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Integrable Systems of Neumann Type

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Abstract We construct families of integrable systems that interpolate between N-dimensional harmonic oscillators and Neumann systems. This is achieved by studying a family of integrable systems generated by the Casimir functions of the Lie algebra of real skew-symmetric matrices and a certain deformation thereof. Involution is proved directly, since the standard involution theorems do not apply to these families. It is also shown that the integrals are independent.

Keywords Integrable system · Casimir function · Integrals in involution · Independence · Skew-symmetric matrices · Lie algebra deformation

Mathematics Subject Classification 37J35 · 53D17

1 Introduction

The theory of integrable systems is a time-honored subject going back to the founders of theoretical mechanics. They are very important because, in principle, the differential or partial differential equations describing them can be solved and because they are key systems around which many other dynamic and geometric phenomena can be studied, such as perturbations,

To the memory of Klaus Kirchgässner, mentor and friend.

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bifurcations, stability, numerics. In addition, integrability for systems on symplectic or Poisson manifolds is characterized by the Liouville–Mineur–Arnold Theorem (see, e.g., [1,4,5]) which states that on a generic leaf of the Poisson manifold, the given Hamiltonian system has half the dimension many integrals that Poisson commute and whose differentials are linearly independent on an open dense set. Many examples of integrable systems are known and have been studied in detail, both in finite and infinite dimensions. The usual approach is to prove involution of the integrals by means of some general involution theorem (such as the Adler–Kostant–Symes theorem [2,7,18], the R-matrix method [17], the method of shifted invariants [10,11], the Thimm method [6,19], or by the use of bi-Hamiltonian structures [9,16]. If these methods do not apply, reduction methods could be employed (for example, in the Calogero and Moser–Sutherland systems [8]), or one is left with the task of a direct proof. The second step in showing that a system is integrable is the proof of the independence of the differentials of the integrals on an open dense set. No general methods are known for this and, with the exception of some algebraic geometric methods that also give the linearization of the flow (see, e.g., [3] and references therein), the proof of independence has to be done on a case by case basis.

In this paper we study a family of systems that interpolate between harmonic oscillators and Neumann systems in arbitrary dimensions on a specific symplectic leaf in the dual of the Lie algebra of skew-symmetric matrices viewed as upper triangular matrices endowed with the "constant coefficient" Poisson bracket. The idea of considering these systems comes from [14] who studied similar systems in the complex setting and for matrices with a different internal structure. It is remarkable that for the systems we consider, even though they are induced from Lie–Poisson systems, the general known involution theorems do not apply to our knowledge, so we give a direct proof of involution. It turns out that we need two families of functions in order to build a complete set of integrals: these are the usual ones given by traces of powers and a new family of quadratic integrals. We consider two families of integrable systems: one induced from the Lie algebra of real skew-symmetric matrices, the other one from a special deformation of this Lie algebra. The generic independence of the differentials of the integrals of motion is proved by a direct verification.

The paper is organized as follows. In Sect. 2 we determine the symplectic leaf of the "constant coefficient" bracket in the space of upper triangular matrices. Then we construct the first family of integrable systems, a hierarchy generated by the Casimir functions on the dual of the Lie algebra of skew-symmetric matrices and certain quadratic functions. A very particular case of this family is the harmonic oscillator. In Sect. 3 we construct another hierarchy of functions in involution generated by the Casimir functions on a special Lie algebraic deformation of the space of skew symmetric matrices and the same family of quadratic functions. A special case of these systems is the classical Neumann system. The involution of the integrals of both families is proved in the respective sections. Section 4 is devoted to the proof of independence of their differentials on an open dense set of phase space. Several examples are discussed throughout the paper.

2 Hierarchy Generated by Casimir Functions of $\mathfrak{so}(n)$

2.1 A Symplectic Leaf

Let $\mathcal{L}_+$ be the vector space of strictly upper triangular $(n \times n)$-matrices. Relative to the non-degenerate pairing

$$\langle \chi, \rho \rangle := \mathrm{Tr}(\rho\chi), \quad \rho \in \mathcal{L}_+, \quad \chi \in \mathfrak{so}(n), \tag{2.1}$$

the space $\mathcal{L}_+$ is the dual of the Lie algebra $\mathfrak{so}(n)$. We shall write a general element $\rho \in \mathcal{L}_+$ as

$$\rho = \begin{pmatrix} \widetilde{A} & B \\ 0 & \widetilde{C} \end{pmatrix}, \tag{2.2}$$

where $\widetilde{A} \in \mathfrak{gl}(2, \mathbb{R})$ and $\widetilde{C} \in \mathfrak{gl}(n-2, \mathbb{R})$ are strictly upper triangular and $B \in \mathrm{Mat}_{2\times(n-2)}(\mathbb{R})$ (the vector space of $2 \times (n-2)$ real matrices).

Thus, if $f \in C^\infty(\mathcal{L}_+)$, we have

$$\frac{\partial f}{\partial \rho} = \begin{pmatrix} \dfrac{\partial f}{\partial A} & \dfrac{\partial f}{\partial B} \\ -\dfrac{\partial f}{\partial B^\top} & \dfrac{\partial f}{\partial C} \end{pmatrix}, \tag{2.3}$$

where $A = \widetilde{A} - \widetilde{A}^\top$, $C = \widetilde{C} - \widetilde{C}^\top$, and $\frac{\partial f}{\partial B^\top} = \left(\frac{\partial f}{\partial B}\right)^\top$. Having Lie algebra $\mathfrak{so}(n)$ with dual $\mathcal{L}_+$ one defines the Lie–Poisson bracket on $C^\infty(\mathcal{L}_+)$ by

$$\{f, g\}_1 = \mathrm{Tr}\left(\rho\left[\frac{\partial f}{\partial \rho}, \frac{\partial g}{\partial \rho}\right]\right), \quad f, g \in C^\infty(\mathcal{L}_+). \tag{2.4}$$

For any fixed $\rho_0 \in \mathcal{L}_+$, the Poisson bracket

$$\{f, g\}_2 = \mathrm{Tr}\left(\rho_0\left[\frac{\partial f}{\partial \rho}, \frac{\partial g}{\partial \rho}\right]\right), \quad f, g \in C^\infty(\mathcal{L}_+) \tag{2.5}$$

is compatible with the Lie–Poisson bracket. In what follows we shall choose

$$\rho_0 := \begin{pmatrix} \widetilde{A}_0 & 0 \\ 0 & 0 \end{pmatrix}, \tag{2.6}$$

where the 2×2 matrix $\widetilde{A}_0$ is given by

$$\widetilde{A}_0 := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{2.7}$$

A simple calculation shows that the Poisson bracket (2.5) can be written in the form

$$\{f, g\}_2 = \mathrm{Tr}\left(\frac{\partial f}{\partial B^\top} A_0 \frac{\partial g}{\partial B} - \frac{\partial g}{\partial A} A_0 \frac{\partial f}{\partial A}\right), \tag{2.8}$$

where $A_0 = \widetilde{A}_0 - \widetilde{A}_0^\top$. The matrix $\frac{\partial g}{\partial A} A_0 \frac{\partial f}{\partial A}$ is antisymmetric because A and A_0 are (2×2) antisymmetric matrices (and these always commute). Therefore,

$$\{f, g\}_2(B) = \mathrm{Tr}\left(\frac{\partial f}{\partial B^\top} A_0 \frac{\partial g}{\partial B}\right). \tag{2.9}$$

Thus, we can think of this bracket as being defined on $C^\infty\left(\mathrm{Mat}_{2\times(n-2)}(\mathbb{R})\right)$, i.e., the injective smooth embedding

$$\left(\mathrm{Mat}_{2\times(n-2)}(\mathbb{R}), \{\,,\,\}_2\right) \ni B \longmapsto \rho = \begin{pmatrix} \widetilde{A} & B \\ 0 & \widetilde{C} \end{pmatrix} \in (\mathcal{L}_+, \{\,,\,\}_2),$$

with $\widetilde{A} \in \mathfrak{gl}(2, \mathbb{R})$, $\widetilde{C} \in \mathfrak{gl}(n-2, \mathbb{R})$ strictly upper triangular fixed matrices, is Poisson. Since A_0 is invertible, the Poisson bracket (2.9) is also invertible, i.e., it is a symplectic form on the

$2(n-2)$-dimensional vector space $\left(\mathrm{Mat}_{2\times(n-2)}(\mathbb{R}), \{\,,\,\}_2\right)$. We have proved hence the first statement in the following result.

Proposition 2.1 *The $2(n-2)$-dimensional vector space $\left(\mathrm{Mat}_{2\times(n-2)}(\mathbb{R}), \{\,,\,\}_2\right)$ is a symplectic leaf of the Poisson manifold $(\mathcal{L}_+, \{\,,\,\}_2)$. The vectors in $\mathbb{R}^{n-2}$ representing the two lines of the matrix B give global symplectic coordinates.*

Proof Let $B \in \mathrm{Mat}_{2\times(n-2)}(\mathbb{R})$ have rows $\vec{p} = (p_1, \ldots, p_{n-2}), \vec{q} = (q_1, \ldots, q_{n-2}) \in \mathbb{R}^{n-2}$. A direct verification shows that the Poisson bracket of $f, g \in C^\infty\left(\mathrm{Mat}_{2\times(n-2)}(\mathbb{R})\right)$ given by (2.9) has the expression

$$\{f, g\}_2(\vec{p}, \vec{q}) = \sum_{i=1}^{n-2}\left(\frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i}\right),$$

i.e., it is the canonical bracket on $\mathbb{R}^{n-2} \times \mathbb{R}^{n-2}$. □

2.2 The First Family of Functions in Involution

Next, we show that the Casimir functions

$$H_k(B) := \mathrm{Tr}\left(\widetilde{\rho}^{2k}\right), \quad k \in \mathbb{N}, \quad \widetilde{\rho} := \rho - \rho^\top = \begin{pmatrix} A & B \\ -B^\top & C \end{pmatrix}, \tag{2.10}$$

of the Poisson bracket (2.4) are in involution with respect to the Poisson bracket (2.9). Since the derivative of H_k is $DH_k = 2k\widetilde{\rho}^{2k-1}$, we get

$$\frac{\partial H_k}{\partial B} = -4k P_+ \widetilde{\rho}^{2k-1} P_-, \tag{2.11}$$

$$\frac{\partial H_k}{\partial B^\top} = 4k P_- \widetilde{\rho}^{2k-1} P_+ \tag{2.12}$$

respectively, where P_+, P_- are the orthogonal projectors given, in block matrix notation, by

$$P_+ = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & 0 \end{pmatrix}, \quad P_- = \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{1} \end{pmatrix}. \tag{2.13}$$

A direct calculation, using $P_+ + P_- = \mathbf{1}$, $P_+P_- = \mathbf{0}$, $P_+^2 = P_+$, and $P_-^2 = P_-$, yields

$$\begin{aligned}\{H_k, H_l\}_2 &= \mathrm{Tr}\left(\frac{\partial H_k}{\partial B^\top} A_0 \frac{\partial H_l}{\partial B}\right) = -16kl\,\mathrm{Tr}\left(P_-\widetilde{\rho}^{2k-1}P_+A_0P_+\widetilde{\rho}^{2l-1}P_-\right)\\ &= -16kl\,\mathrm{Tr}\left(\widetilde{\rho}^{k+l-1}P_+A_0P_+\widetilde{\rho}^{k+l-1}\right) + 16kl\,\mathrm{Tr}\left(P_+\widetilde{\rho}^{2k-1}P_+A_0P_+\widetilde{\rho}^{2l-1}P_+\right)\\ &= 0,\end{aligned} \tag{2.14}$$

because the matrix in the first term is antisymmetric and the matrix in the second term, as a product of three antisymmetric 2×2 matrices, is also antisymmetric.

