Resource Verification for Higher-order Functions with Memoization

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Abstract
We present a new approach for specifying and verifying resource utilization of higher-order functional programs that use lazy evaluation and memoization. In our approach, users can specify the desired resource bound as templates with numerical holes e.g. as $\exists \forall$ in the contracts of functions, as well as express invariants necessary for establishing the bounds that may possibly depend on the state of memoization. Our approach operates in two phases: first generating an instrumented first-order program that accurately models the higher-order control flow, and the effects of memoization on resources, using sets, algebraic datatypes, and mutual recursion, and then verifying the contracts of the first-order program by producing verification conditions (VCs) of the form $\exists \forall$ using an extended assume/guarantee reasoning. We use our approach to verify precise bounds on resources such as evaluation steps, and number of heap-allocated objects, on 17 challenging data structures and algorithms. Our benchmarks, comprising of 5K lines of functional Scala code, include lazy mergesort, Okasaki’s real-time queue and dequeue data structures that rely on aliasing of references to first-class functions; lazy data structures based on numerical representations such as the conqueue data structure of Scala’s data-parallel library, cyclic streams such as hamming number sequence, as well as dynamic programming algorithms like Levenshtein distance, Viterbi algorithm, and packrat parsing. Our evaluations show that, when averaged over all benchmarks, the actual runtime resource consumption is at least 80% of the value inferred by our tool when estimating the number of evaluation steps, and is at least 88% for the number of heap-allocated objects.

1. Introduction
Static estimation of performance properties of program is an important problem that has attracted great deal of research, and has resulted in techniques ranging from estimation of resource usage in terms of concrete physical quantities [54], to static analysis tools that derive upper bounds on the abstract complexities of programs [1, 21, 24]. Recent advances [4, 16, 21, 24, 50, 56] have shown that automatically inferring bounds on more algorithmic metrics of resource usage, such as the number of steps in the evaluation of an expression (commonly referred to as steps or ticks), or the number of memory allocations (alloca), is feasible on programs that use higher-order functions, and datatypes, especially in the context of functional programs. However, most existing approaches aim for complete automation, but trade off expressive power in the process. Many of these techniques offer little provision for users to specify the bounds they are interested in, or to provide invariants needed to prove bounds of complex computation. This prevents many of the techniques from being applicable to formally reason about resource properties of implementations that have theoretically complex proofs for resource bounds, such as balanced trees where the time depends on the height or weight invariants that ensure balance. This is in stark contrast to the situation in verification of correctness properties, where large-scale software and hardware verification efforts are commonplace [23, 27, 28, 33]. Alternative approaches [13, 35] have started incorporating user specifications to target more precise bounds, as well as more complex programs by making use of user-supplied program specific invariants.

In this paper, we show that such contract-based approach can be extended to verify complex resource bounds in a challenging domain: higher-order functional programs that rely on memoization and lazy evaluation. We use the term memoization to refer to caching of outputs of a function for each distinct input encountered during an execution. By lazy evaluation we mean the usual combination of call-by-name and memoization [39], which we can view as wrapping values into memoized closures with a parameter of unit type. These features are important as they improve the running time (as well as other resources), often by orders of magnitude, while preserving the functional model for the purpose of reasoning about the result of the computation. For instance, dynamic programming algorithms, which have numerous practical applications e.g. in parsers such as packrat parser, are based on memoization. Other examples include lazy data structures proposed by Okasaki [39] that use lazy evaluation to support persistent operations, such as enqueue, and dequeue, in worst-case constant time. A related, but more complicated data structure Conqueue has been used to implement data-parallel operations in Scala efficiently [41, 42]. The (anonymous) Appendix C shows the verified implementations of lazy Real-time queue, and Knapsack dynamic programming algorithm, in our system. These features are being increasingly adopted by languages with large user base, such as Java 8, C#, and Scala, through libraries such as LINQ, and Scala Streams, making their use increasingly prevalent. The challenge that arises with these features is that reasoning about resources like running time and memory usage becomes state-dependent and more complex than functional correctness—to the extent that precise running time bounds remain open in some cases (e.g. lazy pairing heaps described in page 79 of [39]). However, reasoning about correctness properties remains purely functional, which makes these features more attractive, in comparison to imperative programming models, from the perspective of software verification.

We therefore believe that it is useful and important to develop algorithms and tools to formally verify resource usage of implementations of such algorithms with respect to abstract metrics such as steps, and alloc. Although our objective is not to compute bounds on physical time, our initial experiments do indicate a strong correlation between the number of steps performed at runtime, and the actual wall-clock execution time for our benchmarks. In particular, on the program lazy, bottom-up merge sort, one step of evaluation at runtime corresponded to 2.35 nanoseconds (ns) on average, with an absolute deviation of 0.01 ns, and in the Real-time queue bench-
sealed abstract class Stream
private case class SCons(x: (BigInt,Bool), tfun:() => Stream) extends Stream{
  lazy val tail = tfun()
}
private case class SNil() extends Stream
private val primes = SCons((1, true), () => nextElem(2))

def nextElem(i: BigInt): Stream = {
  require(i ≥ 2)
  val x = (i, isPrimeNum(i))
  val y = i + 1
  SCons(x, () => nextElem(y))
} ensuring(r ⇒ steps ≤ ? * i + ?)
def isPrimeNum(n: BigInt): Bool = {
  def rec(i: BigInt): Bool = {
    require(i ≥ 1 & i < n)
    if (i == 1) true
    else if (i % i == 0) false
    else rec(i - 1)
  }
  ensuring(r ⇒ steps <= ? * i + ?)
  rec(n - 1)
} ensuring(r ⇒ steps ≤ ? * n + ?)
def isPrimeS(s: Stream, i: BigInt): Bool = {
  def rec(i: BigInt): Bool = {
    require(i ≥ 2)
    s match {
      case SNil() => false
      case SCons(x, tfun) => tfun == ((i) ⇒ nextElem(i))
    }
  }
  ensuring(r ⇒ steps <= ? * i + ?)
  rec(i+2)
} ensuring(r ⇒ steps ≤ ? * i + ?)
def primesUntil(n: BigInt): List = {
  def takePrimes(0, n, primes) = Nil()
  def primesUntil(n: BigInt): List = {
    require(n ≥ 2)
    takePrimes(0, n-2, primes)
  } ensuring(r ⇒ steps <= ? * n² + ?)
  def primesUntil(n: BigInt, s: BigInt, s: Stream): List = {
    require(0 ≤ i & i ≤ n & & isPrimeS(s, i+2))
    s match {
      case s @ SCons(x, b, _) if i < n ⇒
        val t = takePrimes(i+1, n, c.tail)
        if (b) Cons(x, t) else t
      case _ ⇒ Nil()
    }
  }
} ensuring(r ⇒ steps <= ? * (n(n-i)) + ?)

Figure 1. Prime numbers until \( n \) using an infinite stream.

mark it corresponded to 12.25 ns with an absolute deviation of 0.03 ns. These results further add to the importance of proving bounds even if they are with respect to the abstract resource metrics.

In this paper, we propose a system for specifying and verifying abstract resource bounds, such as steps and alloc, of programs written in a pure subset of Scala [37] with added support for memoization and new specification expressions. In our approach, users can specify the desired resource bound as templates with numerical holes e.g. as steps ≤ \( ? * \text{size]() + ? \) in the contracts of functions, along with other invariants necessary for proving the bounds. Our system proves the bound by automatically inferring values for the holes that will make the bound hold for all executions of the function. For instance, our system was able to infer that the number of steps spent in accessing the \( k \)th element of an unsorted list \( l \), using a lazy, bottom-up, merge sort algorithm [2], is bounded by \( 36(k - \log l.size)+6l.size+22 \). We empirically compared the number of steps used by this program at runtime, against the bound inferred by our tool, by varying the size of the list \( l \) from 10 to 10K, and \( k \) from 1 to 100. Our results showed that the inferred values were 90% accurate for this example (section 5 presents more results). We now present an overview of how programs can be specified and verified in our system, using the pedagogical example shown in Fig. I that creates an infinite stream of prime numbers.

**Prime stream example.** The Stream datatype shown in Fig. 1 with two constructors: SCons and SNil shows a definition of a stream, which is similar to a list datatype with constructors: Cons, and Nil. The first argument \( x \) of SCons is a pair of an unbounded integer (BigInt), and a boolean. The second argument tfun is a function from Unit to Stream. The type SCons has a lazy field tail (declared using lazy val in Scala syntax) that lazily evaluates tfun, i.e., computes tfun() once, and caches the result for reuse. The program defines a stream: primes that lazily computes for all natural numbers starting from 1 its primality, by creating an SCons whose second argument is a lambda term (anonymous function) that calls nextElem(2). Note that accessing the tail field of primes for the first time evaluates this call, which returns a new stream s such that s.tail invokes nextElem on the next natural number.

The function isPrimeNum(n) tests the primality of \( n \) by checking if any number greater than 1, and smaller than \( n \) divides \( n \), using an inner function rec. The number of steps it takes is linear in \( n \). The function primesUntil returns all prime numbers until the parameter \( n \) using a helper function takePrimes, which is passed the primes stream, a counter \( i \) set to 0, and the number of elements to access from the primes stream, which is initially \( n-2 \). The function takePrimes recursively calls itself on the tail of the input stream, incrementing the index \( i \), as long as \( i < n \) (line 41). It then constructs an output list of prime numbers, as the recursion unwinds. Consider now the running time of this function. Firstly, if takePrimes is given an arbitrary stream, its time (and hence steps) cannot be bounded, because accessing the field tail at line 41 could take arbitrary amount of time. Therefore, we need to specify the structure of the closure tfun in the stream passed as input to takePrimes in order to prove its running time. This is accomplished by the function isPrimeS(s, i), which returns true if s is a SCons whose tfun parameter is equivalent to \( () ⇒ \text{nextElem}(i) \). Though the comparison at line 30 will always return false under reference equality, in our system, we relax this to allow for structurally equivalence of higher-order functions in order to allow specifying such properties. (We shortly explain the rationale for this choice in more detail, and formalize this equality in definition 1.) Using isPrimeS, we can specify that the stream passed as input to takePrimes is an SCons whose tfun parameter invokes nextElem(i+2), as shown in the Fig. 1, and
bound the steps of the function to $O(n(n - i))$. Consequently, we can establish that \text{primesUntil} takes $O(n^2)$ steps. For \text{primesUntil}, our tool inferred that steps $\leq 16n^2 + 4$.

**Properties depending on memoization table state.** The quadratic bound of \text{primesUntil} is precise only when the function is called for the first time. If \text{primesUntil}(n) is called twice, the time taken by the second call would be linear in n, since every access to tail within \text{takePrimes} will take constant time, as it has been cached during the previous call. The time behavior of the function depends on the state of the memoization table when it is invoked, making the reasoning about resources imperative. To specify such functions we support a built-in operation \text{cached}($f(s)$) that can query the state of the cache. This predicate holds if the function $f$ is a memoized function, and is cached for the value $x$. Note that it does not invoke $f(x)$. The function \text{concUntil}(s, i) shown in Fig. 2 uses this predicate to state a property that holds iff the first i calls to the tail field of the stream s have been cached. (Accessing the lazy field $e.tail$ is similar to calling a function $tail(e)$ that is memoized for its arguments $c$.) This property holds for \text{primes stream} at the end of the stream s as shown in line 9 of Fig. 2. Moreover, if this property holds in the state of the cache at the beginning of the function, the number of steps executed by the function would be linear in $n$. This is expressed as shown in line 10 using an disjunctive resource bound. Observe that in the postcondition of the function, we need to refer to the state of the cache at the beginning of the function, as it changes during the execution of the function. For this purpose, we support a built-in construct “inSt” that can be used in the postcondition to refer to the state at the beginning of the function, and an “in” construct which can be used to evaluate an expression in the given state. These expressions are meant only for use in contracts. Note that unlike in a imperative language, here the state, namely the cache, is implicit, and cannot be directly accessed by the programmers to specify properties on it. However, the upside is that the knowledge that the state behaves like a cache can be exploited to reason functionally about the result of the functions, which results in fewer contracts, and more efficient verification (see section 4).

**Equality of closures.** Supporting equality of closures is important for two reasons. (a) Firstly, interesting data structures based on lazy evaluation use aliasing references to closures. Expressing invariants of such data structures requires equating closures. Fig. 3 pictorially depicts the invariants of the concrete data structure $[42]$. In the figure, inc represents a closure whose captured arguments are a digit (1 or 0), and a reference to a stream. sched is a list of references to closures reachable from head. (b) Secondly, it is convenient in specifications to state that two closures have the same behavior, as was required in the example shown in Fig. 1. While reference equality is too restrictive for convenient specification, full semantic equality between closures is undecidable, and tricky, especially, considering that functions can have contracts in our language. Therefore, we resort to structural equivalence of closures, wherein two closures are equivalent iff their abstract syntax trees are identical, without unfolding named functions. This is formally defined in Def 1 in section 2. This also has the advantage that it allows modeling reference equality of closures by associating unique identifiers with closures that are incremented as the closures are created in the program. In fact, our system supports reference equality for closures as well.

