xz}]_{\mathcal{B}} \right) \\
&= \frac{1}{4} \det \left([\overrightarrow{xy}]_{\mathcal{B}} \mid [\overrightarrow{xz}]_{\mathcal{B}} \right) \\
&= \frac{1}{4} \det \begin{pmatrix} m_u(y) - m_u(x) & m_u(z) - m_u(x) \\ n_u(y) - n_u(x) & n_u(z) - n_u(x) \end{pmatrix}.
\end{aligned}$$

From Remark 3.2.5, we have

$$m(u,x,\xi) = -m_u(x), \quad m(u,y,\xi) = -m_u(y), \quad m(u,z,\xi) = -m_u(z),$$

and similar equations for the function n. Together with the 1-cocycle relation of Corollary 3.2.9 we obtain

$$\begin{aligned}
\begin{pmatrix} m_u(y) - m_u(x) \\ n_u(y) - n_u(x) \end{pmatrix} &= \begin{pmatrix} -m(u,y,\xi) + m(u,x,\xi) \\ -n(u,y,\xi) + n(u,x,\xi) \end{pmatrix} = \begin{pmatrix} m(y,x,\xi) \\ n(y,x,\xi) \end{pmatrix}, \\
\begin{pmatrix} m_u(z) - m_u(x) \\ n_u(z) - n_u(x) \end{pmatrix} &= \begin{pmatrix} -m(u,z,\xi) + m(u,x,\xi) \\ -n(u,z,\xi) + n(u,x,\xi) \end{pmatrix} = \begin{pmatrix} m(z,x,\xi) \\ n(z,x,\xi) \end{pmatrix}.
\end{aligned}$$

Using anti-symmetry, we conclude

$$\mathrm{vol}_X(x,y,z)(\xi) = \frac{1}{4} \det \begin{pmatrix} m(y,x,\xi) & m(z,x,\xi) \\ n(y,x,\xi) & n(z,x,\xi) \end{pmatrix}.$$

Notice how the signs in the last equality depend on the dimension of the building. □

Remark 3.3.5. — It is worth mentioning a formula making the link with the specifications of the Busemann cocycle. With the same notations and hypothesis, the change of basis implies

$$e_1 \wedge e_2 = \det(M)^{-1} v_1 \wedge v_2 = 2 \cdot v_1 \wedge v_2.$$

Consequently,

$$\begin{aligned}
\mathrm{vol}_X(x,y,z)(\xi) &= v_1 \wedge v_2(\overrightarrow{xy}, \overrightarrow{xz}) \\
&= \det \begin{pmatrix} \langle \overrightarrow{xy}, v_1 \rangle & \langle \overrightarrow{xy}, v_2 \rangle \\ \langle \overrightarrow{xz}, v_1 \rangle & \langle \overrightarrow{xz}, v_2 \rangle \end{pmatrix} \\
&= \det \begin{pmatrix} B(x,y)(\xi_1) & B(x,y)(\xi_2) \\ B(x,z)(\xi_1) & B(x,z)(\xi_2) \end{pmatrix}.
\end{aligned}$$

This formula follows from Example 1.2.31 and was first observed by Klingler [**Kli03**, Appendix A, Proposition 12].

Lemma 3.3.6. — *Let A be an apartment of X containing two vertices x, y and let ξ, ξ' be two opposite chambers at infinity, i.e. $\delta(\xi, \xi') = s_1 s_2 s_2$. Then,*

$$m(y, x, \xi') = n(x, y, \xi) \quad \text{and} \quad n(y, x, \xi') = m(x, y, \xi).$$

Proof of Lemma 3.3.6. — Let u be a vertex in the intersection of $\mathrm{Sect}_x(\xi)$ and $\mathrm{Sect}_y(\xi)$. Consider the opposite sector $\mathrm{Sect}_u(\xi')$. For $R \in \mathbf{N}$ large enough, the vertex $u' \in A$ with sector coordinates (R, R) in $\mathrm{Sect}_u(\xi')$ will be such that $u' \in \mathrm{Sect}_x(\xi') \cap \mathrm{Sect}_y(\xi')$, and $x, y \in \mathrm{Conv}^{(0)}(u, u')$. In the latter parallelogram, we have the following equations

$$m_x(u) + n_x(u') = R \quad \text{and} \quad n_x(u) + m_x(u') = R,$$

and the same equations holds for y in place of x. Lemma 3.2.7 implies

$$m(y, x, \xi') = m_x(u') - m_y(u') = R - n_x(u) - (R - n_y(u)) = n_y(u) - n_x(u) = n(x, y, \xi),$$

and similarly $n(y, x, \xi') = m(x, y, \xi)$. $\square$

Proof of Proposition 3.3.3. — Given $x, y, z \in X^{(0)}$ and $\xi \in \Omega$, pick $u \in \mathrm{Sect}_x(\xi)$ (or equivalently $\xi \in \Omega_x(u)$) such that

$$m_x(u), n_x(u) \geq \max\{d_1(x, y), d_1(x, z)\}.$$

Lemma 3.2.6 implies $u \in \mathrm{Sect}_t(\xi)$ for all $t \in \{x, y, z\}$. For every $t \in \{x, y, z\}$, let A_t be an apartment containing $\mathrm{Sect}_t(\xi)$, let ξ_t be the chamber opposite[(1)] to ξ in A_t, and denote by $\rho = \rho_{A_x,\xi}$ the canonical retraction onto A_x centered at ξ. The retraction $\rho|_{A_t} : A_t \to A_x$ is an isometry fixing $\mathrm{Sect}_x(\xi) \cap \mathrm{Sect}_t(\xi)$ for all $t \in \{x, y, z\}$, hence fixing u. Consequently $\mathrm{Sect}_u(\xi_t)$ is mapped isometrically by ρ onto $\mathrm{Sect}_u(\xi_x)$. We denote by $\tilde{t}$ the image under ρ of t, so that

$$x = \tilde{x} = \rho(x), \quad \tilde{y} = \rho(y), \quad \text{and} \quad \tilde{z} = \rho(z).$$

It follows from $t \in \mathrm{Sect}_u(\xi_t)$ that $\tilde{t} \in \mathrm{Sect}_u(\xi_x)$, where t stands for either x, y, or z. Notice the relation between the orientations,

$$\sigma(A_x, \xi)\sigma(A_x, \xi_x) = (-1)^{l(s_1 s_2 s_1)} = -1.$$

Applying the definition of vol_X twice,

$$\mathrm{vol}_X(x, y, z)(\xi) = \mathrm{vol}_{A_x,\sigma(A_x,\xi)}(\tilde{x}, \tilde{y}, \tilde{z}) = -\mathrm{vol}_{A_x,\sigma(A_x,\xi_x)}(\tilde{x}, \tilde{y}, \tilde{z}) = -\mathrm{vol}_X(\tilde{x}, \tilde{y}, \tilde{z})(\xi_x).$$

By construction of $\tilde{x}, \tilde{y}, \tilde{z}$ and ξ_x, we have using Lemmas 3.3.4 and 3.3.6

$$\begin{aligned}
\mathrm{vol}_X(x, y, z)(\xi) &= -\mathrm{vol}_X(\tilde{x}, \tilde{y}, \tilde{z})(\xi_x) \\
&= -C_1 \cdot \det \begin{pmatrix} m(\tilde{y}, \tilde{x}, \xi_x) & m(\tilde{z}, \tilde{x}, \xi_x) \\ n(\tilde{y}, \tilde{x}, \xi_x) & n(\tilde{z}, \tilde{x}, \xi_x) \end{pmatrix} \\
&= -C_1 \cdot \det \begin{pmatrix} n(\tilde{x}, \tilde{y}, \xi) & n(\tilde{x}, \tilde{z}, \xi) \\ m(\tilde{x}, \tilde{y}, \xi) & m(\tilde{x}, \tilde{z}, \xi) \end{pmatrix} \\
&= C_1 \cdot \det \begin{pmatrix} m(\tilde{x}, \tilde{y}, \xi) & m(\tilde{x}, \tilde{z}, \xi) \\ n(\tilde{x}, \tilde{y}, \xi) & n(\tilde{x}, \tilde{z}, \xi) \end{pmatrix} \\
&= C_1 \cdot \det \begin{pmatrix} m(x, y, \xi) & m(x, z, \xi) \\ n(x, y, \xi) & n(x, z, \xi) \end{pmatrix},
\end{aligned}$$

where the last equality follows from Lemma 3.2.7 (or from the 1-cocycle identity via u) and that ρ maps isometrically $\mathrm{Sect}_u(\xi_t)$ onto $\mathrm{Sect}_u(\xi_x)$. The result holds by setting $C_{\mathrm{vol}} = C_1 = \frac{1}{4}$. $\square$

[(1)] In the spherical building ∂X, we have $\delta(\xi, \xi_t) = s_1 s_2 s_1$, the longest element of W. In particular, $\mathrm{Sect}_u(\xi_t) \subset A_t$.

A comment on the proof. Similarly to the previous lemma, we used the fact that $\dim(X) = 2$ to multiply both columns by (-1), emphasizing the dependency on the type of X. On the other hand, it is not clear how the antepenultimate equality should behave in other buildings, *e.g.* if X is of type $\widetilde{A}_n$.

As a general philosophy the integers $m(x, y, \xi), n(x, y, \xi)$ can be computed with the cocycle relation via any $u \in \mathrm{Sect}_x(\xi) \cap \mathrm{Sect}_y(\xi)$. It is therefore clear that they are constant on $\Omega_x(u)$ as functions of ξ. We then write $m(x, y, u), n(x, y, u)$ for their values at some $\xi \in \Omega_x(u)$. As always $m_x(u), n_x(u)$ being greater than $d_1(x, y)$ is a sufficient condition, see Lemma 3.2.7.

Corollary 3.3.7. — *For every $x, y, z \in X^{(0)}$, the function $\mathrm{vol}_X(x, y, z)$ factors throught $S_{R,R}(x)$ for all natural numbers $R \geq \max\{d_1(x, y), d_1(x, z)\}$. In other words, $\mathrm{vol}_X(x, y, z)$ is constant on $\Omega_x(u)$ for each $u \in S_{R,R}(x)$, and*

$$\mathrm{vol}_X(x, y, z)(u) = C_{\mathrm{vol}} \cdot \det \begin{pmatrix} m(x, y, u) & m(x, z, u) \\ n(x, y, u) & n(x, z, u) \end{pmatrix}.$$

In particular, $\mathrm{vol}_X(x, y, z) : \Omega \to \mathbf{R}$ is locally constant.

3.4. Visual measures and the Poisson transform

This section introduces two kind of Borel measures on Ω called *visual measures* in order to generalize the Poisson transform.

3.4.1. Visual measures on Ω with respect to a point. — The characterization of Ω as a projective limit of finite sets, namely $\varprojlim \mathscr{S}(x)$, is very useful to define a Borel probability measure ν_x on Ω associated to x called the *visual measure* at x. Start by endowing each sector sphere $S_{m,n}(x)$ with the uniform probability measure $\nu_{m,n}$ giving each singleton $\{u\}$ weight

$$\nu_{m,n}(u) = N_{m,n}^{-1} = \frac{1}{|S_{m,n}(x)|}.$$

Lemma 3.4.1. — *The measures $\nu_{m,n}$ form a projective system of measures that is*

$$(p^x_{m',n'})_*(\nu_{m,n}) = (p^{x,m,n}_{m',n'})_*(\nu_{m,n}) = \nu_{m',n'},$$

for all $(m', n') \leq (m, n) \in \mathbf{N}^2$, where $(p^{x,m,n}_{m',n'})_*$ is the map that sends a measure on $S_{m,n}(x)$ to its image measure[(2)] *(or pushforward measure) on $S_{m',n'}(x)$. Consequently, there is a unique Borel probability measure ν_x on Ω satisfying*

$$(p^x_{m,n})_*(\nu_x) = \nu_{m,n},$$

for all $(m, n) \in \mathbf{N}^2$. In particular,

$$\nu_x(\Omega_x(u)) = \nu_{m,n}(u) = N_{m,n}^{-1},$$

for all $(m, n) \in \mathbf{N}^2$ and $u \in S_{m,n}(u)$.

[(2)]Let $p : X \to Y$ be a measurable map between two measurable spaces. For every measure ν on X, the *image measure* of ν under p is defined by the formula $p_*(\nu)(B) = \nu(p^{-1}(B))$ for all measurable $B \subset Y$.

Proof. — The reference for the projective limit measure is [**Bou04**, Integration, Chapitre III, §4.5]. It suffices to prove $\nu_{m',n'}(u) = (p^{x,m,n}_{m',n'})_*(\nu_{m,n})(u)$ for all $u \in S_{m',n'}(x)$. On the one hand, $\nu_{m',n'}(u) = N^{-1}_{m',n'}$. On the other hand, we have a partition

$$S_{m,n}(x) = \bigsqcup_{u \in S_{m',n'}(x)} (p^{x,m,n}_{m',n'})^{-1}(u),$$

into subsets of cardinal $N_{m,n} \cdot N^{-1}_{m',n'}$. We conclude that

$$(p^{x,m,n}_{m',n'})_*(\nu_{m,n})(u) = \nu_{m,n}\left((p^{x,m,n}_{m',n'})^{-1}(u)\right) = \frac{N_{m,n} \cdot N^{-1}_{m',n'}}{N_{m,n}} = N^{-1}_{m',n'}.$$

□

Remark 3.4.2. — Let x be a vertex of X and ν_x the corresponding visual measure. The group of automorphisms of X fixing x acts on each sector sphere $S_{m,n}(x)$ preserving the probability measure $\nu_{m,n}$. Consequently ν_x is also invariant under this action.

The topology on Ω was seen to be invariant of the point of reference in the building X, (Section 3.2). The visual measures however depend fundamentally on the point of reference. Nevertheless they are all pairwise absolutely continuous.

Proposition 3.4.3. — [**CMS94**, Lemma 2.5] *For every $x, y \in X^{(0)}$, the measures ν_x and ν_y are mutually absolutely continuous, i.e. they share the same null sets (subsets of measure 0). Moreover, the Radon-Nikodym derivative of ν_y with respect to ν_x is given by*

$$\frac{d\nu_x}{d\nu_y}(\xi) = q^{2(m(x,y,\xi)+n(x,y,\xi))},$$

for all $\xi \in \Omega$. In other words the following 'change of variable' formula holds:

$$\nu_x(f) = \int_\Omega f(\xi)d\nu_x(\xi) = \int_\Omega f(\xi)\frac{d\nu_x}{d\nu_y}(\xi)d\nu_y(\xi) = \nu_y(\frac{d\nu_x}{d\nu_y}f),$$

for all measurable $f : \Omega \to \mathbf{C}$. □

3.4.2. Visual measure on Ω with respect to a chamber. — In Chapter 2, we presented the Poisson transform as defined by Klingler in [**Kli04**]. The construction of a B-invariant measure ν_B was achieved by an alternating sum of B-invariant Borel probability measures on each B-orbit of Ω. There B denoted the standard Iwahori subgroup of a group G of F-points of connected, simply connected, almost F-simple algebraic group over a local field F. In this algebraic setting, X was Bruhat-Tits building of G on which it acts strongly transitively, and B was the stabilizer of a chamber C of X. Not all buildings admit a strong transitive group of automorphisms. For instance, one can build[(3)] a locally finite tree with trivial automorphism group by carefully choosing the valency of each vertex. In this case the orbits in Ω would be trivial whereas in the former case the B-orbits were in bijection with the finite Weyl group W thanks to the Iwasawa decomposition of Proposition 2.3.1. The proof showed that the B-orbits were characterized locally in the link of a vertex containing the chamber C. We use this idea to generalize the measures $g_*\nu_B$ by working in the link of a vertex.

Let X be a (thick) locally finite Euclidean building of type $\widetilde{A}_2$. We start by recalling the structure of the link of a vertex x, say of type 0. In the classical case of the Bruhat-Tits building

[(3)]or grow if you prefer

of $\mathrm{SL}_3(F)$ over a local field F, the link of a vertex is isomorphic to the incidence geometry of the projective plane $\mathbf{P}^2(k)$ where k is the residue field of F, *i.e.* a finite field of order q. In the general case, the link is also the incidence geometry of an abstract finite projective plane. More precisely, $\mathrm{lk}(x)$ is isomorphic to a bi-partite $(q+1)$-regular graph where the chambers of $\mathrm{lk}(x)$ correspond to the edges of the graph and the partition is given by the type of the vertices. They form two sets of cardinal q^2+q+1 and the graph has *girth* $= 6 = \mathrm{card}(W)$, *i.e.* the shortest length of a loop is 6. One concludes that

$$\mathrm{card}(\mathrm{lk}(x)) = (q+1)(q^2+q+1) = (q^3+2q^2+2q+1).$$

The link of X is a finite building of type $(W, \{s_1, s_2\})$. Given a chamber $C \in \mathrm{lk}(x)$, we have

$$\mathrm{card}(\{C' \in \mathrm{lk}(x) \mid \delta(C, C') = w\}) = q^{\ell(w)},$$

for all $w \in W$, hence,

$$\mathrm{card}(\mathrm{lk}(x)) = \sum_{w \in W} q^{\ell(w)}$$

Since the cardinal of $\mathrm{lk}(x)$ is independent of x we shall abbreviate $|\,\mathrm{lk}\,|$. We now define the analogue of the B-orbits $\mathcal{O}_w$ of Section 2.3.

Notation 3.4.4. — For $\xi \in \Omega$, we denote by $C_x(\xi)$ the unique chamber in the intersection $\mathrm{Sect}_x(\xi) \cap \mathrm{lk}(x)$, that is the initial chamber of $\mathrm{Sect}_x(\xi)$. We may refer to it as the *top* of $\mathrm{Sect}_x(\xi)$.

Definition 3.4.5. — Given $C \in \mathrm{lk}(x)$, define the following open subset of Ω:

$$\Omega_x(C) := \{\xi \in \Omega \mid C_x(\xi) = C\} = \Omega_x(u_1) \cap \Omega_x(u_2),$$

where u_1, u_2 are the vertices of type 1 and 2 of C respectively.

Notation 3.4.6. — Let C be a chamber whose vertex of type 0 is x. Let $\xi \in \Omega$, we denote:

- $w_C(\xi) := \delta(C, C_x(\xi)) \in W$.
- For $w \in W$, we define the *w-orbit* from C by $\mathcal{O}_w(C) = \{\xi \in \Omega \mid w_C(\xi) = w\}$.

Immediately, we have the obvious decompositions

$$\Omega = \bigsqcup_{C \in \mathrm{lk}(x)} \Omega_x(C) = \bigsqcup_{w \in W} \mathcal{O}_w(C),$$

which imply the following lemma.

Lemma 3.4.7. — *For every $w \in W$,*

$$\mathcal{O}_w(C) = \bigsqcup_{\substack{C' \in \mathrm{lk}(x) \\ \delta(C, C') = w}} \Omega_x(C').$$

Moreover, for all $C' \in \mathrm{lk}(x)$ and all $w \in W$,

$$\nu_x(\Omega_x(C')) = \frac{1}{|\,\mathrm{lk}\,|} \quad \text{and} \quad \nu_x(\mathcal{O}_w(C)) = \frac{q^{\ell(w)}}{|\,\mathrm{lk}\,|}.$$

□

Proof. — The first part is clear from the definitions. We show that $\nu_x(\Omega_x(C')) = \nu_x(\Omega_x(C))$, for all $C' \in \mathrm{lk}(x)$. By finite additivity,

$$1 = \nu_x(\Omega) = \nu_x\left(\bigsqcup_{\tilde{C}\in \mathrm{lk}(x)} \Omega_x(\tilde{C})\right) = \sum_{\tilde{C}\in \mathrm{lk}(x)} \nu_x(\Omega_x(\tilde{C})) = |\,\mathrm{lk}\,| \cdot \nu_x(\Omega_x(C')),$$

for all $C' \in \mathrm{lk}(x)$. The last statement follows as easily. □

We are ready to define the measure ν_C with the w-orbits $\mathcal{O}_w$ generalizing the B-orbits of Section 2.3.

Definition 3.4.8. — Let C be a chamber in X and x be its vertex of type 0.

- The Borel probability measure $\nu_{C,w}$ is defined by restriction to $\mathcal{O}_w(C)$ and normalisation of ν_x. More precisely,
$$\nu_{C,w}(E) = \frac{\nu_x(E \cap \mathcal{O}_w(C))}{\nu_x(\mathcal{O}_w(C))},$$
for all measurable $E \subset \Omega$ or equivalently,
$$\int_\Omega f(\xi)d\nu_{C,w}(\xi) = \fint_{\mathcal{O}_w(C)} f(\xi)d\nu_x(\xi) = \frac{1}{\nu_x(\mathcal{O}_w(C))}\int_{\mathcal{O}_w(C)} f(\xi)d\nu_x(\xi),$$
for all continuous $f : \Omega \to \mathbf{C}$.
- The signed Borel measure ν_C is defined by the formula
$$\nu_C \coloneqq \sum_{w\in W} \varepsilon(w)\nu_{C,w} = \sum_{w\in W} (-1)^{\ell(w)}\nu_{C,w}.$$
As ε is a character of W, we have in particular that $\sum_{w\in W}\varepsilon(w) = 0$, so that
$$\nu_C(\Omega) = \sum_{w\in W} \varepsilon(w)\nu_{C,w}(\Omega) = \sum_{w\in W} \varepsilon(w)\underbrace{\nu_{C,w}(\mathcal{O}_w(C))}_{=1} = 0$$
- Finally $|\nu_C| = \sum_{w\in W}\nu_{C,w}$, so that $|\nu_C|(\Omega) = |W| = 6$.

Definition 3.4.9 (Poisson transform). — Let $f : \Omega \to \mathbf{C}$ be a measurable function. The *Poisson transform* of f is the map $\mathcal{P}f : \mathrm{Ch}(X) \to \mathbf{C}$ define by integration via

$$\mathcal{P}f(C) = \langle f, \nu_C\rangle = \int_\Omega f(\xi)d\nu_C(\xi),$$

for all $C \in \mathrm{Ch}(X)$.

3.4.3. A Formula for the Poisson transform. — Given a chamber C of X with x_C as its type 0 vertex[(4)], recall the associated measure ν_C is given by

$$\nu_C = \sum_{w\in W} (-1)^{\ell(w)}\nu_{C,w},$$

where each $\nu_{C,w}$ is a normalization of $\nu_{x_C}|_{\mathcal{O}_w(C)}$.

For every $x \in X^{(0)}$, the map $C_x : \Omega \to \mathrm{lk}(x)$ factors through $S_{1,1}(x)$ simply because it assigns to $\xi \in \Omega$ the top of $\mathrm{Sect}_x(\xi)$. (The point $u \in \mathrm{Sect}_x(\xi)$ of coordinates $(1,1)$ clearly determines those

[(4)] We warn the reader that x_C was denoted by the letter x in the previous section introducing ν_C. Here x is a variable of vol_X.

of coordinates $(1,0)$ and $(0,1)$ hence the top of $\mathrm{Sect}_x(\xi)$.) Since the map $w_C : \Omega \to W$ is defined by

$$w_C(\xi) = \delta(C, C_{x_C}(\xi)),$$

it also factors through the sphere $S_{1,1}(x_C)$.

Now the Radon-Nikodym derivative of ν_C with respect to ν_{x_C} becomes

$$\frac{d\nu_C}{d\nu_{x_C}}(\xi) = (-1)^{\ell(w_C(\xi))} q^{-\ell(w_C(\xi))} |\,\mathrm{lk}\,|,$$

for all $\xi \in \Omega$. To be precise :

$$\frac{d\nu_C}{d\nu_{x_C}}(\xi) = \sum_{w \in W} \delta_{\{w_C(\xi)=w\}} \varepsilon(w) q^{-\ell(w)} |\,\mathrm{lk}\,|.$$

For any $u \in S_{m,n}(x_C)$ with $m, n \geq 1$, we write $w_C(u)$ and $\frac{d\nu_C}{d\nu_{x_C}}(u)$ for the corresponding images. Also Example 3.3.2 showed the Radon-Nikodym derivative of ν_{x_C} with respect to ν_x,

$$\frac{d\nu_{x_C}}{d\nu_x}(\xi) = q^{-2(m(x,x_C,\xi)+n(x,x_C,\xi))},$$

to factor through $S_{m,n}(x)$ provided $m, n \geq d_1(x, x_C)$.

Theorem 3.4.10. — *Let $\underline{x} = (x, y, z) \in X^3$ be a triple of vertices, $C \in \mathrm{Ch}(X)$, and let x_C be vertex of C of type 0. Then for every natural number $R \in \mathbf{N}$, satisfying*

$$R \geq \max\{d_1(x, y), d_1(x, z), 2d_1(x, x_C)\},$$

we have

$$\mathcal{P}\mathrm{vol}_X(\underline{x})(C) = C_{\mathrm{vol}} \cdot N_{R,R}^{-1} \cdot \sum_{u \in S_{R,R}(x)} \det \begin{pmatrix} m(x,y,u) & m(x,z,u) \\ n(x,y,u) & n(x,z,u) \end{pmatrix} \cdot \frac{d\nu_C}{d\nu_{x_C}}(u) \cdot \frac{d\nu_{x_C}}{d\nu_x}(u), \quad (3.3)$$

where C_{vol} is the constant of Proposition 3.3.3.

Proof. — Under the hypothesis

$$R \geq \max\{d_1(x, y), d_1(x, z)\},$$

Corollary 3.3.7 implies that $\mathrm{vol}_X(\underline{x})$ factors through $S_{R,R}(x)$, *i.e.* is constant on each $\Omega_x(u)$ when u ranges in the sphere. Thus

$$\begin{aligned} \mathcal{P}\mathrm{vol}_X(\underline{x})(C) &= \int_\Omega \mathrm{vol}_X(\underline{x})(\xi) d\nu_C(\xi) \\ &= \sum_{u \in S_{R,R}(x)} \mathrm{vol}_X(\underline{x})(u) \int_{\Omega_x(u)} d\nu_C(\xi) \\ &= \sum_{u \in S_{R,R}(x)} \mathrm{vol}_X(\underline{x})(u) \cdot \nu_C(\Omega_x(u)). \end{aligned}$$

If moreover[5] $R \geq 2d_1(x, x_C)$, we have $\Omega_x(u) = \Omega_{x_C}(u)$ thanks to Corollary 3.2.8. But also the Radon-Nikodym derivatives appearing in the next computations are constant on $\Omega_x(u)$. Therefore,

$$\begin{aligned}
\nu_C(\Omega_x(u)) &= \int_{\Omega_x(u)} d\nu_C(\xi) \\
&= \int_{\Omega_x(u)} \frac{d\nu_C}{d\nu_x}(\xi) d\nu_x(\xi) \\
&= \int_{\Omega_x(u)} \frac{d\nu_C}{d\nu_{x_C}}(\xi) \frac{d\nu_{x_C}}{d\nu_x}(\xi) d\nu_x(\xi) \\
&= \frac{d\nu_C}{d\nu_{x_C}}(u) \frac{d\nu_{x_C}}{d\nu_x}(u) \int_{\Omega_x(u)} d\nu_x(\xi) \\
&= \frac{d\nu_C}{d\nu_{x_C}}(u) \cdot \frac{d\nu_{x_C}}{d\nu_x}(u) \cdot \nu_x(\Omega_x(u)) \\
&= \frac{d\nu_C}{d\nu_{x_C}}(u) \cdot \frac{d\nu_{x_C}}{d\nu_x}(u) \cdot N_{R,R}^{-1}
\end{aligned}$$

for all $u \in S_{R,R}(x)$. A shorter proof of this computation could look like :

$$\nu_C(\Omega_x(u)) = \frac{d\nu_C}{d\nu_x}(u) \cdot \nu_x(\Omega_x(u)) = \frac{d\nu_C}{d\nu_x}(u) N_{R,R}^{-1},$$

for all $u \in S_{R,R}(x)$. The result follows from the formula for $\mathrm{vol}_X(x, y, z)(u)$ given in Proposition 3.3.3. But here is the final computation nevertheless:

$$\begin{aligned}
\mathcal{P}\mathrm{vol}_X(\underline{x})(C) &= \sum_{u \in S_{R,R}(x)} \mathrm{vol}_X(\underline{x})(u) \cdot \nu_C(\Omega_x(u)) \\
&= \sum_{u \in S_{R,R}(x)} \mathrm{vol}_X(\underline{x})(u) \cdot \frac{d\nu_C}{d\nu_{x_C}}(u) \cdot \frac{d\nu_{x_C}}{d\nu_x}(u) \cdot N_{R,R}^{-1} \\
&= C_{\mathrm{vol}} \cdot N_{R,R}^{-1} \cdot \sum_{u \in S_{R,R}(x)} \det \begin{pmatrix} m(x, y, u) & m(x, z, u) \\ n(x, y, u) & n(x, z, u) \end{pmatrix} \cdot \frac{d\nu_C}{d\nu_{x_C}}(u) \cdot \frac{d\nu_{x_C}}{d\nu_x}(u).
\end{aligned}$$

□

In this construction, the observer was located at x. Using y and z in the role of x one obtains an averaging formula provided R satisfies the hypothesis

$$R \geq \max\{\mathrm{diam}(\{x, y, z\}), 2d_1(x, x_C), 2d_1(y, x_C), 2d_1(z, x_C)\},$$

where the diameter is with respect to the sector distance d_1. We could take the less precise but uniform bound

$$R \geq 2\,\mathrm{diam}(\{x, y, z, x_C\}).$$

In the next corollary, we denote # the counting measure of a finite set and $⨍ d\#$ denotes the average summation on it.

