Proof of Lemma 2. Notice first that we can express the estimated state \( z_{t,l}^i \) as the average of the estimated states plus an error \( Y_{t,l} \), i.e. \( z_{t,l}^i = \sum_{j \in N} \frac{1}{N} z_{t,l}^j + Y_{t,l} \), where \( Y_{t,l} \) is the component of \( Y_{t,l} \) corresponding to the node \( i \). From the fact that the consensus algorithm preserves averages we have that \( z_{t,l}^i = \sum_{j \in N} \frac{1}{N} z_{t,l}^j + Y_{t,l} \). Then from the state dynamics and filter update equations (1) and (5), and the definitions of \( \Phi^i \), \( W_i \) and \( \Gamma_i \) we obtain equation (14) as follows:

\[
e_{t+1,0}^i = A(x_t - z_{t,l}^i) - L^i(C^i x_t + v_t^i - C^i z_{t,l}^i) + w_t
\]

\[
= A(x_t - \frac{1}{N} \sum_{j \in N} z_{t,l}^j - Y_{t,l}^i)
- L^i(C^i x_t + v_t^i - C^i \frac{1}{N} \sum_{j \in N} z_{t,l}^j - C^i Y_{t,l}^i) + w_t
= \Phi^i(x_t - \frac{1}{N} \Phi^j z_{t,l}^j) - \Gamma_t + W_t
= \sum_{j \in N} \frac{1}{N} \Phi^i e_{t,0}^j - \Gamma_t + W_t.
\]

From the definitions of \( \Phi \), \( \Gamma_t \) and \( W_t \) we obtain directly equation (15)

\[
e_{t+1,0} = \Phi e_{t,0} - \Gamma_t + W_t
= \frac{1}{N} \text{col}(\Phi^i) 1^T \otimes I_n e_{t,0} - \Gamma_t + W_t,
\]

Since we can observe that \( \text{col}(\Phi^i) \) is equal to \( \text{diag}(\Phi^i) 1 \otimes I_n \) the previous equation is equivalent to

\[
e_{t+1,0} = \frac{1}{N} \text{diag}(\Phi^i) 1 \otimes I_n 1^T \otimes I_n e_{t,0} - \Gamma_t + W_t.
\]

Using the former equation, the mixed-product property of the Kronecker product\(^1\) and the definition of \( e_{t,0}^\text{avg} \) we obtain equation (16) as follows

\[
e_{t+1,0} = \text{diag}(\Phi^i) \frac{1}{N} (11^T) \otimes I_n e_{t,0} - \Gamma_t + W_t
= \text{diag}(\Phi^i) e_{t,0}^\text{avg} - \Gamma_t + W_t.
\]

Finally, from the definition of \( e_{t+1,0}^\text{avg} \) and equation (16) we have

\[
e_{t+1,0}^\text{avg} = \frac{1}{N} (11^T) \otimes I_n e_{t+1,0}
= \frac{1}{N} (11^T) \otimes I_n \left( \text{diag}(\Phi^i) e_{t,0}^\text{avg} - \Gamma_t + W_t \right).
\]

Since \( 1^T \otimes I_n \text{diag}(\Phi^i) \) is equal to \( \text{row}(\Phi^i) \) and from the mixed-product property of the Kronecker product we have

\[
e_{t+1,0}^\text{avg} = \frac{1}{N} \text{row}(\Phi^i) e_{t,0}^\text{avg}
+ \frac{1}{N} (11^T) \otimes I_n (W_t - \Gamma_t).
\]

Noting that \( \frac{1}{N} (11^T) \otimes I_n e_{t,0}^\text{avg} \) is equal to \( e_{t,0}^\text{avg} \) we have

\[
e_{t+1,0} = \frac{1}{N} \text{row}(\Phi^i) \frac{1}{N} (11^T) \otimes I_n e_{t,0}^\text{avg}
+ \frac{1}{N} (11^T) \otimes I_n (W_t - \Gamma_t).
\]