**Approach and Contributions.** Our approach operates in two phases. In the first phase we generate an instrumented first-order program with specifications, referred to as the model, that accurately captures the higher-order control flow using defunctionalization $[4, 44]$, and the effects of memoization on resources using sets, and algebraic datatypes. During this translation process we ensure that the resource usage of the input program is modeled accurately, without any abstraction (described in more detail in section 3). In the subsequent phase, we convert the problem of verifying contracts of the generated first-order programs to checking assertions using an assume-guarantee reasoning that exploits predicates that evolve monotonically with changes to the cache. We then encode the assume/guarantee assertions as $\exists \forall$ formulas in theories that can be efficiently decided by state-of-the-art SMT solvers. We explain the verification approach in section 4. To summarize, the following are the contributions of this paper:

- We propose a specification approach for expressing resource bounds of programs, and the necessary invariants, in the presence of memoization, and higher-order functions. We formally define an operational semantics for the constructs of our language that is parametric with respect to the resource usage (section 2).
- We propose a system for verifying the contracts of programs expressed in our language, by combining, and extending existing techniques from resource bound inference, and software verification (sections 3, and 4).
- We use our system to prove asymptotically precise resource bounds of 17 benchmarks, expressed in an functional subset of Scala $[37]$, implementing complex lazy data structures, and dynamic programming algorithms, comprising 4.5K lines of Scala code, and 123 resource templates (section 5).
- We experimentally evaluate the accuracy of the inferred bounds by rigorously comparing them with the runtime values for the resources on large inputs. Our results show that, while the inferred values always upper bound the runtime values, the runtime values for steps is at least 80% of the value inferred by the tool, and is at least 88% for alloc. (section 5).

**2. Language and Semantics**

Fig. 4 show the syntax of a simple higher-order function language, extended with memoization, contracts and specification constructs, that we will use to formalize our approach. Expressions of the language consists of variables, constants ($Cst$), primitive operations ($Prim$) on integers and booleans, let expressions, match expressions, lambda terms, and applications $x y$. We distinguish between direct calls to named functions $f x$, and indirect calls $x y$. We require that every direct call invokes a function defined in the program, and indirect calls need not. $Tdef$ shows the syntax
of user-defined algebraic datatypes. The datatypes use structural equality, formalized in definition 1. As a syntactic sugar, we consider tuples as a special datatype, and denote tuple construction using \((x_1, \cdots, x_n)\), and selecting the \(i^{th}\) element of a tuple using \(x_i\). The annotation \(\mathfrak{O}\) serves to mark functions that have to be memoized. Such functions are evaluated exactly once for each distinct input passed to them at run time. The language does not support lazy evaluation, but it can simulated using a function of the form: \(\mathfrak{O}\) def lazy(T(f)) { \(f\) \(\Rightarrow\) \(f\) }. Invoking \(\mathfrak{O}(f) \Rightarrow e\) evaluates \(e\) lazily. Expressions that are bodies of functions can have contracts (or specifications). Such expressions have the form: \(\{e_1\} \in \{e_2\}\), where \(e_1\) and \(e_2\) are the pre and postcondition of \(e\), respectively. The syntax of specification expressions is given by \(E_{spec}\). The postcondition of an expression \(e\) can refer to the result of \(e\) using the variable \(res\), and to the resource usage of \(e\) using steps, and alloc. Users can specify upper bounds on resources as templates \(e_i \in E_{tmp}\) with holes, where the holes always appear as coefficients of variables defined in the program. Nonetheless, the variables themselves could be bound to more complex expressions through let binders.

For ease of formalization, we enforce the following syntactic constraints without reducing generality: all expressions, except lambda terms, are in \(A\)-normal form i.e., the arguments of all operations/functions are variables; and all lambdas are of the form: \(\lambda x.f \ (x, y)\), where \(f\) is a named function whose argument is a pair (a two element tuple), and \(y\) is a captured variable. Every expression \(e\) has a unique static label \(\text{label}(e)\). Fig. 5 defines the big-step operational semantics, and resource consumption of the constructs of our language. We use a big-step semantics, along the lines of Lauchbury’s operational semantics for lazy evaluation [30], as it naturally lends itself to a compositional reasoning which our approach is based upon.

**Notation.** Given a domain \(A\), we use \(\bar{a} \in A^+\) to denote a sequence of elements in \(A\), and \(a_i\) to refer to the \(i^{th}\) element. (Note that this is different from tuple selector \(x_i\), which is an expression of the language). We use \(A \nrightarrow B\) to denote a partial function from \(A\) to \(B\). Given a partial function \(h\), \(h(\bar{x})\) denotes the function that applies \(h\) point-wise on each element of \(\bar{x}\), and \([a \nrightarrow b]\) denotes the function that maps \(a\) to \(b\) and every other value in the domain of \(h\) to \(b\). We use \([\bar{a} \nrightarrow \bar{b}]\) to denote \([a_1 \nrightarrow b_1] \cdots [a_n \nrightarrow b_n]\). We omit \(h\) in the above notation if \(h\) is an empty function. We define a partial function \(h_1 \uplus h_2\) as \((h_1 \uplus h_2)(x) = x\) if \((x \in \textit{dom}(h_2))\) \(h_2(x)\) else \(h_1(x)\).

Let type \(e\) denotes the type of an expression \(e\). Given a lambda \(l\), we use \(FV(l)\) to denote free variable captured by \(l\), and \(target(l)\) to denote the function called in the body of the lambda. The operation \(e[e'/x]\) denotes the syntactic replacement of the free occurrences of \(x\) in \(e\) by \(e'\). We use \([a, b]\) to denote a closed integer interval from \(a\) to \(b\). Given a substitution \(\iota : TVars \nrightarrow V\), we use \(e_{\iota}\) to represent substitution of the holes by the values given by the assignment. We also extend this notation to formulas later.

**Semantic domains.** Let \(Adr\) denote the addresses of heap-allocated structures, namely closures and datatypes. The state of an interpreter evaluating expressions of our language is a quadruple consisting of a cache \(C\), a heap \(\mathcal{H}\), an assignment of variables to values \(\sigma\), and a set of function definitions. Formally, \(u, v \in Val = \mathbb{Z} \cup \text{Bool} \cup Adr\) \(FVal = Fids \times Val \) \(DVal = Cids \times Val\) \(Clo = Lam \times Store\) \(\mathcal{H} \in \mathcal{Heap} \Rightarrow Adr \nrightarrow (DVal \cup Clo)\) \(\sigma \in Store = Vars \nrightarrow Val \) \(C \in Cache = FVal \nrightarrow Val \) \(\Gamma \in Env \subseteq Cache \times Heap \times Store \times 2^{\textit{Env}}\)

The cache component \(C\) of the environments have the property that every key of the cache, which is a concrete function call in \(FVal\), is mapped to the result of the call (Definition 2 formally

<table>
<thead>
<tr>
<th>CST</th>
<th>VAR</th>
<th>PRIM</th>
<th>EQUAL</th>
</tr>
</thead>
<tbody>
<tr>
<td>(c \in Cst)</td>
<td>(x \in Vars)</td>
<td>(op \in Prim)</td>
<td>(v = \sigma(x) \tau \sigma(y))</td>
</tr>
</tbody>
</table>

**Figure 5.** Resource annotated operational semantics for the language defined in Fig. 4.
defines this property). We define a few helper functions to operate on the semantic domains. Let $fresh(H)$ denote an element $a \in A \land a \not\in dom(H)$. Let $body_{\eta}(f)$ and $param_{\eta}(f)$ denote the body and parameter of the functions defined in the environment $\Gamma$, and $Mem_{\eta} \subseteq Fids$ denote the set of memoized functions in the function definitions in $\Gamma$.

**Structural equivalence.** Below we formalize the notion of structural equivalence of datatypes and closures. Two datatypes are structurally equivalent iff they use the same constructor, and their fields are equivalent. We define structural equivalence of closures similar to syntactic equality of lambda modulo alpha renaming (but extended to captured variables). Two closures are structurally equivalent iff their bodies are of the form $\lambda x.f \ (x,y)$ and $\lambda w.f \ (u,z)$, where the captured variables $y$ and $z$ are bound to structurally equivalent values. Two addresses are equivalent iff they are bounded to structurally equivalent values in the heap. Formally,

**Def 1.** Define a relation $\approx$ with respect to $H \in Heap$ as: (subscript omitted for clarity)

$\forall a \in Z \cup \mathit{Bool}. a \approx a$

$\forall \{a,b\} \subseteq \mathit{Adr}. a \approx b \iff \mathit{dom}(H)(a) \approx \mathit{dom}(H)(b)$

$\forall f \in \mathit{Fids}, \{a,b\} \subseteq \mathit{Val}. (f \ a) \approx (f \ b) \iff a \approx b$

$\forall c \in \mathit{Cids}, \{a,b\} \subseteq \mathit{Val}$. ($c \ a$) $\approx$ ($c \ b$) $\forall i \in \{1,n\}. a_i \approx b_i$

$\forall \{1,\ldots,x,y,\ldots,z\} \subseteq \mathit{Lamb}. (\lambda x.f \ (x,y)) \approx \lambda w.f \ (w,z)$

$\forall \{1,\ldots,x,y,\ldots,z\} \subseteq \mathit{Store}. (\mathit{Mem}((1,\ldots,x),\{\ldots\}) \approx (\mathit{Mem}((1,\ldots,z),\{\ldots\}) \iff \sigma_1(y) \approx \sigma_2(z)$

Note that the above definition uses a simple structural equivalence over semantic domains, and does not use the operational semantics.

**Judgements.** We use judgements of the form $\Gamma \vdash e \mathit{v} p \nu, \Gamma'$ to denote that under an environment $\Gamma' \in Env$, an expression $e$ evaluates to a value $v$ in $Val$, and results in a new environment $\Gamma'' \in Env$, while consuming $p \in \mathit{Units}$ of a resource. When necessarily, we expand $\Gamma$ as $\Gamma' : \{C, H, \sigma, F\}$ to highlight its individual components of the environment. We also ignore any component of the judgement that is not relevant to the context when there is no ambiguity. In Fig. 5, we ignore the function definitions from the environment as they do not change during the evaluation of expressions. Below we formally characterize the environment of our semantics.

**Def 2 (Environment).** A quadruple $\Gamma : \{C, H, \sigma, F\}$ belongs to the environment $Env$ iff (a) $\forall k \in \mathit{dom}(C), \Gamma \vdash k \mathit{v} p \nu, (C', H', \sigma') \Rightarrow v \approx C(k)$, and (b) for every closure in $H$ containing a lambda: $\forall x, f \ (x,y), f$ is defined in $F$.

**Resource parameterization.** We parameterize the operational semantics in a way that it can be instantiated on multiple resources using the following parameterization functions: (a) A cost function $\mathrm{cost}$ that returns the resource requirement of an operation $op$ such as $\mathrm{cons}$ or $\mathrm{app}$. $\mathrm{cost}$ may possibly have parameters. In particular, we use $\mathrm{cmatch}(i)$ to denote the cost of a match operation when the $i^{th}$ case was taken, which should include the cost of failing all the previous cases. (b) A resource combinator $\oplus : \mathit{Z} \rightarrow \mathit{Z}$ that computes the resource usage of an expression by combining the resource usages of the sub-expressions. Typically, $\oplus$ is either $+$ or $\max$.

We specifically consider two resources in this paper: (a) the number of steps in the evaluation of an expression, denoted steps, and (b) the number of heap-allocated objects (viz. a closure, datatype, or a cache entry) created by an expression, denoted alloc. In the case of steps, $\mathrm{cost}$ is 1 for almost every operation (except $\mathrm{cmiss}$ and $\mathrm{cmatch}(i)$). We consider construction of datatypes, and primitive operations, including those on big integers, as unitary steps. We define $\mathrm{cmatch}(i) = i$, as we need to include the cost of failing all the $i - 1$ match cases. In the case of alloc, $\mathrm{cost}$ is 1 for datatype, and closure creations, and also for a cache miss since it allocates a cache entry, and zero otherwise. For both resources, $\oplus$ is $+$. Our implementation, however, supports other resources such as abstract stack space usage, and number of recursions, which can also be defined using the parameters.

**Semantic rules.** For brevity, we skip the discussion of straightforward semantic rules (such as let and match), and focus on rules that are atypical. The rule $\mathit{FVAL}$ defines the semantics of a call whose arguments have been evaluated to concrete values (in $Val$). It models the call-by-value parameter passing mechanism: it binds the parameters to argument values, and evaluates the body, which can be an expression with contracts, under the new binding. A call evaluates to a value only if the contracts of the callee are satisfied, as given by the rule: $\mathit{Contract}$ (discussed shortly).

**Memoized Call Semantics.** The semantics of calling a memoized function is defined by the rules: $\mathit{MEMOCALLHIT}$ and $\mathit{MEMOCALLMISS}$. Calling a memoized function involves as a first step querying the cache for the result of the call. In case the result is not found, the callee is invoked, and the cache is updated once (and if) the callee returns a value. Querying the cache involves comparing arguments of the call for equality. We define a lookup relation $\in_H$ that uses structural equivalence to lookup the cache as follows: ($f \ u \in_H \in_H (f \ u') \iff \mathit{dom}(C) \wedge \mathit{Val}(f \ u') \wedge \mathit{Val}(f \ u)$

We parametrize the cost of searching, and updating the cache using the parameters $\mathrm{chit}$ and $\mathrm{cmiss}$. In particular, to calculate the steps resource, we consider lookup and update as unitary steps, and hence, define $\mathrm{cmiss} = 2$ (as it involves a lookup and an update operation), and $\mathrm{chit} = 1$. However, in general, $\mathrm{cmiss}$, and $\mathrm{chit}$ may depend on the values of the arguments.

**Specifications.** The construct $\mathit{cached}(f \ x)$ evaluates to true in an environment $\Gamma$ iff the call $f \ c$ is cached for the value of $x$ in $\Gamma$. Observe that the resource consumption of this construct is zero. This is because the construct is syntactically excluded from being part of the implementation of functions (see Fig. 4), which renders its resource usage irrelevant. The rule $\mathit{CONTRACTION}$ defines the semantics of an expression $\hat{e}$ of the form $\{\mathit{pre}\} e \{\mathit{post}\}$. The expression evaluates to a value $v$ only if $\mathit{pre}$ holds in the input environment, and $\mathit{post}$ holds in the environment resulting after evaluating $e$. Observe that the value, cache effects and resource usage of $\hat{e}$ are equal to that of $e$. Also note that the resource variables steps, and alloc are bound to the resource consumption of $e$ before evaluating the postcondition. We show the semantics of the constructs: in, and $\mathit{inSt}$, along with a couple of other specification constructs in Appendix A. Informally, $\mathit{inSt}$ is used by expressions in the postcondition to refer to the state of the cache at the beginning of the function, and $\mathit{in}(e, x)$ evaluates an expression $e$ in the cache state given by $x$, as illustrated by the example shown in Fig. 2.

