Resisting and Eliminating Smoothness Heuristics

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ÉCOLE POLYTECHNIQUE
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Overview

Basics

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Breaking supersingular binary curves

Eliminating smoothness heuristics

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The Index Calculus Method

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Consider the DLP in $\mathbb{F}_{q^n}$. The ICM consists of two stages:

1. Choose a factor base $\mathcal{F}$, find relations between elements and then compute their logarithms.
2. For an arbitrary element, express it as a product of lower degree elements; recurse until all leaves are in $\mathcal{F}$.

Smoothness and the F.T.C.

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Theorem (*Odlyzko '84, Lovorn '92*)

For $m^{1/100} \leq B \leq m^{99/100}$, the probability that a polynomial $f \in \mathbb{F}_q[X]$ of degree m chosen uniformly at random is B-smooth, is

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'The Fundamental Theorem of Cryptography'

"If we have no clue about something, then we can safely assume that it behaves as a uniformly distributed random variable."

– Igor Shparlinski

The Joux-Lercier FFS variation [JL06]

To find factor base relations in $\mathbb{F}_{q^n}$ one uses the following setup.

- Choose $g_1, g_2 \in \mathbb{F}_q[X]$ of degrees d_1, d_2 s.t. $X - g_1(g_2(X))$ has a degree n irreducible factor $I(X)$ over $\mathbb{F}_q$, so that $\mathbb{F}_{q^n} = \mathbb{F}_q[X]/(I(X)) = \mathbb{F}_q(x)$
- Let $y = g_2(x)$; then $x = g_1(y)$ and $\mathbb{F}_{q^n} \cong \mathbb{F}_q(x) \cong \mathbb{F}_q(y)$
- In best case factor base is $\{x - a \mid a \in \mathbb{F}_q\} \cup \{y - b \mid b \in \mathbb{F}_q\}$

Relation generation:

- Considering elements $xy + ay + bx + c$ with $a, b, c \in \mathbb{F}_q$, one obtains the $\mathbb{F}_{q^n}$-equality

$$xg_2(x) + ag_2(x) + bx + c = yg_1(y) + ay + bg_1(y) + c$$

- When both sides split over $\mathbb{F}_q$ one obtains a relation

Optimising d_1 and d_2 in [JL06]

F.T.C. $\Longrightarrow$ that as $q \to \infty$ each side of $xy + ay + bx + c$ splits over $\mathbb{F}_q$ with probability $1/(d_2 + 1)!$ and $1/(d_1 + 1)!$ respectively.

- $\Longrightarrow$ Choose $d_1 \approx d_2 \approx \sqrt{n}$
- For $q = L_{q^n}(1/3, 3^{-2/3})$ algorithm is $L_{q^n}(1/3, 3^{1/3})$

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A Counterpoint to the F.T.C.

Fortunately, in one sub-case of the [JL06] setup, we do have a clue.

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Resisting smoothness heuristics

'On the Function Field Sieve and the Impact of Higher Splitting Probabilities: Application to Discrete Logarithms in $\mathbb{F}_{2^{1971}}$ and $\mathbb{F}_{2^{3164}}$'

Faruk Göloğlu, Robert Granger, Gary McGuire and Jens Zumbrägel.
(Best Paper Award at CRYPTO 2013)

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CASL

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- The first *polynomial time* relation generation method for degree one elements
- The first *polynomial time* elimination method for degree two elements

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An auspicious choice for g_2 in [JL06]

Assume now that the base field is $\mathbb{F}_{q^k}$ for $k \geq 2$.

- Let $y = g_2(x) = x^q$
- Eliminates half of the factor base since

$$(y + b) = (x + b^{1/q})^q \Longrightarrow \log(y + b) = q \log(x + b^{1/q})$$

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- The l.h.s. of $xy + ay + bx + c$ becomes

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- This polynomial *provably* splits over $\mathbb{F}_{q^k}$ with probability

$$\approx 1/q^3 \gg 1/(q+1)!$$

Bluher polynomials

Let $k \geq 3$ and consider the polynomial $X^{q+1} + aX^q + bX + c$.