We obtain a hierarchy of Hamiltonian equations generated by the Hamiltonians H_k, $k \in \mathbb{N}$, for the Poisson bracket (2.9), namely

$$\frac{\partial B}{\partial t_k} = A_0 \frac{\partial H_k}{\partial B}. \tag{2.15}$$

The Hamiltonians for $k = 1, 2, 3$ have the expressions

$$H_1 = \operatorname{Tr} A^2 + \operatorname{Tr} C^2 - 2 \operatorname{Tr} B^\top B, \tag{2.16}$$

$$\begin{aligned} H_2 = {} & \operatorname{Tr} A^4 + \operatorname{Tr} C^4 - 4 \operatorname{Tr} A^2 B B^\top - 4 \operatorname{Tr} C^2 B^\top B \\ & + 2 \operatorname{Tr} B B^\top B B^\top - 4 \operatorname{Tr} A B C B^\top, \end{aligned} \tag{2.17}$$

$$\begin{aligned} H_3 = {} & \operatorname{Tr} A^6 + \operatorname{Tr} C^6 - \operatorname{Tr} A^4 B B^\top - 6 \operatorname{Tr} C^4 B^\top B \\ & - 2 \operatorname{Tr} B^\top B B^\top B B^\top B + 6 \operatorname{Tr} A^2 B B^\top B B^\top + 6 \operatorname{Tr} C^2 B^\top B B^\top B - \\ & - 6 \operatorname{Tr} A^3 B C B^\top - 6 \operatorname{Tr} C^3 B^\top A B - 6 \operatorname{Tr} A^2 B C^2 B^\top + 3 \operatorname{Tr} A B B^\top A B B^\top \\ & + 3 \operatorname{Tr} C B^\top B C B^\top B + \operatorname{Tr} A B B^\top B C B^\top + \operatorname{Tr} A B C B^\top B B^\top. \end{aligned} \tag{2.18}$$

For these Hamiltonians, the equations of motion are

$$\frac{\partial B}{\partial t_1} = -4 A_0 B, \tag{2.19}$$

$$\frac{\partial B}{\partial t_2} = 8 \left(a^2 A_0 B - A_0 B C^2 + A_0 B B^\top B + a B C \right), \tag{2.20}$$

$$\begin{aligned} \frac{\partial B}{\partial t_3} = {} & 12 \Big(-a^4 A_0 B - A_0 B C^4 - A_0 B B^\top B B^\top B - 2a^2 A_0 B B^\top B + \\ & + A_0 B C^2 B^\top B + A_0 B B^\top B C^2 - a^3 B C + a B C^3 + a^2 A_0 B C^2 - \\ & - a B B^\top A B + A_0 B C B^\top B C - a B B^\top B C + A_0 B B^\top A B C + \\ & + A_0 B C B^\top A B - a B C B^\top B \Big), \end{aligned} \tag{2.21}$$

respectively.

2.3 The Second Family of Functions in Involution

This hierarchy has an extra family of integrals of motion in involution given by

$$\delta_k(B) := \operatorname{Tr} \left(B C^{2k-1} B^\top A_0 \right), \quad k \in \mathbb{N}. \tag{2.22}$$

To prove involutivity, note that

$$\begin{aligned} \{\delta_k, \delta_l\}_2 &= \operatorname{Tr} \left(\frac{\partial \delta_k}{\partial B^\top} A_0 \frac{\partial \delta_l}{\partial B} \right) \\ &= 4 \operatorname{Tr} \left((A_0 B C^{2k-1})^\top A_0 (C^{2l-1} B^\top A_0)^\top \right) \\ &= 4 \operatorname{Tr} \left(C^{2k-1} B^\top A_0^3 B C^{2l-1} \right) = -4 \operatorname{Tr} \left(C^{k+l-1} B^\top A_0 B C^{k+l-1} \right) = 0, \end{aligned} \tag{2.23}$$

because $A_0^2 = -\mathbf{1}$ and the matrix $C^{k+l-1} B^\top A_0 B C^{k+l-1}$ is antisymmetric.

2.4 Involution of All Functions

Finally, we prove that δ_k commutes with H_l. This will be done in two steps. First, we prove a following recursion formula. Since $C = P_- \widetilde{\rho} P_-$, $B^\top = -P_- \widetilde{\rho} P_+$, and $A_0^2 = -\mathbf{1}$, using

the relations $P_+ + P_- = \mathbf{1}$, $P_+^2 = P_+$, $P_-^2 = P_-$, and $P_+P_- = \mathbf{0}$, we get for any $k, l \in \mathbb{N}$, $k \geq 2$,

$$\begin{aligned}
\{\delta_k, H_l\}_2 &= \operatorname{Tr}\left(\frac{\partial \delta_k}{\partial B^\top} A_0 \frac{\partial H_l}{\partial B}\right) = -8l \operatorname{Tr}\left(C^{2k-1} B^\top A_0^2 P_+ \widetilde{\rho}^{2l-1} P_-\right) \\
&= -8l \operatorname{Tr}\left((P_- \widetilde{\rho} P_-)^{2k-1} P_- \widetilde{\rho} P_+ \widetilde{\rho}^{2l-1}\right) \\
&= -8l \operatorname{Tr}\left(P_- \widetilde{\rho} P_- (P_- \widetilde{\rho} P_-)^{2k-2} P_- \widetilde{\rho} P_+ \widetilde{\rho}^{2l-1}\right) \\
&= -8l \operatorname{Tr}\left(\widetilde{\rho}(P_- \widetilde{\rho} P_-)^{2k-2} P_- \widetilde{\rho} P_+ \widetilde{\rho}^{2l-1}\right) \\
&\quad +8l \operatorname{Tr}\left(P_+ \widetilde{\rho}(P_- \widetilde{\rho} P_-)^{2k-2} P_- \widetilde{\rho} P_+ \widetilde{\rho}^{2l-1} P_+\right) \\
&= -8l \operatorname{Tr}\left((P_- \widetilde{\rho} P_-)^{2k-2} P_- \widetilde{\rho} P_+ \widetilde{\rho}^{2l}\right) \\
&\quad +8l \operatorname{Tr}\left((P_- \widetilde{\rho} P_-)^{k-1} \widetilde{\rho} P_+ \widetilde{\rho}^{2l-1} P_+ \widetilde{\rho}(P_- \widetilde{\rho} P_-)^{k-1}\right) \\
&= -8l \operatorname{Tr}\left(P_- \widetilde{\rho}(P_- \widetilde{\rho} P_-)^{2k-3} \widetilde{\rho} P_+ \widetilde{\rho}^{2l}\right) \\
&= -8l \operatorname{Tr}\left(\widetilde{\rho}(P_- \widetilde{\rho} P_-)^{2k-3} P_- \widetilde{\rho} P_+ \widetilde{\rho}^{2l}\right) + 8l \operatorname{Tr}\left(P_+ \widetilde{\rho}(P_- \widetilde{\rho} P_-)^{2k-3} P_- \widetilde{\rho} P_+ \widetilde{\rho}^{2l} P_+\right) \\
&= -8l \operatorname{Tr}\left((P_- \widetilde{\rho} P_-)^{2(k-1)-1} P_- \widetilde{\rho} P_+ \widetilde{\rho}^{2(l+1)-1}\right) \\
&\qquad +8l \operatorname{Tr}\left((P_+ \widetilde{\rho} P_-)(P_- \widetilde{\rho} P_-)^{2k-3} (P_- \widetilde{\rho} P_+)(P_+ \widetilde{\rho}^{2l} P_+)\right) \\
&= -8l \operatorname{Tr}\left(C^{2(k-1)-1} B^\top A_0^2 P_+ \widetilde{\rho}^{2(l+1)-1} P_-\right) = \frac{l}{l+1}\{\delta_{k-1}, H_{l+1}\}_2.
\end{aligned} \tag{2.24}$$

because $(P_+ \widetilde{\rho} P_-)(P_- \widetilde{\rho} P_-)^{2k-3}(P_- \widetilde{\rho} P_+)$ is an antisymmetric 2×2 matrix and $P_+ \widetilde{\rho}^{2l} P_+$ is a symmetric 2×2 matrix. Note that many times we used the fact that a trace is invariant under cyclic permutations and the matrix $(P_- \widetilde{\rho} P_-)^{n-1} \widetilde{\rho} P_+ \widetilde{\rho}^{2m-1} P_+ \widetilde{\rho}(P_- \widetilde{\rho} P_-)^{n-1}$ is antisymmetric. Second, we show that δ_1 commutes with H_l:

$$\begin{aligned}
\{\delta_1, H_l\}_2 &= -8l \operatorname{Tr}\left(C B^\top A_0^2 P_+ \widetilde{\rho}^{2l-1} P_-\right) = -8l \operatorname{Tr}\left(P_- \widetilde{\rho} P_- \widetilde{\rho} P_+ \widetilde{\rho}^{2l-1}\right) \\
&= -8l \operatorname{Tr}\left(\widetilde{\rho} P_- \widetilde{\rho} P_+ \widetilde{\rho}^{2l-1}\right) + 8l \operatorname{Tr}\left(P_+ \widetilde{\rho} P_- \widetilde{\rho} P_+ \widetilde{\rho}^{2l-1} P_+\right) \\
&= -8l \operatorname{Tr}\left(P_- \widetilde{\rho} P_+ \widetilde{\rho}^{2l}\right) - 8l \operatorname{Tr}\left((P_- \widetilde{\rho} P_+) \widetilde{\rho}^{2l-1} (P_- \widetilde{\rho} P_+)^\top\right) \\
&= -8l \operatorname{Tr}\left(\widetilde{\rho} P_+ \widetilde{\rho}^{2l}\right) + 8l \operatorname{Tr}\left((P_+ \widetilde{\rho} P_+)(P_+ \widetilde{\rho}^{2l} P_+)\right) \\
&= -8l \operatorname{Tr}\left(P_+ \widetilde{\rho}^{2l+1} P_+\right) = 0,
\end{aligned} \tag{2.25}$$

because $(P_- \widetilde{\rho} P_+) \widetilde{\rho}^{2l-1} (P_- \widetilde{\rho} P_+)^\top$ is antisymmetric, $P_+ \widetilde{\rho} P_+$ is antisymmetric and $P_+ \widetilde{\rho}^{2l} P_+$ is symmetric, and $P_+ \widetilde{\rho}^{2l+1} P_+$ is antisymmetric.

Formulas (2.24) and (2.25) immediately imply that $\{\delta_k, H_l\}_2 = 0$ for any $k, l \in \mathbb{N}$. Thus the family of functions $\{H_l, \delta_k \mid k, l \in \mathbb{N}\}$ are in involution.

The candidates for the independent integrals depend on whether n is even or odd. If $n = 2p+1$, we take $H_1, \ldots, H_p, \delta_1, \ldots, \delta_{p-1}$ as the system of integrals in involution. Hence we have $2p-1 = n-2$ integrals in involution. If $n = 2p$, we take $H_1, \ldots, H_{p-1}, \delta_1, \ldots, \delta_{p-1}$ as the system of integrals in involution. Hence we have $2p-2 = n-2$ integrals in involution.

