Compressive PCA for Low-Rank Matrices on Graphs

Nauman Shahid*, Nathanael Perraudin, Gilles Puy†, Pierre Vandergheynst
Email: {nauman.shahid, nathanael.perraudin, pierre.vandergheynst}@epfl.ch, † gilles.puy@gmail.com

Abstract

We introduce a novel framework for an approximate recovery of data matrices which are low-rank on graphs, from sampled measurements. The rows and columns of such matrices belong to the span of the first few eigenvectors of the graphs constructed between their rows and columns. We leverage this property to recover the non-linear low-rank structures efficiently from sampled data measurements, with a low cost (linear in \( n \)). First, a Restricted Isometry Property (RIP) condition is introduced for efficient uniform sampling of the rows and columns of such matrices based on the cumulative coherence of graph eigenvectors. Secondly, a state-of-the-art fast low-rank recovery method is suggested for the sampled data. Finally, several efficient, parallel and parameter-free decoders are presented along with their theoretical analysis for decoding the low-rank and cluster indicators for the full data matrix. Thus, we overcome the computational limitations of the standard linear low-rank recovery methods for big datasets. Our method can also be seen as a major step towards efficient recovery of non-linear low-rank structures. On a single core machine, our method gains a speed up of \( p^2/k \) over Robust PCA, where \( k \ll p \) is the subspace dimension. Numerically, we can recover a low-rank matrix of size \( 10304 \times 1000 \) in 15 secs, which is 100 times faster than Robust PCA.

Index Terms

Robust PCA, graph Laplacian, spectral graph theory, compressive sampling

I. INTRODUCTION

In many applications in signal processing, computer vision and machine learning, the data has an intrinsic low-rank structure. One desires to extract this structure efficiently from the noisy observations. Robust PCA [4], a linear dimensionality reduction algorithm can be used to exactly describe a dataset lying on a single linear low-dimensional subspace. Low-rank Representation (LRR) [15], on the other hand can be used for data drawn from multiple linear subspaces. However, these methods suffer from two prominent problems:

1) They do not recover non-linear low-rank structures.
2) They do not scale for big datasets \( Y \in \mathbb{R}^{p \times n} \) (large \( p \) and large \( n \), where \( p \) is the number of features).

The recovery of non-linear structures requires the assumption that the data lies on a smooth non-linear low-dimensional manifold. Extensions of RPCA and LRR such as Robust PCA on Graphs (RPCAG) [28] and Graph Regularized LRR (GLRR) [16] propose to incorporate graph regularization as a method to recover non-linear low-rank structures. These method still suffer from the scalability problem for big datasets.

Randomized techniques come into play to deal with the scalability problem associated with very high dimensional data (the case of large \( p \)) [35, 21, 37, 14, 4, 19, 24, 25, 9] using the tools of compression [5]. These works improve upon the computational complexity by reducing only the data dimension \( p \) but still scale in the same manner w.r.t \( n \). The case of large \( n \) can be tackled by using the sampling schemes accompanied with Nystrom method [36, 33]. However, these methods work efficiently only for low-rank kernel matrices and do not recover the low-rank data matrix itself. How to tackle the case of big \( n \) then?

For many machine learning applications involving big data, such as clustering, an approximate low-rank representation might suffice. The recently introduced Fast Robust PCA on Graphs (FRPCAG) [29] approximates a recovery method for non-linear low-rank datasets, which are called Low-rank matrices on graphs. Inspired by the underlying stationarity assumption [22],

Affiliation: Signal Processing Laboratory 2 (LTS2), EPFL STI IEL, Lausanne, CH-1015, Switzerland. Phone: +41 21 69 34754. † G. Puy contributed to this work while he was at INRIA (Rennes - Bretagne Atlantique, Campus de Beaulieu, FR-35042 Rennes Cedex, France). N.Shahid and N.Perraudin are supported by the SNF grant no. 200021_154350/1 for the project “Towards signal processing on graphs”. G.Puy was funded by the European Research Council, PLEASE project (ERC-StG-2011-277906).
the authors introduce a joint notion of low-rankness for the features and samples (rows and columns) of a data matrix. More specifically, a low-rank matrix on graphs is defined as a matrix whose rows and columns belong to the span of the first few eigenvectors of the graphs constructed between its rows and columns.

FRPCAG does not require an SVD and scales linearly with $n$. However, it still suffers from 1) memory requirements 2) non-parallel implementation 3) cost of k-means for clustering and 4) the cost of parameter tuning for large $p$ and large $n$. This said, sometimes one might not even have access to the full dataset $Y$. This is typical, for instance for the biomedical applications, such as MRI and tomography. In such applications the number of observations are limited by the data acquisition protocols. In MRI, the number of observations is proportional to the time and dose required for the procedure. In tomography one might have access to the projections only. Thus, FRPCAG is not be usable if 1) the dataset is large and 2) only a subset of the dataset is available.

Despite the above limitations of the data acquisition, one might have access to some additional information about the unobserved samples. In MRI for instance, sparsity of the samples in the Fourier domain serves as a good prior. We limit ourselves to the case where a graphical prior is available for the complete set of observations for the application under consideration. A graph, encoding pairwise similarity of images of database, friendships in a social network, videos on youtube or the articles on arXiv are a few common examples.

A. The Problem Statement

In this work we answer the following question: Given a few randomly sampled observations and features from a data matrix $Y \in \mathbb{R}^{p \times n}$ which is jointly low-rank on two manifolds, is it possible to efficiently recover the complete low-rank? Alternatively, what would be an efficient and highly scalable low-rank recovery framework using compression?

B. Contributions & Highlights

We call our proposed framework “Compressive PCA on graphs” (CPCA). Throughout, we assume that one of the following two is available for low-rank recovery: 1) a graphical prior for the dataset and sampled / compressed observations or 2) the complete dataset. Following are the main contributions and highlights of this work:

1. Sampling & RIP for low-rank matrices on graphs: To solve the scalability problem of FRPCAG we propose to perform a dual uniform sampling of the data matrices, along rows and columns. We present a restricted isometry property (RIP) for low-rank matrices on graphs and relate it to the cumulative coherence of the graph eigenvectors. FRPCAG is then used to recover the low-rank for the sampled data.

2. Decoders for low-rank recovery:
   1) We present two (ideal and alternate) convex and efficient decoders for recovering the full low-rank from the corresponding low-rank matrix of the sampled data.
   2) Our main contribution comprises the set of 3 additional parallel, low-cost and parameter-free approximate decoders, which significantly boost the speed of our framework by introducing a few approximations.
   3) Our rigorous theoretical analysis proves that the recovery error of the above decoders depends on the spectral gaps of the row and column graph Laplacians.

3. Semi-supervised label propagation for clustering: We propose a low-cost and parallel semi-supervised label propagation scheme for decoding the cluster labels of the complete dataset from the cluster labels of the sampled matrix.

4. Extensive Experimentation: Low-rank recovery experiments on 3 real video datasets and clustering experiments on 5 benchmark datasets reveal that the performance of our model is better or comparable to 10 different state-of-the-art PCA and non-PCA based methods.

C. Advantages of Compressive PCA

1. Adaptability: Our proposed framework is directly applicable even for the case where only a fraction of rows and columns of a data matrix are observed and a graphical prior (both for rows and columns) is available from an external source.

2. Scalability: CPCA scales linearly with $n$, i.e., $O(nkK)$ complexity per iteration, where $k(\ll n)$ is the rank of low-dimensional subspace and $K(\ll n)$ is the number of nearest neighbors for graphs construction. A speed-up of $p/k$ per-iteration
is obtained over FRPCAG and \( p^2/k \) over RPCA solely for low-rank recovery and clustering tasks. Numerically, CPCA can recover a low-rank matrix of size \( 10304 \times 1000 \) in 15 secs, which is 6 and 100 times faster than FRPCAG and RPCA.

3. Robustness to Fast Graph Construction: We specifically assume that the graphs are available from an external source. Alternatively, they can be constructed with \( (\mathcal{O}(np\log(n))) \) cost using approximate \( K \)-nearest neighbors search using FLANN \([17]\) which supports a high degree of parallelism. The performance of CPCA is robust to the approximate search.

4. Robustness to Noise & Errors: Our experimentation shows that CPCA is quite robust to noise and errors in the dataset.

D. Similarities & Differences with the state-of-the-art

Our proposed framework is inspired by the recently introduced sampling of band-limited signals on graphs \([23]\). While we borrow several concepts from here, our framework is significantly different from \([23]\) in many contexts. We target the low-rank recovery and clustering tasks related to PCA. Thus, contrary to \([23]\) our proposed decoders are designed for these applications. A major contribution of our work in contrast to \([23]\) is the design of approximate sampling schemes in \([23]\). A major difference lies in the application domain and hence the experiments. Unlike \([23]\), we target two applications related to PCA: 1) low-rank recovery and 2) clustering. Thus, contrary to \([23]\) our proposed framework is significantly different from \([23]\) in many contexts. We target the low-rank recovery and clustering tasks related to PCA. Thus, contrary to \([23]\) our proposed decoders are designed for these applications. A major contribution of our work in contrast to \([23]\) is the design of approximate sampling schemes in \([23]\).

II. GRAPHS & LOW-RANK MATRICES ON GRAPHS

A. Graph Nomenclature

For a matrix \( Y \in \mathbb{R}^{p \times n} \), a \( K \)-nearest neighbor undirected graph between the rows or columns of \( Y \) is denoted as \( G = (\mathcal{V}, \mathcal{E}) \), where \( \mathcal{E} \) is the set of edges and \( \mathcal{V} \) is the set of vertices. The first step in the construction of \( G \) consists of connecting each \( y_i \) to its \( K \) nearest neighbors \( y_j \) (using Euclidean distance), resulting in \( |\mathcal{E}| \) connections. The \( K \)-nearest neighbors are non-symmetric but a symmetric weighted adjacency matrix \( W \) is computed via a Gaussian kernel as \( W_{ij} = \exp(-\|y_i - y_j\|_2^2/\sigma^2) \) if \( y_j \) is connected to \( y_i \) and 0 otherwise. Let \( D \) be the diagonal degree matrix of \( G \) which is given as: \( D_{ii} = \sum_j W_{ij} \).

Then, the combinatorial Laplacian that characterizes the graph \( G \) is defined as \( L = D - W \) and its normalized form as \( L_n = D^{-1/2}(D - W)D^{-1/2} \). Throughout this work we use the approximate nearest neighbor algorithm (FLANN \([17]\)) for graph construction whose complexity is \( \mathcal{O}(np\log(n)) \) for \( p \ll n \) \([27]\) (and it can be performed in parallel).

B. Low-rank matrices on graphs

Let \( L_c \in \mathbb{R}^{n \times n} \) be the Laplacian of the graph \( G_c \) connecting the different columns of \( Y \) and \( L_r \in \mathbb{R}^{p \times p} \) the Laplacian of the graph \( G_r \) that connects the rows of \( Y \). Furthermore, let \( L_c = Q \Lambda_c Q^\top = Q_kc \Lambda_{ckc} Q_kc^\top + \bar{Q}_c \Lambda_{ckc} \bar{Q}_k^\top \), where \( \Lambda_{ckc} \in \mathbb{R}^{k_c \times k_c} \) is a diagonal matrix of lower eigenvalues and \( \bar{\Lambda}_{ckc} \in \mathbb{R}^{(n-k_c) \times (n-k_c)} \) is a diagonal matrix of higher graph eigenvalues. Similarly, let \( L_r = P \Lambda_r P^\top = P_kc \Lambda_{rkc} P_k^\top + \bar{P}_c \Lambda_{rkc} \bar{P}_k^\top \). All the values in \( \Lambda_r \) and \( \Lambda_c \) are sorted in increasing order. For a \( K \)-nearest neighbors graph constructed from \( k_c \)-clusterable data (along columns) one can expect \( \lambda_{k_c} / \lambda_{k_c+1} \approx 0 \) as \( \lambda_{k_c} \approx 0 \) and \( \lambda_{k_c+1} \ll \lambda_{k_c} \). We refer to the ratio \( \lambda_{k_c} / \lambda_{k_c+1} \) as the spectral gap of \( L_c \). The same holds for the Laplacian \( L_r \).

Definition 1. A matrix \( Y^* \) is \((k_r, k_c)\)-low-rank on the graphs \( L_r \) and \( L_c \) if \( (Y^*)_{ij} \in \text{span}(P_{k_r}) \) for all \( j = 1, \ldots, n \) and \( (Y^*)_{ij}^\top \in \text{span}(Q_{k_c}) \) for all \( i = 1, \ldots, p \). The set of \((k_r, k_c)\)-low-rank matrices on the graphs \( L_r \) and \( L_c \) is denoted by \( \mathcal{L}(P_{k_r}, Q_{k_c}) \).

III. A GLIMPSE OF COMPRESSIVE PCA (CPCA)

In the sequel we present the full compressive PCA framework for two different cases:
1) Case I: When Y is available: Given a data matrix \( Y \in \mathbb{R}^{p \times n} = \bar{X} + \tilde{E} \), where \( \bar{X} \in \mathbb{L}(P_k, Q_k) \) and \( \tilde{E} \) models the errors, the goal is to develop a method to efficiently recover \( \bar{X} \). We propose to:

1) Construct Laplacians \( \mathcal{L}_r \) and \( \mathcal{L}_c \) between the rows and columns of \( Y \) using the scheme of Section II-A.
2) Sample the rows and columns of \( Y \) to get a subsampled matrix \( \bar{Y} = \bar{Y}^* + E \) using the sampling scheme of Section IV.
3) Construct the compressed Laplacians \( \mathcal{L}_r^c, \mathcal{L}_c^c \) from \( \mathcal{L}_r, \mathcal{L}_c \) (Section V).
4) Determine a low-rank \( \bar{X} \) for \( \bar{Y} \) with \( \mathcal{L}_r^c, \mathcal{L}_c^c \) in algorithm 1 of FRPCG:

\[
\min_{\bar{X}} \phi(\bar{Y} - \bar{X}) + \gamma_r \text{tr}(\bar{X}^\top \mathcal{L}_c \bar{X}) + \gamma_r \text{tr}(\bar{X}^\top \mathcal{L}_r \bar{X}),
\]

where \( \phi \) is a loss function (possibly \( l_p \) norm), \( \bar{X} = \bar{X}^* + \tilde{E} = M_r \bar{X} M_c + \tilde{E} \), \( \tilde{E} \) models the errors in the recovery of the subsampled low-rank \( \bar{X} \) and \( M_r, M_c \) are the row and column sampling matrices whose design is discussed in Section IV.