If $ab \neq c$ and $a^q \neq b$, this may be transformed into

$$F_B(\overline{X}) = \overline{X}^{q+1} + B\overline{X} + B\,, \quad \text{with} \quad B = \frac{(b-a^q)^{q+1}}{(c-ab)^q}\,,$$

via $X = \frac{c-ab}{b-a^q}\,\overline{X} - a$.

Theorem (*Bluher '04*)

The number of elements $B \in \mathbb{F}_{q^k}^{\times}$ s.t. the polynomial $F_B(\overline{X}) \in \mathbb{F}_{q^k}[\overline{X}]$ splits completely over $\mathbb{F}_{q^k}$ equals

$$\frac{q^{k-1}-1}{q^2-1} \quad \textit{if } k \textit{ is odd}\,, \qquad \frac{q^{k-1}-q}{q^2-1} \quad \textit{if } k \textit{ is even}\,.$$

Polynomial time relation generation: $k \geq 3$

Assume that g_1 can be found s.t. $X - g_1(X^q) \equiv 0 \pmod{I(X)}$ with $\deg(I) = n \leq qd_1$. Then we have the following method:

- Compute $\mathcal{B} = \{B \in \mathbb{F}_{q^k}^{\times} \mid X^{q+1} + BX + B \text{ splits over } \mathbb{F}_{q^k}\}$
- Since $B = (b - a^q)^{q+1}/(c - ab)^q$, for any $a, b \in \mathbb{F}_{q^k}$ s.t. $b \neq a^q$, and $B \in \mathcal{B}$, there exists a unique $c \in \mathbb{F}_{q^k}$ s.t. $x^{q+1} + ax^q + bx + c$ splits over $\mathbb{F}_{q^k}$
- For each such (a, b, c), test if r.h.s. $yg_1(y) + ay + bg_1(y) + c$ splits; if so then have a relation
- If $q^{3k-3} > q^k(d_1 + 1)!$ then for $d_1 \geq 1$ constant we expect to compute logs of degree 1 elements of $\mathbb{F}_{q^{kn}}$ in time

$$O(q^{2k+1})$$

Degree 2 elimination

Let $Q(y) = y^2 + q_1 y + q_0 \in \mathbb{F}_{q^{kn}}$ be an element to be eliminated, i.e., written as a product of linear elements.

- Recall that in $\mathbb{F}_{q^{kn}}$ we have $y = x^q$ and $x = g_1(y)$, so for any univariate polynomials w_0, w_1 we have

$$w_0(x^q)\,x + w_1(x^q) = w_0(y)\,g_1(y) + w_1(y)$$

- Compute a reduced basis of the lattice

$$L_Q = \{(w_0(Y), w_1(Y)) \in \mathbb{F}_{q^k}[Y]^2 : w_0(Y)\,g_1(Y) + w_1(Y) \equiv 0 \pmod{Q(Y)}\}$$

- In general we have $(u_0, Y + u_1), (Y + v_0, v_1)$, with $u_i, v_i \in \mathbb{F}_{q^k}$, and for $s \in \mathbb{F}_{q^k}$ we have $(Y + v_0 + su_0, sY + v_1 + su_1) \in L_Q$
- r.h.s. $(y + v_0 + su_0)\,g_1(y) + (sy + v_1 + su_1)$ has degree $d_1 + 1$, so cofactor splits with probability $\approx 1/(d_1 - 1)!$
- l.h.s. is $(x^q + v_0 + su_0)x + (sx^q + v_1 + su_1)$ which is of the form

$$x^{q+1} + ax^q + bx + c$$

Degree 2 elimination

Consider the l.h.s. $x^{q+1} + sx^q + (v_0 + su_0)x + (v_1 + su_1)$.