Example 2.2 In this example we consider the case when $\widetilde{\rho}$ is 3×3-matrix and assumes the form

$$\widetilde{\rho} = \begin{pmatrix} 0 & a & p_1 \\ -a & 0 & q_1 \\ -p_1 & -q_1 & 0 \end{pmatrix}. \tag{2.26}$$

The Poisson bracket (2.9) becomes

$$\{f, g\}_2(p_1, q_1) = \frac{\partial f}{\partial p_1}\frac{\partial g}{\partial q_1} - \frac{\partial f}{\partial q_1}\frac{\partial g}{\partial p_1} \tag{2.27}$$

for any $f, g \in C^\infty(\mathbb{R}^2)$. In this case we have only one Hamiltonian

$$H_1 = -2(a^2 + p_1^2 + q_1^2), \tag{2.28}$$

which generates Hamilton's equations for the harmonic oscillator

$$\frac{\partial p_1}{\partial t} = \{p_1, H_1\}_2 = -4q_1, \tag{2.29}$$

$$\frac{\partial q_1}{\partial t} = \{q_1, H_1\}_2 = 4p_1. \tag{2.30}$$

Example 2.3 In this example we consider the case when $\widetilde{\rho}$ is 4×4-matrix and assumes the form

$$\widetilde{\rho} = \begin{pmatrix} 0 & a & p_1 & p_2 \\ -a & 0 & q_1 & q_2 \\ -p_1 & -q_1 & 0 & c \\ -p_2 & -q_2 & -c & 0 \end{pmatrix}. \tag{2.31}$$

The Poisson bracket (2.9) becomes

$$\{f, g\}_2(p_i, q_i) = \frac{\partial f}{\partial p_1}\frac{\partial g}{\partial q_1} - \frac{\partial f}{\partial q_1}\frac{\partial g}{\partial p_1} + \frac{\partial f}{\partial p_2}\frac{\partial g}{\partial q_2} - \frac{\partial f}{\partial q_2}\frac{\partial g}{\partial p_2} \tag{2.32}$$

for $f, g \in C^\infty(\mathbb{R}^4)$. The integrals in involution are

$$H_1 = -2(a^2 + c^2 + p_1^2 + p_2^2 + q_1^2 + q_2^2), \tag{2.33}$$

$$\delta_1 = 2c(p_2q_1 - p_1q_2). \tag{2.34}$$

Hamilton's equations for H_1 give again the equations for the harmonic oscillator.

Since $\widetilde{\rho} \in \mathfrak{so}(4)$, the last invariant is the Pfaffian (the square root of the determinant), which for the writing (2.31) equals $ac + p_2q_1 - p_1q_2 = ac + \frac{1}{2c}\delta_1$, i.e., the Pfaffian is equal to δ_1 up to multiplication and addition with constants. If we take $H_2 = 2(a^2 + c^2 + p_1^2 + p_2^2 + q_1^2 + q_2^2)^2 - 4(ac + p_2q_1 - p_1q_2)^2 = \frac{1}{2}H_1^2 - \left(2ac + \frac{1}{c}\delta_1\right)^2$, the same situation occurs: H_2 is a function of H_1 and δ_1.

Example 2.4 In this example we consider the case when $\widetilde{\rho}$ is a 5×5-matrix and assumes the form

$$\widetilde{\rho} = \begin{pmatrix} 0 & a & p_1 & p_2 & p_3 \\ -a & 0 & q_1 & q_2 & q_3 \\ -p_1 & -q_1 & 0 & -c_3 & c_2 \\ -p_2 & -q_2 & c_3 & 0 & -c_1 \\ -p_3 & -q_3 & -c_2 & c_1 & 0 \end{pmatrix}. \tag{2.35}$$

As usual, (2.9) is the standard Poisson bracket

$$\{f, g\}_2(p_i, q_i) = \sum_{i=1}^{3} \left(\frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} \right) \tag{2.36}$$

for $f, g \in C^\infty(\mathbb{R}^6)$. The integrals in involution are

$$H_1 = -2(a^2 + \|\vec{C}\|^2 + \|\vec{p}\|^2 + \|\vec{q}\|^2), \tag{2.37}$$

$$H_2 = \frac{1}{2} H_1^2 - 4(\vec{q} \cdot \vec{C})^2 - 4(\vec{p} \cdot \vec{C})^2 - 4 \left\| -a\vec{C} + \vec{q} \times \vec{p} \right\|^2, \tag{2.38}$$

$$\delta_1 = -2\vec{C} \cdot (\vec{q} \times \vec{p}), \tag{2.39}$$

where $\vec{p} = (p_1, p_2, p_3)$, $\vec{q} = (q_1, q_2, q_3)$, and $\vec{C} = (c_1, c_2, c_3)$. Hamilton's equations for H_2 are

$$\frac{\partial \vec{p}}{\partial t} = 8 \left(-\frac{1}{2} H_1 \vec{q} - \left(\vec{q} \cdot \vec{C} \right) \vec{C} + \vec{p} \times (\vec{p} \times \vec{q}) - a\vec{C} \times \vec{p} \right), \tag{2.40}$$

$$\frac{\partial \vec{q}}{\partial t} = 8 \left(\frac{1}{2} H_1 \vec{p} + \left(\vec{p} \cdot \vec{C} \right) \vec{C} + \vec{q} \times (\vec{p} \times \vec{q}) - a\vec{C} \times \vec{q} \right). \tag{2.41}$$

3 Hierarchy generated by Casimir functions of $\mathfrak{so}_\epsilon(n)$

In this section we consider some deformation the Lie algebra $\mathfrak{so}(n)$ and the hierarchy generated by the Casimir functions for this deformation.

3.1 The Deformed Lie Algebra

Consider the deformation of the Lie algebra $\mathfrak{so}(n)$ given by

$$\begin{pmatrix} 0 & \epsilon a & \epsilon \vec{p} \\ -a & 0 & \vec{q} \\ -\vec{p}^\top & -\vec{q}^\top & C \end{pmatrix} \in \mathfrak{so}_\epsilon(n), \quad \epsilon, a \in \mathbb{R}, \quad \vec{p}, \vec{q} \in \mathbb{R}^{n-2}, \quad C \in \mathfrak{so}(n-2). \tag{3.1}$$

Since

$$\left[\begin{pmatrix} 0 & \epsilon a_1 & \epsilon \vec{p}_1 \\ -a_1 & 0 & \vec{q}_1 \\ -\vec{p}_1^\top & -\vec{q}_1^\top & C_1 \end{pmatrix}, \begin{pmatrix} 0 & \epsilon a_2 & \epsilon \vec{p}_2 \\ -a_2 & 0 & \vec{q}_2 \\ -\vec{p}_2^\top & -\vec{q}_2^\top & C_2 \end{pmatrix} \right]$$

$$= \begin{pmatrix} 0 & \epsilon(\vec{p}_2 \cdot \vec{q}_1 - \vec{p}_1 \cdot \vec{q}_2) & \epsilon\,(a_1\vec{q}_2 - a_2\vec{q}_1 + \vec{p}_1 C_2 - \vec{p}_2 C_1) \\ -(\vec{p}_2 \cdot \vec{q}_1 - \vec{p}_1 \cdot \vec{q}_2) & 0 & \epsilon\,(a_2\vec{p}_1 - a_1\vec{p}_2) + \vec{q}_1 C_2 - \vec{q}_2 C_1 \\ \begin{array}{c} -(a_1\vec{q}_2 - a_2\vec{q}_1 \\ + \vec{p}_1 C_2 - \vec{p}_2 C_1)^\top \end{array} & \begin{array}{c} -(\epsilon\,(a_2\vec{p}_1 - a_1\vec{p}_2) \\ + \vec{q}_1 C_2 - \vec{q}_2 C_1)^\top \end{array} & \begin{array}{c} \epsilon\left(\vec{p}_2^\top \vec{p}_1 - \vec{p}_1^\top \vec{p}_1\right) + \vec{q}_2^\top \vec{q}_1 \\ -\vec{q}_1^\top \vec{q}_1 + [C_1, C_2] \end{array} \end{pmatrix}$$

it follows that the set of elements of the form (3.1) is a Lie algebra relative to the standard matrix commutator. This was proved in [13] in a more general situation. The space $\mathcal{L}_+$ is also dual to the Lie algebra $\mathfrak{so}_\epsilon(n)$ by the pairing given by the trace of the product of matrices (2.1). Thus $C^\infty(\mathcal{L}_+)$ is endowed with the Lie–Poisson bracket

$$\{f, g\}_{1,\epsilon} = \mathrm{Tr}\left(\rho \left[\frac{\partial f}{\partial \rho}, \frac{\partial g}{\partial \rho} \right] \right), \tag{3.2}$$

where

$$\frac{\partial f}{\partial \rho} = \begin{pmatrix} 0 & \epsilon \frac{\partial f}{\partial a} & \epsilon \frac{\partial f}{\partial \vec{p}} \\ -\frac{\partial f}{\partial a} & 0 & \frac{\partial f}{\partial \vec{q}} \\ -\frac{\partial f}{\partial \vec{p}^\top} & -\frac{\partial f}{\partial \vec{q}^\top} & \frac{\partial f}{\partial C} \end{pmatrix}. \tag{3.3}$$

It easy to see that if $\epsilon = 1$ then the Poisson bracket (3.2) coincides with (2.4). If $\epsilon = 0$, the Poisson bracket (3.2) coincides with the Lie–Poisson bracket on the dual of the Euclidean algebra $\mathfrak{se}(n-1)$. Considering the second compatible Poisson bracket associated to (3.2) for the constant matrix ρ_0 given in (2.6), we obtain the Poisson bracket (2.9).

If $\rho \in \mathcal{L}_+$, define

$$\widetilde{\rho}_\epsilon := \begin{pmatrix} 0 & a & \vec{p} \\ -\frac{1}{\epsilon} a & 0 & \vec{q} \\ -\frac{1}{\epsilon} \vec{p}^\top & -\vec{q}^\top & C \end{pmatrix} \in \mathfrak{so}_\epsilon(n).$$

It is easy to see that the elements of the form $\widetilde{\rho}_\epsilon$ are characterized among the matrices in $\mathfrak{sl}(n, \mathbb{R})$ by the condition

$$\widetilde{\rho}_\epsilon \eta + \eta \widetilde{\rho}_\epsilon^\top = 0, \tag{3.4}$$

where

$$\eta := \begin{pmatrix} \epsilon & 0 & \vec{0} \\ 0 & 1 & \vec{0} \\ \vec{0}^\top & \vec{0}^\top & \mathbf{1} \end{pmatrix}. \tag{3.5}$$

3.2 The First Family of Functions in Involution

Let us show that the Casimir functions of (3.2) (see [13])

$$H_{k,\epsilon} = \epsilon^k \operatorname{Tr} \widetilde{\rho}_\epsilon^{2k} = \epsilon^k \operatorname{Tr} \begin{pmatrix} 0 & a & \vec{p} \\ -\frac{1}{\epsilon} a & 0 & \vec{q} \\ -\frac{1}{\epsilon} \vec{p}^\top & -\vec{q}^\top & C \end{pmatrix}^{2k}, \quad k \in \mathbb{N}, \tag{3.6}$$

are in involution with respect to the Poisson bracket (2.9). We begin by noticing that if $\widetilde{\rho}_\epsilon \in \mathfrak{so}_\epsilon(n)$ then $\widetilde{\rho}_\epsilon^{2k-1} \in \mathfrak{so}_\epsilon(n)$. Indeed, since

$$\widetilde{\rho}_\epsilon = -\eta \widetilde{\rho}_\epsilon^\top \eta^{-1}, \tag{3.7}$$

we get

$$\widetilde{\rho}_\epsilon^{2k-1} = -\eta \left(\widetilde{\rho}_\epsilon^{2k-1} \right)^\top \eta^{-1} \tag{3.8}$$

which means that $\widetilde{\rho}_\epsilon^{2k-1} \in \mathfrak{so}_\epsilon(n)$. The even powers of $\widetilde{\rho}_\epsilon$ are deformed symmetric matrices, i.e., they satisfy the conditions

$$\widetilde{\rho}_\epsilon^{2k} \eta - \eta \left(\widetilde{\rho}_\epsilon^{2k} \right)^\top = 0 \tag{3.9}$$

as an easy verification shows.