Corollary 3.4.11. — *Let $\underline{x} = (x, y, z) \in X^3$ be a triple of vertices, $C \in \mathrm{Ch}(X)$, and let x_C be the vertex of C of type* 0. *Then for every $R \in \mathbf{N}$ satisfying*

$$R \geq \max\{\mathrm{diam}(\{x, y, z\}), 2d_1(x, x_C), 2d_1(y, x_C), 2d_1(z, x_C)\},$$

[5] Without loss of generality, suppose R is greater than 42, to avoid degeneracies.

we have

$$\mathcal{P}\mathrm{vol}_X(\underline{x})(C) = \fint_{t\in\{x,y,z\}} \fint_{S_{R,R}(t)} \mathrm{vol}_X(x,y,z)(u)\frac{d\nu_C}{d\nu_t}(u)d\#(u)d\#(t),$$

□

Comments 3.4.12. — We make two comments:

(i) Trying to apply some sort of Fubini's theorem to this last formula is desirable and could be one way to investigate the values taken by $\mathcal{P}\mathrm{vol}_X(\underline{x})$. However this requires us to compare the various spheres $S_{R,R}(t)$ with $t \in \{x,y,z\}$. We fear that after an exhaustive analysis one may fall back on the original formula due to the 1-cocycle identity satisfied by m and n. Another path of investigation is to try to group the positive and negative terms in the summation over $u \in S_{R,R}(x)$.

(ii) The formula above indeed generalizes the Poisson transform of Klingler in the case of a Bruhat-Tits building, except maybe up to a permutation of the types, hence up to a sign. This has no real consequence for $\widetilde{A}_2$ buildings since all vertices are special. We did not discuss whether the Poisson transform ranges in the harmonic functions on $\mathrm{Ch}(X)$, nor did we approach the question of square integrability. The formula we obtained seems not to depend on the isomorphism class of the building, but rather on the regularity parameter and on the cardinal of the intersections of some sectors spheres. Intuitively, it should be that the formula depends only on the relative positions of x, y, z, x_C, and on q. Should this be true, we could say that the value of $\mathcal{P}\mathrm{vol}_X(\underline{x})(C)$ does not depend on the building under investigation! This would imply that it is indeed harmonic and L^2 for <u>all</u> $\widetilde{A}_2$ buildings whose regularity parameter $q \in \mathbf{N}$ is a prime power, because we know it to be true for Bruhat-Tits buildings.

CHAPTER 4

THE RANK ONE CASE

In this chapter we compute the growth of Klingler's 1-cocycle. The Bruhat-Tits building X associated to $\mathrm{SL}_2(F)$ over a local field is a $(q+1)$-regular tree, where q is the cardinal of the residue field of F. In particular, q is a power of a prime number. In this case, the cocycle coincides with the Busemann cocycle which exists for any CAT(0) space. In fact, the present context can be extended to any regular tree without restriction on the regularity parameter q, except maybe $q \geq 2$ to ensure thickness. Independently, Gournay and Jolissaint [**GJ15**] obtained an explicit bound for the norm of any harmonic 1-cocycle. Their result applies to all harmonic 1-cocycle and makes use of the Green kernel and its inverse, something that seems unavailable for higher rank buildings. The method presented in this chapter yields a sublinear bound as that of *loc. cit.* We hope nevertheless that our explicit calculations shed light on the combinatorics of Klingler's cocycle.

4.1. Homogeneous trees and extended Poisson transform

Notation 4.1.1. — For $q \in \mathbf{N}$, let X be the $(q+1)$-regular (unoriented) tree identified as usual with its geometric realization endowed with the CAT(0) metric d for which the edges have length 1. We denote by:

- $X^{(0)}$ the set of vertices of X with type function $\tau : X^{(0)} \to \mathbf{Z}/2\mathbf{Z}$.
- $X^{(1)} = \mathrm{Ch}(X)$ the set of (open) edges.
- $\Omega = \partial X = \mathrm{Ch}(\partial X)$ the visual boundary of X as a CAT(0) space.
- r_x^ξ the unique geodesic ray issuing at $x \in X$ pointing toward $\xi \in \Omega$.
- $B : X^2 \to \mathrm{C}(\Omega)$ the Busemann cocycle which maps $(x, y) \in X^2$ to the function

$$\xi \mapsto B(x,y)(\xi) = B_\xi(x,y) := \lim_{t\to\infty} d(y, r_x^\xi(t)) - t.$$

More generally

$$B(x,y)(\xi) = \lim_{z\to\xi} d(y,z) - d(x,z),$$

in the CAT(0) compactification of X.

The boundary Ω is endowed with the topology generated by subsets of the form

$$\Omega_x(u) = \{\xi \in \Omega \mid \text{ the geodesic ray } r_x^\xi \text{ passes through } u\},$$

where $u \in X$. For every $R \in \mathbf{N}$, let $S_R(x)$ denote the sphere in X of radius R about a vertex x. The visual measure ν_x centered at a vertex x is the Borel probability measure such that $\Omega_x(u)$ has measure $\mathrm{card}(S_R(x))^{-1}$, with $R = d(x,u)$.

The construction of the Poisson transform in an $\widetilde{A}_2$ building is easily adapted to the present setting[(1)]. We now describe the content of this Section which is a slight generalization of our previous constructions. In the general case we associate to each edge C a signed Borel measure ν_C on Ω, following the ideas of Section 3.4. The Poisson transform of a measurable function $f : \Omega \to \mathbf{C}$ is then the map $\mathcal{P}f : \mathrm{Ch}(X) \to \mathbf{C}$ given by

$$\mathcal{P}f(C) = \langle f, \nu_C \rangle = \int_\Omega f(\xi) d\nu_C(\xi).$$

Assuming $\mathcal{P}f$ to be square summable on the set of edges, its ℓ^2-norm is invariant under pointwise changes of sign. In other words, changing the sign of $\mathcal{P}f(C)$ at arbitrarily many C does not change $\|\mathcal{P}f\|_{\ell^2}$. Such a change amounts to replacing the measure ν_C by $-\nu_C$. Thus one can adapt the signs of the measure ν_C in order to get a uniform configuration. Up to a sign, the measure ν_C can be described as follows. If we remove the edge C from the tree X, we are left with two connected components T^+ and T^- whose visual boundaries partition $\partial X = \partial T^+ \sqcup \partial T^-$. Up to a sign, the measure ν_C is the alternating sum of Borel probability measures on ∂T^+ and ∂T^-, see Notation 4.1.5 and Lemma 4.1.7. In order to put this in a slightly more general context, we introduce oriented edges and corresponding measures $e \mapsto \nu_e$ such that if $e, \bar{e}$ are the two opposite orientations of a common edge then $\nu_e = -\nu_{\bar{e}}$.

Notation 4.1.2. — We denote by:

- $\mathbb{E}$ the set of all[(2)] *oriented edges* of X endowed with the involution $e \mapsto \bar{e}$ sending an edge e to its opposite orientation,
- $o, t : \mathbb{E} \to X^{(0)}$ the maps sending an edge e to its *origin* $o(e)$ and *target* $t(e)$ respectively.
- $|\mathbb{E}| := X^{(1)}$ the set of *unoriented edges*, *i.e.* the image under the 2-to-1 map

$$|\cdot| : \mathbb{E} \to |\mathbb{E}|$$

  forgetting the orientation, namely sending e to $|e| = \{o(e), t(e)\}$.

To define the measure ν_e associated to an oriented edge $e \in \mathbb{E}$, we need the following lemma.

Lemma 4.1.3. — *Let v be a vertex in X and T be the connected component containing v of the forest obtained from X by removing at most q edges of the link of v. Then for every $x, y \notin T$, we have*

$$\Omega_x(u) = \Omega_y(u),$$

for all $u \in T \smallsetminus \{v\}$. Moreover the Busemann cocycle satisfies

$$B(x,y)(\xi) = d(y,v) - d(x,v),$$

for all $\xi \in \Omega_x(u)$.

[(1)] The reader is invited to do it as an exercise.
[(2)] two for each edge $C \in X^{(1)}$,

Proof. — Let $u \in T \smallsetminus \{v\}$, $x, y \notin T$, and $\xi \in \Omega_x(u)$. Since r_x^ξ passes through u, the geodesic must enter T at v, and, hence, never leave T thereafter because geodesics may not backtrack in a tree. On the other hand we know that $r_x^\xi(R) \in r_y^\xi$ for all $R > 0$ large enough. Therefore r_y^ξ also enters T at v. This shows $\Omega_x(\xi) \subset \Omega_y(\xi)$, and the equality holds by switching roles of x and y. The Busemann cocycle at $\xi \in \Omega$ is given by

$$B(x,y)(\xi) = \lim_{z \to \xi} d(y,z) - d(x,z).$$

Equivalently $B(x,y)(\xi)$ is the unique integer satisfying $r_x^\xi(R) = r_y^\xi(R + B(x,y)(\xi))$, for all sufficiently large $R > 0$. In fact, as soon as $r_x^\xi(R)$ sits in the intersection of r_x^ξ and r_y^ξ, then the previous identity holds. Since v is in that intersection the last assertion follows. □

If ν_x denotes the visual measure at x, we have

$$\nu_x(\Omega_x(u)) = \Big((q+1)q^{d(x,u)-1}\Big)^{-1},$$

for all $u \neq x$. Recall that the Radon-Nikodym derivatives for visual measures is given by

$$\frac{d\nu_x}{d\nu_y}(\xi) = q^{B(x,y)(\xi)},$$

for all $x, y \in X$ and $\xi \in \Omega$, see Proposition 3.4.3.

Corollary 4.1.4. — *Let v and T be as in Lemma 4.1.3. Then for all vertices $x, y \notin T \smallsetminus \{v\}$*

$$\frac{d\nu_x}{d\nu_y}(\xi) = q^{d(y,v)-d(x,v)},$$

for all $u \in T \smallsetminus \{v\}$ and $\xi \in \Omega_x(u)$. □

Notation 4.1.5. — For every $e \in \mathbb{E}$, we denote:

- T_e^+ and T_e^-, the two connected components of $X \smallsetminus \{|e|\}$, such that the origin $o(e)$ is in T_e^+ and its target $t(e)$ is in T_e^-, see Figure 1,
- $\partial T_e^\pm$, the visual boundary of $T_e^\pm$, so that $\Omega = \partial X = \partial T_e^+ \sqcup \partial T_e^-$,
- $\nu_e^\pm$, the *visual measure* on $\partial T_e^\pm$, *i.e.* the Borel probability measure on $\partial T_e^\pm$ proportional to the (vertex) visual measure $\nu_{o(e)}$.
- $\nu_e \coloneqq \nu_e^+ - \nu_e^-$ and $|\nu_e| \coloneqq \nu_e^+ + \nu_e^-$.

The mnemonic is to picture the edge e as being the neck of an hourglass. We think of T_e^+ being filled with sand flowing into T_e^- in the direction determined by e, again see Figure 1.

Remark 4.1.6. — The boundaries $\partial T_e^\pm$ are well defined thanks to Lemma 4.1.3. More precisely, ∂T_e^- contains the classes of rays issuing at $o(e)$ and passing through $t(e)$, *i.e.*

$$\partial T_e^- = \Omega_{o(e)}(t(e)),$$

whereas ∂T_e^+, the complement in Ω, can be written as

$$\partial T_e^+ = \bigsqcup_{\substack{u \in S_1(o(e)) \\ u \neq t(e)}} \Omega_{o(e)}(u).$$

The normalizations of $\nu_e^\pm$ are done using

$$\nu_{o(e)}(\partial T_e^-) = \frac{1}{q+1} \quad \text{and} \quad \nu_{o(e)}(\partial T_e^+) = \frac{q}{q+1}.$$

The following lemma is now clear.

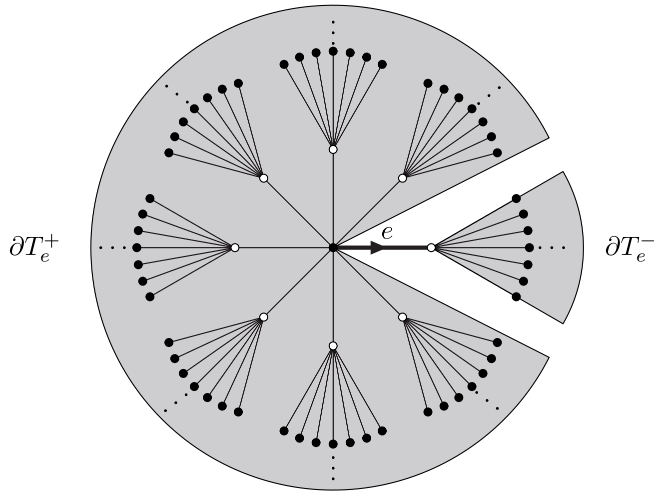

FIGURE 1. Decomposition $\Omega = \partial T_e^+ \sqcup \partial T_e^-$.

Lemma 4.1.7. — *Consider an edge $C = |e|$ for some $e \in \mathbb{E}$. Then the signed measure ν_C defined as in Definition 3.4.8 is $\pm\nu_e$ with $+$ sign if and only if the orientation corresponding to the labelling of C is that of e, that is if and only if $o(e)$ is of type* 0. □

Definition 4.1.8 (Poisson transform). — The *Poisson transform* of a measurable function $f : \Omega \to \mathbf{C}$ is the map $\mathcal{P}f : \mathbb{E} \to \mathbf{C}$ given by

$$\mathcal{P}f(e) = \langle f, \nu_e \rangle = \int_\Omega f(\xi) d\nu_e(\xi).$$

Readily the above Poisson transform is merely an alternating version of the Poisson transform of previous chapters, so that the next Proposition holds.

Proposition 4.1.9. — *Let $f : \Omega \to \mathbf{C}$ be a measurable function. Then $\mathcal{P}f$ is antisymmetric with respect to the map reversing orientations, that is*

$$\mathcal{P}f(\bar{e}) = -\mathcal{P}f(e)$$

for all $e \in \mathbb{E}$, thus

$$|\mathcal{P}f(e)| = |\mathcal{P}f(|e|)|.$$

If moreover $\mathcal{P}f \in \ell^2(\mathbb{E})$, then

$$\|\mathcal{P}f\|^2_{\ell^2(\mathbb{E})} = \sum_{e\in\mathbb{E}} \mathcal{P}f(e)^2 = 2 \cdot \sum_{|e|\in|\mathbb{E}|} \mathcal{P}f(|e|)^2 = 2\|\mathcal{P}f\|^2_{\ell^2(|\mathbb{E}|)} = 2\|\mathcal{P}f\|^2_{\ell^2(\mathrm{Ch}(X))}.$$

□

Definition 4.1.10. — An *orientation of the edges* of X is a section of the forgetful map $|\cdot|$. It is equivalent to the choice of a fundamental domain $\mathbb{E}^+$ of the involution $e \mapsto \bar{e}$. Clearly $\mathbb{E} = \mathbb{E}^+ \sqcup \mathbb{E}^-$, where $\mathbb{E}^- = \overline{\mathbb{E}^+}$.

As mentioned above, the labeling induces an orientation of the edges of X. More precisely each edge C is mapped to the oriented edge $e \in \mathbb{E}$ such that $o(e)$ is of type 0 and $t(e)$ of type 1. We will later choose adequate orientations to perform computations.

Corollary 4.1.11. — *Given an orientation of the edges with fundamental domain* $\mathbb{E}^+$, *one has* $\|\mathcal{P}f\|^2_{\ell^2(\mathbb{E}^+)} = \|\mathcal{P}f\|^2_{\ell^2(|\mathbb{E}|)}$. □

To compute the norm of a Poisson transform, we are free to pick any orientation of each edge. The rest of the chapter aims at computing an upper bound for $\|\mathcal{P}B(x,y)\|^2_{\ell^2(\mathbb{E})}$ which yields the asymptotic growth when $d(x,y)$ tends to infinity.

4.2. Strategy and results

Let G be the group of type-preserving automorphisms of X. It acts naturally on $\mathbb{E}$ as well. The G-equivariance of the Klingler cocycle, *i.e.* the Busemann cocycle B, implies that $\mathcal{P}B(x,y)$ is invariant under the action of the intersection $K_{[x,y]} := K_x \cap K_y$ of the stabilisers of x and y. The latter coincides with $\mathrm{Fix}_G([x,y])$, the pointwise stabiliser of the geodesic segment $[x,y]$. Let R be set of representatives for the action of $K_{[x,y]}$ onto $\mathbb{E}$. We conclude

$$\|\mathcal{P}B(x,y)\|^2_{\ell^2(\mathbb{E})} = \sum_{e\in R} \mathcal{P}B(x,y)(e)^2 \cdot \mathrm{card}(K_{[x,y]}e). \tag{4.1}$$

Consequently, it is desirable to identify the various $K_{[x,y]}$-orbits and determine their cardinal as well as the value of the Poisson transform of $B(x,y)$ at those points. This is the strategy we adopt.

4.2.1. Projection. — Prior to describing the orbits precisely, we recall the notion of projection. Let $e \in \mathbb{E}$ be an oriented edge, the distance between e and the segment $[x,y]$ is given by

$$d(e,[x,y]) := d(|e|,[x,y]) = \inf_{t\in[x,y],t'\in|e|} d(t,t').$$

The group G acts on X by isometries, consequently this distance is constant on the whole $K_{[x,y]}$-orbit of e. Similarly $d(x,e)$ and $d(y,e)$ enjoy the same property. These quantities determine the relative position of x, y and e in X and should intuitively determine the value of $\mathcal{P}B(x,y)(e)$ up to a sign as we shall see in the next section.

In a tree, and more generally in a CAT(0) space [**BH99**, Chapter II.2, Proposition 2.4], there is a well defined notion of *projection* onto a closed convex subset. The projection $p : X \to [x,y]$ has the property that the distance between any point $v \in X$ and the segment $[x,y]$ is realized by $d(v,p(v))$. Let $e \in \mathbb{E}$ and suppose $|e|$ is not contained in $[x,y]$. The distance $d(|e|,[x,y])$ is realized by $d(v,p(v))$ where v is the vertex of $|e|$ closest to $[x,y]$. In fact, v is itself the projection of x, or equivalently of any point of $[x,y]$, onto the closure of $|e|$. We shall write p_e for the projection of e onto $[x,y]$, see Figure 10 for an example.

4.2.2. Orbits in $\mathbb{E}$ under the action of $\mathrm{Fix}_G([x,y])$. — When sorting the possible configurations of $e \in \mathbb{E}$ with respect to $[x,y]$, it seems natural to distinguish the case where x, y and e lie in a common apartment, *i.e.* on a geodesic line. This amounts to saying that either $|e| \subset [x,y]$ or, else, that p_e is equal to x or y. Recall that the projection of an edge onto $[x,y]$ is defined only for edges not in $[x,y]$. For simplicity we assume $d(x,y) \geq 2$ so that the following configurations occur and yield disjoint subsets of $\mathbb{E}$:

$$\begin{aligned} e \in \mathbb{A}_1 &\iff p_e = x, \\ e \in \mathbb{A}_2 &\iff |e| \subset [x,y], \\ e \in \mathbb{A}_3 &\iff p_e = y, \\ e \in \mathbb{B} &\iff p_e \neq x, y. \end{aligned}$$

We imply that p_e is defined if printed. Set $\mathbb{A} := \cup_i \mathbb{A}_i$; we have a partition

$$\mathbb{E} = \mathbb{A}_1 \sqcup \mathbb{A}_2 \sqcup \mathbb{A}_3 \sqcup \mathbb{B}, \tag{4.2}$$

into stable subsets under $e \mapsto \bar{e}$.

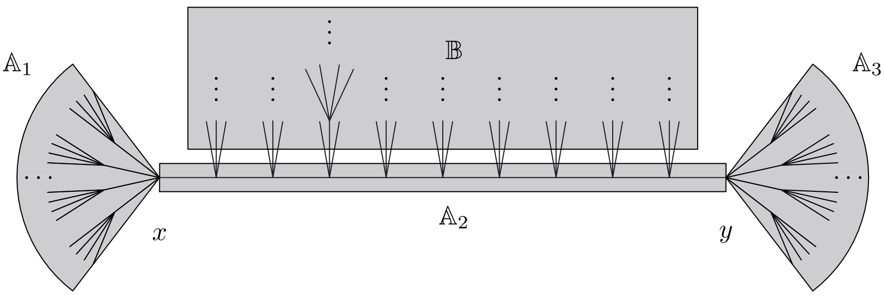

FIGURE 2. An arrangement of X emphasizing the decomposition of $\mathbb{E}$ with $q = 4$.

The motivation behind this sorting is the transitivity of $\mathrm{Fix}_G([x,y])$ on the set of apartments of X containing $[x,y]$, *e.g.* Corollary 1.2.23. Consequently, the partition (4.2) is $K_{[x,y]}$-invariant. The orbit of an edge in $\mathbb{A}$ is easy to determine, whereas the case $\mathbb{B}$ will require more work, see Section 4.4. We nevertheless state the main resuls and prove the case $\mathbb{A}$ modulo the proof of Lemma 4.2.3 which appears in Section 4.3.

Lemma 4.2.1. — *The cardinal of the $K_{[x,y]}$-orbit of $e \in \mathbb{A}$ is given by:*

$$\mathrm{card}(K_{[x,y]}e) = \begin{cases} q^{d(e,[x,y])+1} & \text{if } e \in \mathbb{A}_1 \cup \mathbb{A}_3, \\ q^{d(e,[x,y])} = 1 & \text{if } e \in \mathbb{A}_2. \end{cases}$$

Proof. — Let $e \in \mathbb{A}$. It is clear that the right hand side depends only on $|e|$. If $e \in \mathbb{A}_2$, then $|e| \subset [x,y]$ is fixed by $K_{[x,y]}$. Assume $e \in \mathbb{A}_1$ and let $w \in W_{\mathrm{aff}}$ be the W-distance between the edge C_x in $[x,y]$ having x as a vertex and $|e|$ that is $w = \delta(C_x, |e|)$. Since $K_{[x,y]}$ acts transitively on the set of apartments containing $[x,y]$, the $K_{[x,y]}$-orbit of $|e|$ is $B(C_x, w)$, which has cardinal $q^{\ell(w)}$. (We warn that we are using the fact that X is a tree and not only a building.) The result follows from $\ell(w) = d(C_x, |e|) = d(|e|, [x,y]) + 1$. The proof for $e \in \mathbb{A}_3$ is the same using C_y the unique edge of $[x,y]$ having y as a vertex. $\square$

Remark 4.2.2. — If we fix an apartment A containing $[x,y]$, that is a geodesic line, and let $R_{\mathbb{A}}$ denote the set of all oriented edges $e \in \mathbb{E}$ with $|e| \in A$, the previous lemma shows $R_{\mathbb{A}}$ is a set of representatives for the orbits contained in $\mathbb{A}$. In addition, we set $R_{\mathbb{A}_i} = \mathbb{A}_i \cap R_{\mathbb{A}}$ for $i = 1,2,3$.

Regarding the value of $\mathcal{P}B(x,y)$ at edges in $\mathbb{A}$, the following lemma is proved in Section 4.3.

Lemma 4.2.3. — *For every $e \in \mathbb{A}_1 \cup \mathbb{A}_3$,*

$$|\mathcal{P}B(x,y)(e)| \leq \frac{2}{(q-1)} q^{-d(e,[x,y])};$$

whereas for every $e \in \mathbb{A}_2$,

$$|\mathcal{P}B(x,y)(e)| \leq \frac{2(q+1)}{(q-1)}.$$

The main result is the following theorem.