- Compute the set $\mathcal{B}$ of elements $B \in \mathbb{F}_{q^k}$ such that $X^{q+1} + BX + B$ splits over $\mathbb{F}_{q^k}$
- For each $B \in \mathcal{B}$ we try to solve $B = (b - a^q)^{q+1}/(c - ab)^q$ for s, i.e., find $s \in \mathbb{F}_{q^k}$ that satisfies

$$B = \frac{(-s^q + u_0 s + v_0)^{q+1}}{(-u_0 s^2 + (u_1 - v_0)s + v_1)^q}$$

  by taking GCD with $s^{q^k} - s$: Cost is $O(q^2 \log q^k)$ $\mathbb{F}_{q^k}$-ops

- Probability of success is $\approx 1 - \left(1 - \frac{1}{(d_1-1)!}\right)^{q^{k-3}}$
- Hence need $q^{k-3} > (d_1 - 1)!$ to eliminate $Q(y)$ with good probability: Expected cost is

$$O(q^2(d_1 - 1)! \log q^k) \ \ \mathbb{F}_{q^k}\text{-ops}$$

Joux's insights

- Independently of [GGMZ13], Joux discovered an isomorphic polynomial time degree one relation generation method.
- For $\mathbb{F}_{q^{2n}}$, assume $h_1(X), h_0(X) \in \mathbb{F}_{q^2}[X]$ of very small degree exist s.t. $h_1(X)X^q - h_0(X)$ has an irreducible factor $I(X)$ of degree n.

For $Q \in \mathbb{F}_{q^2}[X]$ of degree D let F, G have degree $< D$. Consider

$$G \cdot \prod_{\alpha \in \mathbb{F}_q} (F - \alpha G) = F^q G - F G^q$$

- Since $X^q \equiv h_0(X)/h_1(X) \pmod{I(X)}$, F^q & G^q have small degree
- Joux insists that r.h.s. is divisible by $Q \implies$ results in a bilinear quadratic system, and that the cofactor is $(D-1)$-smooth

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Balancing classical descent with this elimination results in an algorithm with heuristic complexity

$$L_{q^{2n}}(1/4 + o(1))$$

New method DLP solutions in 2013

- 11th Feb'13, Joux: $\mathbb{F}_{2^{1778}}$ in 220 core hours
- 19th Feb'13, GGMZ: $\mathbb{F}_{2^{1971}}$ in $3,132$ core hours
- 3rd May'13, GGMZ: $\mathbb{F}_{2^{3164}}$ in $107,000$ core hours
- 22nd Mar'13, Joux: $\mathbb{F}_{2^{4080}}$ in $14,100$ core hours
- 11th Apr'13, GGMZ: $\mathbb{F}_{2^{6120}}$ in 750 core hours
- 21st May'13, Joux: $\mathbb{F}_{2^{6168}}$ in 550 core hours

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Supersingular binary curves: genus 1

For $i \in \mathbb{F}_2$ consider the elliptic curves

$$E_i/\mathbb{F}_2 : Y^2 + Y = X^3 + X + i$$

- Both E_i are supersingular ($E_i(\overline{\mathbb{F}}_2)$ has no points of order 2)
- For prime p we have

$$\#E_i(\mathbb{F}_{2^p}) = \begin{cases} 2^p + 1 + (-1)^i 2^{(p+1)/2} & \text{for } p \equiv 1, 7 \pmod 8 \\ 2^p + 1 - (-1)^i 2^{(p+1)/2} & \text{for } p \equiv 3, 5 \pmod 8 \end{cases}$$

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Lesson 1 (*MOV '93*)

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Lesson 2 (*Pairing-based cryptography '00/01*)

Provided that the applications are good enough, ignore Lesson 1.