Now we prove that $\{H_{k,\epsilon}, H_{l,\epsilon}\}_2 = 0$. Since

$$\frac{\partial H_{k,\epsilon}}{\partial B} = -4k\epsilon^k \begin{pmatrix} \frac{1}{\epsilon} & 0 \\ 0 & 1 \end{pmatrix} P_+ \widetilde{\rho}_\epsilon^{2k-1} P_-, \tag{3.10}$$

$$\frac{\partial H_{k,\epsilon}}{\partial B^\top} = 4k\epsilon^k P_- \widetilde{\rho}_\epsilon^{2k-1} P_+, \tag{3.11}$$

substitution in the Poisson bracket (2.9) yields

$$\begin{aligned}
\{H_{k,\epsilon}, H_{l,\epsilon}\}_2 &= \operatorname{Tr}\left(\frac{\partial H_{k,\epsilon}}{\partial B^\top} A_0 \frac{\partial H_{l,\epsilon}}{\partial B}\right) \\
&= -16kl\epsilon^{k+l} \operatorname{Tr}\left(P_- \widetilde{\rho}_\epsilon^{2k-1} P_+ A_0 \begin{pmatrix} \frac{1}{\epsilon} & 0 \\ 0 & 1 \end{pmatrix} P_+ \widetilde{\rho}_\epsilon^{2l-1} P_-\right) \\
&= -16kl\epsilon^{k+l} \operatorname{Tr}\left(\widetilde{\rho}_\epsilon^{2k-1} P_+ \begin{pmatrix} 0 & 1 \\ -\frac{1}{\epsilon} & 0 \end{pmatrix} P_+ \widetilde{\rho}_\epsilon^{2l-1}\right) \\
&\qquad +16kl \operatorname{Tr}\left(P_+ \widetilde{\rho}_\epsilon^{2k-1} P_+ \begin{pmatrix} 0 & 1 \\ -\frac{1}{\epsilon} & 0 \end{pmatrix} P_+ \widetilde{\rho}_\epsilon^{2l-1} P_+\right) \\
&= -16kl\epsilon^{k+l} \left(\operatorname{Tr}\begin{pmatrix} 0 & 1 \\ -\frac{1}{\epsilon} & 0 \end{pmatrix} P_+ \widetilde{\rho}_\epsilon^{2l+2k-2} P_+\right) \\
&\qquad +16kl \operatorname{Tr}\left(P_+ \widetilde{\rho}_\epsilon^{2k-1} P_+ \begin{pmatrix} 0 & 1 \\ -\frac{1}{\epsilon} & 0 \end{pmatrix} P_+ \widetilde{\rho}_\epsilon^{2l-1} P_+\right) = 0.
\end{aligned} \tag{3.12}$$

The first summand vanishes because it is the product of two 2×2 matrices, one of them a deformed antisymmetric matrix and the second a deformed symmetric matrix

$$\begin{pmatrix} 0 & 1 \\ -\frac{1}{\epsilon} & 0 \end{pmatrix} \begin{pmatrix} b & c \\ \frac{1}{\epsilon} c & d \end{pmatrix} = \begin{pmatrix} \frac{1}{\epsilon} c & d \\ -\frac{1}{\epsilon} b & -\frac{1}{\epsilon} c \end{pmatrix} \tag{3.13}$$

and hence the trace vanishes. The second summand vanishes because it is the product of three deformed antisymmetric 2×2 matrices.

3.3 Involutivity of the Full Set of Functions

In addition to the involutive family of integrals $H_{k,\epsilon}$ we have the involutive family of integrals δ_k given by (2.22), as we have seen in §2. Next, we prove that δ_1 commutes with $H_{l,\epsilon}$. Indeed,

$$\begin{aligned}
\{\delta_1, H_{l,\epsilon}\}_2 &= \operatorname{Tr}\left(\frac{\partial \delta_1}{\partial B^\top} A_0 \frac{\partial H_{l,\epsilon}}{\partial B}\right) = -8l\epsilon^l \operatorname{Tr}\left(C B^\top A_0^2 \begin{pmatrix} \frac{1}{\epsilon} & 0 \\ 0 & 1 \end{pmatrix} P_+ \widetilde{\rho}_\epsilon^{2l-1} P_-\right) \\
&= -8l\epsilon^l \operatorname{Tr}\left(P_- \widetilde{\rho}_\epsilon P_- \widetilde{\rho}_\epsilon P_+ \widetilde{\rho}_\epsilon^{2l-1} P_-\right) \\
&= -8l\epsilon^l \operatorname{Tr}\left(\widetilde{\rho}_\epsilon P_- \widetilde{\rho}_\epsilon P_+ \widetilde{\rho}_\epsilon^{2l-1}\right) + 8l\epsilon^l \operatorname{Tr}\left(P_+ \widetilde{\rho}_\epsilon P_- \widetilde{\rho}_\epsilon P_+ \widetilde{\rho}_\epsilon^{2l-1} P_+\right) \\
&= -8l\epsilon^l \left[\operatorname{Tr}\left(\widetilde{\rho}_\epsilon P_+ \widetilde{\rho}_\epsilon^{2l}\right) - \operatorname{Tr}\left(P_+ \widetilde{\rho}_\epsilon P_+ \widetilde{\rho}_\epsilon^{2l} P_+\right) - \operatorname{Tr}\left(P_- \widetilde{\rho}_\epsilon P_+ \widetilde{\rho}_\epsilon^{2l-1} P_+ \widetilde{\rho}_\epsilon P_-\right)\right] \\
&= -8l\epsilon^l \operatorname{Tr}\left(\widetilde{\rho}_\epsilon P_+ \widetilde{\rho}_\epsilon^{2l}\right) = -8l\epsilon^l \operatorname{Tr}\left(P_+ \widetilde{\rho}_\epsilon^{2l+1} P_+\right) = 0.
\end{aligned} \tag{3.14}$$

In the proof we used the following properties:

1. in the third equality we used the identity

$$B^\top \begin{pmatrix} \frac{1}{\epsilon} & 0 \\ 0 & 1 \end{pmatrix} = -P_- \widetilde{\rho}_\epsilon P_+; \tag{3.15}$$

2. the matrix $P_-\widetilde{\rho}_\epsilon P_+\widetilde{\rho}_\epsilon^{2l-1}P_+\widetilde{\rho}_\epsilon P_-$ is antisymmetric because it is of the form

$$\left(\tfrac{1}{\epsilon}\vec{p}^\top\ \vec{q}^\top\right)\begin{pmatrix}0 & c\\ -\frac{1}{\epsilon}c & 0\end{pmatrix}\begin{pmatrix}\vec{p}\\ \vec{q}\end{pmatrix} = -\frac{1}{\epsilon}c\left(\vec{q}^\top\vec{p}-\vec{p}^\top\vec{q}\right); \tag{3.16}$$

3. the matrix $P_+\widetilde{\rho}_\epsilon P_+\widetilde{\rho}_\epsilon^{2l}P_+$ is a product of a deformed antisymmetric and a deformed symmetric matrix like in (3.13) which has zero trace.

Next, we show that $\{\delta_k, H_{l,\epsilon}\}_2 = 0$. Indeed,

$$\begin{aligned}
\{\delta_k, H_{l,\epsilon}\}_2 &= \operatorname{Tr}\left(\frac{\partial\delta_k}{\partial B^\top}A_0\frac{\partial H_{l,\epsilon}}{\partial B}\right)\\
&= -8l\epsilon^l\operatorname{Tr}\left(C^{2k-1}B^\top A_0^2\begin{pmatrix}\frac{1}{\epsilon} & 0\\ 0 & 1\end{pmatrix}P_+\widetilde{\rho}_\epsilon^{2l-1}P_-\right)\\
&\overset{(3.15)}{=} -8l\epsilon^l\operatorname{Tr}\left((P_-\widetilde{\rho}_\epsilon P_-)^{2k-1}P_-\widetilde{\rho}_\epsilon P_+\widetilde{\rho}_\epsilon^{2l-1}P_-\right)\\
&= -8l\epsilon^l\operatorname{Tr}\left(P_-\widetilde{\rho}_\epsilon P_-(P_-\widetilde{\rho}_\epsilon P_-)^{2k-2}P_-\widetilde{\rho}_\epsilon P_+\widetilde{\rho}_\epsilon^{2l-1}P_-\right)\\
&= -8l\epsilon^l\operatorname{Tr}\left(\widetilde{\rho}_\epsilon(P_-\widetilde{\rho}_\epsilon P_-)^{2k-2}P_-\widetilde{\rho}_\epsilon P_+\widetilde{\rho}_\epsilon^{2l-1}\right)\\
&\qquad +8l\epsilon^l\operatorname{Tr}\left(P_+\widetilde{\rho}_\epsilon P_-(P_-\widetilde{\rho}_\epsilon P_-)^{2k-2}P_-\widetilde{\rho}_\epsilon P_+\widetilde{\rho}_\epsilon^{2l-1}P_+\right)\\
&= -8l\epsilon^l\operatorname{Tr}\left((P_-\widetilde{\rho}_\epsilon P_-)^{2k-2}P_-\widetilde{\rho}_\epsilon P_+\widetilde{\rho}_\epsilon^{2l}\right)\\
&\qquad +8l\epsilon^l\operatorname{Tr}\left((P_-\widetilde{\rho}_\epsilon P_+)(P_+\widetilde{\rho}_\epsilon^{2l-1}P_+)(P_+\widetilde{\rho}_\epsilon P_-)(P_-\widetilde{\rho}_\epsilon P_-)^{2k-2}\right)\\
&= -8l\epsilon^l\operatorname{Tr}\left(P_-\widetilde{\rho}_\epsilon(P_-\widetilde{\rho}_\epsilon P_-)^{2k-3}\widetilde{\rho}_\epsilon P_+\widetilde{\rho}_\epsilon^{2l}\right)\\
&= -8l\epsilon^l\operatorname{Tr}\left(\widetilde{\rho}_\epsilon(P_-\widetilde{\rho}_\epsilon P_-)^{2k-3}P_-\widetilde{\rho}_\epsilon P_+\widetilde{\rho}_\epsilon^{2l}\right)\\
&\qquad +8l\epsilon^l\operatorname{Tr}\left(\left(P_+\widetilde{\rho}_\epsilon P_-(P_-\widetilde{\rho}_\epsilon P_-)^{2k-3}P_-\widetilde{\rho}_\epsilon P_+\right)P_+\widetilde{\rho}_\epsilon^{2l}P_+\right)\\
&\overset{(3.15)}{=} -8l\epsilon^l\operatorname{Tr}\left((P_-\widetilde{\rho}_\epsilon P_-)^{2(k-1)-1}P_-\widetilde{\rho}_\epsilon P_+\widetilde{\rho}_\epsilon^{2(l+1)-1}\right)\\
&= -8l\epsilon^l\operatorname{Tr}\left(C^{2(k-1)-1}B^\top A_0^2\begin{pmatrix}\frac{1}{\epsilon} & 0\\ 0 & 1\end{pmatrix}P_+\widetilde{\rho}_\epsilon^{2(l+1)-1}P_-\right)\\
&= \frac{l}{(l+1)\epsilon}\{\delta_{k-1}, H_{l+1,\epsilon}\}_2.
\end{aligned} \tag{3.17}$$

In the seventh equality we use $\operatorname{Tr}\left((P_-\widetilde{\rho}_\epsilon P_+)(P_+\widetilde{\rho}_\epsilon^{2l-1}P_+)(P_+\widetilde{\rho}_\epsilon P_-)(P_-\widetilde{\rho}_\epsilon P_-)^{2k-2}\right) = 0$, because the matrix $(P_-\widetilde{\rho}_\epsilon P_+)(P_+\widetilde{\rho}_\epsilon^{2l-1}P_+)(P_+\widetilde{\rho}_\epsilon P_-)$ is antisymmetric (like in (3.16)) and the matrix $(P_-\widetilde{\rho}_\epsilon P_-)^{2k-2} = C^{2k-2}$ is symmetric. In the ninth equality, we use the vanishing of the trace of the matrix $\left(P_+\widetilde{\rho}_\epsilon P_-(P_-\widetilde{\rho}_\epsilon P_-)^{2k-3}P_-\widetilde{\rho}_\epsilon P_+\right)P_+\widetilde{\rho}_\epsilon^{2l}P_+$. Indeed,

$$\begin{aligned}
-P_+\widetilde{\rho}_\epsilon P_-(P_-\widetilde{\rho}_\epsilon P_-)^{2k-3}P_-\widetilde{\rho}_\epsilon P_+ &= \begin{pmatrix}\vec{p}\\ \vec{q}\end{pmatrix}C^{2k-3}\left(\tfrac{1}{\epsilon}\vec{p}^\top\ \vec{q}^\top\right)\\
&= \begin{pmatrix}\frac{1}{\epsilon}\vec{p}C^{2k-3}\vec{p}^\top & \vec{p}C^{2k-3}\vec{q}^\top\\ \frac{1}{\epsilon}\vec{q}C^{2k-3}\vec{p}^\top & \vec{q}C^{2k-3}\vec{q}^\top\end{pmatrix}
\end{aligned} \tag{3.18}$$

is deformed antisymmetric and $P_+\widetilde{\rho}_\epsilon^{2l}P_+$ is deformed symmetric, like in (3.13).