**Contract verification problem.** Given a (possibly open) program $P$, the contract verification problem is to decide for every function $\mathit{def} f = \{\mathit{pre}\} e \{\mathit{post}\}$ defined in the program $P$, whether in every environment in which $\mathit{pre}$ does not evaluate to $false$, $\{\mathit{pre}\} e \{\mathit{post}\}$ evaluates to a value. Formally, $\forall \Gamma \in Env. \exists \mathit{nu} (\Gamma \vdash \mathit{nu} \downarrow false) \lor \Gamma \vdash \{\mathit{pre}\} e \{\mathit{post}\} \downarrow v$. (We omit the quantification on $v$, when there is no ambiguity.) Since contracts in our programs can specify bounds on resources, the above definition also guarantees that the properties on resources hold.

**Resource inference problem.** Recall that we allow the resource bounds of functions to be templates. In this case, the problem is to find an assignment $\iota$ for the holes such that in the program obtained by substituting the holes with their assignment, the contracts of all functions are verified, as formalized below. Let $e \ i$ denotes substituting the holes in an expressions $e$ with the assignment given by $i$. The resource bound inference problem is to find an assignment
i, such that for every function \( \text{def } f = \{ \text{pre} \} e \{ \text{post} \} \), where \( \text{post} \) may contain holes, \( \forall \Gamma \in \text{Env} \). (\( \Gamma \vdash \text{pre} \downarrow \neg \) \( \Gamma \vdash \{ \text{pre} \} e \{ \text{post} \} \downarrow \neg \).

3. Generating Model Programs

In the following sections, we describe our approach in two phases: model generation phase (discussed in this section), and verification phase (discussed in section 4). The goal of the model generation phase is to generate a first-order program with recursion, that accurately models the resource usage of the source program without any abstraction, only using theories suitable for automated reasoning. We refer to output of this phase as the model. In particular, there are three reductions that are handled by this phase: (a) Defunctionalization of higher-order functions to first-order functions [44].

(b) Encoding of cache as an expression that changes during the execution of the program, and (c) Instrumentation of expressions with their resource usage while accounting for the effects of memoization. We formally establish the soundness and completeness of the translation with respect to the operational semantics shown in Fig. 5, by establishing a bisimulation between the source program and the model (Theorem 2). In contrast to related works [4], which use defunctionalization as a means to estimate the resource usage of source programs, here we are only interested in the values (and not resources) of expressions of the model. The expressions of the model themselves track the resource usages.

Model Language.

The model language does not use higher-order features, memoization, or a special specification construct. However, we introduce two features that were not a part of the source language: (a) set values, and set primitives such as union, and inclusion; and (b) an error construct that halts the evaluation.

\( e_m \in \text{Eval} ::= x | \text{Cst} | \text{prim}(x | f \times C \bar{x}) | \text{Comp}_m \{ \{ e_m \} \} \). The values of the model language includes sets (\( \text{Set} = 2^{\text{Val}} \subseteq \text{Val} \)). The environments of the model does not have the cache component: \( \Gamma \in \text{Env}_{\text{mod}} = \text{Heap} \times \text{Store} \times 2^{\text{PDef}} \).

Figures 6 formally defines the translation function \( \text{Eval} \) that translates expressions of a source program \( P \) to the model program \( P^\text{m} \). Fig. 7(b) illustrates the translation on an implementation of a lazy take operation show in Fig. 7(a), which is a part of the scheduling-based Dcache data structure described in [39], Pg. 112, Fig. 8.4. The take operation returns the first \( n \) elements of the input stream, and requires that the input stream is memoized at least until \( n \) in order to achieve a constant time bound. The function \( \text{concUntil} \) is shown in Fig. 2.

Closure encoding. To represent closures using algebraic datatypes in a way that preserves the structural equivalence of closures. We say two lambdas \( \lambda x.f_1(x,y) \) and \( \lambda x.f_2(x,z) \) are compatible, denoted \( l_1 \equiv l_2 \), iff they invoke the same targets i.e, \( f_1 = f_2 \). This relation is interesting because, during any evaluation, two closures could be structurally equivalent if their lambdas are compatible i.e, \( l_1 \equiv l_2 \) iff \( \exists h, \sigma_1, \sigma_2 \text{ s.t. } (l_1, \sigma_1) \approx (l_2, \sigma_2) \).

In the generated model, we ensure that the closures with lambdas that are compatible are represented using the same datatype. For each lambda \( l \) let \( l/\approx \) denote the representative of the equivalence class with respect to \( \approx \). For each function type \( \tau \) used in \( P \), we add a datatype \( d_\tau \) to the model defined as follows. Let \( \{l_i | i \in \{1, n\}\} \) be the labels of the representatives, with respect to \( \approx \), of the lambda terms in the program that are of type \( \tau \). The datatype \( d_\tau \) has \( n+1 \) constructors denoted \( C_{l_1}, i \in \{1, n\}, \) and \( C_\tau \). That is, \( d_\tau ::= (C_{l_1} \tau_1, \ldots, C_{l_n} \tau_n, C_\tau \text{ Int}) \). The \( i^{\text{th}} \) constructor \( C_{l_i} \) represents the closure of the \( i^{\text{th}} \) lambda term, whose parameter is the free variable of the lambda term. The \( \{n+1\}^{\text{th}} \)

\begin{align*}
\text{Expression Translation} \\
\{[e]_p \}st = (x, st, c_{\text{st}}) \\
\{op \}p st = (op \times st, c_{\text{op}}) \quad \text{if } op \in \text{Prim} \\
\{C \}p st = (C \bar{x}, st, c_{\text{cons}}) \quad \text{if } C \in \text{Cids} \\
\{let \ x := e \in \{e\}_p \} st = \\
\quad \text{let } u := \{e_1\}_p \text{ in } \\
\quad \text{let } w := \{e_2[u_1/x]_p \} u_2 \text{ in } \{w.1, w.2, c_{\text{let}} \oplus u.3 \oplus w.3\} \\
\{x \text{ match } \} \{C_i \bar{x} = e \} \_1 \_n \} st = x \text{ match } \\
\{C_i \bar{x} \Rightarrow \text{let } u := \{e_1\}_p \text{ st } \} \{u.1, u.2, c_{\text{match}}(i) \oplus u.3\} \\
\text{Call and Lambda Translation} \\
\{[f]_p \} st = \quad \text{if } f \text{ does not have } \text{@memoize} \text{ annotation} \\
\quad \text{let } w := \{f \} \_m \text{ st in } \{w.1, w.2, c_{\text{call}} \oplus w.3\} \\
\{f \} \_p st = \quad \text{if } f \text{ has } \text{@memoize} \text{ annotation} \\
\quad \text{let } w := \{f \} \_m \text{ st in } \\
\quad \text{let } x_{\text{cost}} = \{f \} \_m \text{ st in } \{c_{\text{call}}\} \text{ else } \{c_{\text{miss}} \oplus c_{\text{call}} \oplus w.3\} \\
\{f : \lambda x. f(x,y)\} \_p st = (C_i y, st, c_{\lambda}) \text{, where, } i = \text{label}(f/\approx) \\
\{x \_n \} st = \\
\quad \text{let } w := \text{App}(\_; \_; \_; \_; \text{st}, y) \text{ in } \{w.1, w.2, c_{\text{app}} \oplus w.3\} \\
\text{Specification Construct Translation} \\
\{\text{cached}(f)\} \_p st = \{\{f\} \_m \text{ st s.t. } st, 0\} \\
\{\text{stream}(x)\} \_p st = \{[e]_p \times x\} \\
\text{Dispatch Functions} \\
\text{def } \text{App}_r (cl, x, st) = \{ \\
\quad \text{cl match}\{C_i y = \{e_1\}_p \text{ st } \cdots C_i y = \{e_n\}_p st \} \\
\quad \text{C}_r y = \text{error } \}
\text{Contract Translation} \\
\{\text{pre}\} e \{\text{post}\}_p st = \quad \text{if } R \in \{\text{steps}, \text{alloc}\} \\
\quad \{\{\text{pre}\}_p st.1\}_p \\
\quad \{[e]_p \}_p st \\
\quad \{\text{let } y := \{\text{post}_{\text{res.1}} \_q / \text{res.2}_{\text{res.3}} \} \_r / \text{inSt}\}_p \{\text{res.2}_{\text{res.3}} \} / \text{R} \_y \}
\text{Function Definition Translation} \\
\text{def } f x = e_p \_p = \text{def } \{f \} \_m (x, st) = \{[e]_p \}_p st
\end{align*}
mimics the computation of resources in the operation semantics. It proceeds bottom-up; first instrumenting the sub-expressions of an expression \( e \), and then, using the resource usage components of the instrumented sub-expressions \( c_{m-1} \) to instrument \( e \). However, instrumentation of a call to a memoized function is handled differently, and is explained shortly.

Fig. 7(b) shows the translated code for the function takeLazy as outputted by our tool. The code shown in Fig. 7 is obtained after a few straightforward static simplifications. For instance, the constants such as 9 and 3 that appear in the resource expressions are the result of summing up the operation costs along the static path containing the resource expressions.

**Cache-state propagation.** The instrumentation of cache state proceeds top down following the control flow of the program. To every function definition in the model, we add a fresh parameter \( st \) (of type Set[Dcache]) that represents the state of the cache at the beginning of the function (see translation of function definitions). This parameter is propagated through the bodies of the function, recording all the calls that are be memoized along the way. The state parameter is used in two places: by calls to memoized functions to model their resource usage, and by the cached construct to check whether the call given as argument is memoized. Consider the translation of a call to a memoized function: \( f \ x \) shown in Fig. 6. It uses the input state parameter \( st \) to check whether the call would be a cache hit, by testing if \( st \) contains \( C_f \ x \) which represents the call \( f \ x \). The resource usage in the cache hit case is given by \( c_{\text{hit}} \), whereas in the miss case it is a combination of \( c_{\text{miss}} \), the cost of the call \( c_{\text{call}} \), and the resource usage of the callee: \( w_3 \). Finally, \( (C_f \ x) \) is added to the output state to record that the call is memoized (regardless of whether or not it was memoized before). Notice that, as in the operational semantics, during the translation of contracts, the precondition is translated using the initial state \( st \), and the postcondition using the state resulting after the translation of body \( res \). Moreover, any changes to the state caused by the contracts are discarded at the end of the contracts. Any uses of \( res \) in the postcondition is replaced with \( res_{\text{1}} \), uses of a resource \( R \) by \( res_{\text{2}} \), and uses of \( nSt \) by \( st \), which represents the input state.

Fig. 7 shows the result of propagating the state through the body of takeLazy function as outputted by our tool. Our tool eliminates propagation through expressions, and function that are statically inferred as not affecting the state. For instance, concrUntil does not return a state as it was statically determined to not have any effect on the state. Note that after the call tail(c, st) at line 36, an instance of Tail(c) is added to the output state to record that the call is memoized, and that the computation of steps at line 40 depends on whether or not Tail(c) belongs to the input state.

**Defunctionalization.** We translate all function applications to first-order calls by creating a dispatch function \( App_c \) for each function type \( \tau \) in the program. The function takes a closure \( cl \), the argument of the application \( x \), and a state parameter \( st \). The dispatch function pattern matches on the closure \( cl \). Each constructor pattern of the form \( C_\lambda \), where \( \lambda \) is the label of a lambda \( \lambda a.e \), dispatches to the expression \( [e] \), where \( e \) is the result of replacing \( e \) the parameter of the lambda \( a \) with \( x \), and the free variables of the lambda with the fields of the closure. If the closure matches \( C_f \) the model halts with an error, as this case corresponds to the scenario where a function not defined within the program being analyzed, is applied to an argument. Such a function, being arbitrary, can have a precondition that is violated by the arguments it is applied to. The model soundly flags this case as an error. We eliminate this case if we can statically infer (based on encapsulation) that the targets of the closures are strictly within the program under analysis.

We replace every application of the form \( x \ y \), where \( \text{type}(x) = \tau \) by a call to \( App_c \) as shown in Fig. 6. Notice that in the illustration

\begin{figure}[h]
\centering
\begin{minted}{java}
sealed abstract class Stream
private case class SCons(x: BigInt, tfun: () ⇒ Stream) extends Stream{
  lazy val tail = tfun()
}
private case class SNil() extends Stream

def takeLazy(n: BigInt, s: Stream): Stream = {
  require(concrUntil(s, n))
  s match {
    case c: SCons ⇒
      val t = c.tail
      val n1 = n - 1
      SCons(c.x, () ⇒ takeLazy(n1, t))
    case SNil() ⇒ s
  }
}
estroking(r ⇒ steps ≤ ?)
sealed abstract class tStream

case class TakeLazy(y: BigInt, s: Stream) extends tStream
sealed abstract class Stream

\end{minted}
\caption{Illustration of the translation shown in Fig. 6.}
\end{figure}

\begin{itemize}
\item Parameter \( f_i \). In the illustration shown in Fig. 7 the datatype Dcache with one constructor: Tail(c) corresponds to this datatype. Though tail is a lazy field of the Stream datatype, we treat it as a memoized function with a single argument that refers to the receiver.
\item **Translation of expressions.** Fig. 6 shows the translation of expressions and types in the program. The translation of an expression \( e \) is given by \( [e]_\phi \), which takes a state expression \( st \) representing the keys of the cache before the evaluation of \( e \), and returns the translated expression (say \( c_{\text{mem}} \)), which is a triple, where the first part \( c_{\text{mem1}} \) corresponds to the value of \( e \), the second part \( c_{\text{mem2}} \) corresponds to the keys of the cache after evaluation of \( e \), and the last \( c_{\text{mem3}} \) corresponds to the resource usage of \( e \). Similar to the operational semantics shown in Fig. 5, we parametrize the computation of resources of an expression \( e \) using the cost function \( c_{\text{res}} \), and the combinator \( \oplus \). However, here \( \oplus \) is applied over expressions of the model that track resource usages, instead of integers. The instrumentation of resource usage of expressions closely
\end{itemize}
shown in Fig. 7, the application of \( f \) inside the function `tail` is translated to a call to the dispatch function `app`.