Theorem 4.2.4. — *Let X be the $(q+1)$-regular tree with $q \geq 4$. There exist constants $C_{\mathbb{A}_1}, C_{\mathbb{A}_2}, C_{\mathbb{A}_3}, C_{\mathbb{B}} > 0$ depending only on the regularity parameter q such that for every pair $x, y \in X$ of vertices with $d(x,y) \geq 2$, if $\mathbb{A}_i$ for $i = 1,2,3$, and $\mathbb{B}$ are the corresponding subsets of oriented edges as defined in Paragraph 4.2.2, we have*

$$\sum_{e \in \mathbb{A}_1} |\mathcal{P}B(x,y)(e)|^2 \leq C_{\mathbb{A}_1},$$

$$\sum_{e \in \mathbb{A}_2} |\mathcal{P}B(x,y)(e)|^2 \leq C_{\mathbb{A}_2} \cdot d(x,y),$$

$$\sum_{e \in \mathbb{A}_3} |\mathcal{P}B(x,y)(e)|^2 \leq C_{\mathbb{A}_3},$$

$$\sum_{e \in \mathbb{B}} |\mathcal{P}B(x,y)(e)|^2 \leq C_{\mathbb{B}}.$$

Moreover,

$$C_{\mathbb{A}_1} = C_{\mathbb{A}_3} = \frac{8q^2}{(q-1)^3}, \quad C_{\mathbb{A}_2} = \frac{8(q+1)^2}{(q-1)^2}, \quad \textit{and} \quad C_{\mathbb{B}} = \frac{16q^3}{(q-1)^3(q+1)}.$$

Corollary 4.2.5. — *Let X be the $(q+1)$-regular tree with $q \geq 4$. Then there is a constant $C > 0$*[(3)] *depending only on the regularity parameter q such that*

$$\|\mathcal{P}B(x,y)\|^2_{\ell^2(\mathbb{E})} \leq C \cdot d(x,y),$$

for all vertices $x, y \in X$, with $d(x,y) \geq 2$. □

Proof of Theorem 4.2.4. — Recall that we assume $d(x,y) \geq 2$ to guaranty the nonemptyness of $\mathbb{B}$. The proof of the case $\mathbb{B}$ is postponed to Section 4.4 where we make use of the assumptions $q \geq 4$. There, we prove some lemmas similar to those for the Case $\mathbb{A}$, namely Lemma 4.2.1 and Lemma 4.2.3, and we complete the proof of the present theorem. The proof for the $\mathbb{A}_i$'s follows easily by inserting the results of Lemma 4.2.1 and Lemma 4.2.3 into equation (4.1). For instance, using the

(3) $C := \max\{C_{\mathbb{A}_i}, C_{\mathbb{B}} \mid i = 1,2,3\}$.

representative set $R_{\mathbb{A}_1}$ defined in Remark 4.2.2,

$$\begin{aligned}\sum_{e\in\mathbb{A}_1}\mathcal{P}B(x,y)(e)^2 &= \sum_{e\in R_{\mathbb{A}_1}}\mathcal{P}B(x,y)(e)^2\cdot\operatorname{card}(K_{[x,y]}e)\\ &\le \sum_{e\in R_{\mathbb{A}_1}}\left(\frac{2}{q-1}q^{-d(e,[x,y])}\right)^2\cdot q^{d(e,[x,y])+1}\\ &= q\left(\frac{2}{q-1}\right)^2\sum_{e\in R_{\mathbb{A}_1}}q^{-d(e,[x,y])}\\ &= \frac{4q}{(q-1)^2}\cdot 2\cdot\sum_{i=0}^{\infty}q^{-i}\\ &= \frac{8q^2}{(q-1)^3} =: C_{\mathbb{A}_1}.\end{aligned}$$

The case $\mathbb{A}_3$ is proved in the same way. The proof for $\mathbb{A}_2$ is simple:

$$\begin{aligned}\sum_{e\in\mathbb{A}_2}\mathcal{P}B(x,y)(e)^2 &= \sum_{e\in R_{\mathbb{A}_2}}\mathcal{P}B(x,y)(e)^2\cdot\underbrace{\operatorname{card}(K_{[x,y]}e)}_{=1}\\ &\le \sum_{e\in R_{\mathbb{A}_2}}\left(\frac{2(q+1)}{(q-1)}\right)^2\\ &= \left(\frac{2(q+1)}{(q-1)}\right)^2\cdot 2\cdot d(x,y) = \underbrace{\frac{8(q+1)^2}{(q-1)^2}}_{=:C_{\mathbb{A}_2}}\cdot d(x,y).\end{aligned}$$

□

4.3. The case $\mathbb{A}$

This section contains the proof of Lemma 4.2.3. Here is a brief summary of the section. In order to compute the value of the Poisson transform of $B(x,y)$ at an edge $e\in\mathbb{A}$, we decompose the boundary Ω into a countable union of disjoint sets on which $B(x,y)$ is constant. This is done by removing the edges of a geodesic line σ containing e and $[x,y]$ to X. We are left with a countable forest $\{T_k\}_{k\in\mathbf{Z}}$ of trees rooted at the vertices of σ, whose boundaries partition $\Omega\setminus\{\sigma(\infty),\sigma(-\infty)\}$. The ν_e-measure ∂T_k is easily computed and then

$$\mathcal{P}B(x,y)(e)=\sum_{k\in\mathbf{Z}}f(k)\nu_e(\partial T_k),$$

where $f(k)$ is the value of $B(x,y)$ on ∂T_k. The question is therefore transposed into a technical problem on $\mathbf{Z}$.

Assumptions 4.3.1. — For the remainder of the section, we fix the vertices $x,y\in X$ at distance $d=d(x,y)$ and a geodesic line $\sigma:\mathbf{R}\to X$ passing through $[x,y]$ parametrized so that $\sigma(0)=x$ and $\sigma(d)=y$.

As noticed in Remark 4.2.2, we only need to estimate the value of $\mathcal{P}B(x,y)$ at edges sitting on σ and we may choose their orientation thanks to Proposition 4.1.9. For every $i\in\mathbf{Z}$, let e_i be the oriented edge with origin $\sigma(i)$ and target $\sigma(i+1)$. Thus it suffices to establish Lemma 4.2.3 for the e_i's only. Accordingly we take $R_{\mathbb{A}}$ to be the set of all oriented edges supported on σ, thus our choice of orientation is $R^+_{\mathbb{A}} = \{e_i \mid i \in \mathbf{Z}\}$ and $R^-_{\mathbb{A}} = \overline{R^+_{\mathbb{A}}}$. The sets $R^+_{\mathbb{A}_i}$, $R^-_{\mathbb{A}_i}$ are defined similarly[(4)]. For every $e_i \in R^+_{\mathbb{A}}$, we have

$$\begin{aligned} e_i \in R_{\mathbb{A}_1} &\iff i < 0, \\ e_i \in R_{\mathbb{A}_2} &\iff 0 \le i < d, \\ e_i \in R_{\mathbb{A}_3} &\iff d \le i. \end{aligned}$$

The $K_{[x,y]}$-orbits of $R^+_{\mathbb{A}}$ form our choice of orientation $\mathbb{A}^+$, idem for $\mathbb{A}^+_i$.

Proposition 4.3.2. — *Let r^ξ_x denote the unique geodesic ray issuing at x pointing toward $\xi \in \Omega$. For every $k \in \mathbf{Z}$, let Ω_k be the subset of $\xi \in \Omega$ such that the intersection of r^ξ_x and σ is the segment $[x, \sigma(k)]$, see Figure 3. Then $B(x,y)$ is constant on Ω_k for all $k \in \mathbf{Z}$. Moreover, if $f(k)$ denotes its value on Ω_k, we have*

$$f(k) = \begin{cases} d & \text{if } k \le 0, \\ d - 2k & \text{if } 0 \le k \le d, \\ -d & \text{if } d \le k. \end{cases}$$

Proof. — For every $k \in \mathbf{Z}$, let T_k be the tree rooted at $\sigma(k)$ obtained by removing from X all (open) edges of σ, thus we have a forest

$$X \smallsetminus \bigcup_{k \in \mathbf{Z}} (\sigma(k), \sigma(k+1)) = \bigsqcup_{k \in \mathbf{Z}} T_k.$$

Assume $\xi \in \Omega$ is not an end of σ; there exists $k \in \mathbf{Z}$ such that $r^\xi_x \cap \sigma = [x, \sigma(k)]$. Thus r^ξ_x enters T_k, namely $r^\xi_x(|k|+1) \in T_k$. Since geodesics cannot backtrack r^ξ_x stays in T_k thereafter. We are in the situation of Lemma 4.1.3, hence $\sigma(k)$ sits on both r^ξ_x and r^ξ_y, so that

$$B(x,y)(\xi) = d(y, \sigma(k)) - d(x, \sigma(k)). \tag{4.3}$$

Finally:

$$B(x,y)(\xi) = d(y,\sigma(k)) - d(x,\sigma(k)) = \begin{cases} (d + |k|) - |k| = d & \text{if } k \le 0, \\ (d-k) - k = d - 2k & \text{if } 0 \le k \le d, \\ (k-d) - k = -d & \text{if } d \le k. \end{cases}$$

□

The proof shows that we could have picked any vertices of σ instead of x to define the Ω_k. Moreover the formula (4.3) for the Busemann cocycle holds not only for x, y but also for any pair of vertices of σ. From this, one can easily deduce the following corollary.

Corollary 4.3.3. — *Under the conditions of Proposition 4.3.2, the subset $\Omega_k \subset \Omega$ can be written as*

$$\Omega_k = \bigsqcup_{u \in T_k \smallsetminus \{\sigma(k)\}} \Omega_x(u) = \bigsqcup_{\substack{u \in S_1(\sigma(k)) \\ u \neq \sigma(k \pm 1)}} \Omega_x(u),$$

and its visual measure with respect to any vertex $\sigma(i)$ is given by $\nu_{\sigma(i)}(\Omega_k) = \frac{q-1}{q+1} q^{-|k-i|}$. □

(4) $R^+_{\mathbb{A}_i} = R_{\mathbb{A}_i} \cap \mathbb{E}^+ = R_{\mathbb{A}_i \cap \mathbb{E}^+} = R_{\mathbb{A}^+_i}$ and also $R^+_{\mathbb{A}} = R_{\mathbb{A}^+}$

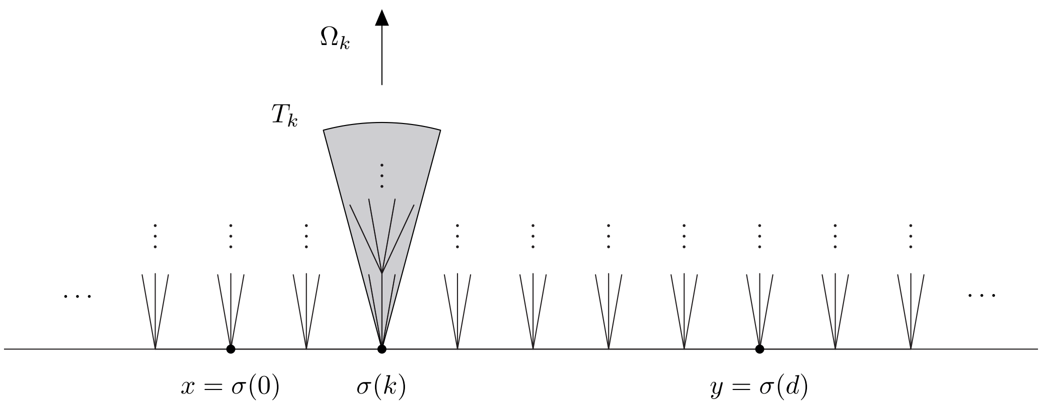

FIGURE 3. Situation of Proposition 4.3.2, with $q = 4$.

The countable family $\{\Omega_k \mid k \in \mathbf{Z}\}$ is an open cover of $\Omega \smallsetminus \{\sigma(\pm)\}$ by level sets of $B(x, y)$. We now compute the ν_e-measure of Ω_k.

Proposition 4.3.4. — *For every $i \in \mathbf{Z}$, let $e_i \in R^+_{\mathbb{A}}$ be the oriented edge with origin $\sigma(i)$ and target $\sigma(i+1)$. The ν_{e_i}-measure of Ω_k defined in Proposition 4.3.2 is given by:*

$$\nu_{e_i}(\Omega_k) = \frac{(q-1)}{q} \cdot \begin{cases} q^{-|k-i|} & \text{if } k - i \leq 0, \\ -q^{-(k-i)+1} & \text{if } k - i > 0. \end{cases}$$

Proof. — This follows from the definitions of Ω_k and ν_e, and the previous corollary. □

Three real continuous functions are useful for the upcoming analysis and have their graphs pictured in Figure 4, 5, and 6.

Definition 4.3.5. — Let $f, g : \mathbf{R} \to \mathbf{R}$ be the continuous real functions defined by:

$$f(x) = \begin{cases} d & \text{if } x \leq 0, \\ d - 2x & \text{if } 0 \leq x \leq d, \\ -d & \text{if } d \leq x, \end{cases}$$

and $g(x) = q^{-|x|}$. Define furthermore $g_{\frac{1}{2}} : \mathbf{R} \to \mathbf{R}$ by:

$$g_{\frac{1}{2}}(x) = \begin{cases} g(x) & \text{if } x \leq 0, \\ 1 - 2x & \text{if } 0 \leq x \leq 1, \\ -g(x-1) & \text{if } 1 \leq x. \end{cases}$$

The function $g_{\frac{1}{2}}$ has a symmetry about $\frac{1}{2}$ as the notation suggests, see Figures 5 and 6. The present definition of f extends that of Proposition 4.3.2. Similarly for $g_{\frac{1}{2}}$, we have

$$\nu_{e_i}(\Omega_k) = \frac{(q-1)}{q} g_{\frac{1}{2}}(k - i)$$

for all $k, i \in \mathbf{Z}$ by Proposition 4.3.4. The reasons behind the definition of $g_{\frac{1}{2}}$ on the unit interval $[0, 1]$ is to have a continuous symmetric function, see below. The following corollary summarizes the above discussion.

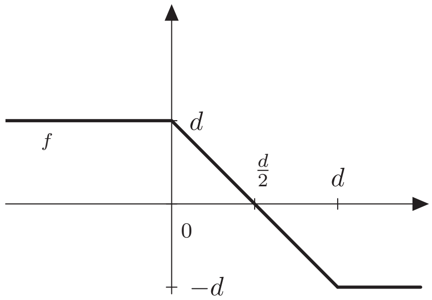

FIGURE 4. Graph of f.

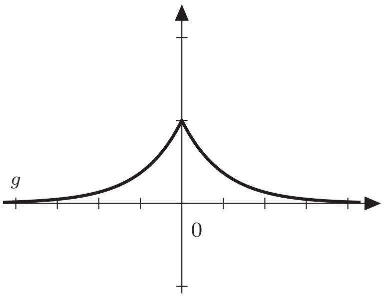

FIGURE 5. Graph of g.

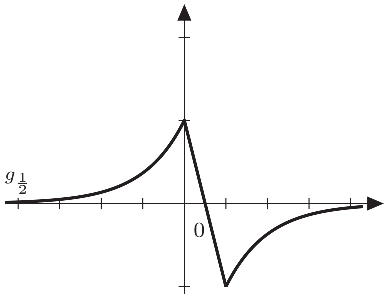

FIGURE 6. Graph of $g_{\frac{1}{2}}$.

Corollary 4.3.6. — *Let $i \in \mathbf{Z}$ and $e_i \in R^+_{\mathbb{A}}$. Then the evaluation of the Poisson transform of $B(x, y)$ at e_i is given by:*

$$\mathcal{P}B(x,y)(e_i) = \frac{(q-1)}{q} \sum_{k \in \mathbf{Z}} f(k) g_{\frac{1}{2}}(k-i). \tag{4.4}$$

Proof. — The singletons $\{\sigma(\pm\infty)\}$ have ν_x-measure 0, hence are null sets with respect to ν_{e_i} as well. Using σ-additivity,

$$\begin{aligned}
\mathcal{P}B(x,y)(e_i) &= \int_{\Omega} B(x,y)(\xi) d\nu_{e_i}(\xi) \\
&= \sum_{k \in \mathbf{Z}} \int_{\Omega_k} B(x,y)(\xi) d\nu_{e_i}(\xi) \\
&= \sum_{k \in \mathbf{Z}} f(k) \nu_{e_i}(\Omega_k) \\
&= \frac{(q-1)}{q} \sum_{k \in \mathbf{Z}} f(k) g_{\frac{1}{2}}(k-i),
\end{aligned}$$

where the last equality uses Proposition 4.3.4.

□

Remark 4.3.7. — Looking at the the formula of the previous corollary the reader may wonder why we did not group all terms in the tails of the series where f is constant equal to d for $k < 0$

and $-d$ for $k \geq d$. The author concedes that he found no intelligent way to treat the resulting finite sum and preferred summing over all $\mathbf{Z}$ to use the invariance under translations of its counting measure as done in the upcoming paragraph.

4.3.1. Analysis on Z. — To finalise the case $\mathbb{A}$, we need to estimate the value of the series obtained in Corollary 4.3.6. In additions to the real functions in one variable $f, g, g_{\frac{1}{2}}$ introduced in the previous section, we use the following notations.

Notation 4.3.8. — We denote:
- $\langle f_1, f_2 \rangle = \sum_{k \in \mathbf{Z}} f_1(k) f_2(k)$, for all f_1, f_2 real valued for which the series is well defined,
- $\tau_t f(x) = f(x-t)$ for all $f : \mathbf{R} \to \mathbf{R}$ and $t \in \mathbf{R}$,
- $\check{f}(x) = (f)\check{}(x) = f(-x)$ for all $f : \mathbf{R} \to \mathbf{R}$,
- $\check{\tau}_t f(x) = f(x+t) = \tau_{-t} f(x)$ for all $f : \mathbf{R} \to \mathbf{R}$ and $t \in \mathbf{R}$,
- $\mathbb{1}_I$ the characteristic function of an interval $I \subset \mathbf{R}$.

The linear operators τ_t and $\check{}$ represent the action of t and -1 respectively for the natural action of $\mathrm{Isom}(\mathbf{R}) = \mathbf{R} \rtimes \mathrm{O}(1) = \mathbf{R} \rtimes \{\pm 1\}$ on the space of real valued functions on the real line.

Proposition 4.3.9. — *The following identities hold when meaningful.*

(i) $\langle \tau_t f_1, \tau_t f_2 \rangle = \langle f_1, f_2 \rangle = \langle \check{f}_1, \check{f}_2 \rangle$,
(ii) $\tau_t \circ \tau_s = \tau_{t+s} = \tau_s \circ \tau_t$,
(iii) $\check{\check{f}} = f$,
(iv) $(\tau_t f)\check{} = \tau_{-t} \check{f} = \check{\tau}_t \check{f}$,
(v) $\tau_t \mathbb{1}_I = \mathbb{1}_{I+t}$,
(vi) $(\mathbb{1}_I)\check{} = \mathbb{1}_{-I}$. □

As Proposition 4.3.2 shows, the values of the Busemann cocycle yield the piecewise affine function f. Here are more notations for affine functions and their relations with the operators $\check{}$ and τ.

Definition 4.3.10 (Affine functions). — Let $p \in \mathbf{R}$.
- For every interval I of the form $[a,b]$ or $]a,b]$ with $-\infty < a < b < \infty$, or $]a,b[$ or $[a,b[$ with $-\infty < a < b \leq \infty$, we denote by A_I^p the affine function with slope p supported on I such that $A_I^p(a) = 0$. In other words,
$$A_I^p(x) = \mathbb{1}_I(x) \cdot p(x-a),$$
for all $x \in \mathbf{R}$.
- For every interval I of the form $[a,b]$ or $[a,b[$ with $-\infty < a < b < \infty$, or $]a,b[$, $]a,b]$ with $-\infty \leq a < b < \infty$, we denote by B_I^p the affine function with slope p supported on I such that $B_I^p(b) = 0$. In other words,
$$B_I^p(x) = \mathbb{1}_I(x) \cdot p(x-b),$$
for all $x \in \mathbf{R}$.

In proofs, we may use implicitly the properties below.

Proposition 4.3.11. — *For every $p \in \mathbf{R}$ and every $I \subset \mathbf{R}$ for which A_I^p is defined, we have*
$$(A_I^p)\check{} = B_{-I}^{-p}.$$
Furthermore,
$$\lambda A_I^p = A_I^{\lambda p} \quad \text{and} \quad \tau_\lambda A_I^p = A_{\lambda+I}^p,$$

for all $\lambda \in \mathbf{R}$. *Similar statements hold for* B_I^p. $\square$

Definition 4.3.12. — Let $f : \mathbf{R} \to \mathbf{R}$ (or $f : \mathbf{Z} \to \mathbf{R}$) be a real valued function.

- We say that f *has a symmetry about* $h \in \mathbf{R}$ (resp. $h \in \frac{1}{2}\mathbf{Z}$) if its graph is invariant under the central symmetry at the point $(h, 0) \in \mathbf{R}^2$. Equivalently f satisfies $-\check{f} = \tau_{-2h} f$.
- We say that f *has an axial symmetry about* $y = h$ if its graph is invariant under the reflexion through the vertical line $y = h$. Equivalently f satsifies $\check{f} = \tau_{-2h} f$.

Remark 4.3.13. — Using the formulas of Proposition 4.3.9, one can show that f is symmetric about 0 if and only if $\tau_h f$ is symmetric about $h \in \mathbf{R}$. The same holds for axial symmetries.

Proposition 4.3.14. — *Let* $f, g, g_{\frac{1}{2}}$ *be as in Definition 4.3.5. Then* f *can be written as*

$$f = d \cdot \mathbb{1}_{]-\infty,0[} + A^{-2}_{[0,d[} + d \cdot \mathbb{1}_{[0,d[} - d \cdot \mathbb{1}_{[d,\infty[},$$

and has a symmetry at $d/2$. *Moreover* $g_{\frac{1}{2}}$ *has a symmetry at* $\frac{1}{2}$ *and* g *has an axial symmetry about* $y = 0$. *In equations, this amounts to*

$$-\check{f} = \tau_{-d} f, \quad -\check{g}_{\frac{1}{2}} = \tau_{-1} g_{\frac{1}{2}}, \quad \text{and} \quad \check{g} = g.$$

Proof. — It suffices to stare at the graphs of Figures 4, 5, and 6. $\square$

Proposition 4.3.15. — *For every* $i \in \mathbf{Z}$, *define a linear operator*

$$T_i = \frac{1}{2}(\tau_{-i} + \tau_{-d+i+1}).$$

Then $T_i f$ *is symmetric about* $\frac{1}{2}$ *and has same sign as* $g_{\frac{1}{2}}$, *consequently*

$$\langle f, \tau_i g_{\frac{1}{2}} \rangle = \langle T_i f, g_{\frac{1}{2}} \rangle = \langle |T_i f|, |g_{\frac{1}{2}}| \rangle = \|T_i f \cdot g_{\frac{1}{2}}\|_{\ell^1(\mathbf{Z})} = 2\|T_i f \cdot g_{\frac{1}{2}}\|_{\ell^1(\mathbf{N}^*)}.$$

Proof. — Using (i) of Proposition 4.3.9,

$$\langle f, \tau_i g_{\frac{1}{2}} \rangle = \langle \tau_{-i} f, g_{\frac{1}{2}} \rangle.$$

The absolute convergence is guaranteed as f is bounded and $g_{\frac{1}{2}}$ is of geometric type. On the other hand, the identities of Proposition 4.3.9 yield

$$\begin{aligned}
\langle f, \tau_i g_{\frac{1}{2}} \rangle &= \langle \tau_{-d} f, \tau_{-d+i+1} \tau_{-1} g_{\frac{1}{2}} \rangle && \text{by (i) and (ii)},\\
&= \langle -\check{f}, -\tau_{-d+i+1} \check{g}_{\frac{1}{2}} \rangle && \text{symmetries of } f \text{ and } g_{\frac{1}{2}},\\
&= \langle \check{f}, (\tau_{d-i-1} g_{\frac{1}{2}})^{\check{}} \rangle && \text{by (iv)},\\
&= \langle f, \tau_{d-i-1} g_{\frac{1}{2}} \rangle && \text{by (i)},\\
&= \langle \tau_{-d+i+1} f, g_{\frac{1}{2}} \rangle && \text{by (i)}.
\end{aligned}$$

Taking the average, we get $\langle f, \tau_i g_{\frac{1}{2}} \rangle = \langle T_i f, g_{\frac{1}{2}} \rangle$. One easily verifies

$$(T_i f)^{\check{}} = -\tau_{-1} T_i f.$$

Since f and its translates are continuous and change signs only at their point of symmetry, the same holds for the average of two translates of f. We deduce that $T_i f$ has the same sign as $g_{\frac{1}{2}}$. The remaining equalities follow easily from the signs and the symmetry about $\frac{1}{2}$. $\square$

Remark 4.3.16. — Since for every, $i \in \mathbf{Z}$,

$$\mathcal{P}B(x,y)(e_i) = \frac{(q-1)}{q}\langle f, \tau_i g_{\frac{1}{2}}\rangle = \frac{(q-1)}{q}\|T_i f \cdot g_{\frac{1}{2}}\|_{\ell^1(\mathbf{Z})} \geq 0,$$

our choice of orientation of the edges $\mathbb{A}^+$ is the set of oriented edges on which the Poisson transform of $B(x,y)$ is non-negative.

So far no majoration has been performed and we are still carrying the exact value of the Poisson transform for edges in $R_{\mathbb{A}}^+$. The next proposition computes explicitly $T_i f$ and bounds $|T_i f|$ by affine functions.

Remark 4.3.17. — The formulation and proof of the next proposition have some redundancy. Indeed since $T_i f = T_{d-i-1} f$ for all $i \in \mathbf{Z}$, the computations need only be performed for $i \geq \frac{d-1}{2}$ which is the fixed point of $i \mapsto d-i-1$. We included both cases for the sake of completeness. The geometric interpretation is that edges opposite to the midpoint of $[x,y]$ should intuitively yield similar value, maybe up to a sign.

Proposition 4.3.18. — *We have upper bounds:*

$$|T_i f| \leq B^{-1}_{]-\infty,i+1]} + A^{1}_{[-i,\infty[}, \qquad \text{if } i < 0, \tag{4.5}$$

$$|T_i f| \leq B^{-2}_{]-\infty,\frac{1}{2}]} + A^{2}_{[\frac{1}{2},\infty[}, \qquad \text{if } 0 \leq i < d, \tag{4.6}$$

$$|T_i f| \leq B^{-1}_{]-\infty,d-i]} + A^{1}_{[-d+i+1,\infty[}, \qquad \text{if } d \leq i. \tag{4.7}$$

Moreover multiplying by $|g_{\frac{1}{2}}|$ and taking the $\ell^1(\mathbf{Z})$-norm on both sides, we obtain:

$$\|T_i f \cdot g_{\frac{1}{2}}\|_{\ell^1(\mathbf{Z})} \leq 2\frac{q}{(q-1)^2} q^{-(|i|-1)}, \qquad \text{if } i < 0, \tag{4.8}$$

$$\|T_i f \cdot g_{\frac{1}{2}}\|_{\ell^1(\mathbf{Z})} \leq 2\frac{q(q+1)}{(q-1)^2}, \qquad \text{if } 0 \leq i < d, \tag{4.9}$$

$$\|T_i f \cdot g_{\frac{1}{2}}\|_{\ell^1(\mathbf{Z})} \leq 2\frac{q}{(q-1)^2} q^{-(i-d)}, \qquad \text{if } d \leq i. \tag{4.10}$$

Proof. — The statement and the proof are in two parts. We first give explicit formulas for $T_i f$ and prove the bounds for $|T_i f|$. Then multiplying those bounds by $|g_{\frac{1}{2}}|$ and compute corresponding $\ell^1(\mathbf{Z})$-norms.

Part 1. — The support of the piecewise affine function f consists of three intervals

$$]-\infty, 0[, \quad [0,d[, \quad \text{and} \quad [d,\infty[,$$

with cut points at 0 and d. Their translates under τ_{-i} and τ_{-d+i+1} yield four cut points namely $-i, d-i, -d+i+1$ and $i+1$ with possible repetitions. Using Proposition 4.3.14 and Proposition 4.3.11, we can write

$$\tau_{-i} f = d \cdot \mathbb{1}_{]-\infty,-i[} + A^{-2}_{[-i,d-i[} + d \cdot \mathbb{1}_{[-i,d-i[} - d \cdot \mathbb{1}_{[d-i,\infty[},$$

$$\tau_{-d+i+1} f = d \cdot \mathbb{1}_{]-\infty,-d+i+1[} + A^{-2}_{[-d+i+1,i+1[} + d \cdot \mathbb{1}_{[-d+i+1,i+1[} - d \cdot \mathbb{1}_{[i+1,\infty[}.$$

Therefore $T_i f$ is piecewise affine with support split into at most five intervals. Their configuration depends on $i \in \mathbf{Z}$ and leads to four cases, see Figure 7. In each case the real line is partitioned into five intervals I_1, I_2, I_3, I_4, I_5 with possible repetitions. We replace the intervals by their index as follows. For every subset $L \subset \{1,2,3,4,5\}$ we write

$$I_L := \cup_{\ell \in L} I_\ell, \quad \mathbb{1}_L := \mathbb{1}_{I_L}, \quad \text{and} \quad A^p_L := A^p_{I_L},$$

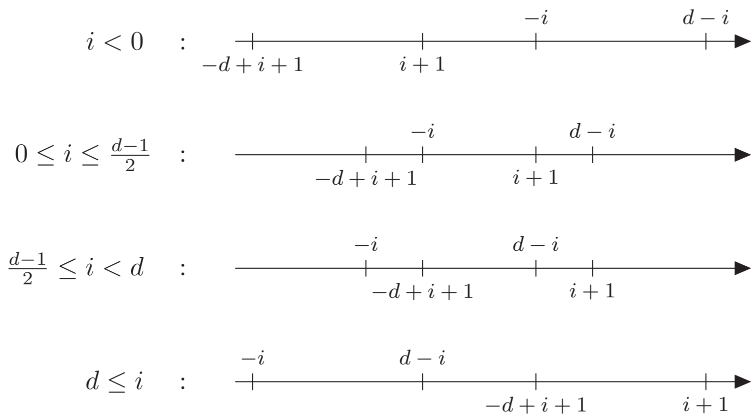

FIGURE 7. The cut points of $\tau_{-i}f$ and $\tau_{-d+i+1}f$.

for $p \in \mathbf{R}$ and being careful that $A^p_{I_L}$ is defined, *i.e.* when I_L is an interval. In addition, a singleton $L = \{\ell\} \subset \{1, 2, 3, 4, 5\}$ is abbreviated $L = \ell$, for instance $\mathbb{1}_{2\cup 3} = \mathbb{1}_{\{2,3\}} = \mathbb{1}_{I_{\{2,3\}}} = \mathbb{1}_{I_2 \cup I_3}$.