Equations (3.14) and (3.17) show that $\{\delta_k, H_{l,\epsilon}\}_2 = 0$ for all $k, l \in \mathbb{N}$. Thus the family of functions $\{H_{l,\epsilon}, \delta_k \mid k, l \in \mathbb{N}\}$ are in involution.

The candidates for the independent integrals depend on whether n is even or odd. If $n = 2p + 1$, we take $H_{1,\epsilon}, \ldots, H_{p,\epsilon}, \delta_1, \ldots, \delta_{p-1}$ as the system of integrals in involution. Hence we have $2p - 1 = n - 2$ integrals in involution. If $n = 2p$, we take $H_{1,\epsilon}, \ldots, H_{p-1,\epsilon}, \delta_1, \ldots, \delta_{p-1}$ as the system of integrals in involution. Hence we have $2p - 2 = n - 2$ integrals in involution.

Example 3.1 For $k = 1, 2$ the Hamiltonians $H_{k,\epsilon}$ are

$$H_{1,\epsilon} = -2\left(a^2 + \|\vec{p}\|^2 + \epsilon\|\vec{q}\|^2\right) + \epsilon \operatorname{Tr} C^2 \tag{3.19}$$

$$\begin{aligned} H_{2,\epsilon} = {} & 2a^4 + \epsilon^2 \operatorname{Tr} C^4 + 2\|\vec{p}\|^4 + 2\epsilon^2\|\vec{q}\|^4 + 4a^2\|\vec{p}\|^2 + 4\epsilon a^2\|\vec{q}\|^2 \\ & + 4\epsilon\,(\vec{p}\cdot\vec{q})^2 + 8\epsilon a\,\vec{p}C\cdot\vec{q} - 4\epsilon^2\vec{q}C^2\cdot\vec{q} - 4\epsilon\,\vec{p}C^2\cdot\vec{p}. \end{aligned} \tag{3.20}$$

Let us consider the case when $\widetilde{\rho}_\epsilon$ is 5×5 matrix, so $\vec{p}, \vec{q} \in \mathbb{R}^3$,

$$C = \begin{pmatrix} 0 & -c_3 & c_2 \\ c_3 & 0 & -c_1 \\ -c_2 & c_1 & 0 \end{pmatrix} \in \mathfrak{so}(3),$$

and define $\vec{C} := (c_1, c_2, c_3)$. Thus, $\operatorname{Tr} C^2 = -2\|\vec{C}\|^2$, $\operatorname{Tr} C^4 = 2\|\vec{C}\|^4$, $\vec{q}C = \vec{q} \times \vec{C}$, and $\vec{q}C^2 = \left(\vec{C}\cdot\vec{q}\right)\vec{C} - \left\|\vec{C}\right\|^2\vec{q}$.

The integrals (3.19) and (3.20) have thus the form

$$H_{1,\epsilon} = -2(a^2 + \epsilon\|\vec{C}\|^2 + \|\vec{p}\|^2 + \epsilon\|\vec{q}\|^2), \tag{3.21}$$

$$H_{2,\epsilon} = \frac{1}{2}H_{1,\epsilon}^2 - 4\epsilon\left(\epsilon(\vec{q}\cdot\vec{C})^2 + (\vec{p}\cdot\vec{C})^2 + \left\|-a\vec{C} + \vec{q}\times\vec{p}\right\|^2\right), \tag{3.22}$$

and we have an additional integral in involution

$$\delta_1 = -2\vec{C}\cdot(\vec{q}\times\vec{p}). \tag{3.23}$$

Combining the Hamiltonians $H_{1,\epsilon}$, $H_{2,\epsilon}$ with δ_1, we find that the three following functions are also in involution

$$h_{1,\epsilon} = \frac{1}{2}\left(\|\vec{p}\|^2 + \epsilon\|\vec{q}\|^2\right), \tag{3.24}$$

$$h_{2,\epsilon} = \frac{1}{2}\epsilon(\vec{q}\cdot\vec{C})^2 + \frac{1}{2}(\vec{p}\cdot\vec{C})^2 + \frac{1}{2}\|\vec{q}\times\vec{p}\|^2 \tag{3.25}$$

and δ_1. Hamilton's equations for $h_{2,\epsilon}$ are

$$\frac{\partial\vec{p}}{\partial t} = \epsilon\left(\vec{q}\cdot\vec{C}\right)\vec{C} - \vec{p}\times(\vec{p}\times\vec{q}), \tag{3.26}$$

$$\frac{\partial\vec{q}}{\partial t} = -\left(\vec{p}\cdot\vec{C}\right)\vec{C} - \vec{q}\times(\vec{p}\times\vec{q}). \tag{3.27}$$

If we let $\epsilon = 0$ then the functions

$$h_{1,0} = \frac{1}{2}\|\vec{p}\|^2, \tag{3.28}$$

$$h_{2,0} = \frac{1}{2}(\vec{p}\cdot\vec{C})^2 + \frac{1}{2}\|\vec{q}\times\vec{p}\|^2 \tag{3.29}$$

and δ_1 are also in involution. Hamilton's equations for $h_{2,0}$ are

$$\frac{\partial \vec{p}}{\partial t} = -\vec{p} \times (\vec{p} \times \vec{q}) = \|\vec{p}\|^2 \vec{q} - (\vec{p} \cdot \vec{q})\, \vec{p}, \tag{3.30}$$

$$\frac{\partial \vec{q}}{\partial t} = -\left(\vec{p} \cdot \vec{C}\right) \vec{C} - \vec{q} \times (\vec{p} \times \vec{q}) = -\left(\vec{p} \cdot \vec{C}\right) \vec{C} + (\vec{p} \cdot \vec{q})\, \vec{q} - \|\vec{q}\|^2 \vec{p}. \tag{3.31}$$

The Hamiltonian $h_{2,0}$ has the symmetry $\vec{q} \mapsto \vec{q} + \lambda \vec{p}$ and $\vec{p} \mapsto \vec{p}$ whose momentum map is $h_{1,0}$ We can perform a Hamiltonian reduction at the value $h_{1,0} = 1/2$. The reduced equations of motion are equivalent to the equation of motion for the classical Neumann system. Indeed, the phase space of the reduced system is TS^2, i.e., it is given by

$$\left\{(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid \|\mathbf{x}\| = 1, \mathbf{x} \cdot \mathbf{y} = 0\right\} \tag{3.32}$$

because the map $h_{1,0}^{-1}(1/2) \ni (\vec{p}, \vec{q}) \mapsto (\mathbf{x}, \mathbf{y}) := ((\vec{p}, \vec{q} - (\vec{p} \cdot \vec{q})\vec{p}) \in TS^2$ descends to a symplectic diffeomorphism of the reduced phase space $h_{1,0}^{-1}(1/2)/\mathbb{R}$ onto TS^2. A direct verification shows that the reduced equations of motion induced by $h_{2,0}$ are

$$\frac{\partial \mathbf{x}}{\partial t} = \mathbf{y}, \tag{3.33}$$

$$\frac{\partial \mathbf{y}}{\partial t} = -\left(\mathbf{x} \cdot \vec{C}\right) \vec{C} - \left(\|\mathbf{y}\|^2 - \left(\mathbf{x} \cdot \vec{C}\right)^2\right) \mathbf{x}, \tag{3.34}$$

i.e., the classical Neumann system equations describing the motion of a particle on S^2 under the influence of the quadratic potential $\frac{1}{2}\mathbf{x}\left(C^2 + \|\vec{C}\|^2 \mathbf{1}\right) \cdot \mathbf{x}$ ([12]; for the N-dimensional generalization see [15]).

We have three functions $h_{1,\epsilon}, h_{2,\epsilon}, \delta_1$ and we calculate

$$\frac{\partial h_{1,\epsilon}}{\partial \vec{p}^\top} = \vec{p}^\top, \tag{3.35}$$

$$\frac{\partial h_{2,\epsilon}}{\partial \vec{p}^\top} = (\vec{p} \cdot \vec{C})\vec{C}^\top + \vec{q}^\top \times \left(\vec{p}^\top \times \vec{q}^\top\right) = (\vec{p} \cdot \vec{C})\vec{C}^\top + ||\vec{q}||^2 \vec{p}^\top - (\vec{p} \cdot \vec{q})\vec{q}^\top, \tag{3.36}$$

$$\frac{\partial \delta_1}{\partial \vec{p}^\top} = -2\vec{C}^\top \times \vec{q}^\top. \tag{3.37}$$

The Jacobian of these functions is given by

$$\begin{aligned} J &= \left| \frac{\partial \left(h_{1,\epsilon}, h_{2,\epsilon}, \delta_1\right)}{\partial \vec{p}^\top} \right| \\ &= 2\left((\vec{p} \cdot \vec{C})\vec{C}^\top - (\vec{p} \cdot \vec{q})\vec{q}^\top\right) \cdot \left(\vec{p}^\top \times \left(\vec{C}^\top \times \vec{q}^\top\right)\right) \\ &= 2\left((\vec{p} \cdot \vec{C})\vec{C}^\top - (\vec{p} \cdot \vec{q})\vec{q}^\top\right) \cdot \left((\vec{p} \cdot \vec{q})\vec{C}^\top - (\vec{p} \cdot \vec{C})\vec{q}^\top\right). \end{aligned} \tag{3.38}$$

Setting the expression in (3.38) equal to zero yields a hypersurface in $\mathbb{R}^6$ and hence on its complement, which is a Zariski open set hence dense in $\mathbb{R}^6$, the Jacobian does not vanish. This proves the independence of the functions $h_{1,\epsilon}, h_{2,\epsilon}, \delta_1$.

Example 3.2 In this example we consider the case when $\widetilde{\rho}_\epsilon$ is 6×6-matrix then the Hamiltonians exactly have the form

$$H_{1,\epsilon} = -2\left(a^2 + \|\vec{p}\|^2 + \epsilon\|\vec{q}\|^2 + \epsilon(c_1^2+c_2^2+c_3^2+c_4^2+c_5^2+c_6^2)\right), \tag{3.39}$$

$$H_{2,\epsilon} = \frac{1}{2}H_{1,\epsilon}^2 - 4\epsilon\left(\epsilon(c_1^2+c_2^2+c_3^2+c_4^2+c_5^2+c_6^2)\|\vec{q}\|^2 + \epsilon\vec{q}C^2\vec{q}^\top + \frac{1}{2}\sum_{k\neq l}J_{kl}^2 + \right.$$
$$\left. +(c_1^2+c_2^2+c_3^2+c_4^2+c_5^2+c_6^2)\|\vec{p}\|^2 + \vec{p}C^2\vec{p}^\top + \epsilon(c_1c_6+c_3c_4-c_2c_5)^2\right), \tag{3.40}$$

and we have additional integrals in involution

$$\delta_1 = 2\vec{q}C\vec{p}^\top, \tag{3.41}$$

$$\delta_2 = 2\vec{q}C^3\vec{p}^\top = -2(c_1^2+c_2^2+c_3^2+c_4^2+c_5^2+c_6^2)\delta_1$$
$$+2(c_1c_6+c_3c_4-c_2c_5)\vec{q}\begin{pmatrix} 0 & c_6 & -c_5 & c_4 \\ -c_6 & 0 & c_3 & -c_2 \\ c_5 & -c_3 & 0 & c_1 \\ -c_4 & c_2 & -c_1 & 0 \end{pmatrix}\vec{p}^\top, \tag{3.42}$$

where

$$C := \begin{pmatrix} 0 & c_1 & c_2 & c_3 \\ -c_1 & 0 & c_4 & c_5 \\ -c_2 & -c_4 & 0 & c_6 \\ -c_3 & -c_5 & -c_6 & 0 \end{pmatrix}, \tag{3.43}$$

$$J_{kl} := aC_{kl} + p_lq_k - p_kq_l, \tag{3.44}$$

and C_{kl} is (k,l) entry of the matrix C. Hamilton's equations for $H_{2,\epsilon}$ are again connected to the Neumann model.