**Soundness and Completeness of the Model.** We now establish the soundness and completeness of the model for verification of contracts of the source program. We start by defining a relation \( \sim \) that relates an environment \( \Gamma_1 \in \text{Env} \) of the source program, with an environment \( \Gamma_2 \in \text{Env}_{\text{mod}} \) of the model, much like a bisimulation relation between transition systems. However, some what unique to our setting, \( \Gamma_1 \) is actually simulated by a pair \((\Gamma_2, S)\), where \( S \in \text{Set} \) denotes the keys of the cache, as formalized below. Let \( P \) be a program, and \( \{H_1, H_2\} \subseteq \text{Heap} \). We define a relation \( \sim_{H_1, H_2} \) on the semantic domain as follows: (subscriptions omitted below for clarity)

1. \( \forall a \in \text{Z} \cup \text{Bool}. \ a \sim a \)
2. \( \forall c \in \text{Cds}, \{\bar{a}, \bar{b}\} \subseteq \text{Val}^n, \ c \sim \bar{a} \iff \forall i \in [1, n], a_i \sim b_i \)
3. \( \forall (l, \sigma) \in \text{Closure}, \ v \in \text{Val}(l, \sigma) \iff C_v \ v \)
4. \( \forall f \in \text{Fids}, \{a, b\} \subseteq \text{Val}. \ f(a) \sim b \iff a \sim b \)
5. \( \forall c \in \text{Cache}, S \in \text{Set}. \ C \sim S \)

\( \begin{align*}
   &\text{dom(C)} = |S| \Land \Box \exists v \in \text{dom(C)}. \exists y \in S. \ x \sim y \\
   &\forall \varphi, \chi \subseteq \text{Adr}. \ a \sim b \Leftarrow \exists f, \exists h_1(a) \sim h_2(a) \\
   &\forall \varphi, \varphi_1, \varphi_2 \subseteq \text{Store}. \ s_1 \sim s_2 \Leftarrow \exists \varphi_1 \subseteq \text{dom}(\varphi_1), \varphi_2(x) \sim \varphi_2(x)
\end{align*} \)

The relation formally captures that a cache is simulated by a set of instances of `Datcache` (rules 5 and 6), and a closure is simulated by an instance of the datatype representing closures if the lambda \( l \) of the closure is compatible to a lambda in the program \( P \) (rule 3). We define a relation \( \sim \) between an environment \( \Gamma_1 \in \text{Env} \) and a pair in \( \text{Env}_{\text{mod}} \times \text{Set} \) as follows:

**Def 3 (Simulation Relation).** For all \( \Gamma_1 : (C_1, \underbrace{H_1, H_2}_1, \sigma_1) \in \text{Env}, \) and \( S \subseteq \text{Set}. \ \Gamma_1 \sim (\Gamma_2, S) \iff C_2 \sim C_1 \Land S \subseteq \sigma_1 \).

The following theorem states that for every environment \( \Gamma_1 \in \text{Env} \) and expression \( e \), the translation of \( e | s \) \( s \) with respect to a state \( s \) at that evaluates to a value \( s \) under an environment \( \Gamma_2 \in \text{Env}_{\text{mod}} \), correctly models (a) the resource usage of \( e \), (b) the set of keys of the cache at the end of \( e \), and (c) preserves the \( \sim \) relation between the output environments and the output cache state, when evaluated under \( \Gamma_2 \) and \( \Gamma_1 \sim (\Gamma_2, S) \).

**Theorem 1 (Bisimulation).** Let \( \Gamma_1 \subseteq \text{Env}_{\text{mod}}, \Gamma_2 \subseteq \text{Env}, \) and \( \forall e \in \text{Env}_{\text{mod}} \text{ such that } \Gamma_2 \vdash s \Downarrow S, \Gamma_1 \sim (\Gamma_2, S) \).

\( \begin{align*}
   &\text{(a)} \text{If } \Gamma_2 \vdash e \Downarrow v, \Gamma_1 \text{ then } \exists \Gamma'_2 \in \text{Env}_{\text{mod}}. \exists u \in \text{DVal} \text{ such that } \\
   &\begin{align*}
   &\Gamma_2 \vdash (\{e\} s) \Downarrow u, \Gamma'_2, \text{ and} \\
   &\bullet \Gamma'_1 \sim (\Gamma'_2, u_2) \Land v \sim_{H_1, H_2} u_1 \Land p = u_3
   \end{align*}
\end{align*} \)

\( \begin{align*}
   &\text{(b)} \text{If } \Gamma_2 \vdash (\{e\} s) \Downarrow u, \Gamma'_2, \text{ and } u \in \text{DVal} \text{ then } \exists \Gamma'_1 \in \text{Env} \text{ such that } \\
   &\begin{align*}
   &\Gamma_1 \vdash e \Downarrow v, \Gamma'_1, \text{ and} \\
   &\bullet \Gamma'_1 \sim (\Gamma'_2, u_2) \Land v \sim_{H_1, H_2} u_1 \Land p = u_3
   \end{align*}
\end{align*} \)

Appendix B shows proof sketches of the theorems. Using the above property, we now establish that for every function \( f \) in the program \( P \), verifying the contracts of its translation \( f_m \) will imply that the contracts of \( f \) hold, and vice versa. A tricky aspect here is that there exists environments \( \Gamma \in \text{Env} \) that binds variables to lambdas that are not in the scope of the program \( P \) under which \( f \) evaluates to a value. Such environments do not have any counterparts (with respect to \( \sim \)) in the model of \( P \). The following theorem holds despite this because in such cases it can be shown that neither the contracts of \( f \) or \( f_m \) hold for all environments, as there exists an environment each in \( \text{Env} \) and \( \text{Env}_{\text{mod}} \) that results in a contract violation in \( f \), and enforces the error condition in \( f_m \).

**Theorem 2 (Model Soundness and Completeness).** Let \( P \) be a program. Let \( \bar{e} = \{p\} \epsilon \{s\} \text{ and } \bar{e}_m = \{p_m\} \epsilon \{s_m\} \).

\( \begin{align*}
   &\forall \Gamma_2 \in \text{Env}_{\text{mod}}. \exists \Gamma_1 \vdash p \Downarrow f_m \Downarrow \langle \bar{e} \rangle \Downarrow v \Downarrow v' \iff \forall \Gamma_1 \in \text{Env}_{\text{mod}}. \exists \Gamma_1 \vdash p \Downarrow f_m \Downarrow \langle \bar{e} \rangle \Downarrow v \Downarrow v'.
\end{align*} \)

A corollary of the above theorem is that the model is sound, and complete for the inference of resource bounds. That is, for any assignment to holes \( s \), for every function \( \bar{e} \), for every function \( f_m(x, st) = \bar{e}_m \).

\( \forall \Gamma_2 \in \text{Env}_{\text{mod}}. \exists \Gamma_2 \vdash p \Downarrow f_m \Downarrow \langle \bar{e} \rangle \Downarrow v' \iff \forall \Gamma_1 \in \text{Env}_{\text{mod}}. \Gamma_1 \vdash p \Downarrow f_m \Downarrow \langle \bar{e} \rangle \Downarrow v. \)

4. **Model Verification and Inference**

In this section, we discuss our approach for verifying the contracts, and inferring the constants in the resource bounds of the model program.

**Modular reasoning with higher-order functions.** Approaches based on function-level modular reasoning verify the postcondition of each function in the program under the assumption that the precondition of the function, and the contracts of the callees (including itself) hold. The precondition of each function is verified at their call sites, independently. (The recursive functions in the program have to be well-founded.) We now formalize this reasoning, and subsequently present an extension for handling higher-order functions more effectively. We use the model shown in Fig. 7(b) as the running example.

Let \( \Delta \) denote the set of function definitions in the program, and \( e_1, e_2 \) be two properties i.e, boolean-valued expressions. Let \( e_1 \rightarrow e_2 \) denote that whenever \( e_1 \) does not evaluate to false, \( e_2 \) evaluates to true i.e, \( \forall \Gamma \in \text{Env}_{\text{mod}}. \Gamma \vdash e_1 \Downarrow \text{false} \Downarrow \Gamma \vdash e_2 \Downarrow \text{true} \).

(1) For each call site \( s \) of the form \( f \) in the program,

**Function-level modular reasoning:**

\( \begin{align*}
   &\vdash_A \text{path}(c) \rightarrow \text{pre}(c) \)
\end{align*} \)

where the variable \( c \) refers to the result of the expression \( e \) in the postcondition of \( e \), and \( \text{path}(c) \) denotes the static path (possibly with disjunctions) to \( c \) from the entry of the function containing \( c \). For instance, the path condition of the call `tail(c, st)` at line 36 of the program shown in Fig. 7(b) is `concrUntil(s, n, st) \wedge s = c \wedge c.instanceOf[SCons]`. For convenience, we use `pre(c)` where \( c \) is a call of the form `f x`, to denote the precondition of `f` (say `pre`) translated to arguments of the call i.e, `pre[x/params(f, p)]`.

When the resource bounds in the program have holes (in TVars), the assume/guarantee assertions generated as above would also have holes. The goal is to find an assignment \( t \) for holes such that all the assume/guarantee assertions of all functions are valid.

While this modular reasoning is applicable to the first-order model described in section 3, it dramatically increases the specification/verification overhead when applied as such to the model. For instance, consider the call to `takeLazzy` within app at line 30 in the example shown in Fig. 7(b). With a plain modular reasoning, the precondition of `app` is not strong enough to show that the precondition of `takeLazzy`, namely `concrUntil(s1, n1, st)`, holds for the call. In order to verify this example, we need to assert that the property `concrUntil(s, n, st)` holds for the values stored in every instance of `TakeLazzy` transitively reachable from the re-
cursive datatype Stream, which is stated by following function.

\[
\text{def pre(cl, st): Boolean \{ cl match \}
\text{\quad case TakeLazy(n1,s1) \Rightarrow concrUntil}(s1,n1,st) \&\&\n\text{\quad \{ case SCons(_t) \Rightarrow pre(t, st); case \_ \Rightarrow true \}}\}
\text{case \_ \Rightarrow true \} }
\]

What complicates this further is that to ensure this precondition at the call to app at line 27, the precondition of the function tail, and all its transitive callers (including `takeLazy`) should be modified similarly. This scenario happens very often when dealing with lazy data structures, for instance in the `Real-time queue` data structure shown in Appendix C. Our initial attempts to use a precondition such as the above resulted in formulas too complicated for the state-of-the-art SMT solvers to solve. In the sequel, we discuss an approach to alleviate this specification overhead, based on the observation that the property `concrUntil` actually holds at the points where the closure `TakeLazy` is created (namely at line 39 in Fig. 7(b)), and is monotonic with respect to the changes in the cache that happen during the program.

**Cache Monotonic Properties.** To mitigate the specification burden, we verify and utilize properties that are cache monotonic. Informedly, a property \( p \in E_{spec} \) is cache monotonic iff whenever it holds in an environment with cache \( C_1 \), it also holds in all environments where the cache has more entries than \( C_1 \). These properties are interesting because once established they can be assumed to hold at any subsequent point in the evaluation (similar to heap-monotonic type states introduced by Fähndrich and Leino [19]).

We find that in almost all cases, the properties that are needed to establish resource bounds are (or can be converted to) cache monotonic properties, e.g. the `concrUntil` property. Intuitively, this phenomenon seems to result from anti-monotonicity of resource usage: resource usage of an expression cannot increase when it is evaluated under a cache that has more entries. Below, we formalize cache monotonicity, and later describe how we exploit it in verification.

Let \( \Gamma \subseteq \text{Env} \) be a partial order on \( \Gamma \subseteq \text{Env} \) defined by: \( \Gamma_1 \sqsubseteq \Gamma_2 \) iff \( \text{dom}(C_1) \subseteq \text{dom}(C_2) \), where \( C_1 \) and \( C_2 \) are the cache components of the environments, respectively. A property \( p \in E_{spec} \) is cache monotonic iff \( \forall (\Gamma_1, \Gamma_2) \subseteq \text{Env}. \Gamma_1 \sqsubseteq \Gamma_2 \Rightarrow \Gamma_1 \vdash p \Downarrow \text{true} \Rightarrow \Gamma_2 \vdash p \Downarrow \text{true} \). To check if a property \( p \in E_{spec} \) is cache monotonic it suffices to check the following property on the translation of \( p \) with respect to \( [\cdot] \) defined in Fig. 6: \( \models_A (st_1 \subseteq st_2 \& \text{pre}(st_1) \rightarrow \text{pre}(st_2)) \).