Case (i): Suppose $i < 0$, the cut points are ordered as

$$-d + i + 1 < i + 1 < -i < d - i,$$

and yield the intervals:

$$I_1 =]-\infty, -d+i+1[, \qquad I_2 = [-d+i+1, i+1[,$$
$$I_3 = [i+1, -i[, \qquad I_4 = [-i, d-i[,$$
$$I_5 = [d-i, \infty[.$$

To write $\tau_{-i}f$ and $\tau_{-d+i+1}f$ in terms of these intervals, notice that

$$]-\infty, -i[= I_1 \cup I_2 \cup I_3, \quad [-i, d-i[= I_4 \quad \text{and} \quad [d-i, \infty[= I_5,$$

so that

$$\tau_{-i}f = d \cdot \mathbb{1}_{1\cup 2\cup 3} + A_4^{-2} + d \cdot \mathbb{1}_4 - d \cdot \mathbb{1}_5,$$

and similarly

$$]-\infty, -d+i+1[= I_1, \quad [-d+i+1, i+1[= I_2, \quad \text{and} \quad [i+1, \infty[= I_3 \cup I_4 \cup I_5,$$

so that

$$\tau_{-d+i+1}f = d \cdot \mathbb{1}_1 + A_2^{-2} + d \cdot \mathbb{1}_2 - d \cdot \mathbb{1}_{3\cup 4\cup 5}.$$

Summing the two previous equations yields

$$2T_i f = 2d \cdot \mathbb{1}_1 + A_2^{-2} + 2d \cdot \mathbb{1}_2 + A_4^{-2} - 2d \cdot \mathbb{1}_5,$$

thus,

$$T_i f = d \cdot \mathbb{1}_1 + A_2^{-1} + d \cdot \mathbb{1}_2 + A_4^{-1} - d \cdot \mathbb{1}_5.$$

We see that $T_i f$ is non-negative on $I_1 \cup I_2$, vanishes on I_3, and is non-positive on $I_4 \cup I_5$. Indeed $A_2^{-1} + d \cdot \mathbb{1}_2 \geq 0$ since I_2 is of length d. In fact, $A_2^{-1} + d \cdot \mathbb{1}_2 = B_2^{-1}$, see Figure 8. Consequently,

$$\begin{aligned} |T_i f| &= d \cdot \mathbb{1}_1 + A_2^{-1} + d \cdot \mathbb{1}_2 + A_4^1 + d \cdot \mathbb{1}_5, \\ &\leq B_{1\cup 2}^{-1} + A_{4\cup 5}^1, \end{aligned}$$

which proves (4.5).

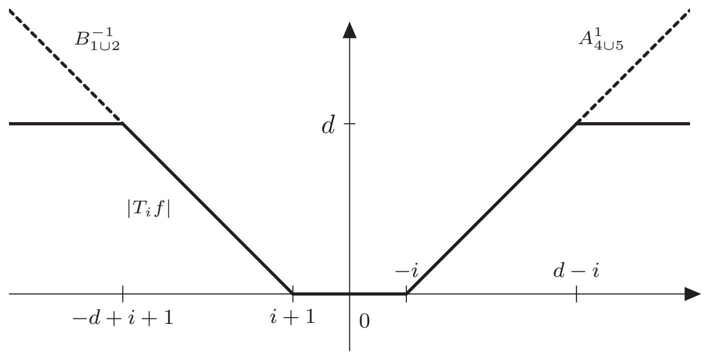

FIGURE 8. Case (i): graph of $|T_i f|$.

Case (ii): Suppose $0 \leq i \leq \frac{d-1}{2}$, the cut points are ordered as

$$-d + i + 1 \leq -i < i + 1 \leq d - i,$$

with possible equalities provided d is odd and $i = \frac{d-1}{2}$. But this is not a problem since we consider semi-open intervals and $[a, a[= \emptyset$ for all $a \in \mathbf{R}$. The partition with possible empty intervals is the following:

$$\begin{aligned} &I_1 =]-\infty, -d+i+1[, && I_2 = [-d+i+1, -i[, \\ &I_3 = [-i, i+1[, && I_4 = [i+1, d-i[, \\ &I_5 = [d-i, \infty[. \end{aligned}$$

The function $\tau_{-i} f$ and $\tau_{-d+i+1} f$ can be written as

$$\begin{aligned} \tau_{-i} f &= d \cdot \mathbb{1}_{1\cup 2} + A_{3\cup 4}^{-2} + d \cdot \mathbb{1}_{3\cup 4} - d \cdot \mathbb{1}_5, \\ \tau_{-d+i+1} f &= d \cdot \mathbb{1}_1 + A_{2\cup 3}^{-2} + d \cdot \mathbb{1}_{2\cup 3} - d \cdot \mathbb{1}_{4\cup 5}, \end{aligned}$$

the mean of which is

$$T_i f = d \cdot \mathbb{1}_1 + (d + A_{2\cup 3}^{-1})\mathbb{1}_2 + (A_{3\cup 4}^{-1} + d + A_{2\cup 3}^{-1})\mathbb{1}_3 + A_{3\cup 4}^{-1} \cdot \mathbb{1}_4 - d \cdot \mathbb{1}_5.$$

The function $(A_{3\cup 4}^{-1} + d + A_{2\cup 3}^{-1})\mathbb{1}_3$ changes sign at $\frac{1}{2}$ and is bounded in absolute value by $B_{]-\infty,\frac{1}{2}]}^{-2} + A_{[\frac{1}{2},\infty[}^{2}$, see Figure 9. Therefore one can check that

$$\begin{aligned} |T_i f| &= d \cdot \mathbb{1}_1 + (d + A_{2\cup 3}^{-1})\mathbb{1}_2 + |A_{3\cup 4}^{-1} + d + A_{2\cup 3}^{-1}| \cdot \mathbb{1}_3 + A_{3\cup 4}^{1} \cdot \mathbb{1}_4 + d \cdot \mathbb{1}_5 \\ &\leq B_{]-\infty,\frac{1}{2}]}^{-2} + A_{[\frac{1}{2},\infty[}^{2}, \end{aligned}$$

as desired to prove (4.6) for $0 \leq i \leq \frac{d-1}{2}$.

The other two cases follows from the first two by switching the roles of $-i$ and $-d + i + 1$. We includ the proofs for the sake of completeness.

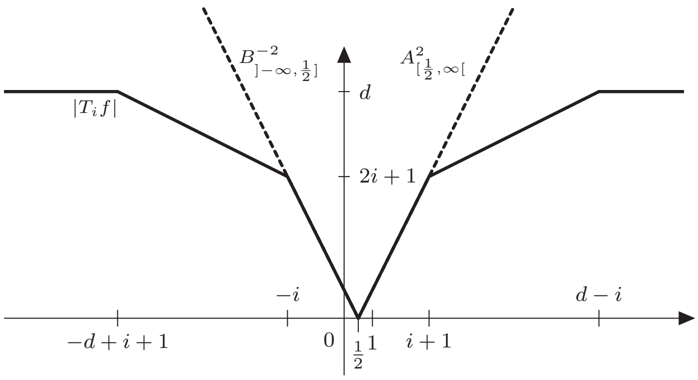

FIGURE 9. Case (ii): graph of $|T_i f|$.

Case (iii): Suppose $0 \leq i \leq \frac{d-1}{2}$. The proof is similar to Case (ii), we have cut points ordered as

$$-i \leq -d+i+1 < d-i \leq i+1.$$

The partition with possible empty intervals is the following:

$$I_1 =]-\infty, -i[, \qquad I_2 = [-i, -d+i+1[,$$
$$I_3 = [-d+i+1, d-i[, \qquad I_4 = [d-i, i+1[,$$
$$I_5 = [i+1, \infty[.$$

The function $\tau_{-i} f$ and $\tau_{-d+i+1} f$ can be written as

$$\tau_{-i} f = d \cdot \mathbb{1}_1 + A^{-2}_{2\cup 3} + d \cdot \mathbb{1}_{2\cup 3} - d \cdot \mathbb{1}_{4\cup 5},$$
$$\tau_{-d+i+1} f = d \cdot \mathbb{1}_{1\cup 2} + A^{-2}_{3\cup 4} + d \cdot \mathbb{1}_{3\cup 4} - d \cdot \mathbb{1}_5,$$

which yields a similar formula to that of Case (ii):

$$T_i f = d \cdot \mathbb{1}_1 + (A^{-1}_{2\cup 3} + d)\mathbb{1}_2 + (A^{-1}_{2\cup 3} + d + A^{-1}_{3\cup 4})\mathbb{1}_3 + A^{-1}_{3\cup 4} \cdot \mathbb{1}_4 - d \cdot \mathbb{1}_5.$$

By the same argument as in Case (ii), we obtain

$$\begin{aligned} |T_i f| &= d \cdot \mathbb{1}_1 + (A^{-1}_{2\cup 3} + d)\mathbb{1}_2 + |A^{-1}_{2\cup 3} + d + A^{-1}_{3\cup 4}| \cdot \mathbb{1}_3 + A^{1}_{3\cup 4} \cdot \mathbb{1}_4 + d \cdot \mathbb{1}_5. \\ &\leq B^{-2}_{]-\infty,\frac{1}{2}]} + A^{2}_{[\frac{1}{2},\infty[}, \end{aligned}$$

as desired to prove (4.6) for $\frac{d-1}{2} \leq i < d$.

Case (iv): Suppose $d \leq i$. The proof is similar to Case (i); we have cut points ordered as

$$-i < d-i < -d+i+1 < i+1,$$

which yield the intervals:

$$I_1 =]-\infty, -i[, \qquad I_2 = [-i, d-i[,$$
$$I_3 = [d-i, -d+i+1[, \qquad I_4 = [-d+i+1, i+1[,$$
$$I_5 = [i+1, \infty[.$$

We write $\tau_{-i} f$ and $\tau_{-d+i+1} f$ in terms these intervals,

$$\tau_{-i} f = d \cdot \mathbb{1}_1 + A^{-2}_2 + d \cdot \mathbb{1}_2 - d \cdot \mathbb{1}_{3\cup 4\cup 5},$$
$$\tau_{-d+i+1} f = d \cdot \mathbb{1}_{1\cup 2\cup 3} + A^{-2}_4 + d \cdot \mathbb{1}_4 - d \cdot \mathbb{1}_5.$$

The mean of which is

$$T_i f = d \cdot \mathbb{1}_1 + A_2^{-1} + d \cdot \mathbb{1}_2 + A_4^{-1} - d \cdot \mathbb{1}_5,$$

thus,

$$\begin{aligned} |T_i f| &= d \cdot \mathbb{1}_1 + A_2^{-1} + d \cdot \mathbb{1}_2 + A_4^{1} + d \cdot \mathbb{1}_5, \\ &\leq B_{1\cup 2}^{-1} + A_{4\cup 5}^{1}, \end{aligned}$$

proving (4.7).

Part 2. — We now prove the upper bounds (4.8),(4.9),(4.10). Since the functions

$$B^{-1}_{]-\infty,-j+1]} + A^1_{[j,\infty[}, \quad B^{-2}_{]-\infty,\frac{1}{2}]} + A^2_{[\frac{1}{2},\infty[} \quad \text{and} \quad |g_{\frac{1}{2}}|,$$

have an axial symmetry about $y = \frac{1}{2}$, for $j \geq 1$, we have

$$\begin{aligned} \left\| \left(B^{-1}_{]-\infty,-j+1]} + A^1_{[j,\infty[} \right) \cdot |g_{\frac{1}{2}}| \right\|_{\ell^1(\mathbf{Z})} &= 2 \cdot \left\| A^1_{[j,\infty[} \cdot |g_{\frac{1}{2}}| \right\|_{\ell^1(\mathbf{N}^*)}, \\ \left\| \left(B^{-2}_{]-\infty,\frac{1}{2}]} + A^2_{[\frac{1}{2},\infty[} \right) \cdot |g_{\frac{1}{2}}| \right\|_{\ell^1(\mathbf{Z})} &= 2 \cdot \left\| A^2_{[\frac{1}{2},\infty[} \cdot |g_{\frac{1}{2}}| \right\|_{\ell^1(\mathbf{N}^*)}. \end{aligned}$$

The computation will give the desired bounds by replacing $j = -i = |i|$ for $i < 0$ and $j = -d+i+1$ for $d \leq i$. For the first norm we compute:

$$\begin{aligned} \left\| A^1_{[j,\infty[} \cdot |g_{\frac{1}{2}}| \right\|_{\ell^1(\mathbf{N}^*)} &= \sum_{k\geq 0} A^1_{[j,\infty]}(k) |g_{\frac{1}{2}}(k)| \\ &= \sum_{k\geq 0} \mathbb{1}_{[j,\infty[}(k)(k-j)q^{-(k-1)} \\ &= \sum_{k\geq j} (k-j)q^{-(k-1)} \\ &= \sum_{k\geq 0} kq^{-(k+j-1)} && \text{by change of variable } k \to k+j, \\ &= q^{-(j-1)} \sum_{k\geq 0} kq^{-k} \\ &= q^{-(j-1)} \frac{q}{(q-1)^2}, \end{aligned}$$

whereas for the second norm we obtain:

$$\begin{aligned} \left\| A^2_{[\frac{1}{2},\infty[} \cdot |g_{\frac{1}{2}}| \right\|_{\ell^1(\mathbf{N}^*)} &= \sum_{k\geq 0} A^2_{[\frac{1}{2},\infty[}(k) |g_{\frac{1}{2}}(k)| \\ &= \sum_{k\geq 1} 2(k - \frac{1}{2})q^{-(k-1)} \\ &= \sum_{k\geq 0} 2(k + \frac{1}{2})q^{-k} && \text{by change of variable } k \to k+1, \\ &= \sum_{k\geq 0} (2k+1)q^{-k} \\ &= \frac{q(q+1)}{(q-1)^2}, \end{aligned}$$

as desired. □

At last we complete the case $\mathbb{A}$ with the proof of Lemma 4.2.3.

Proof of Lemma 4.2.3. — It suffices to prove the lemma for edges in $R_{\mathbb{A}}^{+}$, which is by definition the set of oriented edges of the form $(\sigma(i), \sigma(i+1))$ with $i \in \mathbf{Z}$, where σ is a fixed geodesic with $\sigma(0) = x$ and $\sigma(d) = y$ (cf. the paragraph below the Assumption 4.3.1). Recall that

$$\begin{aligned} e_i \in R_{\mathbb{A}_1}^{+} &\iff i < 0, \\ e_i \in R_{\mathbb{A}_2}^{+} &\iff 0 \leq i < d, \\ e_i \in R_{\mathbb{A}_3}^{+} &\iff d \leq i. \end{aligned}$$

In any case we proved:

$$\begin{aligned} \mathcal{P}B(x,y)(e_i) &= \frac{(q-1)}{q} \langle \tau_{-i} f, g_{\frac{1}{2}} \rangle && \text{equation (4.4) of Corollary 4.3.6,} \\ &= \frac{(q-1)}{q} \| T_i f \cdot g_{\frac{1}{2}} \|_{\ell^1(\mathbf{Z})}. && \text{by Proposition 4.3.15.} \end{aligned}$$

If $0 \leq i < d$, the bound (4.9) of Proposition 4.3.18 gives

$$\begin{aligned} \mathcal{P}B(x,y)(e_i) &= \frac{(q-1)}{q} \| T_i f \cdot g_{\frac{1}{2}} \|_{\ell^1(\mathbf{Z})} \\ &\leq \frac{(q-1)}{q} \cdot 2 \cdot \frac{q(q+1)}{(q-1)^2} = \frac{2(q+1)}{(q-1)}. \end{aligned}$$

If $i < 0$, the vertex x sits between e_i and y on the geodesic σ, so that the distance $d(e_i, [x,y])$ is given by $d(\sigma(i+1), x) = d(\sigma(i+1), \sigma(0)) = |i+1| = |i| - 1$. Applying the bound (4.8) of Proposition 4.3.18 yields

$$\begin{aligned} \mathcal{P}B(x,y)(e_i) &= \frac{(q-1)}{q} \| T_i f \cdot g_{\frac{1}{2}} \|_{\ell^1(\mathbf{Z})} \\ &\leq \frac{(q-1)}{q} \cdot 2 \cdot \frac{q}{(q-1)^2} q^{-(|i|-1)} \\ &= 2 \cdot \frac{1}{(q-1)} q^{-d(e,[x,y])}. \end{aligned}$$

If $d \leq i$, it is y that sits on $[x, \sigma(i)]$, hence $d(e_i, [x,y]) = d(\sigma(i), y) = d(\sigma(i), \sigma(d)) = i - d$. The bound (4.10) of the same proposition yields

$$\begin{aligned} \mathcal{P}B(x,y)(e_i) &= \frac{(q-1)}{q} \| T_i f \cdot g_{\frac{1}{2}} \|_{\ell^1(\mathbf{Z})} \\ &\leq \frac{(q-1)}{q} \cdot 2 \cdot \frac{q}{(q-1)^2} q^{-(i-d)} \\ &= 2 \cdot \frac{1}{(q-1)} q^{-d(e,[x,y])}. \end{aligned}$$

□

4.4. The case $\mathbb{B}$

This section contains the proof of the last estimate of Theorem 4.2.4, which states the existence of a constant $C_{\mathbb{B}} > 0$ depending on the regularity parameter q such that

$$\sum_{e\in\mathbb{B}} \mathcal{P}B(x,y)(e)^2 \leq C_{\mathbb{B}}.$$

Recall that an oriented edge e is in $\mathbb{B}$ if and only if there is no geodesic line containing both $|e|$ and $[x,y]$, which is equivalent to having a well defined projection p_e onto $[x,y]$ such that $p_e \neq x, y$. For this situation to occur we assume $d(x,y) \geq 2$.

We briefly give the strategy before proceeding step by step. In the case $\mathbb{A}$, the $K_{[x,y]}$-orbits in $\mathbb{A}$ were level sets for $\mathcal{P}B(x,y)$ easily described thanks to the transitivity of $K_{[x,y]}$ on the set of apartments containing $[x,y]$. We indeed showed that the value at $e \in \mathbb{A}$ depends, up to a sign, only on its distance to $[x,y]$ and the number of edges at a given distance was easy to deduce. In the present case, the value depends not only on the distance $d(e,[x,y])$ but also on the position of the projection p_e of $|e|$ on the segment $[x,y]$. It is a priori not clear that all edges at a given distance and projecting on a given $p \in [x,y]$, lie in the same $K_{[x,y]}$-orbit. We mean that it is not clear if strong transitivity suffices to prove it as we did for $\mathbb{A}$. In fact, they do sit in the same $K_{[x,y]}$-orbit. To see this, consider the tree T_p rooted at p in the forest obtained by removing the all edges of $[x,y]$ from X. The automorphisms of X fixing pointwise $[x,y]$ and all the other trees in the forest, acts on T_p as the full group of automorphisms of the rooted q-ary tree[(5)] T_p. The latter is certainly transitive on each level, *i.e.* on the subset of vertices at given distance. In any case, we compute the value of $\mathcal{P}B(x,y)$ for an arbitrary edge $e \in \mathbb{B}$ see Remark 4.4.11, identify the level sets denoted $\mathbb{B}_{k,l}$ in Notation 4.4.13 below, and use them (instead of $K_{[x,y]}$-orbits) to estimate:

$$\sum_{e\in\mathbb{B}} \mathcal{P}B(x,y)(e)^2 = \sum_{k,l} \mathcal{P}B(x,y)(e_{k,l})^2 \operatorname{card}(\mathbb{B}_{k,l}),$$

where $e_{k,l}$ is any edge in $\mathbb{B}_{k,l}$, see the proof after Notation 4.4.13.

Continuing the comparison with the case $\mathbb{A}$, we fix again a geodesic line σ containing $[x,y]$ and defining a partition $\{\Omega_k^\sigma \mid k \in \mathbf{Z}\}$ of $\Omega \smallsetminus \{\sigma(\infty), \sigma(-\infty)\}$, for which $B(x,y)$ takes value $f(k)$ on each Ω_k^σ. We then consider a second geodesic line τ containing $|e|$ and p_e that intersects $[x,y]$ only at the point p_e. The situation is pictured in Figure 10. This obviously requires $q \geq 3$, but we assume $q \geq 4$ for a general configuration. We thus obtain another partition $\{\Omega_l^\tau \mid l \in \mathbf{Z}\}$ of $\Omega \smallsetminus \{\tau(\infty), \tau(-\infty)\}$. Intersecting the latter with the former partition yields a countable family $\{\Omega_{k,l} \mid k, l \in \mathbf{Z}\}$ of ν_e-measurable subsets covering $\Omega \smallsetminus \{\sigma(\pm\infty), \tau(\pm\infty)\}$. The intersection $\Omega_{k,l} = \Omega_k^\sigma \cap \Omega_l^\tau$ is frequently empty and has computable ν_e-measure otherwise, see Proposition 4.4.6. The end points of τ and σ are negligible and consequently,

$$\mathcal{P}B(x,y)(e) = \sum_{k\in\mathbf{Z}} f(k) \sum_{l\in\mathbf{Z}} \nu_e(\Omega_{k,l}),$$

which results in the formula of Corollary 4.4.7. From here, we can work the value of the formula and proceed to an analysis of the problem transported in $\mathbf{Z}$. It is without a doubt possible to adapt the method to $q = 2, 3$.

After this description we proceed to concretize this strategy rigorously.

Assumptions 4.4.1. — For the remainder of the section, we assume $q \geq 4$.

(5) In fact T_p starts with $q-1$ edges, then is q-ary thereafter.

- We fix vertices $x, y \in X$ at distance $d = d(x, y) \geq 2$ and a geodesic line $\sigma : \mathbf{R} \to X$ passing through $[x, y]$ parametrized so that $\sigma(0) = x$ and $\sigma(d) = y$.
- Also fixed is an edge $e \in \mathbb{B}$ oriented so that its target $t(e)$ realizes the distance of $|e|$ to $[x, y]$. The edge e is *pointing* toward $[x, y]$.

The projection p_e of e onto $[x, y]$ satisfies $d(e, [x, y]) = d(p_e, t(e))$.

- Let τ be a geodesic line passing through $|e|$, parametrized so that $e = (\tau(0), \tau(1))$, and whose intersection with σ is reduced to the point p_e, see Figure 10.
- Let k_e denote the distance between x and the projection p_e, and l_e denote the distance between $o(e)$ and p_e.

Therefore k_e, l_e satisfy $\sigma(k_e) = p_e = \tau(l_e)$, as well as

$$d(x, p_e) = k_e, \quad d(y, p_e) = d - k_e, \quad \text{and} \quad d(e, [x, y]) = d(t(e), d_e) = l_e - 1.$$

Thus $1 \leq k_e \leq d - 1$ and $l_e \geq 1$, see Figure 10.

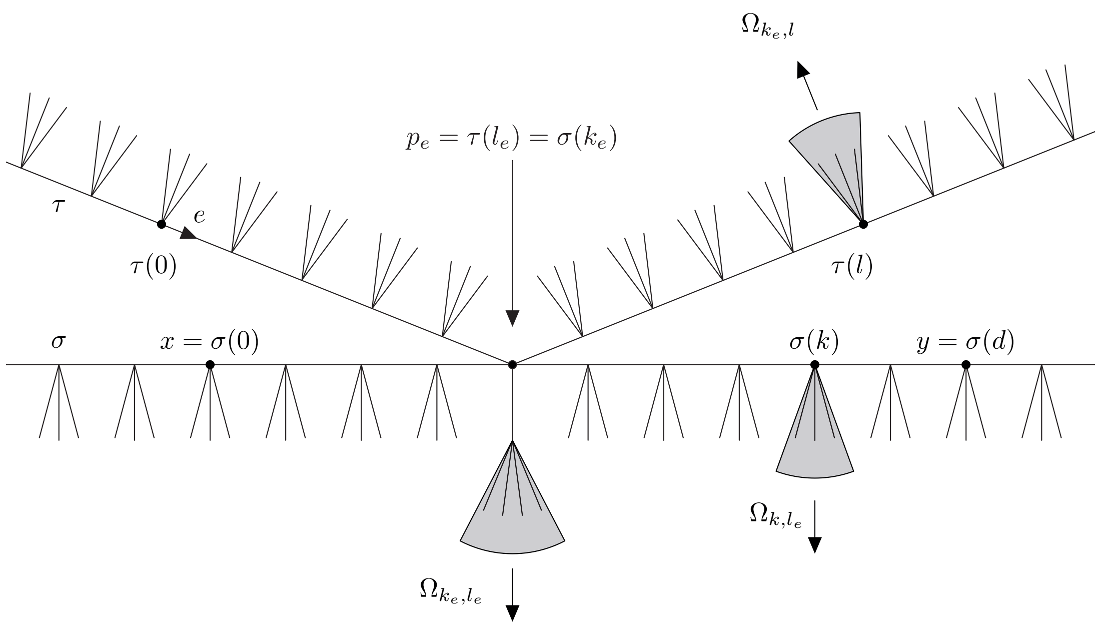

FIGURE 10. The geodesics σ, τ in X.

Definition 4.4.2 (Trees T_k^σ and T_l^τ). — For every $k \in \mathbf{Z}$, let T_k^σ be the connected component containing $\sigma(k)$ of the space obtained by removing the edges of σ to X. Similarly for $l \in \mathbf{Z}$ define T_l^τ to be the connected component containing $\tau(l)$ of the space obtained by removing the edges of τ.

Remark 4.4.3. — For $l \neq l_e$, the tree T_l^τ rooted at $\tau(l)$ is a subset of $T_{k_e}^\sigma$ so that

$$T_{k_e}^\sigma = (\bigsqcup_{l \neq l_e} T_l^\tau) \sqcup (T_{l_e}^\tau \cap T_{k_e}^\sigma).$$

Symmetrically, for $k \neq k_e$, the tree T_k^σ rooted at $\sigma(k)$ is a subset of $T_{l_e}^\tau$ and

$$T_{l_e}^\sigma = (\bigsqcup_{k \neq k_e} T_k^\sigma) \sqcup (T_{l_e}^\tau \cap T_{k_e}^\sigma).$$

The tree $T_{l_e}^\tau \cap T_{k_e}^\sigma$ is nonempty since $q \geq 4$.

Notice that for the trees defined above Lemma 4.1.3 applies. Since the measure ν_e is defined in terms of $\nu_{o(e)}$, the point $o(e) = \tau(0)$ will serve as a reference centre. One should keep in mind $p_e = \tau(l_e) = \sigma(k_e)$.

Proposition 4.4.4. — *For every $l \in \mathbf{Z}$, let Ω_l be the subset of $\xi \in \Omega$ such that the intersection of $r_{\tau(0)}^\xi$ and τ is the segment $[\tau(0), \tau(l)]$. In Figure 10, they consist of the small trees facing up touching τ but not σ when $l \neq l_e$, whereas Ω_{l_e} is the entire bottom part. For every $k \in \mathbf{Z}$, let $\Omega_{l_e,k}$ be the subset of $\xi \in \Omega_{l_e}$ such that the intersection of $r_{\tau(0)}^\xi$ and σ is $[\sigma(k_e), \sigma(k)]$. In Figure 10, they consist of the small trees facing down. Then $B(x, y)$ is constant on Ω_l for $l \neq k_e$, where it takes value $f(k_e)$, f being the function of Definition 4.3.5, and also constant on each $\Omega_{l_e,k}$ where it takes value $f(k)$.*

Proof. — It is a simple application of Lemma 4.1.3. We remark that $x, y \notin T_l^\tau$ for all $l \neq l_e$ and $x, y \notin T_k^\sigma$ for all $k \neq k_e$. One may need to be careful with Ω_{l_e,k_e}. $\square$

Regarding the ν_e-measure of the set defined above, the upcoming proposition makes use of the functions $g, g_{\frac{1}{2}}$ of Definition 4.3.5, and of h defined below.