Supersingular binary curves: genus 2

For $i \in \mathbb{F}_2$ let

$$H_i/\mathbb{F}_2 : Y^2 + Y = X^5 + X^3 + i$$

- Both H_i are supersingular (Jac_{H_i} is isogenous to a product of two supersingular elliptic curves)
- We have $\#\mathrm{Jac}(H_i)(\mathbb{F}_{2^p}) =$

$$\begin{cases} 2^{2p} + (-1)^i 2^{(3p+1)/2} + 2^p + (-1)^i 2^{(p+1)/2} + 1 & \text{for } p \equiv 1, 7, 17, 23 \pmod{24} \\ 2^{2p} - (-1)^i 2^{(3p+1)/2} + 2^p - (-1)^i 2^{(p+1)/2} + 1 & \text{for } p \equiv 5, 11, 13, 19 \pmod{24} \end{cases}$$

- $\#\mathrm{Jac}(H_i)(\mathbb{F}_{2^p}) \mid (2^{12p} - 1) \Longrightarrow \mathrm{Jac}(H_i)$ has embedding degree 12.

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Only genus 1 and 2 seriously considered $\Longrightarrow$ we are interested in the DLPs in (the prime order $r \mid \#\mathrm{Jac}$ subgroups of) $\mathbb{F}_{2^{4p}}^{\times}$ and $\mathbb{F}_{2^{12p}}^{\times}$.

Concrete security of small characteristic pairings

Adj, Menezes, Oliveira and Rodríguez-Henríquez used the techniques from [Joux13] and [BGJT13] to analyse the concrete security of the DLP in pairing fields once thought to be 128-bit secure.

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In particular, they showed that:

- The DLP in the 804-bit order r subgroup of $\mathbb{F}_{3^{6 \cdot 509}}^{\times}$ can be solved in time $2^{73.7} M_r$, using $q = 3^6$ and $k = 2$
- The DLP in the 698-bit order r subgroup of $\mathbb{F}_{2^{12 \cdot 367}}^{\times}$ can be solved in time $2^{94.6} M_r$, using $q = 2^{12}$ and $k = 2$
- The DLP in the 1221-bit order r subgroup of $\mathbb{F}_{2^{4 \cdot 1223}}^{\times}$ can be solved in time $\approx 2^{128} M_r$, using $q = 2^{12}$ and $k = 2$

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Consider the following:

- $h_1(X)X^q - h_0(X) \equiv 0 \pmod{I(X)} \Longrightarrow n \leq q + \deg(h_1)$
- The descent cost is lower for smaller q

Our contributions

We exploited the following observations/principles/techniques:

- $h_1(X^q)X - h_0(X^q) \equiv 0 \pmod{I(X)} \Longrightarrow n \leq q \cdot \deg(h_1) + 1$
- *Principle of parsimony:* always try to work in the target field; only when this fails should one embed into an extension
- A bonus of solving factor base logs in an extension is that one can factor elements over the extension during the descent
- We can also use $k = 1$ for the GB phase, eliminating higher degrees *& postponing the need for the QPA*

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'Breaking '128-bit Secure' Supersingular Binary Curves (or how to solve discrete logarithms in $\mathbb{F}_{2^{4 \cdot 1223}}$ and $\mathbb{F}_{2^{12 \cdot 367}}$)'

Robert Granger, Thorsten Kleinjung and Jens Zumbrägel.
`eprint.iacr.org/2014/119`

Solving the DLP in $\mathbb{F}_{2^{12\cdot 367}}$

Over $\mathbb{F}_{2^{367}}$ the Jacobian of $H_0/\mathbb{F}_2 : Y^2 + Y = X^5 + X^3$ has a subgroup of prime order $r = (2^{734} + 2^{551} + 2^{367} + 2^{184} + 1)/(13 \cdot 7170258097)$.