**Creation-dispatch rule for encapsulated types.** Let \( \tau \) be the set of encapsulated function types in a program which cannot be assigned closures created outside the program. For instance, the type \( (\cdot) \Rightarrow \text{Stream} \) in Fig. 7(a) is a encapsulated function type. Let \( \{ l_i \mid i \in [1..n] \} \) be the lambdas in the program of type \( \tau \), and \( \{ C_i, x_i \mid i \in [1..n] \} \) be the closure constructions in the model representing the lambdas. Let \( \text{DispCalls} = \{ f ; y_i \mid i \in [1..n] \} \) denote the calls made within the dispatch functions, representing the indirect calls to the lambdas. For instance, for the model shown in Fig. 7(b), \( l_1 \) is \( (\cdot) \Rightarrow \text{takeLazy}(n1,t) \), \( (C_1 \ x_1) \) is `TakeLazy(n1,r1)` at line 39, and \( (f_1 ; y_1) \) is `takeLazy(n1,s1,st)` at line 30. Let \( \text{Props} = \{ P_i \mid i \in [1..n] \} \), where \( \text{FV}(P_i) \subseteq x_i \cup (\{ st \}) \), be a collection of properties that are defined on the captured arguments of the closures (namely \( C_i, x_i \)), and on a set-valued variable \( st \) representing the keys of the cache. Given an expression \( e \), let `stevar(e)` be the state expression that statically reaches \( e \). Note that there is exactly on state expression reaching every point in the program by the definition of the translation shown in Fig. 6. For instance, the state expression reaching the line 30 of Fig. 7(b) is \( st \), whereas the state expression reaching the line 39 is `n1st`. We now extend the function-level assume/guarantee rules to include the following rule: if each of the properties \( P_i \) are cache monotonic, and hold at the point of creation of the closure \( C_i \) for the state reaching the creation point, then it can be assumed to hold at the point of dispatch. This is formally expressed as shown below:

**Modular reasoning with creation-dispatch rule**

(I) For each def \( f = \{ \text{pre} \} e \{ \text{post} \} \models_A \text{pre} \rightarrow \text{post} (\{ e / \text{res} \}) \)

(II) For each call site \( c \notin \text{DispCalls} \) of the form \( f \)

\( \models_A (\text{path}(c) \rightarrow \text{pre}(c)) \)

(III) **Cache monotonicity** For each \( P_i \in \text{Props} \)

\( \models_A (st_1 \subseteq st_2 \& \text{Paths}(\{ st_1 / st_2 \}) \rightarrow P_i(st_2 / st_1) \)

(V) For each closure construction site \( c \) of the form \( C_i \)

\( \models_A \text{path}(c) \rightarrow P_i(st \{ \text{stevar}(c) / st \}) \)

(IV) For each call site \( c \in \text{DispCalls} \) of the form \( f ; y_i \)

\( \models_A \text{path}(c) \& P_i(y_i / x_i) \{ \text{stevar}(c) / st \} \rightarrow \text{pre}(c) \)

While the above is reasoning holds irrespectively of how the properties \( P_i \) are chosen for each lambda \( l_i \), we use a particular strategy in our implementation. For each \( l_i = \lambda x.f \), we choose the property \( P_i \) to be the disjunctions of the precondition of the call \( f \) that only refer to the captured variables \( FV(l_i) \).

For the example shown in Fig. 7(b), our approach would verify that (a) `concrUntil` is a cache monotonic property. That is, \( \models_A st_1 \subseteq st_2 \& \text{concrUntil}(s, i, st_1) \rightarrow \text{concrUntil}(s, i, st_2) \) is valid. (b) that the property `concrUntil(u1,n1,nst)` holds at point of creation of the closure `TakeLazy(n1,u1)` at line 39. That is, \( \models_A \{ \text{concrUntil}(s, n, st) \& s = c \& \text{c.isInstanceOf}(\text{SCons}) \& u = \text{tail}(c.st) \& \text{nst} = u = u \cup (\{ \text{Tail}(c) \} \& n1 \& n1 = n-1) \} \rightarrow \text{concrUntil}(u1,n1,nst) \).

The property `concrUntil(s1,n1,nst)` is assumed to hold while checking the precondition of call to `TakeLazy` at line 30. With this extension, we do not need any more preconditions than what is stated in the program to verify it.

**Solving parametric verification conditions.** To solve the assertions generated by assume/guarantee reasoning, and infer values for the holes, we extend the template inference algorithm proposed in previous research [35], and implemented in the publicly available system [9, 48]. Fig. 8 shows a block diagram of the inference algorithm, which we briefly describe in the sequel.

Given an assume/guarantee assertion \( \models_A e_1 \rightarrow e_2 \) the VC generation phase converts it to a quantifier-free formula (VC) of the form \( \phi(\bar{x}, \bar{a}) \), where the variables \( \bar{a} \) corresponds to the numerical holes, such that the assume/guarantee assertion holds if there exists an assignment \( \bar{a} \) for \( \bar{a} \) such that \( \phi(\bar{x}, \bar{a}) \) is unsatisfiable. (The VC could be thought of as a formula in the form: \( 3\bar{x} \exists \bar{a} \neg \phi, \bar{a} \) where the uninterpreted function symbols are universally quantified.) The VC's belong to the theory \( T \) of uninterpreted functions, algebraic data types, sets, nonlinear arithmetic. But, due to the syntactic restrictions on the templates (shown in Fig. 4), the VC's would be linear parametric formulas [34], where every nonlinear term is of the form \( a \cdot x, \) where \( a \) belongs to \( \bar{a} \), and \( x \) belongs to \( \bar{x} \).

Converting an assume/guarantee assertion to a formula is fairly straightforward. Every primitive operation is mapped to a corresponding theory operation. Match expressions are converted to disjunctions, and let expressions to equalities. The bodies (with con-
tracts) of function calls in the expressions are unfolded up to a certain depth, and treated uninterpreted. Nonlinear operation over \( x \) are axiomatized in the VC. The resulting VC is solved using a counter-example guided algorithm (discussed shortly). If the solving fails, a new VC is generated by further unfolding recursive functions, and instantiating nonlinear axioms, and the process is repeated, until a solution is found or a timeout is reached.

**Preserving source-level properties.** Functions in the source-program (not specifications) are referentially transparent (i.e., functional) with respect to the result of evaluation. However, in the model program, this is not immediately obvious as the functions are instrumented with the state. In fact, the axiom \( \forall x, s_1, s_2, f_m \ (x, s_1),_1 = f_m \ (x, s_2),_1 \), holds in the model, but is not explicitly instantiated. We instantiate this property automatically in a complete way by encoding it using uninterpreted functions in the generated VC. This helps achieve a functional reasoning for properties on the results of expressions. We also automatically instantiate that the state at the end of a function, is a super set of the state at the beginning.

**Solving linear parametric formulas with sets.** Given a linear parametric VC of the form: \( \phi(x, \bar{a}) \), the solution for \( \bar{a} \) that will make \( \phi \) unsatisfiable is computed using an iterative, but terminating algorithm that progresses in two phases: an existential solving phase (phase I), and a universal solving phase (phase II). Phase I discovers candidate assignments \( \iota \) for the free variables \( \bar{a} \). It initially starts with an arbitrary guess, and subsequently refines it based on the counter-examples produced by Phase II. Phase II checks if the candidate assignment \( \iota \) makes \( \phi \) unsatisfiable. That is, if \( \phi \iota \) is unsatisfiable. If not, it chooses a disjunct \( d \bar{x}, \bar{a} \) satisfiable under \( \iota \) that only has numerical variables, by axiomatizing uninterpreted functions, and algebraic datatypes in a complete way [35]. This numerical disjunct is then given back to phase I. Phase I generates and solves a quantifier-free nonlinear constraint \( C(\bar{a}) \), based on Farkas’ Lemma [46], to obtain the next candidate assignment for \( \bar{a} \) that will make \( d \bar{x}, \bar{a} \), and other disjuncts previously seen, unsatisfiable. Each phase invokes the Z3 [17] and CVC4 [5] SMT solvers in portfolio mode on quantifier-free formulas. This algorithm was shown to be complete for linear parametric formulas belonging to the combined theory of real arithmetic, uninterpreted functions, and algebraic datatypes [35]. Below we extend this result to include sets. (Appendix B shows the proof sketch.)

**Theorem 3.** Given a linear parametric formula \( \phi(x, \bar{a}) \) with free variables \( \bar{x} \) and \( \bar{a} \), belonging to a theory \( T \) that is a combination of quantifier-free theories of uninterpreted functions, algebraic datatypes, and sets, and either integer linear arithmetic or real arithmetic, finding a assignment \( \iota \) such that \( \text{dom}(\iota) = |\bar{a}| \), and \( (\phi \iota) \) is \( T \)-unsatisfiable is decidable.

**5. Evaluation**

We implemented the approach described in the previous sections, and used our system to verify resource bounds of the many data structures and algorithms. In this section, we summarize the results of our experiments. All evaluations presented in this section were performed on a machine with a 4 core, 3.60 GHz, Intel Core i7 processor, having 32GB RAM, running Ubuntu operating system.

**Benchmark statistics.** Fig. 9 shows selected benchmarks that were verified by our approach. Each benchmark was implemented, and specified in a purely functional subset of Scala, extended with our specification constructs. We carefully picked some of the most challenging benchmarks from the literature of lazy data-structures, and dynamic programming algorithms. For instance, the benchmark \( \text{rtq} \) (shown in Appendix C) has been mentioned as being outside the reach of prior works [16, 50]. For each benchmark, the figure shows the total lines of Scala code, and the size of the compiled JVM byte code in columns \( \text{LOC} \), and \( \text{BC} \). The benchmarks comprise a total of 4.5K lines of Scala code, and 1.2MB of bytecodes. The column \( T \) shows the number of functions with resource bound templates, and the column \( S \) the number of specification functions. (We do not verify resource bounds of specification functions but only check their termination [91].) The column \( AT \) shows the time taken by our system, rounded off to minutes, to verify the specifications, and infer the constants. As shown by the figure, all benchmarks were verified within a few minutes. The column \( \text{Resource bounds} \) shows a sample bound for steps, and alloc resource. The constants in the bound were automatically inferred by the tool. We verified a total of 123 bounds each for steps, and alloc. Many bounds used recursive functions, and almost 20 bounds had nonlinear operations. A few bounds were disjunctive (like the bound shown in Fig. 1, and the bound of \( \text{concatNonEmpty} \)). However, in our experience, the most challenging bounds to prove were the constant time bounds of scheduling-based lazy data structures viz. \( \text{rtq}, \text{deque}, \text{num}, \) and \( \text{conq} \), due to their complexity.

**Evaluation of accuracy of the inferred bounds.** We instrumented the benchmarks for tracking steps, and alloc resources as defined by the operational semantics, and executed them on concrete inputs that were likely to expose the worst case behavior. We varied the sizes of the inputs in fixed intervals up to 10k for most benchmarks, but for those with nonlinear behavior, we used smaller inputs that scaled within a cutoff time of 5 min, as tabulated in the column \( I \) of Fig. 10. For scheduling based data structures (discussed shortly) we varied the input in powers of two until \( 2^{20} \), which results in their worst-case behavior. For every top-level (externally accessible) function in a benchmark, we computed the mean ratio between the runtime resource usage, and the static resource usage predicted by our tool, using the following formula:

\[
M_{\text{err}} \left( \frac{\text{resource consumed by the } i^{th} \text{ input}}{\text{static estimate for } i^{th} \text{ input}} \times 100 \right)
\]

\( M_{\text{err}} \) is the mean error, and \( i \) represents the input. The results indicate that our approach is very accurate in inferring bounds.
<table>
<thead>
<tr>
<th>Benchmark</th>
<th>LOC</th>
<th>BC</th>
<th>T</th>
<th>S</th>
<th>AT</th>
<th>Resource bounds</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lazy data-structures</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Lazy Selection Sort (sel)</td>
<td>70</td>
<td>36kb</td>
<td>4</td>
<td>1</td>
<td>1m</td>
<td>steps ≤ 15k · l.size + 8k + 13 alloc ≤ 2k · l.size + 2k + 2</td>
</tr>
<tr>
<td>Prime Stream (prims)</td>
<td>95</td>
<td>51kb</td>
<td>7</td>
<td>2</td>
<td>1m</td>
<td>steps ≤ 16n² + 4 alloc ≤ 6n − 11</td>
</tr>
<tr>
<td>Fibonacci Stream (fibs) [8]</td>
<td>199</td>
<td>59kb</td>
<td>5</td>
<td>5</td>
<td>1m</td>
<td>steps ≤ 45n + 4 alloc ≤ 4n</td>
</tr>
<tr>
<td>Hamming Stream (hams) [8]</td>
<td>223</td>
<td>78kb</td>
<td>8</td>
<td>6</td>
<td>2m</td>
<td>steps ≤ 129n + 4 alloc ≤ 16n</td>
</tr>
<tr>
<td>Stream library (slib) [49]</td>
<td>408</td>
<td>0.1mb</td>
<td>22</td>
<td>5</td>
<td>1m</td>
<td>steps ≤ 25l.size + 6 alloc ≤ 4l.size</td>
</tr>
<tr>
<td>Real time queue (rtq) [38, 39]</td>
<td>207</td>
<td>69kb</td>
<td>5</td>
<td>6</td>
<td>1m</td>
<td>steps ≤ 37, steps ≤ 40 alloc ≤ 8, alloc ≤ 7</td>
</tr>
<tr>
<td></td>
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<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Lazy Mergesort (msort) [2]</td>
<td>290</td>
<td>0.1mb</td>
<td>6</td>
<td>8</td>
<td>1m</td>
<td>steps ≤ 36(k · [log (l.size)]) + 5l.size + 22 alloc ≤ 6(k · [log (l.size)]) + 6l.size + 3</td>
</tr>
<tr>
<td>Deque (deq) [38, 39]</td>
<td>426</td>
<td>0.1mb</td>
<td>16</td>
<td>7</td>
<td>3m</td>
<td>steps ≤ 580, steps ≤ 970 alloc ≤ 50, alloc ≤ 78</td>
</tr>
<tr>
<td>Lazy Numerical Rep. (num) [39]</td>
<td>546</td>
<td>0.1mb</td>
<td>6</td>
<td>25</td>
<td>1m</td>
<td>steps ≤ 106 alloc ≤ 15</td>
</tr>
<tr>
<td>Conqueue (conq) [41, 42]</td>
<td>880</td>
<td>0.2mb</td>
<td>12</td>
<td>33</td>
<td>3m</td>
<td>steps ≤ 124 alloc ≤ 23</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>steps ≤ 29[</td>
</tr>
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<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Dynamic Programming</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>LCS (lex)</td>
<td>121</td>
<td>37kb</td>
<td>4</td>
<td>4</td>
<td>1m</td>
<td>steps ≤ 33mn + 33m + 33n + 31 alloc ≤ 2mn + 2m + 2n + 3</td>
</tr>
<tr>
<td>Levenshtein Distance (levd)</td>
<td>110</td>
<td>37kb</td>
<td>4</td>
<td>4</td>
<td>1m</td>
<td>steps ≤ 39mn + 39m + 39n + 37 alloc ≤ 2mn + 2m + 2 + 3</td>
</tr>
<tr>
<td>Hamming Numbers (hm)</td>
<td>105</td>
<td>44kb</td>
<td>3</td>
<td>3</td>
<td>3m</td>
<td>steps ≤ 71n + 70 alloc ≤ 3n + 4</td>
</tr>
<tr>
<td>Weight Scheduling (ws)</td>
<td>133</td>
<td>44kb</td>
<td>3</td>
<td>5</td>
<td>1m</td>
<td>steps ≤ 20jobi + 19 alloc ≤ 2jobi + 3</td>
</tr>
<tr>
<td>Knapsack (kr)</td>
<td>122</td>
<td>48kb</td>
<td>5</td>
<td>4</td>
<td>1m</td>
<td>steps ≤ 17(w · i.size) + 18w + 17i.size + 18 alloc ≤ 2w + 3</td>
</tr>
<tr>
<td>Packrat Parsing (pp) [20]</td>
<td>249</td>
<td>73kb</td>
<td>7</td>
<td>5</td>
<td>1m</td>
<td>steps ≤ 73n + 70 alloc ≤ 11n + 11</td>
</tr>
<tr>
<td>Viterbi (viti) [52]</td>
<td>191</td>
<td>63kb</td>
<td>6</td>
<td>7</td>
<td>2m</td>
<td>steps ≤ 34k²t + 38k² − 8kt + 26k + 276t + 30 alloc ≤ 2kt + 2k + 4t + 5</td>
</tr>
</tbody>
</table>