Definition 4.4.5. — The function $h : \mathbf{R} \to \mathbf{R}$ defined by

$$h(x) = \begin{cases} g(x+1) & \text{if } x \leq -1, \\ 1 & \text{if } -1 \leq x \leq 1 \text{ and } x \neq 0, \\ 0 & \text{if } x = 0, \\ g(x-1) & \text{if } x \geq 1, \end{cases}$$

is continuous except at $x = 0$ and has an axial symmetry about $y = 0$, namely $\check{h} = h$. The value of h at $x = 0$ will be useful for later purposes.

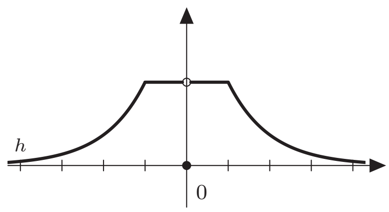

FIGURE 11. Graph of h.

Proposition 4.4.6. — *We have*

$$\Omega_{l_e,k_e} = \bigsqcup_{u \in T_{l_e}^\tau \cap T_{k_e}^\sigma} \Omega_{o(e)}(u),$$

and for every $l \neq l_e$ and $k \neq k_e$,

$$\Omega_l = \bigsqcup_{u \in T_l^\tau} \Omega_{o(e)}(u) \quad \text{and} \quad \Omega_{l_e,k} = \bigsqcup_{u \in T_k^\sigma} \Omega_{o(e)}(u).$$

Moreover, the ν_e-measures of those sets are given by

$$\begin{aligned} \nu_e(\Omega_l) &= \frac{(q-1)}{q} \cdot g_{\frac{1}{2}}(l), && \text{if } l \neq l_e, \\ \nu_e(\Omega_{l_e,k}) &= \frac{(q-1)}{q} \cdot g_{\frac{1}{2}}(l) \cdot \frac{1}{q} \cdot h(k-k_e), && \text{if } k \neq k_e, \\ \nu_e(\Omega_{l_e,k_e}) &= \frac{(q-3)}{q} \cdot g_{\frac{1}{2}}(l_e). \end{aligned}$$

Proof. — This follows easily using Lemma 4.1.3 and the Radon-Nikodym derivative of the visual measures. □

Corollary 4.4.7. — *Under Assumptions 4.4.1 the Poisson transform of $B(x,y)$ at e is given by*

$$\mathcal{P}B(x,y)(e) = \frac{(q-1)}{q^2} g_{\frac{1}{2}}(l_e) \left(\sum_{k \in \mathbf{Z}} f(k)h(k-k_e) \right) - \frac{2}{q} f(k_e) g_{\frac{1}{2}}(l_e).$$

Proof. — The boundary Ω is decomposed as the disjoint union of the sets Ω_l for $l \neq l_e$, the sets $\Omega_{l_e,k}$ for $k \in \mathbf{Z}$, and the four points $\sigma(\pm\infty)$, $\tau(\pm\infty)$ which form a null set with respect to $\nu_{o(e)}$. Proposition 4.4.4 guarantees $B(x,y)$ to be constant on each set of this partition. Consequently, using σ-additivity

$$\begin{aligned} \mathcal{P}B(x,y)(e) &= \int_\Omega B(x,y)(\xi) d\nu_e(\xi) \\ &= \int_{\Omega_{l_e,k_e}} B(x,y)(\xi) d\nu_e(\xi) + \sum_{k \neq k_e} \int_{\Omega_{l_e,k}} B(x,y)(\xi) d\nu_e(\xi) + \sum_{l \neq l_e} \int_{\Omega_l} B(x,y)(\xi) d\nu_e(\xi) \\ &= f(k_e)\nu_e(\Omega_{l_e,k_e}) + \sum_{k \neq k_e} f(k)\nu_e(\Omega_{l_e,k}) + f(k_e) \sum_{l \neq l_e} \nu_e(\Omega_l) \\ &= \frac{(q-3)}{q} f(k_e) g_{\frac{1}{2}}(l_e) + \frac{(q-1)}{q^2} g_{\frac{1}{2}}(l_e) \sum_{k \in \mathbf{Z}} f(k)h(k-k_e) + \frac{(q-1)}{q} f(k_e) \sum_{l \neq l_e} g_{\frac{1}{2}}(l), \end{aligned}$$

where the last equality is due to Proposition 4.4.6 and we use $h(0) = 0$ to make the notation uniform. Since $g_{\frac{1}{2}}$ is symmetric about $\frac{1}{2}$ by Proposition 4.3.14, we have $-g_{\frac{1}{2}}(-l) = g_{\frac{1}{2}}(l+1)$, thus the terms in the last series cancel out except for $l = -l_e + 1$:

$$\sum_{l \neq l_e} g_{\frac{1}{2}}(l) = g_{\frac{1}{2}}(-l_e + 1) = -g_{\frac{1}{2}}(l_e).$$

Therefore,

$$\begin{aligned} \mathcal{P}B(x,y)(e) &= \frac{(q-3)}{q} f(k_e) g_{\frac{1}{2}}(l_e) + \frac{(q-1)}{q^2} g_{\frac{1}{2}}(l_e) \sum_{k \in \mathbf{Z}} f(k)h(k-k_e) - \frac{(q-1)}{q} f(k_e) g_{\frac{1}{2}}(l_e) \\ &= \frac{(q-1)}{q^2} g_{\frac{1}{2}}(l_e) \sum_{k \in \mathbf{Z}} f(k)h(k-k_e) - \frac{2}{q} f(k_e) g_{\frac{1}{2}}(l_e), \end{aligned}$$

as desired. □

This corollary provides an explicit formula that we will estimate with tools of Paragraph 4.3.1.

4.4.1. From X to $\mathbf{Z}$. — We now proceed to the analysis of the formula obtain in Corollary 4.4.7. Replacing the integers k_e, l_e by arbitrary $k, l \in \mathbf{Z}$, we define

$$S_l(k) := \frac{(q-1)}{q^2} g_{\frac{1}{2}}(l)\langle f, \tau_k h\rangle - \frac{2}{q} f(k) g_{\frac{1}{2}}(l).$$

The absolute convergence of $\Delta(k) := \langle f, \tau_k h\rangle$ is clear. We apply similar averaging technics to those performed in the case $\mathbb{A}$.

Lemma 4.4.8. — *For every $k \in \mathbf{Z}$, define the operator $\tilde{T}_k = \frac{1}{2}(\tau_{-k} - \tau_{k-d})$. Then:*

(i) the function Δ has a symmetry about $d/2$ and satisfies

$$\Delta(k) = \langle \tilde{T}_k f, h\rangle.$$

(ii) The function S_l also has a symmetry about $d/2$ and $|S_l|$ has an axial symmetry about $y = d/2$ for all $l \in \mathbf{Z}$.

Proof. — In order to prove (i), recall that $-\check{f} = \tau_{-d} f$ and $\check{h} = h$. Hence,

$$\Delta(k) = \langle f, \tau_k h\rangle = \langle \tau_{-k} f, h\rangle = \langle (\tau_{-k} f)\check{}, \check{h}\rangle = \langle \tau_k \check{f}, h\rangle = -\langle \tau_{k-d} f, h\rangle,$$

which proves that $\Delta(k) = \langle \tilde{T}_k f, h\rangle$ at once. But continuing the development,

$$\Delta(k) = -\langle \tau_{k-d} f, h\rangle = -\langle f, \tau_{d-k} h\rangle = -\Delta(d-k) = -\check{\Delta}(k-d) = -\tau_d \check{\Delta}(k)$$

proves the claimed symmetry about $d/2$.

For (ii), notice that S_l is a linear combination of Δ and f,

$$S_l = \frac{(q-1)}{q^2} g_{\frac{1}{2}}(l)\Delta - \frac{2}{q} g_{\frac{1}{2}}(l) f.$$

Since both f and Δ are symmetric at $d/2$, so is S_l. The axial symmetry of $|S_l|$ is immediate. □

Recall from the Assumptions 4.4.1 that we wish to evaluate $S_l(k)$ for $1 \leq k \leq d-1$. The symmetry of S_l allows us to focus on $1 \leq k \leq d/2$ only.

Lemma 4.4.9. — *For every integer $1 \leq k \leq d/2$, the function $\tilde{T}_k f$ has finite support, an axial symmetry about $y = 0$ and the has same sign as h, that is $\tilde{T}_k f \geq 0$. Therefore,*

$$\Delta(k) = \langle \tilde{T}_k f, h\rangle = \|\tilde{T}_k f \cdot h\|_{\ell^1(\mathbf{Z})} = 2\|\tilde{T}_k f \cdot h\|_{\ell^1(\mathbf{N}^*)},$$

using again $h(0) = 0$.

Proof. — The symmetry is true in general as shown by

$$(\tilde{T}_k f)\check{} = \frac{1}{2}(\tau_{-k} f - \tau_{k-d} f)\check{} = \frac{1}{2}(\check{\tau}_{-k}\check{f} - \check{\tau}_{k-d}\check{f}) = \frac{1}{2}(-\tau_{k-d} f + \tau_{-k} f) = \tilde{T}_k f.$$

For the analysis of $\tilde{T}_k f$, we proceed as in the proof of Proposition 4.3.18 and use its conventions. The notable cut points of the piecewise affine function f are 0 and d. Therefore those of $\tau_{-k} f$ and $\tau_{k-d} f$ are $-k$, $d-k$ and $k-d$, k respectively. Since $1 \leq k \leq d/2$ they are ordered as

$$k - d \leq -k < k \leq d - k,$$

with possible equalities if $k = d/2$. The cut points define five intervals

$$I_1 =]-\infty, k-d[, \qquad I_2 = [k-d, -k[,$$
$$I_3 = [-k, k[, \qquad I_4 = [k, d-k[,$$
$$I_5 = [d-k, \infty[,$$

with I_2 and I_4 possibly empty. Using the notations of the aforementioned proof, we can write

$$\begin{aligned}\tau_{-k}f &= d\cdot \mathbb{1}_{1\cup 2} + A^{-2}_{3\cup 4} + d\cdot \mathbb{1}_{3\cup 4} - d\cdot \mathbb{1}_5\\ \tau_{k-d}f &= d\cdot \mathbb{1}_1 + A^{-2}_{2\cup 3} + d\cdot \mathbb{1}_{2\cup 3} - d\cdot \mathbb{1}_{4\cup 5}\end{aligned}$$

Consequently

$$\begin{aligned}\tau_{-k}f - \tau_{k-d}f &= (d-d)\mathbb{1}_1 + (d - A^{-2}_{2\cup 3} - d)\mathbb{1}_2 + (A^{-2}_{3\cup 4} + d - A^{-2}_{2\cup 3} - d)\mathbb{1}_3\\ &\quad + (A^{2}_{3\cup 4} + d + d)\mathbb{1}_4 + (-d+d)\mathbb{1}_5\\ &= A^{2}_{2\cup 3}\mathbb{1}_2 + (A^{-2}_{3\cup 4} + A^{2}_{2\cup 3})\mathbb{1}_3 + (2d + A^{-2}_{3\cup 4})\mathbb{1}_4.\end{aligned}$$

Therefore,

$$\begin{aligned}\tilde{T}_k f &= A^{1}_{2\cup 3}\mathbb{1}_2 + (A^{-1}_{3\cup 4} + A^{1}_{2\cup 3})\mathbb{1}_3 + (d + A^{-1}_{3\cup 4})\mathbb{1}_4\\ &= A^1_2 + (d-2k)\mathbb{1}_3 + B^{-1}_4\\ &= A^1_2 + f(k)\mathbb{1}_3 + B^{-1}_4.\end{aligned}$$

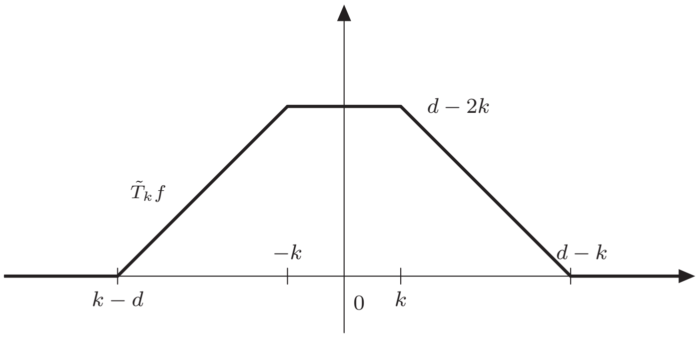

FIGURE 12. Graph of $\tilde{T}_k f$.

From this formula and the graph of $\tilde{T}_k f$, Figure 12, we conclude that it has an axial symmetry at $y=0$ and $\tilde{T}_k f \geq 0$. The degenerate case $k = d/2$ occurs when $\tilde{T}_k f$ vanishes. □

Lemma 4.4.10. — *For every* $1 \leq k \leq d/2$,

$$S_l(k) = \frac{-2}{(q-1)} q^{-k} g_{\frac{1}{2}}(l)(1 - q^{-f(k)}),$$

for all $l \in \mathbf{Z}$.

Proof. — In order to simplify notations we set $x \coloneqq q^{-1}$ and shall uses both variables simultaneously. Also the reader should keep in mind that $f(k) = d - 2k \geq 0$ for $1 \leq k \leq d/2$. Since

$$S_l = \frac{(q-1)}{q^2} g_{\frac{1}{2}}(l)\Delta - \frac{2}{q} g_{\frac{1}{2}}(l) f = \frac{2(q-1)}{q^2} g_{\frac{1}{2}}(l)\left(\frac{\Delta}{2} - \frac{q}{(q-1)} f\right),$$

we define

$$\Sigma \coloneqq \frac{\Delta}{2} - \frac{q}{(q-1)} f = \frac{\Delta}{2} - \frac{1}{(1-x)} f.$$

Lemma 4.4.9 implies

$$\frac{\Delta(k)}{2} = \|\tilde{T}_k f \cdot h\|_{\ell^1(\mathbf{N}^*)} = \langle \tilde{T}_k f \cdot \mathbb{1}_{[1,\infty[}, h\rangle.$$

From the proof of that lemma and Figure 12, we have

$$\tilde{T}_k f \cdot \mathbb{1}_{[1,\infty[} = f(k) \cdot \mathbb{1}_{[1,k[} + B^{-1}_{[k,d-k[} = f(k) \cdot \mathbb{1}_{[1,d-k]} - A^{1}_{[k,d-k]},$$

using, in the last equality,

$$A^{1}_{[k,d-k]}(d-k) = f(k) = B^{-1}_{[k,d-k[}(k) \quad \text{and} \quad A^{1}_{[k,d-k]}(k) = 0 = B^{-1}_{[k,d-k[}(d-k).$$

This form of $\tilde{T}_k f$ will simplify computations. Recall the well-known identities:

$$\mathcal{E}_1(m) := \sum_{n=1}^{m} x^{n-1} = \sum_{n=0}^{m-1} x^n = \frac{1-x^m}{1-x},$$

and

$$\mathcal{E}_2(m) := \sum_{n=0}^{m} n x^{n-1} = x^{-1} \sum_{n=0}^{m} n x^n = \frac{1}{(1-x)^2}(m x^{m+1} - (m+1)x^m + 1),$$

for all $m \in \mathbf{N}$.

Consequently,

$$\begin{aligned}
\frac{\Delta(k)}{2} &= \langle \tilde{T}_k f \cdot \mathbb{1}_{[1,\infty[}, h\rangle \\
&= \sum_{n=1}^{d-k} f(k) q^{-n+1} - \sum_{n=k}^{d-k} (n-k) q^{-n+1} \\
&= f(k) \sum_{n=1}^{d-k} x^{n-1} - x^k \sum_{n=0}^{f(k)} n x^{n-1} \\
&= f(k)\mathcal{E}_1(d-k) - x^k \mathcal{E}_2(f(k)),
\end{aligned}$$

hence,

$$\begin{aligned}
\Sigma(k) &= \frac{\Delta(k)}{2} - \frac{1}{(1-x)} f(k) \\
&= f(k)\mathcal{E}_1(d-k) - x^k \mathcal{E}_2(f(k)) - \frac{1}{(1-x)} f(k) \\
&= \frac{1-x^{d-k}}{(1-x)} f(k) - \frac{1}{(1-x)} f(k) - \frac{x^k}{(1-x)^2}\Big(f(k)x^{f(k)+1} - (f(k)+1)x^{f(k)} + 1\Big) \\
&= \frac{-x^{d-k}}{(1-x)} f(k) - \frac{x^k}{(1-x)^2}\Big(f(k)x^{f(k)+1} - (f(k)+1)x^{f(k)} + 1\Big) \\
&= \frac{-x^k}{(1-x)^2}\Big(f(k)x^{f(k)}(1-x) + f(k)x^{f(k)+1} - (f(k)+1)x^{f(k)} + 1\Big) \\
&= \frac{-x^k}{(1-x)^2}\Big(-x^{f(k)} + 1\Big) = \frac{-q^{-k}q^2}{(q-1)^2}\left(1 - q^{-k}\right).
\end{aligned}$$

The conclusion is

$$S_l(k) = \frac{2(q-1)}{q^2} g_{\frac{1}{2}}(l)\Sigma(k) = \frac{-2q^{-k}}{(q-1)} g_{\frac{1}{2}}(l)\left(1 - q^{-k}\right).$$

□

Remark 4.4.11. — Going back to the geometric meaning of $S_l(k)$, recall that $\mathcal{P}B(x,y)(e) = S_{l_e}(k_e)$, see Corollary 4.4.7. Since $g_{\frac{1}{2}}(l) < 0$ for $l \geq 1$ the previous lemma shows that the Poisson transform of $B(x, y)$ takes positive value on the edge $e \in \mathbb{B}$ pointing toward $[x, y]$ whenever $k_e = d(x, p_e) \leq d/2$. Thanks to the symmetry of S_l, we know that if $d/2 \leq k_e \leq d - 1$ then

$$\mathcal{P}B(x, y)(e) = S_{l_e}(k_e) = -S_{l_e}(d - k_e) = -S_{l_e}(d(y, p_e)) < 0.$$

Corollary 4.4.12. — *Let $e \in \mathbb{B}$ be an oriented edge and p_e its projection onto $[x, y]$. Then,*

$$|\mathcal{P}B(x, y)(e)| \leq \frac{2}{(q-1)} \cdot q^{-(m_e + d(e,[x,y]))},$$

where $m_e := \min(d(p_e, x), d(p_e, y))$.

Proof. — Let $e \in \mathbb{B}$ be such that $k_e = d(x, p_e) \leq d/2$ so that $m_e = k_e$. By Corollary 4.4.7 and Lemma 4.4.10 we have

$$\begin{aligned} |\mathcal{P}B(x, y)(e)| &= S_{l_e}(m_e) \\ &= \frac{2}{(q-1)} q^{-m_e} |g_{\frac{1}{2}}(l_e)|(1 - q^{-m_e}) \\ &\leq \frac{2}{(q-1)} q^{-m_e} q^{-l_e+1} \\ &= \frac{2}{(q-1)} \cdot q^{-m_e - d(e,[x,y])}, \end{aligned}$$

because $d(e, [x, y]) = l_e - 1$. Assume now $d/2 \leq k_e \leq d-1$. Then $m_e = d(y, p_e) = d - k_e$ and again

$$|\mathcal{P}B(x, y)(e)| = |S_{l_e}(k_e)| = |S_{l_e}(d - k_e)| = |S_{l_e}(m_e)|,$$

which can be estimated as in the first case. □

Notation 4.4.13. — The set of edges $e \in \mathbb{B}$ oriented as in Assumptions 4.4.1 is denoted $\mathbb{B}^+$, it consists of all edges *pointing* toward $[x, y]$. For every integer $1 \leq k \leq d - 1$ and $l \geq 1$ define $\mathbb{B}^+_{k,l}$ to be the set of edges e pointing toward $[x, y]$ such that $k_e = k$ and $l_e = l$. We have

$$\mathbb{B}^+ = \bigsqcup_{\substack{l \geq 1 \\ 1 \leq k \leq d-1}} \mathbb{B}^+_{k,l}.$$

Subsets $\mathbb{B}^-$, $\mathbb{B}^-_{k,l}$ are defined by taking the opposite orientations.

As mentioned in Remark 4.4.11, $\mathcal{P}B(x, y)(e)$ needs not be positive for $e \in \mathbb{B}^+$, but it is if $e \in \mathbb{B}^+_{k,l}$ with $k \leq d/2$ or $e \in \mathbb{B}^-_{k,l}$ with $k \geq d/2$. This however is not relevant to our task. At last, we are ready to prove the last estimate of Theorem 4.2.4.

Proof of Theorem 4.2.4 (continued). — In what follows, summing over $1 \leq k \leq d/2$ or over $d/2 \leq k \leq d - 1$ yield the same result by symmetry, see Remark 4.4.11. We shall sum twice over $1 \leq k \leq d/2$ possibly repeating the term $d/2$. Recall Corollary 4.4.12 states

$$|\mathcal{P}B(x, y)(e)| \leq \frac{2}{(q-1)} q^{-k} q^{-l+1},$$

for all $e \in \mathbb{B}_{k,l}$ with $1 \leq k \leq d/2$. An argument similar to the proof of Lemma 4.2.1 shows

$$\mathrm{card}(\mathbb{B}_{k,l}) = (q-1)q^{l-1}.$$

All together,

$$\begin{aligned}
\sum_{e\in\mathbb{B}} \mathcal{P}B(x,y)(e)^2 &= 2\sum_{e\in\mathbb{B}^+} \mathcal{P}B(x,y)(e)^2 \\
&= 2\sum_{l\geq 1}\sum_{k=1}^{d-1}\sum_{e\in\mathbb{B}^+_{k,l}} \mathcal{P}B(x,y)(e)^2 \\
&\leq 4\sum_{l\geq 1}\sum_{1\leq k\leq d/2} \operatorname{card}(\mathbb{B}_{k,l})\left(\frac{2}{(q-1)}\right)^2 q^{-2k}q^{-2(l-1)} \\
&= \frac{16}{(q-1)^2}\sum_{l\geq 1}\sum_{1\leq k\leq d/2}(q-1)q^{l-1}q^{-2k}q^{-2(l-1)} \\
&= \frac{16}{(q-1)}\sum_{l\geq 0}\sum_{1\leq k\leq d/2} q^{-2k}q^{-l} && \text{by changing } l\to l-1 \\
&\leq \frac{16}{(q-1)}\sum_{l\geq 0} q^{-l}\sum_{k\geq 0} q^{-2k} \\
&= \frac{16}{(q-1)}\left(\frac{q}{(q-1)}\right)\left(\frac{q^2}{(q^2-1)}\right) = \frac{16q^3}{(q-1)^3(q+1)} =: C_{\mathbb{B}}.
\end{aligned}$$

□

CHAPTER 5

GEOMETRY OF $\widetilde{A}_2$-BUILDINGS

5.1. Retractions and arrowings

In an attempt to understand retractions centered at a chamber at infinity we came across an enlightening description of what happen in an $\widetilde{A}_2$ building given by Ramagge, Robertson and Steger [**RRS98**]. In an apartment, a sector can be thought of as pointing up to the chamber at infinity it determines. One can represent this by drawing a small vector pointing 'up' in each chamber of that apartment, see Figure 1. The same can be done globally in the building since every chamber is contained in an apartment containing an equivalent sector. Thus each chamber is endowed with a 'small vector pointing up'. This defines a global arrowing of the chambers pointing toward the chamber at infinity. The authors of *loc. cit.* indicate that we may think of the building as hanging from the chamber at infinity with all the arrows pointing up.

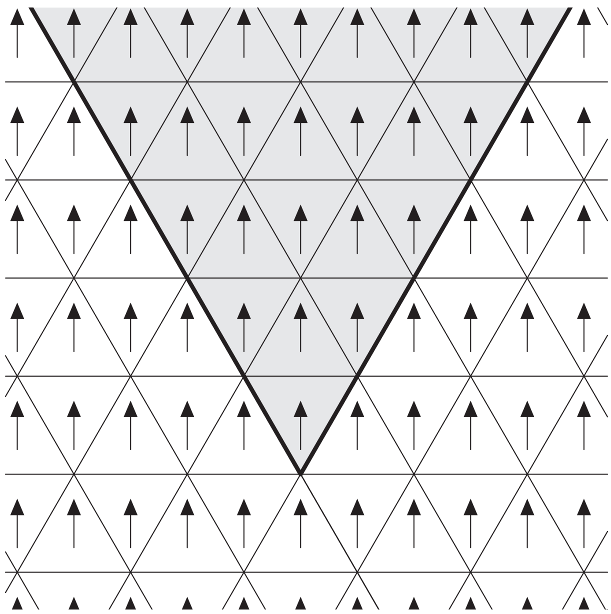

FIGURE 1. Arrowing of an apartment with respect to a sector.

Let X be an $\widetilde{A}_2$ building, and let $\xi \in \Omega$ be a chamber at infinity. Suppose A is an apartment containing the sector $\mathrm{Sect}_x(\xi)$ for some vertex $x \in A$. The chambers of A in the link of x are the possible directions of $\mathrm{Sect}_x(\xi)$ in A. They correspond naturally with the elements of the finite Weyl group W. In the $\widetilde{A}_2$ Coxeter complex, one could use the root system of type A_2 to describe these directions, but we shall not do so here as this would not be true in $\widetilde{A}_n$, $n \geq 3$. Rather we think of an arrow as a vector in a chamber C either pointing toward a vertex of C along the bisector or pointing in the opposite direction.

Definition 5.1.1. — An *arrow* is by definition a triple (C, i, ε) where C is a chamber of X, $i \in \mathbf{Z}/3\mathbf{Z}$ a type and $\varepsilon \in \{\pm 1\}$ a sign. If x is the vertex of C of type i, the arrow is geometrically represented in C on the bisector at x, pointing toward x if $\varepsilon = -1$ (attracting) and pointing in the opposite direction if $\varepsilon = +1$ (repulsing). An *arrowing of* X is by definition a set-theoretic section s of the projection map $(C, i, \varepsilon) \mapsto C$.

In Figure 2 we represented a chamber C and the arrow $(C, 1, +)$, where the vertex in grey is of type 1, and Figure 3 shows $(C, 0, -)$.

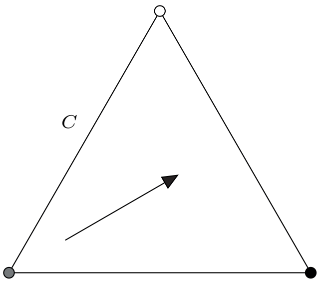

FIGURE 2. The arrow $(C, 1, +)$.

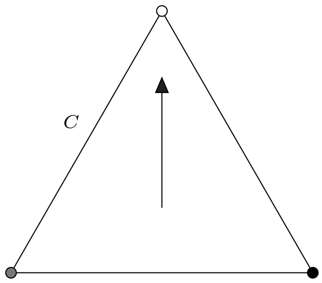

FIGURE 3. The arrow $(C, 0, -)$.