- Let $\mathbb{F}_{2^{12}} = \mathbb{F}_2[U]/(U^{12} + U^3 + 1) = \mathbb{F}_2(u)$
- Let $\mathbb{F}_{2^{367}} = \mathbb{F}_2[X]/(I(X)) = \mathbb{F}_2(x)$ where $I(X)$ the irreducible degree 367 divisor of $h_1(X^{64})X - h_0(X^{64})$, with

$$h_1 = X^5 + X^3 + X + 1, \;\; h_0 = X^6 + X^4 + X^2 + X + 1$$

- $\mathbb{F}_{2^{12\cdot 367}}$ is then the compositum of $\mathbb{F}_{2^{12}}$ and $\mathbb{F}_{2^{367}}$
- We chose as our generator $g' = g^{(2^{4404}-1)/r}$ where $g = x + u^7$, and target element $x'_\pi = x_\pi^{(2^{4404}-1)/r}$ where

$$x_\pi = \sum_{i=0}^{4403} (\lfloor \pi \cdot 2^{i+1} \rfloor \bmod 2) \cdot u^{11-(i \bmod 12)} \cdot x^{\lfloor i/12 \rfloor}$$

Factor base logs and initial descent

We also represent $\mathbb{F}_{2^{12}}$ as $\mathbb{F}_{q^2}$ with $q = 2^6$ and $k = 2$:

- Let $\mathbb{F}_{2^6} = \mathbb{F}_2[U]/(T^6 + T + 1) = \mathbb{F}_2(t)$
- Let $\mathbb{F}_{2^{12}} = \mathbb{F}_{2^6}[V]/(V^2 + tV + 1) = \mathbb{F}_{2^6}(v)$

Since $q^{2k-3} \not> (6+1)!$ we consider relations over $\mathbb{F}_{q^4}$ instead:

- Let $\mathbb{F}_{2^{24}} = \mathbb{F}_{2^6}[W]/(W^4 + W^3 + W^2 + t^3) = \mathbb{F}_{2^6}(w)$

For the factor base $\{x + a \mid a \in \mathbb{F}_{2^{24}}\}$ we have:

$$(x+a)^{2^{367}} = x + a^{2^{367}} = x + a^{2^7}$$

$\Longrightarrow$ reduced factor base has $699,252$ elements and linear system was solved in 4896 core hours on a 24 core cluster.

Factor base logs and initial descent

We also represent $\mathbb{F}_{2^{12}}$ as $\mathbb{F}_{q^2}$ with $q = 2^6$ and $k = 2$:

- Let $\mathbb{F}_{2^6} = \mathbb{F}_2[U]/(T^6 + T + 1) = \mathbb{F}_2(t)$
- Let $\mathbb{F}_{2^{12}} = \mathbb{F}_{2^6}[V]/(V^2 + tV + 1) = \mathbb{F}_{2^6}(v)$

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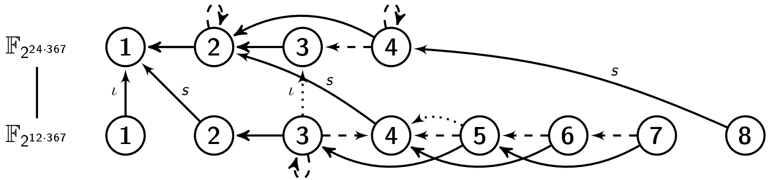
Initial descent: We performed a continued fraction initial split, then degree-balanced classical descent to degrees ≤ 8 in 38224 core hours.