Figure 9. Selected benchmarks comprising of ~4.5K lines of Scala code and 123 resource bounds each for steps and alloc.

The column dynamic/static * 100 of Fig. 10 shows this metric for each benchmark, when averaged over all top-level functions in the benchmark. As shown in the figure, when averaged across all benchmarks, the runtime resource usage was 80% of what was inferred statically for steps, and is 88% for alloc. In all cases, the inferred resource usage were sound upper bounds for the runtime resource usage. We now discuss the reasons for some of the inaccuracy in the inferred bounds.

In our system, there are two factors that influence the overall accuracy of the bound: (a) the constants inferred by tool, and (b) the resource templates provided by the user. For instance, in the prims benchmark shown in Fig. 1, the function isPrimeNum(n) has a worst-case steps count of $11n − 7$, which will be reached only if $i$ is prime. (It varies between $O(\sqrt{n})$ and $O(i)$ otherwise.) Hence, for the function primesUntil(n), which transitively invokes isPrimeNum function on all numbers until $n$, no solution for the template: $? + n² + ?$ can accurately match its worst-case, runtime steps count. Another example is the $O(k · [\log(l.size)])$ resource bound of msort benchmark. In any actual run, as $k$ increases, the size of the stream that is accessed (which is initially $l$) decreases, and hence $[\log(l.size)]$ term decreases in steps.

To provide more insights into the contribution of each of these factors to the inaccuracy, we performed the following experiment. For each function, we reduced each constant in its resource bound, keeping the other constants fixed, until the bound violated the resources usage of at least one dynamic run. We call such a bound a pareto optimal bound with respect to the dynamic runs. Note that if there are $n$ constants in the resource bound of a function, there would be $n$ pareto optimal bounds for the function. We measured the mean ratio between the resource usage predicted by the pareto optimal bound, and that predicted by the bound inferred by the tool.

The column optimal/static * 100 of Fig. 10 shows this metric for each benchmark, when averaged over all pareto optimal bounds of all top-level functions in the benchmark. A high percentage for this metric is an indication that any inaccuracy is due to imprecise templates, whereas, a low percentage indicates a possible incompleteness in the resource inference algorithm, which is often due to non-linearity, or absence of sufficiently strong invariants. As shown in Fig. 10, the constants inferred by the tool were 91% accurate for steps, and 94% accurate for alloc, when compared to the pareto optimal values that fits the runtime data. Furthermore, the imprecision due to templates is a primary contributor for inaccuracy, especially in benchmarks where the accuracy is lower than 80% (such as Viterbi and prims). In the sequel, we discuss the benchmarks, and the results of their evaluation, in more detail.

Cyclic streams. The benchmarks fibs, and hams implement infinite fibonacci, and hamming sequences as cyclic streams using lazy zipWith, and merge functions. Their implementations were based on the related work of [50]. In comparison to the related work, in which the alloc bounds computed for hams were 64% accurate for inputs smaller than 10, our system was able to infer bounds that were 83% accurate for inputs up to 10K.

<table>
<thead>
<tr>
<th>Function</th>
<th>Accuracy (in wc)</th>
<th># of worst-case calls</th>
</tr>
</thead>
<tbody>
<tr>
<td>rotateRev (deq)</td>
<td>92%</td>
<td>24 out of 31</td>
</tr>
<tr>
<td>takeLazy (deq)</td>
<td>94%</td>
<td>45 out of 148</td>
</tr>
<tr>
<td>rotateDrop (deq)</td>
<td>84%</td>
<td>2 out of 72</td>
</tr>
<tr>
<td>pAdd (pp)</td>
<td>100%</td>
<td>100 out of 10K</td>
</tr>
</tbody>
</table>

Figure 11. Behavior of inner functions of deq and pp benchmarks.
Scheduling-based lazy data structures. The benchmarks \(rtq\), \(deq\), \(num\) and \(conq\) use lazy evaluation to implement worst-case constant time persistent queues, and dequeues, using a strategy called scheduling. The data structures differ significantly in their internal representation, invariants, resource usage, and the operations they support. These are one of the most efficient persistent data structures. For instance, the \(rtq\) benchmark takes a few nanoseconds to persistently enqueue an element into a queue of size \(2^{30}\). The \(conq\) data structure, proposed by [41], is used to implement data-parallel operations of the standard Scala library efficiently.

The data structures consist of one or two streams, referred to as \(spine\), that store the content, and a \(schedule\), which is a list of references to closures possibly nested deep within the spine. The content can be other data structures. In the case \(conq\) the content is a AVL-like balanced tree called \(ConcTree\) [41, 42]. The schedules correspond to unfinished operations, like \(enqueue\), initiated previously. Every operation on the data structure is performed lazily in increments that complete in a constant number of steps. Whenever a new operation is initiated, the schedules are forced so that an increment of a previous operation is performed. A complex invariant ensures that the pending operations do not cascade to result in non-constant time worst-case behavior. A complete implementation of the real-time queue data structure is shown in the Appendix C. We are not aware of any prior approach that proves the resource bounds of these benchmarks. We also discovered and fixed a missing corner case of the \(rotateDrop\) function shown in Fig. 8.4 of [39], which was unraveled by the system.

As the results in Fig. 10 show, the inferred bounds were at least 83% accurate for \(rtq\), and \(num\) benchmarks, but have low accuracy for \(deq\), and \(conq\). On further analysis of \(deq\), we found that the bounds inferred by our system for all inner functions (like \(rotateDrop\) and \(rotateRev\) functions of \(deq\)) were, in fact, reached in the dynamic runs, but manifested only occasionally when invoked from the top-level functions. Fig. 11 shows the accuracy i.e., the worst-case runtime step count as a percentage of the inferred bound, for the inner functions of \(deq\), and the number of calls from top-level functions that resulted in the worst-case behavior. The low accuracy seems to result from the lack of sufficient invariants for the top-level functions that prohibits the calls to inner functions from consistently exhibiting worst-case behavior.

Other lazy benchmarks. The benchmark \(slib\) is a collection of operations over streams such as \(map\), \(scan\), \(takeWhile\), \(cycle\). The operations were chosen from the Haskell stream library [49]. (We excluded functions such as \(filter\), and \(drop\) that can potentially diverge on infinite streams.) The benchmarks \(msort\), and \(sel\) implement lazy sorted streams that allows accessing the \(k^{th}\) minimum without performing the entire sorting. In particular, \(msort\) uses a lazy bottom-up merge sort [2], wherein a logical tree of closures of the merge function are created, and forced on demand.

Dynamic programming algorithms. We verified the resource bounds of dynamic programming algorithms [15] shown in Fig. 9 by expressing them as memoized recursive functions. The benchmarks \(lcs\) and \(levd\) implement the algorithms for finding the longest common subsequence, and Levenshtein distance between two strings (represented as lists). The benchmark \(ks\) implements the algorithm for packing a list of items, each of value \(v_i\) and weight \(w_i\), into a knapsack of capacity \(w\) in a way that maximizes the total value of the items in the knapsack. (We show an implementation of the benchmark in Appendix C) \(hs\) is a memoized version of the hamming stream benchmark that computes a sorted list of numbers of the form \(2^{3/5^k}\). \(ws\) implements a weighted scheduling algorithm that optimally schedules \(n\) jobs with (overlapping) start and finish times so that the total value of the scheduled jobs is maximized. The benchmark \(pp\) is a memoized implementation of a packrat parser presented by [20], and uses the same parsing expression grammar as in that work. \(vit\) is an implementation of the Viterbi algorithm for finding the most likely sequence of hidden states in the hidden Markov models. As show in Fig. 10, the inferred bounds for steps are on average 90% accurate for the dynamic programming algorithms except \(pp\), and \(vit\), and is 100% accurate in the case of \(alloc\), for all benchmarks except \(pp\). In the case of \(vit\) the main reason for inaccuracy stems from the cubic template (shown in Fig. 9), as highlighted by the results of comparison with the pareto optimal bound shown in Fig. 10. In the case of \(pp\), the evaluations were performed on random strings, as we were unable to precisely deduce the worst-case input. Nevertheless, the bounds inferred were accurate for the inner functions: \(pAdd\), \(pMul\), and \(pPrim\), as shown by Fig. 11.

6. Related Work

Static Resource Analysis for Lazy Evaluation. Danielsson [16] present a lightweight type-based analysis for verifying time complexity of lazy functional programs, and applies it to implicit queues. As noted in the paper, the approach is limited in handling aliasing of lazy references, which makes them ineffective on our benchmarks. Vasconcelos et al. [47, 50] present a typed-based analysis for inferring bounds on memory allocations of Haskell programs. They evaluated their system on cyclic hamming and fibonacci stream, which were included in our benchmarks, and discussed in section 5. A main difference between our approach and the above works is that, our approach is targeted at verifying user-specified bounds and has been benchmarked for relatively large input sizes. The benchmarks listed also highlight the fact that this is demonstrated on complex, real-world programs.

Static Resource Bounds Analysis. Resource-Aware ML [24, 26] is a type-driven approach for inferring resource bounds of ML programs. Other automated systems for resource bounds inference include Speed [21, 41, 56], and Costa [1]. These approaches do not seem to directly support lazy evaluation or memoization. While fully automated, these approaches target simpler programs and simpler bounds that depend on less complex invariants compared to our approach. Carboneaux et al. [13] present an system to verify stack space bounds of C programs, without recursive calls and function pointers, written for embedded systems. Madhavan and Kunck [35] present an approach that infers resource bounds from user-defined templates for first-order, non-lazy functional programs.


Imperative and Higher-Order Verification. Verification Systems such as [12, 18, 25, 31, 40, 45, 55], and interactive theorem provers [6, 14, 36] have been used to verify complex, imperative programs. Automation in our system appears above the one in interactive provers, and we believe it could be further improved using quantifier instantiation, induction, and static analysis [7, 22, 35, 43]. Most approaches for imperative programs target a homogeneous, mutable heap. Using mutation directly to model caches dramatically increases the contract overhead in our benchmarks, especially due the presence of higher-order functions. In this work, we consider an almost immutable heap except for the cache, and devise a verification approach to handle mutations to the cache efficiently. We believe that similar separation of heap into mutable, and immutable parts, to reduce contract overhead, can be extended to other forms of restricted mutation such as write-once fields [3], and Unique references. LiquidHaskell [51] supports verification of complex specifications expressed as type refinements. However, it does not allow specifying resource properties. The Leon system supports verifica-
tion of programs with higher-order functions [53], but it does not support lazy evaluation or memoization in its present form.

References


### A. Additional Specification Constructs

We support a few more specification constructs in our language for expressing properties about the implementation, which we discuss below. The semantics of these constructs are shown in Fig. 12. As mentioned in section 2, we support a construct `inSt` to access the state of the cache at the beginning of a function in the postcondition of the function. Analogously, we support a construct `outSt` to refer to the state of the cache at the end of the function in the postcondition. However, to define the semantics of these expressions, we slightly modify the domain of `Val` to also include a cache. That is, `Cache ⊆ Val`. Below we describe a couple of other specification constructs. The construct `e?` computes the result of an expression `e` without caching the result of `e` for reuse. This is a side-effect-free operation that is to be used in places where only the result of the expression is relevant. We support a construct `fmatch` of the form: `x fmatch \{ \lambda x. f_1 (x, y), \ldots \lambda x. f_n (x, y) \} \Rightarrow e_i \}_{i=1}^n` that performs structural matching on closures, i.e., matching based on structural equality. For instance, this expression matches `x` to the first case if `x` evaluates to a closure of the form: `\lambda x. f_1 (x, y), \ldots \lambda x. f_n (x, y)`, `[y \mapsto \nu]`. It binds the variable `y` in the match case to the value `\nu`, and evaluates `e_i` using the new binding. Fig. 12 shows the semantics of these constructs, and redefines the semantics of the contracts in the presence of these constructs.