Definition 5.1.2. — The *arrowing* s_ξ *associated to a chamber* $\xi \in \Omega$ at infinity is defined as follows. Let A be an apartment containing $\mathrm{Sect}_x(\xi)$ with $x \in X^{(0)}$ and denote $C_x(\xi)$ its initial chamber[(1)], see Figure 4. The arrow of $C_x(\xi)$ should geometrically point away from x in the direction of $\mathrm{Sect}_x(\xi)$, hence we set $s_\xi(C_x(\xi)) := (C_x(\xi), i, +)$ where $i = \tau(x)$ is the type of x. The arrow of the chamber i-adjacent to $C_x(\xi)$, say $C' \in A$, points in the same direction but this time toward the vertex of C' of type i, consequently $s_\xi(C') = (C', i, -)$. On the other hand, in the chamber C'', j-adjacent to C with $i \neq j$, the arrow points toward the vertex of C'' of type $k \neq i, j$, so that $s_\xi(C'') = (C'', k, -)$. If p_i denote the bijection of the indices fixing i and exchanging the other two, the above amounts to

$$s_\xi(C) = (C, i, \varepsilon) \implies s_\xi(C') := (C', p_i(j), -\varepsilon),$$

for every chamber C' j-adjacent to $C = C_x(\xi)$. We can extend this to the entire apartment A yielding parallel arrows in the geometric sense. For if $i_1, \ldots, i_n$ denote the successive types

(1)In general one would need a special vertex, say of type 0.

appearing in a minimal gallery from $C = C_x(\xi)$ to an arbitrary chamber C' of A, then

$$s_\xi(C) = (C, i, \varepsilon) \implies s_\xi(C') \coloneqq (C', p_{i_n} \circ \cdots \circ p_{i_1}(i), (-1)^n \varepsilon).$$

One verifies that this depends not on the minimal gallery but only on the W-distance between C and C', namely $w = \delta(C_x(\xi), C')$. If $p_w(i)$ denotes the index of the right hand side, the latter becomes

$$s_\xi(C') = (C', p_w(i), (-1)^{\ell(w)} \varepsilon).$$

One checks that $p_{ww'}(i) = p_w \circ p_{w'}(i)$ for all w, w', elements of the affine Weyl group W_{aff}, hence the previous formula is compatible with the W-metric. Consequently we have a well defined arrowing on A independent of the $C_x(\xi)$ we started with. Furthermore this definition extends to the whole building because any chamber is contained in an apartment containing a sector in the class ξ, and any two such apartments are isometric via a retraction centered at ξ fixing their intersection pointwise.

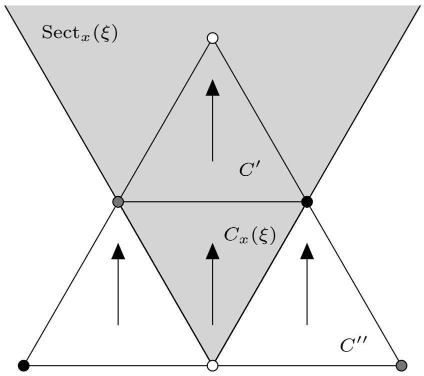

FIGURE 4. Arrows in the chambers neighboring $C_x(\xi)$.

Definition 5.1.3. — Two arrows (C, i, ε), (C', j, ε') are called *parallel* if

$$(C', j, \varepsilon') = (C', p_w(i), (-1)^{\ell(w)} \varepsilon), \tag{5.1}$$

where $w = \delta(C, C')$, or equivalently

$$(C, i, \varepsilon) = (C, p_{w^{-1}}(j), (-1)^{\ell(w)} \varepsilon'). \tag{5.2}$$

If C and C' are i-adjacent, we say the arrows are *symmetric* if $i = j$ and $\varepsilon = \varepsilon'$.

Since the retraction $\rho_{(A,\xi)}$ maps isometrically any apartment containing ξ as a chamber at infinity onto A by fixing their common subsector pointwise, the arrowing is equivariant with respect to $\rho_{(A,\xi)}$. Precisely we mean that if $s_\xi(C) = (C, i, \varepsilon)$, then

$$s_\xi(\rho_{(A,\xi)}(C)) = (\rho_{(A,\xi)}(C), i, \varepsilon). \tag{5.3}$$

From there we can determine local conditions that the arrows of adjacent chambers must satisfy.

Proposition 5.1.4. — *Let (C, i, ε) and (C', j, ε') be the arrows given by s_ξ of two adjacent chambers. Then they are either parallel or symmetric.*

Proof. — Say C, C' are k-adjacent, and consider a sector $\mathrm{Sect}_x(\xi)$ representing ξ containing C and a chamber D which is k-adjacent to C[2]. If $C' = D$ we are done by definition, therefore we assume $C' \neq D$. The image of C' under the retraction $\rho_{(A,\xi)}$ is either C or D. In the first case, the relation (5.3) yields $s_\xi(C) = (C, j, \varepsilon')$, meaning C and C' are symmetric. In the other case, $s_\xi(C')$ is likewise symmetric to $s_\xi(D)$ which is parallel to $s_\xi(C)$. We conclude that C and C' are parallel. □

Corollary 5.1.5. — [**RRS98**, §1.5] *For every panel F there is a unique chamber D containing F such that the arrow $s_\xi(D)$ is parallel to $s_\xi(C)$ for all distinct chambers C containing F as a face. In particular, C and D sit in an apartment having ξ as a chamber at infinity, and the arrows of chambers containing F distinct from D are all pairwise symmetric.*

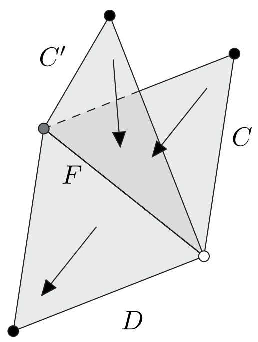

FIGURE 5. Arrows of adjacent chambers.

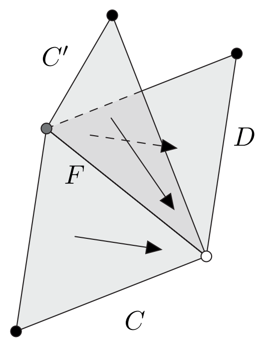

FIGURE 6. Other possible arrows.

Remark 5.1.6. — In the link of a vertex x, the initial chamber of $\mathrm{Sect}_x(\xi)$ is the only one with an arrow of type i and sign $+1$. This is clear by unicity of the sector issuing at x pointing toward ξ.

From the previous corollary and remark, we have strong local information determining the arrows on all chambers of the link of x.

Question 5.1.7. — *Suppose an arrowing s satisfies the property of Corollary 5.1.5, and that for every vertex x of type* 0*, there is a unique arrow, in the link of x, with type* 0 *and sign* $+1$*. Can we establish the existence of a chamber at infinity $\xi \in \Omega$ such that $s = s_\xi$?*

If answered affirmatively, then the chamber is unique. Indeed any other chamber ξ' would share a spherical apartment at infinity with ξ, which must be of the form ∂A for some Euclidean apartment A of X. There, the arrows coincide if and only if $\xi = \xi'$.

Question 5.1.8. — *Let s_ξ be the arrowing associated to $\xi \in \Omega$. Is is true that two chambers C, C' are contained in a common apartment containing a sector of ξ if and only if $s_\xi(C)$ and $s_\xi(C')$ are parallel ?*

[2] This is always possible starting with a sector representing ξ in an apartment containing C. Then pick a suitable translate of the sector.

5.1.1. Folding Diagram. — Fix a chamber at infinity ξ and its corresponding arrowing; an arbitrary apartment A may not contain a sector of ξ. In this case, it must have a pair of adjacent non-parallel arrows by construction. Using the local conditions, one can see that the arrows, are arranged in a specific way, forming the *folding diagram* of A. The latter determines completely the images of A under the retraction centered at ξ onto any apartment containing ξ at infinity. This was introduced in [**RRS98**, §1.5], where they explain that the local conditions force the existence of two focal points sitting on a wall of A, as pictured in Figure 7. In fact we could speak of the folding diagram of any convex flat subset, that is a convex subset contained in at least one apartment.

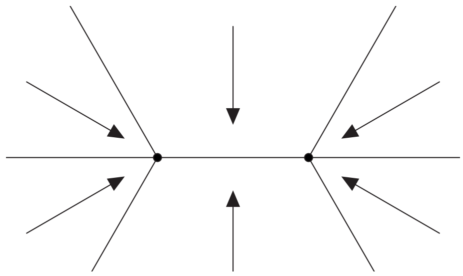

FIGURE 7. A folding diagram.

5.2. Convex hull of three points

In any building a pair of points is always contained in an apartment and their vertex convex hull, *i.e.* convex hull with respect to the 1-skeleton metric d_1, is contained in any such apartment. However three points are generically not contained in a common flat, for example in a tree they would form a tripod. Nevertheless, given a chamber at infinity ξ of a building of type $\widetilde{A}_2$, the information on the relative position of three points with respect to ξ is partially contained in the integers $m(x,y,\xi), m(x,z,\xi), n(x,y,\xi), n(x,z,\xi)$ of Chapter 3. A description of the vertex convex hull and the relative positions of three vertices in an $\widetilde{A}_2$ building remains desirable for our study.

In [**Laf00**], Lafforgue gives a result 'extracted' from [**RRS98**] and proves a similar statement for the symmetric spaces of $\mathrm{SL}_3(\mathbf{R})$ and $\mathrm{SL}_3(\mathbf{C})$. Though Lafforgue claims the former result is easily deducible from [**RRS98**] its proof remained unclear to us. The result can be stated as follows.

Theorem 5.2.1. — [**Laf00**, Theorem 3.1] *Let X be a Euclidean building of type $\widetilde{A}_2$ and d_1 the graph theoretic distance on the* 1*-skeleton $X^{(1)}$. For every triple (x_0, x_1, x_2) of vertices of X, there exist vertices $t_0, t_1, t_2 \in X^{(0)}$ forming an equilateral flat triangle, possibly reduced to a point, such that the pairs (t_i, t_{i+1}) have the same shape*[(3)] *$(0, p)$ or $(p, 0)$ for $p \in \mathbf{N}$, and satisfy*

$$d_1(x_i, t_i) + d_1(t_i, t_{i+1}) + d_1(t_{i+1}, x_{i+1}) = d_1(x_i, x_{i+1}), \tag{5.4}$$

for all $i \in \mathbf{Z}/3\mathbf{Z}$, and

$$\max_i(d_1(t_i, t_{i+1})) \leq \min_i(d_1(x_i, x_{i+1})). \tag{5.5}$$

(3) The shape of a pair of vertices (u, v) is a pair $(m, n) \in \mathbf{N}^2$ such that $v \in S_{m,n}(x)$. We defined it as $(m_u(v), n_u(v))$ in Chapter 3

Remark 5.2.2. — With the notations of Lafforgue, the theorem above is obtained by replacing his X by the set of vertices of the building, θ by the identity and Γ by the trivial group. Then (5.4) and (5.5) correspond to the condition (K_0a) in the paper. Moreover the condition (H_0) there is clear, whereas (K_0b) seems unnecessary.

Already if the vertices x_i lie in a common apartment such a triangle needs not be degenerate and seems moreover to be unique, see Figure 13 page 99. Lafforgue does not address the uniqueness of such triangle.

Remark 5.2.3. — If the three pairwise vertex convex hulls intersect non-trivially, then any point of the intersection satisfies the conclusion of Theorem 5.2.1. Indeed if C_i denotes the vertex convex hull $\mathrm{Conv}^{(0)}(x_{i-1}, x_{i+1})$ then a vertex t is in C_i if and only if

$$d_1(x_{i-1}, t) + d_1(t, x_{i+1}) = d_1(x_{i+1}, x_{i-1}).$$

In a regular tree, where the 1-skeleton metric and the CAT(0) metric coincide, geodesic segments $[x, y], [y, z], [z, x]$ always intersect non-trivially and the intersection is reduced to a point say p. The latter is characterized by the fact that p is the projection of each of the three points on their opposite segment. We try to implement similar ideas in the case of $\widetilde{A}_2$ buildings.

Setting 5.2.4. — For the remainder of this section, X is a locally finite $\widetilde{A}_2$ building of which x_0, x_1, x_2 are vertices indexed over $\mathbf{Z}/3\mathbf{Z}$. Moreover let C_i denote the vertex convex hull of $\{x_{i+1}, x_{i-1}\}$ for $i = 0, 1, 2$, see Figure 8. The unindexed intersection $\bigcap C_i$ means it is taken over $i = 0, 1, 2$, whereas U_i will denote the intersection $C_{i-1} \cap C_{i+1}$ so that $x_i \in U_i$.

The next lemma is the first of a series meant to describe the possible configurations of x_0, x_1, x_2 in the building. It implies that the intersection $\bigcap C_i$ is a horizontal segment in each convex hull C_i as soon as it is not empty, nor a singleton.

Lemma 5.2.5. — *For every $i \neq j \in \{0, 1, 2\}$ and every pair $(u, v) \in U_i \times U_j$, we have*

$$d_1(x_i, u) \leq d_1(x_i, v).$$

Thus for every $u, v \in \bigcap C_i$, we have equalities $d_1(x_i, u) = d_1(x_i, v)$ for all $i = 0, 1, 2$.

Conversely if $(u, v) \in U_i \times U_j$ with $i \neq j \in \{0, 1, 2\}$, then the equality $d_1(x_i, u) = d_1(x_i, v)$ is equivalent to $d_1(x_j, u) = d_1(x_j, v)$ and implies $u, v \in \bigcap C_i$.

Proof. — We consider the convex hull $C_2 = \mathrm{Conv}^{(0)}(x_0, x_1)$ for definiteness and suppose $u \in U_0$ and $v \in U_1$. In this convex hull, we have the equivalence

$$d_1(x_0, u) \leq d_1(x_0, v) \iff d_1(x_1, v) \leq d_1(x_1, u).$$

To show both inequalities at once, we prove the equivalent condition

$$d_1(x_0, u) + d_1(x_1, v) \leq d_1(x_0, v) + d_1(x_1, u).$$

Consider two d_1-geodesic paths in the 1-skeleton, namely γ_0 from x_0 to x_2 and γ_1 from x_1 to x_2, passing through u and v respectively. Their existence is clear as $u \in U_0 \subset \mathrm{Conv}^{(0)}(x_0, x_2)$ and $v \in U_1 \subset \mathrm{Conv}^{(0)}(x_1, x_2)$. In Figure 9, we pictured only the initial segments of γ_1, γ_2 in C_2. The sum of their lengths is

$$\ell(\gamma_0) + \ell(\gamma_1) = d_1(x_0, x_2) + d_1(x_1, x_2) = d_1(x_0, u) + d_1(u, x_2) + d_1(x_1, v) + d_1(v, x_2).$$

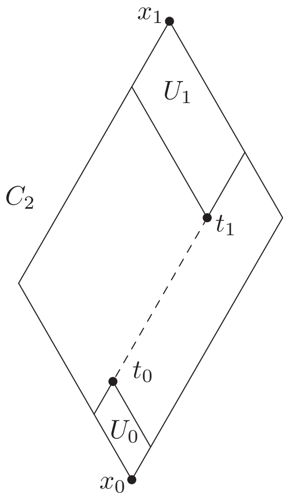

FIGURE 8. $C_2 = \mathrm{Conv}^{(0)}(x_0, x_1)$.

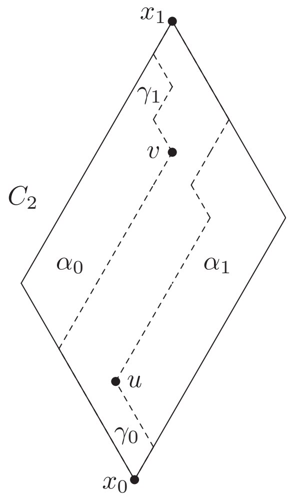

FIGURE 9. Paths in C_2.

Moreover consider α_0, α_1, two d_1-geodesic paths from x_0 to v, and x_1 to u respectively, see Figure 9. The concatenation of α_0 with the end of γ_1 yields a path β_0 from x_0 to x_2 passing through v of length

$$\ell(\beta_0) = d_1(x_0, v) + d_1(v, x_2).$$

Similarly the concatenation of α_1 with the end of γ_0 yields a path β_1 from x_1 to x_2 passing through u of length

$$\ell(\beta_1) = d_1(x_1, u) + d_1(u, x_2).$$

These concatenations need not be geodesic for the d_1 metric, hence their length is at least that of γ_0 and γ_1 respectively. Therefore, the inequality $\ell(\beta_0) + \ell(\beta_1) \geq \ell(\gamma_0) + \ell(\gamma_1)$ implies

$$d_1(x_0, v) + d_1(x_1, u) \geq d_1(x_0, u) + d_1(x_1, v).$$

The converse is clear for if $d_1(x_0, u) = d_1(x_0, v)$, then $\ell(\beta_0) + \ell(\beta_1) = \ell(\gamma_0) + \ell(\gamma_1)$ so that $\ell(\beta_0) = \ell(\gamma_0)$ and $\ell(\beta_1) = \ell(\gamma_1)$. This means that β_0 and β_1 are geodesic. Hence $u \in \mathrm{Conv}^{(0)}(x_1, x_2) = C_0$ and $v \in \mathrm{Conv}^{(0)}(x_0, x_2) = C_1$, which implies $u, v \in \bigcap C_i$. □

Lemma 5.2.6. — *Assume $\bigcap C_i$ is empty or reduced to a point. Then there exist $t_i \in U_i$, for $i = 0, 1, 2$, such that*

$$d_1(t_i, x_{i\pm1}) = d_1(U_i, x_{i\pm1}) \coloneqq \min_{u \in U_i} d_1(u, x_{i\pm1}), \tag{5.6}$$

and (t_0, t_1, t_2) is uniquely determined by this property. Equivalently the t_i's are determined by the fact that $U_i = \mathrm{Conv}^{(0)}(x_i, t_i)$.

Proof. — First notice that $\bigcap C_i$ and the U_i's are flat (contained in an apartment) convex subsets because they are intersections of such. Each U_i is in particular an n-gon[4] for some $n = 1, \ldots, 6$.

[4] A 1-gon is a vertex, a 2-gon a segment, a 3-gon a triangle, etc.

Indeed the walls of the $\widetilde{A}_2$ Coxeter complex form angles $\pi/3$ or $2\pi/3$, and the Euclidean formula for the sum of angles in a Euclidean n-gon with $n \geq 3$ yields

$$\frac{n\pi}{3} \leq (n-2)\pi \leq \frac{2n\pi}{3},$$

hence $3 \leq n \leq 6$. We immediately rule out the possibility $n = 6$ because x_i is a vertex of U_i at which the angle is $\frac{\pi}{3}$. Indeed U_i is a subset of the parallelogram $\mathrm{Conv}^{(0)}(x_i, x_{i+1})$ and they share the vertex x_i.

Claim. — *U_i is not a 3 nor a 5-gon.*

Proof of the Claim. — Suppose for contradiction that U_0 is, the proof for U_1, U_2 is the same. The parallelograms $C_2 = \mathrm{Conv}^{(0)}(x_0, x_1)$ and $C_1 = \mathrm{Conv}^{(0)}(x_0, x_2)$ are therefore non-degenerate and one of the sides of U_0 is horizontal, see Figure 10, in both convex hulls. We show this side to be contained in the intersection $\bigcap C_i$ contradicting the hypothesis. Let u, v be distinct vertices of the aforementioned side of U_0 with $d_1(u, v) = 1$, and let F be the face they form. We show that u sits in the vertex convex hull of x_1, x_2; the same argument applies to v and yields the desired contradiction. In order to prove

$$d_1(x_2, u) + d_1(u, x_1) = d_1(x_2, x_1),$$

we consider the chambers D_0, D_1 of $\mathrm{Conv}(x_0, x_1)$ having F as a codimension 1 face with $D_0 \subset U_0$. Let $D_2 \neq D_0$ be the chamber in $\mathrm{Conv}(x_0, x_2)$ also having F as a face. For $i = 0, 1, 2$, let v_i be the vertex of D_i distinct from u, v. The convex hull of v_0, x_2 is the union of D_0 and a convex pentagon, represented pointing upward in Figure 10. The latter together with D_1 forms the convex hull of v_1, x_2. Fix an apartment A containing x_0, x_1; the retraction $\rho \coloneqq \rho_{A,D_1}$ maps isometrically $\mathrm{Conv}(v_1, x_2)$ onto A so that $\rho_{A,D_1}(D_2) = D_0$. Moreover, the image $\rho(\mathrm{Conv}(v_1, x_2))$ stays in the sector of A issuing at x_1 and containing $\mathrm{Conv}(x_0, x_1)$. It is mapped into the half space containing D_0 delimited by the support of F. Consequently,

$$d_1(x_1, u) + d_1(u, \rho(x_2)) = d_1(x_1, \rho(x_2)).$$

On the one hand, the retraction ρ contracts the d_1 metric[5], thus

$$d_1(x_1, x_2) \geq d_1(\rho(x_1), \rho(x_2)) = d_1(x_1, \rho(x_2)).$$

On the other hand, $\mathrm{Conv}(v_1, x_2)$ is mapped isometrically onto its image, so that $d_1(u, x_2) = d_1(u, \rho(x_2))$. Hence

$$d_1(x_1, x_2) \geq d_1(x_1, u) + d_1(u, x_2),$$

which is an equality thanks to the triangle inequality. □

Consequently U_i is either reduced to $\{x_i\}$, or a segment, or a parallelogram. In all cases it is of the form $\mathrm{Conv}^{(0)}(x_i, t_i)$ for some t_i. Consequently

$$d_1(x_i, t_i) = \max_{u \in U_i} d_1(x_i, u),$$

but since

$$d_1(t_i, x_{i\pm1}) = d_1(x_i, x_{i\pm1}) - d_1(x_i, t_i),$$

[5] A finite d_1-geodesic is mapped to a path of the same length. Any geodesic between the images must have at most this length.

we conclude

$$d(t_i, x_{i\pm1}) = \min_{u \in U_i} d_1(u, x_{i\pm1}).$$

The vertex t_i is uniquely determined by this property because $U_i = \mathrm{Conv}^{(0)}(x_i, t_i)$. □

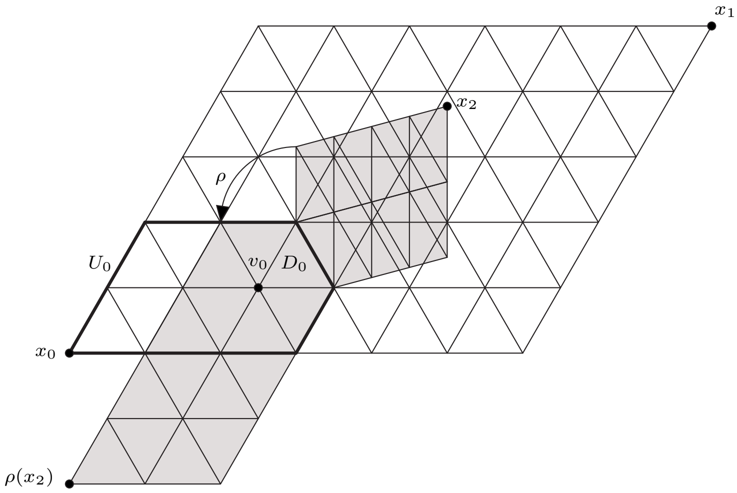

FIGURE 10. The retraction of $\mathrm{Conv}(v_1, x_2)$ onto A.

The next lemma gives a characterization of equilateral triangles in an $\widetilde{A}_2$ building. We recall first the notion of *Alexandrov angle*, an important feature of CAT(0) spaces [**BH99**, Chapter II.3, Proposition 3.1]. The CAT(0) metric on X is denoted d and the Alexandrov angle at $x \in X$ between two geodesic segments $[x, y], [x, z]$ is given by

$$\angle_x(y, z) \coloneqq \inf_{t, t' > 0} \overline{\angle}_x(\sigma_y(t), \sigma_z(t')),$$

where σ_y, σ_z are the geodesics from x to y and z respectively parametrized with $\sigma_y(0) = \sigma_z(0) = x$, and $\overline{\angle}$ denotes the comparison angle. The infimum on the right hand side is in fact a limit since the argument of the infimum is a non-decreasing function of both variables t, t'. The Alexandrov angles of a geodesic triangle $\Delta(x, y, z)$ are known to satisfy

$$\angle_x(y, z) + \angle_y(z, x) + \angle_z(x, y) \leq \pi, \tag{5.7}$$

with equality if and only if the triangle is flat, *i.e.* isometric to a Euclidean triangle [**BH99**, Chapter II.2, Proposition 2.9].

Lemma 5.2.7. — *In the Setting 5.2.4, the following are equivalent:*

(i) The geodesic triangle $\Delta(x_0, x_1, x_2)$ is a flat equilateral triangle with sides sitting on walls,
(ii) $U_i = \mathrm{Conv}^{(0)}(x_i, x_{i+1}) \cap \mathrm{Conv}^{(0)}(x_i, x_{i-1}) = \{x_i\}$ for all $i = 0, 1, 2$.

Proof. — That (i) implies (ii) is clear. Thus suppose (ii) holds and consider the intersections of C_1 and C_2 with the link of x_0. Since the intersection of the latter is reduced to x_0, there exist distinct non-adjacent chambers $D_1, D_2 \in \mathrm{lk}(x_0)$, the closure of which contain $C_1 \cap \mathrm{lk}(x_0)$ and $C_2 \cap \mathrm{lk}(x_0)$

respectively. By the building axioms, D_1, D_2 lie in a common apartment. Its intersections with the geodesic segments $[x_0, x_1]$ and $[x_0, x_2]$ are contained in $\overline{D_1}, \overline{D_2}$ respectively, hence the segments must form an Alexandrov angle at least $\frac{\pi}{3}$. By switching roles, the same hold for the Alexandrov angles at x_1 and x_2. Thus equation (5.7) for x_0, x_1, x_2 is an equality and the geodesic triangle $\Delta(x_0, x_1, x_2)$ is flat equilateral, *i.e.* sits in some apartment of X. To conclude, the only possibility for D_1 and D_2 to be non-adjacent and to have $[x_0, x_1] \subset \overline{D_1}$, $[x_0, x_2] \subset \overline{D_2}$ forming an Alexandrov angle of $\frac{\pi}{3}$ is that the segments are facets of a common chamber $D' \in \mathrm{lk}(x_0)$ distinct and adjacent to both D_1, D_2. □

Theorem 5.2.8. — *Let x_0, x_1, x_2 be vertices of X such that $\bigcap C_i$ is empty or a singleton. Then there exists a unique flat equilateral triangle $\Delta(t_0, t_1, t_2)$, with $t_i \in U_i$ and sides sitting on walls, such that equation* (5.6) *holds. In other words, the vertices t_i are uniquely determined by*

$$t_i = \operatorname*{argmin}_{u \in U_i} d(u, x_{i\pm1}), \tag{5.8}$$

or by

$$U_i = \mathrm{Conv}^{(0)}(x_i, t_i).$$

Proof. — For $i = 0, 1, 2$ let $t_i \in U_i$ be the vertices given by Lemma 5.2.6. We use the notations

$$\tilde{C}_i := \mathrm{Conv}^{(0)}(t_{i+1}, t_{i-1}) \quad \text{and} \quad \tilde{U}_i := \tilde{C}_{i+1} \cap \tilde{C}_{i-1},$$

so that $\tilde{C}_i \subset C_i$ and $\tilde{U}_i \subset U_i$. The former inclusion implies that $\bigcap \tilde{C}_i$ is either empty or a singleton. Lemma 5.2.6 applied to (t_0, t_1, t_2) yields vertices $s_i \in \tilde{U}_i$ such that $\tilde{U}_i = \mathrm{Conv}^{(0)}(t_i, s_i)$. On the one hand $s_i \in \tilde{U}_i \subset U_i$ hence $d_1(x_i, s_i) \leq d_1(x_i, t_i)$ by maximality of t_i in U_i, see Lemma 5.2.6. On the other hand, the inequality

$$d_1(x_i, t_{i\pm1}) \geq d_1(x_i, t_i),$$

of Lemma 5.2.5 shows that the convex hulls $\mathrm{Conv}^{(0)}(t_i, t_{i\pm1})$ consist of points at distance at least $d_1(x_i, t_i)$ from x_i. We conclude that $d_1(x_i, s_i) = d_1(x_i, t_i)$, hence $s_i = t_i$, by the characterization of t_i of Lemma 5.2.6. This shows that $\tilde{U}_i = \{t_i\}$ and we can therefore apply Lemma 5.2.7 to deduce that $\Delta(t_0, t_1, t_2)$ is a flat equilateral triangle whose sides are contained on walls. □

Remark 5.2.9. — In the previous theorem, if $\Delta(t_0, t_1, t_2)$ is not reduced to a singleton then its sides are not horizontal in each corresponding convex hull. But since they sit on walls we must have

$$d_1(x_i, x_j) = d_1(x_i, t_i) + d_1(t_i, t_j) + d_1(t_j, x_j), \tag{5.9}$$

for all $i \neq j \in \{0, 1, 2\}$, see Figure 8. On the other hand the triangle $\Delta(t_0, t_1, t_2)$ is equilateral thus

$$\max_i d_1(t_i, t_{i+1}) \leq \min_i d_1(x_i, x_{i+1}),$$

by looking at the pair x_i, x_{i+1} minimizing the right hand side.