Eliminating small degree elements in $\mathbb{F}_{2^{12\cdot 367}}/\mathbb{F}_{2^{12}}$



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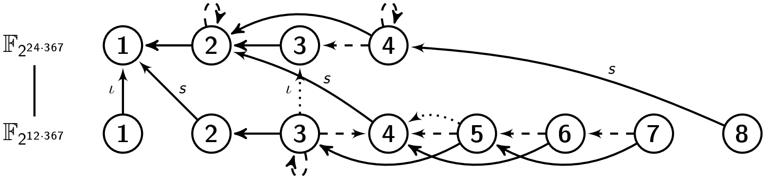

The GB phase cost 8432 core hours on Magma V2.20-1, for a total of approximately 52240 core hours. On 30/1/14 we announced that $x'_\pi = g'^{\,\log}$, with $\log =$

4093208920214235164093447733900702563725614097945142354192285387447360
4390153516847214082336876895639025110622309801452728710173825428267646
9559843114767895545475795766475848754227211594761182312814017076893242

Overview

Basics

Resisting smoothness heuristics

Breaking supersingular binary curves

Eliminating smoothness heuristics

Eliminating irreducibles of degree a power of 2 in $\mathbb{F}_{q^{kn}}/\mathbb{F}_{q^k}$

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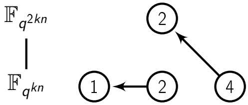

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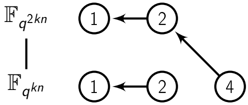

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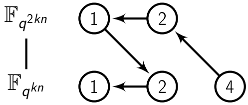

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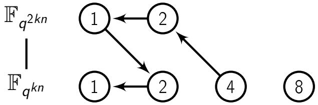

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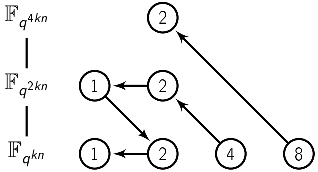

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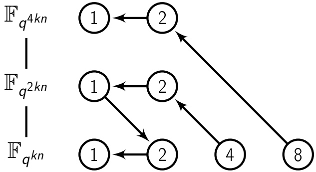

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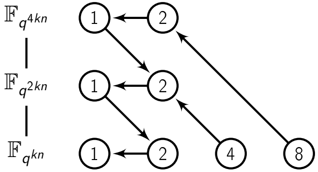

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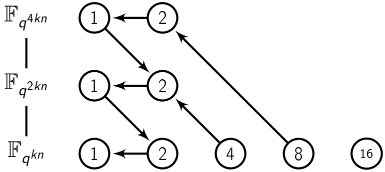

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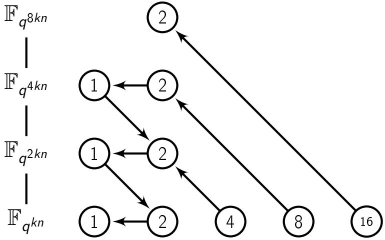

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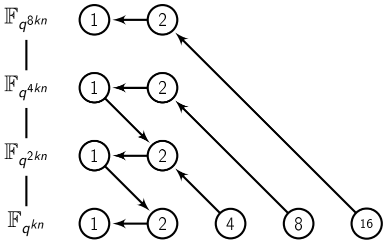

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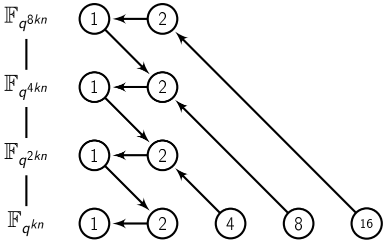

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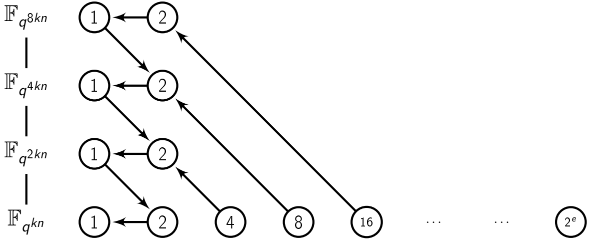