### B. Proofs Sketches of Theorems

#### Theorem 1. (Bisimulation)

Let `e \in E_{rec}`, `st \in E_{mod}`, `Γ_1 \in Env`, and `Γ_2 \in E_{mod}` such that `Γ_2 \vdash st \Downarrow S \Rightarrow Γ_1 \sim (Γ_2, S)`. (a) If `Γ_1 \vdash e \Downarrow p v`, `Γ_1 \vDash \exists \bar{u} \in E_{mod}, \exists v \in DVal` s.t. `Γ_2 \vdash ([\bar{e}] st) \Downarrow u, Γ_2`, and `Γ_1 \sim (Γ_2, u, 2)`: 

\[ v \sim_{H_2} u, 1 \quad p = u_3 \]

(b) If `Γ_2 \vdash ([\bar{e}] st) \Downarrow u, Γ_2` and `u \in DVal` then `Γ_1 \vdash e \Downarrow p v`, `Γ_1`:

\[ Γ_1 \sim (Γ_2, u, 2) \]

\[ v \sim_{H_2} u, 1 \quad p = u_3 \]

*Proof Sketch.* To prove this we use structural induction over the expressions of the language. We prove that the claim holds for every expression, assuming that it holds for the subexpressions. We sketch the proof for one semantic rule. Others can be proved similarly. The proof depends on following two important properties: (a) the heaps in the environments evolve monotonically with respect to the evaluation of expressions. That is, the evaluation can only add more entries to the heap, and existing entries cannot be updated. This follows from the operational semantics, and more intuitively because the language is functional. (Recall that the caches are modeled separately.) Given two heaps `H_1` and `H_2`, we say `H_1 \triangledown H_2` if `H_2` has more entries than `H_1`. (b) The relation `\sim_{H_1, H_2}` is monotonic with respect to `\sim`. That is, if `x \sim_{H_1, H_2} y`, and `H_1 \triangledown H_2` then, `x \sim_{H_1, H_2} y`.

Now consider the part(a) of the theorem. We refer to the individual components of the environments using the corresponding subscripts and primes. Careful the rule `MEMOALLMISS` shown in Fig. 5. In this case, `e = f x`. Let `v = [\bar{e}] st`. Consider the property `Γ_1 \sim (Γ_2', r, 2)` which is one of the three properties that are needed to be established. By the definition of the semantics, the store components of `Γ_1` and `Γ_2` are same as that of `Γ_1` and `Γ_2`, respectively. This implies that `σ_1 \sim_{H_1, H_2}', σ_2`, as both `H_1 \triangledown H_1', H_2 \triangledown H_2'`, and `\sim` is monotonic with respect to the heaps. Notice that the bisimulation relation between two environments `\sim` defined in Def. 3 does not constrain the heap components of the environments. Therefore let us focus on the cache component.

In the case of `Γ_1'`, the cache component is obtained by the evaluation of `f \sigma_1(x)` on `Γ_1`, and then adding a mapping `(f \nu) \mapsto v` (see Fig. 5). In the case of `Γ_2', r, 2` is defined as the state obtained by the evaluation of `f_m \sigma_2(x)` on `Γ_2`, and the union of `C_1 \sigma_2(x)` (see Fig. 6). However, by inductive hypothesis, we can assume that the theorem holds for the body of the function calls `f` and `f_m` (given that parameter translation preserves the claim of the theorem). Therefore, we can assume that `C \sim_{H_2} r, 2` with respect to the heaps at the exit point of the functions `f` and `f_m`, respectively. Now, we need to show that the newly added key `(f \sigma_1(x)) \sim C_1 \sigma_2(x)`. But, this follows from the definition of `\sim_{H_1, H_2}'` that the function `σ_1(x)` `\sim_{H_1, H_2}'` `σ_2(x)`, and the monotonicity of the relation `\sim` with respect to the heaps. Hence, `Γ_1 \sim (Γ_2', r, 2)`. Similarly, other parts of the claim can be proven.

#### Theorem 2. (Model Soundness and Completeness)

Let `P` be a program. Let `e = \{ p \} e_s` and `e_m = \{ p_m \} e_m`. Let `def f = x be` be a function definition in `P`, and let `def f_m(x, st) = e_m` be the translation of `f`, where `st` is the state parameter added by the translation. 

\[ ∀P_2 \in E_{mod}, \exists \bar{u}, \bar{v} \Rightarrow Γ_2 \vdash p \Downarrow false \lor Γ_2 \vdash e_m \Downarrow \bar{v} \]

\[ iff \quad Γ_1 \vdash p \Downarrow false \lor Γ_1 \vdash e \Downarrow \bar{v} \]

*Proof.* The proof this theorem is more involved as not every `Γ_1 \in Env` can be simulated by a `Γ_2` and a set pair. This is because, the definition of `\sim` is with respect to lambdas and functions defined
in a program \( P \), and \( \Gamma_2 \) may contain bindings that involves lambdas created outside the program. However, if this not the case, for example, in closed programs, or programs with complete encapsulation where closures cannot be assigned values from outside, the proof follows directly from the Theorem 2 as explained below.

**Proof for closed programs, and programs with encapsulated function types.** Let us consider the if direction. That is, say \( \forall \Gamma_1 : \in Env \). \( \Gamma_1 \vdash p \Downarrow false \lor \Gamma_1 \vdash e \Downarrow \nu \). Consider a \( \Gamma_2 \in Env_{mod} \). Let \( \sigma_2(st) = S \), where \( st \) is the state parameter of the function \( f_m \). Let \( \Gamma_1 \in Env \) such that \( \Gamma_1 \sim (\Gamma_2, S) \). Such a \( \Gamma_1 \) can be constructed by mapping all constructors in the \( dom(\Gamma_2) \) of the form \( C_1 u \) representing closure of a lambda \( lam \) with label \( l \) to their corresponding concrete closures: \( (lam, [FV(lam) \rightarrow u]) \), and all constructors of the form \( (C_1 u) \in S \) to a binding in the cache for the key \((f u) \in FVal \). (This is in essence the concretization function.) We are given that \( \Gamma_1 \vdash p \Downarrow false \land \Gamma_1 \vdash e \Downarrow \nu \). By Theorem 1, \( \Gamma_1 \vdash p \Downarrow v_1 \) implies that \( \Gamma_2 \vdash [p]_{\nu} st \Downarrow v_2 \), and \( v_1 \sim v_2 \) with respect to the heaps at the end of the evaluation. However, by the definition of translation, \([p]_{\nu} st = p_{mod} \) (see Fig. 6), and since \( p \) is boolean valued, by the definition of \( \sim \), \( v_1 \sim v_2 \). Similarly, it can be shown that, given \( \Gamma_1 \vdash e \Downarrow \nu \), \( \Gamma_2 \vdash e_{\nu} st \Downarrow \nu \). By definition, the body of a function \( f_{\nu} \) is \( e_{\nu} \). Hence, the claim holds along the if direction. The other direction can be similarly proved.

**Proof of more general case.** To argue about the more general case, where closures can be assigned from outside the program, \( P \), we need to reason about potentially diverging computations, and those to crash. For brevity, we only describe the proof informally for this case. Consider the only if direction. The other direction is similar. That is, say \( \forall \Gamma_1 : \in Env_{mod} \) \( \exists \nu \). \( \Gamma_1 \vdash p \Downarrow false, \Gamma_1 \vdash e \Downarrow \nu \). Now for every \( \Gamma_1 \in Env \) such that \( \Gamma_1 \sim (\Gamma_2, S) \), where \( S \subseteq S_{keys} \), it can be shown that the claim holds, as illustrated above, using Theorem 1. Now, consider a \( \Gamma_1 \in Env \) that is not related by \( \sim \). The only way this could happen is either the keys of the cache \( C_1 \) or the closures in \( H_1 \) do not belong to the program \( P \). Both cases are somewhat similar, since the syntactic restriction of the language requires that all named functions directly invoked are defined in program. So the only way a function \( f \) not defined in the program could be invoked is indirectly through a lambda application. Therefore, any cache entries \( f_u \), where \( f \) does not belong to the program would either not be used, or be looked up only from a lambda created outside the program. So let us only consider cases where closures could be created outside the program, and invoke functions not in the scope of the program. We refer to such closures external closures. Now, consider a \( \Gamma_1 \in Env \) such that evaluation of some expression \( e \) in \( P \) is not defined, either because no semantics rule applies, or because the expression \( e \) can diverge. However, we can still reason about the steps in the evaluation of \( e \), though they may unbounded. We now shown that when the contracts of \( f_m \) hold for all inputs, evaluation of \( e \) on \( \Gamma_1 \) cannot be undefined, by deriving a contradiction in all the three following cases.

Case (a). An expression \( e \) belonging to \( P \) does not use any external closure when evaluated on \( \Gamma_1 \). This is the simplest case. Clearly, we can remove the binding for external closures from \( \Gamma_1 \), and still execute \( e \), and produce the same behavior. Moreover, such a reduced \( \Gamma_1 \) would be related to \( \Gamma_2 \), and \( S \subseteq S_{keys} \) by \( \sim \), and hence by Theorem 1, and the given facts, execution of \( e \) on \( \Gamma_1 \) must produce a value, which is a contradiction.

Case (b). An expression \( e \) belonging to \( P \) uses an external closure when evaluated on \( \Gamma_1 \), but does not apply the lambda at any point. This case would result if the closures are used only in equalities, but never applied to a value. Let \( x \) and \( y \) be two external closures that are ever compared during the evaluation of \( e \) on \( \Gamma_1 \), and have the same type \( \tau \) (otherwise type checking will fail). In the model, this environment \( \Gamma_1 \) can be simulated by an environment \( \Gamma_2 \) where \( x \) and \( y \) are mapped to instances of \( C_r \). If the \( x \) and \( y \) are equal in \( \Gamma_1 \) then we make the instances of \( C_r \) have the same parameters (note that \( C_r \) takes one integer parameter as defined in section 3), otherwise we assign different parameters to the constructors \( C_r \).

Now, execution of \( \Gamma_2 \) on \([e]_{\nu} st \) will simulate the execution of \( \Gamma_1 \). Hence from the given facts, execution of \( e \) on \( \Gamma_1 \) must produce a value, which is again a contradiction. Note that equating an external closure \( x \) (calling a function defined outside the program) with a closure created inside the program \( y \) is bound to fail both in \( P \) and \( P_{mod} \) as we use structural equality for closures.

Case (c). An expression \( e \) belonging to \( P \) applies an external closure when evaluated on \( \Gamma_1 \). Let \( y \) be the external closure that is applied first, before any other external closure. Clearly, the (dynamic) path \( p \) from the beginning of \( e \) to the point of application did not have any other external application. Therefore, as described in the cases (a) and (b), there exists a \( \Gamma_2 \) such that \( \sigma_2(y) \) is an instance of \( C_r \), where \( r \) is the type of the closure \( y \), and evaluating \( \Gamma_2 \) on \([p]_{\nu} st \) is defined.

That is, it produces an output state, say \( \Gamma_2 \). By the definition of the translation shown in Fig. 6, this application is modeled using \( Appr \), which on seeing \( C_r \) will halt the execution with an error, contradicting the assumption that the contracts hold for all \( \Gamma_2 \in Env_{mod} \) for the translated expression \([e]_{\nu} st \).

**Theorem 3.** Given a linear parametric formula \( \phi(\vec{x}, \vec{a}) \) with free variables \( \vec{x} \) and \( \vec{a} \), belonging to a theory \( T \) that is a combination of quantifier-free theories of uninterpreted functions, algebraic datatypes, and sets, and either linear arithmetic or real arithmetic, finding an assignment \( \nu \) such that \( dom(\nu) = [\vec{a}] \), and \( (\phi \nu) \) is \( T \)-unsatisfiable is decidable.

**Proof Sketch.** We express the problem as trying to decide the validity of a formula of the form: \( \exists \nu.\forall \vec{x}.(\forall f.\phi(\vec{x}', f, \vec{a})) \land \)
\( (\forall \bar{x}, \phi_{\text{set}}(\bar{x'}, \bar{s}), \circ) \) where, \( \bar{f} \) are the uninterpreted function symbols in \( \phi \), \( \bar{s} \) are variables of set sort, \( \bar{x'} \) are variables of other sorts, and \( \phi_{\text{set}} \) is a formula in \( \mathcal{T}_{\text{set}} \) that has only set operations. This is possible because the existentially quantified variables \( \bar{x} \) are only numerical variables. Since the theory of set admits decidable quantifier elimination [29], the above formula could be reduced to an equivalent formula of the form \( \exists \bar{x}, \forall \bar{x'}, \bar{f}, \phi''(\bar{x'}, \bar{s}) \), which is decided using the algorithm presented in [35], and depicted in Fig. 8. \( \square \)

C. Sample Implementations and Specifications of Benchmarks

The following are two selected benchmarks verified by our tool. Okasaki’s Real-time queue falls in the category of lazy data structure and Knapsack falls in the category of dynamic programming problems.

Okasaki’s Real-Time Queue. Fig. 13 shows a complete implementation of the Okasaki’s Real-time queue data structure [38, 39] in our syntax. Consider the function rotate. It reverses the list \( r \) and appends it to the lazy stream \( f \), using the stream \( a \) as a temporary storage, which is initially set to empty. Essentially, \( \text{rotate}(r, f, a) = f + + \text{reverse}(r) + + a \). (and \( r \) actually represent the front and rear parts of the real-time queue data structure). However, the function performs its work lazily; every call to rotate constructs the first element of the result, and returns a stream whose tail is a suspended recursive call.