We conjecture that the union of $\Delta(t_0, t_1, t_2)$ and $\bigcup C_i$ should be the vertex convex hull $\mathrm{Conv}^{(0)}(x_0, x_1, x_2)$.

Question 5.2.10. — *Is the union $\bigcup C_i \cup \Delta(t_0, t_1, t_2)$ convex for the graph theoretic distance d_1 on the 1-skeleton?*

5.3. Relative directions

Let x, y, z be vertices of a locally finite $\widetilde{A}_2$ building X. The formula (3.3) of Theorem 3.4.10 for the Poisson transform of vol_X, evaluated at a chamber C, averages a combination of

$$m(x, y, u), n(x, y, u), m(x, z, u), n(x, z, u)$$

with u ranging in a large sector sphere $S_{R,R}(x)$ centered at x, and the Radon-Nikodym derivative with respect to C. In order to obtain quantitative values similar to those of Chapter 4, the first step is to know how many vertices u of the large sphere yield given values of $m(x, y, u)$, $n(x, y, u)$. As observed in Lemma 3.2.7, the latter are calculated by simply using the sector coordinates $m_y(u), n_y(u)$, knowing $m_x(u) = n_x(u) = R$. This amounts to understanding the cardinal of the intersection of two sector spheres. In [**CMS94**], the authors relate the cardinal of the intersection to some structure constants of an algebra of averaging operators on the vertices, which is related to the classical Hecke algebra. In his PhD dissertation, [**Par05**], Parkinson pushed the method of the former and extended this to arbitrary buildings, see his article [**Par06**]. Interestingly, for fixed $i, j, k, l, m, n \in \mathbf{N}$, the cardinal of $S_{m,n}(x) \cap S_{i,j}(y)$ does not depend on the choice of $y \in S_{k,l}(x)$, see [**CMS94**, Lemma 2.4]. The second step is to include z in the picture which increases the difficulty greatly, as the above articles testify. Eventually one will have to work with the chamber C as well, but we did not bring the discussion this far.

From the previous section we understand better the configuration of x, y, z in the building. The present section aims at describing the intersection of the aforementioned spheres with respect to a given configuration. For every $u \in S_{R,R}(x)$, the sectors at x, y, z containing u point in various directions, but, in their links, the initial chambers of these sectors are contained, pairwise, in common apartments. We believe that this information should suffice to determine the sector coordinates $m_y(u), n_y(u), m_z(u), n_z(u)$. In the next paragraphs we partially implement this idea on particular configurations of x, y, z starting with the very natural case where the three points sit in a common apartment.

5.3.1. Two points in an apartment. — Let X be a locally finite $\widetilde{A}_2$ building and let $\xi \in \Omega$ be a chamber in the spherical building at infinity. Recall that, for every $x \in X^{(0)}$, we denoted $C_x(\xi)$ the unique chamber of $\mathrm{lk}(x)$ in the sector $\mathrm{Sect}_x(\xi)$. For a pair of vertices x, y, we would like to understand the relative position of $C_x(\xi)$, $C_y(\xi)$ in an apartment containing them. Even if there is in general no apartment containing both sectors, we may get informations from the position of the above chambers in an apartment, using the fact that the panels of the closed sectors $\overline{\mathrm{Sect}}_x(\xi), \overline{\mathrm{Sect}}_y(\xi)$ are geodesic rays pairwise asymptotic. A consequence of the CAT(0) inequality is the convexity of the distance function between two geodesics.

Proposition 5.3.1. — [**BH99**, Chapter II.2, Proposition 2.2] *Let $\sigma, \sigma' : [0, d] \to X$ be geodesic segments, then $t \mapsto d(\sigma(t), \sigma'(t))$ is a convex function.*

Corollary 5.3.2. — *For $i = 1, 2$, let ξ_i denote the vertex of ξ of type i and let $r_x^{\xi_i}, r_y^{\xi_i}$ denote the corresponding geodesic rays starting at x and y respectively. Then the function $t \mapsto d(r_x^{\xi_i}(t), r_y^{\xi_i}(t))$ is convex and non-increasing.*

Proof. — Since the convexity needs only be checked on compact intervals, the previous proposition ensures it. A pair of geodesic rays being asymptotic means the function $t \mapsto d(r_x^{\xi_i}(t), r_y^{\xi_i}(t))$ is bounded. But any convex bounded function $f : \mathbf{R}_+ \to \mathbf{R}$ is non-increasing. □

Consequently, for two chambers $C_x \in \mathrm{lk}(x)$ and $C_y \in \mathrm{lk}(y)$, to possibly be the top of equivalent sectors, their sides must satisfy a similar non-increasing condition, which can readily be checked in an apartment containing them. Fortunately in a Euclidean space this condition is easily tested.

Setting 5.3.3. — For the rest of this paragraph we fix an apartment A identified with a Euclidean space with scalar product $\langle \cdot\,, \cdot \rangle$ inducing the Euclidean metric d, and two vertices $x, y \in A$. Let C_x, C_y be a pair of chambers of A containing x and y respectively. They will be thought of as variables for possible configurations in what follows. The vertices of C_x and C_y are denoted x, x', x'' and y, y', y'' respectively so that their types are given by

$$\tau(x') = \tau(x) + 1, \quad \tau(x'') = \tau(x) + 2, \quad \tau(y') = \tau(y) + 1, \quad \text{and} \quad \tau(y'') = \tau(y) + 2,$$

modulo 3. Furthermore for $i = 1, 2$, let $r_x^i : [0, 1] \to X$ denote the geodesic segment from x to the vertex of C_x of type $\tau(x) + i$, so that $r_x^1 = [x, x']$ and $r_x^2 = [x, x'']$. Similarly let r_x^i denote the corresponding geodesic segments for y.

We wish to determine when the real functions $f_i : [0, 1] \to \mathbf{R}_+$, for $i = 1, 2$, defined by

$$t \mapsto d(r_x^i(t), r_y^i(t)),$$

are non-increasing, depending on x, y and on the chambers C_x, C_y, (convexity is clear by Proposition 5.3.1).

Proposition 5.3.4. — *The function f_1 is non-increasing if and only if*

$$y \in H_v^x \coloneqq \{z \in A \mid \langle v, v\rangle \leq \langle \overrightarrow{xz}, v\rangle\},$$

where $v = \overrightarrow{xx}' - \overrightarrow{yy}'$. This is equivalent to $x \in H_{-v}^y$. The same holds for f_2 replacing v by $v' = \overrightarrow{xx}'' - \overrightarrow{yy}''$. The subset H_v^x is a closed half-space of A if and only if $v \neq 0$.

Proof. — Everything takes place in the Euclidean space A. We can write

$$f_1(t) = d(r_x^1(t), r_y^1(t)) = \|(x + t \cdot \overrightarrow{xx}') - (y + t \cdot \overrightarrow{yy}')\| = \|\overrightarrow{yx} + tv\|,$$

and therefore its square is a polynomial in t of degree 2, namely

$$f_1^2(t) = \|\overrightarrow{yx}\|^2 + 2t\langle \overrightarrow{yx}, v\rangle + t^2\|v\|^2.$$

Therefore f_1 being non-increasing is equivalent to f_1^2 being so which in turn is equivalent to $(f_1^2)'(t) \leq 0$ for all $t \in [0, 1]$. The latter inequality, namely

$$2\langle \overrightarrow{yx}, v\rangle + 2t\|v\|^2 \leq 0,$$

is true for all $t \in [0, 1]$ if and only if $\|v\|^2 \leq \langle \overrightarrow{xy}, v\rangle$, holds. □

Definition 5.3.5. — We say that (C_x, C_y) is a *possible configuration* if the two functions f_1, f_2 satisfy the conditions of previous proposition, that is

$$y \in H_v^x \cap H_{v'}^x \iff x \in H_{-v}^y \cap H_{-v'}^y,$$

where $v = \overrightarrow{xx}' - \overrightarrow{yy}'$ and $v' = \overrightarrow{xx}'' - \overrightarrow{yy}''$.

For definiteness and computation purposes, we consider x as the origin of A and we fix a chamber C_1 at x. Furthermore, we consider the unit vectors $\{w_1, w_2\}$ formed by the sides of C_1, so that w_1 is the vector starting at x pointing towards the vertex of type $\tau(x) + 1$ and w_2 is the vector starting at x pointing toward the vertex of type $\tau(x) + 2$. For example if $C_x = C_1$ then $w_1 = \overrightarrow{xx}'$ and $w_2 = \overrightarrow{xx}''$. On the chambers of $\mathrm{lk}(x) \cap A$ we write labels from 1 to 6 *clockwise* so that C_1 is labeled with 1 and shares w_1 as a face with the chamber labeled with 2, see Figure 11. According to the label of C_x, the vectors $\overrightarrow{xx}'$ and $\overrightarrow{xx}''$ take the values given by Table 1.

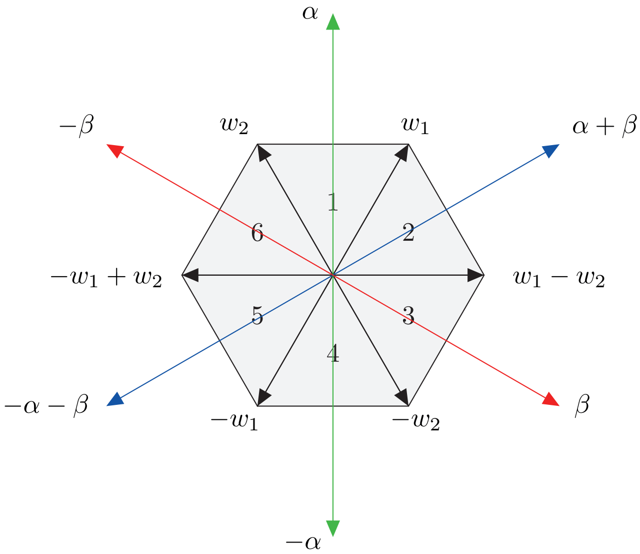

FIGURE 11. Labelling of the chambers in $\mathrm{lk}(x)\cap A$.

Label of C_x	$\overrightarrow{xx}'$	$\overrightarrow{xx}''$
1	w_1	w_2
2	w_1	w_1-w_2
3	$-w_2$	w_1-w_2
4	$-w_2$	$-w_1$
5	$-w_1+w_2$	$-w_1$
6	$-w_1+w_2$	w_2

TABLE 1. Sides of C_x.

By translation we label the chambers of $\mathrm{lk}(y)\cap A$ which determines a similar table for the sides of C_y. Proposition 5.3.4 tells us to look at the vectors $v=\overrightarrow{xx}'-\overrightarrow{yy}'$ and $v'=\overrightarrow{xx}''-\overrightarrow{yy}''$ which is done in Table 2 and Table 3.

$\overrightarrow{yy}' \setminus \overrightarrow{xx}'$	w_1	$-w_2$	$-w_1+w_2$
w_1	0	$-w_1-w_2$	$-2w_1+w_2$
$-w_2$	w_1+w_2	0	$-w_1+2w_2$
$-w_1+w_2$	$2w_1-w_2$	w_1-2w_2	0

TABLE 2. Values of v.

We now rewrite this in a suitable basis linked with the root system A_2 by setting

$$\alpha := w_1+w_2 \quad \text{and} \quad \beta := w_1-2w_2.$$

$\overrightarrow{yy}'' \setminus \overrightarrow{xx}''$	w_2	$w_1 - w_2$	$-w_1$
w_2	0	$w_1 - 2w_2$	$-w_1 - w_2$
$w_1 - w_2$	$-w_1 + 2w_2$	0	$-2w_1 + w_2$
$-w_1$	$w_1 + w_2$	$2w_1 - w_2$	0

TABLE 3. Values of v'.

The vector α is the sum of the sides of the chamber labeled with 1, and β is the sum of that labeled with 3, see Figure 11. The entries of the two previous tables are linear combinations of α and β with coefficients plus or minus 1, see Tables 4 and 5 below.

$\overrightarrow{yy}' \setminus \overrightarrow{xx}'$	w_1	$-w_2$	$-w_1 + w_2$
w_1	0	$-\alpha$	$-\alpha - \beta$
$-w_2$	α	0	$-\beta$
$-w_1 + w_2$	$\alpha + \beta$	β	0

TABLE 4. Values of v in terms of $\{\alpha, \beta\}$.

$\overrightarrow{yy}'' \setminus \overrightarrow{xx}''$	w_2	$w_1 - w_2$	$-w_1$
w_2	0	β	$-\alpha$
$w_1 - w_2$	$-\beta$	0	$-\alpha - \beta$
$-w_1$	α	$\alpha + \beta$	0

TABLE 5. Values of v' in terms of $\{\alpha, \beta\}$.

As C_x, C_y ranges through the chambers of $\mathrm{lk}(x) \cap A$ and $\mathrm{lk}(y) \cap A$ respectively, the vectors v, v' take various values gathered in Table 6. Conversely, Table 7 shows the inverse; given values of v, v' it yields the set of configurations of C_x, C_y realizing them. The top-left entry of the latter contains all pairs (i, i) where $i = 1, \dots, 6$, this corresponds to C_y being a translate of C_x in A. Also the symbol $\emptyset$ was used to denote that no pair C_x, C_y yields these vectors, *e.g.* it is impossible to have $v = \alpha$ and $v' = -\alpha$.

$C_x \setminus C_y$	1	2	3	4	5	6
1	$(0,0)$	$(0,\beta)$	$(-\alpha,\beta)$	$(-\alpha,-\alpha)$	$(-\alpha-\beta,-\alpha)$	$(-\alpha-\beta,0)$
2	$(0,-\beta)$	$(0,0)$	$(-\alpha,0)$	$(-\alpha,-\alpha-\beta)$	$(-\alpha-\beta,-\alpha-\beta)$	$(-\alpha-\beta,-\beta)$
3	$(\alpha,-\beta)$	$(\alpha,0)$	$(0,0)$	$(0,-\alpha-\beta)$	$(-\beta,-\alpha-\beta)$	$(-\beta,-\beta)$
4	(α,α)	$(\alpha,\alpha+\beta)$	$(0,\alpha+\beta)$	$(0,0)$	$(-\beta,0)$	$(-\beta,-\alpha)$
5	$(\alpha+\beta,\alpha)$	$(\alpha+\beta,\alpha+\beta)$	$(\beta,\alpha+\beta)$	$(\beta,0)$	$(0,0)$	$(0,\alpha)$
6	$(\alpha+\beta,0)$	$(\alpha+\beta,\beta)$	(β,β)	$(\beta,-\alpha)$	$(0,-\alpha)$	$(0,0)$

TABLE 6. Values of (v, v') from the labels of C_x, C_y.

Table 7 allow us to somehow reverse the previous discussion. Indeed, the vectors v, v' must be either 0 or elements of

$$\Phi := \{\alpha, \beta, \alpha + \beta, -\alpha, -\beta, -\alpha - \beta\}.$$

Vectors $v \setminus v'$	0	α	$\alpha+\beta$	β	$-\alpha$	$-\alpha-\beta$	$-\beta$
0	(i,i)	$(2,3)$	$(1,6)$	$(4,5)$	$(3,2)$	$(6,1)$	$(5,4)$
α	$(6,5)$	$(1,4)$	$(1,5)$	$\emptyset$	$\emptyset$	$\emptyset$	$(6,4)$
$\alpha+\beta$	$(3,4)$	$(2,4)$	$(2,5)$	$(3,5)$	$\emptyset$	$\emptyset$	$\emptyset$
β	$(2,1)$	$\emptyset$	$(2,6)$	$(3,6)$	$(3,1)$	$\emptyset$	$\emptyset$
$-\alpha$	$(5,6)$	$\emptyset$	$\emptyset$	$(4,6)$	$(4,1)$	$(5,1)$	$\emptyset$
$-\alpha-\beta$	$(4,3)$	$\emptyset$	$\emptyset$	$\emptyset$	$(4,2)$	$(5,2)$	$(5,3)$
$-\beta$	$(1,2)$	$(1,3)$	$\emptyset$	$\emptyset$	$\emptyset$	$(6,2)$	$(6,3)$

Table 7. Labels of (C_x, C_y) realizing (v, v').

Thus we consider all possible intersections of the half-spaces H_r^x for r ranging in Φ, since $H_0^x = A$, and see how they cut A into convex zones, Figure 12. Any such zone Z is of the form

$$Z = \bigcap_{r\in R} H_r^x \cap \bigcap_{r\in R^c} (H_r^x)^c,$$

where $R \subset \Phi$. Note that Z is closed because the complement of H_r^x strictly contains H_{-r}^x. Then, thanks to Proposition 5.3.4, if $y \in Z$, the possible configurations for (C_x, C_y) are read in Table 7 by taking the entries with v, v' varying in $R \cup \{0\}$. Here is an example that we shall use again later.

Example 5.3.6. — Let Z be the zone defined by $R = \{\alpha, \alpha+\beta, -\beta\}$, it is the shaded sector of Figure 12. More precisely,

$$Z = H_\alpha^x \cap H_{-\beta}^x \cap H_{\alpha+\beta}^x.$$

The possible configurations are given by the sub-table of Table 7 obtained by keeping only the rows and columns labeled by $0, \alpha, \alpha+\beta, -\beta$, that is Table 8.

Vectors $v \setminus v'$	0	α	$\alpha+\beta$	$-\beta$
0	(i,i)	$(2,3)$	$(1,6)$	$(5,4)$
α	$(6,5)$	$(1,4)$	$(1,5)$	$(6,4)$
$\alpha+\beta$	$(3,4)$	$(2,4)$	$(2,5)$	$\emptyset$
$-\beta$	$(1,2)$	$(1,3)$	$\emptyset$	$(6,3)$

Table 8. Possible configurations if $y \in Z$.

5.3.2. Three points in an apartment. — The complexity of the position of three points in an $\widetilde{A}_2$-building was discussed in Section 5.2. It is natural to ask what can be said about the values of $\mathrm{vol}_X(x, y, z)$ on a large sector sphere $S_{R,R}(x)$ when the three vertices x, y, z sit in a common apartment, say A. The discussion of the previous section is certainly useful in this setting. However given $\xi \in \Omega$, the chambers $C_x(\xi), C_y(\xi), C_z(\xi)$ need not lie in a common apartment, but they do pairwise by the building axioms. In this paragraph, we discuss this case via an example, see Setting 5.3.7, in which we determine all possible configurations of the chambers at x, y, z using the results of the previous paragraph. We proceed in two parts:

- Describe all possible relative positions of $C_x(\xi), C_y(\xi), C_z(\xi)$ using the previous paragraph and something we introduce below called *transitions*.

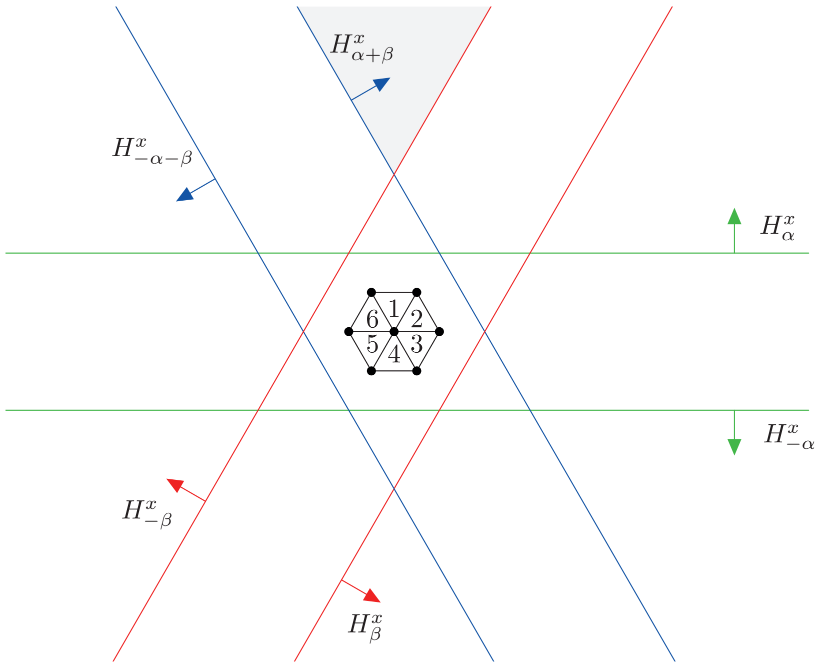

FIGURE 12. Half-spaces about x.

- Show each of these configurations to be realized by a folding diagram of $\mathrm{Conv}(x, y, z)$.

Our interest is the asymptotic behavior of the Poisson transform of $\mathrm{vol}_X(x, y, z)$ as the points x, y, z get far from each other. Taking x as the reference point in Figure 12, the open strips of the form $(H^x_r \cup H^x_{-r})^c$, with $r = \alpha, \beta, \alpha+\beta$, are unbounded. The vertices in their union is the union of the sector spheres $S_{m,n}(x) \cap A$ with m or n strictly smaller than 2. Therefore the asymptotic behavior may vary depending on whether y, z stay in those strips or not. The situation we consider for the rest of this section avoids this by staying in the complement of the strips.

Setting 5.3.7. — For the remainder of this section we assume x, y, z to be vertices sitting in a common apartment A such that x, y, z, pairwise, have sector coordinates at least 2. Without loss of generality, suppose that y is in the shaded area of Figure 12. Further we assume z to sit in the sector *parallel* to the chamber at x labeled with 2. More precisely,

$$y \in H^x_\alpha \cap H^x_{-\beta} \cap H^x_{\alpha+\beta} \quad \text{and} \quad z \in H^x_\alpha \cap H^x_\beta \cap H^x_{\alpha+\beta}.$$

Finally suppose that the chambers at y in $\mathrm{Conv}(y, x)$ and $\mathrm{Conv}(y, z)$ respectively are adjacent, thus labeled in A with 4 and 3 respectively. Consequently, those at z in $\mathrm{Conv}(z, x)$ and $\mathrm{Conv}(z, y)$ are also adjacent and labeled with 5 and 6 respectively. The configuration is pictured in Figure 13. This last assumption is equivalent in A to both

$$z \in H^y_{-\alpha} \cap H^y_\beta \cap H^y_{\alpha+\beta} \iff y \in H^z_\alpha \cap H^z_{-\beta} \cap H^z_{-\alpha-\beta}.$$

We chose a reference apartment A but in fact the setting above takes place in $\mathrm{Conv}(x, y, z)$ and depends not on the choice of an apartment containing x, y, z. In Example 5.3.6, we calculated the possible configuration of pairs C_x, C_y, which we recall in Table 9. We can proceed similarly and

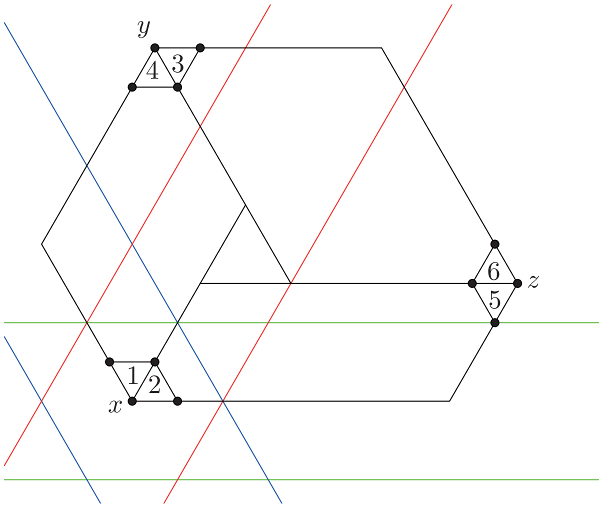

FIGURE 13. Configuration in A.

establish possible configurations for the ordered pairs C_y, C_z and C_z, C_x, see Table 10 and Table 11, using the fact that

$$z \in H^y_{-\alpha} \cap H^y_{\beta} \cap H^y_{\alpha+\beta} \quad \text{and} \quad x \in H^z_{-\alpha} \cap H^z_{-\beta} \cap H^z_{-\alpha-\beta}.$$

From x to y	0	α	$\alpha+\beta$	$-\beta$
0	(i,i)	$(2,3)$	$(1,6)$	$(5,4)$
α	$(6,5)$	$(1,4)$	$(1,5)$	$(6,4)$
$\alpha+\beta$	$(3,4)$	$(2,4)$	$(2,5)$	$\emptyset$
$-\beta$	$(1,2)$	$(1,3)$	$\emptyset$	$(6,3)$

TABLE 9. Labels of possible (C_x, C_y).

From y to z	0	$\alpha+\beta$	β	$-\alpha$
0	(i,i)	$(1,6)$	$(4,5)$	$(3,2)$
$\alpha+\beta$	$(3,4)$	$(2,5)$	$(3,5)$	$\emptyset$
β	$(2,1)$	$(2,6)$	$(3,6)$	$(3,1)$
$-\alpha$	$(5,6)$	$\emptyset$	$(4,6)$	$(4,1)$

TABLE 10. Labels of possible (C_y, C_z).

Remark 5.3.8. — Let $D \in \mathrm{lk}(x)$ be a chamber adjacent to the two chambers labeled with 1 and 2 in Figure 13 but not contained in A. In any apartment containing D and y, this chamber would have label 2. However if D sits in an apartment containing z, it will have label 1. So we have to keep in mind that Table 9 was established for pairs of chambers C_x, C_y in a common apartment regardless of z and C_z. The same thing applies to Tables 10 and 11.

From z to x	0	$-\alpha$	$-\alpha-\beta$	$-\beta$
0	(i,i)	$(3,2)$	$(6,1)$	$(5,4)$
$-\alpha$	$(5,6)$	$(4,1)$	$(5,1)$	$\emptyset$
$-\alpha-\beta$	$(4,3)$	$(4,2)$	$(5,2)$	$(5,3)$
$-\beta$	$(1,2)$	$\emptyset$	$(6,2)$	$(6,3)$

TABLE 11. Labels of possible (C_z, C_x).