5) Use the decoders presented in Section VII to decode the low-rank \( \bar{X} = \bar{X}^* + E^* \) (for the unsampled \( Y \) on graphs \( \mathcal{L}_r, \mathcal{L}_c \) if the task is low-rank recovery, or perform k-means on \( \bar{X} \) to get cluster labels \( \bar{C} \) and use the semi-supervised label propagation (presented in Section X) to get the cluster labels \( \bar{C} \) for the full \( X \).

2) Case II: When Y is not available: We assume for this case that the sampled data matrix \( \bar{Y} \) and the prior information in the form of Laplacians \( \mathcal{L}_r, \mathcal{L}_c \) for the unsampled \( Y \) is available. One can apply steps 3 to 5 to recover the low-rank.

We directly state the computational complexities of several problems due to the space constraints. The details of calculations are presented in Appendix G. In the sequel we also use the terms compression, sampling and downsampling alternatively in the same context.

IV. HOW MUCH TO SAMPLE? RIP FOR LOW-RANK MATRICES ON GRAPHS

A. The Uniform Sampling Matrices

Let \( M_c \in \mathbb{R}^{n \times p} \) be the subsampling matrix for sampling the rows and \( M_r \in \mathbb{R}^{p \times n} \) for sampling the columns of \( Y \). \( M_c \) and \( M_r \) are constructed by drawing \( \rho_c \) and \( \rho_r \) indices \( \Omega_c = \{\omega_1, \cdots, \omega_{p_c}\} \) and \( \Omega_r = \{\omega_1, \cdots, \omega_{p_r}\} \) uniformly without replacement from the sets \( \{1, 2, \cdots, n\} \) and \( \{1, 2, \cdots, p\} \) and satisfy:

\[
M_{c}^{ij} = \begin{cases} 
1 & \text{if } i = \omega_j \\
0 & \text{otherwise}
\end{cases} \quad M_{r}^{ij} = \begin{cases} 
1 & \text{if } j = \omega_i \\
0 & \text{otherwise}
\end{cases}
\]

Now, the subsampled data matrix \( \bar{Y} \in \mathbb{R}^{n_c \times p_r} \) can be written as \( \bar{Y} = M_r Y M_c \). CPCA requires \( M_r \) and \( M_c \) to be constructed such that the “low-rankness” property of the data \( Y \) is preserved under sampling. Before discussing this, we introduce a few basic definitions in the context of graphs \( G_c \) and \( G_r \).

B. Graph Cumulative Coherence

Definition 2. (Graph cumulative coherence). The cumulative coherence of order \( k_c, k_r \) of \( G_c \) and \( G_r \) is:

\[
\nu_{k_c} = \max_{1 \leq i \leq n} \sqrt{n} \| Q_{k_c}^\top \Delta_i \|_2 \quad \text{and} \quad \nu_{k_r} = \max_{1 \leq j \leq p} \sqrt{p} \| P_{k_r}^\top \Delta_j \|_2,
\]

where \( \Delta_i \) is 1 for a node \( i \) and 0 otherwise, i.e., \( \Delta_i \) is the localized node of the graph. In the above equations \( Q_{k_c}^\top \Delta_i \) and \( P_{k_r}^\top \Delta_j \) characterize the first \( k_c \) and \( k_r \) fourier modes \([31]\) of \( \Delta_i, \Delta_j \) on the graphs \( G_c \) and \( G_r \) respectively. Thus, the cumulative coherence is a measure of how well the energy of the \((k_r, k_c)\) low-rank matrices spreads over the nodes of the graphs. These quantities exactly control the number of vertices \( \rho_c \) and \( \rho_r \) that need to be sampled from the graphs \( G_r \) and \( G_c \) such that the properties of the graphs are preserved \([23]\). Consider the example of a single node \( i \), if the coherence for this node is high then it implies that their exist some low-rank signals whose energy is highly concentrated on the node \( i \). Removing this node would result in a loss of information in the data. If the coherence of this node is low then removing it in the sampling process would result in no loss of information. We already mentioned that we are interested in the case of uniform sampling. Therefore in order to be able to sample a small number of nodes uniformly from the graphs, the cumulative coherence should be as low as possible.
C. Implications of the Graph Cumulative Coherence

We remind that for our application we desire to sample the data matrix \( Y \) such that its low-rank structure is preserved under this sampling. How can we ensure this via the graph cumulative coherence? This follows directly from the fact that we are concerned about the data matrices which are also low-rank with respect to the two graphs under consideration \( Y \in \mathcal{LR}(P_{k_r}, Q_{k_c}) \). In simple words, the columns of the data matrix \( Y \) belong to the span of the eigenvectors \( P_{k_r} \) and the rows to the span of \( Q_{k_c} \). Thus, the coherence conditions for the graph directly imply the coherence condition on the data matrix \( Y \) itself. Therefore, using these quantities to sample the data matrix \( Y \) will ensure the preservation of two properties under sampling: 1) the structure of the corresponding graphs and 2) the low-rankness of the data matrix \( Y \). Given the above definitions, we are now ready to present the restricted-isometry theorem for the low-rank matrices on the graphs.

D. RIP for Low-rank Matrices on Graphs

Theorem 1. (Restricted-isometry property (RIP) for low-rank matrices on graphs) Let \( M_c \) and \( M_r \) be two random subsampling matrices as constructed in (1). For any \( \delta, \epsilon \in (0, 1) \), with probability at least \( 1 - \epsilon \),

\[
(1 - \delta)\|Y\|_F^2 \leq \frac{np}{\rho_r \rho_c} \|M_r Y M_c\|_F^2 \leq (1 + \delta)\|Y\|_F^2
\]

(2)

for all \( Y \in \mathcal{LR}(P_{k_r}, Q_{k_c}) \) provided that

\[
\rho_c \geq \frac{27}{\delta^2} r_c^2 \log \left(\frac{4k_c}{\epsilon}\right) \quad \text{and} \quad \rho_r \geq \frac{27}{\delta^2} r_r^2 \log \left(\frac{4k_r}{\epsilon}\right),
\]

(3)

where \( \nu_{k_c}, \nu_{k_r} \) characterize the graph cumulative coherence as in Definition 2 and \( \frac{np}{\rho_r \rho_c} \) is just a normalization constant which quantifies the norm conservation in (2).

Proof. Please refer to Appendix A.

Theorem 1 is a direct extension of the RIP for \( k \)-bandlimited signals on one graph [23]. It states that the information in \( Y \in \mathcal{LR}(P_{k_r}, Q_{k_c}) \) is preserved with overwhelming probability if the sampling matrices (1) are constructed with a uniform sampling strategy satisfying (3). Note that \( \rho_r \) and \( \rho_c \) depend on the cumulative coherence of the graph eigenvectors. The better the spread the eigenvectors are, the smaller is the number of vertices that need to be sampled. As proved in [23], we have \( \nu_{k_c}^2 \geq k_r \) and \( \nu_{k_r}^2 \geq k_c \). Therefore, \( k_c \) and \( k_r \) are the minimum number of columns and rows that we need to sample. Our proposed framework in this paper builds on the case of uniform sampling but it can be easily extended for other sampling schemes [23].

V. Graphs for Compressed Data

To ensure the preservation of algebraic and spectral properties one can construct the compressed Laplacians \( \tilde{L}_r \in \mathbb{R}^{\rho_r \times \rho_r} \) and \( \tilde{L}_c \in \mathbb{R}^{\rho_c \times \rho_c} \) from the Kron reduction of \( L_r \) and \( L_c \) [6]. Let \( \Omega \) be the set of sampled nodes and \( \bar{\Omega} \) the complement set and let \( \mathcal{L}(A_r, A_c) \) denote the (row, column) sampling of \( \mathcal{L} \) w.r.t sets \( A_r, A_c \) then the Laplacian \( \tilde{L}_c \) for the columns of compressed matrix \( \tilde{Y} \) is:

\[
\tilde{L}_c = \mathcal{L}_c(\Omega, \Omega) - \mathcal{L}_c(\Omega, \bar{\Omega}) \mathcal{L}_c^{-1}(\bar{\Omega}, \bar{\Omega}) \mathcal{L}_c(\bar{\Omega}, \Omega).
\]

Let \( \mathcal{L}_c \) has \( k_c \) connected components or \( \lambda_{k_c}/\lambda_{k_c+1} \approx 0 \). Then, as argued in theorem III.4 of [6] two nodes \( \alpha, \beta \) are not connected in \( \tilde{L}_c \) if there is no path between them in \( L_c \) via \( \Omega \). Thus, if the sampling is done uniformly then one can expect \( \tilde{L}_c \) to have \( k_c \) connected components as well. The same holds for \( \tilde{L}_r \) as well. This method involves the multiplication of 3 sparse matrices. The only expensive operation above is the inverse of \( \mathcal{L}(\bar{\Omega}, \bar{\Omega}) \) which can be performed with \( \mathcal{O}(\Omega_l K_n) \) cost using the Lanczos method [32], where \( \Omega_l \) is the number of iterations for Lanczos approximation.

VI. FRPCA on the Compressed Data

Once the compressed dataset \( \tilde{Y} \in \mathbb{R}^{\rho_r \times \rho_c} \) and the Laplacians \( \tilde{L}_r \in \mathbb{R}^{\rho_r \times \rho_r}, \tilde{L}_c \in \mathbb{R}^{\rho_c \times \rho_c} \) are obtained, the next step is to recover the low-rank \( \hat{X} \in \mathbb{R}^{\rho_r \times \rho_c} \). Let \( \tilde{L}_c = \bar{Q}_c \tilde{\Lambda}_c \bar{Q}_c^T = \bar{Q}_c \bar{\Lambda}_{k_c} \bar{Q}_c^T + \bar{Q}_c \bar{\Lambda}_{\bar{k}_c} \bar{Q}_c^T \) and \( \bar{\Lambda}_{k_c} \in \mathbb{R}^{k_c \times k_c} \) is a diagonal matrix of lower eigenvalues and \( \bar{\Lambda}_{\bar{k}_c} \in \mathbb{R}^{(\rho_r - k_c) \times (\rho_r - k_c)} \) is a diagonal matrix of higher graph eigenvalues. Similarly, let \( \tilde{L}_r = \bar{P}_r \bar{\Lambda}_r \bar{P}_r^T = \bar{P}_r \bar{\Lambda}_{r_k} \bar{P}_r^T + \bar{P}_r \bar{\Lambda}_{r_{\bar{k}_r}} \bar{P}_r^T \). Furthermore assume that all the values in \( \bar{\Lambda}_r \) and \( \bar{\Lambda}_c \) are sorted in increasing order.
Assume $\hat{Y} = \hat{Y}^* + E$, where $E$ models the noise in the compressed data and $\hat{Y}^* \in \mathcal{LR}(\tilde{P}_{k_r}, \tilde{Q}_{k_c})$. The low-rank $\hat{X} = \hat{X}^* + \hat{E}$ can be recovered by solving the FRPCAG problem as proposed in [29] and re-written below:

$$\min_{\hat{X}} \phi(\hat{Y} - \hat{X}) + \gamma_c \text{tr}(\hat{X} \tilde{L}_c \hat{X}^\top) + \gamma_r \text{tr}(\hat{X}^\top \tilde{L}_r \hat{X}),$$

(4)

where $\phi$ is a proper, positive, convex and lower semi-continuous loss function (possibly $l_p$ norm). From Theorem 1 in [29], the low-rank approximation error comprises the orthogonal projection of $\hat{X}^*$ on the complement graph eigenvectors $(\tilde{Q}_{k_c}, \tilde{P}_{k_r})$ and depends on the spectral gaps $\hat{\lambda}_{k_r}/\hat{\lambda}_{k_r+1}, \hat{\lambda}_{k_c}/\hat{\lambda}_{k_c+1}$ as following:

$$||\hat{X}^* \tilde{Q}_{k_c}||_F^2 + ||\tilde{P}_{k_r} \hat{X}^*||_F^2 = ||\hat{E}||_F^2 \leq \frac{1}{\gamma} \phi(E) + ||\hat{Y}^*||_F^2 \left(\frac{\hat{\lambda}_{k_c}}{\hat{\lambda}_{k_c+1}} + \frac{\hat{\lambda}_{k_r}}{\hat{\lambda}_{k_r+1}}\right),$$

(5)

where $\gamma$ depends on the signal-to-noise ratio. Clearly, if $\lambda_{k_c}/\lambda_{k_c+1} \approx 0$ and $\lambda_{k_r}/\lambda_{k_r+1} \approx 0$ and the compressed Laplacians are constructed using the kron reduction then $\hat{\lambda}_{k_c}/\hat{\lambda}_{k_c+1} \approx 0$ and $\hat{\lambda}_{k_r}/\hat{\lambda}_{k_r+1} \approx 0$. Thus, exact recovery is attained.

Let $g(Z) = \gamma_c \text{tr}(Z \tilde{L}_c Z^\top) + \gamma_r \text{tr}(Z \tilde{L}_r Z^\top)$, then $\nabla g(Z) = 2(\gamma_c Z \tilde{L}_c + \gamma_r \tilde{L}_r Z)$. Also define $\text{prox}_{\lambda h}(Z) = Y + \text{sgn}(Z - Y) \circ \max(|Z - Y| - \lambda, 0)$, $\lambda$ as the step size (we use $\lambda = 1/\beta'$), where $\beta \leq \beta' = 2\gamma_c \|L_c\|_2 + 2\gamma_r \|L_r\|_2$ and $\|L\|_2$ is the spectral norm (or maximum eigenvalue) of $L$, $\epsilon$ as the stopping tolerance and $J$ the maximum number of iterations. Then FRPCAG can be solved by the FISTA in Algorithm 1.