Note that

$$H_{3,\epsilon} = \left(\frac{1}{4} - \frac{3}{8}\epsilon\right)H_{1,\epsilon}^3 + \frac{3}{4}\epsilon H_{2,\epsilon}H_{1,\epsilon}$$
$$-\epsilon^2\left[a(c_1c_6+c_3c_4-c_2c_5) + \frac{1}{2}(\delta_3 + (c_1^2+c_2^2+c_3^2+c_4^2+c_5^2+c_6^2)\delta_1)\right]^2. \tag{3.45}$$

4 Independence of integrals

4.1 Independence of the Integrals $\delta_1, \ldots, \delta_{p-1}$

Let $n = 2p+1$ or $n = 2p$. We shall prove that the functions $\delta_1, \ldots, \delta_{p-1}$ are independent. Note that δ_k given by formula (2.22) can be rewritten in the form

$$\delta_k(\vec{p},\vec{q}) = -2\vec{p}C^{2k-1}\vec{q}^\top. \tag{4.1}$$

From the above we obtain

$$\frac{\partial\delta_k}{\partial\vec{p}} = 2\vec{q}C^{2k-1}. \tag{4.2}$$

The $(n-2)\times(n-2)$ matrix C is antisymmetric, so it can be presented in the form

$$\begin{aligned} C &= Q^\top D Q = Q^\top \begin{pmatrix} 0 & \Lambda \\ -\Lambda & 0 \end{pmatrix} Q \qquad \text{if } n = 2p, \\ C &= Q^\top D Q = Q^\top \begin{pmatrix} 0 & \Lambda & 0 \\ -\Lambda & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} Q \text{ if } n = 2p+1, \end{aligned} \tag{4.3}$$

where $Q \in SO(n-2)$ and $\Lambda = \mathrm{diag}(\lambda_1, \dots, \lambda_{p-1})$. Therefore we have

$$\frac{\partial \delta_k}{\partial \vec{p}} = 2\vec{q}\, Q^\top D^{2k-1} Q = 2\mathbf{v} D^{2k-1} Q, \tag{4.4}$$

where $\mathbf{v} := \vec{q}\, Q^\top$. We consider the generic case $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_{p-1}$. Now we check the linear independence of $\delta_1, \dots, \delta_{p-1}$. Suppose

$$\left(\alpha_1 \mathbf{v} + \alpha_2 \mathbf{v} D^2 + \cdots + \alpha_{p-1} \mathbf{v} D^{2p-4}\right) D Q = 0 \tag{4.5}$$

for $\alpha_1, \dots, \alpha_{p-1} \in \mathbb{R}$. Because

$$\begin{aligned} D^{2m} &= (-1)^m \begin{pmatrix} \Lambda^{2m} & 0 \\ 0 & \Lambda^{2m} \end{pmatrix} \qquad \text{if } n = 2p, \\ D^{2m} &= (-1)^m \begin{pmatrix} \Lambda^{2m} & 0 & 0 \\ 0 & \Lambda^{2m} & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ if } n = 2p+1, \end{aligned} \tag{4.6}$$

if we denote $\mathbf{v} = (\mathbf{u}, \mathbf{w})$ or $\mathbf{v} = (\mathbf{u}, \mathbf{w}, v_3)$, respectively, then the equality (4.5) implies

$$\alpha_1 \mathbf{u} - \alpha_2 \mathbf{u} \Lambda^2 + \cdots + (-1)^p \alpha_{p-1} \mathbf{u} \Lambda^{2(p-2)} = 0, \tag{4.7}$$

or in matrix form

$$\begin{pmatrix} u_1 & \lambda_1^2 u_1 & \dots & \lambda_1^{2(p-2)} u_1 \\ u_2 & \lambda_2^2 u_2 & \dots & \lambda_2^{2(p-2)} u_2 \\ \dots & \dots & \dots & \dots \\ u_{p-1} & \lambda_{p-1}^2 u_{p-1} & \dots & \lambda_{p-1}^{2(p-2)} u_{p-1} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ -\alpha_2 \\ \vdots \\ (-1)^p \alpha_{p-1} \end{pmatrix} = 0, \tag{4.8}$$

where $\mathbf{u} = \left(u_1, \dots, u_{p-1}\right)$. This is a $(p-1)\times(p-1)$ linear homogeneous system with unknowns $\alpha_1, -\alpha_2, \alpha_3, \dots, (-1)^p \alpha_{p-1}$. The determinant of this system is

$$u_1 \cdot \dots \cdot u_{p-1} \begin{vmatrix} 1 & \lambda_1^2 & \dots & \lambda_1^{2(p-2)} \\ 1 & \lambda_2^2 & \dots & \lambda_2^{2(p-2)} \\ \dots & \dots & \dots & \dots \\ 1 & \lambda_{p-1}^2 & \dots & \lambda_{p-1}^{2(p-2)} \end{vmatrix} = u_1 \cdot \dots \cdot u_{p-1} \prod_{1 \le i < j \le p-1} \left(\lambda_j^2 - \lambda_i^2\right). \tag{4.9}$$

Generically, all $u_i \neq 0$ for $i = 1, \dots, p-1$, which implies that $\alpha_i = 0$ for all $i = 1, \dots p-1$, thus proving that $\delta_1, \dots, \delta_{p-1}$ are independent.

4.2 Independence of the Integrals $H_1, \dots H_p$

Let $n = 2p+1$ or $n = 2p$. We shall prove that the functions $H_1, \dots, H_p$ are independent (we will consider only the case when $\epsilon = 1$; the proof when the $\epsilon \neq 1$ is similar). Since H_k

and $\frac{\partial H_k}{\partial B}$ are given by formulas (2.10) and (2.11), respectively, we get

$$\frac{\partial H_k}{\partial \vec{p}} = -4k P_+^1 \widetilde{\rho}^{2k-1} P_- \,, \tag{4.10}$$

where P_+^1, P_- are the orthogonal projectors given, in block matrix notation, by

$$P_+^1 = \left(\begin{array}{cc|c} \mathbf{1} & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & \mathbf{0} \end{array}\right), \quad P_- = \left(\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & \mathbf{1} \end{array}\right). \tag{4.11}$$

The $n \times n$ matrix $\widetilde{\rho}$ is antisymmetric, so it can be presented in the form

$$\begin{aligned} \widetilde{\rho} = R^\top E R = R^\top \left(\begin{array}{cc|cc} 0 & \omega_1 & 0 & 0 \\ -\omega_1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \Omega \\ 0 & 0 & -\Omega & 0 \end{array}\right) R \quad \text{if } n = 2p, \\ \widetilde{\rho} = R^\top E R = R^\top \left(\begin{array}{cc|ccc} 0 & \omega_1 & 0 & 0 & 0 \\ -\omega_1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \Omega & 0 \\ 0 & 0 & -\Omega & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) R \text{ if } n = 2p+1, \end{aligned} \tag{4.12}$$

where $R \in SO(n)$ and $\Omega = \mathrm{diag}(\omega_2, \ldots, \omega_p)$. Therefore, by (4.10), we have

$$\frac{\partial H_k}{\partial \vec{p}} = -4k P_+^1 R^\top E^{2k-1} R P_-. \tag{4.13}$$

We consider the generic case $0 < \omega_1 < \omega_2 < \cdots < \omega_p$ and we write the matrix $R^\top$ in block form as

$$R^\top = \left(\begin{array}{cc|c} a_1 & a_2 & \mathbf{b_1} \\ a_3 & a_4 & \mathbf{b_2} \\ \hline \mathbf{d}_1^\top & \mathbf{d}_2^\top & \mathbf{G} \end{array}\right), \tag{4.14}$$

with $\mathbf{b_1} = \left(b_{11}, \cdots, b_{1,n-2}\right)$, $\mathbf{b_2} = \left(b_{21}, \cdots, b_{2,n-2}\right)$, $\mathbf{d_1} = \left(d_{11}, \cdots, d_{1,n-2}\right)$, $\mathbf{d_2} = \left(d_{21}, \cdots, d_{2,n-2}\right)$ and $\mathbf{G} \in \mathfrak{gl}(n-2, \mathbb{R})$. After a simple calculation, formula (4.13) can be rewritten in the form

$$\begin{aligned} \frac{\partial H_k}{\partial \vec{p}} = (-1)^k 4k \Bigg(& (-\omega_1^k a_2, \omega_1^k a_1) \begin{pmatrix} \mathbf{d_1} \\ \mathbf{d_2} \end{pmatrix} \\ & + (-\omega_2^k b_{1p}, -\omega_3^k b_{1,p+1}, \ldots, -\omega_p^k b_{1,2p-2}, \omega_2^k b_{11}, \omega_3^k b_{12}, \ldots, \omega_p^k b_{1,p-1}) \mathbf{G}^\top \Bigg), \end{aligned} \tag{4.15}$$

if $n = 2p$ and in the form

$$\begin{aligned} \frac{\partial H_k}{\partial \vec{p}} = (-1)^k 4k \Bigg(& (-\omega_1^k a_2, \omega_1^k a_1) \begin{pmatrix} \mathbf{d_1} \\ \mathbf{d_2} \end{pmatrix} \\ & + (-\omega_2^k b_{1p}, -\omega_3^k b_{1,p+1}, \ldots, -\omega_p^k b_{1,2p-2}, \omega_2^k b_{11}, \omega_3^k b_{12}, \ldots, \omega_p^k b_{1,p-1}, 0) \mathbf{G}^\top \Bigg), \end{aligned} \tag{4.16}$$

if $n = 2p + 1$. Assume, generically, that $\mathbf{G}^\top$ has an inverse and denote

$$\begin{pmatrix} \mathbf{d_1'} \\ \mathbf{d_2'} \end{pmatrix} = \begin{pmatrix} \mathbf{d_1} \\ \mathbf{d_2} \end{pmatrix} \left(\mathbf{G}^\top\right)^{-1}. \tag{4.17}$$

Now we check the linear independence of the differentials of $H_1, \ldots, H_p$ at a generic point $\begin{pmatrix} \vec{p} \\ \vec{q} \end{pmatrix} \in \mathbb{R}^{2(n-2)}$. Suppose

$$\beta_1 \frac{\partial H_1}{\partial \vec{p}} + \beta_2 \frac{\partial H_2}{\partial \vec{p}} + \cdots + \beta_p \frac{\partial H_p}{\partial \vec{p}} = 0. \tag{4.18}$$