The specifications of rotate assert properties of the function that hold before and after the execution. Consider the property on the sizes of the arguments. This property is independent of the data type i.e, it does not depend on whether the closures in the input list are forced or not. In contrast, the property \( \text{isConcrete} \) is state dependent: it returns true if every node of the argument stream has been forced, false otherwise. Notice that the postcondition also asserts a constant time bound for the function rotate. The requirement that \( \text{isConcrete}(f) \) holds at the beginning of the function is crucial for proving the time bound. Otherwise, forcing \( f \) at line 65 may invoke a previously suspended call to rotate, thus resulting in a cascade of forces.

As shown in Fig. 13, the real time queue data structure has three components: a lazy stream \( f \) denoting the front of the queue, a list \( r \) denoting the rear of the queue, and a lazy stream \( s \) denoting the schedule. We define the data structure invariants using the boolean-valued function valid. Every public queue operation, namely enqueue and dequeue, require that the valid property holds for the input queue, and also ensures that the property holds for the output queue (see the definitions of the functions in Fig. 13).

Consider the property \( \text{firstUneval}(f) == \text{firstUneval}(s) \) that relates the schedule and the front streams that is a part of the definition of valid. The definition of firstUneval is shown in Fig. 13. It returns the first node in the stream that has not been forced. This property states that the unevaluated nodes of \( f \) and \( s \) are equal. In addition to this, the data structure also maintains the invariant that the size of the front is greater than the size rear, and that the size of the schedule is equal to the difference between the sizes of the front and the rear. These are succinctly captured by the second predicate of the function valid. The specification of the firstUneval function asserts a few interesting properties of the function that are needed for verification.

The data structure uses the same idea as a simple immutable queue that uses two lists, namely front and rear, that has a constant, amortized running time for ephemeral (i.e., non-persistent) usage. The elements are enqueued to the rear list and dequeued from the front list. Once in a while, when there are very few or no elements in the front list, the dequeue operation would reverse the rear and append it to the front. This is captured by the rotate function of Fig. 13. The real time queue data structure uses a similar strategy, but it exploits lazy evaluation to perform the costly rotate operation incrementally, alongside the enqueue and dequeue operations. It thus achieves constant running time, in the worst case, for all operations even under persistent usage. For this purpose, it augments the queue with a schedule which is a reference to a closure that corresponds to the next step of an unfinished rotate operation. The rotate operation itself is performed lazily: every call to rotate constructs the first element of the result, and returns a stream whose tail is a suspended recursive call.

During every enqueue and dequeue operation, if the schedule is non-empty, the head of the schedule is forced (line 127 of the function createQ). This corresponds to performing one step of the rotate operation. On the other hand, if the schedule is empty, which implies that there are no pending rotate operations, a new rotate operation is initiated (lines 129 to 130). Hence, whenever a rotate operation is initiated every node of the argument \( f \) is forced. This is asserted by the isConcrete(f) predicate used in the precondition of the rotate function, which is critical for proving the O(1) time bound of rotate. Our system verifies the complete program shown in Fig. 13.

Memoized Knapsack Program. Knapsack is a standard problem with a dynamic programming solution. Given a list of items with values and weight pairs. The goal is to maximize the values subject to total weight of the sack. Figure 16 shows the verified implementation of the Knapsack program. The entry point of this program is the function \( \text{knapscak}(w, \text{items}) \) shown in Fig. 16. It computes the optimal values of filling knapsacks of capacity lesser than or equal to \( w \) using items. In each iteration, it invokes the function solveForWeight (through a helper function). The function requires that the optimal solutions of smaller weights have been computed and cached using the predicate \( \text{dpms} \). Given this property, it computes the optimal way of filling a knapsack of weight \( w \) by traversing the list of items, and choosing the items to include in the list. The time complexity of the algorithm is quadratic in \( \text{items.size} \) and \( w \). Fig. 9 presents values inferred by the tool for the top-level function in this program.
object RealTimeQueue {
  sealed abstract class Stream[T] {
    @inline
def isEmpty: Boolean = this == SNil[T]()
  }

  lazy val tail: Stream[T] = {
    require(isEmpty)
    this match {
      case SCons(_, tailFun, _) => tailFun()
    }
  }

  def size = this match {
    case SCons(_, r) => r.size
    case SNil() => BigInt(0)
  }

  /**
   * A property that is true if 'sz' field decreases for the tail of the stream. 'sz' is a well-founded ordering.
   */
  def valid: Boolean = this match {
    case c @ SCons(_, _, _) =>
      val s = size
      s > 0 && s.size == (c.tail).size + 1 && (c.tail).valid
      case _ => true
  }

  private class SCons[T](x: T, tailFun: () => Stream[T], sz: BigInt) extends Stream[T]
  private class SNil[T]() extends Stream[T]

  /**
   * A property that holds for stream where all elements have been memoized.
   */
  def isConcrete[T](l: Stream[T]): Boolean = {
    def @inline firstUneval(l) == firstUneval(l.tail) && isConcrete(l.tail)
    ensure(res.size == f.size + r.size + a.size)
    val res = f ++ reverse(r) ++ a // invariant: res.size = f.size + r.size + a.size
    val steps = BigInt(0)
  }

  sealed abstract class List[T] {
    val size: BigInt = {
      this match {
        case Nil() => BigInt(0)
        case Cons(_, t) => 1 + t.size
      }
    }

    /**
     * Returns the first element of the stream whose tail is not memoized.
     */
    def firstUneval[T]: Stream[T] = {
      require(l.valid)
      l match {
        case c @ SCons(_, _, _) =>
          if (cached(c.tail))
            firstUneval(c.tail)
          else l
        case _ => l
      }
    }
  }

  case class Queue[T](f: Stream[T], r: List[T], s: Stream[T]) {
    @inline
def isEmpty = f.isEmpty
    def valid = {
      f.valid && s.valid &&
      // (b) no lazy closures implies stream is concrete
      if (res.isEmpty) isConcrete(l) &&
      // (c) after evaluating the firstUneval closure in 'f'
      // we can access the next unevaluated closure
      res match {
        case c @ SCons(_, _, _) =>
          firstUneval(l) == firstUneval(c.tail)
        case _ => true
      }
    }

    /**
     * A helper function for enqueue and dequeue methods that forces the schedule once
     */
    @inline
def createQ[T](f: Stream[T], r: List[T], s: Stream[T]) = {
      s match {
        case c @ SCons(_, _, _) => Queue(f, r, c.tail) // force
        case SNil() =>
          val rotes = rotate(f, r, SNil[T]())
          Queue(rotes, Nil(), rotes)
      }
    }

    /**
     * Creates an empty queue, with an empty schedule
     */
    def empty[T] = {
      val a: Stream[T] = SNil()
      Queue(a, Nil(), a)
    }

    ensuring (res => res.valid && steps <= ?)
  }

  @invstate def rotate[T](T)(f: Stream[T], r: List[T], a: Stream[T]): Stream[T] = {
    require(r.size == f.size + 1 && f.valid &&
      a.valid && isConcrete(f))
    (f, r) match {
      case (SCons(y, Cons(_, _)), Cons(_, t)) =>
        SCons[T](y, lift(a), a.size + 1)
      case (c @ SCons(x, Cons(y, r1)), Cons(_, r)) =>
        val newa = SCons[T](y, (1) => a, a.size + 1)
        val tail = c.tail
        val rot = () => rotate(tail, r1, newa)
        SCons[T](x, rot, f.size + r.size + a.size)
    }
  }

    ensuring (res => res.valid &&
      res.size == f.size + t.size + a.size &&
      res.isEmpty && steps <= ?)
}
```scala
/**
 * Reads the first elements of the queue without removing it.
 */
def head[T](q: Queue[T]): T = {
  require(!q.isEmpty && q.valid)
  case SCons(x, _) => x
} ensuring (res => steps <= ?)

/**
 * Append an element to the end of the queue
 */
def enqueue[T](x: T, q: Queue[T]): Queue[T] = {
  case SCons(x, _) => q
  case SCons(x, q.r) => createQ(q.f, Cons(x, q.r), q.s)
} ensuring (res => steps <= ?)

/**
 * Removes the element at the beginning of the queue
 */
def dequeue[T](q: Queue[T]): Queue[T] = {
  require(!q.isEmpty && q.valid)
  case SCons(x, q.r) => createQ(q.f, q.r, q.s)
} ensuring (res => steps <= ?)

// We use 'fune' as a shorthand for 'firstUneval'
require(funeMonotone(q.f, q.s, inSt[T], outSt[T]) && createQ(q.f, Cons(x, q.r), q.s))

funeCompose(l2, st1, st2) \implies fune(l1, st2) == fune(fune(l1, st1), st2)

st1.subsetOf(st2) \implies fune(l, st2) == fune(fune(l, st1), st2)

∀ x \in st2. x, x \in st1 \implies fune(l, st2) == fune(fune(l, st1), st2)

@invisibleBody
@traceInduct
def funeCompose[T][T1: Stream[T], T2: Stream[T],
st1: Set[Fun[T1]],
st2: Set[Fun[T2]]]: Boolean = {
  require(l1.valid && l2.valid &&
  (firstUneval(l1 in st1) == (firstUneval(l2 in st1) &&
  st1 subsetOf st2))
  \implies fune(l1, st2) == fune(fune(l1, st1), st2)
  \implies funeCompose(l1, st1, st2) \&
  funeCompose(l2, st1, st2) \&
  (firstUneval(l1 in st2) == (firstUneval(l2 in st2))
  \implies funeCompose(l1, st2, st2)
  \implies funeCompose(l2, st1, st2) \&
  (firstUneval(l1) in st2) == (firstUneval(l2) in st2)
  \implies funeCompose(l1, st2, st2)
} holds

object Knapsack {
  sealed abstract class ILList[+L] { // a list of pairs: (weight, value)
    def size: BigInt = {
      this match {
        case Cons(_ tail) => 1 + tail.size
        case Nil() => BigInt(0)
      } ensuring (_) => 0
    }
    override def toString: String = {
      if (tail !== Nil()) x.toString
      else x.toString + :: y + tail.toString
    }
    case class Nil() extends ILList {
      @ignore
      override def toString = ""
    }
  }

  // Monotonicity of 'fune' with respect to the state
  @invisibleBody
  @traceInduct
def funeMono(i: BigInt, items: ILList): Boolean = {
    require(i >= 0)
    cached(solveForWeight(i, items)) &&
    (i <= 0) BigInt(0)
  }

  else {
    maxOverItems(items, w, items)
  }

  ensuring (_) => steps <= ? items.size + ?
}

// Monotonicity of 'firstUneval' with respect to the state
@invisibleBody
@traceInduct
def depsMono(i: BigInt, items: ILList, st1: Set[Fun[BigInt]],
st2: Set[Fun[BigInt]]) = {
  require(i >= 0)
  (st1 subsetOf st2) \&
  (deps(i, items) in st1) \implies (deps(i, items in st2))
} holds

// ∀ x, y \leq y \&
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// \& \implies
}

// Computes the optimal value of filling a knapsacks of
// weight 'w' using 'items'
@memozize
def solveForWeight(w: BigInt, items: ILList): BigInt = {
  require(w >= 0 \&\& (w == 0 || deps(w - 1, items)))
  if (w == 0) BigInt(0)
  else {
    maxOverItems(items, w, items)
  }
  ensuring (_) => steps <= ? items.size + ?
}
```

Figure 13. Okasaki’s Real-time queue data structure

Figure 14. Knapsack program part I.
/**
 * Computes the optimal value of filling a knapsack of weight 'w' using 'remItems'
 */
@invariant
def maxOverItems(items: IList, w: BigInt, remItems: IList): BigInt = {
  require((w == 0 || (w > 0 && deps(w - 1, items))) &&
    // lemma inst
    remItems match {
      case Cons((wi, vi), _) =>
        if (wi <= w && vi > 0)
          depsLem(w - wi, w - 1, items)
        else true
      case Nil() =>
        BigInt(0)}
    remItems match {
      case Cons((wi, vi), tail) =>
        val maxWithoutItem = maxOverItems(items, w, tail)
        if (wi <= w && vi > 0)
          val maxWithItem = vi + solveForWeight(w - wi, items)
          if (maxWithItem >= maxWithoutItem)
            maxWithItem
          else maxWithoutItem
        else maxWithoutItem
      case Nil() =>
        BigInt(0)
    })
  ensuring (
    steps <= ? * items.size + ? * w + ?
  )
}
@invisibleBody
def solveForWeightHelper(i: BigInt, items: IList) = {
  require(i == 0 || (i > 0 && deps(i - 1, items)))
  solveForWeight(i, items)
} ensuring (
  steps <= ? * items.size + ?
)

Figure 15. Knapsack program part II.

// Computes the optimal solution for all weights upto 'w'.
def solveUptoWeight(w: BigInt, items: IList): IList = {
  require(w >= 0)
  if (w == 0)
    Cons((w, solveForWeightHelper(w, items)), Nil())
  else {
    val tail = solveUptoWeight(w - 1, items)
    Cons((w, solveForWeightHelper(w, items)), tail)
  }
} ensuring {
  steps <= ? * items.size + ? * w + ?
}

// Computes the list of optimal solutions
// for all weights up to 'w'
def knapsack(w: BigInt, items: IList) = {
  solveUptoWeight(w, items)
} ensuring {
  steps <= ? * items.size + ? * w + ?
}

@ignore
def main(args: Array[String]) {
  import scala.util.Random
  // pick some random weights and values
  val input1 = (1 to 10).foldRight(Nil(): IList) {
    case (i, acc) =>
      Cons((i, i), acc)
  }
  val reslist1 = knapsack(100, input1)
  println("Optimal solutions: " + reslist1.toString)

  val input2 = ((1 to 10) zip (10 to 1 by -1)).foldRight(Nil(): IList) {
    case ((i, j), acc) =>
      Cons((i, j), acc)
  }
  val reslist2 = knapsack(100, input2)
  println("Optimal solutions for set 2: " + reslist2.toString)
}

Figure 16. Knapsack program part III.