**Algorithm 1** FISTA for FRPCAG

**INPUT:** $Z_1 = Y$, $S_0 = 0$, $t_1 = 1$, $\epsilon > 0$

for $j = 1, \ldots J$ do

S$_j$ = prox$_{\lambda h}(Z_j - \lambda_j \nabla g(Z_j))$

t$_{j+1} = 1 + \sqrt{1 + 4t_j^2}$

Z$_{j+1} = S_j + \frac{t_j - 1}{t_{j+1}} (S_j - S_{j-1})$

if $\|Z_{j+1} - Z_j\|_F^2 < \epsilon \|Z_j\|_F^2$ then

BREAK

end if

end for

**OUTPUT:** $U_{j+1}$

**VII. DECODERS FOR LOW-RANK RECOVERY**

Let $\hat{X} \in \mathbb{R}^{p_r \times p_c}$ be the low-rank solution of (4) with the compressed graph Laplacians $\tilde{L}_r, \tilde{L}_c$ and sampled data $\hat{Y}$. The goal is to decode the low-rank $X \in \mathbb{R}^{p \times n}$ for the full $Y$. We assume that $\hat{X} = M_r \hat{X} M_c + \hat{E}$, where $\hat{E} \in \mathbb{R}^{p_r \times p_c}$ models the noise incurred by [4].

**A. Ideal Decoder**

A straight-forward way to decode $X$ on the original graphs $L_r$ and $L_c$, when one knows the basis $P_{k_r}, Q_{k_c}$ involves solving the following optimization problem:

$$\min_{\hat{X}} \|M_r X M_c - \hat{X}\|_F^2, \text{ s.t.: } (X)_i \in \text{span}(P_{k_r}), \ (X^\top)_j \in \text{span}(Q_{k_c}).$$

(6)

**Theorem 2.** Let $M_r$ and $M_c$ be such that (4) holds and $X^*$ be the solution of (6) with $\hat{X} = M_r \hat{X} M_c + \hat{E}$, where $\hat{X} \in \mathcal{LR}(P_{k_r}, Q_{k_c})$ and $\hat{E} \in \mathbb{R}^{p_r \times p_c}$. We have:

$$\|X^* - \hat{X}\|_F \leq 2 \sqrt{n p_{r_c} \rho_r (1 - \delta)} \|\hat{E}\|_F,$$

(7)

where $\sqrt{n p_{r_c} \rho_r (1 - \delta)}$ is a constant resulting from the norm preservation in (2) and $\|\hat{E}\|_F$ is bounded by eq. (5).

**Proof.** Please refer to Appendix B.
Thus, the error of the ideal decoder is only bounded by the error $\hat{E}$ in the low-rank $\hat{X}$ obtained by solving $\ref{eq:ideal_decoder}$. In fact $\hat{E}$ depends on the spectral gaps of $\mathcal{L}_c, \mathcal{L}_r$, as given in eq. $\ref{eq:ideal_decoder_error}$. Hence, the ideal decoder itself does not introduce any error in the decode stage. The solution for this decoder requires projecting over the eigenvectors $P$ and $Q$ of $\mathcal{L}_r$ and $\mathcal{L}_c$. This is computationally expensive because diagonalization of $\mathcal{L}_r$ and $\mathcal{L}_c$ cost $\mathcal{O}(p^3)$ and $\mathcal{O}(n^3)$ respectively. Moreover, the constants $k_r, k_c$ are not known beforehand and require tuning.

B. Alternate Decoder

As the ideal decoder is computationally costly, we propose to decode $X$ from $\hat{X}$ by using a convex and computationally tractable problem which involves the minimization of graph dirichlet energies.

$$\min_{\hat{X}} \| M_r M_c - \hat{X} \|^2_F + \gamma_c \text{tr}(X_L C X^\top) + \gamma_r \text{tr}(X^\top L_r X).$$  \hspace{1cm} \hspace{1cm} (8)

**Theorem 3.** Let $M_r$ and $M_c$ be such that $\ref{eq:ideal_decoder} \hspace{1cm}$ holds and $\gamma > 0$. Let also $X^*$ be the solution of $\ref{eq:ideal_decoder}$ with $\tilde{\gamma}_c = \gamma / \lambda_{k_c + 1}$, $\tilde{\gamma}_r = \gamma / \lambda_{k_r + 1}$, and $\hat{X} = M_r \hat{X} M_c + \hat{E}$, where $\hat{X} \in \mathcal{LR}(P_{k_r}, Q_{k_r})$ and $\hat{E} \in \mathbb{R}^{p_r \times p_c}$. We have:

$$\| \hat{X}^* - \hat{X} \|_F \leq \sqrt{\frac{np}{p_{c}p_{r}(1 - \delta)}} \left[ 2 + \left( \frac{1}{\sqrt{2}} + \sqrt{\gamma} \right) \sqrt{\frac{\lambda_{k_c}}{\lambda_{k_c + 1}}} \right] \| \hat{E} \|_F + \frac{1}{\sqrt{2}} \sqrt{\frac{\lambda_{k_r}}{\lambda_{k_r + 1}}} \| \hat{X} \|_F,$$

and

$$\| E^* \|_F \leq \sqrt{\frac{np}{p_{c}p_{r}(1 - \delta)}} \left[ 2 + \left( \frac{1}{\sqrt{2}} + \sqrt{\gamma} \right) \sqrt{\frac{\lambda_{k_c}}{\lambda_{k_c + 1}}} \right] \| \hat{E} \|_F + \frac{1}{\sqrt{2}} \sqrt{\frac{\lambda_{k_r}}{\lambda_{k_r + 1}}} \| \hat{X} \|_F,$$

where $\hat{X}^* = \text{Proj}_{\mathcal{LR}(P_{k_r}, Q_{k_r})}(X)$ and $E^* = X^* - \hat{X}^*$. $\text{Proj}_{\mathcal{LR}(P_{k_r}, Q_{k_r})}(\cdot)$ denotes the orthogonal projection onto $\mathcal{LR}(P_{k_r}, Q_{k_r})$ and $\gamma$ depends on the signal to noise ratio.

**Proof.** Please refer to Appendix C.

Theorem 3 states that in addition to the error $\hat{E}$ in $\hat{X}$ incurred by $\ref{eq:ideal_decoder}$ and characterized by the bound in eq. $\ref{eq:ideal_decoder_error}$, the error of the alternate decoder $\ref{eq:alternate_decoder}$ also depends on the spectral gaps of the Laplacians $\mathcal{L}_r$ and $\mathcal{L}_c$ respectively. This is the price that one has to pay in order to avoid the expensive ideal decoder. For a $k_r, k_c$ clusterable data $Y$ across the rows and columns, one can expect $\lambda_{k_r} / \lambda_{k_r + 1} \approx 0$ and $\lambda_{k_c} / \lambda_{k_c + 1} \approx 0$ and the solution is as good as the ideal decoder. Nevertheless, it is possible to reduce this error by using graph filters $g$ such that the ratios $g(\lambda_{k_r}) / g(\lambda_{k_r + 1})$ and $g(\lambda_{k_c}) / g(\lambda_{k_c + 1})$ approach zero. However, we do not discuss this approach in our work. It is trivial to solve $\ref{eq:alternate_decoder}$ using a conjugate gradient scheme that costs $\mathcal{O}(InpK)$, where $I$ is the number of iterations for the algorithm to converge.

VIII. APPROXIMATE DECODER: SUBSPACE UPSAMPLING

The alternate decoder proposed above has the following disadvantages:

1) It is almost as computationally expensive as FRPCAG.

2) It requires tuning two model parameters.

In this section we describe the step-by-step construction of an approximate decoder which overcomes these limitations. The main idea is to breakdown the decode phase of low-rank $X$ into its left and right singular vectors or subspaces. Let $X = U \Sigma V^\top$ and $\hat{X} = \hat{U} \hat{\Sigma} \hat{V}^\top$ be the SVD of $X$ and $\hat{X}$. We propose to recover $U$ from $\hat{U}$ and $V$ from $\hat{V}$.

A. Construction of the Alternate Decoder

1) Step 1: Splitting the alternate decoder: Using the SVD of $X$ and $\hat{X}$ and the invariance property of the trace under cyclic permutations, we can replace $\ref{eq:alternate_decoder}$ by:

$$\min_{U, V} \| M_r U \Sigma V^\top M_c - \hat{U} \hat{\Sigma} \hat{V}^\top \|^2_F + \tilde{\gamma}_c \text{tr}(\Sigma^2 V^\top \mathcal{L}_c V) + \tilde{\gamma}_r \text{tr}(U^\top \mathcal{L}_r U \Sigma^2)$$

\hspace{1cm} s.t. $U^\top U = I_k$, $V^\top V = I_k$.

The above eq. introduces several new variables based on the SVD of $X$ and $\hat{X}$. Clearly, one needs to specify $k$ as the dimension of the subspaces $U$ and $V$. First, $\hat{\Sigma}$ can be determined by one inexpensive SVD ($\mathcal{O}(p_r^2 p_c)$ for $p_r < p_c$) of $\hat{X}$ and
then $k$ can be set equal to the number of entries in $\tilde{\Sigma}$ which are above a threshold. It is important to note that this procedure is inexpensive.

Now, if (9) holds for the alternate decoder then $\|\Sigma^* - \tilde{\Sigma}\|_F$ (where $\Sigma^*, \tilde{\Sigma}$ are the singular values of $X^*, \tilde{X}$) is also bounded as argued in the discussion of Appendix C. Thus, the singular values $\Sigma$ and $\tilde{\Sigma}$ of $X$ and $\tilde{X}$ differ approximately by the normalization constant of theorem 1.

Assuming $\tilde{U} = M_r \tilde{U} + \tilde{E}^u$ and $\tilde{V} = V M_c + \tilde{E}^v$, where $(\tilde{U})_i \in \text{span}(P_{k_i})$, $i = 1, \cdots, p$, $(\tilde{V})_j \in \text{span}(Q_{k_i})$, $j = 1, \cdots, n$ and $\tilde{E}^u \in \mathbb{R}^{\rho_r \times r_c}$, $\tilde{E}^v \in \mathbb{R}^{\rho_c \times r_c}$ and $\tilde{X} = U \Sigma V^\top$, where $\tilde{X} \in \mathcal{LR}(P_{k_i}, Q_{k_i})$, we can propose an approximate decoder which separately solves the subspaces $(U, V)$ learning problems.

$$\min_U \|M_r U - \tilde{U}\|_F^2 + \gamma_r^\alpha \text{tr}(U^\top \mathcal{L}_r U) \quad \text{s.t.:} \quad U^\top U = I_k,$$

$$\min_V \|V^\top M_c - \tilde{V}\|_F^2 + \gamma_c^\alpha \text{tr}(V^\top \mathcal{L}_c V) \quad \text{s.t.:} \quad V^\top V = I_k.$$

(10)

Now using $X = U \tilde{\Sigma} V^\top \sqrt{np/r_c (1 - \delta)}$ gives a good approximate solution.

2) Step 2: Dropping Orthonormality Constraints: Solving (10) is as expensive as (9) due to the orthonormality constraints (as explained in appendix D). Therefore, we drop the constraints and get

$$\min_U \|M_r U - \tilde{U}\|_F^2 + \gamma_r^\alpha \text{tr}(U^\top \mathcal{L}_r U),$$

(11)

$$\min_V \|V^\top M_c - \tilde{V}\|_F^2 + \gamma_c^\alpha \text{tr}(V^\top \mathcal{L}_c V).$$

(12)

The solution to (11) & (12) is not orthonormal anymore. The deviation from the orthonormality depends on the constants $\gamma_r^\alpha$ and $\gamma_c^\alpha$, but $X = U \tilde{\Sigma} V^\top \sqrt{np/r_c (1 - \delta)}$ is still low-rank. The above two problems can be solved using traditional iterative methods for linear systems. In fact, the closed form solutions can be written as:

$$U = (M_r^\top M_r + \gamma_r^\alpha \mathcal{L}_r)^{-1} M_r^\top \tilde{U},$$

(13)

$$V = (M_c M_c^\top + \gamma_c^\alpha \mathcal{L}_c)^{-1} M_c \tilde{V}.$$  

(14)

Thus, problems (11) & (12) decode the subspaces $U$ and $V$ such that they are smooth on their respective graphs $\mathcal{L}_r$ and $\mathcal{L}_c$. This can also be referred to as a simultaneous decoding and subspace denoising stage. The columns of $U$ and $V$ are not normalized with the above solution, therefore, a unit norm normalization step is needed at the end.

**Theorem 4.** Let $M_r$ and $M_c$ be such that (2) holds and $\gamma_r^\alpha, \gamma_c^\alpha > 0$. Let also $U^*$ and $V^*$ be respectively the solutions of (11) and (12) with $\tilde{U} = M_r \tilde{U} + \tilde{E}^u$ and $\tilde{V} = V M_c + \tilde{E}^v$, where $(\tilde{U})_i \in \text{span}(P_{k_i})$, $i = 1, \cdots, p$, $(\tilde{V})_j \in \text{span}(Q_{k_i})$, $j = 1, \cdots, n$, $\tilde{E}^u \in \mathbb{R}^{\rho_r \times r_c}$, $\tilde{E}^v \in \mathbb{R}^{\rho_c \times r_c}$. We have:

$$\|\tilde{U}^* - \tilde{U}\|_F \leq \sqrt{\frac{2p}{r_c (1 - \delta)}} \left[ 2 + \frac{1}{\sqrt{\gamma_r^\alpha \lambda_{k_r+1}}} \right] \|\tilde{E}^u\|_F + \left( \frac{\lambda_{k_r}}{\lambda_{k_r+1}} + \sqrt{\gamma_r^\alpha \lambda_{k_r}} \right) \|\tilde{U}\|_F,$$

and

$$\|E^*\|_F \leq \sqrt{\frac{2}{\gamma_r^\alpha \lambda_{k_r+1}}} \|\tilde{E}^u\|_F + \sqrt{\frac{2 \lambda_{k_r}}{\lambda_{k_r+1}}} \|\tilde{U}\|_F.$$ 

where $\tilde{U}^* = P_{k_r} P_{k_r}^\top X$ and $E^* = U^* - \tilde{U}^*$. The same inequalities with slight modification also hold for $V^*$, which we omit because of space constraints.