For $n = 2p$, this is equivalent to the linear homogeneous system

$$\begin{pmatrix} \omega_1 d_1'' - \omega_2 b_{1p} & \omega_1^3 d_1'' - \omega_2^3 b_{1p} & \ldots\ \omega_1^{2p-1} d_1'' - \omega_2^{2p-1} b_{1p} \\ \omega_1 d_2'' - \omega_3 b_{1,p+1} & \omega_1^3 d_2'' - \omega_3^3 b_{1,p+1} & \ldots\ \omega_1^{2p-1} d_2'' - \omega_3^{2p-1} b_{1,p+1} \\ \ldots & \ldots & \ldots\ldots \\ \omega_1 d_{p-1}'' - \omega_p b_{1,2p-2} & \omega_1^3 d_{p-1}'' - \omega_p^3 b_{1,2p-2} & \ldots\ \omega_1^{2p-1} d_{p-1}'' - \omega_p^{2p-1} b_{1,2p-2} \\ \omega_1 d_p'' + \omega_2 b_{11} & \omega_1^3 d_p'' + \omega_2^3 b_{11} & \ldots\ \omega_1^{2p-1} d_p'' + \omega_2^{2p-1} b_{11} \\ \omega_1 d_{p+1}'' + \omega_3 b_{12} & \omega_1^3 d_{p+1}'' + \omega_3^3 b_{12} & \ldots\ \omega_1^{2p-1} d_{p+1}'' + \omega_3^{2p-1} b_{12} \\ \ldots & \ldots & \ldots\ldots \\ \omega_1 d_{2p-2}'' + \omega_p b_{1,p-1} & \omega_1^3 d_{2p-2}'' + \omega_p^3 b_{1,p-1} & \ldots\ \omega_1^{2p-1} d_{2p-2}'' + \omega_p^{2p-1} b_{1,p-1} \end{pmatrix}$$
$$\times \begin{pmatrix} \beta_1 \\ -\beta_2 \\ \vdots \\ (-1)^{p+1}\beta_p \end{pmatrix} = 0 \tag{4.19}$$

and if $n = 2p + 1$ to the linear homogeneous system

$$\begin{pmatrix} \omega_1 d_1'' - \omega_2 b_{1p} & \omega_1^3 d_1'' - \omega_2^3 b_{1p} & \ldots\ \omega_1^{2p-1} d_1'' - \omega_2^{2p-1} b_{1p} \\ \omega_1 d_2'' - \omega_3 b_{1,p+1} & \omega_1^3 d_2'' - \omega_3^3 b_{1,p+1} & \ldots\ \omega_1^{2p-1} d_2'' - \omega_3^{2p-1} b_{1,p+1} \\ \ldots & \ldots & \ldots\ldots \\ \omega_1 d_{p-1}'' - \omega_p b_{1,2p-2} & \omega_1^3 d_{p-1}'' - \omega_p^3 b_{1,2p-2} & \ldots\ \omega_1^{2p-1} d_{p-1}'' - \omega_p^{2p-1} b_{1,2p-2} \\ \omega_1 d_p'' + \omega_2 b_{11} & \omega_1^3 d_p'' + \omega_2^3 b_{11} & \ldots\ \omega_1^{2p-1} d_p'' + \omega_2^{2p-1} b_{11} \\ \omega_1 d_{p+1}'' + \omega_3 b_{12} & \omega_1^3 d_{p+1}'' + \omega_3^3 b_{12} & \ldots\ \omega_1^{2p-1} d_{p+1}'' + \omega_3^{2p-1} b_{12} \\ \ldots & \ldots & \ldots\ldots \\ \omega_1 d_{2p-2}'' + \omega_p b_{1,p-1} & \omega_1^3 d_{2p-2}'' + \omega_p^3 b_{1,p-1} & \ldots\ \omega_1^{2p-1} d_{2p-2}'' + \omega_p^{2p-1} b_{1,p-1} \\ \omega_1 d_{2p-1}'' & \omega_1^3 d_{2p-1}'' & \ldots\ \omega_1^{2p-1} d_{2p-1}'' \end{pmatrix}$$
$$\times \begin{pmatrix} \beta_1 \\ -\beta_2 \\ \vdots \\ (-1)^{p+1}\beta_p \end{pmatrix} = 0, \tag{4.20}$$

where $d_i'' := -a_2 d_{1i}' + a_1 d_{2i}'$. Note that the big matrices have p columns and $2p - 2$, respectively $2p - 1$, rows. For the proof, it suffices to show that the $p \times p$ determinant

$$\begin{vmatrix} \omega_1 d_1'' - \omega_2 b_{1p} & \omega_1^3 d_1'' - \omega_2^3 b_{1p} & \ldots\ \omega_1^{2p-1} d_1'' - \omega_2^{2p-1} b_{1p} \\ \omega_1 d_2'' - \omega_3 b_{1,p+1} & \omega_1^3 d_2'' - \omega_3^3 b_{1,p+1} & \ldots\ \omega_1^{2p-1} d_2'' - \omega_3^{2p-1} b_{1,p+1} \\ \ldots & \ldots & \ldots\ldots \\ \omega_1 d_{p-1}'' - \omega_p b_{1,2p-2} & \omega_1^3 d_{p-1}'' - \omega_p^3 b_{1,2p-2} & \ldots\ \omega_1^{2p-1} d_{p-1}'' - \omega_p^{2p-1} b_{1,2p-2} \\ \omega_1 d_p'' + \omega_2 b_{11} & \omega_1^3 d_p'' + \omega_2^3 b_{11} & \ldots\ \omega_1^{2p-1} d_p'' + \omega_2^{2p-1} b_{11} \end{vmatrix} \tag{4.21}$$

does not vanish for generic values of b_{1j}. Operating on the columns gives the determinant

$$\begin{vmatrix} \omega_1 d_1'' - \omega_2 b_{1p} & \left(-\omega_2^3 + \omega_2\omega_1^2\right) b_{1p} & \dots \left(-\omega_2^{2p-1} + \omega_2^{2p-3}\omega_1^2\right) b_{1p} \\ \omega_1 d_2'' - \omega_3 b_{1,p+1} & \left(-\omega_3^3 + \omega_3\omega_1^2\right) b_{1,p+1} & \dots \left(-\omega_3^{2p-1} + \omega_3^{2p-3}\omega_1^2\right) b_{1,p+1} \\ \dots & \dots & \dots\dots \\ \omega_1 d_{p-1}'' - \omega_p b_{1,2p-2} & \left(-\omega_p^3 + \omega_p\omega_1^2\right) b_{1,2p-2} & \dots \left(-\omega_p^{2p-1} + \omega_p^{2p-3}\omega_1^2\right) b_{1,2p-2} \\ \omega_1 d_p'' + \omega_2 b_{11} & -\left(-\omega_2^3 + \omega_2\omega_1^2\right) b_{11} & \dots -\left(-\omega_2^{2p-1} + \omega_2^{2p-3}\omega_1^2\right) b_{11} \end{vmatrix} \tag{4.22}$$

and, assuming that $b_{1p} \neq 0$ and operating on the rows, yields

$$\begin{aligned} &\begin{vmatrix} \omega_1 d_1'' - \omega_2 b_{1p} & \left(-\omega_2^3 + \omega_2\omega_1^2\right) b_{1p} & \dots \left(-\omega_2^{2p-1} + \omega_2^{2p-3}\omega_1^2\right) b_{1p} \\ \omega_1 d_2'' - \omega_3 b_{1,p+1} & \left(-\omega_3^3 + \omega_3\omega_1^2\right) b_{1,p+1} & \dots \left(-\omega_3^{2p-1} + \omega_3^{2p-3}\omega_1^2\right) b_{1,p+1} \\ \dots & \dots & \dots\dots \\ \omega_1 d_{p-1}'' - \omega_p b_{1,2p-2} & \left(-\omega_p^3 + \omega_p\omega_1^2\right) b_{1,2p-2} & \dots \left(-\omega_p^{2p-1} + \omega_p^{2p-3}\omega_1^2\right) b_{1,2p-2} \\ \omega_1 \left(d_1'' \frac{b_{11}}{b_{1p}} + d_p''\right) & 0 & \dots 0 \end{vmatrix} \\ &= (-1)^{p+1}\omega_1 \left(d_1'' \frac{b_{11}}{b_{1p}} + d_1''\right) \begin{vmatrix} \left(-\omega_2^3 + \omega_2\omega_1^2\right) b_{1p} & \dots \left(-\omega_2^{2p-1} + \omega_2^{2p-3}\omega_1^2\right) b_{1p} \\ \left(-\omega_3^3 + \omega_3\omega_1^2\right) b_{1,p+1} & \dots \left(-\omega_3^{2p-1} + \omega_3^{2p-3}\omega_1^2\right) b_{1,p+1} \\ \dots & \dots\dots \\ \left(-\omega_p^3 + \omega_p\omega_1^2\right) b_{1,2p-2} & \dots \left(-\omega_p^{2p-1} + \omega_p^{2p-3}\omega_1^2\right) b_{1,2p-2} \end{vmatrix} \\ &= (-1)^{p+1}\omega_1 \left(d_1'' \frac{b_{11}}{b_{1p}} + d_1''\right) b_{1p} \cdots b_{1,2p-2} \begin{vmatrix} -\omega_2^3 + \omega_2\omega_1^2 & \dots -\omega_2^{2p-1} + \omega_2^{2p-3}\omega_1^2 \\ -\omega_3^3 + \omega_3\omega_1^2 & \dots -\omega_3^{2p-1} + \omega_3^{2p-3}\omega_1^2 \\ \dots & \dots\dots \\ -\omega_p^3 + \omega_p\omega_1^2 & \dots -\omega_p^{2p-1} + \omega_p^{2p-3}\omega_1^2 \end{vmatrix} \\ &= \omega_1 \left(d_1'' \frac{b_{11}}{b_{1p}} + d_1''\right) b_{1p} \cdots b_{1,2p-2} \left(\omega_2^3 - \omega_2\omega_1^2\right) \cdots \left(\omega_p^3 - \omega_p\omega_1^2\right) \begin{vmatrix} 1 & \omega_2^2 & \dots & \omega_2^{2(p-2)} \\ 1 & \omega_3^2 & \dots & \omega_3^{2(p-2)} \\ \dots & \dots & \dots & \dots \\ 1 & \omega_p^2 & \dots & \omega_p^{2(p-2)} \end{vmatrix} \\ &= \omega_1 \cdots \omega_p \left(\omega_2^2 - \omega_1^2\right) \cdots \left(\omega_p^2 - \omega_1^2\right) \prod_{2 \le i < j \le p} \left(\omega_j^2 - \omega_i^2\right) \left(d_1'' b_{11} + d_1'' b_{1p}\right) b_{1,p+1} \cdots b_{1,2p-2}. \end{aligned} \tag{4.23}$$

Because of the generic choice $0 < \omega_1 < \cdots < \omega_p$, all factors involving the ω_k are strictly positive. Thus, assuming the generic conditions $b_{1j} \neq 0$ for all $j = p, \dots, 2p-2$ and $d_1'' b_{11} + d_1'' b_{1p} \neq 0$, this determinant does not vanish.

This proves that the functions $H_1, \dots, H_p$ are independent.

4.3 Independence of the Functions $H_1, \dots, H_p, \delta_1, \dots \delta_{p-1}$ if $n = 2p+1$ and $H_1, \dots, H_{p-1}, \delta_1, \dots \delta_{p-1}$ if $n = 2p$