Using the mixed-product property and the fact that \( \text{row}(\Phi^i) \frac{1}{N} 1 \otimes I_n = \frac{1}{N} \sum_{j \in N} \Phi^j = A - LC \) the former equation is equivalent to

\[
e_{t+1,0}^\text{avg} = \frac{1}{N} 1 \otimes I_n (A - LC) 1^T \otimes I_n e_{t,0}^\text{avg}
+ \frac{1}{N} (11^T) \otimes I_n (W_t - \Gamma_t).
\]

Again, using the mixed-product property we have that

\[
1 \otimes I_n (A - LC) = I_n \otimes (A - LC) 1 \otimes I_n.
\]

And therefore it follows that

\[
e_{t+1,0}^\text{avg} = I_n \otimes (A - LC) \frac{1}{N} 1 \otimes I_n 1^T \otimes I_n e_{t,0}^\text{avg}
+ \frac{1}{N} (11^T) \otimes I_n (W_t - \Gamma_t).
\]

And finally, from the former equation, the definition of \( e_{t,0}^\text{avg} \) and the mixed-product property we obtain equation (17).

---

1. Given four matrices \( M_1, M_2, M_3, M_4 \) of proper size, the mixed-product property consists of the fact that \( (M_1 \otimes M_2)(M_3 \otimes M_4) = (M_1 M_3) \otimes (M_2 M_4) \).
of Lemma 1, and that assumption A2 holds, then noting that \(\|e_{0,0}\| \leq \|e_{0,0}\|\) and that \(\|e_{0,0}\| \leq \max \left(1, \frac{\bar{z}}{\beta} \right)\|e_{0,0}\|\) applying equations (19) and (20) recursively we obtain

\[
\|e_{p+1,0}\| \leq \beta \|e_{p,0}\| + \Phi \epsilon_{p,0} + \|e_{p,0}\| + \tilde{\Phi} \|e_{p-1,0}\| + \Phi \tilde{\Phi} k_0 a \beta^{-\tau} + \epsilon + \tilde{\Phi} \tilde{\Phi} k_0 a \beta^{-\tau} + \epsilon
\]

where \(\tilde{\beta}\) is defined in (21) and is strictly positive and smaller than 1 by assumption. Repeating this step \(p\) times we have

\[
\|e_{p+1,0}\| \leq \beta^{p+1} \|e_{0,0}\| + \sum_{\tau=0}^{\beta} \beta \tilde{\beta} \|e_{\tau,0}\| + \Phi \tilde{\Phi} k_0 a \beta^{-\tau} + \epsilon
\]

Since \(0 < \beta < 1\), by using the property of the geometric series, we get that the expression above is equal to

\[
\|e_{p+1,0}\| \leq \beta^{p+1} \left(\|e_{0,0}\| + \sum_{\tau=0}^{\beta} \beta \tilde{\beta} \tilde{\beta} \|e_{\tau,0}\| + \Phi \tilde{\Phi} k_0 a \beta^{-\tau} + \epsilon\right)
\]

3) We have from (18) that

\[
\|Y_{p,0}\| \leq \|e_{p,0}\| + \max \left(1, \frac{\bar{z}}{\beta} \right) \|e_{p,0}\| + \tilde{\Phi} \|e_{p-1,0}\| + \Phi \tilde{\Phi} k_0 a \beta^{-\tau} + \epsilon + \tilde{\Phi} \tilde{\Phi} k_0 a \beta^{-\tau} + \epsilon
\]

Moreover we have

\[
\|Y_{p,t}\| \leq \alpha \|Y_{p,0}\| + k_0 a \beta^{-\tau} + \epsilon + \tilde{\Phi} \|e_{\tau,0}\| + \Phi \tilde{\Phi} k_0 a \beta^{-\tau} + \epsilon + \tilde{\Phi} \tilde{\Phi} k_0 a \beta^{-\tau} + \epsilon
\]

\[
\forall t \geq 0, t \geq l \geq 0.
\]

from Lemma 1.