**Proof.** Please refer to Appendix E. \qed

As $\tilde{X} = \tilde{U} \tilde{\Sigma} \tilde{V}^\top$, we can say that the error with this approximate decoder is upper bounded by the product of the errors of the individual subspace decoders. Also note that the error again depends on the spectral gaps defined by the ratios $\lambda_{k_r}/\lambda_{k_r+1}$ and $\lambda_{k_c}/\lambda_{k_c+1}$. This decoder is less expensive as compared to the alternate decoder and costs $O(\text{IK}kn)$. Furthermore, each of the vectors of the subspaces $U$ and $V$ can be determined in parallel.
3) Step 2: Subspace Upsampling: Since our subspaces \( \tilde{U} \) and \( \tilde{V} \) are determined by the SVD of \( \tilde{X} \) which is noise and outlier free, we can directly upsample them without needing a margin for the noise. Thus, we propose to re-formulate the approximate decoder as follows:

\[
\begin{align*}
\min_U & \quad \text{tr}(U^\top L_r U) \\
\text{s.t.} & \quad M_r U = \tilde{U},
\end{align*}
\]

\[
\begin{align*}
\min_V & \quad \text{tr}(V^\top L_c V) \\
\text{s.t.} & \quad M_c^\top V = \tilde{V}.
\end{align*}
\]

Note that now we have a parameter-free decode stage.

B. Solution of the Subspace Upsampling Decoder

The solution to the above problems is simply a graph upsampling operation as explained in Lemma 1.

**Lemma 1.** Let \( S \in \mathbb{R}^{c \times r} \) and \( R \in \mathbb{R}^{d \times r} \) be the two matrices such that \( d < r \) and \( d < c \). Furthermore, let \( M \in \mathbb{R}^{d \times c} \) be a sampling matrix as constructed in (1) and \( L \in \mathbb{R}^{c \times c} \) be a symmetric positive semi-definite matrix. We can write \( S = [S^\top_a | S^\top_b] \), where \( S_b \in \mathbb{R}^{d \times r} \) and \( S_a \in \mathbb{R}^{(c-d) \times r} \) are the known and unknown submatrices of \( S \). Then the exact and unique solution to the following problem:

\[
\min_{S_a} \text{tr}(S^\top L S), \quad \text{s.t.} \quad MS = R
\]

is given by \( S_a = -L^{-1}_{aa} L_{ab} R \).

**Proof.** Please refer to Appendix F.

Using Lemma 1 and the notation of Section V we can write:

\[
\begin{align*}
U &= \begin{bmatrix}
-L^{-1}_r(\bar{\Omega}_r, \bar{\Omega}_r) L_r(\bar{\Omega}_r, \bar{\Omega}_r) \tilde{U} \\
\tilde{U}
\end{bmatrix} \\
V &= \begin{bmatrix}
-L^{-1}_c(\bar{\Omega}_c, \bar{\Omega}_c) L_c(\bar{\Omega}_c, \bar{\Omega}_c) \tilde{V} \\
\tilde{V}
\end{bmatrix} \tag{17}
\end{align*}
\]

Eqs. (17) involves solving a sparse linear system. If each connected component of the graph has at least one labeled element, \( L_r(\bar{\Omega}_r, \bar{\Omega}_r) \) is full rank and invertible. If the linear system above is not large then one can directly use eq. (17). However, to avoid inverting the big matrix we can use the standard Projected Conjugate Gradient method to solve it. Note that the eqs. (17) and even PCG can be implemented in parallel for every column of \( U \) and \( V \). This gives a significant advantage over the alternate decoder in terms of computation time. The cost of this decoder is \( O(\tilde{O} K n) \) where \( \tilde{O} \) is the number of iterations for the PCG method. The decoder for approximate recovery is presented in Algorithm 2.

**Algorithm 2** Subspace Upsampling based Approximate Decoder for low-rank recovery

**INPUT:** \( \tilde{X} \in \mathbb{R}^{p \times c}, L_r \in \mathbb{R}^{p \times p}, L_c \in \mathbb{R}^{n \times n} \)

1. do SVD(\( \tilde{X} \)) = \( \tilde{U} \tilde{\Sigma} \tilde{V}^\top \)
2. find \( k \) such that \( \tilde{\Sigma}_{k,k}/\tilde{\Sigma}_{1,1} < 0.1 \)
3. Solve eqs. (15) for every column of \( U, V \) as following:

for \( i = 1, \ldots, k \) do

    solve \( \min_{u_i} u_i^\top L_r u_i \) s.t \( M_r u_i = \tilde{u}_i \) using PCG

    solve \( \min_{v_i} v_i^\top L_c v_i \) s.t \( M_c^\top v = \tilde{v}_i \) using PCG

end for

4. Set \( u_i = u_i/\|u_i\|_F, v_i = v_i/\|v_i\|_F, \forall i = 1, \ldots, k \)
5. Set \( \Sigma = \sqrt{\frac{np}{p_r p_c (1-d)}} \Sigma \)
6. Set \( X = U \Sigma V^\top \)

**OUTPUT:** The full low-rank \( X \in \mathbb{R}^{p \times n} \)
IX. OTHER APPROXIMATE DECODERS

Alternatively, if the complete data matrix $Y$ is available then we can reduce the complexity further by performing a graph- upsampling for only one of the two subspaces $U$ or $V$.

A. Approximate decoder 2

Suppose we do the upsampling only for $U$, then the approximate decoder 2 can be written as:

$$\min_U \text{tr}(U^\top L_r U) \quad \text{s.t.} \quad M_r U = \hat{U}.$$  \hspace{1cm} (17)

The solution for $U$ is given by eq. [17] Then, we can write $V$ as:

$$V = Y^\top U \hat{\Sigma}^{-1} \sqrt{\frac{\rho_c \rho_r (1 - \delta)}{np}}.$$  \hspace{1cm} (18)

However, we do not need to explicitly determine $V$ here. Instead the low-rank $X$ can be determined directly from $U$ with the projection given below:

$$X = U \hat{\Sigma} \sqrt{\frac{np}{\rho_c \rho_r (1 - \delta)}} V^\top = U U^\top Y.$$  \hspace{1cm} (19)

B. Approximate decoder 3

Similar to the approximate decoder 2, we can propose another approximate decoder 3 which performs a graph upsampling on $V$ and then determines $U$ via matrix multiplication operation.

$$\min_V \text{tr}(V^\top L_c V) \quad \text{s.t.} \quad M_r^\top V = \hat{V}.$$  \hspace{1cm} (20)

The solution for $V$ is given by eq. [17] Using the similar trick as for the approximate decoder 2, we can compute $X$ without computing $U$. Therefore, $X = Y V V^\top$.

For the proposed approximate decoders, we would need to do one SVD to determine the singular values $(\hat{\Sigma})$. However, note that this SVD is on the compressed matrix $\hat{X} \in \mathbb{R}^{\rho_r \times \rho_c}$. Thus, it is inexpensive $O(\rho_r^2 \rho_c)$ assuming that $\rho_r < \rho_c$.

X. DECODER FOR CLUSTERING: SEMI-SUPERVISED LABEL PROPAGATION

For the clustering application we do not need the full low-rank matrix $X$. Thus, we propose to do k-means on the low-rank representation of the sampled data $\hat{X}$ obtained using (4), extract the cluster labels $\tilde{C}$ and then decode the cluster labels $C$ for $X$ on the graphs $L_r$ and $L_c$. This scheme is similar to the standard semi-supervised label propagation.

Let $\tilde{C} \in \{0, 1\}^{\rho_r \times k}$ be the cluster labels of $\hat{X}$ (for $k$ clusters) which are obtained by performing k-means. Then $\tilde{C}_{ij} = 1$ if $\hat{x}_i \in j^{th}$ cluster and 0 otherwise. We use the same strategy as for the approximate low-rank decoder and propose to solve the following problem:

$$\min_C \text{tr}(C^\top L_c C) \quad \text{s.t.} \quad M_c^\top C = \tilde{C}.$$  \hspace{1cm} (21)

According to Lemma [1] the solution is given by:

$$C = \begin{bmatrix} -L_c^{-1}(\bar{\Omega}_c, \bar{\Omega}_c) L_c (\bar{\Omega}_c, \Omega_c) \tilde{C} \end{bmatrix} \hspace{1cm} (22)$$

The solution $C$ obtained by solving the above problem is not binary and some maximum pooling thresholding needs to be done to get the cluster labels, i.e,

$$C_{ij} \left\{ \begin{array}{ll} 1 & \text{if } C_{ij} = \max\{C_{ij} \forall j = 1 \cdots k\} \\ 0 & \text{otherwise.} \end{array} \right.$$  \hspace{1cm} (23)

Algorithm 3 summarizes this procedure.
**Algorithm 3** Approximate Decoder for clustering

```
INPUT: \( \hat{X} \in \mathbb{R}^{p \times r} \), \( \mathcal{L}_r \in \mathbb{R}^{n \times n} \)
1. do k-means on \( \hat{X} \) to get the labels \( \hat{C} \in \{0, 1\}^{p \times k} \)
2. Solve eqs. (18) for every column of \( C \) as following:
   for \( i = 1, \ldots, k \)
   solve \( \min_{c_i} c_i^T \mathcal{L}_r c_i \) s.t. \( M_r^T c_i = \hat{c}_i \) using PCG
   end for
3. Set \( C_{ij} = 1 \) if \( \max\{C_{ij} \forall j = 1 \cdots k\} \) and 0 otherwise.
OUTPUT: cluster indicators for \( X: C \in \{0, 1\}^{n \times k} \)
```

**Table I:** Summary of CPCA and its computational complexity for a dataset \( Y \in \mathbb{R}^{p \times n} \). Throughout we assume that \( K, k, p, r, c, p \ll n \).

<table>
<thead>
<tr>
<th>Steps</th>
<th>The Complete CPCA Algorithm</th>
<th>Complexity</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Construct graph Laplacians between the rows ( \mathcal{L}_r ) and columns ( \mathcal{L}_c ) of ( Y ) using Section II-A</td>
<td>( \mathcal{O}(np \log(n)) )</td>
</tr>
<tr>
<td>2</td>
<td>Construct row and column sampling matrices ( M_r \in \mathbb{R}^{p \times r} ) and ( M_c \in \mathbb{R}^{n \times r} ) satisfying ( \mathcal{I} ) and theorem ( \mathcal{I} )</td>
<td>–</td>
</tr>
<tr>
<td>3</td>
<td>Sample the data matrix ( Y ) as ( Y = M_r Y M_c )</td>
<td>–</td>
</tr>
<tr>
<td>4</td>
<td>Construct the new graph Laplacians between the rows ( \mathcal{L}_r ) and columns ( \mathcal{L}_c ) of ( Y ) using Section V</td>
<td>( \mathcal{O}(O_{1K}) )</td>
</tr>
<tr>
<td>5</td>
<td>Solve FRPCAG ( {4} ) using Algorithm 1 to get the low-rank ( X )</td>
<td>( \mathcal{O}(I_{r1K}) )</td>
</tr>
<tr>
<td>6</td>
<td><strong>For low-rank recovery:</strong> Decode ( \hat{X} ) from ( X ) using the approximate decoder Algorithm 2</td>
<td>( \mathcal{O}(O_{1nkK}) )</td>
</tr>
<tr>
<td>7</td>
<td><strong>For clustering:</strong> Decode the cluster labels ( C ) for ( X ) using the semi-supervised label propagation (Algorithm 3)</td>
<td>( \mathcal{O}(O_{1nk}) )</td>
</tr>
</tbody>
</table>

**XI. COMPLETE CPCA ALGORITHM & COMPLEXITY**

A summary of all the decoders and their computational complexities is presented in Table XII of Appendix G. The complete CPCA algorithm and the computational complexities of different steps are presented in Table I. For \( K, k, p, r, c, p \ll n \) CPCA algorithm scales as \( \mathcal{O}(nkK) \) per iteration. Thus, assuming that the row and column graphs are available from external source, a speed-up of \( p/k \) per iteration is obtained over FRPCAG and \( p^2/k \) over RPCA. A detailed explanation regarding the calculation of complexities of CPCA and other models is presented in Table XI and Appendix G.

**XII. EXPERIMENTAL RESULTS**

We perform two types of experiments corresponding to two applications of PCA 1) Data clustering and 2) Low-rank recovery using two open-source toolboxes: the UNLocBoX [21] and the GSPBox [20]. It is crucial to point out here that the clustering is not a standard application of PCA, because PCA is just a feature extraction method. However, the clustering experiments had been widely adopted as a standard procedure to demonstrate the quality of the feature extraction methods [8], [34], [38], [11], [12], [3], [30], [11]. Thus, we present clustering experiments to be consistent with the state-of-the-art procedures.

A. Clustering

1) **Experimental Setup:** **Datasets:** We perform our clustering experiments on 5 benchmark databases (as in [28], [29]): CMU PIE, ORL, YALE, MNIST and USPS. For the USPS and ORL dataset, we further run two types of experiments 1) on subset of datasets and 2) on full datasets. The experiments on the subsets of the datasets take less time so they are used to show the efficiency of our model for a wide variety of noise types. The details of all datasets used are provided in Table XII of Appendix G.

**Noise & Errors:** To evaluate the robustness of CPCA to corruptions we add 3 different types of noise in all the samples of datasets in different experiments: 1) Gaussian noise and 2) Laplacian noise with standard deviation ranging from 5% to 20% of the original data 3) Sparse noise (randomly corrupted pixels) occupying 5% to 20% of each data sample.