We begin with the case $n = 2p+1$ and show that $H_1, \dots, H_p, \delta_1, \dots \delta_{p-1}$ are independent functions. As we shall see, we shall use in this proof the separate independence of the sets of functions $\{\delta_1, \dots, \delta_{p-1}\}$ and $\{H_1, \dots, H_p\}$ proved so far. Suppose that

$$\beta_1 \frac{\partial H_1}{\partial \vec{p}} + \beta_2 \frac{\partial H_2}{\partial \vec{p}} + \cdots + \beta_p \frac{\partial H_p}{\partial \vec{p}} + \alpha_1 \frac{\partial \delta_1}{\partial \vec{p}} + \alpha_2 \frac{\partial \delta_2}{\partial \vec{p}} + \cdots + \alpha_{p-1} \frac{\partial \delta_{p-1}}{\partial \vec{p}} = 0, \tag{4.24}$$

where $\frac{\partial H_k}{\partial \vec{p}}$ is given by formula (4.13) and $\frac{\partial \delta_1}{\partial \vec{p}}$ by formula (4.2). This is a $(2p-1)\times(2p-1)$ linear homogeneous system with unknowns $\beta_1, -\beta_2, \beta_3, \ldots, (-1)^{p-1}\beta_p, \alpha_1, -\alpha_2, \alpha_3, \ldots, (-1)^p\alpha_{p-1}$ given by the matrix

$$\left(\begin{array}{ll|l}
\omega_1 d_1'' - \omega_2 b_{1p} & \ldots\ \omega_1^{2p-1} d_1'' - \omega_2^{2p-1} b_{1p} & \begin{pmatrix} u_1 & \ldots & \lambda_1^{2(p-2)} u_1 \\ u_2 & \ldots & \lambda_2^{2(p-2)} u_2 \\ \ldots & \ldots & \ldots \\ u_{p-1} & \ldots & \lambda_{p-1}^{2(p-2)} u_{p-1} \end{pmatrix} \frac{(-1)}{2k} DQ(\mathbf{G}^\top)^{-1} \\
\omega_1 d_2'' - \omega_3 b_{1,p+1} & \ldots\ \omega_1^{2p-1} d_2'' - \omega_3^{2p-1} b_{1,p+1} & \\
\ldots & \ldots\ \ldots & \\
\omega_1 d_{p-1}'' - \omega_p b_{1,2p-2} & \ldots\ \omega_1^{2p-1} d_{p-1}'' - \omega_p^{2p-1} b_{1,2p-2} & \\
\hline
\omega_1 d_p'' + \omega_2 b_{11} & \ldots\ \omega_1^{2p-1} d_p'' + \omega_2^{2p-1} b_{11} & \begin{pmatrix} w_1 & \ldots & \lambda_1^{2(p-2)} w_1 \\ w_2 & \ldots & \lambda_2^{2(p-2)} w_2 \\ \ldots & \ldots & \ldots \\ w_{p-1} & \ldots & \lambda_{p-1}^{2(p-2)} w_{p-1} \end{pmatrix} \frac{1}{2k} DQ(\mathbf{G}^\top)^{-1} \\
\omega_1 d_{p+1}'' + \omega_3 b_{12} & \ldots\ \omega_1^{2p-1} d_{p+1}'' + \omega_3^{2p-1} b_{12} & \\
\ldots & \ldots\ \ldots & \\
\omega_1 d_{2p-2}'' + \omega_p b_{1,p-1} & \ldots\ \omega_1^{2p-1} d_{2p-2}'' + \omega_p^{2p-1} b_{1,p-1} & \\
\hline
\omega_1 d_{2p-1}'' & \ldots\ \omega_1^{2p-1} d_{2p-1}'' & 0 \ldots 0
\end{array}\right),$$

where D, Q are introduced in (4.3) and $\mathbf{G}$ in (4.14). We must show that its determinant does not vanish, generically. Operating on the columns and rows yields

$$\left|\begin{array}{ll|l}
\omega_1 d_1'' - \omega_2 b_{1p} & \ldots \left(-\omega_2^{2p-1} + \omega_2^{2p-3}\omega_1^2\right) b_{1p} & \begin{pmatrix} u_1 & \ldots & \lambda_1^{2(p-2)} u_1 \\ u_2 & \ldots & \lambda_2^{2(p-2)} u_2 \\ \ldots & \ldots & \ldots \\ u_{p-1} & \ldots & \lambda_{p-1}^{2(p-2)} u_{p-1} \end{pmatrix} \frac{(-1)}{2k} DQ(\mathbf{G}^\top)^{-1} \\
\omega_1 d_2'' - \omega_3 b_{1,p+1} & \ldots \left(-\omega_3^{2p-1} + \omega_3^{2p-3}\omega_1^2\right) b_{1,p+1} & \\
\ldots & \ldots\ \ldots & \\
\omega_1 d_{p-1}'' - \omega_p b_{1,2p-2} & \ldots \left(-\omega_p^{2p-1} + \omega_p^{2p-3}\omega_1^2\right) b_{1,2p-2} & \\
\hline
\omega_1 d_p'' + \omega_2 b_{11} & \ldots -\left(-\omega_2^{2p-1} + \omega_2^{2p-3}\omega_1^2\right) b_{11} & \begin{pmatrix} w_1 & \ldots & \lambda_1^{2(p-2)} w_1 \\ w_2 & \ldots & \lambda_2^{2(p-2)} w_2 \\ \ldots & \ldots & \ldots \\ w_{p-1} & \ldots & \lambda_{p-1}^{2(p-2)} w_{p-1} \end{pmatrix} \frac{1}{2k} DQ(\mathbf{G}^\top)^{-1} \\
\omega_1 d_{p+1}'' + \omega_3 b_{1,2} & \ldots -\left(-\omega_3^{2p-1} + \omega_3^{2p-3}\omega_1^2\right) b_{1,2} & \\
\ldots & \ldots\ \ldots & \\
\omega_1 d_{2p-2}'' + \omega_p b_{1,p-1} & \ldots -\left(-\omega_p^{2p-1} + \omega_p^{2p-3}\omega_1^2\right) b_{1,p-1} & \\
\hline
\omega_1 d_{2p-1}'' & \ldots\ 0 & 0 \ldots 0
\end{array}\right|$$

$$= \omega_1 d_{2p-1}'' \left|\begin{array}{ll|l}
\omega_1 g_1 - \omega_2 f_1 & \ldots \left(-\omega_2^{2p-1} + \omega_2^{2p-3}\omega_1^2\right) f_1 & \mathbf{0} \\
\omega_1 g_2 - \omega_3 f_2 & \ldots \left(-\omega_3^{2p-1} + \omega_3^{2p-3}\omega_1^2\right) f_2 & \\
\ldots & \ldots\ \ldots & \\
\omega_1 g_{p-1} - \omega_p f_{p-1} & \ldots \left(-\omega_p^{2p-1} + \omega_p^{2p-3}\omega_1^2\right) f_{p-1} & \\
\hline
\omega_1 d_p'' + \omega_2 b_{11} & \ldots -\left(-\omega_2^{2p-1} + \omega_2^{2p-3}\omega_1^2\right) b_{11} & \begin{pmatrix} w_1 & \ldots & \lambda_1^{2(p-2)} w_1 \\ w_2 & \ldots & \lambda_2^{2(p-2)} w_2 \\ \ldots & \ldots & \ldots \\ w_{p-1} & \ldots & \lambda_{p-1}^{2(p-2)} w_{p-1} \end{pmatrix} \frac{1}{2k} DQ(\mathbf{G}^\top)^{-1} \\
\omega_1 d_{p+1}'' + \omega_3 b_{1,2} & \ldots -\left(-\omega_3^{2p-1} + \omega_3^{2p-3}\omega_1^2\right) b_{1,2} & \\
\ldots & \ldots\ \ldots & \\
\omega_1 d_{2p-2}'' + \omega_p b_{1,p-1} & \ldots -\left(-\omega_p^{2p-1} + \omega_p^{2p-3}\omega_1^2\right) b_{1,p-1} &
\end{array}\right|$$

$$= \omega_1 d_{2p-1}'' \left|\begin{array}{ll}
\omega_1 g_1 - \omega_2 f_1 & \ldots \left(-\omega_2^{2p-1} + \omega_2^{2p-3}\omega_1^2\right) f_1 \\
\omega_1 g_2 - \omega_3 f_2 & \ldots \left(-\omega_3^{2p-1} + \omega_3^{2p-3}\omega_1^2\right) f_2 \\
\ldots & \ldots\ \ldots \\
\omega_1 g_{p-1} - \omega_p f_{p-1} & \ldots \left(-\omega_p^{2p-1} + \omega_p^{2p-3}\omega_1^2\right) f_{p-1}
\end{array}\right| \left|\begin{array}{lll}
w_1 & \ldots & \lambda_1^{2(p-2)} w_1 \\
w_2 & \ldots & \lambda_2^{2(p-2)} w_2 \\
\ldots & \ldots & \ldots \\
w_{p-1} & \ldots & \lambda_{p-1}^{2(p-2)} w_{p-1}
\end{array}\right| \left|\frac{1}{2k} DQ(\mathbf{G}^\top)^{-1}\right|,$$

where $g_i = d_i'' + \frac{u_i}{w_i} d_{p+i-1}''$, $f_i = b_{1,p+i-1} - \frac{u_i}{w_i} b_{1,i}$, $i = 1, \ldots, p-1$ and we required the generic condition that all $w_i \neq 0$. These types of determinants were calculated in the previous two special cases and shown that they do not vanish, generically.

If $n = 2p$, the independence of the functions $H_1, \ldots, H_{p-1}, \delta_1, \ldots, \delta_{p-1}$ is shown in a similar manner. The relevant determinant is now

$$\left|\begin{array}{ll|l}
\omega_1 d_1'' - \omega_2 b_{1p} & \dots\ \omega_1^{2(p-2)} d_1'' - \omega_2^{2(p-2)} b_{1p} & \begin{pmatrix} u_1 & \dots & \lambda_1^{2(p-2)} u_1 \\ u_2 & \dots & \lambda_2^{2(p-2)} u_2 \\ \dots & \dots & \dots \\ u_{p-1} & \dots & \lambda_{p-1}^{2(p-2)} u_{p-1} \end{pmatrix} \frac{(-1)}{2k} DQ(\mathbf{G}^\top)^{-1} \\
\omega_1 d_2'' - \omega_3 b_{1,p+1} & \dots\ \omega_1^{2(p-2)} d_2'' - \omega_3^{2(p-2)} b_{1,p+1} & \\
\dots & \dots\ \dots & \\
\omega_1 d_{p-1}'' - \omega_p b_{1,2p-2} & \dots\ \omega_1^{2(p-2)} d_{p-1}'' - \omega_p^{2(p-2)} b_{1,2p-2} & \\
\hline
\omega_1 d_p'' + \omega_2 b_{11} & \dots\ \omega_1^{2(p-2)} d_p'' + \omega_2^{2(p-2)} b_{11} & \begin{pmatrix} w_1 & \dots & \lambda_1^{2(p-2)} w_1 \\ w_2 & \dots & \lambda_2^{2(p-2)} w_2 \\ \dots & \dots & \dots \\ w_{p-1} & \dots & \lambda_{p-1}^{2(p-2)} w_{p-1} \end{pmatrix} \frac{1}{2k} DQ(\mathbf{G}^\top)^{-1} \\
\omega_1 d_{p+1}'' + \omega_3 b_{12} & \dots\ \omega_1^{2(p-2)} d_{p+1}'' + \omega_3^{2(p-2)} b_{12} & \\
\dots & \dots\ \dots & \\
\omega_1 d_{2p-2}'' + \omega_p b_{1,p-1} & \dots\ \omega_1^{2(p-2)} d_{2p-2}'' + \omega_p^{2(p-2)} b_{1,p-1} &
\end{array}\right|$$

and the proof proceeds following analogous steps as in the case $n = 2p + 1$.

Note that there is no contradiction of the independence of the complete set of functions $\{H_1, \dots, H_{p-1}, \delta_1, \dots \delta_{p-1}\}$ with the fact that the two sets of functions $\{H_1, \dots, H_p\}$ and $\{\delta_1, \dots \delta_{p-1}\}$ are *separately* independent. As pointed out in examples, H_p and the Pfaffian are both expressible in terms of the other integrals.

Theorem 4.1 *The collection of functions* $\{H_1, \dots, H_p, \delta_1, \dots \delta_{p-1}\}$ *if* $n = 2p + 1$ *and* $\{H_1, \dots, H_{p-1}, \delta_1, \dots \delta_{p-1}\}$ *if* $n = 2p$ *is an integrable system on the* $2(n-2)$*-dimensional vector space* $\mathrm{Mat}_{2\times(n-2)}(\mathbb{R})$ *endowed with the canonical symplectic structure.*

The case for general $\epsilon \in \mathbb{R}$ is treated in a similar manner. The proofs are identical but the writing is more cumbersome.

Theorem 4.2 *The collection of functions* $\{H_{1,\epsilon}, \dots, H_{p,\epsilon}, \delta_1, \dots, \delta_{p-1}\}$ *if* $n = 2p + 1$ *and* $\{H_{1,\epsilon}, \dots, H_{p-1,\epsilon}, \delta_1, \dots, \delta_{p-1}\}$ *if* $n = 2p$ *is an integrable system on the* $2(n-2)$*-dimensional vector space* $\mathrm{Mat}_{2\times(n-2)}(\mathbb{R})$ *endowed with the canonical symplectic structure.*

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