4) Then we note that since \(z_{p,t} = Y_{p,t} + z_{p,0}\), from the fact that the consensus algorithm preserves averages, and \(x_p = \frac{1}{N} \sum_{i \in N} e_{p,0} + z_{p,0}\) we have

\[
z_{p+1,0} = A_{p+1} + L I \left(\gamma_{p} - C f_{p+1}\right)
\]

\[
= \Phi \tilde{\Phi} + L I \left(\gamma_{p} - C f_{p+1}\right)
\]

\[
= \Phi \tilde{\Phi} + L I C f_{p+1} + L I C f_{p+1}
\]

\[
= \Phi \tilde{\Phi} + L I C f_{p+1} + L I C f_{p+1}
\]

\[
= \Phi \tilde{\Phi} + L I C f_{p+1} + L I C f_{p+1}
\]

Therefore for the vector \(z_{p+1,0}\) we have

\[
z_{p+1,0} = \text{diag} (\Phi^i) Y_{p,t} + I_N \otimes \tilde{A}_{p,0} + \text{diag} (L_i C_i) \frac{1}{N} (11^T) \otimes I_N e_{p,0} + \text{col} (L_i^T)
\]

and, noting that \(\sum_{i \in N} (11^T) = \text{NLC}\), we have

\[
z_{p+1,0} = \frac{1}{N} (11^T) \otimes I_N \text{diag} (\Phi^i) Y_{p,t} + I_N \otimes \tilde{A}_{p,0} + I_N \otimes (L_i C_i) \frac{1}{N} (11^T) \otimes I_N e_{p,0} + I_N \text{col} (L_i^T)
\]

For the vector \(z_{p+1,0}\) we have

\[
z_{p+1,0} = I_N \otimes A Q_{p,t-1} - z_{p,t-1}
\]

\[
= I_N \otimes A Q_{p,t-1} - z_{p,t-1}
\]

\[
= I_N \otimes A Q_{p,t-1} - z_{p,t-1}
\]

\[
= I_N \otimes A Q_{p,t-1} - z_{p,t-1}
\]

\[
= I_N \otimes A Q_{p,t-1} - z_{p,t-1}
\]

and finally

\[
\|z_{p+1,0} - z_{p,0}\| \leq A (a \beta^{p+1} + b \beta^{p+1}) + \sqrt{N} a \beta^{p+1} + \sqrt{N} \max_{i \in N} |L_i^T| e_p
\]

\[
\leq c_5 \beta^p \|e_{0,0}\| + c_6 a \beta^p + d_5 + d_6 \frac{b}{\epsilon}.
\]
5) Since \( z_{p,t_f} = Y_{p,t_f} + z_{p,0} \), which, subtracting both sides by \( 1 \otimes x_p \), is equivalent to \( e_{p,t_f} = Y_{p,t_f} + e_{p,0} \), we have for the norm of \( e_{p,t_f} \)
\[
\| e_{p,t_f} \| \leq \| Y_{p,t_f} \| + \| e_{p,0} \| \\
\leq \beta_p \left[ (1 + \alpha^l t_f \max \left(1, \frac{\bar{\Phi}}{2 \Phi} \right)) \| e_{0,0} \| + \right. \\
+ \left( c_8 + \alpha^l t_f \right) a \frac{\alpha}{2 \pi \tau} \\
+ \left( 1 + \alpha^l t_f \max \left(1, \frac{\bar{\Phi}}{2 \Phi} \right) \right) \frac{\epsilon}{1 - \beta} \\
+ \left. \left( d_8 + \alpha^l t_f \right) b \frac{\alpha}{2 \pi \tau}, \forall t + 1 \geq p \geq 1, \right]
\]

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