**Comparison with other methods:** We compare the clustering performance of CPCA with 11 other models including: 1) k-means on original data 2) Laplacian Eigenmaps (LE) [1] 3) Locally Linear Embedding (LLE) [26] 4) Standard PCA 5) Graph Laplacian PCA (GLPCA) [11] 6) Manifold Regularized Matrix Factorization (MMF) [38] 7) Non-negative Matrix Factorization (NMF) [13] 8) Graph Regularized Non-negative Matrix Factorization (GNMF) [3] 9) Robust PCA (RPCA) [4] 10) Robust...
Table II: Clustering error of USPS datasets for different PCA based models. The best results per column are highlighted in bold and the 2nd best in blue. NMF and GNMF require non-negative data so they are not evaluated for USPS because USPS is also negative.

<table>
<thead>
<tr>
<th>Dataset</th>
<th>Model</th>
<th>Gaussian noise</th>
<th>Laplacian noise</th>
<th>Sparse noise</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>no noise</td>
<td>5% 10% 15% 20%</td>
<td>5% 10% 15% 20%</td>
<td>5% 10% 15% 20%</td>
</tr>
<tr>
<td>USPS</td>
<td>k-means</td>
<td>0.31 0.31 0.33 0.32</td>
<td>0.32 0.30 0.36 0.37</td>
<td>0.40 0.45 0.53 0.73</td>
</tr>
<tr>
<td>small</td>
<td>LLE</td>
<td>0.40 0.34 0.32 0.24</td>
<td>0.40 0.40 0.33 0.36</td>
<td>0.23 0.30 0.33 0.37</td>
</tr>
<tr>
<td>(n = 3500</td>
<td>LE</td>
<td>0.38 0.38 0.36 0.35</td>
<td>0.38 0.38 0.38 0.38</td>
<td>0.32 0.33 0.36 0.48</td>
</tr>
<tr>
<td>p = 256</td>
<td>PCA</td>
<td>0.27 0.29 0.28 0.26</td>
<td>0.29 0.29 0.28 0.24</td>
<td>0.29 0.26 0.26 0.28</td>
</tr>
<tr>
<td></td>
<td>MMF</td>
<td>0.21 0.21 0.21 0.21</td>
<td>0.21 0.21 0.22 0.21</td>
<td>0.21 0.23 0.23 0.27</td>
</tr>
<tr>
<td></td>
<td>GLPCA</td>
<td>0.20 0.20 0.21 0.23</td>
<td>0.21 0.21 0.22 0.21</td>
<td>0.26 0.24 0.24 0.28</td>
</tr>
<tr>
<td></td>
<td>RPCA</td>
<td>0.26 0.25 0.23 0.24</td>
<td>0.26 0.26 0.25 0.24</td>
<td>0.26 0.24 0.23 0.30</td>
</tr>
<tr>
<td></td>
<td>FRPCAAG</td>
<td>0.20 0.20 0.21 0.20</td>
<td>0.20 0.21 0.21 0.21</td>
<td>0.21 0.22 0.23 0.25</td>
</tr>
<tr>
<td></td>
<td>CPCA (2,1)</td>
<td>0.20 0.20 0.20 0.22</td>
<td>0.20 0.22 0.22 0.21</td>
<td>0.23 0.23 0.25 0.28</td>
</tr>
<tr>
<td></td>
<td>CPCA (10,1)</td>
<td>0.15 0.14 0.15 0.14</td>
<td>0.16 0.16 0.16 0.17</td>
<td>0.18 0.21 0.21 0.21</td>
</tr>
<tr>
<td></td>
<td>CPCA (5,1)</td>
<td>0.39 0.39 0.38 0.38</td>
<td>0.39 0.39 0.36 0.37</td>
<td>0.39 0.40 0.41 0.50</td>
</tr>
</tbody>
</table>

PCA on Graphs (RPCAG) [28 and 11] Fast Robust PCA on Graphs (FRPCAAG) [29]. RPCA and RPCAG are not used for the evaluation of MNIST, USPS large and ORL large datasets due to computational complexity of these models.

Pre-processing: All datasets are transformed to zero-mean and unit standard deviation along the features / rows. For MMF the samples are additionally normalized to unit-norm. For NMF and GNMF only the unit-norm normalization is applied to all the samples of the dataset as NMF based models can only work with non-negative data.

Evaluation Metric: We use clustering error as a metric to compare the clustering performance of various models. The clustering error for LE, PCA, GLPCA, MMF, NMF and GNMF is evaluated by performing k-means on the principal components V (note that these models explicitly learn V, where X = USV^T). The clustering error for RPCA, RPCAG and FRPCAAG is determined by performing k-means directly on the low-rank X. For our CPCA method, k-means is performed on the small low-rank X and then the labels for full X are decoded using Algorithm 3.

Parameter Selection: To perform a fair validation for each of the models we use a range of values for the model parameters as presented in Table XIII of Appendix C. For a given dataset, each of the models is run for each of the parameter tuples in this table and the parameters corresponding to minimum clustering error are selected for testing purpose. Furthermore, PCA, GLPCA, MMF, NMF and GNMF are non-convex models so they are run 10 times for each of the parameter tuple. RPCA, RPCAG, FRPCAAG and CPCA based models are convex so they are run only once. For our proposed CPCA, we use a convention CPCA(a,b), where a and b denote the downsampling factors on the columns and rows respectively. A uniform sampling strategy is always used for CPCA.

Graph Construction: The K-nearest neighbor graphs G_r, G_c are constructed using FLANN [18] as discussed in Section I-A. The small graphs G_r, G_c can also be constructed using FLANN or the kron reduction strategy of Section V. For all the experiments reported in this paper we use K-nearest neighbors = 10 and Gaussian kernel for the adjacency matrices W. The smoothing parameters \sigma^2 for the Gaussian kernels are automatically set to the average distance of the K-nearest neighbors.
Table III: Clustering error of ORL datasets for different PCA based models. The best results per column are highlighted in bold and the 2nd best in blue.

<table>
<thead>
<tr>
<th>Data set</th>
<th>Model</th>
<th>no noise</th>
<th>Gaussian noise</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>5%</td>
</tr>
<tr>
<td>OR</td>
<td>k-means</td>
<td>0.40</td>
<td>0.43</td>
</tr>
<tr>
<td></td>
<td>LLE</td>
<td>0.26</td>
<td>0.26</td>
</tr>
<tr>
<td></td>
<td>LE</td>
<td>0.21</td>
<td>0.18</td>
</tr>
<tr>
<td></td>
<td>PCA</td>
<td>0.30</td>
<td>0.30</td>
</tr>
<tr>
<td></td>
<td>MMF</td>
<td>0.21</td>
<td>0.20</td>
</tr>
<tr>
<td></td>
<td>GLPCA</td>
<td>0.14</td>
<td>0.13</td>
</tr>
<tr>
<td></td>
<td>NMF</td>
<td>0.31</td>
<td>0.34</td>
</tr>
<tr>
<td></td>
<td>GNMF</td>
<td>0.29</td>
<td>0.29</td>
</tr>
<tr>
<td></td>
<td>RPCA</td>
<td>0.36</td>
<td>0.34</td>
</tr>
<tr>
<td></td>
<td>RPCAG</td>
<td>0.17</td>
<td>0.17</td>
</tr>
<tr>
<td></td>
<td>CPCA (2,2)</td>
<td>0.23</td>
<td>0.23</td>
</tr>
<tr>
<td></td>
<td>CPCA (1,2)</td>
<td>0.17</td>
<td>0.17</td>
</tr>
<tr>
<td>OR</td>
<td>k-means</td>
<td>0.49</td>
<td>0.50</td>
</tr>
<tr>
<td></td>
<td>LLE</td>
<td>0.28</td>
<td>0.27</td>
</tr>
<tr>
<td></td>
<td>LE</td>
<td>0.24</td>
<td>0.25</td>
</tr>
<tr>
<td></td>
<td>PCA</td>
<td>0.35</td>
<td>0.34</td>
</tr>
<tr>
<td></td>
<td>MMF</td>
<td>0.23</td>
<td>0.23</td>
</tr>
<tr>
<td></td>
<td>GLPCA</td>
<td>0.18</td>
<td>0.18</td>
</tr>
<tr>
<td></td>
<td>NMF</td>
<td>0.36</td>
<td>0.33</td>
</tr>
<tr>
<td></td>
<td>GNMF</td>
<td>0.34</td>
<td>0.37</td>
</tr>
<tr>
<td></td>
<td>FRPCAG</td>
<td>0.17</td>
<td>0.17</td>
</tr>
<tr>
<td></td>
<td>CPCA (2,2)</td>
<td>0.21</td>
<td>0.21</td>
</tr>
</tbody>
</table>

2) Discussion: We point out here that the purpose of our clustering experiments is two-fold: 1) To show the efficiency of CPCA for a wide variety of noise and errors and downsampling. 2) To also study the conditions under which CPCA performs worse than the other models. For this purpose, we test CPCA under a variety of downsampling for different datasets. Cases with $p \ll n$ and $n \ll p$ carry special interest. Therefore, we present our discussion below in the light of above two goals.

Clustering Results: Tables II, III, IV & V present the clustering results for USPS small, USPS large, MNIST, MNIST small, ORL small, ORL large, CMU PIE and YALE datasets. Note that not all the models are run for all the datasets due to computational constraints. The best results are highlighted in bold and the second best in blue. From Tables III & V it is clear that our proposed CPCA model attains comparable clustering results to the state-of-the-art RPCAG and FRPCAG models and better than the others in most of the cases for the USPS dataset.

It is important to note that for the USPS and MNIST datasets $p \ll n$. Thus, for the USPS dataset, the compression is only applied along the columns ($n$) of the dataset. This compression results in clustering error which is comparable to the other state-of-the-art algorithms. As $p = 256$ for the USPS dataset, it was observed that even a 2 times downsampling results in a loss of information and the clustering quality deteriorates. The same observation can be made about ORL small, ORL large, CMU PIE and YALE datasets from Tables III & IV for CPCA (2,2), i.e, two times downsampling on both rows and columns. On the other hand the performance of CPCA (1,2) is reasonable for the ORL small dataset. Recall that CPCA (a,b) means a downsampling by $a$ and $b$ across columns and rows (samples and features). Also note that for ORL dataset $n \ll p$.

Finally, we comment about the results on MNIST small dataset ($p = 784$, $n = 1000$) from Table II. It is clear that FRPCAG (no compression) results in the best performance. CPCA (5,1) results in a highly undersampled dataset which does not capture enough variations in the MNIST small dataset to deliver a good clustering performance.

When should the compression be done? The above findings are intuitive as it only makes sense to compress both rows and columns in our CPCA based framework if a reasonable speed-up can be obtained without compromising the performance, i.e, if both $n$ and $p$ are large. If either $n$ or $p$ is small then one might only apply compression along the larger dimension, as the compression on the smaller dimension would not speed up the computations significantly. For example, for the USPS dataset, a speed-up of $p/k = 256/(10 \times 10) \approx 25$ times would be obtained over FRPCAG by compressing along the samples (columns...
Table IV: Clustering error of CMU PIE and YALE datasets for different PCA based models. The best results per column are highlighted in bold and the 2nd best in blue.

<table>
<thead>
<tr>
<th>Data set</th>
<th>Model</th>
<th>no noise</th>
<th>Gaussian noise</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>5%</td>
</tr>
<tr>
<td>C</td>
<td>k-means</td>
<td>0.76</td>
<td>0.76</td>
</tr>
<tr>
<td></td>
<td>LLE</td>
<td>0.47</td>
<td>0.50</td>
</tr>
<tr>
<td></td>
<td>LE</td>
<td>0.60</td>
<td>0.58</td>
</tr>
<tr>
<td></td>
<td>PCA</td>
<td>0.27</td>
<td>0.27</td>
</tr>
<tr>
<td>M</td>
<td>MMF</td>
<td>0.67</td>
<td>0.67</td>
</tr>
<tr>
<td></td>
<td>GLPCA</td>
<td>0.37</td>
<td>0.39</td>
</tr>
<tr>
<td>U</td>
<td>NMF</td>
<td>0.24</td>
<td>0.27</td>
</tr>
<tr>
<td>P</td>
<td>GNMF</td>
<td>0.58</td>
<td>0.59</td>
</tr>
<tr>
<td>I</td>
<td>RPCA</td>
<td>0.39</td>
<td>0.38</td>
</tr>
<tr>
<td>E</td>
<td>RPCAG</td>
<td>0.24</td>
<td>0.24</td>
</tr>
<tr>
<td></td>
<td>FRPCAG</td>
<td>0.23</td>
<td>0.24</td>
</tr>
<tr>
<td></td>
<td>CPCA (2,2)</td>
<td>0.26</td>
<td>0.26</td>
</tr>
<tr>
<td></td>
<td>CPCA (2,1)</td>
<td>0.28</td>
<td>0.29</td>
</tr>
<tr>
<td>Y</td>
<td>k-means</td>
<td>0.76</td>
<td>0.76</td>
</tr>
<tr>
<td></td>
<td>LLE</td>
<td>0.51</td>
<td>0.47</td>
</tr>
<tr>
<td></td>
<td>LE</td>
<td>0.52</td>
<td>0.56</td>
</tr>
<tr>
<td></td>
<td>PCA</td>
<td>0.53</td>
<td>0.52</td>
</tr>
<tr>
<td>A</td>
<td>MMF</td>
<td>0.58</td>
<td>0.59</td>
</tr>
<tr>
<td>L</td>
<td>GLPCA</td>
<td>0.47</td>
<td>0.46</td>
</tr>
<tr>
<td>E</td>
<td>NMF</td>
<td>0.57</td>
<td>0.58</td>
</tr>
<tr>
<td></td>
<td>GNMF</td>
<td>0.59</td>
<td>0.59</td>
</tr>
<tr>
<td></td>
<td>RPCA</td>
<td>0.45</td>
<td>0.48</td>
</tr>
<tr>
<td></td>
<td>RPCAG</td>
<td>0.40</td>
<td>0.40</td>
</tr>
<tr>
<td></td>
<td>FRPCAG</td>
<td>0.40</td>
<td>0.37</td>
</tr>
<tr>
<td></td>
<td>CPCA (2,2)</td>
<td>0.43</td>
<td>0.45</td>
</tr>
</tbody>
</table>

n) only without a loss of clustering quality. Our experiments showed that this speed up increased up to 30 by compressing along the features but with a loss of performance (The results are not presented for brevity).