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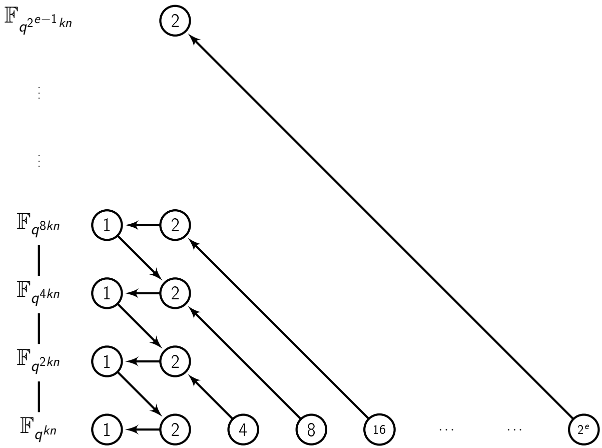

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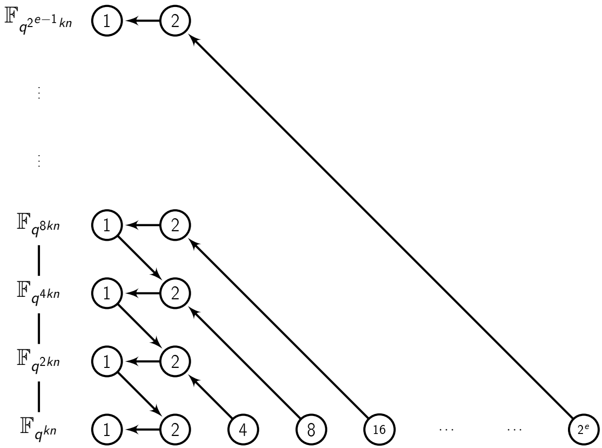

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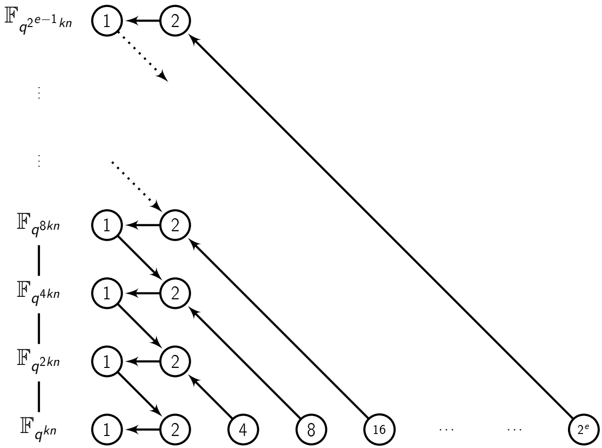

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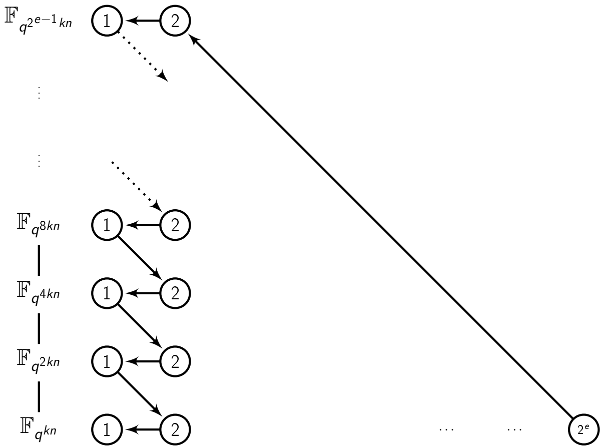

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Given a prime p and an integer n, for q the smallest power of p greater than n and for an integer $k = O(1)$, there exist polynomials $h_0, h_1 \in \mathbb{F}_{q^k}[X]$ of degree at most two s.t. $h_1(X^q)X - h_0(X^q)$ has an irreducible factor of degree n (or the equivalent for $h_1(X)X^q - h_0(X)$).

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Heuristic 2

There exists a polynomial time algorithm for obtaining the logarithms of polynomials of bounded degree using the parameters from Heuristic 1.

A new quasi-polynomial algorithm

Theorem (*G.-Kleinjung-Zumbrägel '14*)

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