To treat the general case, we will encode this change of labels by transitions. Nevertheless we start by searching for possible configurations of three *coplanar* chambers C_x, C_y, C_z, *i.e.* contained in a common apartment. Let N_{xy} be the set of labels from x to y, *i.e.* the set of entries of Table 9, N_{yz} that of Table 10, and N_{zx} that of Table 11. In the framework of Setting 5.3.7, we are looking for all triples of labels of a possible configuration of C_x, C_y, C_z. A triple $(a_x, a_y, a_z) \in \{1\dots, 6\}^3$ is a possible configuration if and only if

$$(a_x, a_y) \in N_{xy}, \quad (a_y, a_z) \in N_{yz}, \quad \text{and} \quad (a_z, a_x) \in N_{zx}.$$

Proposition 5.3.9. — *The possible configurations of three chambers C_x, C_y, C_z contained in a common apartment is given by the list of labels of Table 12.*

Labels (a_x, a_y, a_z)	
$a_x = 1$	$(1,1,1),(1,1,6),(1,2,1),(1,2,5)$ $(1,2,6),(1,3,1),(1,3,4),(1,3,5)$ $(1,3,6),(1,4,1),(1,4,4),(1,4,5)$ $(1,4,6),1,5,5),(1,5,6),(1,6,6)$
$a_x = 2$	$(2,2,1),(2,2,2),(2,2,5),(2,2,6)$ $(2,3,1),(2,3,2),(2,3,3),(2,3,4)$ $(2,3,5),(2,3,6),(2,4,1),(2,4,4)$ $(2,4,5),(2,4,6),(2,5,5),(2,5,6)$
$a_x = 3$	$(3,3,3),(3,3,4),(3,3,5),(3,3,6)$ $(3,4,4),(3,4,5),(3,4,6)$
$a_x = 4$	$(4,4,4),(4,4,5)$
$a_x = 5$	$(5,4,5),(5,5,5)$
$a_x = 6$	$(6,3,5),(6,3,6),(6,4,5),(6,4,6)$ $(6,5,5),(6,5,6),(6,6,6)$

TABLE 12. Labels of possible configurations of coplanar C_x, C_y, C_z.

Proof. — The list is established by considering the oriented graph with vertices in $\{1,\dots,6\}$ and edges $N_{xy} \cup N_{yz} \cup N_{zx}$. A possible configuration is the same as an oriented loop of lenght 3 such that the first edge is in N_{xy}, the second in N_{yz} and the last in N_{zx}. For a better readability, we draw the graph of Figure 14 instead. Loops can be read by starting on the left at some integer between 1 to 6 and looking at all paths going to the right ending at the same integer. $\square$

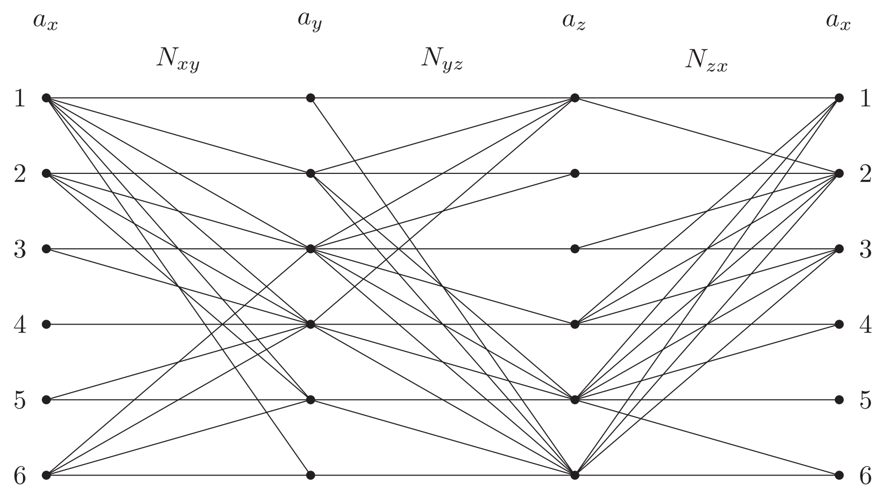

FIGURE 14. A graph to determine the loops.

5.3.3. Transitions. — In most cases the chambers C_x, C_y, C_z, are not coplanar. As mentioned in Remark 5.3.8, $C_x \in \mathrm{lk}(x)$ could have a certain label with respect to an apartment containing $\mathrm{Conv}(x, y)$ and a different one with respect to $\mathrm{Conv}(x, z)$. Looking carefully at the chambers in the links of x, y, z we can deduce a set of rules codifying the possible transitions.

Definition 5.3.10. — For every $C \in \mathrm{lk}(x)$, let $(\lambda_x^z(C), \lambda_x^y(C))$ be the pair of labels of C taken with respect to an apartment containing $\mathrm{Conv}(x, z)$ and to one containing $\mathrm{Conv}(x, y)$, the pair is called the *transition of* C at x. The labeling in each apartment is done according to Setting 5.3.7. Similarly we define $(\lambda_y^x, \lambda_y^z)$ and $(\lambda_z^y, \lambda_z^x)$, the transition at y and z respectively.

Below the link of x is treated in detail assuming for simplicity that x is of type 0. The labels were introduced for computation purposes but they simply translate the value of the W-metric. Let C_1, C_2 be the chambers of $\mathrm{lk}(x) \cap \mathrm{Conv}(x, y)$ and $\mathrm{lk}(x) \cap \mathrm{Conv}(x, z)$ respectively. Since $\tau(x) = 0$, the link of x is a finite building for the W-metric δ of X restricted to $\mathrm{lk}(x)$. It takes values in the subgroup $W < W_{\mathrm{aff}}$ isomorphic to the symmetric group on three elements, here generated by s_1, s_2. By construction $\delta(C_1, C_2) = s_1$, see Figure 11.

Lemma 5.3.11. — *For $w_1, w_2 \in W$, the cardinal of the intersection of the δ-balls*

$$B(C_1, w_1) \cap B(C_2, w_2) = \{C \in \mathrm{lk}(x) \mid \delta(C_1, C) = w_1, \delta(C_2, C) = w_2\},$$

is given by

$$\operatorname{card}(B(C_1,w_1)\cap B(C_2,w_2)) = \begin{cases} 1 & \textit{if } (w_1,w_2)=(e,s_1) \textit{ or } (s_1,e),\\ q-1 & \textit{if } w_1=w_2=s_1,\\ q & \textit{if } (w_1,w_2)=(s_2,s_1s_2) \textit{ or } (s_1s_2,s_2),\\ (q-1)q & \textit{if } w_1=w_2=s_1s_2,\\ q^2 & \textit{if } (w_1,w_2)=(s_2s_1,s_1s_2s_1) \textit{ or } (s_1s_2s_1,s_2s_1),\\ (q-1)q^2 & \textit{if } w_1=w_2=s_1s_2s_1,\\ 0 & \textit{else,} \end{cases}$$

where q is the regularity parameter of X.

Proof. — Since $\delta(C_1,C_2)=s_1$, the axiom (W2) for the W-metric, Proposition 1.2.14, implies that $B(C_1,w_1)\cap B(C_2,w_2)$ is nonempty if and only if $w_2=w_1$ or $w_2=s_1w_1$. Moreover if $\ell(s_1w_1)=\ell(w_1)+1$, then the intersection is nonempty if and only if $w_2=s_1w_1$. We can deduce the above cases by distinguishing whether C, C_1, C_2 lie in a common apartment of $\operatorname{lk}(x)$ or not, see Figure 16. In the first case, we can look at the q chambers s_2-adjacent to C_1 which covers the case $(w_1,w_2)=(s_2,s_1s_2)$. Similarly the q-chambers s_2-adjacent to C_2 are in the intersection $B(C_1,w_1)\cap B(C_2,w_2)$ with $(w_1,w_2)=(s_1s_2,s_2)$. Continuing so we obtain the q^2 chambers in the intersection with $(w_1,w_2)=(s_2s_1,s_1s_2s_1)$ and the q^2 others with parameters $(w_1,w_2)=(s_1s_2s_1,s_2s_1)$. On the other hand, there are $q-1$ chambers s_1-adjacent to both C_1,C_2 but distinct from the two, that is $w_1=w_2=s_1$. These chambers are each s_2-adjacent to q other chambers, they are in the intersection given by $w_1=w_2=s_1s_2$. The latter are in turn each s_1-adjacent to q chambers in the intersection $w_1=w_2=s_1s_2s_1$. □

The previous lemma translates in terms of relative labels.

Corollary 5.3.12. — *The values of $(\lambda_x^z(C),\lambda_x^y(C))$ and the number of chambers $C\in\operatorname{lk}(x)$ realizing it are given in Table 13. The corresponding statement for $(\lambda_y^x(C),\lambda_y^z(C))$ is in Table 14 and that for $(\lambda_z^y(C),\lambda_z^x(C))$ in Table 15.* □

Transition labels $(\lambda_x^z(C)),\lambda_x^y(C)$	number of chambers C
$(1,1)$	1
$(2,2)$	1
$(1,2)$	$q-1$
$(6,6)$	q
$(3,3)$	q
$(6,3)$	$(q-1)q$
$(5,5)$	q^2
$(4,4)$	q^2
$(5,4)$	$(q-1)q^2$

Table 13. Transitions at x and the number of chamber realizing them.

Let T_x, T_y, T_z be the set of transitions given by Table 13, Table 14, and Table 15 respectively. In order for three C_x, C_y, C_z chambers, in the links of x, y, z respectively, to be a possible configuration,

Transition labels $(\lambda_y^x(C), \lambda_y^z(C))$	number of chambers C
$(3,3)$	1
$(4,4)$	1
$(3,4)$	$q-1$
$(2,2)$	q
$(5,5)$	q
$(2,5)$	$(q-1)q$
$(1,1)$	q^2
$(6,6)$	q^2
$(1,6)$	$(q-1)q^2$

Table 14. Transitions at y and the number of chamber realizing them.

Transition labels $(\lambda_z^y(C), \lambda_z^x(C))$	number of chambers C
$(5,5)$	1
$(6,6)$	1
$(5,6)$	$q-1$
$(1,1)$	q
$(4,4)$	q
$(4,1)$	$(q-1)q$
$(2,2)$	q^2
$(3,3)$	q^2
$(3,2)$	$(q-1)q^2$

Table 15. Transitions at z and the number of chamber realizing them.

the six labels

$$\lambda_x^z(C_x), \lambda_x^y(C_x), \lambda_y^x(C_y), \lambda_y^z(C_y), \lambda_z^y(C_z), \lambda_z^x(C_z)$$

must satisfy the necessary conditions imposed by the sets N_{xy}, N_{yz}, N_{zx}, namely

$$(\lambda_x^y(C_x), \lambda_y^x(C_y)) \in N_{xy}, \quad (\lambda_y^z(C_y), \lambda_z^y(C_z)) \in N_{yz} \quad \text{and} \quad (\lambda_z^x(C_z), \lambda_x^z(C_x)) \in N_{zx}.$$

In other words we are looking for sextuples $(a_x, b_x, a_y, b_y, a_z, b_z)$ of labels satisfying the conditions imposed by the sets N_{xy}, N_{yz}, N_{zx} and the transition conditions T_x, T_y, T_z, namely

$$\begin{array}{llll} (a_x, b_x) \in T_x, & (a_y, b_y) \in T_y, & (a_z, b_z) \in T_z & \text{and} \\ (b_x, a_y) \in N_{xy}, & (b_y, a_z) \in N_{yz}, & (b_z, a_x) \in N_{zx}. & \end{array}$$

Proposition 5.3.13. — *The possible configurations of three chambers C_x, C_y, C_z with labels as above are given by Table 16 which lists equivalently all sequences of the form $a_y b_y a_z b_z a_x b_x$.*

Proof. — The proof goes as in Proposition 5.3.9, except that we insert the transitions between the sets N_{xy}, N_{yz}, N_{zx}. Consider the oriented graph with vertices $\{1, \dots, 6\}$ and edges E, the union of the transitions T_x, T_y, T_z and of the constraint sets N_{xy}, N_{yz}, N_{zx}. The sequences of labels $(a_x, b_x, a_y, b_y, a_z, b_z)$ correspond to oriented paths of length 6, such that the first edge is in T_x, the second in N_{xy}, the third T_y and so on. During the computation we equivalently extracted the sequences $a_y b_y a_z b_z a_x b_x$, we hope it will not cause confusion. We concatenated in that order the oriented edges of $T_y, N_{yz}, T_z, N_{zx}, T_x, N_{xy}$. This amounts to finding all paths in the graph of Figure

15 from left to right starting and finishing at the same integer $k \in \{1, \ldots, 6\}$. A simple script in Python [**Pyt**] using the package NetworkX [**HSS08**] yields the list of Table 16. $\square$

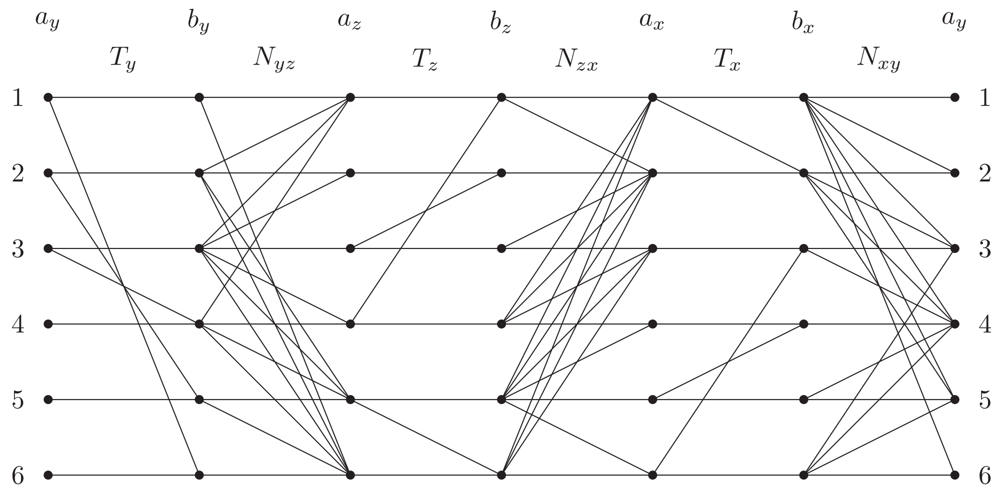

FIGURE 15. A graph to determine the possible sequences $a_y b_y a_z b_z a_x b_x$.

$a_y b_y a_z b_z a_x b_x$	
$a_y = 1$	166611, 116611, 111111
$a_y = 2$	256611, 256612, 256622, 255511, 255512, 255522, 255611, 255612 255622, 222222, 226611, 226612, 226622, 225511, 225512, 225522 225611, 225612, 225622, 221111, 221112, 221122
$a_y = 3$	344111, 344112, 344122, 344411, 344412, 344422, 344433, 345511 345512, 345522, 345533, 345566, 345563, 345611, 345612, 345622 345633, 345666, 345663, 346611, 346612, 346622, 346633, 346666 346663, 341111, 341112, 341122, 332222, 333222, 333322, 333333 334111, 334112, 334122, 334411, 334412, 334422, 334433, 335511 335512, 335522, 335533, 335566, 335563, 335611, 335612, 335622 335633, 335666, 335663, 336611, 336612, 336622, 336633, 336666 336663, 331111, 331112, 331122
$a_y = 4$	444111, 444112, 444122, 444411, 444412, 444422, 444433, 444444 445511, 445512, 445522, 445533, 445544, 445554, 445555, 445566 445563, 445611, 445612, 445622, 445633, 445666, 445663, 446611 446612, 446622, 446633, 446666, 446663, 441111, 441112, 441122
$a_y = 5$	556611, 556612, 556622, 556666, 555511, 555512, 555522, 555555 555566, 555611, 555612, 555622, 555666
$a_y = 6$	666611, 666666

TABLE 16. Possible sequences $a_y b_y a_z b_z a_x b_x$.

Remark 5.3.14. — The patient reader can verify that the tuples in Table 16 such that $a_x = b_x$, $a_y = b_y$, and $a_z = b_z$ coincide, as expected, with those of Table 12 corresponding to the coplanar case.

Under our assumptions, see Setting 5.3.7, the pairwise convex hulls of x, y, z intersect along their boundaries, therefore Theorem 5.2.8 applies. Let t_x, t_y, t_z be the vertices of the corresponding equilateral triangle, see also Figure 13. Interestingly, the triangle $\Delta(t_x, t_y, t_z)$ seems to be forced to stay in the red zone $(H^x_{-\beta} \cap H^x_{\beta})^c$ independently of y, z. (Its sides have length at most 4.)

The list obtained in Proposition 5.3.13 boils down to three different lists according the shape of the triangle, either it is reduced to a point, or its sides are of shape $(0, p)$, or of $(p, 0)$ for $p \geq 1$. Let L_x be the line in A supporting the geodesic segment $[x, t_x]$ and let L_y, L_z denote the corresponding lines in A for y, z. We have the three cases:

(a) The three lines intersect at $t = t_x, t_y, t_z$, thus the triangle is degenerate.
(b) The point t_y sits on L_x, thus $t_z \in L_y$ and $t_x \in L_z$, see Figure 13.
(c) The point t_z sits on L_x, thus $t_x \in L_y$ and $t_y \in L_y$.

Case (a) is equivalent to

$$x \in S_{R,R}(y), \quad y \in S_{R,R}(z) \quad \text{and} \quad z \in S_{R,R}(x),$$

for some parameter $R \in \mathbf{N}$. In this case we extract an exact list of labels, *i.e.* a list in which all tuples of labels can be realized by a folding diagram. (The same strategy works for the other cases.)

To see this, suppose $C_y = C_y(\xi)$ for some $\xi \in \Omega$, then the arrow of C_y starts at y, see Section 5.1, and C_y is the unique chamber in $\mathrm{lk}(y)$ with this property, by uniqueness of the sector $\mathrm{Sect}_y(\xi)$. A consequence of Lemma 5.3.11 is that C_y belongs to a branching of three roots (half-apartments of $\mathrm{lk}(y)$) as pictured in Figure 16. From the arrow of C_y, we can deduce the others, see the two examples of Figure 17 and Figure 18. In the first, the chamber C_y has labels $(a_y, b_y) = (4, 4)$ whereas in the second C_y has labels $(3, 4)$. The arrows in the link of y impose conditions on the retraction diagram of A, or rather on that of $\mathrm{Conv}(x, y, z)$, because the arrows of the two chambers in $\mathrm{lk}(y) \cap \mathrm{Conv}(x, y, z)$ have been determined. For instance in Figure 18 the arrows tell us that one of the focal points of the folding diagram must sit on L_y, which justifies the choice of the above three cases. We determined all such conditions at each x, y, z for the Case (a) and drew the numerous possible folding diagrams to exclude the tuples of Table 16 that did not satisfy those conditions. This is summed up in the following proposition whose proof is omitted.

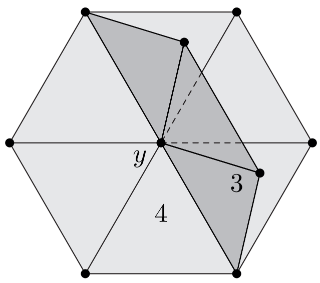

FIGURE 16. Branching containing C_y, where the chambers labeled 3 and 4 are in $\mathrm{Conv}(x, y, z)$.

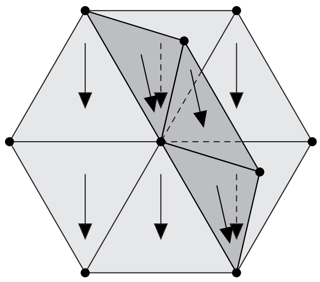

FIGURE 17. Link of y with $a_y b_y = 44$.

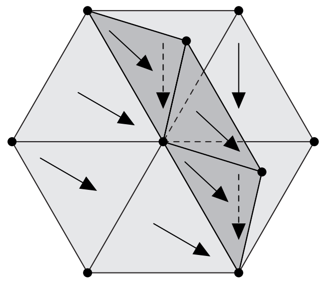

FIGURE 18. Link of y with $a_y b_y = 34$.

$a_y b_y a_z b_z a_x b_x$	
$a_y = 1$	166611
$a_y = 2$	256611, 255522, 255612, 226612, 225622, 221112
$a_y = 3$	344112, 344422, 344433, 345522, 345533, 345612, 345663, 346611 346666, 341111, 333222, 334122, 335622, 335633, 336612, 336663 331112
$a_y = 4$	444111, 444412, 445512, 445554, 445563, 445611, 445666
$a_y = 5$	555512, 555611, 555666
$a_y = 6$	

TABLE 17. Possible labels for non-coplanar chambers in Case (a).

Proposition 5.3.15. — *Suppose we are in Case (a) and let C_x, C_y, C_z be three chambers in the links of x, y, z respectively. Then,*

- *if C_x, C_y, C_z are coplanar, there exists a folding diagram compatible with C_x, C_y, C_z if and only if the labels (a_x, a_y, a_z) of C_x, C_y, C_z appear in the list of Table 12.*
- *If C_x, C_y, C_z are non-coplanar, there is a folding diagram compatible with C_x, C_y, C_z if and only if the labels $(a_x, b_x, a_y, b_y, a_z, b_z)$ of C_x, C_y, C_z appear in the list of Table 17.* □

Examples 5.3.16. — We give here two examples of tuples of labels, one from Table 17 that is realized by a folding diagram and one from Table 16 that has been excluded in the Case (a). Figure 19 shows one possible folding diagram for the labels $(a_y, b_y, a_z, b_z, a_x, b_x) = (3, 4, 5, 6, 6, 3)$, whereas on Figure 20 we can see that the conditions on the links fail to be realized by a folding diagram if $(a_y, b_y, a_z, b_z, a_x, b_x) = (2, 5, 5, 6, 2, 2)$.

Comment 5.3.17. — We only worked with the case of three coplanar points and gave a general strategy to understand better the possible configurations of $C_x(\xi), C_y(\xi), C_z(\xi)$ as ξ ranges in Ω. The triangle $\Delta(t_x, t_y, t_z)$, when non-degenerate, plays a crucial role when determining the folding diagram of the convex hulls $\mathrm{Conv}(x, y)$, $\mathrm{Conv}(y, z)$, $\mathrm{Conv}(z, x)$. The folding diagram of the triangle imposes conditions on the arrows of the chambers of the latter convex hulls. This is of course related to the question of determining the cardinal of the intersection of three sector spheres centered at x, y, z respectively.

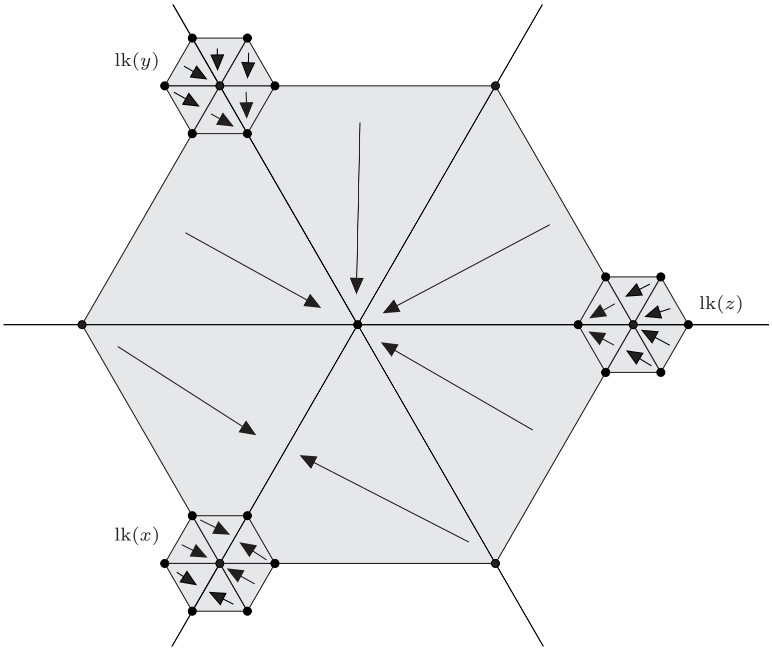

FIGURE 19. A folding diagram for the labels 345663.

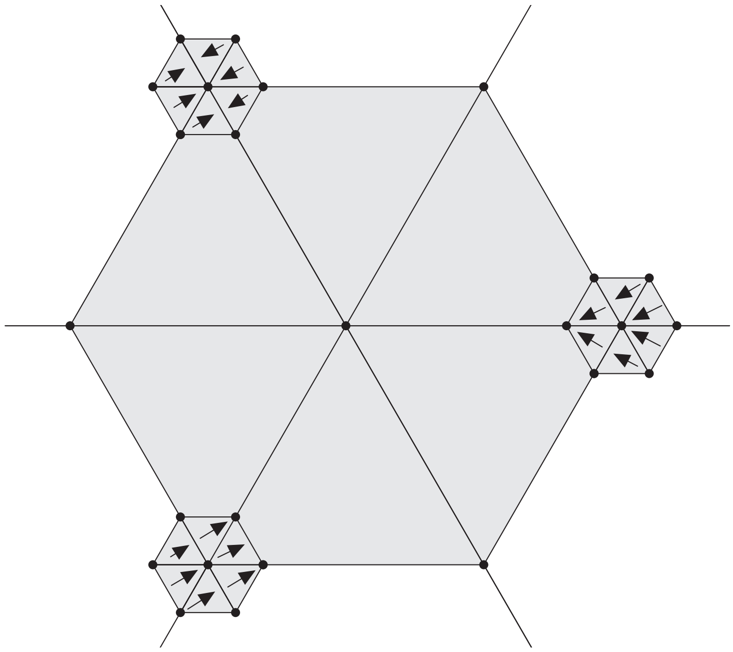

FIGURE 20. The arrows induced by the labels 255622 fail to be realized by a folding diagram.

NOTATION INDEX

SUBJECT INDEX

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Education

Since 2011 **PhD Student at EPFL, Switzerland**
Chair of Ergodic and Geometric Group Theory of Prof. Nicolas Monod
Dissertation: *Cocycle growth for the Steinberg representation*

2006 - 2011 **BSc & MSc at EPFL, Switzerland**
Master thesis performed at the California Institute of Technology, USA
Dissertation: *Spectral decomposition and Eisenstein series*

Future Academic Experience

2016 - 2017 **Postdoc at University of Utah, USA**
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Research Interests

Geometry of Euclidean Buildings
Geometric Group Theory
Continuous and Bounded Cohomology of Groups
Totally Disconnected Locally Compact Groups

Talks

Apr. 2015 Group Theory Seminar, EPFL, Lausanne
Irreducible representations and cohomology of simple algebraic groups over a non-Archimedean local field of characteristic zero

Feb. 2015 14th Graduate Colloquium, EPFL, Lausanne
From Tilings to Buildings

Nov. 2014 Seminar, EPFL, Lausanne
Buildings of type $\widetilde{A}_n$ and the growth of the natural cocycle for the Steinberg representation

Oct. 2014 Arbeitsgemeinschaft on Totally Disconnected Groups, Oberwolfach
Minimal closed normal subgroups in compactly generated t.d.l.c. groups

Conferences Attended

May. 2015	Group Theory Day, Neuchâtel, Switzerland
Feb. 2015	14th Graduate Colloquium, Lausanne, Switzerland
Jan. 2015	4th Young Geometric Group Theory Meeting, Spa, Belgium
Jan. 2015	Winter Meeting on Bruhat-Tits Buildings, London, UK
Oct. 2014	Arbeitsgemeinschaft on Totally Disconnected Groups, Oberwolfach, Germany
Jul. 2014	Borel Seminar, Discrete Group Actions in Geometry and Topology, Les Diablerets, Switzerland
Jun. 2014	Growth in Groups, Neuchâtel, Switzerland
Mar. 2014	Asymptotic Properties of Groups, Paris, France
Jan. 2014	3rd Young Geometric Group Theory Meeting, Luminy, France
Oct. 2013	Masterclass on Ergodic Theory and von Neumann Algebras, Copenhagen, Denmark
Mar. 2013	Kervaire Seminar, Geometry of Groups 2013, Les Diablerets, Switzerland
Dec. 2012	Parole aux jeunes chercheurs sur les groupes, Geneva, Switzerland
Mar. 2012	Kervaire Seminar, CAT(0) Spaces and Groups, Les Diablerets, Switzerland
Mar. 2012	Groups 2012, Bielefeld, Germany

Teaching

2014	Geometry and Groups (head assistant)
2011 - 2014	Geometry I & II (head assistant)
2010	Geometry II
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