3) Robustness to Noise & Errors: Tables II, III, IV & V also show that CPCA is quite robust to a variety of noise and errors in the dataset. Even in the presence of higher levels of Gaussian and Laplacian noise, CPCA performs better than and comparable to other methods for the USPS dataset.

4) Computation Time vs Performance & Sampling: It is interesting to compare 1) the time needed for FRPCAG and CPCA to perform clustering 2) the corresponding clustering error and 3) the sub-sampling rates in CPCA. Table V shows such a comparison for 70,000 digits of MNIST with (10, 2) times downsampling on the (columns, rows) respectively for CPCA. The time needed by CPCA is an order of magnitude lower than FRPCAG. Surprisingly, the error of CPCA is also lower than FRPCAG. Such cases can also be observed in USPS dataset (Table II). Downsampling removes spurious samples sometimes and the voting scheme (Section X) becomes robust to these samples which lie on the cluster borders. Note that the time reported here does not include the construction of graphs $G_r, G_c$ as both methods use the same graphs. Furthermore, these graphs can be constructed in the order of a few seconds if parallel processing is used. The time for CPCA includes steps 2 to 5 of Table I.

Table V: Clustering error and computational times of FRPCAG and CPCA on MNIST dataset (784 × 70,000).

<table>
<thead>
<tr>
<th>Model</th>
<th>FRPCAG</th>
<th>CPCA (10,2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Error</td>
<td>0.25</td>
<td>0.24</td>
</tr>
<tr>
<td>time (secs)</td>
<td>350</td>
<td>58</td>
</tr>
</tbody>
</table>

5) Computation Time for Clustering: Table VI presents the computational time and number of iterations for the convergence of CPCA, FRPCAG, RPCAG & RPCA on different sizes and dimensions of the datasets. We also present the time needed for the graph construction. The computation is done on a single core machine with a 3.3 GHz processor without using any distributed or parallel computing tricks. An $\infty$ in the table indicates that the algorithm did not converge in 4 hours. It is notable
<table>
<thead>
<tr>
<th>Dataset</th>
<th>Samples</th>
<th>Features</th>
<th>Classes</th>
<th>Graphs</th>
<th>FRPCAG</th>
<th>CPCA</th>
<th>RPCAG</th>
<th>RPCA</th>
</tr>
</thead>
<tbody>
<tr>
<td>MNIST</td>
<td>5000</td>
<td>784</td>
<td>10</td>
<td>$G_r$, $G_c$, FRPCAG, CPCA, RPCAG, RPCA and the number of iterations to converge for different datasets. The computation is done on a single core machine with a 3.3 GHz processor without using any distributed or parallel computing tricks. $\infty$ indicates that the algorithm did not converge in 4 hours.</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>MNIST</td>
<td>15000</td>
<td>784</td>
<td>10</td>
<td>10.8</td>
<td>4.3</td>
<td>13.7</td>
<td>27</td>
<td>(5,1)</td>
</tr>
<tr>
<td>MNIST</td>
<td>25000</td>
<td>784</td>
<td>10</td>
<td>32.5</td>
<td>13.3</td>
<td>35.4</td>
<td>23</td>
<td>(5,1)</td>
</tr>
<tr>
<td>ORL</td>
<td>300</td>
<td>10304</td>
<td>30</td>
<td>40.7</td>
<td>22.2</td>
<td>58.6</td>
<td>24</td>
<td>(10,1)</td>
</tr>
<tr>
<td>USPS</td>
<td>3500</td>
<td>256</td>
<td>10</td>
<td>1.8</td>
<td>56.4</td>
<td>24.7</td>
<td>12</td>
<td>(2,1)</td>
</tr>
</tbody>
</table>

Table VII: Preservation of the rank of the datasets in the compressed low-rank $\tilde{X}$ determined by solving FRPCAG (4).

<table>
<thead>
<tr>
<th>Dataset</th>
<th>downsampling factor</th>
<th>actual rank</th>
<th>Rank after FRPCAG on sampled matrix</th>
</tr>
</thead>
<tbody>
<tr>
<td>ORL</td>
<td>(2,1)</td>
<td>10</td>
<td>41</td>
</tr>
<tr>
<td>USPS</td>
<td>(10,2)</td>
<td>10</td>
<td>41</td>
</tr>
<tr>
<td>MNIST</td>
<td>(10,2)</td>
<td>10</td>
<td>41</td>
</tr>
<tr>
<td>CMU PIE</td>
<td>(2,1)</td>
<td>10</td>
<td>41</td>
</tr>
<tr>
<td>YALE</td>
<td>(2,1)</td>
<td>10</td>
<td>41</td>
</tr>
</tbody>
</table>

Table VIII: Variation of clustering error of CPCA with different uniform downsampling schemes / factors across rows and columns of the USPS dataset ($256 \times 10,000$).

<table>
<thead>
<tr>
<th>downsampling (rows / cols)</th>
<th>1</th>
<th>5</th>
<th>10</th>
<th>15</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.16</td>
<td>0.16</td>
<td>0.21</td>
<td>0.21</td>
<td>0.21</td>
</tr>
<tr>
<td>2</td>
<td>0.21</td>
<td>0.23</td>
<td>0.23</td>
<td>0.24</td>
<td>0.25</td>
</tr>
<tr>
<td>4</td>
<td>0.26</td>
<td>0.30</td>
<td>0.31</td>
<td>0.31</td>
<td>0.31</td>
</tr>
</tbody>
</table>

7) Clustering error vs downsampling rate: Table VIII shows the variation of clustering error of CPCA with different downsampling factors across rows and columns of the USPS dataset ($256 \times 10,000$). Obviously, higher downsampling results in an increase in the clustering error. However, note that we can downsample the samples (columns) by a factor of 5 without observing an error increase. The downsampling of features results in an error increase because the number of features for this dataset is only 256 and downsampling results in a loss of information. Similar behavior can also be observed for the ORL small and ORL large datasets in Table III where the performance of CPCA is slightly worse than FRPCAG because the number of samples $n$ for ORL is only 400.
Background separation from videos

![original](original.png)  | RPCA | RPCAG | FRPCAG | CPCA (5,1) | CPCA (10,4)
--- | --- | --- | --- | --- | ---
![RPCA](RPCA.png)  | ![RPCAG](RPCAG.png) | ![FRPCAG](FRPCAG.png) | ![CPCA (5,1)](CPCA_5_1.png) | ![CPCA (10,4)](CPCA_10_4.png)

**Figure 1:** Static background separation from videos using different PCA based models. The first row corresponds to a frame from the video of a shopping mall lobby, the second row to a restaurant food counter and the third row to an airport lobby. The leftmost plot in each row shows the actual frame, the other 5 show the recovered low-rank using RPCA, RPCAG, FRPCAG and CPCA with two different uniform downsampling schemes.

![alternate decoder](alternate_decoder.png)  | approximate decoder 1 | approximate decoder 2 | approximate decoder 3
--- | --- | --- | ---
![alternate decoder](alternate_decoder.png)  | ![approximate decoder 1](approximate_decoder_1.png) | ![approximate decoder 2](approximate_decoder_2.png) | ![approximate decoder 3](approximate_decoder_3.png)

**Figure 2:** A quality comparison of various low-rank decoders discussed in this work.

### B. Low-rank recovery

1) **Background Separation from Videos:** In order to demonstrate the effectiveness of our model to recover low-rank static background from videos we perform experiments on 1000 frames of 3 videos available online. All the frames are vectorized and arranged in a data matrix $Y$ whose columns correspond to frames. The graph $G_c$ is constructed between the columns of $Y$ and the graph $G_r$ is constructed between the rows of $Y$ following the methodology of Section II-A. Fig. 1 shows the recovery of low-rank frames for one actual frame of each of the videos. The leftmost plot in each row shows the actual frame, the other 5 show the recovered low-rank representations using RPCA, RPCAG, FRPCAG and CPCA with two different uniform downsampling rates. For CPCA Algorithm 2 is used and $k$ is set such that $\tilde{\sum}_{k,k}/\tilde{\sum}_{1,1} < 0.1$.

For the 2nd and 3rd rows of Fig. 1 it can be seen that our proposed model is able to separate the static backgrounds very accurately from the moving people which do not belong to the static ground truth. However, the quality is slightly compromised in the 1st row where the shadow of the person appears in the low-rank frames recovered with CPCA. In fact, this person remains static for a long time in the video and the uniform sampling compromises the quality slightly.

2) **Computation Time for Background Separation:** Table IX presents the computational time in seconds of RPCA, RPCAG, FRPCAG and CPCA for low-rank recovery of different videos in Fig. 1. The time reported here corresponds to steps 2 to 5 and 7 of Table I algorithm 1 of [29] for FRPCAG, [4] for RPCA and [28] for RPCAG, excluding the construction of graphs $G_r, G_c$. The speed-up observed for these experiments from Table IX is 10 times over FRPCAG and 100 times over RPCA.
Table IX: Computational time in seconds of RPCA, RPCAG, FRPCAG and CPCA for low-rank recovery of different videos in Fig. 1.

<table>
<thead>
<tr>
<th>Videos</th>
<th>RPCA</th>
<th>RPCAG</th>
<th>FRPCAG</th>
<th>CPCA (5,1)</th>
<th>CPCA (10,4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2700</td>
<td>3550</td>
<td>120</td>
<td>21</td>
<td>8</td>
</tr>
<tr>
<td>2</td>
<td>1650</td>
<td>2130</td>
<td>85</td>
<td>15</td>
<td>6</td>
</tr>
<tr>
<td>3</td>
<td>3650</td>
<td>4100</td>
<td>152</td>
<td>32</td>
<td>11</td>
</tr>
</tbody>
</table>

Figure 3: A comparison of the quality of low-rank for the shopping mall video (1st row of Fig. 1) extracted using the approximate decoder (17) for different downsampling factors on the pixels and frames. It is obvious that the quality of low-rank remains intact even with higher downsampling factors.

3) Quality Comparison of Decoders: Fig. 2 presents a comparison of the quality of the low-rank static background extracted using the alternate (8) and approximate decoders discussed in (17) for a video (1st row) of Fig. 1. Clearly, the alternate decoder performs slightly better than the approximate decoders but at the price of tuning of two model parameters.

4) Quality of Low-Rank with Downsampling: Fig. 3 presents a comparison of the quality of low-rank for the same video extracted using the approximate decoder (17) using different downsampling factors on the pixels and frames. It is obvious that the quality of low-rank remains intact even with higher downsampling factors.

XIII. CONCLUSION

We present Compressive PCA on Graphs (CPCA) which approximates a recovery of low-rank matrices on graphs from their sampled measurements. It is supported by the proposed restricted isometry property (RIP) which is related to the coherence of the eigenvectors of graphs between the rows and columns of the data matrix. Accompanied with several efficient, parallel, parameter free and low-cost decoders for low-rank recovery and clustering, the presented framework gains a several orders of magnitude speed-up over the low-rank recovery methods like Robust PCA. Our theoretical analysis reveals that CPCA targets exact recovery for low-rank matrices which are clusterable across the rows and columns. Thus, the error depends on the spectral gaps of the graph Laplacians. Extensive clustering experiments on 5 datasets with various types of noise and comparison with 11 state-of-the-art methods reveal the efficiency of our model. CPCA also achieves state-of-the-art results for background separation from videos.

REFERENCES


A. Proof of theorem \[2\]

We start with the sampling of the rows. Theorem 5 in \[23\] shows that for any \(\delta_r, \epsilon_r \in (0,1)\), with probability at least \(1 - \epsilon_r\),

\[
(1 - \delta_r)\|z\|_2^2 \leq \frac{p}{\rho_r} \|M_r z\|_2^2 \leq (1 + \delta_r)\|z\|_2^2
\]

for all \(z \in \text{span}(P_{k_r})\) provided that

\[
\rho_r \geq \frac{3}{\delta_r^2} \nu_{k_r}^2 \log \left(\frac{2k_r}{\epsilon_r}\right).
\]

Notice that Theorem 5 in \[23\] is a uniform result. As a consequence, with probability at least \(1 - \epsilon_r\),

\[
(1 - \delta_r)\|y_i\|_2^2 \leq \frac{p}{\rho_r} \|M_r y_i\|_2^2 \leq (1 + \delta_r)\|y_i\|_2^2, \quad i = 1, \ldots, n,
\]

for all \(y_1, \ldots, y_n \in \text{span}(P_{k_r})\) provided that \[20\] holds. Summing the previous inequalities over all \(i\) shows that, with probability at least \(1 - \epsilon_r\),

\[
(1 - \delta_r)\|Y\|_F^2 \leq \frac{p}{\rho_r} \|M_r Y\|_F^2 \leq (1 + \delta_r)\|Y\|_F^2,
\]

for all \(Y \in \mathbb{R}^{p \times n}\) with column-vectors in \(\text{span}(P_{k_r})\).

Let us continue with the sampling of the columns. Again, Theorem 5 in \[23\] shows that for any \(\delta_c, \epsilon_c \in (0,1)\), with probability at least \(1 - \epsilon_c\),

\[
(1 - \delta_c)\|w\|_2^2 \leq \frac{n}{\rho_c} \|w^T M_c\|_2^2 \leq (1 + \delta_c)\|w\|_2^2
\]

for all \(w \in \text{span}(Q_{k_c})\) provided that

\[
\rho_c \geq \frac{3}{\delta_c^2} \nu_{k_c}^2 \log \left(\frac{2k_c}{\epsilon_c}\right).
\]

As a consequence, with probability at least \(1 - \epsilon_c\),

\[
(1 - \delta_c)\|z_i\|_2^2 \leq \frac{n}{\rho_c} \|z_i^T M_c\|_2^2 \leq (1 + \delta_c)\|z_i\|_2^2, \quad i = 1, \ldots, \rho_r,
\]

for all \(z_1, \ldots, z_{\rho_r} \in \text{span}(Q_{k_c})\) provided that \[23\] holds. Summing the previous inequalities over all \(i\) shows that, with probability at least \(1 - \epsilon_c\),

\[
(1 - \delta_c)\|Z\|_F^2 \leq \frac{n}{\rho_c} \|Z M_c\|_F^2 \leq (1 + \delta_c)\|Z\|_F^2
\]

for all \(Z \in \mathbb{R}^{p \times n}\) with row-vectors in \(\text{span}(Q_{k_c})\). In particular, this property holds, with at least the same probability, for all matrices \(Z\) of the form \(M_r Y\) where \(Y \in \mathbb{R}^{p \times n}\) is a matrix with row-vectors in \(\text{span}(Q_{k_c})\).

We now continue by combining \[22\] and \[25\]. We obtain that

\[
(1 - \delta_r)(1 - \delta_c)\|Y\|_F^2 \leq \frac{np}{\rho_r \rho_c} \|M_r Y M_c\|_F^2 \leq (1 + \delta_r)(1 - \delta_c)\|Y\|_F^2
\]

for all \(Y \in \mathbb{R}^{p \times n}\) with column-vectors in \(\text{span}(P_{k_r})\) and row-vectors in \(\text{span}(Q_{k_c})\), provided that \[20\] and \[23\] hold. It remains to compute the probability that \[26\] holds. Property \[26\] does not hold if \[22\] or \[25\] do not hold. Using the union bound, \[26\] does not hold with probability at most \(\epsilon_r + \epsilon_c\). To finish the proof, one just need to choose \(\epsilon_r = \epsilon_c = \epsilon/2\) and \(\delta_r = \delta_c = \delta/3\), and notice that \((1 + \delta/3)^2 \leq 1 + \delta\) and \((1 - \delta/3)^2 \geq 1 - \delta\) for \(\delta \in (0,1)\).
B. Proof of Theorem 2

Using the optimality condition we have, for any $Z \in \mathbb{R}^{p \times n}$,

$$\|M_r X^* M_e - \tilde{X}\|_F \leq \|M_r Z M_e - \tilde{X}\|_F.$$ 

For $Z = \tilde{X}$, we have

$$\|M_r X^* M_e - \tilde{X}\|_F \leq \|M_r \tilde{X} M_e - \tilde{X}\|_F,$$

which gives

$$\|M_r X^* M_e - M_r \tilde{X} M_e - \tilde{E}\|_F \leq \|\tilde{E}\|_F.$$ 

As (2) holds, we have

$$\sqrt{\rho_r \rho_c (1 - \delta)} \|X^* - \tilde{X}\|_F \leq \|\tilde{E}\|_F.$$

Therefore, we get

$$\|X^* - \tilde{X}\|_F \leq 2 \sqrt{\frac{np}{\rho_r \rho_c (1 - \delta)}} \|\tilde{E}\|_F.$$ 

C. Proof of Theorem 4

Using the optimality condition we have for any $Z \in \mathbb{R}^{p \times n}$ and optimal solution $X^* = \tilde{X} + E^*$:

$$\|M_r X^* M_e - \tilde{X}\|_F^2 + \gamma_c \text{tr}(X^* L_e X^{*\top}) + \gamma_r \text{tr}(X^{*\top} L_r X^*) \leq \|M_r Z M_e - \tilde{X}\|_F^2 + \gamma_c \text{tr}(Z^\top L_e Z) + \gamma_r \text{tr}(Z^\top L_r Z)$$ (27)

using $Z = \tilde{X} = P_{k_r} Y_{b_r} Q_{k_c}^\top$ as in the proof of theorem 2 in [29], where $Y_{b_r} \in \mathbb{R}^{k_r \times k_c}$ and it is not necessarily diagonal. Note that $\|Y_{b_r}\|_F = \|\tilde{X}\|_F$, $Q_{k_c}^\top Q_{k_c} = I_{k_c}$, $P_{k_r}^\top P_{k_r} = I_{k_r}$, $\tilde{Q}_{k_r}^\top Q_{k_r} = 0$, $\tilde{P}_{k_r}^\top P_{k_r} = 0$. From the proof of theorem 2 in [29] we also know that:

$$\text{tr}(\tilde{X} L_e (\tilde{X})^\top) \leq \lambda_{k_r} \|\tilde{X}\|_F^2$$

$$\text{tr}(\tilde{X}^\top L_r (\tilde{X})) \leq \lambda_{k_r} \|\tilde{X}\|_F^2$$

$$\text{tr}(X^* L_e (X^*)^\top) \geq \lambda_{k_c+1} \|X^* Q_{k_c}\|_F^2$$

$$\text{tr}(X^* L_r (X^*)^\top) \geq \lambda_{k_{c+1}} \|\tilde{P}_{k_r}^\top X^*\|_F^2$$

Now using all this information in (27) we get

$$\|M_r X^* M_e - \tilde{X}\|_F^2 + \gamma_c \lambda_{k_c+1} \|X^* Q_{k_c}\|_F^2 + \gamma_r \lambda_{k_{c+1}} \|\tilde{P}_{k_r}^\top X^*\|_F^2 \leq \|\tilde{E}\|_F^2 + (\gamma_c \lambda_{k_c} + \gamma_r \lambda_{k_r}) \|\tilde{X}\|_F^2$$

From above we have:

$$\|M_r X^* M_e - \tilde{X}\|_F \leq \|\tilde{E}\|_F + \sqrt{(\gamma_c \lambda_{k_c} + \gamma_r \lambda_{k_r}) \|\tilde{X}\|_F^2}$$ (28)

and

$$\sqrt{(\gamma_c \lambda_{k_c+1} \|X^* Q_{k_c}\|_F^2 + \gamma_r \lambda_{k_{c+1}} \|\tilde{P}_{k_r}^\top X^*\|_F^2)} \leq \|\tilde{E}\|_F + \sqrt{(\gamma_c \lambda_{k_c} + \gamma_r \lambda_{k_r}) \|\tilde{X}\|_F^2}$$ (29)

using

$$\tilde{\gamma}_c = \gamma \frac{1}{\lambda_{k_c+1}} \quad \text{and} \quad \tilde{\gamma}_r = \gamma \frac{1}{\lambda_{k_{c+1}}}.$$ 

and

$$\|E^*\|_F^2 = \|X^* Q_{k_c}\|_F^2 = \|\tilde{P}_{k_r}^\top X^*\|_F^2$$
we get:
\[
\| M_r X^* M_c - \tilde{X} \|_F \leq \| \tilde{E} \|_F + \sqrt{\gamma} \left( \frac{\lambda_{k_c}}{\lambda_{k_c+1}} + \frac{\lambda_{k_r}}{\lambda_{k_r+1}} \right) \| \tilde{X} \|_F
\]
(30)
and
\[
\sqrt{2\gamma} \| E^* \|_F \leq \| \tilde{E} \|_F + \sqrt{\gamma} \left( \frac{\lambda_{k_c}}{\lambda_{k_c+1}} + \frac{\lambda_{k_r}}{\lambda_{k_r+1}} \right) \| \tilde{X} \|_F
\]
(31)
which implies
\[
\| E^* \|_F \leq \frac{\| \tilde{E} \|_F}{\sqrt{2\gamma}} + \frac{1}{\sqrt{2}} \sqrt{\gamma} \left( \frac{\lambda_{k_c}}{\lambda_{k_c+1}} + \frac{\lambda_{k_r}}{\lambda_{k_r+1}} \right) \| \tilde{X} \|_F
\]
(32)
Focus on \( \| M_r X^* M_c - \tilde{X} \|_F^2 \) now. As \( M_r, M_c \) are constructed with a sampling without replacement, we have \( \| M_r E^* M_c \|_F \leq \| E^* \|_F \). Now using the above facts and the RIP we get:
\[
\| M_r X^* M_c - \tilde{X} \|_F = \| M_r (\tilde{X}^* + E^*) M_c - M_r \tilde{X} M_c - \tilde{E} \|_F
\]
\[
\geq \sqrt{\frac{\rho_c \rho_r (1 - \delta)}{np}} \| \tilde{X}^* - \tilde{X} \|_F - \| \tilde{E} \|_F - \| E^* \|_F
\]
this implies
\[
\| \tilde{X}^* - \tilde{X} \|_F \leq \sqrt{\frac{np}{\rho_c \rho_r (1 - \delta)}} \left( \frac{1}{\sqrt{2} \gamma} \| \tilde{E} \|_F + \left( \frac{1}{\sqrt{2} + \sqrt{\gamma}} \right) \left( \frac{\lambda_{k_c}}{\lambda_{k_c+1}} + \frac{\lambda_{k_r}}{\lambda_{k_r+1}} \right) \| \tilde{X} \|_F \right)
\]

**Discussion** Let \( A_1, A_2 \in \mathbb{R}^{p \times n} \) and \( A_1 = U_1 S_1 V_1^T \), \( A_2 = U_2 S_2 V_2^T \) then if \( \| A_1 - A_2 \|_F^2 \to 0 \), then \( S_1 \to S_2 \).

We observe that
\[
\| A_1 - A_2 \|_F^2 = \| U_1 S_1 V_1^T - U_2 S_2 V_2^T \|_F^2 = \| U_2^T U_1 S_1 V_1^T V_2 - S_2 \|_F^2
\]
which implies that \( U_2^T U_1 S_1 V_1^T V_2 \approx S_2 \). This is equivalent to saying that for the significant values of \( S_2 \), the orthonormal matrices \( U_2^T U_1 \) and \( V_1^T V_2 \) have to be almost diagonal. As a result, for the significant values of \( S_2 \), \( U_2 \) and \( V_2 \) have to be aligned with \( U_1 \) and \( V_1 \). The same reason also implies that \( S_1 \approx S_2 \).

**D. Solution of eq. (10)**

Let us examine how to solve (10). The problem can be reformulated as:
\[
\min_U \text{tr}(U^T L_r U) \quad \text{s.t.} \quad U^T U = I_k, \quad \| M_r U - \tilde{U} \|_F^2 < \epsilon
\]

Let \( U' \) be the zero appended matrix of \( \tilde{U} \), then we can re-write it as:
\[
\min_U \text{tr}(U^T L_r U) \quad \text{s.t.} \quad U^T U = I_k, \quad \| M_r (U - U') \|_F^2 < \epsilon
\]
The above problem is equivalent to (10), as the term \( \| M_r (U - U') \|_F^2 \) has been removed from the objective and introduced as a constraint. Note that the constant \( \gamma_r \) is not needed anymore. The new model parameter \( \epsilon \) controls the radius of the \( L_2 \) ball \( \| M_r (U - U') \|_F^2 \). In simple words it controls how much noise is tolerated by the projection of \( U \) on the ball that is centered at \( U' \). To solve the above problem one needs to split it down into two sub-problems and solve iteratively between:

1) The optimization \( \min_U \text{tr}(U^T L_r U) \quad \text{s.t.} \quad U^T U = I_k \). The solution to this problem is given by the lowest \( k \) eigenvectors of \( L_r \). Thus it requires a complexity of \( O((n + p)k^2) \) for solving both problems (10).

2) The projection on the \( L_2 \) ball \( \| M_r (U - U') \|_F^2 \) whose complexity is \( O(\rho_c + \rho_r) \).

Thus the solution requires a double iteration with a complexity of \( O(Ink^2) \) and is almost as expensive as FRPCAG.
E. Proof of Theorem 4

We can write (11) and (12) as following:

\[ \min_{u_{i_1}, \ldots, u_{i_p}} \sum_{i=1}^{p} \left[ \|M_r u_i - \tilde{u}_i\|_2^2 + \gamma' u_i^\top L_r u_i \right] \] (33)

\[ \min_{v_{i_1}, \ldots, v_{i_n}} \sum_{i=1}^{n} \left[ \|M_r^\top v_i - \tilde{v}_i\|_2^2 + \gamma' v_i^\top L_r v_i \right] \] (34)

In this proof, we only treat Problem (33) and the recovery of \( \bar{F} \). The proof for Problem (12) and the recovery of \( \bar{V} \) is identical. The above two problems can be solved independently for every \( i \). From theorem 3.2 of [23] we obtain:

\[ \| \bar{u}_i^* - \bar{u}_i \|_2 \leq \sqrt{\frac{p}{\rho_r (1 - \delta)}} \left[ 2 + \frac{1}{\sqrt{\gamma_r \lambda_{k_r + 1}}} \right] \| \bar{e}_i^* \|_2 + \left( \sqrt{\frac{\lambda_{k_r}}{\lambda_{k_r + 1}}} + \sqrt{\frac{\gamma_r}{\lambda_{k_r}}} \right) \| \bar{u}_i \|_2 \],

and

\[ \| \bar{e}_i^* \|_2 \leq \sqrt{\frac{p}{\gamma_r \lambda_{k_r + 1}}} \| \bar{e}_i^* \|_2 + \sqrt{\frac{\lambda_{k_r}}{\lambda_{k_r + 1}}} \| \bar{u}_i \|_2 , \]

which implies

\[ \| \bar{u}_i^* - \bar{u}_i \|_2^2 \leq \frac{2}{\gamma_r \lambda_{k_r + 1}} \| \bar{e}_i^* \|_2^2 + 2 \sqrt{\frac{\lambda_{k_r}}{\lambda_{k_r + 1}}} \| \bar{u}_i \|_2^2 , \]

Summing the previous inequalities over all \( i \)'s yields

\[ \| \bar{U}^* - \bar{U} \|_F^2 \leq \frac{2}{\rho_r (1 - \delta)} \left[ 2 + \frac{1}{\sqrt{\gamma_r \lambda_{k_r + 1}}} \right] \| \bar{E}^u \|_F^2 + \left( \sqrt{\frac{\lambda_{k_r}}{\lambda_{k_r + 1}}} + \sqrt{\frac{\gamma_r}{\lambda_{k_r}}} \right) \| \bar{U} \|_F^2 , \]

and

\[ \| \bar{E}^* \|_F^2 \leq \frac{2}{\gamma_r \lambda_{k_r + 1}} \| \bar{E}^u \|_F^2 + 2 \sqrt{\frac{\lambda_{k_r}}{\lambda_{k_r + 1}}} \| \bar{U} \|_F^2 . \]

Taking the square root of both inequalities terminates the proof. Similarly, the expressions for \( \bar{V} \) can be derived:

\[ \| \bar{V}^* - \bar{V} \|_F \leq \sqrt{\frac{2}{\rho_c (1 - \delta)}} \left[ 2 + \frac{1}{\sqrt{\gamma_c \lambda_{k_c + 1}}} \right] \| \bar{E}^v \|_F + \left( \sqrt{\frac{\lambda_{k_c}}{\lambda_{k_c + 1}}} + \sqrt{\frac{\gamma_c}{\lambda_{k_c}}} \right) \| \bar{V} \|_F , \]

and

\[ \| \bar{E}^* \|_F \leq \sqrt{\frac{2}{\gamma_c \lambda_{k_c + 1}}} \| \bar{E}^v \|_F + \sqrt{2 \frac{\lambda_{k_c}}{\lambda_{k_c + 1}}} \| \bar{V} \|_F . \]

F. Proof of Lemma 7

Let \( S = [S_a^\top | S_b^\top]^\top \). Further we split \( \mathcal{L} \) into submatrices as follows:

\[ \mathcal{L} = \begin{bmatrix} \mathcal{L}_{aa} & \mathcal{L}_{ab} \\ \mathcal{L}_{ba} & \mathcal{L}_{bb} \end{bmatrix} \]

Now (16) can be written as:

\[ \min_{S_a} \begin{bmatrix} S_a \\ S_b \end{bmatrix}^\top \begin{bmatrix} \mathcal{L}_{aa} & \mathcal{L}_{ab} \\ \mathcal{L}_{ba} & \mathcal{L}_{bb} \end{bmatrix} \begin{bmatrix} S_a \\ S_b \end{bmatrix} \]

s.t: \( S_b = R \)
further expanding we get:
\[
\min_{S_a} S_a^T L_{aa} S_a + S_a^T L_{ab} R + R^T L_{ba} S_a + R L_{ab} R
\]
using \( \nabla S_a = 0 \), we get:
\[
2 L_{ab} R + 2 L_{aa} S_a = 0
\]
\[
S_a = -L_{aa}^{-1} L_{ab} R
\]

G. Computational Complexities & Additional Results

We present the computational complexity of all the models considered in this work. For a matrix \( X \in \mathbb{R}^{p \times n} \), let \( I \) denote the number of iterations for the algorithms to converge, \( p \) is the number of features, \( n \) is the number of samples, \( \rho_r, \rho_c \) are the number of features and samples for the compressed data \( \tilde{Y} \) and satisfy eq. (1) and theorem 1, \( k \) is the rank of the low-dimensional space or the number of clusters, \( K \) is the number of nearest neighbors for graph construction, \( O_l, O_c \) correspond to the number of iterations in the Lanczos and Chebyshev approximation methods. All the models which use the graph \( G_c \) are marked by '+'. The construction of graph \( G_r \) is included only in FRPCAG and CPCA. Furthermore,

1) We assume that \( K, k, \rho_r, \rho_c, p << n \) and \( n + p + k + K + \rho_r + \rho_c \approx n \).

2) The complexity of \( \| Y - X \|_1 \) is \( O(np) \) per iteration and that of \( \| \tilde{Y} - \tilde{X} \|_1 \) is \( O(\rho_c \rho_r) \).

3) The complexity of the computations corresponding to the graph regularization \( \text{tr}(X L_c X^T) + \text{tr}(X^T L_r X) = O(p|E_r| + n|E_c|) = O(pK + npK) \), where \( E_r, E_c \) denote the number of non-zeros in \( L_r, L_c \) respectively. Note that we use the \( K \)-nearest neighbors graphs so \( E_r \approx Kp \) and \( E_c \approx Kn \).

4) The complexity for the construction of \( \tilde{L}_c \) and \( \tilde{L}_r \) for compressed data \( \tilde{Y} \) is negligible if FLANN is used, i.e, \( O(\rho_c \rho_r \log(\rho_c)) \) and \( O(\rho_c \rho_r \log(\rho_r)) \). However, if the kron reduction strategy of Section VIII is used then the cost is \( O(KO_l(n + p)) \approx O(KO_l n) \).

5) We use the complexity \( O(np^2) \) for all the SVD computations on the matrix \( X \in \mathbb{R}^{p \times n} \) and \( O(\rho_c \rho_r^2) \) for \( \tilde{X} \in \mathbb{R}^{\rho_c \times \rho_r} \).

6) The complexity of \( \| M_r X M_c - \tilde{X} \|_F^2 \) is negligible as compared to the graph regularization terms \( \text{tr}(X L_c X^T) + \text{tr}(X^T L_r X) \).

7) We use the approximate decoders for low-rank recovery in the complexity calculations (eq. (17) in Section VIII). All the decoders for low-rank recovery are summarized in Table X.

8) The complexity of k-means [7] is \( O(Ink p) \) for a matrix \( X \in \mathbb{R}^{p \times n} \) and \( O(I \rho_r \rho_c k) \) for a matrix \( \tilde{X} \in \mathbb{R}^{\rho_r \times \rho_c} \).
Table X: A summary and computational complexities of all the decoders proposed in this work. The Lanczos method used here is presented in [32].

<table>
<thead>
<tr>
<th>Type</th>
<th>Low-rank</th>
<th>complexity</th>
<th>Algo</th>
<th>parallel</th>
</tr>
</thead>
<tbody>
<tr>
<td>ideal</td>
<td>( \min_{X} | M_rXc - \tilde{X} |<em>F^2 ) s.t: ( X^\top \in \text{span}(Q</em>{bc}) ) ( X \in \text{span}(P_{kr}) )</td>
<td>( \mathcal{O}(n^3) )</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>alternate</td>
<td>( \min_{X} | M_rXc - \tilde{X} |_F^2 ) ( + \gamma_c \text{tr}(XLcX^\top) ) ( + \gamma_r \text{tr}(X^\top LcX) )</td>
<td>( \mathcal{O}(KnK) )</td>
<td>gradient descent</td>
<td>no</td>
</tr>
<tr>
<td>approximate</td>
<td>( \min_{\bar{U}} | M_r\bar{U} - \bar{\tilde{U}} |_F^2 ) ( + \gamma_r \text{tr}(U^\top LrU) )</td>
<td>( \mathcal{O}(KnK) )</td>
<td>gradient descent</td>
<td>yes</td>
</tr>
<tr>
<td>approximate</td>
<td>( \min_{\bar{V}} | M_c^\top \bar{V} - \bar{\tilde{V}} |_F^2 ) ( + \gamma_c \text{tr}(V^\top LcV) )</td>
<td>( \mathcal{O}(KnK) )</td>
<td>gradient descent</td>
<td>yes</td>
</tr>
<tr>
<td>Subspace-Upsampling</td>
<td>( \min_{\bar{U}} \text{tr}(U^\top LrU) ) s.t: ( M_r\bar{U} = \bar{\tilde{U}} )</td>
<td>( \mathcal{O}(pKO_K) )</td>
<td>PCG</td>
<td>yes</td>
</tr>
<tr>
<td>approximate 2</td>
<td>( \min_{\bar{V}} \text{tr}(V^\top LcV) ) s.t: ( M_c^\top \bar{V} = \bar{\tilde{V}} )</td>
<td>( \mathcal{O}(nKO_K) )</td>
<td>PCG</td>
<td>yes</td>
</tr>
<tr>
<td>approximate 3</td>
<td>( \min_{\bar{U}} \text{tr}(U^\top LrU) ) s.t: ( M_r\bar{U} = \bar{\tilde{U}} ) ( X = \bar{U}\bar{\Sigma}\bar{V}^\top )</td>
<td>( \mathcal{O}(p^2 KO_K) )</td>
<td>SVD</td>
<td>–</td>
</tr>
<tr>
<td>approximate 3</td>
<td>( \min_{\bar{V}} \text{tr}(V^\top LcV) ) s.t: ( M_c^\top \bar{V} = \bar{\tilde{V}} ) ( X = \bar{U}\bar{\Sigma}\bar{V}^\top )</td>
<td>( \mathcal{O}(nkO_K) )</td>
<td>PCG</td>
<td>yes</td>
</tr>
<tr>
<td>approximate 3</td>
<td>( \min_{\bar{V}} \text{tr}(V^\top LcV) ) s.t: ( M_c^\top \bar{V} = \bar{\tilde{V}} ) ( X = \bar{U}\bar{\Sigma}\bar{V}^\top )</td>
<td>( \mathcal{O}(nKO_K) )</td>
<td>PCG</td>
<td>yes</td>
</tr>
</tbody>
</table>
Table XI: Computational complexity of all the models considered in this work

<table>
<thead>
<tr>
<th>Model</th>
<th>Complexity</th>
<th>Complexity</th>
<th>Complexity</th>
<th>Overall Complexity (low-rank)</th>
<th>Overall Complexity (clustering)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$G_r$</td>
<td>$G_t$</td>
<td>Algorithm</td>
<td>Fast SVD for $p \ll n$</td>
<td>k-means</td>
</tr>
<tr>
<td></td>
<td>$O(np \log(n))$</td>
<td>$O(np \log(p))$</td>
<td>for $p \ll n$</td>
<td>Decoder</td>
<td>Total</td>
</tr>
<tr>
<td>LE [1]</td>
<td>+</td>
<td>–</td>
<td>$O(n^3)$</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>LLE [26]</td>
<td>–</td>
<td>–</td>
<td>$O((p+k)n^2)$</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>PCA</td>
<td>–</td>
<td>–</td>
<td>$O(p^2 n)$</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>GLPCA [11]</td>
<td>+</td>
<td>–</td>
<td>$O(n^3)$</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>NMF [13]</td>
<td>–</td>
<td>–</td>
<td>$O(\text{Inpk})$</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>GNMF [3]</td>
<td>+</td>
<td>–</td>
<td>$O(\text{Inpk})$</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>MMF [38]</td>
<td>+</td>
<td>–</td>
<td>$O(((p+k)k^2 + pk)I)$</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>RPCA [4]</td>
<td>–</td>
<td>–</td>
<td>$O(\text{Inp}^2)$</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>RPCAG [28]</td>
<td>–</td>
<td>–</td>
<td>$O(\text{Inp}^2)$</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>FRPCAG [29]</td>
<td>+</td>
<td>+</td>
<td>$O(\text{InpK})$</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>CPCA</td>
<td>+</td>
<td>+</td>
<td>$O(I_{\rho, \rho} \rho)$</td>
<td>$O(nkO_{\rho})$</td>
<td>$O(np \log(n))$</td>
</tr>
</tbody>
</table>
**Table XII:** Details of the datasets used for clustering experiments in this work.

<table>
<thead>
<tr>
<th>Dataset</th>
<th>Samples</th>
<th>Dimension</th>
<th>Classes</th>
</tr>
</thead>
<tbody>
<tr>
<td>ORL large</td>
<td>400</td>
<td>56 × 46</td>
<td>40</td>
</tr>
<tr>
<td>ORL small</td>
<td>300</td>
<td>28 × 23</td>
<td>30</td>
</tr>
<tr>
<td>CMU PIE</td>
<td>1200</td>
<td>32 × 32</td>
<td>30</td>
</tr>
<tr>
<td>YALE</td>
<td>165</td>
<td>32 × 32</td>
<td>11</td>
</tr>
<tr>
<td>MNIST</td>
<td>70000</td>
<td>28 × 28</td>
<td>10</td>
</tr>
<tr>
<td>MNIST small</td>
<td>1000</td>
<td>28 × 28</td>
<td>10</td>
</tr>
<tr>
<td>USPS large</td>
<td>10000</td>
<td>16 × 16</td>
<td>10</td>
</tr>
<tr>
<td>USPS small</td>
<td>3500</td>
<td>16 × 16</td>
<td>10</td>
</tr>
</tbody>
</table>

**Table XIII:** Range of parameter values for each of the models considered in this work. $k$ is the rank or dimension of subspace or the number of clusters. $\lambda$ is the weight associated with the sparse term for Robust PCA framework [4] and $\gamma, \alpha$ are the parameters associated with the graph regularization term.

<table>
<thead>
<tr>
<th>Model</th>
<th>Parameters</th>
<th>Parameter Range</th>
</tr>
</thead>
<tbody>
<tr>
<td>LLE [26], PCA LE [1]</td>
<td>$k$</td>
<td>$k \in {2^{1}, 2^{2}, \ldots , \min(n, p)}$</td>
</tr>
<tr>
<td>GLPCA [11]</td>
<td>$k, \gamma$</td>
<td>$k \in {2^{1}, 2^{2}, \ldots , \min(n, p)}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\gamma \Rightarrow \beta$ using [11] $\beta \in {0.1, 0.2, \ldots , 0.9}$</td>
</tr>
<tr>
<td>MMF [38]</td>
<td>$k, \gamma$</td>
<td>$k \in {2^{1}, 2^{2}, \ldots , \min(n, p)}$</td>
</tr>
<tr>
<td>NMF [13]</td>
<td>$k$</td>
<td></td>
</tr>
<tr>
<td>GNMF [8]</td>
<td>$k, \gamma$</td>
<td>$\gamma \in {2^{-3}, 2^{-2}, \ldots , 2^{10}}$</td>
</tr>
<tr>
<td>RPCA [4]</td>
<td>$\lambda$</td>
<td>$\lambda \in {rac{2{-1}}{\sqrt{\max(n,p)}} : 0.1 : \frac{2^{3}}{\sqrt{\max(n,p)}}}$</td>
</tr>
<tr>
<td></td>
<td>$\gamma$</td>
<td>$\gamma \in {2^{-3}, 2^{-2}, \ldots , 2^{10}}$</td>
</tr>
<tr>
<td>RPCAG [28]</td>
<td>$\gamma_r, \gamma_c$</td>
<td>$\gamma_r, \gamma_c \in (0, 30)$</td>
</tr>
<tr>
<td>FRPCAG [29]</td>
<td>$\gamma_r, \gamma_c$</td>
<td>$\gamma_r, \gamma_c \in (0, 30)$</td>
</tr>
<tr>
<td>CPCA</td>
<td>$k$ (approximate decoder)</td>
<td>$\Sigma_{k,k}/\Sigma_{1,1} &lt; 0.1$</td>
</tr>
</tbody>
</table>